COUNTING

2nd Edition

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World Scientific

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Preface

Combinatorics is a branch of mathematics dealing with discretely structured problems. Its scope of study includes selections and arrangements of objects with prescribed conditions, configurations involving a set of nodes interconnected by edges (called graphs), and designs of experimental schemes according to specified rules. Combinatorial problems and their applications can be found not only in various branches of mathematics, but also in other disciplines such as engineering, computer science, operational research, management sciences and the life sciences. Since computers require discrete formulation of problems, combinatorial techniques have become essential and powerful tools for engineers and applied scientists, in particular, in the area of designing and analyzing algorithms for various problems which range from designing the itineraries for a shipping company to sequencing the human genome in the life sciences.

The *counting problem,* which seeks to find out how many arrangements there are in a particular situation, is one of the basic problems in combinatorics. Counting is used in forensic science to calculate the probability that a sample of biological evidence found at the crime scene matches that of a particular accused person. In Chemistry, Cayley used graphs to count the number of isomers of saturated hydrocarbons, while Harary and Read counted the number of certain organic compounds built up from benzene rings by representing them as configurations of hexagons joined together along a common edge. In Genetics, by counting all possibilities for a DNA chain made up of the four bases, scientists arrive at an astoundingly large number and so are able to understand the tremendous possible variation in genetic makeup. Counting has been used as well to study the primary and secondary structures of RNA.

The second edition of *Counting* brings together the 13 chapters of *Counting* and the 7 chapters of *Counting: Supplementary Chapters and Solutions Manual.* The book is intended as an introduction to basic counting techniques for upper secondary to undergraduate students, and teachers. We believe that it would also be of interest to those who appreciate mathematics and to avid puzzle-solvers.

The variety of problems and applications in this book is not only useful for building up an aptitude in counting but is a rich source for honing basic skills and techniques in general problem-solving. Many of the problems evade routine and, as a desired result, force the reader to think hard in his attempts to solve them. In fact, the diligent reader will often discover more than one way of solving a particular problem, which is indeed an important awareness in problem-solving. This book thus helps to provide students an early start to learning problem-solving heuristics and thinking skills.

The first two chapters cover two basic principles, namely, the Addition Principle and the Multiplication Principle. Both principles are commonly used in counting, even by those who would never count themselves as students of mathematics! However, these principles have equally likely been misunderstood and misused. These chapters help to avoid this by stating clearly the conditions under which the principles can be applied. Chapter 3 introduces the concepts of combinations and permutations by viewing them as subsets and arrangements of a set of objects, while Chapter 4 provides various applications of the concepts learnt.

Many apparently complex counting problems can be solved with just “a change of perspective”. Chapter 5 presents an important principle along this line, i.e. the Bijection Principle; while Chapter 6 introduces a very useful perspective to which many counting problems can be converted to, i.e. the distribution of balls into boxes. The next three chapters flesh out the Bijection Principle and the distribution perspective with a number of applications and variations.

In Chapter 3, we introduce a family of numbers which are denoted by $\binom{n}{r}$ or C^n_r. Chapter 10 to Chapter 12 put this family of numbers in the context of the binomial expansion and Pascal’s Triangle.

A number of useful identities are proven and problems are posed where these identities surprisingly appear.

Chapter 13 and Chapter 14 cover the Principle of Inclusion and Exclusion (PIE), and its general statement. Many situations of counting are complicated by the possibility of double counting. Sometimes, however, while trying to make provisions for double counting, one may overcompensate and deduct more than the number which was double counted. PIE neatly handles this kind of situation.

The quaintly named Pigeonhole Principle is studied in Chapter 15. Unlike the other principles which we have discussed so far in this book, the Pigeonhole Principle does not actually count the number of ways for a particular situation. Instead, the Pigeonhole Principle is used to check the existence of a particular situation. This aspect of "checking existence" together with "counting" and, as yet not discussed in our book, "optimization" are the three main focus areas in Combinatorics. In the Pigeonhole Principle, we try to transform the problem partly into one of distributing a number of objects into a number of boxes. The questions to focus on then become "What are the objects?" and "What are the boxes?"

The next four chapters are on recurrence relations. Some counting problems defy the techniques and principles learnt thus far. Chapter 16 introduces the technique of using recurrence relations. These recurrence relations represent algebraically the situation where the solution of a counting problem of bigger size can be obtained from the solutions of the same problem but of smaller size. It is by writing the recurrence relations and obtaining a number series from them, that the original counting problem can be solved. Chapter 17 to Chapter 19 study three series of numbers which are derived from special recurrence relations. These are the Stirling numbers of the first kind, the Stirling numbers of the second kind and Catalan numbers.

Chapter 20 closes this book with a collection of interesting problems in which the approaches to solving them appear as applications of one or more concepts learnt in all the earlier chapters. Problems in this and other chapters marked with (AIME) are from the American Invitational Mathematics Examination. We would like to express our gratitude to Mathematical Association of America for allowing us to include these problems in the book.

This book is based on a series of articles on counting that first appeared in Mathematical Medley, a publication of the Singapore Mathematical Society. We would like to thank Tan Ban Pin who greatly helped the first author with the original series. Many thanks also to our colleagues, Dong Fengming, Lee Tuo Yeong, Toh Tin Lam and Katherine Goh for reading through the draft and checking through the problems — any mistakes that remain are ours alone.

For those who find this introductory work interesting and would like to know more about the subject, a recommended list of publications for further reading is provided at the end of this book.

Koh Khee Meng
Tay Eng Guan
September 2012

Contents

$\mathbb{N}$	=	$\{1, 2, 3, ...\}$
$\mathbb{N}_k$	=	$\{1, 2, 3, ..., k\}$
(AP)	:	Addition Principle
(MP)	:	Multiplication Principle
(CP)	:	Complementation Principle
(IP)	:	Injection Principle
(BP)	:	Bijection Principle
(PP)	:	Pigeonhole Principle
(GPP)	:	Generalised Pigeonhole Principle
(PIE)	:	Principle of Inclusion and Exclusion
(GPIE)	:	Generalised Principle of Inclusion and Exclusion
(BT)	:	Binomial Theorem
LHS	:	Left-hand side
RHS	:	Right-hand side
$\lvert S \rvert$	=	the number of elements in the set S
$\lfloor x \rfloor$	=	the largest integer less than or equal to x
$\lceil x \rceil$	=	the largest integer less than or equal to x
FTA	:	Fundamental Theorem of Arithmetic
gcd	:	greatest common divisor
B_n	=	Bell number
	=	the number of ways of dividing n distinct objects into (nonempty) groups
$C_{(n)}$	=	Catalannumber
	=	the number of shortest routes from $O(0, 0)$ to $A(n, n)$ which do not cross the diagonal $y = x$ in the rectangular coordinate system
D_n	=	the number of derangements of $\mathbb{N}_n$
$s^*(m, k)$	=	the number of ways of arranging m distinct objects around k identical circles such that each circle has at least one object
$s(m, k)$	=	Stirling number of the first kind
	=	the coefficient of xk in the expansion of $x(x-1)(x-2) \dots (x-(m-1))$
$S(m, k)$	=	Stirling number of the second kind
	=	the number of ways of distributing m distinct objects into k identical boxes such that no box is empty
$C^n_r = \binom{n}{r}$	=	the number of r-element subsets of an n-element set
$P^n{}_r$	=	the number of r-permutations of n distinct objects
AIME	:	American Invitational Mathematics Examination

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Chapter 1

The Addition Principle

In the process of solving a counting problem, there are two very simple but basic principles that we always apply. They are called the *Addition Principle* and the *Multiplication Principle.* In this chapter, we shall introduce the former and illustrate how it is applied.

Let us begin with a simple problem. Consider a 4-element set $A = \{a, b, c, d\}$. In how many ways can we form a 2-element subset of *A*? This can be answered easily by simply listing all the 2-element subsets:

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}.$$

Thus, the answer is 6.

Let us try a slightly more complicated problem.

Example 1.1 *A group of students consists of* 4 *boys and* 3 *girls. How many ways are there to select* 2 *students of the same sex from the group* ?

Solution As the problem requires us to select students of the *same sex*, we naturally divide our consideration into two distinct cases: both of the two students are boys, or, both are girls. For the former case, this is the same as selecting 2 elements from a 4-element set. Thus, as shown in the preceding discussion, there are 6 ways. For the latter case, assume the 3 girls are g_1, g_2 and g_3. Then there are 3 ways to form such a pair; namely,

$$\{g_1, g_2\}, \{g_1, g_3\}, \{g_2, g_3\}.$$

Thus, the desired number of ways is (6 + 3), which is 9.

In dealing with counting problems that are not so straightforward, we quite often have to divide our consideration into cases which are *disjoint* (like boys or girls in Example 1.1) and *exhaustive* (besides boys or girls, no other cases remain). Then the total number of ways would be the sum of the numbers of ways from each case. More precisely, we have:

> **The Addition Principle**
> Suppose that there are n_1 ways for the event E_1 to occur and n_2 ways for the event E_2 to occur. If all these ways are distinct, then the number of ways for E_1 or E_2 to occur is $n_1 + n_2$. (1.1)

For a finite set *A*, the *size* of *A* or the *cardinality* of *A,* denoted by |*A*|, is the number of elements in *A*. For instance, if $A = \{u, v, w, x, y, z\}$, then $|A| = 6$; if *A* is the set of all the letters in the English alphabet, then $|A| = 26$; if ø denotes the *empty* (or *null*) set, then $|\emptyset| = 0$.

Using the language of sets, the Addition Principle simply states the following.

> If A and B are finite sets with $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$. (1.2)

Two sets *A* and *B* are *disjoint* if $A \cap B = \emptyset$. Clearly, the above result can be extended in a natural way to any finite number of pairwise disjoint finite sets as given below.

> (AP) If $A_1, A_2, \ldots, A_n, n \geq 2$, are finite sets which are pairwise disjoint, i.e. $A_i \cap A_j = \emptyset$ for all i, j with $1 \leq i < j \leq n$, then
>
> $$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|, \qquad (1.3)$$
>
> or, in a more concise form:
>
> $$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|.$$

Example 1.2 *From town X to town Y, one can travel by air, land or sea. There are* 3 *different ways* *by air,* 4 *different ways by land and* 2 *different ways by sea as shown in Figure* 1.1.

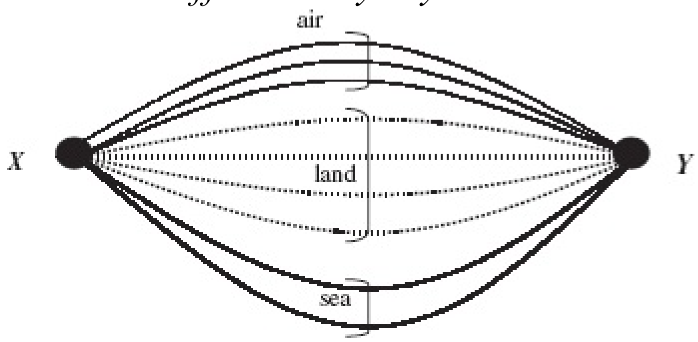


Figure 1.1

How many vjays are there from X to Y?

Let A_1 be the set of ways by air, A_2 the set of ways by land and A_3 the set of ways by sea from X to Y. We are given that

$$|A_1| = 3, \quad |A_2| = 4 \quad \text{and} \quad |A_3| = 2\,.$$

Note that $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \emptyset$ and $A_1 \cup A_2 \cup A_3$ is the set of ways from X to Y. Thus, the required number of ways is $|A_1 \cup A_2 \cup A_3|$, which, by (AP), is equal to

$$|A_1| + |A_1| + |A_1| = 3 + 4 + 2 = 9\,.$$

Example 1.3 *Find the number of squares contained in the* 4 x 4 *array (where each cell is a square) of Figure* 1.2.

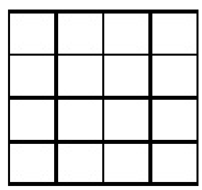


Figure 1.2

Solution The squares in the array can be divided into the following 4 sets:

A_1: the set of 1 × 1 squares,

A_2: the set of 2 × 2 squares,

A_3: the set of 3 × 3 squares, and

A_4: the set of 4 × 4 squares.

There are 16 “1 × 1 squares”. Thus $|A_1| = 16$. There are nine “2 × 2 squares”. Thus $|A_2| = 9$. Likewise, $|A_3| = 4$ and $|A_4| = 1$.

Clearly, the sets A_1, A_2, A_3, A_4 are pairwise disjoint and $A_1 \cup A_2 \cup A_3 \cup A_4$ is the set of the squares contained in the array of Figure 1.2. Thus, by (AP), the desired number of squares is given by

$$\left|\bigcup_{i=1}^{4} A_i\right| = \sum_{i=1}^{4} |A_i| = 16 + 9 + 4 + 1 = 30\,.$$

Example 1.4 *Find the number of routes from X to Y in the oneway system shown in Figure* 1.3.

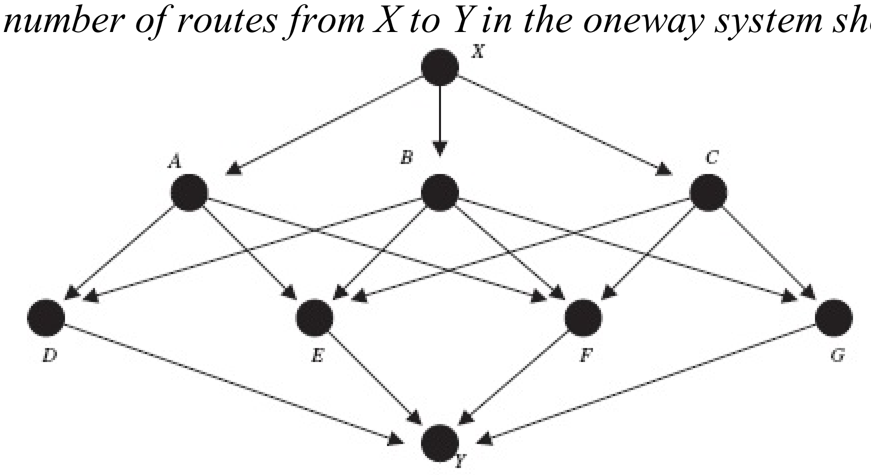


Figure 1.3

Solution Of course, one can count the number of such routes by simply listing all of them: $X \to A \to D \to Y$, $X \to A \to E \to Y, \ldots, X \to C \to G \to Y$.

Let us, however, see how to apply (AP) to introduce a more general method.

Call a route from X to Y an *X–Y route*. It is obvious that just before reaching Y along any X-Y route, one has to reach D, E, F or G. Thus, by (AP), the number of X-Y routes is the sum of the numbers of X-D routes, X-E routes, X-F routes and X-G routes.

How many X-D routes are there? Just before reaching D along any X-D route, one has to reach either A or B, and thus, by (AP), the number of X-D routes is the sum of the numbers of X-Aroutes and X-B routes. The same argument applies to others (X-E routes,...) as well.

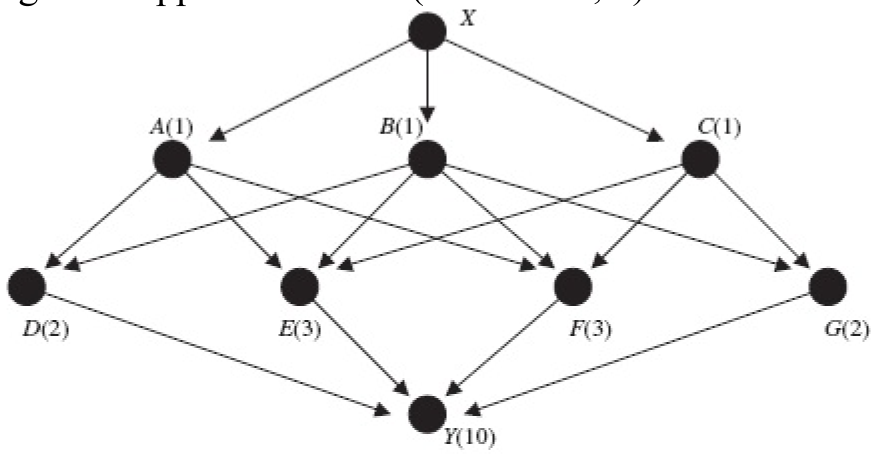
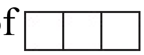
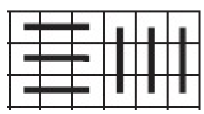


Figure 1.4

It is clear that the number of X-A routes (X-B routes and X-C routes) is 1. With these initial values, one can compute the numbers of X-D routes, X-E routes, etc., using (AP) as explained above. These are shown in brackets at the respective vertices in Figure 1.4. Thus, we see that the total number of possible X-Y routes is $2 + 3 + 3 + 2$, i.e. 10.

Exercise

1.1 We can use 6 pieces of  to cover a 6×3 rectangle, for example, as shown below:

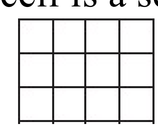


In how many different ways can the 6×3 rectangle be so covered?

1.2 Do the same problem as in Example 1.3 for $1 \times 1, 2 \times 2$, 3×3 and 5×5 square arrays. Observe the pattern of your results. Find, in general, the number of squares contained in an $n \times n$ square array, where $n \geq 2$.

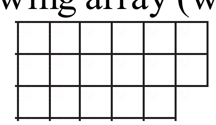
1.3 How many squares are there in

(i) the following 4×3 array (where each cell is a square)?

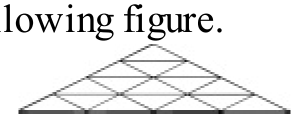


(ii) an $n \times 3$ array (where each cell is a square), with $n \geq 5$?

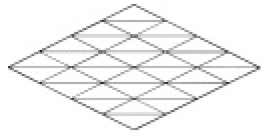
1.4 How many squares are there in the following array (where each cell is a square)?



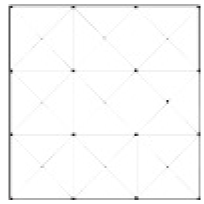
1.5 Find the number of triangles in the following figure.



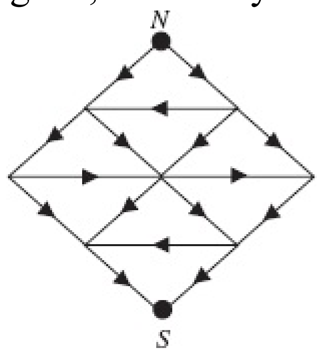
1.6 Find the number of triangles in the following figure.



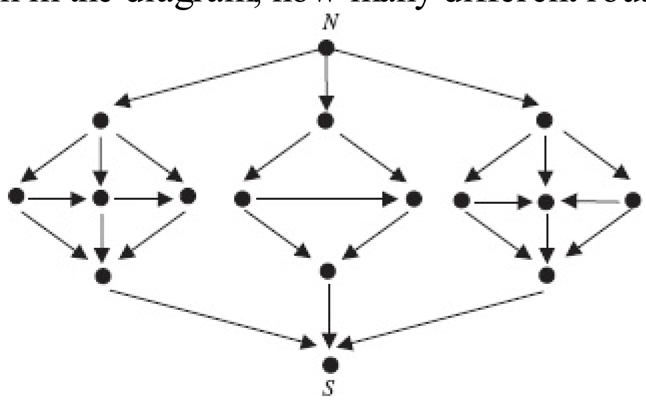
1.7 How many squares are there in the following configuration (where each cell is a square with diagonals)?



1.8 Following the arrows given in the diagram, how many different routes are there from N to S?



1.9 Following the arrows given in the diagram, how many different routes are there from N to S?



Chapter 2

The Multiplication Principle

Mr. Tan is now in town X and ready to leave for town Z by car. But before he can reach town Z, he has to pass through town Y. There are 4 roads (labeled 1, 2, 3, 4) linking X and Y, and 3 roads (labelled as *a, b, c)* linking Y and Z as shown in Figure 2.1. How many ways are there for him to drive from X to Z?

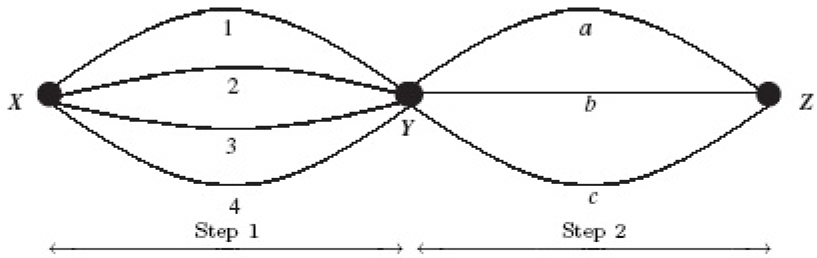


Figure 2.1

Mr. Tan may choose road "1" to leave X for Y, and then select "a" from Y to Z. For simplicity, we denote such a sequence by $(1,a)$. Thus, by listing all possible sequences as shown below:

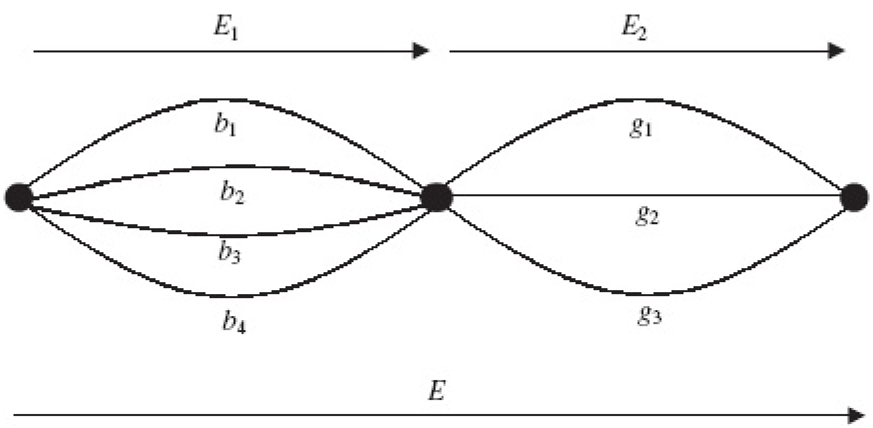
$$(1, a), (1, b), (1, c),$$
$$(2, a), (2, b), (2, c),$$
$$(3, a), (3, b), (3, c),$$
$$(4, a), (4, b), (4, c),$$

we get the answer $(4 \times 3 =)$ 12.

Very often, to accomplish a task, one may have to split it into ordered stages and then complete the stages step by step. In the above example, to leave X and reach Z, Mr. Tan has to split his journey into 2 stages: first from X to Y, and then Y to Z. There are 4 roads to choose from in Step 1: To each of these 4 choices, there are 3 choices in Step 2. Note that the number of choices in Step 2 is independent of the choice in Step 1. Thus, the number of ways from X to Z is given by 4×3 (= 12). This illustrates the meaning of the following principle.

The Multiplication Principle Suppose that an event E can be split into two events E_1 and E_2 in ordered stages. If there are n_1 ways for E_1 to occur and n_2 ways for E_2 to occur, then the number of ways for the event E to occur is $n_1 n_2$.	(2.1)

Example 2.1 *How many ways are there to select* 2 *students of different sex from a group of* 4 *boys and* 3 *girls* ?



E : forming a pair consisting of a boy and a girl;

E_1: selecting a boy;

E_2: selecting a girl.

Figure 2.2

Solution The situation when the 2 students chosen are of the same sex was discussed in Example 1.1. We now consider the case where the 2 students chosen are of different sex. To choose 2 such students, we may first choose a boy and then select a girl. There are 4 ways for Step 1 and 3 ways for Step 2 (see Figure 2.2). Thus, by the Multiplication Principle, the answer is 4 × 3 (= 12).

The Addition Principle can be expressed using set language. The Multiplication Principle can likewise be so expressed. For the former, we make use of the *union* $A \cup B$ of sets A and B. For the latter, we shall introduce the *cartesian product* $A \times B$ of sets A and B. Thus given two sets A and B, let

$$A \times B = \{(x, y) \mid x \in A,\ y \in B\};$$

namely, $A \times B$ consists of all *ordered pairs* (x,y), where the first coordinate, "x", is any member in the first set A, and the second coordinate, "y", is any member in the second set B. For instance, if $A = \{1,2,3,4\}$ and $B = \{a, b, c\}$, then

$$A \times B = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),$$
$$(3,c),(4,a),(4,b),(4,c)\}.$$

Assume that A and B are finite sets. How many members (i.e. ordered pairs) are there in the set $A \times B$? In forming ordered pairs in $A \times$ B, a member, say "x" in A is paired up with every member in B. Thus there are $|B|$ ordered pairs having "x" as the first coordinate. Since there are |A| members in A, altogether we have $|A|\,|B|$ ordered pairs in $A \times B$. That is,

$$\boxed{|A \times B| = |A|\,|B|.} \qquad (2.2)$$

Principle (2.1) and result (2.2) are two different forms of the same fact. Indeed, an event E which is split into two events in ordered stages can be regarded as an ordered pair (a, b), where "a" stands for the first event and "b" the second; and vice versa.

Likewise, Principle (2.1) can be extended in a very natural way as follows:

(MP) Suppose that an event E can be split into k events $E_1, E_2, \ldots, E_k$ in ordered stages. If there are n_1 ways for E_1 to occur, n_2 ways for E_2 to occur, ..., and n_k ways for E_k to occur, then the number of ways for the event E to occur is given by $n_1 n_2 \cdots n_k$.	(2.3)

By extending the cartesian product $A \times B$ of two sets to $A_1 \times A_2 \times \ldots \times A_k$ of k sets, we shall also derive an identity which extends (2.2) and expresses (2.3) using set language.

Let $A_1, A_2, \ldots, A_k$ be k finite sets, and let

$$A_1 \times A_2 \times \cdots \times A_k$$
$$= \{(x_1, x_2, \ldots, x_k) \mid x_i \in A_i \quad \text{for each } i = 1, 2, \ldots, k\}.$$

Then

$$\boxed{\text{(MP)} \quad |A_1 \times A_2 \times \cdots \times A_k| = |A_1|\,|A_2| \ldots |A_k|.} \qquad (2.4)$$

Example 2.2 *There are four 2-digit binary sequences:* 00,01,10,11. *There are eight 3-digit binary sequences:* 000,001,010,011,100,101, 110,111. *How many 6-digit binary sequences can we form*?

Solution The event of forming a 6-digit binary sequence can be split into 6 ordered stages as shown in Figure 2.3.

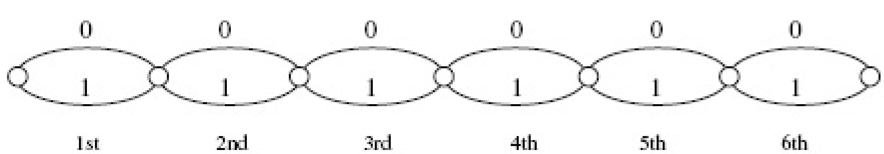


Figure 2.3

Thus, by (2.3), the desired number of sequences is $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$.

Using set language, the same problem can be treated as follows. We have

$$A_1 = A_2 = \cdots = A_6 = \{0, 1\}.$$

The members in $A_1 \times A_2 \times \ldots \times A_6$ can be identified with 6-digit binary sequences in the following way:

$$(1,1,0,1,0,1) \leftrightarrow 110101,$$
$$(0,1,1,0,1,1) \leftrightarrow 011011,$$
$$\text{etc.}$$

Thus, the number of 6-digit binary sequences is given by $| A_1 \times A_2 \times \ldots \times A_6 |$, which, by (2.4), is equal to

$$|A_1||A_2|\ldots|A_6| = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6.$$

From now on, (MP) shall refer to Principle (2.3) or the identity (2.4).

Example 2.3 *Figure 2.4 shows 9 fixed points a,b,c,..., i which are located on the sides of ΔABC. If we select one such point from each side and join the selected points to form a triangle, how many such triangles can be formed*?

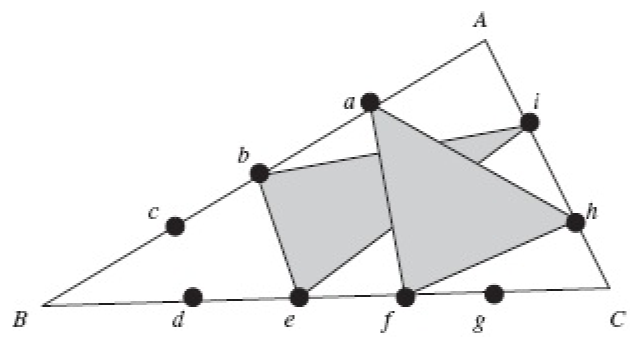


Figure 2.4

Solution To form such a triangle, we first select a point on *AB*, then a point on *BC* and finally a point on CA. There are 3 ways in Step 1 (one of *a, b,c),* 4 ways in Step 2 (one of *d, e, f, g)* and 2 ways in Step 3 (either *h* or i). Thus by (MP), there are $3 \times 4 \times 2$ (= 24) such triangles.

We have seen in both the preceding and the current chapters some problems that can be solved by applying (AP) or (MP) *individually*. Indeed, more often than not, problems that we encounter are more 14 complex and these require that we apply the principles together. The following is an example.

Example 2.4 (Continuation of Example 2.3) *Find the number of triangles that can be formed using the 9 fixed points of Figure 2.4 as vertices*.

Solution This problem is clearly more complex than that of Example 2.3 as there are other triangles whose three vertices are not necessarily chosen from three different sides; but then, where else can they be chosen from? The answer is: two from one side and one from the remaining two sides. In view of this, we shall now classify the required triangles into the following two types.

Type 1 — *Triangles whose three vertices are chosen from three different sides*.

As shown in Example 2.3, there are $3 \times 4 \times 2$ (= 24) such triangles.

Type 2 — *Triangles having two vertices from one side and one from the other two sides*.

We further split our consideration into three subcases.

(i) *Two vertices from AB and one from BC or CA.*

There are 3 ways to choose two from *AB* (namely, $\{a,b\}$, $\{a,c\}$ or $\{b,c\}$) and 6 ways to choose one from the other sides (namely, *d, e, f, g, h, i*). Thus, by (MP), there are 3×6 (= 18) such triangles.

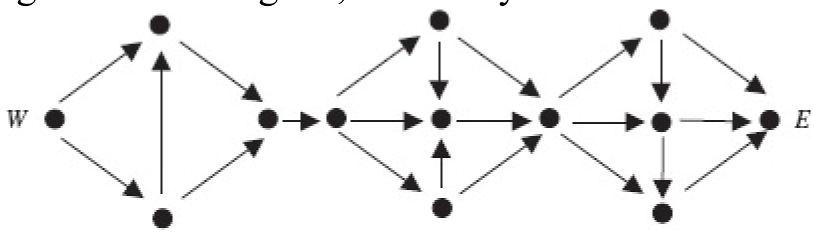
(ii) *Two vertices from BC and one from CA or AB.*
There are 6 ways to choose two from *BC* (why?) and 5 ways to choose one from the other sides (why?). Thus, by (MP), there are 6 × 5 (= 30) such triangles.
(iii) *Two vertices from CA and one from AB or BC.*
There is only one way to choose two from *CA* and there are 7 ways to choose one from the other sides. Thus, by (MP), there are 1 × 7 (= 7) such triangles.

Summing up the above discussion, we conclude that by (AP), there are 18 + 30 + 7 (= 55) triangles of Type 2.

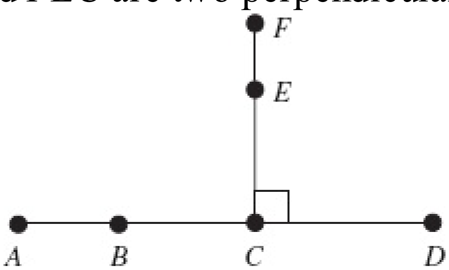
As there are 24 triangles of Type 1 and 55 triangles of Type 2, the required number of triangles is thus, by (AP), 24 + 55 (= 79).

Exercise

2.1 Following the arrows given in the diagram, how many different routes are there from *W* to *E*?



2.2 In the following figure, *ABCD* and *FEC* are two perpendicular lines.



(i) Find the number of right-angled triangles *AXCY* that can be formed with *X*, *Y* taken from *A, B, D, E, F*.

(ii) Find the number of triangles that can be formed with any three points *A, B, C, D, E, F* as vertices.

2.3 There are 2 distinct terms in the expansion of $a(p + q)$:

$$a(p+q) = ap + aq.$$

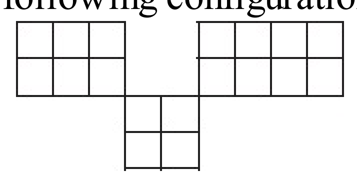
There are 4 distinct terms in the expansion of $(a + b)(p + q)$:

$$(a+b)(p+q) = ap + aq + bp + bq.$$

How many distinct terms are there in each of the expansions of

(i) $(a+b+c+d)(p+q+r+s+t)$,
(ii) $(x_1+x_2+\cdots+x_m)(y_1+y_2+\cdots+y_n)$, and
(iii) $(x_1+x_2+\cdots+x_m)(y_1+y_2+\cdots+y_n)(z_1+z_2+\cdots+z_t)$?

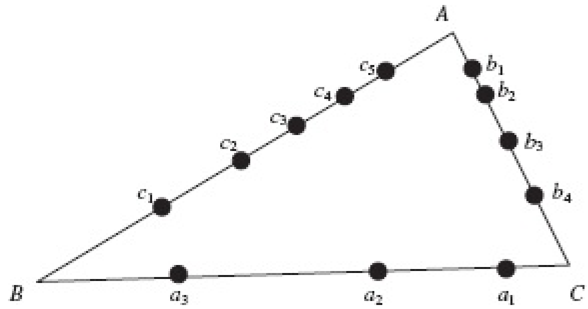
2.4 In how many different ways can the following configuration be covered by nine 2 × 1 rectangles?



2.5 A ternary sequence is a sequence formed by 0, 1 and 2. Let *n* be a positive integer. Find the number of n-digit ternary sequences

(i) with no restrictions;
(ii) which contain no "0"
(iii) which contain at most one "0"
(iv) which contain at most one "0" and at most one "1".

2.6 The following diagram shows 12 distinct points: $a_1, a_2, a_3, b_1, \ldots, b_4, c_1, \ldots, c_5$ chosen from the sides of ΔABC.



(i) How many line segments are there joining any two points, each point being from a different side of the triangle?

(ii) How many triangles can be formed from these points?

(iii) How many quadrilaterals can be formed from these points?

Chapter 3

Subsets and Arrangements

There are 25 students in the class. How many ways are there to choose 5 of them to form a committee? If among the chosen five, one is to be the chairperson, one the secretary and one the treasurer, in how many ways can this be arranged? In this chapter, our attention will be focused on the counting problems of the above types. We shall see how (MP) is used to solve such problems, and how (MP), by incorporating (AP), enables us to solve more complicated problems.

From now on, for each natural number *n*, we shall denote by $\mathbb{N}_n$ the set of natural numbers from 1 to *n* inclusive, i.e.

$$\mathbb{N}_n = \{1, 2, 3, \dots, n\}.$$

Consider the 4-element set $\mathbb{N}_4 = \{1, 2, 3, 4\}$. How many subsets of $\mathbb{N}_4$ are there? This question can be answered readily by listing all the subsets of $\mathbb{N}_4$. Table 3.1 lists all the subsets according to the number of elements they possess: It is now easy to count the total number of subsets of $\mathbb{N}_4$ $(= 16)$.

Table 3.1

Number of elements	Subsets of $\mathbb{N}_4$	Number of subsets of $\mathbb{N}_4$
0	$\emptyset$	1
1	$\{1\}, \{2\}, \{3\}, \{4\}$	4
2	$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$	6
3	$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$	4
4	$\{1, 2, 3, 4\}$	1

We note that the 5 numbers, namely, 1, 4, 6, 4, 1 (whose sum is 16) shown in the rightmost column of Table 3.1 are the corresponding numbers of r-element subsets of $\mathbb{N}_4$, where r = 0,1,2,3,4. These numbers are very interesting, useful and important in mathematics, and mathematicians have introduced special symbols to denote them.

In general, given two integers *n* and *r* with $0 \le r \le n$, we denote by $\binom{n}{r}$ the number of r-element subsets of $\mathbb{N}_n$. Thus, Table 3.1 tells us that

$$\binom{4}{0} = 1, \quad \binom{4}{1} = 4, \quad \binom{4}{2} = 6, \quad \binom{4}{3} = 4, \quad \binom{4}{4} = 1.$$

The symbol $\binom{n}{r}$ is read "n choose r". Some other symbols for this quantity include C_r^n and ${}^nC_r$.

Now, what is the value of $\binom{5}{2}$? Since $\binom{5}{2}$ counts, by definition, the number of 2-element subsets of $\mathbb{N}_5$, we may list all these subsets as shown below:

$$\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\},$$

and see that there are 10 in total. Thus, we have $\binom{5}{2} = 10$.

You may ask: How about $\binom{100}{6}$? We are sure that we are too busy to have time to compute $\binom{100}{6}$ by listing all the 6-element subsets of $\mathbb{N}_{100}$. Thus, a natural question arises: Is there a more efficient way to compute $\binom{100}{6}$? The answer is "Yes", and we are going to show you.

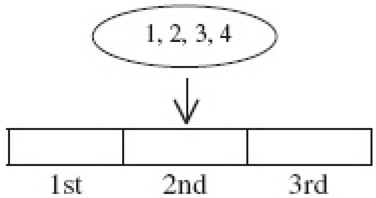
Let us first consider a different but related problem. How many ways are there to arrange any three elements of $\mathbb{N}_4 = \{1, 2, 3, 4\}$. in a row? With a little patience, we can list all the required arrangements as shown in Table 3.2.

Table 3.2

123	132	213	231	312	321
124	142	214	241	412	421
134	143	314	341	413	431
234	243	324	342	423	432

Thus, there are 24 ways to do so. The answer is "correct" but the method is "naive". Is there a cleverer way to get the answer?

Imagine that we wish to choose 3 numbers from $\mathbb{N}_4$ and put them one by one into 3 spaces as shown.



This event can be thought of as a sequence of events: We first select a number from $\mathbb{N}_4$ and place it in the 1st space; we then select another number and place it in the 2nd space; finally, we select another number and place it in the 3rd space. There are 4 choices for the first step, 3 choices (why?) for the second and 2 choices (why?) for the last. Thus, by (MP), there are 4·3·2 (=24) ways to do so. The answer agrees with what we obtained above. Isn't this method better?

This method is better not only in shortening our solution, but also in giving us an idea on how to generalise the above result.

In the above example, we considered the number of ways of arranging 3 elements of $\mathbb{N}_4$ in a row. We now ask a general question: Given integers r, *n* with $0 \le r \le n$, how many ways are there to arrange any *r* elements of $\mathbb{N}_n$ in a row?

Consider the *r* spaces shown in Figure 3.1.

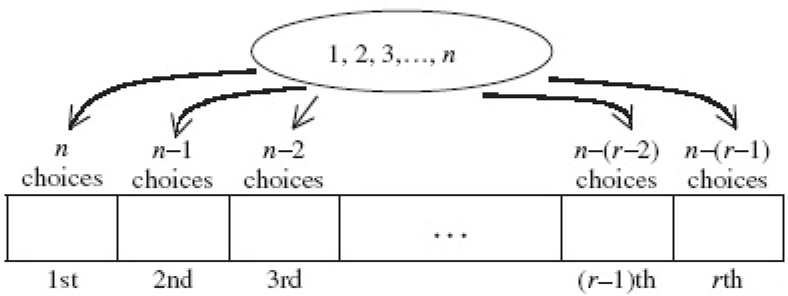


Figure 3.1

We wish to choose *r* elements from {1,…, *n*} to fill the *r* spaces, where the ordering of elements matters. There are *n* choices for the 1st space. After fixing one in the 1st space, there are *n* – 1 choices remaining for the 2nd space. After fixing one in the 2nd space, there are *n* – 2 choices left for the 3rd space, and so on. After fixing one in the (*r* – 1)th space, there are *n* – *(r* – 1) choices left for the rth space. Thus, by (MP), the number of ways to arrange any *r* elements from $\mathbb{N}_n$ in a row is given by

$$n(n-1)(n-2)\cdots(n-r+1).$$

For convenience, let us call an arrangement of any *r* elements from $\mathbb{N}_n$ in a row, an *r-permutation* of $\mathbb{N}_n$, and denote by P_r^n the number of *r*-permutations of $\mathbb{N}_n$. Thus, we have

$$\boxed{P_r^n = n(n-1)(n-2)\cdots(n-r+1).} \qquad (3.1)$$

For instance, all the arrangements in Table 3.2 are 3-permutations of $\mathbb{N}_4$, and, by (3.1), the number of 3-permutations of $\mathbb{N}_4$ is given by

$$P_3^4 = 4 \cdot 3 \cdot 2 = 24,$$

which agrees with what we have counted in Table 3.2.

The expression (3.1) for *Py* looks a bit long. We shall make it more concise by introducing the following useful notation. Given a positive integer n, define n! to be the product of the *n* consecutive integers *n*, *n* – 1,…, 3,2,1; that is,

$$\boxed{n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1.} \qquad (3.2)$$

Thus $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. The symbol "$n!$" is read "$n$ factorial". By convention, we define $0! = 1$.

Using the "factorial" notation, we now have

$$\begin{aligned} P_r^n &= n(n-1)\cdots(n-r+1) \\ &= \frac{n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots 2\cdot 1}{(n-r)(n-r-1)\cdots 2\cdot 1} = \frac{n!}{(n-r)!}. \end{aligned}$$

That is,

$$\boxed{P_r^n = \frac{n!}{(n-r)!}.} \tag{3.3}$$

When $n = 4$ and $r = 3$, we obtain

$$P_3^4 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = \frac{4\cdot 3\cdot 2\cdot 1}{1} = 4\cdot 3\cdot 2 = 24,$$

which agrees with what we found before.

The expression (3.3) is valid when $0 \le r \le n.$ Consider two extreme cases: when $r = 0$ and $r = n$ respectively. When $r = 0$, by (3.3),

$$P_0^n = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1.$$

(How can this be explained?) When $r = n$, an n-permutation of $\mathbb{N}_n$ is simply called a *permutation* of $\mathbb{N}_n$. Thus, by (3.3) and that $0! = 1$, the number of permutations of $\mathbb{N}_n$ is given by

$$P_n^n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!,$$

i.e.

$$\boxed{P_n^n = n!.} \tag{3.4}$$

Thus, for example, P_5^5 counts the number of permutations of $\mathbb{N}_5$, and we have, by (3.4), $P_5^5 = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120.$

Let us now return to the problem of evaluating the quantity $\binom{n}{r}$.

We know from (3.3) that the number *Py* of r-permutations of $\mathbb{N}_n$ is given by $\frac{n!}{(n-r)!}$. We shall now count this number (namely, the number of r-permutations of $\mathbb{N}_n$) in a different way.

To get an r-permutation of $\mathbb{N}_n$, we may proceed in the following manner: first select an r-element subset of $\mathbb{N}_n$, and then arrange the chosen r elements in a row. The number of ways for the first step is, by definition, $\binom{n}{r}$, while that for the second step is, by (3.4), $r!$. Thus, by (MP), we have

$$P_r^n = \binom{n}{r}\cdot r!.$$

As

$$P_r^n = \frac{n!}{(n-r)!},$$

we have

$$\binom{n}{r}\cdot r! = \frac{n!}{(n-r)!},$$

and thus

$$\boxed{\binom{n}{r} = \frac{n!}{r!(n-r)!}.} \tag{3.5}$$

For instance,

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = 10, \quad \text{while}$$

$$\binom{100}{6} = \frac{100!}{6!94!} = 1192052400.$$

Note that when $r = 0$ or n, we have

$$\binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{n} = 1.$$

Again, by convention, we define

$$\binom{0}{0} = 1.$$

By applying (3.5), one can show that the following identity holds (see Problem 3.1):

$$\boxed{\binom{n}{r} = \binom{n}{n-r}.} \tag{3.6}$$

Thus, $\binom{10}{8} = \binom{10}{2} = 45$ and $\binom{100}{94} = \binom{100}{6} = 1192052400$.

We define P_r^n as the number of r-permutations and $\binom{n}{r}$ as the number of r-element subsets of $\mathbb{N}_n$. Actually, in these definitions, $\mathbb{N}_n$ can be replaced by any n-element set since it is the number of the elements but not the nature of the elements in the set that matters. That is, given any n-element set S, P_r^n (respectively, $\binom{n}{r}$) is defined as the number of r-permutations (respectively, r-element subsets) of S. Any r-element subset of S is also called an *r-combination* of S.

We have introduced the notions of r-permutations (or permutations) and r-combinations (or combinations) of a set S. Always remember that these two notions are closely related but different. While a "combination" of S is just a *subset* of S (and thus the ordering of elements is immaterial), a "permutation" of S is an *arrangement* of certain elements of S (and thus the ordering of elements is important).

Exercise

3.1 Show that for non-negative integers r and n, with $r \le n$,

(i) $\binom{n}{r} = \binom{n}{n-r}$;

(ii) $r\binom{n}{r} = n\binom{n-1}{r-1}$, where $r \ge 1$;

(iii) $(n-r)\binom{n}{r} = n\binom{n-1}{r}$;

(iv) $r\binom{n}{r} = (n-r+1)\binom{n}{r-1}$, where $r \ge 1$.

3.2 Show that for $1 \le r \le n$,

(i) $P_r^{n+1} = P_r^n + rP_{r-1}^n$;

(ii) $P_r^{n+1} = r! + r(P_{r-1}^n + P_{r-1}^{n-1} + \cdots + P_{r-1}^r)$;

(iii) $(n-r)P_r^n = nP_r^{n-1}$;

(iv) $P_r^n = (n-r+1)P_{r-1}^n$;

(v) $P_r^n = nP_{r-1}^{n-1}$.

3.3 Prove that the product of any n consecutive integers is divisible by $n!$.

3.4 Find the sum

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!.$$

Chapter 4

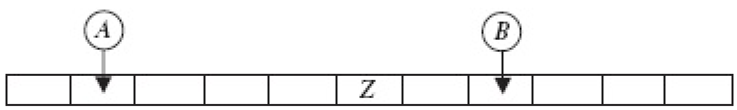
Applications

Having introduced the concepts of r-permutations and r-combinations of an n-element set, and having derived the formulae for P_r^n and $\binom{n}{r}$, we shall now give some examples to illustrate how these can be applied.

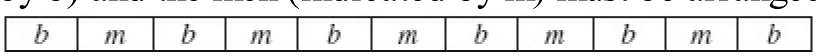
Example 4.1 *There are* 6 *boys and* 5 *men waiting for their turn in a barber shop. Two particular boys are A and B, and one particular man is Z. There is a row of* 11 *seats for the customers. Find the number of ways of arranging them in each of the following cases*:

(i) *there are no restrictions*;
(ii) *A and B are adjacent*;
(iii) *Z is at the centre, A at his left and B at his right (need not be adjacent)*;
(iv) *boys and men alternate.*

Solution (i) This is the number of permutations of the 11 persons. The answer is 11!.
(ii) Treat $\{A, B\}$ as a single entity. The number of ways to arrange the remaining 9 persons together with this entity is $(9 + 1)!$. But A and B can permute themselves in 2 ways. Thus the total desired number of ways is, by (MP), $2 \cdot 10!$·
(iii)



As shown in the diagram above, A has 5 choices and B has 5 choices as well. The remaining 8 persons can be placed in 8! ways. By (MP), the total desired number of ways is $5 \cdot 5 \cdot 8!$.
(iv) The boys (indicated by b) and the men (indicated by m) must be arranged as shown below.



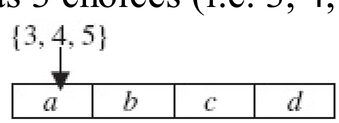
The boys can be placed in 6! ways and the men can be placed in 5! ways. By (MP), the desired number of ways is 6!5!.

Example 4.2 *In each of the following cases, find the number of integers between* 3000 *and* 6000 *in which no digit is repeated*:

(i) *there are no additional restrictions*;
(ii) *the integers are even.*

Solution Let $abcd$ be a required integer.

(i) As shown in the diagram below, a has 3 choices (i.e. 3, 4, or 5), say $a = 3$.



Since no digit is repeated, a way of forming “bcd” corresponds to a 3-permutation from the 9-element set $\{0,1, 2, 4, 5,..., 9\}$. Thus the required number of integers is $3 \cdot P_3^9$.
(ii) Again, $a = 3$, 4 or 5. We divide the problem into two cases.
Case (1) $a = 4$ (even)

4	b	c	d

In this case, d has 4 choices (i.e. 0, 2, 6, 8), say $d = 2$. Then a way of forming "bc" is a 2-permutation from the 8-element set $\{0, 1, 3, 5, 6, 7, 8, 9\}$. Thus the required number of integers is $4 \cdot P_2^8$.

Case (2) $a = 3$ or 5 (odd)

In this case, d has 5 choices, and the number of ways to form *"bc'* is P_2^8. The required number of integers is $2 \cdot 5 \cdot P_2^8$.

By (AP), the desired number of integers is

$$4 \cdot P_2^8 + 2 \cdot 5 \cdot P_2^8 = 14 \cdot P_2^8.$$

Example 4.3 *There are* 10 *pupils in a class.*

(i) *How many ways are there to form a 5-member committee for the class*?

(ii) *How many ways are there to form a 5-member committee in which one is the Chairperson, one is the Vice-Chairperson, one is the Secretary and one is the Treasurer*?

(iii) *How many ways are there to form a 5-member committee in which one is the Chairperson, one is the Secretary and one is the Treasurer*?

Solution (i) This is the same as finding the number of 5- combinations of a 10-element set. Thus the answer is $\binom{10}{5} = 252$.

(ii) This is the same as choosing 5 pupils from the class and then placing them in the following spaces.

Chairperson	V-Chairperson	Secretary	Treasurer	Member

Clearly, this is a "permutation" problem, and the answer is $P_5^{10} = \frac{10!}{5!} = 30240$.

(iii) This problem can be counted in the following procedure: we first select one for Chairperson, then one for Secretary, then one for Treasurer, and finally two from the remainder for committee members as shown below:

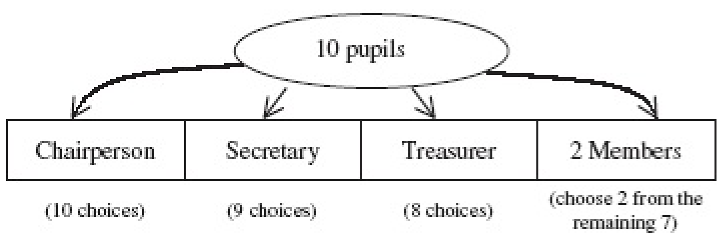


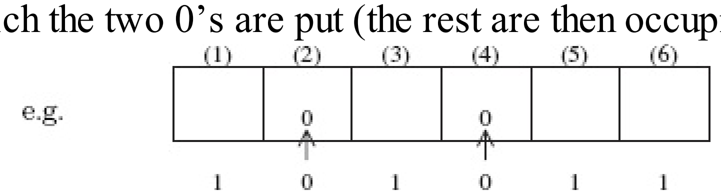
Figure 4.1

Thus, by (MP), the answer is given by $10 \cdot 9 \cdot 8 \cdot \binom{7}{2} = 15120$.

Note There are different ways to solve (iii). You may want to try your own ways.

Example 4.4 *As shown in Example 2.2, the number of* 6-*digit binary sequences is* 2^6. *How many of them contain exactly two* 0*'s* (e.g. 001111,101101,...)?

Solution Forming a 6-digit binary sequence with two 0's is the same as choosing two spaces from the following 6 spaces into which the two 0's are put (the rest are then occupied by 1's) as shown below:



Thus, the number of such binary sequences is $\binom{6}{2}$.

Example 4.5 *Figure* 4.2 *shows* 9 *distinct points on the circumference of a circle.*

(i) *How many chords of the circle formed by these points are there* ?

(ii) *If no three chords are concurrent in the interior of the circle, how many points of intersection of these chords within the circle are there*?

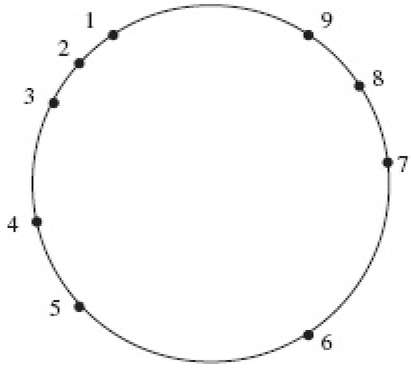
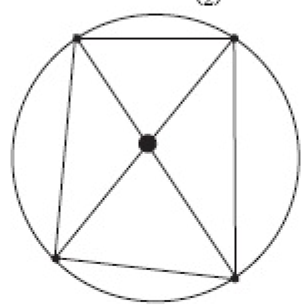


Figure 4.2

Solution (i) Every chord joins two of the nine points, and every two of the nine points determine a unique chord. Thus, the required number of chords is $\binom{9}{2}$.



(ii) Every point of intersection of two chords corresponds to four of the nine points, and every four of the nine points determine a point of intersection. Thus the required number of points of intersection is $\binom{9}{4}$.

Example 4.6 *At a National Wages Council conference, there are* 19 *participants from the government, the unions and the employers. Among them,* 9 *are from the unions.*

In how many ways can a 7-*member committee be formed from these participants in each of the following cases*:

(i) *there are no restrictions*?

(ii) *there is no unionist in the committee*?

(iii) *the committee consists of unionists only*?

(iv) *there is exactly one unionist in the committee* ?

(v) *there is at least one unionist in the committee* ?

Solution (i) This is the number of 7-element subsets of a 19-element set. By definition, the desired number is $\binom{19}{7}$.

(ii) This is the number of ways to form a 7-member committee from the 10 non-unionists. Thus, the desired number is $\binom{10}{7}$.

(iii) Obviously, the desired number is $\binom{9}{7}$.

(iv) We first select a member from the 9 unionists and then select the remaining 6 from the 10 non-unionists. By (MP), the desired number is $\binom{9}{1}\binom{10}{6} = 9\binom{10}{6}$.

(v) There are 7 cases to consider, namely, having r unionists, where $r = 1, 2, 3, 4, 5, 6, 7$. Thus, by (AP) and (MP), the desired number is given by

$$\binom{9}{1}\binom{10}{6} + \binom{9}{2}\binom{10}{5} + \cdots + \binom{9}{6}\binom{10}{1} + \binom{9}{7}.$$

Indeed, we can have a shorter way to solve this part by using the idea of "complementation".

By (i), there are $\binom{19}{7}$ 7-member committees we can form from 19 participants. Among them, there are $\binom{10}{7}$ such committees which contain no unionist by (ii). Thus, the number of 7-member committees which contain at least one unionist should be $\binom{19}{7} - \binom{10}{7}$. (The reader may check that these two answers agree.)

The second solution given in (v) for the above example is just an instance of applying the following principle.

Principle of Complementation (CP)
Let A be a subset of a finite set B.
Then $|B\backslash A| = |B| - |A|$, where $B\backslash A = \{x \mid x \in B \text{ but } x \notin A\}$, called the complement of A in B.

(4.1)

If you revisit Example 2.4 you may then observe that the problem can also be solved by (CP). There are $\binom{9}{3}$ ways to form a 3-vertex subset from the given 9 vertices. Among them, the 3 on AB and any 3 on BC do not form a triangle. Thus, the number of triangles one can form is, by (CP),

$$\binom{9}{3} - \left\{\binom{3}{3} + \binom{4}{3}\right\},$$

which is 79.

We have seen from the above examples how, by applying (CP), we are able to considerably shorten the work needed to solve a counting problem. When a direct approach involves a large number of cases for which a certain condition holds, the complementary view of the *smaller* number of cases in which the condition does NOT hold allows a quicker solution to the problem. What follows then is that we count the number of ways afforded by the smaller number of "complementary" cases and finally obtain the required answer by subtracting this from the total number of ways.

Exercise

4.1 (Continuation from Example 4.1)

(v) A and B are at the two ends;

(vi) Z is at the centre and adjacent to A and B;

(vii) A, B and Z form a single block (i.e. there is no other person between any two of them);

(viii) all men form a single block;

(ix) all men form a single block and all boys form a single block;

(x) no two of A, B and Z are adjacent;

(xi) all boys form a single block and Z is adjacent to A;

(xii) Z is between A and B (need not be adjacent).

4.2 (Continuation from Example 4.2)

(iii) the integers are odd;

(iv) the integers are divisible by 5;

(v) the integers are greater than 3456.

4.3 Four people can be paired off in three ways as shown below:
(1) $\{\{A,B\}, \{C,D\}\}$,
(2) $\{\{A, C\}, \{B,D\}\}$,
(3) $\{\{A,D\}, \{B,C\}\}$.

In how many ways can 10 people be paired off?

4.4 Three girls and seven boys are to be lined up in a row. Find the number of ways this can be done if
(i) there is no restriction;
(ii) the girls must form a single block;
(iii) no two girls are adjacent;
(iv) each boy is adjacent to at most one girl.

4.5 Eight students are in a sailing club. In how many ways can they form a team consisting of 4 Laser pairs, where the order of the pairs does not matter? (Note: A Laser is a sailing boat that takes a crew of two.)

4.6 There are 3 boys and 2 girls.
(i) Find the number of ways to arrange them in a row.
(ii) Find the number of ways to arrange them in a row so that the 2 girls are next to each other.
(iii) Find the number of ways to arrange them in a row so that there is at least 1 boy between the 2 girls.

4.7 In how many ways can a committee of 5 be formed from a group of 11 people consisting of 4 teachers and 7 students if
(i) the committee must include exactly 2 teachers?
(ii) the committee must include at least 3 teachers?
(iii) a particular teacher and a particular student cannot be both in the committee?

4.8 A *palindrome* is a number that remains the same when it is read backward, for example, 2002 is a palindrome. Find the number of n-digit palindromes.

4.9 A team of 6 horses to draw the royal carriage is to be chosen from a group of 10 horses. Find in how many ways this can be done
(i) if the order of the horses in the team does not matter;
(ii) if the team consists of 6 horses in a definite order;
(iii) if the team consists of a first pair, a second pair and a third pair but order within each pair does not matter.

4.10 Find how many four-figure numbers have three and only three consecutive figures identical.

4.11 Find the number of ways in which 9 persons can be divided into
(i) two groups consisting of 6 and 3 persons;
(ii) three groups consisting of 3, 3 and 2 persons with 1 person rejected.

4.12 (i) Find the number of integers from 100 to 500 that do not contain the digit “0”.

(ii) Find the number of integers from 100 to 500 that contain exactly one “0” as a digit.

4.13 Calculate the number of ways of selecting 2 points from 7 distinct points. Seven distinct points are marked on each of two parallel lines. Calculate the number of

(i) distinct quadrilaterals which may be formed using 4 of the 14 points as vertices;

(ii) distinct triangles which may be formed using 3 of the 14 points as vertices.

·.14 (a) A team to climb Mount Everest consisting of 3 teachers and 3 students is to be picked from 5 teachers and 10 students of a university. Find the number of ways in which this can be done.

(b) It was decided that 2 of the 10 students must either be selected together or not selected at all. Find how many possible teams could be selected in these circumstances. The selected team is arranged into 3 pairs, each consisting of a teacher and a student. Find the number of ways in which this can be done.

Chapter 5

The Bijection Principle

We have introduced three basic principles for counting, namely, the (AP), the (MP) and the (CP). In this chapter, we shall introduce another basic principle for counting which we call the *Bijection Principle,* and discuss some of its applications.

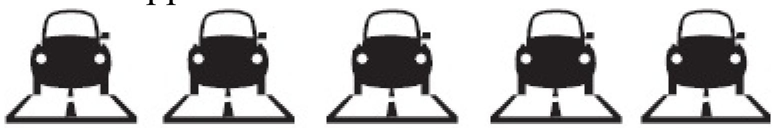


Figure 5.1

Suppose that there are 200 parking lots in a multi-storey carpark. The carpark is full with each vehicle occupying a lot and each lot being occupied by a vehicle (see Figure 5.1). Then we know that the number of vehicles in the carpark is 200 without having to count the vehicles one by one. The number of vehicles and the number of lots are the same because there is a one-to-one correspondence between the set of vehicles and the set of lots in the carpark. This is a simple illustration of the Bijection Principle that we will soon state.

Let us first recall some concepts on mappings of sets. Suppose A and B are two given sets. A *mapping* f from A to B, denoted by

$$f : A \to B,$$

is a rule which assigns to each element a in A a *unique* element, denoted by $f(a)$, in B. Four examples of mappings are shown pictorially in Figure 5.2.

Certain kinds of mappings are important. Let $f : A \to B$ be a mapping. We say that f is *injective* (or *one-to-one*) if $f(x) \neq f(y)$ in B whenever $x \neq y$ in A. Thus, in Figure 5.2, f_2 and f_4 are injective, while f_1 and f_3 are not (why?). We say that f is *surjective* (or *onto*) if for any b in B, there exists an a in A such that $f(a) = b$. Thus, in Figure 5.2, f_3 and f_4 are surjective whereas f_1 and f_2 are not (why?). We call f a *bijection* from A to B if f is both injective and surjective. (Sometimes, a bijection from A to B is referred to as a *one-to-one correspondence* between A and B.) Thus, in Figure 5.2, f_4 is the only bijection. These observations on the four mappings are summarised in Table 5.1.

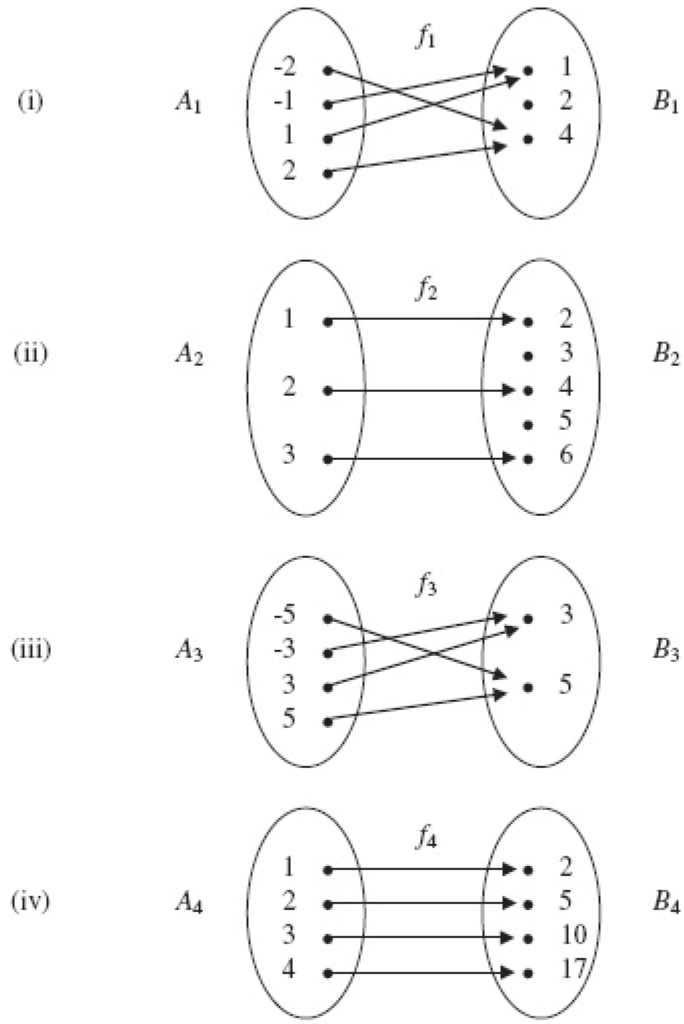


Figure 5.2

Table 5.1

	Injection	Surjection	Bijection
f_1	×	×	×
f_2	✓	×	×
f_3	×	✓	×
f_4	✓	✓	✓

Let A and B be two finite sets. Suppose there is a mapping f: $A \to B$ that is injective. Then, by definition, each element a in A corresponds to its image $f(a)$ in B, and distinct elements in A correspond to distinct images in B. Thus, we have:

The Injection Principle (IP)
Let A and B be finite sets. If there exists a one-to-one mapping $f : A \to B$, then

$$|A| \leq |B|. \tag{5.1}$$

Suppose further, that the one-to-one mapping f:$A \to B$ is onto. Then each element b in B has a unique preimage a in A such that $f(a) = b$. In this case, we clearly have:

The Bijection Principle (BP)
Let A and B be finite sets. If there exists a bijection f: $A \to B$, then

$$|A| = |B|. \tag{5.2}$$

In this chapter, we shall focus on (BP). Through the discussions on a number of problems, we shall show you how powerful this principle is.

First of all, let us revisit a problem we studied in Chapter 4. In Example 4.5, we counted the number of chords and the number of points of intersection of the chords joining some fixed points on the circumference of a circle. Let us consider a similar problem. Figure 5.3 shows five distinct points on the circumference of a circle.

How many chords can be formed by these points?

Let A be the set of such chords, and B, the set of 2-element subsets of $\{1,2,3,4,5\}$. Given a chord α in A, define $f(\alpha) = \{p,q\}$, where p, q are the two points (on the circumference) which determine the chord α (see Figure 5.4). Then β is a mapping from A to B. Clearly, if α and β are two distinct chords in A, then $f(\alpha) = f(\beta)$. Thus, f is injective. On the other hand, for any 2-element subset $\{p, q\}$ in B (say, $p = 2$ and $q = 5$), there is a chord α in A (in this instance, α is the chord joining points 2 and 5) such that $f(\alpha) = \{p,q\}$. Thus, f is onto.

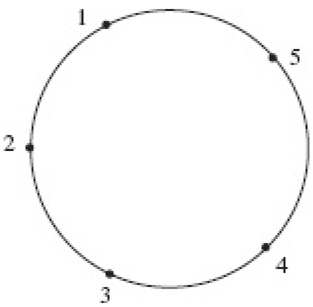


Figure 5.3

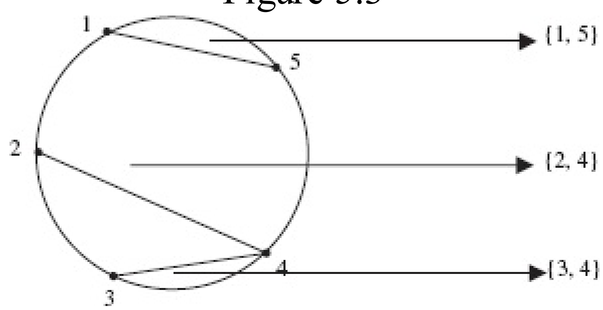


Figure 5.4

Hence, $f : A \to B$ is a bijection and, by (BP), we have $|A| = |B|$. As B is the set of all 2-element subsets of $\{1,2,3,4,5\}$, $|B| = \binom{5}{2}$. We thus conclude that $|A| = |B| = \binom{5}{2}$.

Next we ask: How many points of intersection (of these $\binom{5}{2}$ chords) that lie in the interior of the circle are there if no three chords are concurrent in the interior of the circle?

Let A be the set of such points of intersection and B, the set of 4-element subsets of $\{1,2,3,4,5\}$. Figure 5.5 exhibits a bijection between A and B (figure out the rule which defines the bijection!). Thus, by (BP), $|A| = |B|$. Since $|B| = \binom{5}{4}$ by definition, we have $|A| = \binom{5}{4}$.

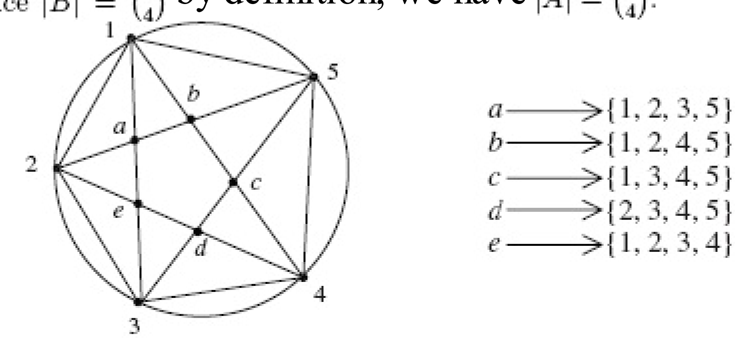


Figure 5.5

Let us proceed to show some more applications of (BP).

Example 5.1 *Figure 5.6 shows a 2 × 4 rectangular grid with two specified corners P and Q. There are 12 horizontal segments and 10 vertical segments in the grid. A shortest P-Q route is a continuous path from P to Q consisting of 4 horizontal segments and 2 verticalsegments. An example is shown in Figure 5.6. How many shortest P-Q routes in the grid are there*

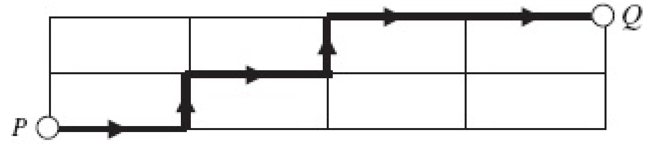
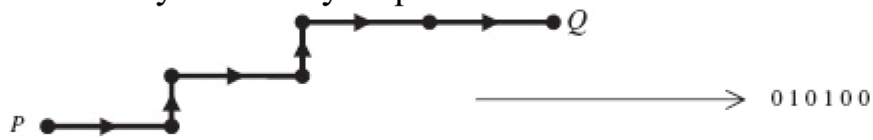


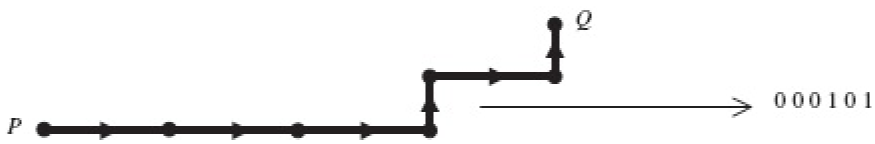
Figure 5.6

Solution Certainly, we can solve the problem directly by listing all the possible shortest routes. This, however, would not be practical if we wish to solve the same problem in, say, a 190 × 100 rectangular grid. We look for a more efficient way.

There are two types of segments: horizontal and vertical. Let us use a “0” to represent a horizontal segment, and a “1” to represent a vertical segment. Thus, the shortest *P–Q* route shown in Figure 5.6 can accordingly be represented by the binary sequence with four “0”s and two “1”s as shown below:

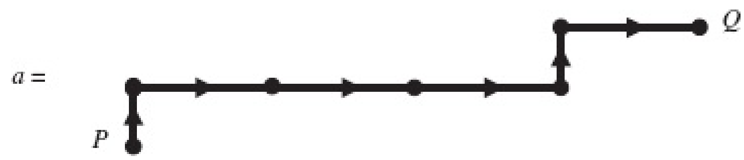


Likewise, we can have:



and so on.

Now, let *A* be the set of all shortest *P-Q* routes and *B,* the set of all 6-digit binary sequences with two 1’s. Then we see that the above way of representing a shortest *P-Q* route in *A* by a binary sequence in *B* defines a mapping $f : A \rightarrow B$. Clearly, different shortest *P-Q* routes in *A* correspond to different sequences in *B* under *f*. Thus, *f*is one-to-one. Further, for any sequence *b* in *B*, say, *b* = 100010, one can find a shortest *P-Q* route, *a* in *A*, in this case,so that $f(a) = b$. Thus *f* is onto, and so it is a bijection. Now, by (BP), we conclude that $|A| = |B|$. But how does this simplify our effort to find the number of shortest *P-Q* routes?



Let us explain. What is the set B? *B* is the set of all 6-digit binary sequences with two 1’s. Can we count $|B|$? Oh, yes! We have already solved it in Example 4.4. The answer is $|B| = \binom{6}{2}$. Accordingly, we have $|A| = |B| = \binom{6}{2}$.

Example 5.2 *The power set of a set S, denoted by* $\mathcal{P}(S)$, *is the set of all subsets of S, inclusive of S and the empty set ø. Thus, for* $\mathbb{N}_n = \{1, 2, \ldots, n\}$, $1 \leq n \leq 3$, *we have*

$$\mathcal{P}(\mathbb{N}_1) = \{\emptyset, \{1\}\},$$

$$\mathcal{P}(\mathbb{N}_2) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\},$$

$$\mathcal{P}(\mathbb{N}_3) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Note that $|\mathcal{P}(\mathbb{N}_1)| = 2, |\mathcal{P}(\mathbb{N}_2)| = 4, |\mathcal{P}(\mathbb{N}_3)| = 8$. Table 3.1 shows that $|\mathcal{P}(\mathbb{N}_4)| = 16$. What is the value of $|\mathcal{P}(\mathbb{N}_5)|$?

Solution For convenience, let $A = \mathcal{P}(\mathbb{N}_5)$; that is, *A* is the power set of {1, 2, 3, 4, 5}. Represent these subsets by 5-digit binary sequences as follows:

$$\begin{array}{lcl}
\emptyset & \longrightarrow & 00000 \\
\{1\} & \longrightarrow & 10000 \\
\{2\} & \longrightarrow & 01000 \\
\vdots & & \\
\{5\} & \longrightarrow & 00001 \\
\{1,2\} & \longrightarrow & 11000 \\
\vdots & & \\
\{4,5\} & \longrightarrow & 00011 \\
\vdots & & \\
\{1,3,5\} & \longrightarrow & 10101 \\
\vdots & & \\
\{1,2,3,4,5\} & \longrightarrow & 11111
\end{array}$$

The rule is that the ith digit of the corresponding binary sequence is "1" if "i" is in the subset; and "0" otherwise. Let B be the set of all 5-digit binary sequences. Clearly, the above rule establishes a bijection between A and B. Thus, by (BP), $|A| = |B|$. Since $|B| = 2^5$ (see Example 2.2), $|A| = 2^5$.

Note that $|\mathcal{P}(\mathbb{N}_1)| = 2 = 2^1$, $|\mathcal{P}(\mathbb{N}_2)| = 4 = 2^2$, $|\mathcal{P}(\mathbb{N}_3)| = 8 = 2^3$, $|\mathcal{P}(\mathbb{N}_4)| = 16 = 2^4$, and now $|\mathcal{P}(\mathbb{N}_5)| = 2^5$. What is $|\mathcal{P}(\mathbb{N}_n)|$ for $n \geq 1$? (See Exercise 5.3.)

Finally, let us introduce a counting problem related to the notion of divisors of natural numbers. We shall denote by $\mathbb{N}$, the set of natural numbers; i.e.

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

Assume that $d, n \in \mathbb{N}$. We say that d is a *divisor* of n if when n is divided by d, the remainder is zero. Thus, 3 is a divisor of 12, 5 is a divisor of 100, but 2 is not a divisor of 9.

Let $n \in \mathbb{N}$, $n \geq 2$. Clearly, n has at least two divisors, namely 1 and n. How many divisors (inclusive of 1 and n) does n have? This is a type of problem that can often be found in mathematical competitions. We shall tackle this problem and see how (MP) and (BP) are used in solving the problem.

To understand the solution, we first recall a special type of numbers called prime numbers and state an important result relating natural numbers and prime numbers.

A natural number p is said to be *prime* (or called a *prime*) if $p \geq 2$ and the only divisors of p are 1 and p. All prime numbers less than 100 are shown below:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41,$$
$$43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.$$

The primes are often referred to as building blocks of numbers because every natural number can always be expressed uniquely as a product of some primes. For example,

$$108 = 2^2 \times 3^3, \qquad 1620 = 2^2 \times 3^4 \times 5,$$
$$1815 = 3 \times 5 \times 11^2, \qquad 215306 = 2 \times 7^2 \times 13^3.$$

This fact is so basic and important to the study of numbers that it is called the *Fundamental Theorem of Arithmetic* (FTA).

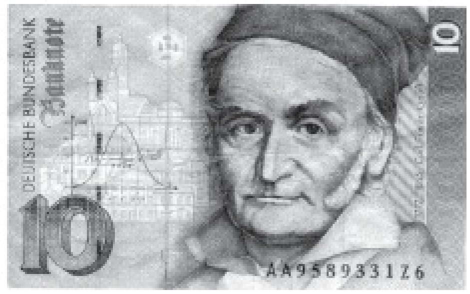
> (FTA) Every natural number $n \geq 2$ can be factorised as
>
> $$n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$$
>
> for some distinct primes $p_1, p_2, \ldots, p_k$ and for some natural numbers $m_1, m_2, \ldots, m_k$. Such a factorisation is unique if the order of primes is disregarded.

(FTA) was first studied by the Greek mathematician, Euclid (c. 450–380 BC) in the case where the number of primes is at most 4. It was the German mathematician, Carl Friedrich Gauss (1777–1855), known as the Prince of Mathematicians, who stated and proved the full result in 1801.

Let us now return to the problem of counting the number of divisors of n. How many divisors does the number 72 have? Since 72 is not a big number, we can get the answer simply by listing all the divisors of 72:

$$1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.$$

The way of counting the divisors of n by listing as shown above is certainly impractical when n gets larger. We look for a more efficient way.



The images above are those of Euclid on a stamp of the Maldives and Gauss on a German banknote.

Let us look at the example when n = 72 again and try to get some information about 72 and its divisors by (FTA).

Observe that $72 = 2^3 \times 3^2$. Suppose x is a divisor of 72. Clearly, x does not contain prime factors other than 2 and 3. That is, x must be of the form

$$x = 2^p \times 3^q$$

where, clearly, $p \in \{0,1, 2, 3\}$ and $q \in \{0,1, 2\}$. On the other hand, any such number $2^p \times 3^q$ is a divisor of 72. Indeed,

$$\begin{aligned} 1 &= 2^0 \times 3^0, & 9 &= 2^0 \times 3^2, \\ 2 &= 2^1 \times 3^0, & 12 &= 2^2 \times 3^1, \\ 3 &= 2^0 \times 3^1, & 18 &= 2^1 \times 3^2, \\ 4 &= 2^2 \times 3^0, & 24 &= 2^3 \times 3^1, \\ 6 &= 2^1 \times 3^1, & 36 &= 2^2 \times 3^2, \\ 8 &= 2^3 \times 3^0, & 72 &= 2^3 \times 3^2. \end{aligned}$$

Let A be the set of divisors of 72 and $B = \{(p,q) \mid 0 \le p \le 3, 0 \le q \le 2\} = \{0,1,2,3\} \times \{0,1,2\}$. Then the above list implies that the mapping f defined by

$$\begin{aligned} f(1) &= (0,0), & f(9) &= (0,2), \\ f(2) &= (1,0), & f(12) &= (2,1), \\ f(3) &= (0,1), & f(18) &= (1,2), \\ f(4) &= (2,0), & f(24) &= (3,1), \\ f(6) &= (1,1), & f(36) &= (2,2), \\ f(8) &= (3,0), & f(72) &= (3,2), \end{aligned}$$

is a bijection from A to B. Thus, by (BP) and (MP), $|A| = |B| = |\{0,1,2,3\} \times \{0,1,2\}| = |\{0,1,2,3\}| \times |\{0,1,2\}| = 4 \times 3 = 12$, which agrees with the above listing.

The following example extends what we discussed above.

Example 5.3 *Find the number of divisors of* 12600.

Solution Observe that $12600 = 2^3 \times 3^2 \times 5^2 \times 7^1$.

Thus a number z is a divisor of 12600 if and only if it is of the form

$$z = 2^a \times 3^b \times 5^c \times 7^d$$

where a, b, c, d are integers such that $0 \le a \le 3, 0 \le b \le 2, 0 \le c \le 2$ and $0 \le d \le 1$.

Let A be the set of divisors z of 12600 and $B = \{(a,b,c,d) \mid a = 0,1,2,3; b = 0,1,2; c = 0,1,2; d = 0,1\}$. Clearly, the mapping f defined by

$$f(z) = (a, b, c, d),$$

is a bijection from A to B. Then, by (BP) and (MP), $|A| = |B| = 4 \cdot 3 \cdot 3 \cdot 2 = 72$.

We have seen from the above examples how crucial applying (BP) is as a step towards solving a counting problem. Given a finite set A, the objective is to enumerate $|A|$, but unfortunately, the straightforward approach is often not easy. In the course of applying (BP), we look for a more familiar finite set B and try to establish a bijec-tion between these two sets. Once this is done, the harder problem of counting $|A|$ is transformed to an easier problem (hopefully) of counting $|B|$. It does not matter how different the members in A and those in B are in nature. As long as there exists a bijection between them, we get $|A| = |B|$.

Exercise

5.1 (a) Find the number of positive divisors of n if

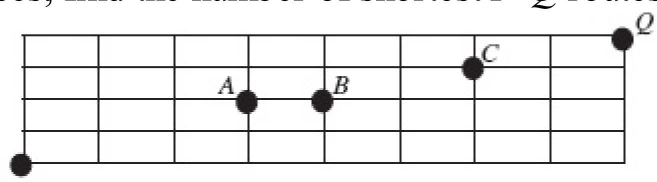
(i) $n = 31752$;

(ii) $n = 55125$.

(b) In general, given an integer $n > 2$, how do you find the number of positive divisors of n?

5.2 Find all positive integers that are divisible by 105 and have exactly 105 different positive divisors.

5.3 In each of the following cases, find the number of shortest P-Q routes in the grid below:



(i) the routes must pass through A;

(ii) the routes must pass through AB;

(iii) the routes must pass through A and C;

(iv) the segment AB is deleted.

5.4 For each positive integer n, show that $|\mathcal{P}(\mathbb{N}_n)| = 2^n$ by establishing a bijection between $\mathcal{P}(\mathbb{N}_n)$ and the set of n-digit binary sequences.

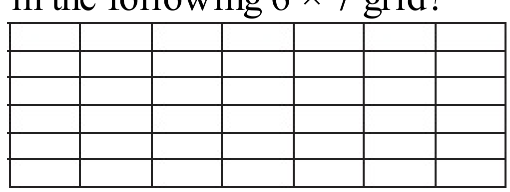
5.5 Let n and r be integers with $1 \le r \le n$. Prove that $\binom{n}{r} = \binom{n}{n-r}$ by establishing a bijection between the set of r-element subsets of $\mathbb{N}_n$ and the set of $(n - r)$-element subsets of $\mathbb{N}_n$.

5.6 The number 4 can be expressed as a sum of one or more positive integers, taking order into account, in the following 8 ways:

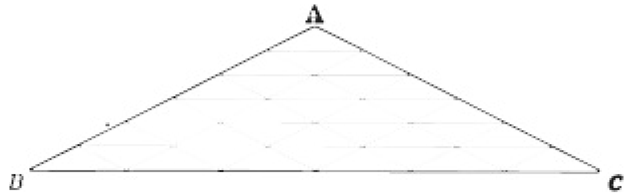
$$\begin{aligned} 4 &= 4 = 1 + 3 \\ &= 3 + 1 = 2 + 2 \\ &= 1 + 1 + 2 = 1 + 2 + 1 \\ &= 2 + 1 + 1 = 1 + 1 + 1 + 1. \end{aligned}$$

Show that every natural number n can be so expressed in 2^{n-1} ways.

5.7 How many rectangles are there in the following 6×7 grid?



5.8 Find the number of parallelograms which are contained in the configuration below and which have no sides parallel to BC. (Hint: Adjoin a new row at the base of the triangle.)



5.9 If n points on the circumference of a circle are joined by straight lines in all possible ways and no three of these lines meet at a single point inside the circle, find

(i) the number of triangles formed with all vertices lying inside the circle;

(ii) the number of triangles formed with exactly two vertices inside the circle;

(iii) the number of triangles formed with exactly one vertex inside the circle;

(iv) the total number of triangles formed.

Chapter 6

Distribution of Balls into Boxes

Figure 6.1 shows three distinct boxes into which seven identical (indistinguishable) balls are to be distributed. Three different ways of distribution are shown in Figure 6.2. (Note that the two vertical bars at the two ends are removed.)

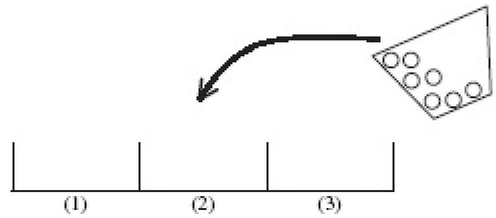


Figure 6.1

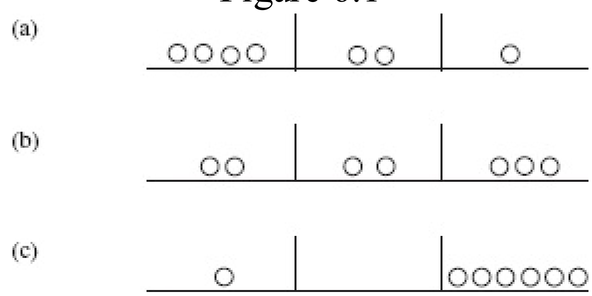


Figure 6.2

In how many different ways can this be done? This is an example of the type of problem we shall discuss in this chapter. We shall see how problems of this type can be solved by applying (BP).

In Figure 6.2, by treating each vertical bar as a “1” and each ball as a “0”, each way of distribution becomes a 9-digit binary sequence with two 1’s. For instance,

$$\begin{aligned}
(a) &\longrightarrow 0\,0\,0\,0\,1\,0\,0\,1\,0,\\
(b) &\longrightarrow 0\,0\,1\,0\,0\,1\,0\,0\,0,\\
(c) &\longrightarrow 0\,1\,1\,0\,0\,0\,0\,0\,0.
\end{aligned}$$

Obviously, this correspondence establishes a bijection between the set of ways of distributing the balls and the set of 9-digit binary sequences with two 1’s. Thus, by (BP), the number of ways of distributing the seven identical balls into three distinct boxes is $\binom{9}{2}$.

In general, we have:

> The number of ways of distributing r *identical* balls into n *distinct* boxes is given by $\binom{r+n-1}{n-1}$, which is equal to $\binom{r+n-1}{r}$, by (3.6). (6.1)

In the distribution problem discussed above, some boxes may be vacant at the end. Supposing no box is allowed to be vacant, how many ways are there to distribute the seven identical balls into three distinct boxes?

To meet the requirement that no box is vacant, we first put a ball in each box and this is counted as one way because the balls are identical. We are then left with 4 (= 7 – 3) balls, but we are now free to distribute these 4 balls into any box. By the result (6.1), the number of ways this can be done is $\binom{4+3-1}{3-1} = \binom{6}{2}$. Thus, the number of ways to distribute 7 identical balls into 3 distinct boxes so that no box is empty is $\binom{6}{2}$.

In general, suppose we wish to distribute r *identical* balls into n *distinct* boxes, where $r \geq n$, in such a way that no box is vacant. This can be done in the following steps: First, we put one ball in each box; and then distribute the remaining $r - n$ balls to the n boxes in any arbitrary way. As the balls are identical, the number of ways for the first step to be done is 1. On the other hand, by the result (6.1), the number of ways to do the second step is

$$\binom{(r-n)+n-1}{n-1}.$$

Thus, by (MP) and upon simplification, we arrive at the following result.

$$\boxed{\text{The number of ways to distribute } r \text{ \textit{identical} balls into } n \text{ \textit{distinct} boxes, where } r \geq n, \text{ so that no box is empty is given by } \binom{r-1}{n-1}, \text{ which is equal to } \binom{r-1}{r-n}.} \quad (6.2)$$

Example 6.1 *There rare 11 men waiting for their turn in a barber shop. Three particular men are A,B and C. There is a row of 11 seats for the customers. Find the number of ways of arranging them so that no two of A, B and C are adjacent.*

Solution There are different ways to solve this problem. We shall see in what follows that it can be treated as a distribution problem.

First of all, there are 3! ways to arrange *A, B* and *C*. Fix one of the ways, say *A*—*B*—*C*. We then consider the remaining 8 persons. Let us imagine tentatively that these 8 persons are identical, and they are to be placed in 4 distinct boxes as shown in Figure 6.3 so that boxes (2) and (3) are not vacant (since no two of *A, B* and *C* are adjacent). To meet this requirement, we place one in box (2) and one in box (3). Then the remaining six can be placed freely in the boxes in $\binom{6+4-1}{4-1} = \binom{9}{3}$ ways by (6.1). (Figure 6.4 shows a way of distribution.)

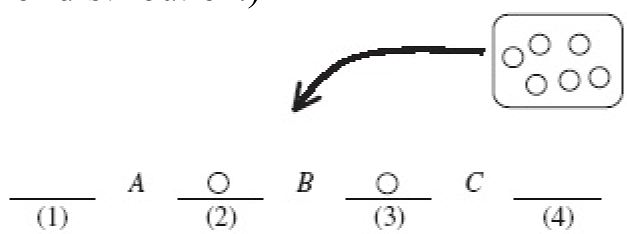


Figure 6.3

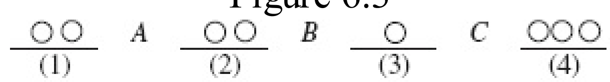


Figure 6.4

But the eight persons are actually distinct. Thus, to each of these $\binom{9}{3}$ ways, there are 8! ways to arrange them.

Hence by (MP), the required number of ways is $3!\binom{9}{3}8!$, which is $8!\ 9 \cdot 8 \cdot 7$.

Remark The answer, $8!\ 9 \cdot 8 \cdot 7$, suggests that the problem can be solved in the following way. We first arrange the 8 persons (excluding *A, B* and *C*) in a row in 8! ways. Fix one of these ways, say

$$\underset{(1)}{\_\_\_}\ X_1\ \underset{(2)}{\_\_\_}\ X_2\ \underset{(3)}{\_\_\_}\ X_3\ \underset{(4)}{\_\_\_}\ X_4\ \underset{(5)}{\_\_\_}\ X_5\ \underset{(6)}{\_\_\_}\ X_6\ \underset{(7)}{\_\_\_}\ X_7\ \underset{(8)}{\_\_\_}\ X_8\ \underset{(9)}{\_\_\_}$$

We now consider *A*. There are 9 ways to place *A* in one of the 9 boxes, say box (4):

$$\underset{(1)}{\_\_\_}\ X_1\ \underset{(2)}{\_\_\_}\ X_2\ \underset{(3)}{\_\_\_}\ X_3\ \underset{(4)}{\underline{\ A\ }}\ X_4\ \underset{(5)}{\_\_\_}\ X_5\ \underset{(6)}{\_\_\_}\ X_6\ \underset{(7)}{\_\_\_}\ X_7\ \underset{(8)}{\_\_\_}\ X_8\ \underset{(9)}{\_\_\_}$$

Next, consider *B*. Since *A* and *B* cannot be adjacent, *B* can be placed only in one of the remaining 8 boxes. Likewise, *C* can be placed only in one of the remaining 7 boxes. The answer is thus $8!\ 9 \cdot 8 \cdot 7$.

Exercise

6.1 There are 4 types of sandwiches. A boy wishes to place an order of 3 sandwiches. How many such orders can he place?

6.2 Calculate the number of distinct 9-letter arrangements which can be made with letters of the word SINGAPORE such that no two vowels are adjacent.

6.3 There is a group of 10 students which includes three particular students *A, B* and *C*. Find the number of ways of arranging the 10 students in a row so that *B* is always between *A* and *C*. (*A* and *B*, or *B* and *C* need not be adjacent.)

6.4 Six distinct symbols are transmitted through a communication channel. *A* total of 18 blanks are to be inserted between the symbols with at least 2 blanks between every pair of symbols. In how many ways can the symbols and blanks be arranged?

Chapter 7

More Applications of (BP)

We shall give additional examples in this chapter to show more applications of (BP).

Consider the following linear equation:

$$x_1 + x_2 + x_3 = 7. \qquad (1)$$

If we put $x_1 = 4$,$x_2 = 1$ and $x_3 = 2$, we see that (1) holds. Since 4,1, 2 are non-negative integers, we say that $(x_1,x_2,x_3) = (4,1,2)$ is a *non-negative integer solution* to the linear equation (1). Note that $(x_1, x_2, x_3) = (1,2,4)$ is also a non-negative integer solution to (1), and so are (4, 2, 1) and (1, 4, 2). Other non-negative integer solutions to (1) include

$$(0,0,7), (0,7,0), (1,6,0), (5,1,1), \ldots.$$

Example 7.1 *Find the number of non-negative integer solutions to* (1).

Solution Let us create 3 distinct "boxes" to represent x_1,x_2 and x_3, respectively. Then each non-negative integer solution $(x_1 ,x_2,x_3) = (a,b,c)$ to (1) corresponds, in a natural way, to a way of distributing 7 identical balls into boxes so that there are *a, b* and *c* balls in boxes (1), (2) and (3) respectively (see Figure 7.1).

This correspondence clearly establishes a bijection between the set of non-negative integer solutions to (1) and the set of ways of distributing 7 identical balls in 3 distinct boxes. Thus, by (BP) and the result of (6.1), the number of non-negative integer solutions to $\binom{7+3-1}{3-1} = \binom{9}{2}$.

$$(4,1,2) \longrightarrow \underset{(1)}{0000} \;|\; \underset{(2)}{0} \;|\; \underset{(3)}{00}$$

$$(2,5,0) \longrightarrow \underset{(1)}{00} \;|\; \underset{(2)}{00000} \;|\; \underset{(3)}{}$$

Figure 7.1

By generalising the above argument and applying the results (6.1) and (6.2), we can actually establish the following general results.

> Consider the linear equation
>
> $$x_1 + x_2 + \cdots + x_n = r \qquad (2)$$
>
> where r is a non-negative integer.
>
> (i) The number of non-negative integer solutions to (2) is given by $\binom{r+n-1}{r}$.
>
> (ii) The number of positive integer solutions $(x_1, x_2, \ldots, x_n)$ to (2), with each $x_i \geq 1$, is given by $\binom{r-1}{r-n}$, where $r \geq n$ and $i = 1, 2, \ldots, n$.

(7.1)

Example 7.2 *Recall that the number of 3-element subsets {a,b,c} of the set* $\mathbb{N}_{10} = \{1, 2, 3, \ldots, 10\}$ *is* $\binom{10}{3}$. *Assume that a < b < c and suppose further that*

$$b - a \geq 2 \quad \text{and} \quad c - b \geq 2 \qquad (3)$$

(i.e. no two numbers in {a,b,c} are consecutive). For instance, {1,3,8} and {3,6,10} satisfy (3) but not {4,6, 7} and {1,2,9}. How many such 3-element subsets of $\mathbb{N}_{10}$ *are there*?

Solution Let us represent a 3-element subset $\{a,b,c\}$ of $\mathbb{N}_{10}$ satisfying (3) by a 10-digit binary sequence as follows:

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
$\{1,3,8\} \longrightarrow$	1	0	1	0	0	0	0	1	0	0
$\{3,6,10\} \longrightarrow$	0	0	1	0	0	1	0	0	0	1

Note that the rule is similar to the one introduced in Example 5.2. Clearly, this correspondence is a bijection between the set A of 3-element subsets of $\mathbb{N}_{10}$ satisfying (3) and the set B of 10-digit binary sequences with three 1's in which no two 1's are adjacent. Thus $|A| = |B|$. But how do we count $|B|$? Using the method discussed in Example 6.1, we obtain

$$|B| = \binom{(7-2)+4-1}{4-1} = \binom{8}{3}.$$

Thus $|A| = \binom{8}{3}$.

Example 7.3 *Two tennis teams A and B, consisting of 5 players each, will have a friendly match playing only singles tennis with no ties allowed. The players in each team are arranged in order*:

$$A : a_1, a_2, a_3, a_4, a_5,$$

$$B : b_1, b_2, b_3, b_4, b_5.$$

The match is run in the following way. First, a_1 plays against b_1. Suppose a_1 wins (i.e. b_1 is eliminated). Then a_1 continues to play against b_2; if a_1 is beaten by b_2 (i.e. a_1 is eliminated), then b_2 continues to play against a_2, and so on. What is the number of possible ways in which all the 5 players in team B are eliminated? (Two such ways are shown in Figure 7.2.)

Solution Let x_i be the number of games won by player a_i, $i = 1, 2, 3, 4, 5$. Thus, in Figure 7.2(i),

$$x_1 = 2, \quad x_2 = 0, \quad x_3 = 3, \quad x_4 = x_5 = 0$$

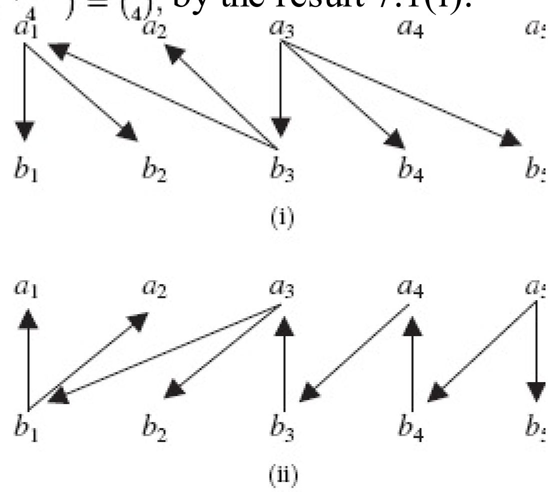
and in Figure 7.2(ii),

$$x_1 = x_2 = 0, \quad x_3 = 2, \quad x_4 = 1, \quad x_5 = 2.$$

In order for the 5 players in team B to be eliminated, we must have

$$x_1 + x_2 + x_3 + x_4 + x_5 = 5 \qquad (4)$$

and the number of ways this can happen is, by (BP), the number of non-negative integer solutions to (4). Thus, the desired answer is $\binom{5+5-1}{4} = \binom{9}{4}$, by the result 7.1(i).



"$a \to b$" means "a beats b"

Figure 7.2

Example 7.4 *Eight letters are to be selected from the five vowels a, e,i,o,u with repetition allowed. In how many ways can this be done if*

(i) *there are no other restrictions*?

(ii) *each vowel must be selected at least once*?

Solution (i) Some examples of ways of the selection are given below:

(1) *a, a, u, u, u, u, u, u*;

(2) *a, e, i, i, i, o, o, u*;

(3) *e, e, i, i, o, o, u, u*.

As shown in Figure 7.3, these selections can be treated as ways of distributing 8 identical objects into 5 distinct boxes.

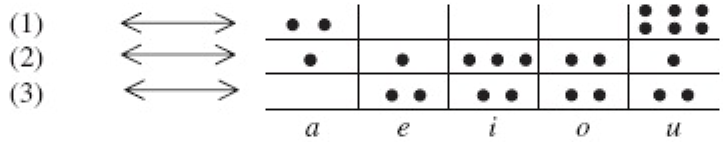


Figure 7.3

Thus, by (BP) and the result (6.1), the number of ways of selection is given by $\binom{8+5-1}{8}$, i.e. $\binom{12}{4}$.

(ii) As shown in the second row of Figure 7.3, a way of selection which includes each vowel can be treated as a way of distribution such that no box is empty. Thus, by (BP) and the result (6.2), the number of ways of selection is given by $\binom{8-1}{8-5}$, i.e. $\binom{7}{3}$.

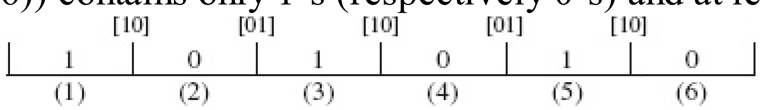
Example 7.5 *Consider the following two 13-digit binary sequences*:

1 1 1 0 1 0 1 1 1 0 0 0 0,

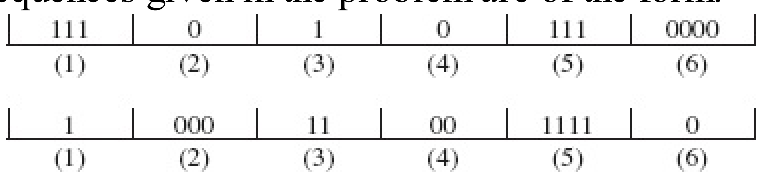
1 0 0 0 1 1 0 0 1 1 1 1 0.

For binary sequences, any block of two adjacent digits is of the form 0, 01,10 or 11. In each of the above sequences, there are three 00, two 01, three 10 and four 11. Find the number of 13-digit binary sequences which have exactly three 00, two 01, three 10 and four 11.

Solution To have exactly three 10 and two 01 in a sequence, such a sequence must begin with 1, end with 0, and have the changeovers of 1 and 0 as shown below, where each of the boxes (1), (3) and (5) (respectively (2), (4) and (6)) contains only 1's (respectively 0's) and at least one 1 (respectively 0).



For instance, the two sequences given in the problem are of the form:



To have three 00 and four 11 in such a sequence, we must

(i) put in three more 0's in boxes (2), (4) or (6) (but in an arbitrary way), and

(ii) put four more 1's in boxes (1), (3) or (5) (also in an arbitrary way).

(Check that there are 13 digits altogether.) The number of ways to do (i) is $\binom{3+3-1}{3}$, i.e. $\binom{5}{2}$; while that of (ii) is $\binom{4+3-1}{4}$, i.e. $\binom{6}{2}$. Thus, by (MP), the number of such sequences is $\binom{5}{2}\binom{6}{2}$ i.e. 150.

Example 7.6 *Consider the following three arrangements of 5 persons A, B, C, D,E in a circle*:

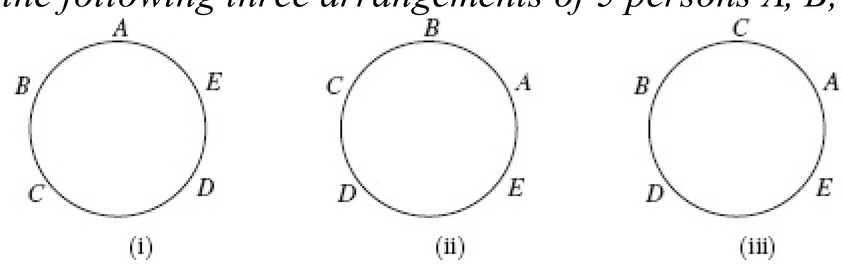


Figure 7.4

Two arrangements of n objects in a circle are considered different if and only if there is at least one object whose neighbour on the right is different in the two arrangements. Thus arrangements (i) and (ii) above are considered identical, while arrangement (iii) is considered different from (i) and (ii). (Note that the right neighbour of A in arrangement (iii) is C while that in both (i) and (ii) is B.) Find the number of arrangements of the 5 persons in a circle.

Solution For each arrangement of the 5 persons in a circle, let us line the 5 in a row as follows: We always start with *A* at the left end. Then we place the right neighbour of *A* (in the circle) to the right of *A* in the row. We continue, in turn, to place the right neighbour (in the circle) of the last placed person to his right in the row until every person is arranged in the row. (We can also visualise this as cutting the circle at A and then unraveling it to form a line.) Then each circular arrangement of the 5 persons corresponds to an arrangement of 5 persons in a row with A at the left end. Now, since A is always fixed at the left end, he can be neglected and the arrangement of 5 persons in a row can be seen to correspond to an arrangement of only 4 persons (B, C, D, E) in a row (see Figure 7.5).

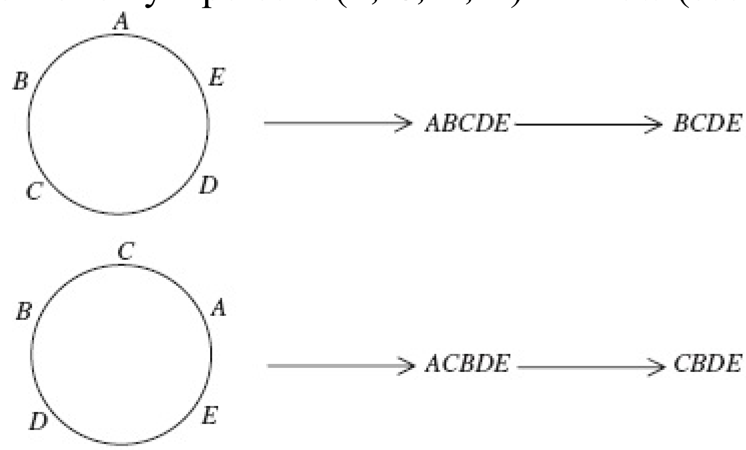


Figure 7.5

This correspondence clearly establishes a bijection between the set of arrangements of 5 persons in a circle and the set of arrangements of 4 persons in a row. Thus, by (BP) and the result of (3.1), the number of arrangements of 5 persons in a circle is 4!. ?

By generalising the above argument, we can establish the following result:

The number of ways of arranging n distinct objects in a circle is given by $(n-1)!$.	(7.2)

Exercise

7.1 Find the number of integer solutions to the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 51$$

in each of the following cases:

(i) $x_i \geq 0$ for each $i = 1,2,\ldots, 5$;

(ii) $x_1 \geq 3, x_2 \geq 5$ and $x_i \geq 0$ for each $i = 3,4, 5$;

(iii) $0 \leq x_1 \leq 8$ and $x_i \geq 0$ for each $i = 2,3,4, 5$;

(iv) $x_1 + x_2 = 10$ and $x_i \geq 0$ for each $i = 1,2,\ldots, 5$;

(v) x_i is positive and odd (respectively, even) for each $i = 1, 2,\ldots,5$.

7.2 An illegal gambling den has 8 rooms, each named after a different animal. The gambling lord needs to distribute 16 tables into the rooms. Find the number of ways of distributing the tables into the rooms in each of the following cases:

(i) Horse Room holds at most 3 tables.

(ii) Each of Monkey Room and Tiger Room holds at least 2 tables.

7.3 The number 6 can be expressed as a product of 3 factors in 9 ways as follows:

$$1\cdot1\cdot6,\ 1\cdot6\cdot1,\ 6\cdot1\cdot1,\ 1\cdot2\cdot3,\ 1\cdot3\cdot2,\ 2\cdot1\cdot3,\ 2\cdot3\cdot1,\ 3\cdot1\cdot2,\ 3\cdot2\cdot1.$$

In how many ways can each of the following numbers be so expressed?

(i) 2592

(ii) 27000

7.4 Find the number of integer solutions to the equation:

$$x_1 + x_2 + x_3 + x_4 = 30$$

in each of the following cases:

(i) $x_i \geq 0$ for each $i = 1, 2, 3, 4$;
(ii) $2 \leq x_1 \leq 7$ and $x_i \geq 0$ for each $i = 2, 3, 4$;
(iii) $x_1 \geq -5, x_2 \geq -1, x_3 \geq 1$ and $x_4 \geq 2$.

7.5 Find the number of quadruples (w,x,y,z) of non-negative integers which satisfy the inequality

$$w + x + y + z \leq 2002.$$

7.6 Find the number of non-negative integer solutions to the equation:

$$5x_1 + x_2 + x_3 + x_4 = 14.$$

7.7 There are five ways to express 4 as a sum of two non-negative integers in which the order matters:

$$4 = 4 + 0 = 3 + 1 = 2 + 2 = 1 + 3 = 0 + 4.$$

Given $r,n \in \mathbb{N}$, what is the number of ways to express r as a sum of n non-negative integers in which the order matters?

7.8 There are six ways to express 5 as a sum of three positive integers in which the order matters:

$$5 = 3+1+1 = 2+2+1 = 2+1+2 = 1+3+1 = 1+2+2 = 1+1+3.$$

Given $r,n \in \mathbb{N}$ with $r \geq n$, what is the number of ways to express r as a sum of n positive integers in which the order matters?

7.9 Find the number of 4-element subsets $\{a, b, c, d\}$ of the set $\mathbb{N}_{20} = \{1, 2, \ldots, 20\}$ satisfying the following condition

$$b - a \geq 2, \ c - b \geq 3 \quad \text{and} \quad d - c \geq 4.$$

7.10 In a sequence of coin tosses, one can keep a record of the number of instances when a tail is immediately followed by a head, a head is immediately followed by a head, etc. We denote these by *TH,HH,* etc. For example, in the sequence *HHTTHHHHTH- HTTTT* of 15 coin tosses, we observe that there are five HH, three *HT*, two *TH* and four *TT* subsequences. How many different sequences of 15 coin tosses will contain exactly two *HH,* three *HT*, four *TH* and five *TT* subsequences?

(AIME)

7.11 Show that the number of ways of distributing r identical objects into n distinct boxes such that Box 1 can hold at most one object is given by

$$\binom{r+n-3}{r-1} + \binom{r+n-2}{r}.$$

7.12 In a new dictatorship, it is decided to reorder the days of the week using the same names of the days. All the possible ways of doing so are to be presented to the dictator for her to decide on one. How many ways are there in which Sunday is immediately after Friday and immediately before Thursday?

7.13 Five couples occupy a round table at a wedding dinner. Find the number of ways for them to be seated if:

(i) every man is seated between two women;
(ii) every man is seated between two women, one of whom is his wife;
(iii) every man is seated with his wife;
(iv) the women are seated on consecutive seats.

7.14 The seats at a round table are numbered from 1 to 10. Find the number of ways in which a family consisting of six adults and four children can be seated at the table

(i) if there are no restrictions;

(ii) if all the children sit together.

7.15 Four policemen, two lawyers and a prisoner sit at a round table. Find the number of ways of arranging the seven people if the prisoner is seated

(i) between the two lawyers;

(ii) between two policemen.

Chapter 8

Distribution of Distinct Objects into Distinct Boxes

We have seen from the various examples given in Chapters 6 and 7 that the *distribution problem,* which deals with the counting of ways of distributing objects into boxes, is a basic model for many counting problems. In distribution problems, objects can be identical or distinct, and boxes too can be identical or distinct. Thus, there are, in general, four cases to be considered, namely

Table 8.1

	Objects	Boxes
(1)	identical	distinct
(2)	distinct	distinct
(3)	distinct	identical
(4)	identical	identical

We have considered Case (1) in Chapters 6 and 7. Case (3) will be discussed in Chapter 18 as Stirling numbers of the second kind, while Case (4) will not be touched upon in this book. In this chapter, we shall consider Case (2).

Suppose that 5 distinct balls are to be put into 7 distinct boxes.

Example 8.1 *In how many ways can this be done if each box can hold at most one ball*

Example 8.2 *In how many ways can this be done if each box can hold any number of balls*

Solution Before we proceed, we would like to point out that the *ordering* of the distinct objects in each box is *not* taken into consideration in the discussion in this chapter.

We first consider Example 8.1. As shown in Figure 8.1, let *a, b, c, d* and *e* denote the 5 distinct balls. First, we put *a* (say) into one of the boxes. There are 7 choices.

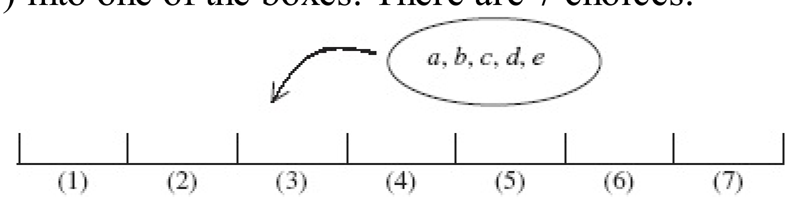


Figure 8.1

Next, we consider *b* (say). As each box can hold at most one ball, and one of the boxes is occupied by *a,* there are now 6 choices for b. Likewise, there are, respectively, 5, 4 and 3 choices for *c,d* and e. Thus, by (MP), the number of ways of distribution is given by $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$.

Note that the above answer can be expressed as P_5^7 which, as defined in Chapter 3, is the number of ways of arranging any 5 objects from 7 distinct objects. The fact that the above answer is P_5^7 does not surprise us as there is a 1-1 correspondence between the distributions of 5 distinct balls into 7 distinct boxes and the arrangements of 5 distinct objects from 7 distinct objects as shown in Figure 8.2. (Find out the rule of the correspondence.)

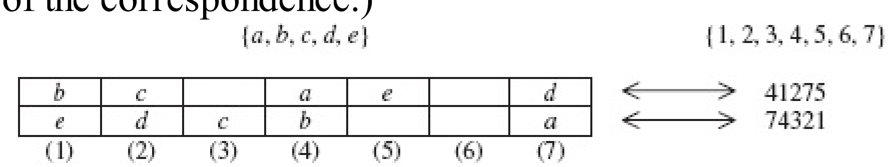


Figure 8.2.

In general, we have:

> The number of ways of distributing r distinct objects into n distinct boxes such that each box can hold at most one object (and thus $r \le n$) is given by P_r^n, which is equal to $n!/(n-r)!$. (8.1)

We now consider Example 8.2. There are 7 ways of putting *a* in the boxes. As each box can hold any number of balls, there are also 7 choices for each of the remaining balls *b, c, d* and *e*. Thus, by (MP), the answer is 7^5.

In general, we have:

The number of ways of distributing r distinct objects into n distinct boxes such that each box can hold any number of objects is given by n^r.	(8.2)

Exercise

8.1 Find the number of ways for a teacher to distribute 6 different books to 9 students if
(i) there is no restriction;
(ii) no student gets more than one book.

8.2 Let A be the set of ways of distributing 5 distinct objects into 7 distinct boxes with no restriction, and let B be the set of 5-digit numbers using 1, 2, 3, 4, 5, 6, 7 as digits with repetition allowed. Establish a bijection between A and B.

8.3 Five friends go to a Cineplex which contains 6 theatres each screening a different movie and 2 other theatres screening the current blockbuster. Find the number of ways the friends can watch a movie in each of the following cases:
(i) two of the friends must be together;
(ii) the theatres do not matter, only the movies do.

8.4 Find the number of ways of distributing 8 distinct objects into 3 distinct boxes if each box must hold at least 2 objects.

8.5 Suppose that m distinct objects are to be distributed into $n$distinct boxes so that each box contains at least one object. State a restriction on m with respect to n. In how many ways can the distribution be done if
(i) $m = n$?
(ii) $m = n + 1$?
(iii) $m = n + 2$?

Chapter 9
Other Variations of the Distribution Problem

Two cases of the distribution problem were discussed in the preceding chapters. In this chapter, we shall study some of their variations.

When *identical* objects are placed in distinct boxes, whether the objects in each box are *ordered* or not makes no difference. The situation is no longer the same if the objects are distinct as shown in Figure 9.1.

Figure 9.1

In Chapter 8, we did not consider the ordering of objects in each box. In our next example, we shall take it into account.

Example 9.1 *Suppose that* 5 *distinct objects a,b,c,d,e are distributed into* 3 *distinct boxes, and that the ordering of objects in each box matters. In how many ways can this be done?*

Solution First, consider a (say). Clearly, there are 3 choices of a box for a to be put in (say, a is put in box (2)). Next, consider b. The object b can be put in one of the 3 boxes. The situation is special if b is put in box (2) because of the existence of a in that box. As the ordering of objects in each box matters, if b is put in box (2), then there are 2 choices for b, namely, left of a or right of a as indicated in Figure 9.2. Thus, altogether, there are 4 choices for b.

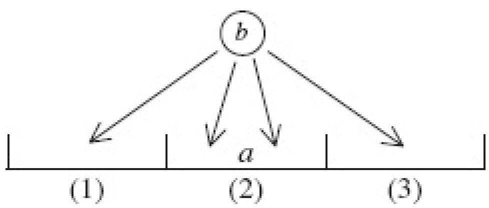


Figure 9.2

Assume that b is put in box (3). Now, consider c. As shown in Figure 9.3, c has 5 choices.

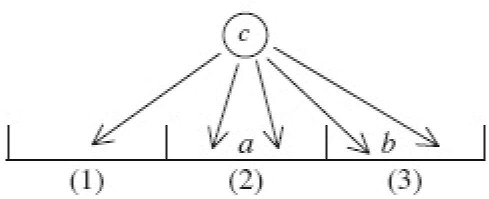


Figure 9.3

Continuing in this manner, we see that d and e have, respectively, 6 and 7 choices. Thus, the answer is given by $3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$.

Let us try a different approach to solve the above problem. First, we pretend that the objects *a, b, c, d,* and *e* are all identical. The number of ways of distributing 5 identical objects into 3 distinct boxes is, by result (6.1), $\binom{5+3-1}{5}$, i.e. $\binom{7}{2}$. Next, take such a way of distribution, say,

Since the 5 objects are actually distinct and the ordering of objects matters, such a distribution for *identical* objects corresponds to 5! different distributions of *distinct* objects. Thus, by (MP), the answer is given by $\binom{7}{2}\cdot 5!$ which agrees with the answer $3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$.

In general, we have:

The number of ways of distributing r *distinct* objects into n *distinct* boxes such that the ordering of objects in each box matters is given by $$\binom{r+n-1}{r}\cdot r! \tag{9.1}$$ which is equal to $$n(n+1)(n+2)\cdots(n+r-1).$$

In our previous discussion on the distribution problem, objects were either *all identical* or *all distinct*. We now consider a case that is a mixture of these two.

Example 9.2 *Four identical objects "a", three identical objects "b" and two identical objects "c" are to be distributed into 9 distinct boxes so that each box contains one object. In how many ways can this be done*?

Solution Let's start with the four a's. Among the 9 boxes, we choose 4 of them, and put one *a* in each chosen box. Next, we consider the three b's. From among the 5 remaining boxes, we choose 3, and put one *b* in each chosen box (see Figure 9.4). Finally, we put one *c* in each of the 2 remaining boxes.

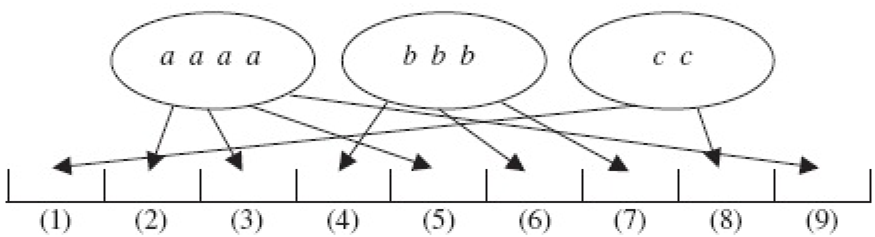


Figure 9.4

There are $\binom{9}{4}$ ways for step 1, $\binom{5}{3}$ ways for step 2 and $\binom{2}{2}(=1)$ way for step 3. Thus, by (MP), the answer is given by

$$\binom{9}{4}\binom{5}{3}\cdot 1 = \frac{9!}{4!5!}\cdot\frac{5!}{3!2!} = \frac{9!}{4!3!2!}.$$

Remark In the above solution, *a* is considered first, followed by *b* and finally c. The answer is independent of this order. For instance, if *b* is considered first, followed by *c* and then *a,* by applying a similar argument we arrive at $\binom{9}{3}\binom{6}{2}\binom{4}{4}$, which is again $\frac{9!}{4!3!2!}$.

There is a 1-1 correspondence between the distributions considered in Example 9.2 and the arrangements of 4 *a*'s, 3 *b*'s and 2 *c*'s in a row as shown in Figure 9.5.

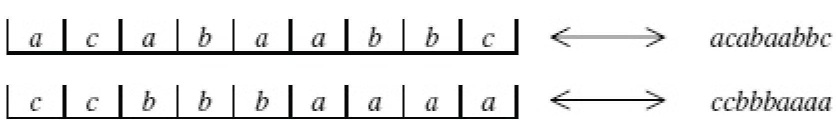


Figure 9.5

Thus, by the result of Example 9.2, the number of arrangements of 4 *a*'s, 3 *b*'s and 2 *c*'s in a row is given by

$$\frac{9!}{4!3!2!}.$$

In general,

> Suppose there are n_1 identical objects of type 1, n_2 identical objects of type 2, …, and n_k identical objects of type k. Let $n = n_1 + n_2 + \cdots + n_k$. Then the number of arrangements of these n objects in a row is given by
> $$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k} \qquad (9.2)$$
> which is equal to
> $$\frac{n!}{n_1!n_2!\ldots n_k!}.$$

Let us reconsider Example 9.1. We observe that there is a 1-1 correspondence between the distributions considered in Example 9.1 and the arrangements of *a, b, c, d, e* and two 1's as shown in Figure 9.6.

By the above result, the number of arrangements of *a, b, c, d, e* and two 1's is given by $\frac{7!}{2!}$, which agrees also with the earlier two answers.

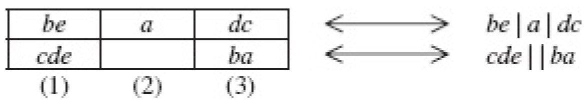


Figure 9.6

Exercise

9.1 Calculate the number of different arrangements which can be made using all the letters of the word BANANARAMA.

9.2 Calculate the number of distinct 9-letter arrangements which can be made with letters of the word PROCESSES. How many of these end with SSS?

9.3 Find the number of arrangements of 4 identical squares, 5 identical pentagons and 6 identical hexagons in a row if

(i) there is no restriction;

(ii) no two pentagons are adjacent;

(iii) any two squares are separated by at least two other polygons.

9.4 Let $A = \{1,2,\ldots, m\}$ and $B = \{1,2,\ldots ,n\}$ where $m,n \geq 1$. Find the number of

(i) mappings from A to B;

(ii) 1–1 mappings from A to B (here $m \leq n$);

(iii) mappings $f: A \to B$ such that $f(i) < f(j)$ in B whenever $i < j$ in A (here $m \leq n$);

(iv) mappings $f: A \to B$ such that $f(1) = 1$.

9.5 Let $A = \{1,2,\ldots, m\}$ and $B = \{1,2,\ldots, n\}$. Find the number of onto mappings from A to B in each of the following cases:

(i) $m = n$;

(ii) $m = n + 1$;

(iii) $m = n + 2$.

(Compare this problem with Problem 8.5.)

9.6 Ten cars take part in an Automobile Association of Singapore autoventure to Malaysia. At the causeway, 4 immigration counters are open. In how many ways can the 10 cars line up in a 4- line queue?

9.7 Solve Problem 8.5 with an additional condition that the ordering of objects in each box counts.

9.8 Show that

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!\cdots n_k!},$$

where $n = n_1 + n_2 + \cdots + n_k$.

Chapter 10

The Binomial Expansion

In Chapter 3, we introduced a family of numbers which were denoted by $\binom{n}{r}$ or C^n_r. Given integers n and r with $0 \le r \le n$, the number $\binom{n}{r}$ is defined as the number of r-element subsets of the set $\mathbb{N}_n = \{1,2,..., n\}$. That is, $\binom{n}{r}$ is the number of ways of selecting r distinct objects from a set of n distinct objects. We also derived the following formula for $\binom{n}{r}$:

$$\boxed{\binom{n}{r} = \frac{n!}{r!(n-r)!}.} \qquad (10.1)$$

By applying (10.1), or otherwise, we can easily derive some interesting identities involving these numbers such as:

$$\boxed{\binom{n}{r} = \binom{n}{n-r}.} \qquad (10.2)$$

$$\boxed{\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.} \qquad (10.3)$$

$$\boxed{r\binom{n}{r} = n\binom{n-1}{r-1}, \quad r \ge 1.} \qquad (10.4)$$

$$\boxed{\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}.} \qquad (10.5)$$

In this chapter, we shall learn more about this family of numbers and derive some other important identities involving them.

In algebra, we learn how to expand the algebraic expression $(1 + x)^n$ for $n = 0,1, 2, 3$. Their expansions are shown below:

$$\begin{aligned}(1+x)^0 &= 1,\\ (1+x)^1 &= 1+x,\\ (1+x)^2 &= 1+2x+x^2,\\ (1+x)^3 &= 1+3x+3x^2+x^3.\end{aligned}$$

Notice that the coefficients in the above expansions are actually numbers of the form $\binom{n}{r}$. Indeed, we have:

$$\begin{aligned}&1=\binom{0}{0},\\ &1=\binom{1}{0},\quad 1=\binom{1}{1},\\ &1=\binom{2}{0},\quad 2=\binom{2}{1},\quad 1=\binom{2}{2},\\ &1=\binom{3}{0},\quad 3=\binom{3}{1},\quad 3=\binom{3}{2},\quad 1=\binom{3}{3}.\end{aligned}$$

What can we say about the coefficients in the expansion of $(1 + x)^4$? Will we obtain

$$(1+x)^4 = \binom{4}{0} + \binom{4}{1}x + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4?$$

Let us try to find out the coefficient of x^2 in the expansion of $(1+x)^4$. We may write

$$(1+x)^4 = \underset{(1)}{(1+x)}\underset{(2)}{(1+x)}\underset{(3)}{(1+x)}\underset{(4)}{(1+x)}.$$

Table 10.1

(1)	(2)	(3)	(4)
x	x		
x		x	
x			x
	x	x	
	x		x
		x	x

Observe that in the expansion, each of the factors (1), (2), (3) and (4) contributes either 1 or x, and they are multiplied together to form a term. For instance, to obtain x^2 in the expansion, two of (1), (2), (3) and (4) contribute x and the remaining two contribute 1. How many ways can this be done? Table 10.1 shows all the possible ways, and the answer is 6.

Thus, there are 6 terms of x^2 and the coefficient of x^2 in the expansion of $(1 + x)^4$ is therefore 6. Indeed, to select two x's from four factors $(1 + x)$, there are $\binom{4}{2}$ ways (while the remaining two have no choice but to contribute "1"). Thus the coefficient of x^2 in the expansion of $(1 + x)^4$ is which is 6. Using a similar argument, one can readily see that

$$(1+x)^4 = \binom{4}{0} + \binom{4}{1}x + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4.$$

In general, what can be said about the expansion of $(1 + x)^n$, where n is any natural number?

Let us write

$$\underset{(1)}{(1+x)^n = (1+x)}\underset{(2)}{(1+x)}\cdots\underset{(n)}{(1+x)}. \qquad (*)$$

To expand $(1 + x)^n$, we first select 1 or x from each of the n factors $(1 + x)$, and then multiply the n chosen 1's and x's together. The general term thus obtained is of the form x^r, where $0 \le r \le n$. What is the coefficient of x^r in the expansion of $(1 + x)^n$ if the like terms are grouped? This coefficient is the number of ways to form the term x^r in the product (*). To form a term x^r, we choose r factors $(1 + x)$ from the n factors $(1 + x)$ in (*) and select x from each of the r chosen factors. Each of the remaining $n - r$ factors $(1 + x)$ has no other option but to contribute 1. Clearly, the above selection can be done in $\binom{n}{r}$ ways. Thus, the coefficient of x^r in the expansion of $(1 + x)^n$ is given by $\binom{n}{r}$. We thus arrive at the following result:

The Binomial Theorem (BT)
For any natural number n,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n. \qquad (10.6)$$

Exercise

10.1 By applying Identity (10.1), or otherwise, derive the following identities:

(i) $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$;

(ii) $\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}$.

10.2 In the expansion of $(1 + x)^{100}$, it is known that the coefficients of x^r and x^{3r}, where $1 \le r \le 33$, are equal. Find the value of r.

10.3 What is the largest value of k such that there is a binomial expansion $(1 + x)^n$ in which the coefficients of k consecutive terms are in the ratio 1:2:3: … : k? Identify the corresponding expansion and the terms.

10.4 Find the terms in the expansion of $(1 + 3x)^{23}$ which have the largest coefficient.

Chapter 11

Some Useful Identities

We gave four simple identities involving binomial coefficients, namely (10.2) -(10.5), in Chapter 10. In this chapter, we shall derive some more identities involving binomial coefficients from (BT). These identities, while interesting in their own right, are also useful in simplifying certain algebraic expressions.

Consider the expansion of $(1 + x)^n$ in (BT). If we let $x = 1$, we then obtain from (BT) the following

$$\boxed{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n.} \qquad \text{(B1)}$$

Example 11.1 *In Example 5.2, we discussedi a counting problem on $P(S)$, the set of all subsets of a finite set S. If S is an n-element set (i.e. $|S| = n$), it can be shown* (*see Problem* 5.4) *by establishing a bijection between $P(S)$ and the set of n-digit binary sequences that there are exactly 2^n subsets of S inclusive of the empty set $\emptyset$ and the set S itself* (*i.e.* $|P(S)| = 2^n$). *We can now* (*give a more* natural *proof for this fact. Assume that $|S| = n$. By definition, the number of*

$$\begin{aligned}
&0\text{-}element\ subsets\ of\ S\ is\ \binom{n}{0},\\
&1\text{-}element\ subsets\ of\ S\ is\ \binom{n}{1},\\
&2\text{-}element\ subsets\ of\ S\ is\ \binom{n}{2},\\
&\qquad\vdots\\
&n\text{-}element\ subsets\ of\ S\ is\ \binom{n}{n}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\mathcal{P}(S)| &= \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}\\
&= 2^n \quad \text{(by (B1))}.
\end{aligned}$$

Example 11.2 *The number* 4 *can be expressed as a sum of one or more positive integers, taking order into account, in the following* 8 *ways*:

$$\begin{aligned}
4 &= 4 = 1+3\\
&= 3+1 = 2+2\\
&= 1+1+2 = 1+2+1\\
&= 2+1+1 = 1+1+1+1.
\end{aligned}$$

Show that every natural number n can be so expressed in 2^{n-1} ways.

Solution This is in fact Problem 5.6. Let us see how (B1) can be used to prove the result. But first of all, consider the special case above when $n = 4$.

We write $4 = 1 + 1 + 1 + 1$ and note that there are three "+"s in the expression. Look at the following relation.

$$4 \longleftrightarrow \underbrace{1+1+1+1}_{4} \qquad \text{(no "+" is chosen)}$$

$$\left.\begin{aligned}
1+3 &\longleftrightarrow \underbrace{1}_{1}\oplus\underbrace{1+1+1}_{3}\\
3+1 &\longleftrightarrow \underbrace{1+1+1}_{3}\oplus\underbrace{1}_{1}\\
2+2 &\longleftrightarrow \underbrace{1+1}_{2}\oplus\underbrace{1+1}_{2}
\end{aligned}\right\} \text{(one "+" is chosen)}$$

$$\left.\begin{array}{l} 1+1+2 \longleftrightarrow \underbrace{1}_{1} \oplus \underbrace{1}_{1} \oplus \underbrace{1+1}_{2} \\ 1+2+1 \longleftrightarrow \underbrace{1}_{1} \oplus \underbrace{1+1}_{2} \oplus \underbrace{1}_{1} \\ 2+1+1 \longleftrightarrow \underbrace{1+1}_{2} \oplus \underbrace{1}_{1} \oplus \underbrace{1}_{1} \end{array}\right\} \quad (2 \text{ “+”s are chosen})$$

$$1+1+1+1 \longleftrightarrow \underbrace{1}_{1} \oplus \underbrace{1}_{1} \oplus \underbrace{1}_{1} \oplus \underbrace{1}_{1} \quad (3 \text{ “+”s are chosen})$$

This correspondence is actually a bijection between the set of all such expressions of 4 and the set of all subsets of three “+”s. Thus, by (BP) and (B1), the required answer is

$$\binom{3}{0}+\binom{3}{1}+\binom{3}{2}+\binom{3}{3}=2^3.$$

In general, write

$$n=\underbrace{1+1+\cdots+1+1}_{n}$$

and note that there are $n-1$ “+”s in the above expression. We now extend the above technique by establishing a bijection between the set of all such expressions of n and the set of all subsets of $n-1$ “+”s. Thus, by (BP) and (B1), the number of all such expressions of n is

$$\binom{n-1}{0}+\binom{n-1}{1}+\cdots+\binom{n-1}{n-1}=2^{n-1}.$$

Consider again the expansion of $(1+x)^n$ in (BT). If we now let $x=-1$, we then have

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}\cdots+(-1)^n\binom{n}{n}=0,$$

where the terms on the LHS alternate in sign. Thus, if n is even, say $n=2k$, then

$$\binom{n}{0}+\binom{n}{2}+\cdots+\binom{n}{2k}=\binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{2k-1};$$

and if n is odd, say $n=2k+1$, then

$$\binom{n}{0}+\binom{n}{2}+\cdots+\binom{n}{2k}=\binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{2k+1}.$$

As

$$\left[\binom{n}{0}+\binom{n}{2}+\cdots\right]+\left[\binom{n}{1}+\binom{n}{3}+\cdots\right]=2^n$$

by (B1), we have:

$$\boxed{\begin{aligned}\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots &= \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots \\ &= \frac{1}{2}(2^n)=2^{n-1}.\end{aligned}} \qquad \text{(B2)}$$

Example 11.3 *A finite set S is said to be “even” (“odd”) if |S| is even (odd). Consider* $\mathbb{N}_8=\{1,2,\ldots,8\}$. *How many even (odd) subsets does* $\mathbb{N}_8$ *have?*

Solution The number of even subsets of $\mathbb{N}_8$ is

$$\binom{8}{0}+\binom{8}{2}+\binom{8}{4}+\binom{8}{6}+\binom{8}{8},$$

and the number of odd subsets of $\mathbb{N}_8$ is

$$\binom{8}{1}+\binom{8}{3}+\binom{8}{5}+\binom{8}{7}.$$

By (B2)

$$\begin{aligned}\binom{8}{0}+\binom{8}{2}+\cdots+\binom{8}{8} &= \binom{8}{1}+\binom{8}{3}+\binom{8}{5}+\binom{8}{7} \\ &= 2^{8-1}=2^7=128.\end{aligned}$$

Consider the following binomial expansion once again:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n.$$

If we treat the expressions on both sides as functions of x, and differentiate them with respect to x, we obtain:

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1}.$$

By letting $x = 1$ in the above identity, we have:

$$\boxed{\begin{aligned}\sum_{k=1}^{n} k\binom{n}{k} &= \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} \\ &= n2^{n-1}.\end{aligned}} \qquad \text{(B3)}$$

Let us try to derive (B3) by a different way. Consider the following problem. Suppose that there are $n(n \geq 1)$ people in a group, and they wish to form a committee consisting of people from the group, including the selection of a leader for the committee. In how many ways can this be done?

Let us illustrate the case when $n = 3$. Suppose that A,B,C are the three people in the group, and that a committee consists of k members from the group, where $1 \leq k \leq 3$. For $k = 1$, there are 3 ways to do so as shown below.

Committee members	Leader
A	A
B	B
C	C

For $k = 2$, there are 6 ways to do so as shown below.

Committee members	Leader
A, B	A
A, B	B
A, C	A
A, C	C
B, C	B
B, C	C

For $k = 3$, there are 3 ways to do so as shown below.

Committee members	Leader
A, B, C	A
A, B, C	B
A, B, C	C

Thus, there are altogether $3 + 6 + 3 = 12$ ways to do so.

In general, from a group of n people, there are $\binom{n}{k}$ ways to form a k-member committee, and k ways to select a leader from the k members in the committee. Thus, the number of ways to form a k-member committee including the selection of a leader is, by (MP), $k\binom{n}{k}$. As k could be $1,2,\ldots,n$, by (AP), the number of ways to do so is given by

$$\sum_{k=1}^{n} k\binom{n}{k}.$$

Let us count the same problem via a different approach as follows. First, we select a leader from the group, and then choose $k - 1$ members, where $k = 1, 2, \ldots, n$, from the group to form a k-member committee. There are n choices for the first step and

$$\binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-1}$$

ways for the second step. Thus, by (MP) and (B1), the required number is

$$n\left[\binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-1}\right] = n2^{n-1}.$$

Since both

$$\sum_{k=1}^{n} k \binom{n}{k} \quad \text{and} \quad n2^{n-1}$$

count the same number, identity (B3) follows.

In the above discussion, we establish identity (B3) by first introducing a "suitable" counting problem. We then count the problem in two different ways so as to obtain two different expressions. These two different expressions must be equal as they count the same quantity. This way of deriving an identity is quite a common practice in combinatorics, and is known as "counting it twice".

Exercise

11.1 By applying Identity (10.5) or otherwise, show that

$$\sum_{k=r}^{n} \binom{n}{k}\binom{k}{r} = \binom{n}{r} 2^{n-r}, \quad \text{where } 0 \le r \le n.$$

11.2 Show that

$$\sum_{k=0}^{n-1} \binom{2n-1}{k} = 2^{2n-2}.$$

11.3 Show that

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}(2^{n+1}-1)$$

by integrating both sides of $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$ with respect to x.

11.4 Show that

$$\sum_{k=1}^{n} k^2 \binom{n}{k} = n(n+1)2^{n-2}.$$

11.5 Solve Example 11.2 by using result (7.1)(ii).

Chapter 12

Pascal's Triangle

In Chapter 10, we established the Binomial Theorem (BT) which states that for all non-negative integers n,

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

Let us display the binomial coefficients row by row following the increasing values of n as shown in Figure 12.1. We observe from Figure 12.1 the following.

1. The binomial coefficient at a lattice point counts the number of shortest routes from the top lattice point (representing $\binom{0}{0}$) to the lattice point concerned. For example, there are $\binom{4}{2}$ (= 6) shortest routes from the lattice point representing $\binom{0}{0}$ to the lattice point $\binom{4}{2}$ (also see Example 5.1).

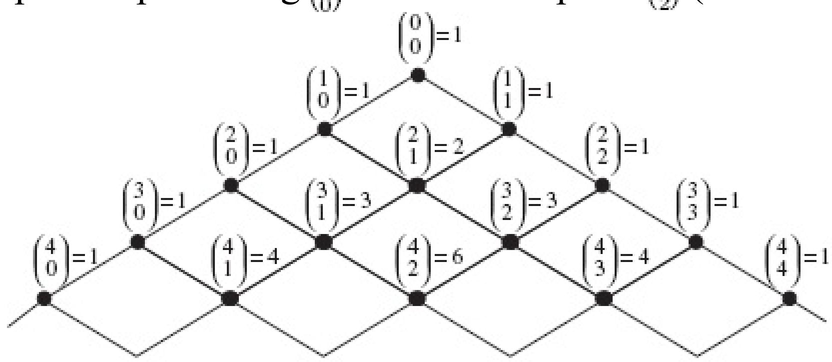


Figure 12.1

2. The number pattern is symmetric with respect to the vertical line through the rop lattice point, and this observation corresponds to the identity $\binom{n}{r} = \binom{n}{n-r}$ (see (10.2)).
3. Any binomial coefficient represented by an interior lattice point is equal to the sum of the two binomial coefficients represented by the lattice points on its "shoulders" (see Figure 12.2). This observation corresponds to the identity $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ (see (10.3)).

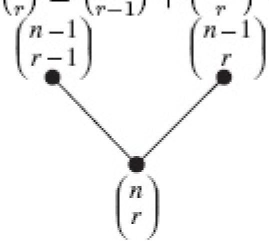


Figure 12.2

4. The sum of the binomial coefficients in the nth row is equal to 2^n and this fact corresponds to the identity

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

The number pattern of Figure 12.1 was known to Omar Khayyam and Jia Xian around 1100 AD. The pattern was also found in the book written by the Chinese mathematician Yang Hui in 1261, in which Yang Hui called it, the Jia Xian triangle. The number pattern in the form of Figure 12.3 was found in another book written by another Chinese mathematician Zhu Shijie in 1303.

However, the number pattern of Figure 12.1 is generally called *Pascal's Triangle* in memory of the great French mathematician Blaise Pascal (1623-1662) who also applied the "triangle" to the study of *probability,* a subject dealing with "chance". For a history of this number pattern, readers are referred to the book *Pascal's Arithmetical Triangle* by A. W. F. Edwards (Oxford University Press

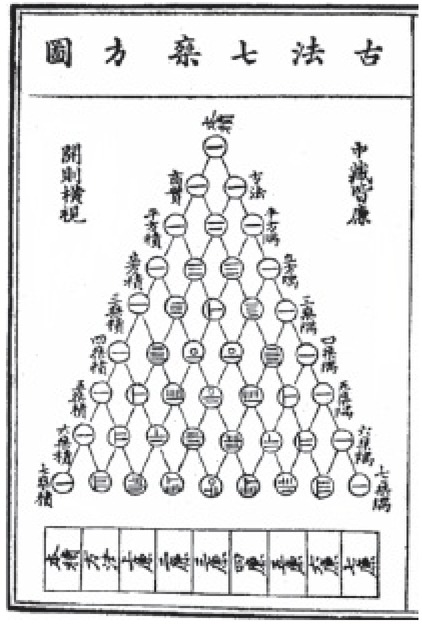


Figure 12.3

Blaise Pascal

Look at Pascal's triangle of Figure 12.4.

What is the sum of the six binomial coefficients enclosed in the shaded rectangle? The answer is 56. Note that this answer appears as another binomial coefficient located on the right of 21 in the next row. Is this situation just a coincidence? Let us take a closer look.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Figure 12.4

Observe that

$$
\begin{aligned}
1+3+6+10+15+21 &= \binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2} \\
&\quad +\binom{6}{2}+\binom{7}{2} \\
&= \binom{3}{3}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2} \\
&\quad +\binom{6}{2}+\binom{7}{2} \\
&\quad \left(\text{as } \binom{2}{2}=\binom{3}{3}\right) \\
&= \binom{4}{3}+\binom{4}{2}+\binom{5}{2}+\binom{6}{2}+\binom{7}{2} \\
&= \binom{5}{3}+\binom{5}{2}+\binom{6}{2}+\binom{7}{2} \\
&= \binom{6}{3}+\binom{6}{2}+\binom{7}{2} \\
&= \binom{7}{3}+\binom{7}{2} \\
&= \binom{8}{3} \quad (=56)
\end{aligned}
$$

by applying the identity $\binom{n-1}{r-1}+\binom{n-1}{r}=\binom{n}{r}$.

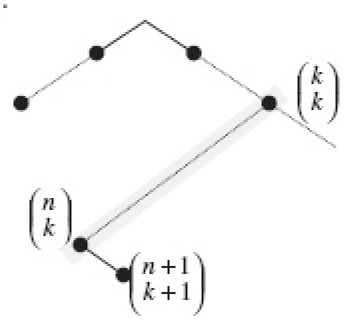


Figure 12.5

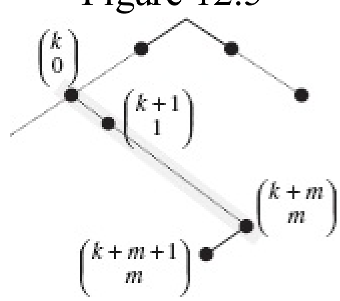


Figure 12.6

The above result is really a special case of a general situation. As a matter of fact, the above argument can also be used to establish the following general result (see also Figure 12.5):

For any non-negative integers n and k with $n \geq k$,

$$\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}. \qquad \text{(B4)}$$

By the symmetry of Pascal's triangle, one obtains the following accompanying identity of (B4) (see also Figure 12.6):

$$\binom{k}{0}+\binom{k+1}{1}+\cdots+\binom{k+m}{m}=\binom{k+m+1}{m}. \qquad \text{(B5)}$$

A second look at Figures 12.5 and 12.6 will make us easily realise why (B4) and (B5) are often called the "Hockey Stick Identities".

To end this chapter, we show an application of Identity (B4) in the solution of the following problem which appeared in International Mathematical Olympiad 1981.

Example 12.1 *Let $1 \le r \le n$ and consider all r-element subsets of the set $\{1,2,\dots,n\}$. Each of these subsets has a smallest member. Let $F(n,r)$ denote the arithmetic mean of these smallest numbers. Prove that*

$$F(n,r) = \frac{n+1}{r+1}.$$

Solution As an illustration of this problem, we consider the case when $n = 6$ and $r = 4$. There are $\binom{6}{4} (= 15)$ 4-element subsets of the set $\{1,2,3,4,5,6\}$. They and their "smallest members" are listed in Table 12.1.

Table 12.1

4-element subsets of $\{1, 2, \dots, 6\}$	Smallest member
{1, 2, 3, 4}	1
{1, 2, 3, 5}	1
{1, 2, 3, 6}	1
{1, 2, 4, 5}	1
{1, 2, 4, 6}	1
{1, 2, 5, 6}	1
{1, 3, 4, 5}	1
{1, 3, 4, 6}	1
{1, 3, 5, 6}	1
{1, 4, 5, 6}	1
{2, 3, 4, 5}	2
{2, 3, 4, 6}	2
{2, 3, 5, 6}	2
{2, 4, 5, 6}	2
{3, 4, 5, 6}	3

By definition,

$$\begin{aligned} F(6,4) &= (10\cdot 1 + 4\cdot 2 + 1\cdot 3) \div 15 \\ &= \frac{7}{5}, \end{aligned}$$

and this is equal to $\frac{n+1}{r+1}$ when $n = 6$ and $r = 4$.

Write $\mathbb{N}_n = \{1,2,\dots, n\}$. To evaluate $F(n, r)$, it is clear that we need to first find out

1. which numbers in $\mathbb{N}_n$ could be the smallest member of an r-element subset of $\mathbb{N}_n$ (in the above example, these are 1, 2, 3 but not 4, 5, 6), and
2. how many times such a smallest member occurs (in the above example, 1 occurs ten times, 2 four times and 3 once);

and then sum these smallest numbers up, and finally divide by $\binom{n}{r}$, the number of r-element subsets of $\mathbb{N}_n$, to obtain the "average".

The last r elements (according to the magnitude) of the set $\mathbb{N}_n$ are:

$$\underbrace{n-r+1, n-r+2, \dots, n-r+r(=n)}_{r}.$$

It follows that $n - r + 1$ is the *largest* possible number to be the smallest member of an r-element subset of $\mathbb{N}_n$. Hence, $1,2,3,\dots,n - r + 1$ are all the possible candidates to be the smallest members of r-element students of $\mathbb{N}_n$.

Let $k \in \{1,2,3,\dots,n - r + 1\}$. Our next task is to find out how many times k occurs as the smallest member. To form an r-element subset of $\mathbb{N}_n$ containing k as the smallest member, we simply form an

$(r-1)$-element subset from the $(n-k)$-element set $\{k+1, k+2, \ldots, n\}$ and then add k to it. The number of $(r-1)$-element subsets of $\{k+1, k+2, \ldots, n\}$ is given by $\binom{n-k}{r-1}$ Thus, k occurs times as the smallest member. Let Σ denote the sum of all these smallest members. Then, as $k = 1,2,\ldots,n-r+1$, we have

$$\begin{aligned}
\Sigma &= 1\binom{n-1}{r-1} + 2\binom{n-2}{r-1} + 3\binom{n-3}{r-1} + \cdots + (n-r+1)\binom{n-(n-r+1)}{r-1} \\
&= (n-r+1)\binom{r-1}{r-1} + \cdots + 3\binom{n-3}{r-1} + 2\binom{n-2}{r-1} + 1\binom{n-1}{r-1} \\
&= \left.\begin{aligned}
&\binom{r-1}{r-1} + \cdots + \binom{n-3}{r-1} + \binom{n-2}{r-1} + \binom{n-1}{r-1} \\
&+\binom{r-1}{r-1} + \cdots + \binom{n-3}{r-1} + \binom{n-2}{r-1} \\
&+\binom{r-1}{r-1} + \cdots + \binom{n-3}{r-1} \\
&\ \vdots \\
&+\binom{r-1}{r-1}.
\end{aligned}\right\} n-r+1 \text{ rows of summands}
\end{aligned}$$

Now, by applying (B4) to each summand above except the last one and noting that $\binom{r-1}{r-1} = \binom{r}{r}$, Σ can be simplified to

$$\Sigma = \underbrace{\binom{n}{r} + \binom{n-1}{r} + \cdots + \binom{r}{r}}_{n-r+1}.$$

By applying (B4) once again, we have

$$\Sigma = \binom{n+1}{r+1}.$$

Finally, by definition of $F(n,r)$, it follows that

$$\begin{aligned}
F(n,r) = \Sigma \div \binom{n}{r} &= \binom{n+1}{r+1} \div \binom{n}{r} \\
&= \frac{(n+1)!}{(r+1)!(n-r)!} \cdot \frac{r!(n-r)!}{n!} \\
&= \frac{n+1}{r+1}
\end{aligned}$$

as desired.

Exercise

12.1 Find the coefficient of x^5 in the expansion of

$$(1+x)^5 + (1+x)^6 + \cdots + (1+x)^{100}.$$

12.2 Find the coefficient of x^3 in the expansion of

$$(1+x)^4 + (1+x)^5 + \cdots + (1+x)^n,$$

where n is a natural number with $n \geq 4$.

12.3 Consider the rows of Pascal's Triangle. Prove that if the nth row is made into a single number by using each element as a digit of the number (carrying over when an element itself has more than one digit), the number is equal to 11^{n-1}. (For example, from the first row $1 = 11^0$, from the second row $11 = 11^1$, from the third row $121 = 11^2$, and from the 6th row $15(10)(10)51 = 15(11)051 = 161051 = 11^5$.)

12.4 On the rth day of an army recruitment exercise, r men register themselves. Each day, the recruitment officer chooses exactly k of the men and line them up in a row to be marched to the barracks. Show that the sum of the numbers of all the possible rows in the first $2k$ days is equal to the number of possible rows in the $(2k+1)$th day.

12.5 The greatest integer not exceeding a real number x is denoted by $\lfloor x \rfloor$. Show that

(i) $\binom{n}{i} < \binom{n}{j}$ if $0 \leq i < j \leq \lfloor \frac{n}{2} \rfloor$;

(ii) $\binom{n}{i} \geq \binom{n}{j}$ if $\lfloor \frac{n}{2} \rfloor \leq i < j \leq n$, with equality if and only if $i = \lfloor \frac{n}{2} \rfloor, j = \lfloor \frac{n}{2} \rfloor + 1$ and n is odd.

12.6 Evaluate $n! + \frac{(n+1)!}{1!} + \frac{(n+2)!}{2!} + \cdots + \frac{(n+r)!}{r!} + \cdots + \frac{(3n)!}{(2n)!}$.

12.7 Find the number of non-negative integer solutions to

$$w + x + y + z \leq 30$$

in the following two ways:

(i) Consider the cases when RHS = 0,1,2,..., 30.

(ii) Consider the situation of distributing a suitable number of identical objects into 5 distinct boxes.

Hence, use a similar approach to prove the Hockey Stick Identity (B4).

12.8 In Pascal's Triangle, there is a row where you can find three consecutive terms *x, y, z* such that *x* : *y* : *z* = 4:5:6. Which row is it? Which terms are they?

Chapter 13

The Principle of Inclusion and Exclusion

In Chapter 1, we introduced the Addition Principle (AP) which was expressed in terms of sets as follows:

$$\boxed{\text{If } A \text{ and } B \text{ are disjoint finite sets, then } |A \cup B| = |A| + |B|.} \tag{13.1}$$

In the statement above, *A* and *B* are assumed to be disjoint, written $A \cap B = \emptyset$, i.e. *A* and *B* have no elements in common. Can we express |*A*∪*B*| in terms of |*A*| and |*B*| regardless of whether *A* and *B* are disjoint? In counting the elements in *A*∪*B*, we may first count those in *A* and then those in *B* .In doing so, any element in *A* ∩ *B* (if there is) is counted exactly twice. Thus, to get the exact count of *A* ∪ *B*|, the number *A* ∩ *B*| should be deducted. It follows that:

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|.} \tag{13.2}$$

This result can also be seen intuitively with the help of the Venn diagram of Figure 13.1.

Note that (13.1) is a special case of (13.2) as (13.1) follows from (13.2) if we assume that $A \cap B = \emptyset$.

Identity (13.2) is a simplest form of a principle called the *Principle of Inclusion and Exclusion* (PIE), which is a very useful and powerful tool in enumeration. First of all, let us show two applications of (13.2).

Example 13.1 *Find the number of integers from the set* {1,2,…, 1000} *which are divisible by* 3 *or* 5.

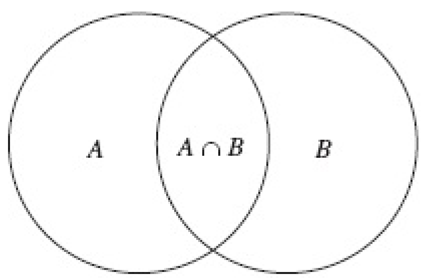


Figure 13.1

Discussion and Solution The integers which we are looking for are 3, 5, 6, 9, 10, 12, 15, 18, 20, … , 999, 1000. How many are there? Let us try to present the solution more formally, and so we let

$$S = \{1, 2, \ldots, 1000\},$$
$$A = \{x \in S \mid x \text{ is divisible by } 3\} \quad \text{and}$$
$$B = \{x \in S \mid x \text{ is divisible by } 5\}.$$

It is now clear that our task is to evaluate $|A \cup B|$ as $A \cup B$ is the set of numbers in *S* which are divisible by 3 or 5.

Before applying (13.2) to evaluate |*A* ∪ *B*|, we recall a useful notation:

For a real number r, let $\lfloor r \rfloor$ denote the greatest integer that is smaller than or equal to r.

Thus $\lfloor 3.15 \rfloor = 3$, $\lfloor \frac{20}{3} \rfloor = 6$, $\lfloor 7 \rfloor = 7$ and so on.

How many integers in {1, 2,…, 10} are there which are divisible by 3? There are three (namely, 3, 6, 9) and note that "three" can be expressed as $\lfloor \frac{10}{3} \rfloor$. The number of integers in {1,2,…, 10} which are divisible by 5 is two (namely, 5, 10) and note that "two" can be expressed as $\lfloor \frac{10}{5} \rfloor$.

Indeed, in general:

For any two natural numbers n, k with k ≤ n, the number of integers in the set {1,2,…, *n}* *which are divisible by k can be expressed as* $\lfloor \frac{n}{k} \rfloor$.

We now return to our original problem of evaluating |*A* ∪ *B*|. To apply (13.2), we need to find |*A*|, |*B*| and |*A* ∩ *B*|. Using the result mentioned above, we see that $|A| = \lfloor \frac{1000}{3} \rfloor = 333$ and |*B*|= $\lfloor \frac{1000}{5} \rfloor = 200$. It remains to find $|A \cap B|$. What does $A \cap B$ represent? Well, $A \cap B$ is the set of integers in S which are divisible by both 3 and 5. How to evaluate $|A \cap B|$? It seems that this problem is as hard as that of evaluating $|A \cup B|$.

Luckily, this is not so as there is a result in Arithmetic that can help us. Let a, b be any two positive integers. It is known that:

An integer is divisible by both a and b when and only when it is divisible by the LCM (least common multiple) of a and b.

It thus follows that $A \cap B$ is the set of numbers in S which are divisible by the LCM of 3 and 5. As the LCM of 3 and 5 is 15, we conclude that

$$A \cap B = \{x \in S \mid x \text{ is divisible by } 15\}.$$

Thus $|A \cap B| = \lfloor \frac{1000}{15} \rfloor = 66.$

Finally, by (13.2), we have

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 333 + 200 - 66 = 467. \end{aligned}$$

Example 13.2 *Find the number of positive divisors of at least one of the numbers* 5400 *and* 18000.

Discussion and Solution In Chapter 5, we discussed the problem of finding the number of positive divisors of a natural number by some examples. The answers obtained can be generalised to lead to the following general result:

> *Let $n \geq 2$ be a natural number. If $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes, then the number of positive divisors of n is $(m_1+1)(m_2+1)\cdots(m_k+1)$.*

We shall see that this result will play an important role in solving our problem.

Let $A = \{x \in \mathbb{N} \mid x$ is a divisor of 5400$\}$ and $B = \{x \in \mathbb{N} \mid x$ is a divisor of 18000$\}$. Clearly, our task is to evaluate $A \cup B$ To apply (13.2) , we need to count $|A|$, $|B|$ and $|A \cap \mathrm{B}|$.

Observe that

$$5400 = 2^3 \cdot 3^3 \cdot 5^2 \quad \text{and} \quad 18000 = 2^4 \cdot 3^2 \cdot 5^3.$$

Thus, by applying the result stated above, we have

$$\begin{aligned} |A| &= (3+1)(3+1)(2+1) = 48 \quad \text{and} \\ |B| &= (4+1)(2+1)(3+1) = 60. \end{aligned}$$

What does $A \cap B$ represent? By definition, $A \cap B$ is the set of common positive divisors of 5400 and 18000, and so it is the set of positive divisors of the Greatest Common Divisor (gcd) of 5400 and 18000. Since

$$\gcd(5400, 18000) = \gcd(2^3 \cdot 3^3 \cdot 5^2, 2^4 \cdot 3^2 \cdot 5^3) = 2^3 \cdot 3^2 \cdot 5^2,$$

it follows that

$$|A \cap B| = (3+1)(2+1)(2+1) = 36.$$

Hence, by (13.2), we have

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 48 + 60 - 36 \\ &= 72. \end{aligned}$$

Formula (13.2) provides an expression for $|A \cup B|$. We shall now apply it to derive an expression for $|A \cup B \cup C|$, where A, B and C are any three finite sets.

Observe that

$$\begin{aligned} &|A \cup B \cup C| \\ &= |A \cup (B \cup C)| \\ &= |A| + |B \cup C| - |A \cap (B \cup C)| \quad (\text{by } (13.2)) \\ &= |A| + |B \cup C| - |(A \cap B) \cup (A \cap C)| \end{aligned}$$

$$\begin{aligned}&= |A| + |B| + |C| - |B \cap C| \\ &\quad - (|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|) \quad \text{(by (13.2))} \\ &= |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|.\end{aligned}$$

That is,

$$\boxed{\begin{aligned}|A \cup B \cup C| = |A| + |B| + |C| - (&|A \cap B| \\ &+ |A \cap C| + |B \cap C|) + |A \cap B \cap C|.\end{aligned}} \qquad (13.3)$$

We shall now show an application of (13.3).

Example 13.3 *Figure 13.2 shows a 4 by 8 rectangular grid with two specified corners p and q and three specified segments uv, wx and yz.*

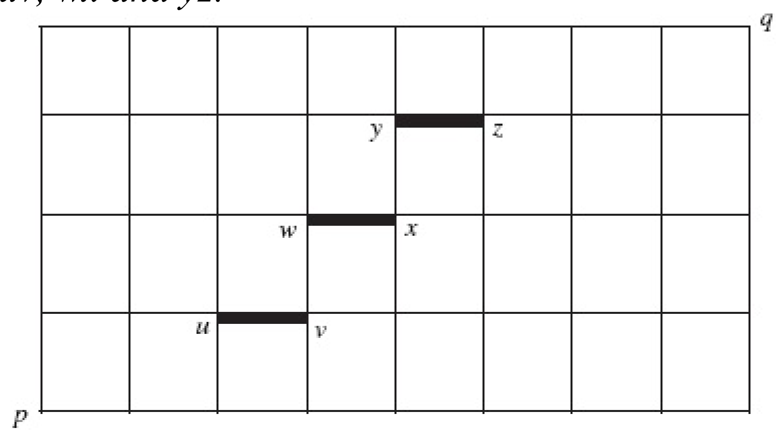


Figure 13.2

Find in the grid

(i) *the number of shortest p-q routes*;
(ii) *the number of shortest p-q routes which pass through*

wx;

(iii) *the number of shortest p-q routes which pass through at least oneof the segments uv, wx and yz;*
(iv) *the number of shortest p-q routes which do not pass through any of the segments uv, wx and yz.*

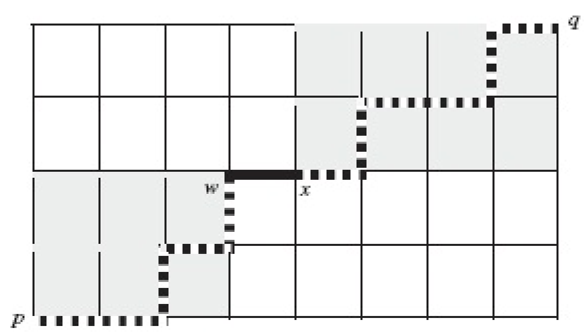


Figure 13.3

Discussion and Solution (i) The problem of counting the number of shortest p-q routes in a rectangular grid was discussed in Example 5.1. Employing the idea developed there, it can be shown that the number of shortest p-q routes in the grid of Figure 13.2 is given by

$$\binom{4+8}{4} = \binom{12}{4}.$$

(ii) As shown in Figure 13.3, a shortest p-q route passing through wx consists of a shortest p-w route (in a 2 by 3 grid), the segment wx and a shortest x-q route (in a 2 by 4 grid). Thus, the number of shortest p-q routes passing through wx is given by

$$\binom{2+3}{2}\cdot 1\cdot\binom{2+4}{2}=\binom{5}{2}\binom{6}{2}.$$

(iii) The counting is more complicated in this case. We introduce three subsets of the set of shortest *p*-*q* routes below.

Let *A* be the set of shortest *p*-*q* routes which pass through *uv*, *B* be the set of shortest *p*-*q* routes which pass through *wx*, and *C* be the set of shortest p-q routes which pass through yz. We note that the answer we are looking for is not $|A| + |B| + |C|$ as the sets A, B, *C* are not pairwise disjoint. The desired answer should be $|A \cup B \cup C|$, and this gives us a chance to apply (13.3). To apply (13.3), we need to evaluate each term on the RHS of (13.3).

First, applying the idea shown in the solution of part (i), we have

$$|A|=\binom{1+2}{1}\binom{3+5}{3}=\binom{3}{1}\binom{8}{3}=3\binom{8}{3},$$

$$|B|=\binom{5}{2}\binom{6}{2}\quad\text{(as shown in part (ii)),}\quad\text{and}$$

$$|C|=\binom{3+4}{3}\binom{1+3}{1}=4\binom{7}{3}.$$

Next, let us compute $|A\cap B|$, $|A\cap C|$ and $|B\cap C|$.

Observe that $A \cap B$ is the set of shortest *p*-*q* routes passing through both *uv* and *wx*. Any such shortest *p*-*q* route consists of a shortest p-q route, the route *uvwx* and a shortest *x*-*q* route. Thus, $|A\cap B|=\binom{3}{1}\cdot 1\cdot\binom{6}{2}=3\binom{6}{2}$.

Likewise, we obtain $|B\cap C|=\binom{5}{2}\binom{4}{1}=4\binom{5}{2}$.

And each route in $A \cap C$ consists of a shortest *p*-*u* route, the segment *uv,* a shortest *v*-*y* route, the segment *yz* and a shortest *z*-*q* route, which gives

$$|A\cap C|=\binom{3}{1}\binom{3}{2}\binom{4}{1}=36.$$

Finally, we evaluate $|A \cap B \cap C|$. Each route in $A \cap B \cap C$ is a *p*-*q* route consisting of a shortest *p*-*u* route, the route *uvwxyz* and a shortest *z*-*q* route. Thus,

$$|A\cap B\cap C|=\binom{3}{1}\binom{4}{1}=12.$$

We are now in a position to evaluate $|A \cup B \cup C|$. By (13.3),

$$|A\cup B\cup C|=3\binom{8}{3}+\binom{5}{2}\binom{6}{2}+4\binom{7}{3}$$
$$-\left(3\binom{6}{2}+4\binom{5}{2}+36\right)+12=349.$$

(iv) Before solving this part, recall the following identity presented in Chapter 4:

$$\boxed{\text{Let } D \text{ be a subset of a finite set } S.\text{ Then, } |S\backslash D|=|S|-|D|.}\tag{13.4}$$

Now, let *S* be the set of shortest *p*-*q* routes in the grid of Figure 13.2. Then we have to evaluate $|S\backslash(A \cup B \cup C)|$, which is equal to

$$|S|-|A\cup B\cup C|\quad\text{by (13.4).}$$

By (i), we have $|S|=\binom{12}{4}$ and by (iii), $|A\cup B\cup C|=349$.

Thus, the desired answer is $\binom{12}{4}-349=495-349=146.$

In the solution of Example 13.3 (iv), we evaluated $|S\backslash(A\cup B \cup C)|$ using (13.3) and (13.4). Now, we shall derive an explicit expression for $|S\backslash(A\cup B \cup C)|$ and show an application of this formula.

In what follows, let *S* be a finite set which is "very large" in the sense that all the sets that we shall consider in a problem are subsets of *S*. In mathematics, we call such a set *S* a *universal* set. For instance, in Example 13.1, the universal set is $\{1, 2,\dots , 1000\}$; in Example 13.2, the universal set is the set of natural numbers; and in Example 13.3, the universal set is the set of shortest *p*-*q* routes in the grid of Figure 13.2.

Let $A \subseteq S$. We write $\bar{A}$ for $S\backslash A$, and call $\bar{A}$ the *complement* of A. In the study of sets, there are two very important laws relating the set operations "union", "intersection" and "complementation". They are called *De Morgan's laws* and are stated below.

$$\boxed{\text{For } A, B \subseteq S,\ \overline{A \cup B} = \bar{A} \cap \bar{B} \quad \text{and} \quad \overline{A \cap B} = \bar{A} \cup \bar{B}.} \tag{13.5}$$

Let A, B, C be any three subsets of S. We shall see that the set $|S\backslash(A \cup B \cup C)|$ that we considered in Example 13.3(iv) can be expressed as $\bar{A} \cap \bar{B} \cap \bar{C}$. Indeed,

$$\begin{aligned} S\backslash(A \cup B \cup C) &= \overline{A \cup B \cup C} \\ &= \overline{(A \cup B) \cup C} \\ &= \overline{A \cup B} \cap \bar{C} \text{ by (13.5)} \\ &= \bar{A} \cap \bar{B} \cap \bar{C} \text{ by (13.5)} \end{aligned}$$

It follows that $|\bar{A} \cap \bar{B} \cap \bar{C}| = |S\backslash(A \cup B \cup C)| = |S| - |A \cup B \cup C|$. Thus, by (13.3), we obtain:

$$\boxed{\begin{aligned} |\bar{A} \cap \bar{B} \cap \bar{C}| = |S| - (|A| + |B| + |C|) + (|A \cap B| + |A \cap C| \\ + |B \cap C|) - |A \cap B \cap C|. \end{aligned}} \tag{13.6}$$

We have just seen how (13.6) was derived from (13.3). It is not difficult to see also that (13.3) can be derived from (13.6). We say that these two identities are *equivalent*.

Now, let $X = \{1,2,\ldots, m\}$ and $Y = \{1,2,\ldots, n\}$, where $m,n \in \mathbb{N}$. The problems of counting the number of mappings and the number of 1–1 mappings from X to Y were proposed in Problem 9.4. Let us reconsider these problems here.

Suppose that $X = \{1, 2, 3\}$ and $Y = \{1, 2, 3, 4, 5\}$. How many mappings are there from X to Y? There are three elements in X, and each of them can be mapped to one of the five elements in Y. Thus the number of mappings from X to Y is given by $5 \cdot 5 \cdot 5 = 5^3$.

How many 1–1 mappings are there from X to Y? The element "1" in X can be mapped to one of the five elements in Y (5 choices). The element "2" in X can be mapped to one of the remaining four elements in Y(4 choices; excluding the image of "1"). Finally, the element "3" in X can be mapped to one of the remaining three elements in Y(3 choices; excluding the images of "1" and "2"). Thus, the number of 1-1 mappings from X to Y is given by $5 \cdot 4 \cdot 3$.

Indeed, in general, we have:

Suppose $X = \{1,2,\ldots,m\}$ and $Y = \{1,2,\ldots,n\}$. Then	
(i) the number of mappings from X to Y is given by n^m;	(13.7)
(ii) the number of 1–1 mappings from X to Y is given by $n(n-1)\cdots(n-m+1)$, where $m \le n$.	(13.8)

What can be said about the number of onto mappings from X to Y? It is interesting to note that this problem is not as straightforward as those of counting the numbers of mappings and 1–1 mappings. In the following example, we shall see how Identity (13.6) is used to tackle this more difficult problem.

Example 13.4 *Let* $X = \{1, 2, 3, 4, 5\}$ *and* $Y = \{1, 2, 3\}$. *Find the number of onto mappings from* X *to* Y.

Discussion and Solution Let S be the set of mappings from X to Y. We shall now introduce three subsets A, B, C of S as follows:

Let A be the set of mappings from X to $Y\backslash\{1\}$,

B be the set of mappings from X to $Y\backslash\{2\}$,

and C be the set of mappings from X to $Y\backslash\{3\}$.

What do the sets $\bar{A}$, $\bar{B}$ and $\bar{C}$ represent? Well, $\bar{A}$ is the set of mappings from X to Y which contain "1" in Y as an image, $\bar{B}$ is the set of mappings from X to Y which contain "2" in Y as an image, and C is the set of mappings from X to Y which contain "3" in Y as an image. It follows that $\bar{B}$ is the set of mappings from X to Y which contain "1", "2" and "3" in Y as images; that is, $\bar{A}\cap\bar{B}\cap\bar{C}$ is the set of *onto* mappings from X to Y. Thus, our task here is to evaluate $|\bar{A}\cap\bar{B}\cap\bar{C}|$. We can therefore apply (13.6)!

Since S is the set of mappings from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3\}$, by 13.7) , $|S| = 3^5$.

Since A is the set of mappings from $\{1, 2, 3, 4, 5\}$ to $\{2, 3\}$, by (13.7) again, $|A| = 2^5$.

Likewise, $|B| = |C| = 2^5$. As $A \cap B$ is the set of mappings from $\{1, 2, 3, 4, 5\}$ to $\{3\}$, by (13.7) again, $A \cap B| = 1^5 = 1$.

Similarly, $|A \cap C| = |B \text{ n } C| = 1$.

Finally, observe that $A \cap B \cap C$ is the set of mappings from X to $Y\setminus\{1,2,3\}$ $(=\emptyset)$. Thus, $A\cap B\cap C=\emptyset$ and so $|A\cap B\cap C|=0$.

Now, by (13.6), we have

$$|\bar{A}\cap\bar{B}\cap\bar{C}| = 3^5 - (2^5+2^5+2^5) + (1+1+1) - 0 = 150.$$

We have seen in this chapter how Addition Principle (13.1) can be generalised to (13.2); and, in turn, (13.2) can be extended to (13.3). Moreover, we have derived an equivalent form (13.6) of (13.3). In the next chapter, we shall introduce a more general form of (PIE) which deals with any n subsets, where $n \geq 2$, and we shall see how it can be applied to solve some interesting and more complicated problems.

Exercise

13.1 Find the number of integers from the set $\{300, 301, \dots , 1000\}$ which are multiples of 6 or 9.

13.2 How many positive integers n are there such that n is a divisor of at least one of the numbers 10^{30} , 20^{20}?

13.3 A group of students took examinations in Chemistry, Mathematics and Physics. Among them, 12 passed Chemistry, 15 Mathematics, and 10 Physics; 8 passed both Chemistry and Mathematics, 5 both Chemistry and Physics, and 6 both Mathematics and Physics. There were 20 students who passed at least one of the three subjects. Find the number of students who passed all three subjects.

13.4 Find the number of integers from the set $\{1,2,---, 1000\}$ which are

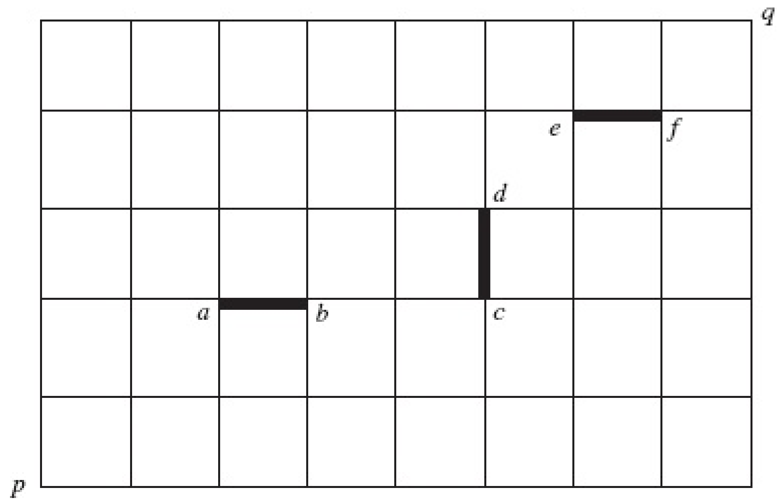
(i) divisible by at least one of 2, 3 and 5;
(ii) divisible by none of 2, 3 and 5.

13.5 Seven distinct objects are to be put into three distinct boxes. Find the number of ways this can be done if

(i) there is no restriction;
(ii) no box is empty.

13.6 The following figure shows a 5 by 8 rectangular grid with two specified corners p and q and three specified segments ab, cd and ef. Find in the grid

(i) the number of shortest p-q routes;
(ii) the number of shortest p-q routes that pass through at least one of the segments ab, cd and ef;
(iii) the number of shortest p-q routes that do not pass through any of the segments ab, cd and ef.



13.7 Let S be the set of 3-digit numbers abc such that $a,b,c \in \{1,2,\ldots, 9\}$ and a, b, c are pairwise distinct. (Thus, $489 \in S$, but $313 \notin S$ and $507 \notin S$.) Find the number of members abc in S such that $a \neq 3$, $b \neq 5$ and $c \neq 7$.

13.8 Find the number of integer solutions to the equation

$$x + y + z = 12,$$

where $0 \leq x \leq 4, 0 \leq y \leq 5$ and $0 \leq 2 \leq 6$. (See Chapter 7.)

13.9 A 5-digit ternary number is a number $x_1x_2x_3x_4x_5$, where $x_i = 0,1$ or 2 for each $i = 1,2,\ldots, 5$. Thus, 00000, 01001, 21022, 11002, etc. are 5-digit ternary numbers. Find the number of 5-digit ternary numbers in which each of the "0", "1" and "2" appears at least once.

13.10 Two scouts x_1, x_2 from School X, 3 scouts y_1, y_2, y_3 from School Y and 4 scouts z_1, z_2, z_3, z_4 from School Z get together in a meeting. In how many ways can they be arranged in a row if not all scouts from the same school are allowed to form a single block in the row? (For instance, $x_1z_3z_2y_1y_3x_2y_2z_1z_4$ is allowed, but $z_1z_4y_1y_2\underline{x_2x_1}y_3z_3z_2$ and $z_4z_3x_1\underline{y_3y_1y_2}x_2z_1z_2$ and $z_4z_3x_1\underline{y_3y_1y_2}x_2z_1z_2$ are not allowed.)

Chapter 14

General Statement of the Principle of Inclusion and Exclusion

In Chapter 13, we introduced the Principle of Inclusion and Exclusion (PIE) by first deriving the identity

$$\boxed{|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|} \tag{14.1}$$

for two finite sets A_1 and A_2, and then extending it to the following identity:

$$\boxed{\begin{aligned}|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_1 \cap A_3| \\ + |A_2 \cap A_3|) + |A_1 \cap A_2 \cap A_3|\end{aligned}} \tag{14.2}$$

for three finite sets A_1, A_2 and A_3. Naturally, one would like to know whether (14.1) and (14.2) could be extended to an identity involving any n (≤ 2) finite sets A_i, $A_2, \ldots, A_n$ and if so, what identity would one get in general. The main objective of this chapter is to deal with this problem. We shall first extend (14.2) to an identity involving four sets, and then by observing these special cases, we will obtain the general statement of the (PIE) for any finite number of finite sets. Finally, two examples will be given to illustrate the application of the general statement of the (PIE).

Suppose that four finite sets A_1, A_2, A_3 and A_4 are given. By applying (14.1), (14.2) and some basic laws for sets, we have

$$\begin{aligned}
&|A_1 \cup A_2 \cup A_3 \cup A_4| \\
&\quad = |(A_1 \cup A_2 \cup A_3) \cup A_4| \\
&= |A_1 \cup A_2 \cup A_3| + |A_4| - |(A_1 \cup A_2 \cup A_3) \cap A_4| \quad (\text{by } (14.1)) \\
&= |A_1 \cup A_2 \cup A_3| + |A_4| - |(A_1 \cap A_4) \cup (A_2 \cap A_4) \cup (A_3 \cap A_4)| \\
&= (|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| \\
&\quad + |A_1 \cap A_2 \cap A_3|) + |A_4| - (|A_1 \cap A_4| + |A_2 \cap A_4| + |A_3 \cap A_4| \\
&\quad - |A_1 \cap A_2 \cap A_4| - |A_1 \cap A_3 \cap A_4| - |A_2 \cap A_3 \cap A_4| \\
&\quad + |A_1 \cap A_2 \cap A_3 \cap A_4|) \quad (\text{by } (14.2)).
\end{aligned}$$

That is,

$$\boxed{\begin{aligned}
&|A_1 \cup A_2 \cup A_3 \cup A_4| \\
&\quad = (|A_1| + |A_2| + |A_3| + |A_4|) \\
&\qquad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| \\
&\qquad + |A_3 \cap A_4|) + (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| \\
&\qquad + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) \\
&\qquad - |A_1 \cap A_2 \cap A_3 \cap A_4|.
\end{aligned}} \tag{14.3}$$

Now, let us look at the identities (14.1)–(14.3) carefully and make some observations on the patterns of the terms on the RHS of the identities.

For the sum of the terms within the first grouping, we have:

n	Sum	Number of terms in the sum
2	$\lvert A_1\rvert + \lvert A_2\rvert$	$2 = \binom{2}{1}$
3	$\lvert A_1\rvert + \lvert A_2\rvert + \lvert A_3\rvert$	$3 = \binom{3}{1}$
4	$\lvert A_1\rvert + \lvert A_2\rvert + \lvert A_3\rvert + \lvert A_4\rvert$	$4 = \binom{4}{1}$

For the sum of terms within the second grouping, we have:

n	Sum	Number of terms in the sum
2	$\lvert A_1 \cap A_2\rvert$	$1 = \binom{2}{2}$
3	$\lvert A_1 \cap A_2\rvert + \lvert A_1 \cap A_3\rvert + \lvert A_2 \cap A_3\rvert$	$3 = \binom{3}{2}$
4	$\lvert A_1 \cap A_2\rvert + \lvert A_1 \cap A_3\rvert + \lvert A_1 \cap A_4\rvert$ $+ \lvert A_2 \cap A_3\rvert + \lvert A_2 \cap A_4\rvert + \lvert A_3 \cap A_4\rvert$	$6 = \binom{4}{2}$

For the sum of terms within the third grouping, we have:

n	Sum	Number of terms in the sum
2	none	0
3	$\lvert A_1 \cap A_2 \cap A_3\rvert$	$1 = \binom{3}{3}$
4	$\lvert A_1 \cap A_2 \cap A_3\rvert + \lvert A_1 \cap A_2 \cap A_4\rvert$ $+ \lvert A_1 \cap A_3 \cap A_4\rvert + \lvert A_2 \cap A_3 \cap A_4\rvert$	$4 = \binom{4}{3}$

We also notice that the groupings alternate in sign, beginning with a (+) sign

Suppose now that we are given n finite sets: A_1, A_2, … , A_n. By generalising the above observations, what identity would you expect for $|A_1 \cup A_2 \cup \cdots \cup A_n|$?

The first grouping should be the sum of $\binom{n}{1} = n$ terms, each involving a single set:

$$|A_1| + |A_2| + \cdots + |A_n|;$$

in abbreviation,

$$\sum_{i=1}^{n} |A_i|.$$

The second grouping should be the sum of $\binom{n}{2}$ terms, each involving the intersection of two sets:

$$|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{n-1} \cap A_n|;$$

in abbreviation,

$$\sum_{i<j} |A_i \cap A_j|.$$

The third grouping should be the sum of $\binom{n}{3}$ terms, each involving the intersection of three sets:

$$|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|;$$

in abbreviation,

$$\sum_{i<j<k} |A_i \cap A_j \cap A_k|.$$

Likewise, the fourth grouping should be the sum of $\binom{n}{4}$ terms, each involving the intersection of four sets:

$$\sum_{i<j<k<l} |A_i \cap A_j \cap A_k \cap A_l|;$$

and so on.

Bearing in mind that the groupings alternate in sign, beginning with a "+" sign, one would expect that the following holds:

$$\boxed{\begin{aligned}&|A_1 \cup A_2 \cup \cdots \cup A_n| \\ &\quad = \sum_{i=1}^{n} A_i - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots \\ &\quad + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.\end{aligned}} \qquad (14.4)$$

Indeed, it can be proved (for instance, by mathematical induction) that (14.4) holds for any n finite sets $A_1, A_2, \ldots, A_n$.

We shall now show an application of (14.4) by considering the following:

Example 14.1 *How many ways are there to arrange n* (≥2) *married couples in a row so that at least one couple are next to each other*?

Discussion and Solution Denote the n husbands and the n wives of the n couples by H_1, W_1, H_2, W_2, …, H_n, W_n. Thus, when $n = 4$ for example, the following arrangements are allowed:

$$W_2 H_1 H_4 \underline{W_3 H_3} H_2 W_1 W_4, \quad H_4 \underline{H_2 W_2} W_4 H_3 \underline{W_1 H_1} W_3.$$

Solving the above problem by dividing it into cases such that exactly one couple are next to each other, exactly two couples are next to each other, and so on would be very complicated. Let us try to apply (14.4).

For each $i = 1,2,\ldots,n$, let A_i be the set of arrangements of the n couples such that H_i and W_i are adjacent (next to each other). The problem is thus to enumerate $|A_1 \cup A_2 \cup \cdots \cup A_n|$.

To apply (14.4), we compute each grouping on its RHS.

To compute $\sum_{i=1}^{n} |A_i|$, we first consider $|A_1|$. A_1 is the set of arrangements of n couples such that H_1 and W_1 are adjacent. This is the same as arranging the $2n - 1$ objects:

$$H_1 W_1, H_2, W_2, \ldots, H_n, W_n$$

in a row where $H_1 W_1$ can be permuted in two ways: $H_1 W_1$ and W1H1. Thus,

$$|A_1| = 2 \cdot (2n-1)!.$$

Similarly, $|A_i| = 2 \cdot (2n-1)!$ for each $i = 2,3,\ldots,n$. Thus,

$$\sum_{i=1}^{n} |A_i| = \binom{n}{1} \cdot 2 \cdot (2n-1)!.$$

To compute $\sum_{i<j} |A_i \cap A_j|$, we first consider $|A_1 \cap A_2|$. $A_1 \cap A_2$ is the set of arrangements of the n couples such that H_1 and W_1 are adjacent and H_2 and W_2 are adjacent. This is the same as arranging the $2n - 2$ objects:

$$H_1 W_1, H_2 W_2, H_3, W_3, \ldots, H_n, W_n$$

in a row where both $H_1 W_1$ and $H_2 W_2$ can be permuted by themselves. Thus,

$$|A_1 \cap A_2| = 2^2 \cdot (2n-2)!.$$

Similarly, for $1 \le i < j \le n$, $|A_i \cap A_j| = 2^2 \cdot (2n-2)!$. Thus,

$$\sum_{i<j} |A_i \cap A_j| = \binom{n}{2} \cdot 2^2 \cdot (2n-2)!.$$

We now leave it to the reader to show that

$$\sum_{i<j<k} |A_i \cap A_j \cap A_k| = \binom{n}{3} \cdot 2^3 \cdot (2n-3)!,$$

and so on to obtain the following final result that

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} \cdot 2^r \cdot (2n-r)!.$$

For the case when $n = 4$, we have

$$\begin{aligned} &|A_1 \cup A_2 \cup A_3 \cup A_4| \\ &= \binom{4}{1} \cdot 2 \cdot (8-1)! - \binom{4}{2} \cdot 2^2 \cdot (8-2)! + \binom{4}{3} \cdot 2^3 \cdot (8-3)! \\ &\quad - \binom{4}{4} \cdot 2^4 \cdot (8-4)! \\ &= 8 \cdot 7! - 24 \cdot 6! + 32 \cdot 5! - 16 \cdot 4! \\ &= 26496. \end{aligned}$$

Next, we shall introduce an old problem regarding decks of cards. Two decks X, Y of cards, with 52 cards each, are given. The 52 cards of X are first laid out. Those of Y are then placed randomly, one each on top of each card of X, so that 52 pairs of cards are formed. The question is: what is the probability that no cards in each pair are identical (i.e. having the same suit and rank)? This problem, known as *"le probleme des rencontres"* (the matching problem), was introduced and studied by the Frenchman Pierre Remond de Montmort (1678–1719) around the year 1708. The number of ways of distributing the cards of Y to form 52 pairs of cards with those in X is clearly 52!. Thus, to find the desired probability, we need to find out the number of ways of distributing the cards of Y such that each card in Y is placed at the top of a different card in X.

Instead of solving the above problem directly, let us generalise it and consider the following more general problem. A permutation $a_1 a_2 \ldots a_n$ of $\mathbb{N}_n$ is called a *derangement* of $\mathbb{N}_n$ if $a_i \neq i$ for each $i \neq 1, 2, \ldots, n$. Thus 54132 is a derangement of $\mathbb{N}_5$ but 51<u>342</u> and 32154 are not. For n = 1, 2, 3, 4, all the derangements of $\mathbb{N}_n$ are shown in the following table.

n	Derangements
1	None
2	21
3	231, 312
4	2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321

Let D_n denote the number of derangements of $\mathbb{N}_n$. It follows from the table above that $D_1 = 0$, $D_2 = 1$, $D_3 = 2$ and $D_4 = 9$. Returning back to the matching problem, it is now clear that its answer is given by $\frac{D_{52}}{52!}$. How do we evaluate D_n for each n? After some thought, you may realise that this is not a trivial problem. Well, we are given a good opportunity to show our second application of (PIE).

Before proceeding any further, let us first derive an equivalent form of (14.4).

For a subset A of a universal set S, recall that A denotes its *complement*. It was pointed out in Chapter 13 that (14.2) is equivalent to the following: For any subsets A_1, A_2, A_3 of S,

$$\begin{aligned} |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| \\ + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

In general, for any n (≥ 2) subsets A_1, A_2, ...,A_n of S, one can show that (14.4) is equivalent to the following:

$$\boxed{\begin{aligned}&|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}|\\&\quad=|S|-|A_1\cup A_2\cup\cdots\cup A_n|\\&\quad=|S|-\sum_{i=1}^{n}|A_i|+\sum_{i<j}|A_i\cap A_j|-\sum_{i<j<k}|A_i\cap A_j\cap A_k|\\&\qquad+\cdots+(-1)^n|A_1\cap A_2\cap\cdots\cap A_n|.\end{aligned}} \tag{14.5}$$

We shall now evaluate D_n by applying (14.5). Let us first identify what the universal set is. We are now concerned with derangements, which are special types of permutations of $\mathbb{N}_n$. So, let the universal set S be the set of all permutations of $\mathbb{N}_n$.

For each $i = 1,2,\ldots,n,$ let A_i be the set of permutations $a_1a_2 \ldots a_n$ in S such that $a_i = i$. Thus, $\overline{A_i}$ is the set of permutations in S such that $a_i \neq i$, and so $\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}$ is the set of permutations in S such that $a_i = i$ for all $i = 1,2,\ldots,n,$ which is exactly the set of derangements of $\mathbb{N}_n$. We thus have

$$D_n=|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}|.$$

To evaluate D_n by (14.5), we evaluate each grouping on the RHS of (14.5). Clearly, as S is the set of all permutations of $\mathbb{N}_n$, we have $|S| = n!$.

Observe that A_1 is the set of permutations of the form $1a_2a_3 \ldots a_n$. Thus, $|A_1| = (n-1)!$ Similarly, $|A_i| = (n-1)!$ for each $i = 2,3,\ldots ,n,$ and so

$$\sum_{i=1}^{n}|A_i|=n\cdot(n-1)!=\binom{n}{1}\cdot(n-1)!.$$

As $A_1\cap A_2$ is the set of permutations of the form $12a_3a_4 \ldots a_n$, we have $|A_1\cap A_2| = (n-2)!$. Similarly, $|A_i\cap A_j| = (n-2)!$ for all $i, j\in \{1,2,\ldots, n\}$ with $i < j$. There are $\binom{n}{2}$ number of ways of choosing i and j from $\{1,2,\ldots, n\}$ with $i<j$, and so

$$\sum_{i<j}|A_i\cap A_j|=\binom{n}{2}\cdot(n-2)!.$$

We now leave it to the reader to show that

$$\sum_{i<j<k}|A_i\cap A_j\cap A_k|=\binom{n}{3}\cdot(n-3)!$$

and so on to obtain the following final result by (14.5) that

$$\begin{aligned}D_n&=|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}|\\&=n!-\binom{n}{1}\cdot(n-1)!+\binom{n}{2}\cdot(n-2)!-\binom{n}{3}\cdot(n-3)!\\&\quad+\cdots+(-1)^n\binom{n}{n}\cdot(n-n)!.\end{aligned}$$

Note that for $r = 1,2,\ldots,n,$

$$\binom{n}{r}\cdot(n-r)!=\frac{n!}{r!(n-r)!}(n-r)!=\frac{n!}{r!}.$$

Thus,

$$\begin{aligned}D_n&=n!-\frac{n!}{1!}+\frac{n!}{2!}-\frac{n!}{3!}+\cdots+(-1)^n\frac{n!}{n!}\\&=n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}\right).\end{aligned}$$

Suppose we generate a permutation of $\mathbb{N}_n$ at random. The probability that this permutation is a derangement is given by $\frac{D_n}{n!}$ which by the above result is

$$\frac{D_n}{n!}=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}.$$

When n gets larger and larger, it is known that the quotient $\frac{D_n}{n!}$ gets closer and closer to $\frac{1}{e}(\approx 0.367)$, where the constant e, called the natural exponential base, is defined by $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$. It is known that $e \approx 2.718281828459045$. (The letter "e" was chosen in honour of the great Swiss mathematician Leonhard Euler (1707–1783) who made some significant contributions to the study of problems related to the limit above.)

We make a final remark which is useful when considering whether to use the general statement of the (PIE) to enumerate $|A|$ for a finite set A. In the event that it is not easy to partition A, i. e. to divide it into cases of mutually exclusive subsets, we may ask the question: Can we find sets A_i, $i = 1,2,\ldots,n$, which are "easy" to count but with no necessity for them to be mutually exclusive, such that either $A = A_1 \cup A_2 \cup \cdots \cup A_n$ or $A = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$?

Exercise

14.1 Show that the number of ways to seat $n(\geq 2)$ married couples round a table so that at least one couple are next to each other is

$$\sum_{r=1}^{n}(-1)^{r+1}\binom{n}{r}\cdot 2^r\cdot(2n-r-1)!.$$

14.2 A lottery is run with each ticket bearing a distinct 7-digit number. Every digit is chosen from the digits 1, 2, 3, 4, 5, 6, 7 and all digits in the number are distinct. Only one prize is given. If all possible tickets are sold, what is the probability that a randomly chosen ticket has at least 4 digits matching those of the winning ticket?

14.3 Show that the number of integer solutions to the equation (see Chapter 7)

$$x_1+x_2+\cdots+x_{11}=50$$

such that $0 \leq x_r \leq 9$ for each $r = 1, 2,\ldots, 11$ is given by

$$\sum_{r=0}^{5}(-1)^r\binom{11}{r}\binom{10(6-r)}{10}.$$

14.4 Each of ten ladies checks her hat and umbrella in a cloakroom and the attendant gives each lady back a hat and an umbrella at random. Show that the number of ways this can be done so that no lady gets back both of her possessions is

$$\sum_{r=0}^{10}(-1)^r\binom{10}{r}\{(10-r)!\}^2.$$

14.5 Show that the number of onto mappings from $\mathbb{N}_m$ to $\mathbb{N}_n$, where $m \geq n \geq 1$, is given by

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}(n-r)^m.$$

14.6 A football team has five different jerseys. The team takes part in a tournament where they have to play 8 matches.

(i) Find the number of ways the team wear their jerseys for the 8 matches

(a) if there are no restrictions on the choices,

(b) if the team never wears the same jersey on consecutive matches,

(c) if the team uses at most three jerseys.

(ii) Given that each jersey is used at least once, show that the number of ways the team can choose their jerseys is

$$5^8-\binom{5}{1}4^8+\binom{5}{2}3^8-\binom{5}{3}2^8+\binom{5}{4}1^8.$$

(iii) State what the following expression represents as far as choosing jerseys is concerned:

$$5^n-\binom{5}{1}4^n+\binom{5}{2}3^n-\binom{5}{3}2^n+\binom{5}{4}1^n.$$

Hence find the value of this expression for $1 \le n < 5$.

(iv) Show that

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^n = n!.$$

Chapter 15

The Pigeonhole Principle

Think of any four integers. We assure that among these four integers, there are two whose difference is divisible by 3. Should you be surprised?

In this chapter, we shall introduce a principle known as the Pigeonhole Principle from which we will be able to deduce the certain existence of objects with some specified properties such as that in the opening assertion.

Suppose 5 pigeons are to be put into two compartments A, B. They may be distributed in the following ways: 5 in compartment A and 0 in compartment B, 4 in A and 1 in B, 3 in A and 2 in B, 2 in A and 3 in B, 1 in A and 4 in B, or 0 in A and 5 in B. We observe that at least one of the compartments will contain at least 3 pigeons. The general statement of this simple observation, known as the Pigeonhole Principle, is given below.

> **The Pigeonhole Principle (PP)**
> Let k and n be any two positive integers. If at least $kn+1$ objects are distributed among n boxes, then one of the boxes must contain at least $k+1$ objects. (15.1)

We shall prove the Pigeonhole Principle by proving the contra- positive statement "If none of the n boxes contain at least $k + 1$ objects, then no more than kn objects are distributed among the n boxes." Thus, if none of the n boxes contain at least $k + 1$ objects, then each box contains at most k objects. Summing up the total number of objects in the n boxes, we have that there are not more than kn objects in total. This proves the contrapositive statement and so proves the Pigeonhole Principle.

Recall that $\lceil x \rceil$ is the least integer not less than x. A more general way of stating the Pigeonhole Principle is as follows.

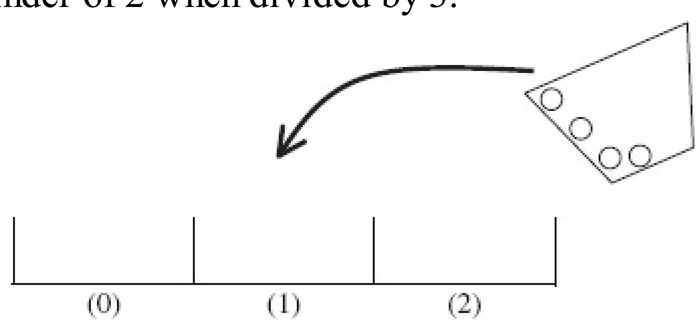
> **The Pigeonhole Principle (PP)**
> Let m and n be any two positive integers. If m objects are distributed among n boxes, then one of the boxes must contain at least $\lceil \frac{m}{n} \rceil$ objects. (15.2)

The proof for (15.2) is similar to that for (15.1). We leave it as a problem in the Exercise.

We shall now show an application of (PP) by posing a problem based on our assertion at the beginning of this chapter.

Example 15.1 *Show that, among any four integers, there are two integers whose difference is divisible by* 3.

Solution We treat the 4 integers as 4 objects, and create three "boxes": Box (0) for integers which are divisible by 3, Box (1) for integers which leave a remainder of 1 when divided by 3, and Box (2) for integers which leave a remainder of 2 when divided by 3.



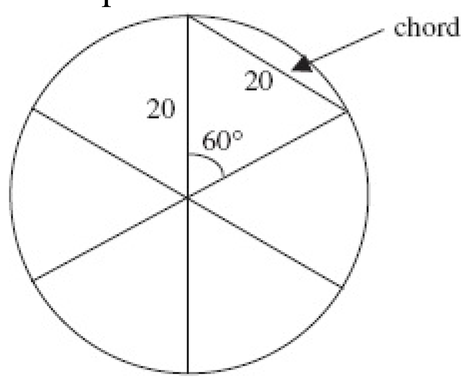
By (PP), there is at least one box with at least two integers. If the two integers are in Box (i), where $i = 0$, 1 or 2, then we may express the integers as $3x + i$ and $3y + i$, where x and y are integers. We may assume that $x \geq y$. Thus, the difference between the two integers is $(3x+i)-(3y+i)=3(x-y)$, which is divisible by 3.

(PP) is also known as the *Dirichlet Drawer Principle,* after the German mathematician Peter G. L. Dirichlet (1805–1859) who used it to prove some results in Number Theory. (PP), like the other counting principles we have encountered, looks almost trivial. However, a Principle makes itself most useful by *suggesting an approach* to attacking a problem. For example, the Addition Principle (AP) suggests that to count the items in a set, it may be better to divide the set into smaller sets which are easier to count by themselves; the Multiplication Principle (MP) suggests that to count the number of ways a process, which is made up of a number of stages, can happen, we may approach the problem by counting the number of ways each stage can happen; the Bijection Principle (BP) suggests an approach where we count an "easier" set of items for which there is a bijection to the set of items we are interested in. The Pigeonhole Principle suggests that to prove some "existence" statements in mathematics, we try to transform the problem partly into one of distributing a number of objects into a number of boxes. The questions to focus on then become "What are the objects?" and "What are the boxes?"

We shall illustrate the use of (PP) with a few more examples.

Example 15.2 *Seven darts are thrown at a dartboard which is circular and of radius* 20 *cm. If all the darts land within the dartboard, show that there are two darts not more than* 20 *cm apart.*

Solution We divide the dartboard into 6 equal sectors as shown in the figure below.



By (PP), there is at least one sector with at least two dart points. It can be shown that the maximum distance between any two points in a sector is either the length of the radius or the length of the chord. In either case, since the arc of the sector subtends an angle of 60° at the centre, the two darts are not more than 20 cm apart.

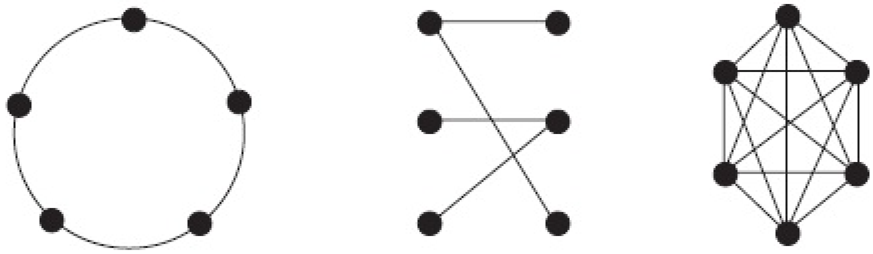
Example 15.3 *Consider a group of* 6 *persons, any two of whom are either mutual acquaintances or do not know each other. Prove that in the group, either there are* 3 *persons who are mutual acquaintances or there are* 3 *persons who do not know each other.*

Solution Let us fix one person, say A. "Distribute" the other 5 persons into two "boxes" — the box containing those who are mutual acquaintances with A and the box containing those who do not know A (and are not known by him). By (PP), at least one of the boxes contains at least $\lfloor \frac{5}{2} \rfloor = 3$ persons.

Suppose the box containing those who are mutual acquaintances with A contains at least 3 persons. If at least two of these persons are mutual acquaintances, then we have 3 persons (these two together with A) who know each other. If none of these persons know each other, then these persons will form a group of at least 3 persons who are complete strangers to one another.

On the other hand, suppose the box containing those that do not know A contains at least 3 persons. If at least two of these persons do not know each other, then we have 3 persons (these two together with A) who do not know each other. If none of these persons do not know each other, then these persons will form a group of at least 3 persons who are mutual acquaintances.

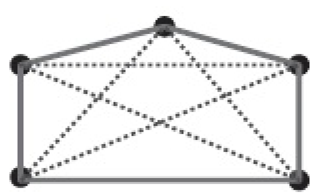
Example 15.3 above can also be proved by modelling the situation with a graph. We shall give a brief introduction of graphs. A *graph G* consists of a set *V* of vertices (or points) and a set *E* of edges, each of which joins a pair of vertices. In this book, we shall restrict ourselves to simple graphs, where a pair of vertices is joined by at most one edge and where each edge joins a pair of distinct vertices. The figure below shows three examples of simple graphs, which, in this book, we shall just call graphs henceforth. Graph Theory is a fascinating and rich field of study in mathematics, with many real-life applications as well as theoretical depth and aesthetic appeal.



We shall now return to Example 15.3 and use graphs to solve the problem. Represent the 6 persons by vertices and join every pair of vertices with an edge. As a result, we have the third graph in the figure above. Given any two vertices, the edge joining them is coloured red if the two persons represented by the vertices know each other; otherwise, the edge is coloured blue.

Consider a vertex *A*. The 5 edges incident with *A* are coloured either red or blue. By (PP), at least 3 of them are coloured one colour, say red (the argument is similar for blue). Let the 3 vertices, other than *A*, incident to these 3 red edges be *B, C* and *D*. If any two of the three, say *B* and *C* are joined by a red edge, we have a red triangle joining *A*, *B* and *C*. Otherwise, there is no red edge between the 3 vertices and this implies that there is a blue triangle joining *B*, *C* and *D*. A red triangle represents 3 persons who are mutual acquaintances and a blue triangle represents 3 persons who do not know each other.

A graph with edges connecting all pairs of vertices is called a complete graph. (How many edges are there in a complete graph with *n* vertices?) Denote a complete graph with *n* vertices as K_n. The problem above shows that if we colour the edges of K_6 using two colours, we cannot avoid either having a subgraph K_3 containing all red edges or a subgraph K_3 containing all blue edges. Can we avoid having a subgraph K_3 containing all red edges or a subgraph K_3 containing all blue edges if we two-colour (i.e. colour with two colours) the edges of K_5? As it turns out, the answer is Yes — see the figure below.



Clearly then, the least integer *p* such that a two-colouring of the edges of K_p will contain either a “red” K_3 or a “blue” K_3 is 6. More generally, we can imagine two-colouring the edges of K_p and asking the question:

Given integers *m* and n, what is the least integer *p* such that a two- colouring of the edges of K_p will contain either a “red” K_m or a “blue” K_n?

This least integer is called the Ramsey number for *m* and *n*, and is denoted by *r(m, n)* in honour of its originator, Frank Ramsey (1903–1930). Frank Ramsey produced work in foundations of mathematics, economics and philosophy. His paper on mathematics, *On a Problem of Formal Logic,* examines methods for determining the consistency of a logical formula and it includes some theorems on combinatorics which have led to the study of a whole new area of mathematics called Ramsey theory. His vast potential was unfortunately not fully realised when he passed away at the early age of 27.

Frank Ramsey

It is always interesting in mathematics to try to extend a concept as far as possible. One way to extend the concept of the Ramsey numbers is the following:

Given integers $n_1, n_2, \ldots, n_r$, what is the least integer p such that an r-colouring of the edges of K_p will contain one of K_{ni} coloured with colour i, where $1 \leq i \leq r$?

We denote this number by $r(n_1, n_2, \ldots, n_s)$. Thus, for example, $r(3, 4, 5)$ is the least integer p such that a 3-colouring (say, red, blue, green) of the edges of K_p will contain either a red K_3, a blue K_4 or a green K_5. The following example illustrates how (PP) can be used to help find $r(3, 3, 3)$.

Example 15.4 *In a more realistic situation, any two persons in a group either know each other, do not know each other, or are such that one knows the other but is not known by her/him. Let us call the third situation a "one-way relationship". Prove that in a group of* 17 *mathematicians chosen randomly from around the globe, there will be three who are mutual acquaintances, or mutual strangers, or are among themselves in one-way relationships.* (A similar problem was posed in IMO 1964/4.)

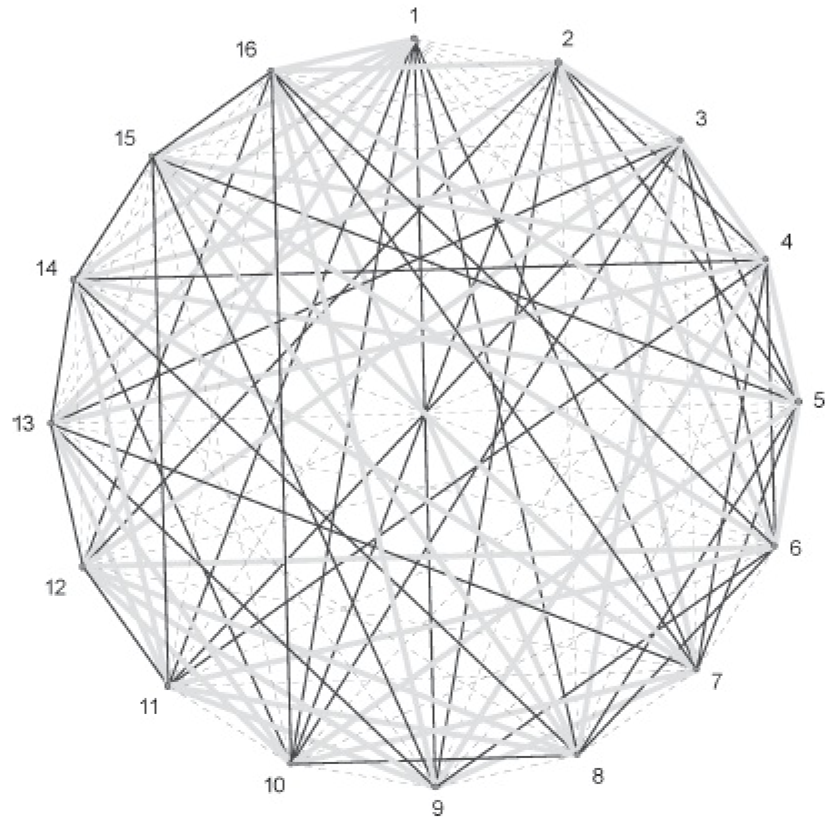
Solution Let the 17 persons be 17 vertices in a complete graph K_{17}. Colour the edges of the graph with 3 colours: red, blue and green to represent mutual friends, mutual strangers and one-way relationship, respectively.

Consider one vertex A. There are 16 edges incident to A, so by (PP), $\lceil \frac{16}{3} \rceil = 6$ edges must be coloured one colour, say red. If among the 6 vertices incident to A via red edges, there is a red edge between two of them, then these two vertices together with A will form a red K_3.

Suppose otherwise. Then the edges between the 6 vertices must be coloured with only two colours, blue or green. This situation is similar to that of Example 15.3. Thus, there must be either a blue K_3 or a green K_3 contained in the subgraph induced by the 6 vertices, and so there must be either a blue K_3 or a green K_3 contained in the original graph K_{17}.

Hence, a 3-colouring of K_{17} contains either a red K_3, a blue K_3 or a green K_3, i.e. among the 17 mathematicians, there are three who are mutual acquaintances, or mutual strangers, or are among themselves in one-way relationships.

Our solution of Example 15.4 above shows that the least integer p such that a 3-colouring of K_p will contain one of a red K_3, a blue K_3 or a green K_3 does not exceed 17, i.e. $r(3, 3, 3) \leq 17$. To show that $r(3, 3, 3)$ is indeed equal to 17, what do we need to do? Just as in showing that $r(3, 3) = 6$ by producing a 2-colouring of K5 that does not contain either a red K3 nor a blue K_3, we now need to produce a 3-colouring of K_{16} that does not contain any red K_3, blue K_3 or green K_3. The following figure shows such a 3-colouring of K_{16}.



3-colouring of K_{16} that does not contain any dark line K_3, grey line K_3 or dash line K_3

(PP) crops up in many areas of mathematics. To end this chapter, we show two more examples which involve sequences and functions. As stated earlier, the Pigeonhole Principle suggests that to prove some "existence" statements in mathematics, we try to transform the problem partly into one of distributing a number of objects into a number of boxes. We then focus on the questions: "What are the objects?" and "What are the boxes?"

Given a sequence of numbers, a *subsequence* of the original sequence is a sequence of numbers obtained by deleting some of the numbers from the original sequence. For example, 1, 7, 6 is a subsequence of 1, 4, 5, 7, 6, 8, 11, 10 which is itself a subsequence of 1, 3, 2, 4, 5, 7, 6, 8, 9, 11, 10, 12, …

Example 15.5 *Prove that in a sequence of* 10 *distinct numbers, there is either an increasing subsequence of* 4 *numbers or a decreasing subsequence of* 4 *numbers.*

Before we begin our solution, we introduce some notation for subsequent ease of explanation. Let $x_1, x_2,\dots ,x_{10}$ be a sequence of 10 distinct numbers. Let $p(x_i)$ be the number of terms in the longest increasing subsequence starting at x_i and $q(x_i)$ be the number of terms in the longest decreasing subsequence starting at x_i. For example, in the sequence of 10 distinct numbers 4, 6, 3, 2, 7, 8, –1, 1, 0, 5, we have: $p(x_1) = |\{4, 6, 7, 8\}| =4$ and $q(x_1) = |\{4, 3, 2,1, 0\}| = 5$; $p(x_2) = |\{6, 7, 8\}| =3$ and $q(x_2) = |\{6, 3, 2,1, 0\}| = 5$; and so on till $p(x1_0) = |\{5\}| = 1$ and $q(x_{10}) = |\{5\} = 1$.

Solution We shall prove the proposition by contradiction. Suppose there exists a sequence of 10 distinct numbers $x_1, x_2,\dots,x_{10}$ where the longest increasing subsequence and the longest decreasing subsequence do not exceed 3 numbers.

It is not obvious at this point what the boxes and the objects are. Since the problem has to do with numbers of terms of subsequences, let us explore in this area.

Now, by the assumption, $1 \le p(x_i) \le 3$ and $1 \le q(x_i) \le 3$ for $i = 1,2,\dots, 10$, i.e. the ordered pairs $(p(x_i),q(x_i))$ can only be from the set $\{(1,1), (1,2), (1,3), (2,1),\dots, (3,3)\}$. We can easily count the number of such possible distinct ordered pairs directly as 9 (= 3 × 3).

At this juncture, the numbers '10' and '9' have appeared and since 10 = 9 + 1, it seems that there is a possibility of using (PP). Are the objects the numbers *xi*? Are the boxes the distinct ordered pairs from the set $\{(1,1), (1,2), (1,3),\dots, (3,3)\}$? Indeed, they are!

We place the numbers x_i into these "boxes" of distinct ordered pairs so that if $(p(x_i),q(x_i)) = (a,b)$, then x_i is placed in the box (a,b). By (PP), there are at least two numbers x_i, x_j, for some i, j, where $1 \le i < j \le 10$, in at least one box. This means that $(p(x_i), q(x_i)) = (p(x_j), q(x_j))$.

Note that $x_i \neq x_j$ since all the numbers are distinct. If $x_i < x_j$, add x_i to the longest increasing subsequence starting with x_j. This will result in an increasing subsequence starting with x_i that has more terms than the longest increasing subsequence starting with x_j. Thus, $p(x_i) > p(x_j)$. (The values of $p(x_1)$ and $p(x_2)$ in the example above show this situation: $p(x_1) = |\{4, 6, 7, 8\}| > |\{6, 7, 8\}| = p(x_2)$.)

On the other hand, if $x_i > x_j$, add x_i to the longest decreasing subsequence starting with x_j. The result will be a decreasing subsequence starting with x_i that has more terms than the longest decreasing subsequence starting with x_j. In this case, $q(x_i) > q(x_j)$. (The values of $q(x_2)$ and $q(x_{10})$ in the example above show this situation: $q(x_2) = |\{6, 3, 2, 1, 0\}| > |\{6, 5\}| > |\{5\}| = q(x_{10})$.)

In either case, the ordered pairs $(p(x_i), q(x_i))$ and $(p(x_j), q(x_j))$ must be distinct, a contradiction. Hence, there is either an increasing subsequence of 4 numbers or a decreasing subsequence of 4 numbers.

Example 15.6 *Let f be a bijection from* $\mathbb{N}_n$ *onto* $\mathbb{N}_n$. *Let* $f^k = \overbrace{f \circ f \circ \cdots \circ f}^{k \text{ terms}}$ *denote the composition of k number of f's. (For example,* $f^3(x) = f(f(f(x)))$.) *Show that there are distinct positive integers i and j such that* $f^i(x) = f^j(x)$ *for all* $x \in \mathbb{N}_n$. *Show also that for some positive integer k,* $f^k(x) = x$ *for all* $x \in \mathbb{N}_n$.

Solution Observe that f^k, for all $k \in \mathbb{N}_n$, is a bijection from $\mathbb{N}_n$ onto $\mathbb{N}_n$. Thus, f^k can be seen as a permutation of $\mathbb{N}_n$. There are *n!* permutations of $\mathbb{N}_n$ and these can be seen as boxes in which we will place the f^k's. However, there is an infinite number of k's, and so by (PP), there is at least one box with at least two f^k's. Hence, there are distinct positive integers i and j such that $f^i(x) = f^j(x)$ for all $x \in \mathbb{N}_n$.

Let i and j, where $i < j$, be such that $f^i(x) = f^j(x)$ for all $x \in \mathbb{N}_n$. Now, $f^j(x) = f^{j-i}(f^i(x)) = f^{j-i}(f^j(x))$ for all $x \in \mathbb{N}_n$. Rewriting, we have $f^{j-i}(f^j(x)) = f^j(x)$ for all $x \in \mathbb{N}_n$. Since f^j is a permutation of N_{ra}, we have that $f^{j-i}(x) = x$ for all $x \in \mathbb{N}_n$.

Exercise

15.1 Each cell in a 6 × 6 grid is filled with one of the numbers 1, 2, 3. Prove that of the sums along the rows, the columns and the diagonals, two sums must be equal.

15.2 Five lattice points are chosen on an $n \times n$ square lattice. Line segments are drawn between every pair of these points. Prove that one of the midpoints of these line segments is also a lattice point.

15.3 Show that for any set of 19 points chosen within a square whose sides are of length 4 units, there are two points in the set whose distance apart is at most $\sqrt{2}$ units. Show also that there is also a triangle formed from three points whose area is at most $\frac{8}{9}$ square units.

15.4 Find the largest integer k in terms of n such that in a sequence of n distinct numbers, there is either an increasing subsequence of k numbers or a decreasing subsequence of k numbers.

15.5 Prove (15.2), a more general statement of the Pigeonhole Principle:

> **The Pigeonhole Principle (PP)**
> Let m and n be any two positive integers. If m objects are distributed among n boxes, then one of the boxes must contain at least $\lceil \frac{m}{n} \rceil$ objects.

15.6 Prove the Generalised Pigeonhole Principle (GPP) as stated below:

> **The Generalised Pigeonhole Principle (GPP)**
> Let $n, k_1, k_2, \ldots, k_n \in \mathbb{N}$. If at least $k_1 + k_2 + \cdots + k_n - (n-1)$ objects are distributed among n boxes, then, for some $i = 1, 2, \ldots, n$, the ith box contains at least k_i objects.

15.7 Given 16 distinct positive integers, each less than 62, show that at least three pairs of them have the same absolute difference (the pairs are distinct but need not be disjoint as sets, for instance, $\{2, 3\}$ and $\{3, 4\}$ are considered as two pairs with the same absolute difference).

15.8 What is the largest number of kings that can be placed on a normal 8 by 8 chessboard so that none of them can take the other in the next move?

15.9 A student gave the following "proof" for the second part of Example 15.6. State whether you agree with the "proof". If not, explain why the "proof" is fallacious.

Problem Let f be a bijection from $\mathbb{N}_n$ onto $\mathbb{N}_n$ Show that for some positive integer k, $f^k(x) = x$ for all $x \in \mathbb{N}_n$.

Proof Observe that f^k, for all $k \in \mathbb{N}$, is a bijection from $\mathbb{N}_n$ onto $\mathbb{N}_n$ Thus, f^k can be seen as a permutation of $\mathbb{N}_n$. There are n! permutations of $\mathbb{N}_n$ and these can be seen as boxes in which we will place the f^k's. Note that the identity permutation is one of these boxes. There is an infinite number of k's, and so by (PP), there is at least one f^k in the box for the identity permutation. Hence, there is a positive integer k such that $f^k(x) = x$ for all $x \in \mathbb{N}_n$.

Chapter 16

Recurrence Relations

Let us begin our discussion by considering the following counting problem.

Example 16.1 *Figure 16.1 shows a 9-step staircase. A boy wishes to climb the staircase up to the highest step. Suppose that each time, the boy either climbs up one step or two steps. How many ways are there for the boy to climb the staircase*?

Discussion and Solution We have learnt a number of principles and techniques to solve some counting problems. Naturally, we would like to try and see if any of these can be applied to solve the above problem without listing all the possible ways. After pondering for a while, however, we may be doubtful about it. Splitting into cases look daunting and so we will not use the Addition Principle. There does not seem to be any fixed set of stages from the ground to Step 9. That eliminates the Multiplication Principle. No bijection is obvious. We cannot easily find sets $A_i, i = 1,2,\ldots,n,$ such that either $A = A_1 \cup A_2 \cup \cdots \cup A_n$ or $A = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n},$ where A is the set to be counted. Thus, the General Principle of Inclusion and Exclusion is not good here either. Is there any new idea available to tackle the problem?

The number of steps of the staircase given in the problem, i.e. 9, may be a little too big. Why don't we try some simpler cases to gain some "feeling" about the problem?

When the staircase consists of 1, 2 and 3 steps, the ways of climbing the staircase are shown in Figure 16.2, and the number of ways is respectively 1, 2 and 3, as well.

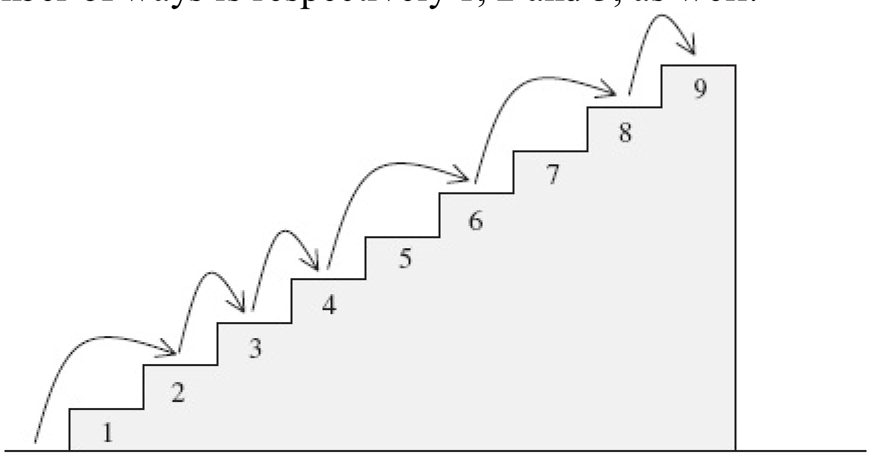


Figure 16.1

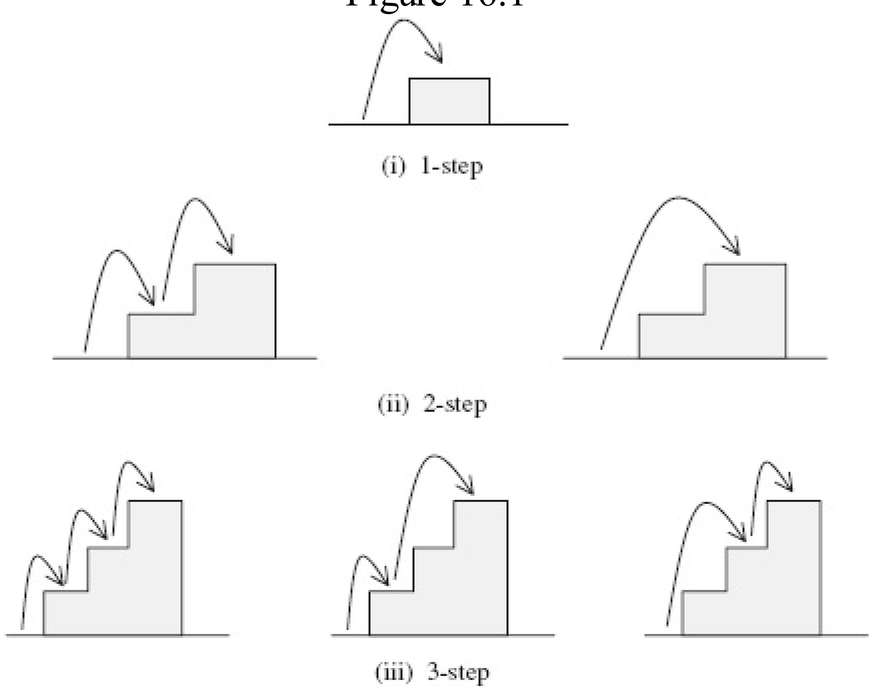


Figure 16.2

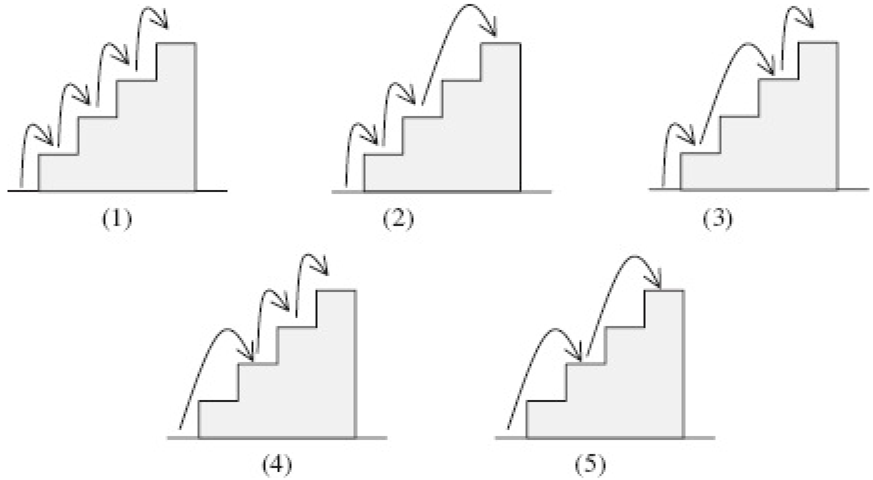


Figure 16.3

How about a 4-step staircase? Will the number be “4” also? No! The number of ways is now “5” and the 5 different ways of climbing are shown in Figure 16.3.

Let us hold the 4-step case for a while and analyse why we have “5” ways here. We begin by asking, “What can the boy do for his first move?” By assumption, he can cover 1 step or 2 steps. We now split our consideration into two cases accordingly.

(i) Suppose the first move covers 1 step. Then there are 3 steps left. How many ways are there to climb the remaining 3 steps? This question is crucial! Can we link it to the 3-step case? There are 3 ways to climb the 3-step staircase as shown in Figure 16.2(iii). Now, if we follow each of these 3 ways to climb the remaining 3 steps of our 4-step staircase, we will get 3 different ways (and no more) to climb the 4-step staircase as shown in (1)–(3) of Figure 16.3.

(ii) Suppose the first move covers 2 steps. Then there are 2 steps left. There are 2 ways to climb the 2-step staircase as shown in Figure 16.2(ii). If we follow each of these 2 ways to climb the remaining 2 steps of our 4-step staircase, we will get 2 different ways (and no more) to climb the 4-step staircase as shown in (4)–(5) of Figure 16.3.

It is now clear that by applying (AP), we will have 3 + 2, i.e. 5 different ways to climb the 4-step staircase.

What have we learnt from the above analysis? We have learnt that the problem of bigger size (4-step) depends on the same problem but of smaller size (3-step and 2-step), and the solution of the problem of bigger size can be obtained from the solutions of the same problem but of smaller size. This is a “new” idea for us. It works for “4-step”. Does it work for any “n-step”?

Now, given any integer $n \geq 3$, for convenience, let us denote by a_n, the number of ways to climb an n-step staircase. Thus, our previous records show that $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_4 = 5$. Indeed, we have just witnessed that $a_4 = a_3 + a_2$ and you may also check that $a_3 = a_2 + a_1$. Can we get a similar “equality” for a_n?

Imagine now the boy is to climb an n-step staircase. His first move can cover, by assumption, either 1 step or 2 steps. Divide our consideration into two cases as follows.

Case (1) The first move covers 1 step.

Then there are $n - 1$ steps left. How many ways are there to climb these remaining $n - 1$ steps? By definition, there are a_{n-1} ways.

Case (2) The first move covers 2 steps.

Then there are $n - 2$ steps left. How many ways are there to climb these remaining $n - 2$ steps? By definition, there are a_{n-2} ways.

Combining the results of these two cases by applying (AP), we conclude that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

The original problem asks for the determination of a_9. We shall evaluate it from the general result "$a_n = a_{n-1} + a_{n-2}$" together with some "initial" values (for instance, $a_1 = 1, a_2 = 2, a_3 = 3$ and $a_4 = 5$). Applying our general result successively, we have:

$$a_5 = a_4 + a_3 = 5 + 3 = 8,$$

$$a_6 = a_5 + a_4 = 8 + 5 = 13,$$

$$a_7 = a_6 + a_5 = 13 + 8 = 21,$$

$$a_8 = a_7 + a_6 = 21 + 13 = 34,$$

and finally

$$a_9 = a_8 + a_7 = 34 + 21 = 55,$$

as required.

In the example above, we obtain a sequence of numbers, namely,

$$a_1 = 1, \quad a_2 = 2, \quad a_3 = 3, \quad a_4 = 5, \quad a_5 = 8, \quad a_6 = 13, \ldots$$

and in general, $a_n = a_{n-1} + a_{n-2}$. The relation $a_n = a_{n-1} + a_{n-2}$which expresses a_n, for a general n, in terms of some preceding numbers in the sequence (in this case, a_{n-1} and a_{n-2}) is called a *recurrence relation.* As we have witnessed just now, deriving a recurrence relation is a way of solving a class of counting problems.

The sequence of numbers 1,2,3, 5,8,... as given above is called the sequence of *Fibonacci* numbers, named after the Italian mathematician Leonardo Fibonacci (1170–1250), a great mathematical innovator during the Middle Ages. Fibonacci was born in Pisa. Around 1192, his father was the director of the Pisan trading colony in Algeria. Hoping that his son would become a businessman, the father brought Fibonacci to Algeria to study mathematics with an Arab master. A few years later, sent by the father on business trips, Fibonacci had several occasions to visit places such as Egypt, Syria, Greece and Sicily, where he took the opportunity to learn various numerical systems and methods of calculation. Around 1200, after returning to Pisa, Fibonacci started to write a book entitled *"Liber abbaci"* (Book of the Abacus). The book was completed in 1202. In this book, one finds the following counting problem about rabbits.

> Beginning with a pair of baby rabbits, and assuming that each pair gives birth to a new pair each month starting from the second month of its life, how many pairs will there be after one year?

If we write F_n to denote the number of pairs of rabbits at the end of the nth month, then one can see from Figure 16.4 that $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, etc. Indeed, it can be shown (see Problem 16.5) in general that $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$, which is essentially the same as the recurrence relation $a_n = a_{n-1} + a_{n-2}$that we derived in Example 16.1. Note that in Example 16.1, our initial values are $a_1 = 1$ and $a_2 = 2$ while in Fibonacci's problem, we have $F_1 = F$move is enough an_2 = 1.

Statue of Fibonacci in Pisa,

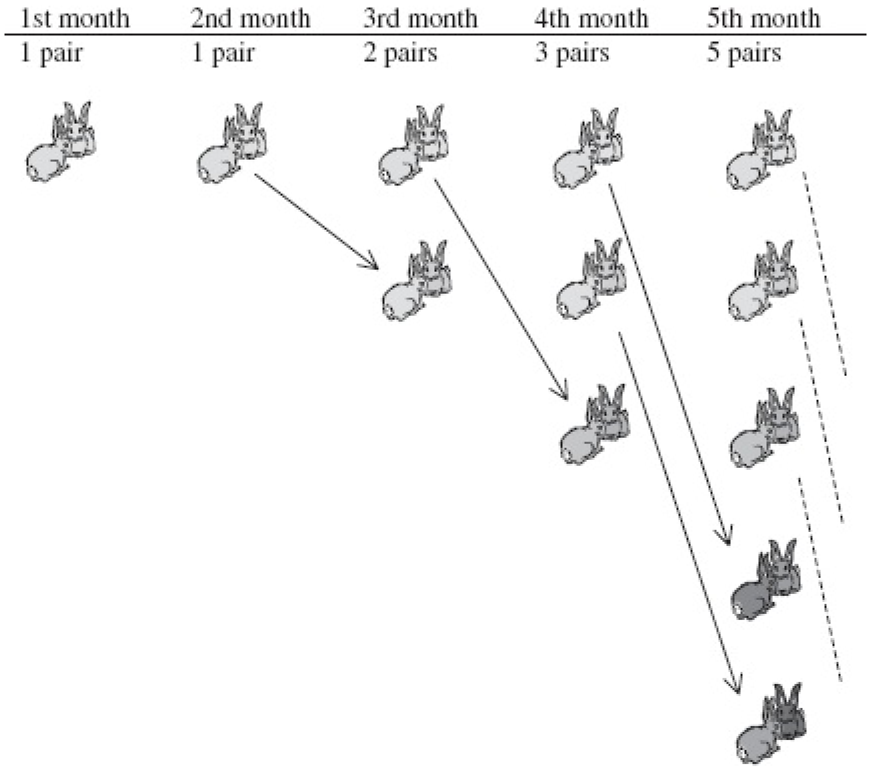


Figure 16.4

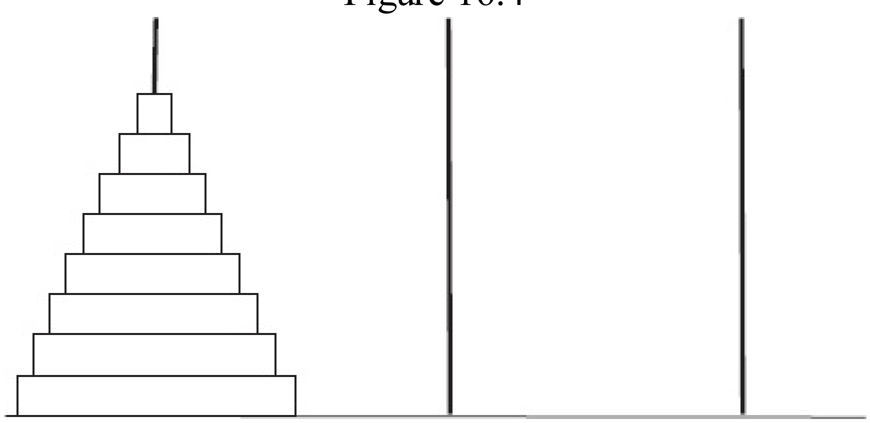


Figure 16.5

Let us proceed to consider our second example.

Example 16.2 *A tower of* 8 *circular discs of decreasing diameters is stacked on one of the three vertical pegs as shown in Figure 16.5.*

The task is to transfer the entire tower to another peg by a number of moves subject to the following rules:

(i) *each move carries exactly one disc; and*

(ii) *no disc can be placed on a smaller one.*

What is the minimum number of moves required to accomplish the task?

Discussion and Solution Again, for convenience, let b_n denote the minimum number of moves required to transfer the entire tower with n discs from one peg to another. The problem is to find the value of b_n.

From the experience we have gained in the preceding example, let us first consider some of the simpler cases. When $n = 1$, it is clear that one move is enough and so $b_1 = 1$. When $n = 2$, a bit of effort will show that two moves are not enough, whereas the following sequence of moves, as shown in Figure 16.6, shows that three moves will do the job. Thus, $b_2 = 3$.

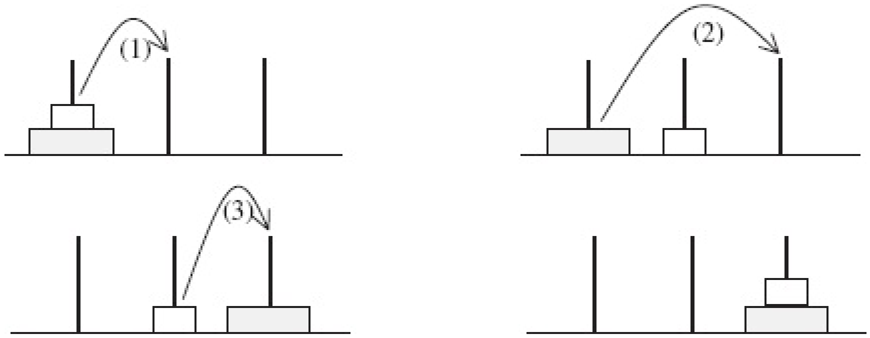


Figure 16.6

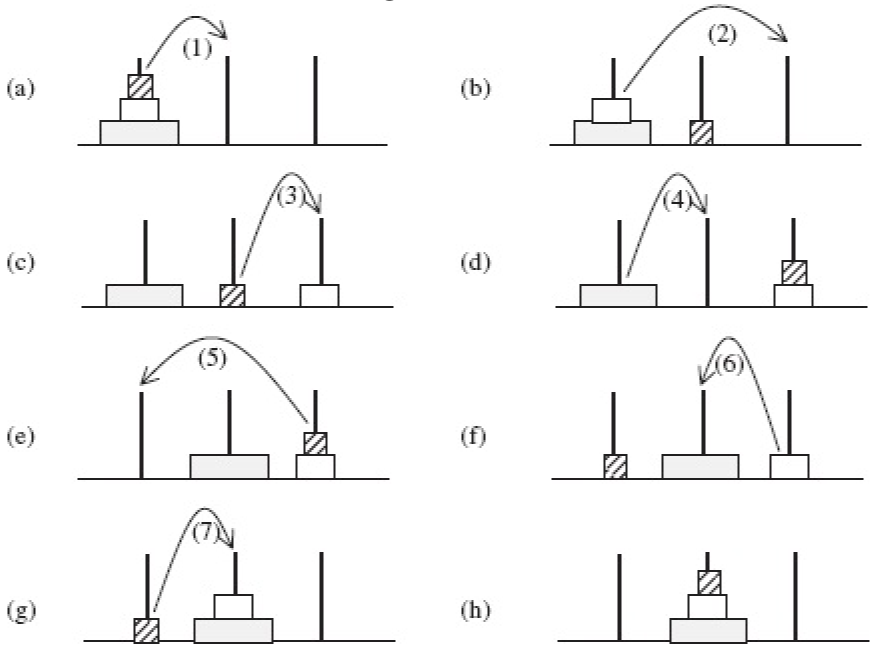


Figure 16.7

Consider now the case when $n = 3$. The sequence of moves shown in Figure 16.7 shows that seven moves are enough to accomplish the task.

Is "7" the minimum number of moves required? As shown in Figure 16.7(a)-(d), before the largest disc can be moved to another peg, we have to transfer the entire tower of two smaller discs to a peg. We know that this requires b_2 (= 3) moves. Next, we move the largest disc to the bottom of the only "empty" peg as shown in Figure 16.7(d), (e). Finally, we have to transfer the entire tower of two smaller discs and place it on the largest disc (Figure 16.7(e)–(h)), and this requires another b_2 (= 3) moves. Thus we need at least $b_2 + 1 + b_2$, i.e. $2b_2 + 1$ (= 7) moves to accomplish the task. This fact, together with the sequence of seven moves shown in Figure 16.7, shows that $b_3 = 7$.

In the discussion above, we have found that $b_3 = 7$. Indeed, we have obtained the relation $b_3 = 2b_2 + 1$, an instance of a recurrence relation. The reader may easily check that $b_2 = 2b_1 + 1$ as well. Can we generalise this relation? More precisely, given $n \geq 2$, is it true that $b_n = 2b_{n-1} + 1$?

Imagine now we have a tower of n (≥ 2) discs stacked on one of the 3 pegs (say peg (a) as shown in Figure 16.8 and we wish to evaluate b_n, the minimum number of moves needed to transfer the entire tower of n discs to another peg.

In the process of transferring the entire tower, it is clear (by rule (ii)) that at a certain stage, we must arrive at the situation, as shown in Figure 16.9, where the entire tower of $n-1$ smaller discs has been transferred to another peg (say peg (c)). This is so because only then we can finally move the largest disc from the original peg to the bottom of another peg (in this case, peg (b)). What is the minimum number of moves needed to transfer the entire tower of $n-1$ smaller discs from peg (a) to peg (c)? By definition, this number is b_{n-1}.

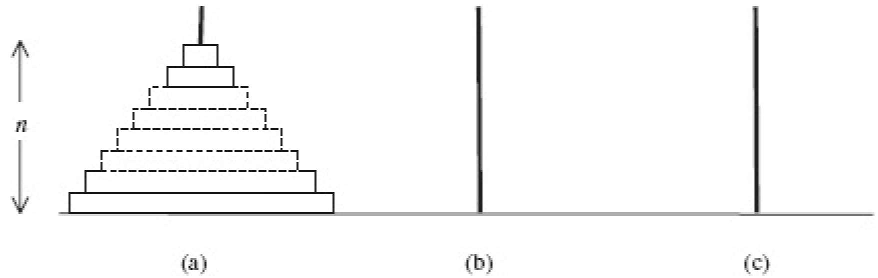


Figure 16.8

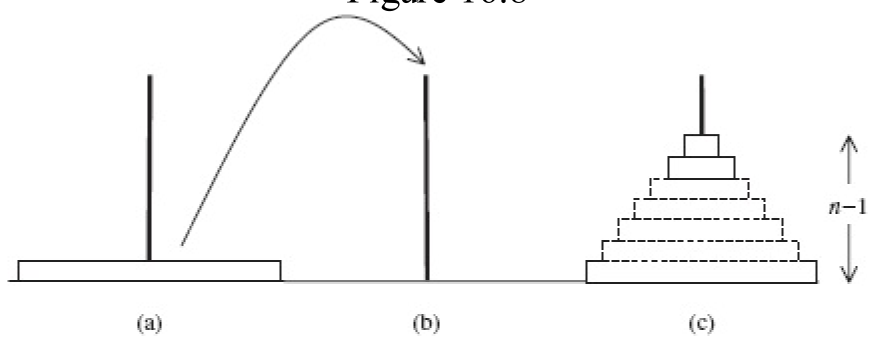


Figure 16.9

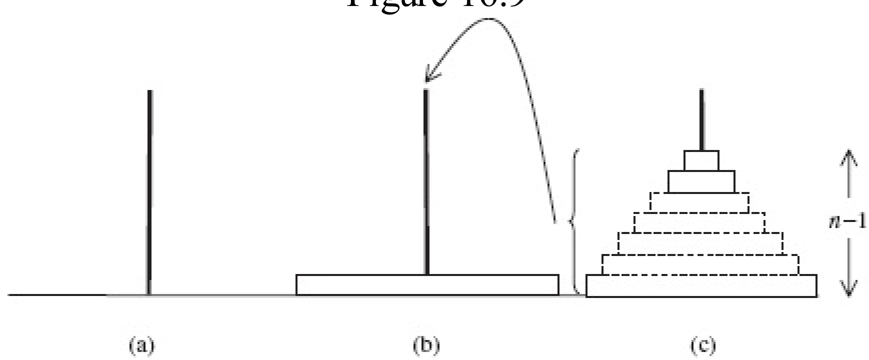


Figure 16.10

After moving the largest disc from peg (a) to peg (b) as shown in Figure 16.10, our final job is to transfer the entire tower of discs at peg (c) on to the top of the largest disc at peg (b). By definition, this number is b_{n-1} again.

Summing up, we see that the minimum number of moves that are needed for the whole task is $b_{n-1} + 1 + b_{n-1}$. Accordingly, by the definition of b_n, we have

$$b_n = 2b_{n-1} + 1,$$

another example of a recurrence relation. Let us return to the problem in Example 16.2, where we were asked to evaluate b_8. Based on the result that bThe problem described $i_1 = 1$, by applying our recurrence relation successively, we obtain

$$b_1 = 1,\ \ b_2 = 3,\ \ b_3 = 7,\ \ b_4 = 15,\ \ b_5 = 31,\ \ b_6 = 63,\ \ b_7 = 127,$$

and finally, $b_8 = 255$, as required.

Observe that we actually have a nice single formula for the value b_n:

$$b_2 = 3 = 2^2 - 1, \quad b_3 = 7 = 2^3 - 1, \quad b_4 = 15 = 2^4 - 1, \ldots,$$

and in general, $b_n = 2^n - 1$ (see Problem 16.9).

The problem described in Example 16.2 is known as the Tower of Hanoi. Why is Hanoi, the capital of Vietnam, associated with this problem? Well, this could have something to do with these two facts: the inventor of the problem was French and the problem was introduced at a time when France began her military involvement in Vietnam.

According to Andreas M. Hinz (in his paper, *The Tower of Hanoi,* published in 1999), the picture shown in Figure 16.11 is of the cover of a box which was found in Paris in 1883. Looking at the picture closely, we find several items therein which are related to tropical Asia, and in particular, Vietnam. These include a Vietnamese, two sites in Vietnam, viz. Tonkin and Annam, and the title "La Tour d'Hanoi". Two special names also appear in the picture. These are Professor N. Claus (de Siam) and his College Li-Sou-Stian. According to the French mathematician de Parville, the two names above are anagrams for Professor Lucas (d'Amiens), the inventor of this problem, and his College Saint Louis. As Lucas was Agrege de l'Universite, it is believed that he is the one carrying the ten-level tower in the picture.

Francois Edouard Anatole Lucas (1842–1891) was a French mathematician who did much work in Number Theory, Recurrent Sequences and Recreational Mathematics. In the "pre-computer age", Lucas was the last "largest prime number record holder" ($2^{127} - 1$). He gave a closed-form expression for the Fibonacci numbers as follows:

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

Figure 16.11

Edouard Lucas (1842–1891)

The associated Lucas sequence was named after him: 2, 1, 3, 4, 7, 11, 18, 29,.... Can you see the recurrence relation that generates the Lucas sequence?

So far, we have discussed two counting problems and introduced a way, called the technique of recursion or the method of recurrence relation, to solve them. The technique of recursion amounts to a derivation of a recurrence relation (such as $a_n = a_{n-1} + a_{n-2}$ and $a_n = 2a_{n-1} + 1$) which expresses the required number of ways, a_n (when the size of the problem is n), in terms of the numbers of ways when the sizes for the problem are smaller than n (such as a_{n-1}, a_{n-2}). It is often very easy to find the number of ways a_1, a_2, a_3 when the sizes for the problem are very small. With these initial values, the recurrence relation that has been established earlier will generate successively the values of the next a_n's. From a computational standpoint, solving a counting problem by the technique of recursion can sometimes be more useful and efficient than by a formula, especially when we need to compute all the values $a_1, a_2, \ldots, a_n$ up to some point.

Exercise

16.1 A man takes a bank loan of \$300,000 that charges 1% interest per year in the first year, 2% interest per year in the second year and 5% interest per year from the third year. Interest is compounded yearly. The man pays \$X each year to service the loan. You may assume that the interest is computed just before the man's yearly payment.

(i) Write a recurrence relation and initial conditions for b_n, the balance of the loan in dollars at the end of n years.

(ii) What should \$$X$, to the nearest dollar, be if the man intends to complete repayment at the end of 10 years?

16.2 Suppose that you have an unlimited supply of red, blue, yellow and green counters, which are indistinguishable except for colour. Write a recurrence relation and initial conditions for the number s_n of ways to stack n counters with no two consecutive green counters.

16.3 There are n lines in a plane. Every pair of lines intersect but no three meet at a common point. How many regions is the plane divided into by these n lines?

16.4 There are n circles in a plane. Every pair of circles intersect at exactly two points but no three meet at a common point. How many regions is the plane divided into by these n circles?

16.5 Let F_n denote the number of pairs of rabbits at the end of the nth month, where $n \geq 1$, as given in Fibonacci's problem of rabbits. Show that $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

16.6 Find a recurrence relation and initial conditions for the number of binary sequences of length n with no consecutive 0's.

16.7 Find a recurrence relation and initial conditions for the number of binary sequences of length n having no k consecutive 0's.

16.8 Find a recurrence relation and initial conditions for the number of binary sequences of length n that do not contain the sequence 101.

16.9 Let b_n denote the minimum number of moves as defined in Example 16.2. Show that $b_n = 2^n - 1$ for all $n \geq 1$.

16.10 A permutation $a_1a_2 \ldots a_n$ of $\mathbb{N}_n$ is called a *derangement* of $\mathbb{N}_n$ if $a_i \neq i$ for each $i = 1, 2, \ldots, n$ (see Chapter 14). For $n \geq 3$, let D_n be the number of derangements of $\mathbb{N}_n$. Show that

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

Chapter 17

The Stirling Numbers of the First Kind

In Chapter 3, we learnt that the number of ways to choose m objects from n distinct objects, where $m \leq n$, and arrange them in a row is given by

$$n(n-1)(n-2)\cdots(n-(m-1)). \qquad (17.1)$$

As shown in Chapter 13 (see (13.8)), the expression (17.1) can also be interpreted as the number of 1-1 mappings from the set $\{1,2,\ldots, m\}$ to the set $\{1,2,\ldots, n\}$.

As (17.1) will be mentioned very often in what follows, for simplicity, we may denote it by $[n]_m$; that is,

$$[n]_m = n(n-1)(n-2)\cdots(n-(m-1)). \qquad (17.2)$$

Let us replace "n" in (17.2) by a real variable "x". Then we have

$$[x]_m = x(x-1)(x-2)\cdots(x-(m-1)), \qquad (17.3)$$

which can be regarded as a polynomial in x of degree m. For instance,

$$\begin{aligned}
[x]_1 &= x,\\
[x]_2 &= x(x-1) = -x + x^2,\\
[x]_3 &= x(x-1)(x-2) = 2x - 3x^2 + x^3,\\
[x]_4 &= x(x-1)(x-2)(x-3) = -6x + 11x^2 - 6x^3 + x^4,\\
[x]_5 &= x(x-1)(x-2)(x-3)(x-4) = 24x - 50x^2 + 35x^3\\
&\quad - 10x^4 + x^5, \quad \text{etc.}
\end{aligned}$$

Just like what we did above, we shall express the polynomial $[x]_m$ in increasing order of powers of x. The following question arises naturally: what can be said about the coefficient of x^k in the expansion of $[x]_m$, where $0 \leq k \leq m$? It is clear from (17.3) that this coefficient depends on both m and k; and so let us, at this moment, denote it by $s(m, k)$. Thus, we have:

$$[x]_m = s(m,0) + s(m,1)x + s(m,2)x^2 + \cdots + s(m,m)x^m. \qquad (17.4)$$

By comparing with the expansions of $[x]_1$, $[x]_2,\ldots$, $[x]_5$ as shown above, we can easily obtain the values of $s(m,k)$, where $0 \leq k \leq m \leq 5$. These are recorded in Table 17.1 (note that we define $s(0, 0)$ to be 1).

It follows from (17.3) and (17.4) that $s(m, 0) = 0$ and $s(m,m) = 1$ for all $m \geq 1$. Also, the sequence of numbers $s(m, 1), s(m, 2),\ldots, s(m, m)$ alternate in sign with $s(m, 1)$ positive when and only when m is odd. It can be shown (see Problem 17.2) that for $m \geq 2$, we have

(i) $s(m,1) = (-1)^{m+1}(m-1)!$ and
(ii) $s(m,m-1) = -\binom{m}{2}$.

How are we going to evaluate $s(m, k)$ in general? We see from the above that when $k = 0$, 1 or $k = m - 1$, m, the values of $s(m, k)$ can be computed by simple formulas. For general k, there is a "recursive" way to evaluate $s(m, k)$ that we shall now present.

By comparing $[x]_m$ and $[x]_{m-1}$ by (17.3), we have

$$\begin{aligned}
[x]_m &= x(x-1)\cdots(x-(m-2))(x-(m-1))\\
&= [x]_{m-1}(x-(m-1)).
\end{aligned}$$

Table 17.1 The values of $s(m, k)$, $0 \leq k \leq m \leq 5$.

$m \backslash k$	0	1	2	3	4	5
0	1					
1	0	1				
2	0	−1	1			
3	0	2	−3	1		
4	0	−6	11	−6	1	
5	0	24	−50	35	−10	1

Thus, by (17.4),

$$\begin{aligned}
&s(m,0)+s(m,1)x+s(m,2)x^2+\cdots+s(m,m)x^m\\
&\quad=[x]_m\\
&\quad=(x-(m-1))[x]_{m-1}\\
&\quad=(x-(m-1))(s(m-1,0)+s(m-1,1)x+s(m-1,2)x^2+\cdots\\
&\qquad+s(m-1,m-1)x^{m-1})\\
&\quad=-(m-1)s(m-1,0)+(s(m-1,0)-(m-1)s(m-1,1))x\\
&\qquad+(s(m-1,1)-(m-1)s(m-1,2))x^2+\cdots\\
&\qquad+s(m-1,m-1)x^m.
\end{aligned}$$

Hence, by equating the coefficients of x^k on both sides of the above equality, we have

$$\begin{aligned}
s(m,0)&=-(m-1)s(m-1,0),\\
s(m,1)&=s(m-1,0)-(m-1)s(m-1,1),\\
s(m,2)&=s(m-1,1)-(m-1)s(m-1,2),\\
&\vdots
\end{aligned}$$

and, in general,

$$\boxed{\begin{array}{l}s(m,k)=s(m-1,k-1)-(m-1)s(m-1,k)\\ \text{with the condition that } s(r,0)=0 \text{ for } r\geq 1\\ \text{and } s(r,r)=1 \text{ for all } r\geq 0.\end{array}} \qquad (17.5)$$

As was pointed out before, the value of $s(m, k)$ depends on two parameters: m and k. In (17.5), we observe that the value of $s(m, k)$ is expressed in terms of the values of $s(m-1,k-1)$ and $s(m-1,k)$, where the values of the parameters do not exceed those in $s(m,k)$. Thus, we can evaluate $s(m, k)$ if we know the values of $s(p, q)$ where $p \leq m$ and $q \leq k$. For instance, when $(m, k) = (6, 3)$, by (17.5)

$$s(6,3)=s(5,2)-5s(5,3).$$

Checking from Table 17.1, we have $s(5,2) = -50$ and $s(5,3) = 35$. Thus

$$s(6,3)=-50-5(35)=-225.$$

In Chapter 16, we introduced the notion of "recurrence relation" with examples such as $a_n = a_{n-1} + a_{n-2}$ and $a_n = 2a_{n-1} + 1$, where the value of a_n is expressed in terms of the values of a_r's where $r < n$. The relation (17.5) is also regarded as a recurrence relation, but it is more complicated as it involves two parameters.

The numbers $s(m,k)$ are called the *Stirling numbers of the first kind* in honour of the Scottish mathematician James Stirling (1692–1770). Inspired by the theory on plane curves due to Isaac Newton, Stirling worked on its extensions and published in 1730 his most influential work *Methodus Differentialis,* where the numbers $s(m, k)$ were introduced.

Combinatorial Interpretation of the Stirling Number $s(m, k)$

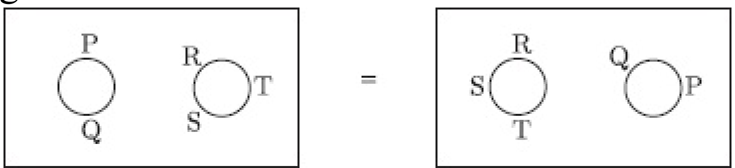
The Stirling number $s(m, k)$ was defined as the coefficient of x^k in the expansion of $[x]_m$, which is purely algebraic in nature. Does it have any combinatorial interpretation? The answer is yes, and we shall now present one.

Let us recall the notion of "circular arrangement" from Chapter 7: two arrangements of n distinct objects in a circle are considered different if and only if there is at least one object whose neighbour on the right is different in the two arrangements. Recall also the following result:

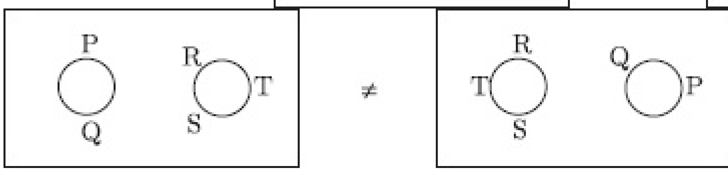
$$\boxed{\begin{array}{l}\text{The number of ways of arranging } n \text{ distinct objects in a circle}\\ \text{is given by } (n-1)!.\end{array}} \qquad (17.6)$$

Let us proceed further to study a variation of circular arrangements. Suppose now there are, say, 5 *distinct* objects to be arranged around, say, 2 *identical* circles with at least one object at each circle. In how many ways can this be done? Again, before we move on, let us agree on what we mean when we say that two arrangements are the same. We will use a definition which extends that for circular arrangements in one circle: two arrangements of n distinct objects in k identical circles are considered different if and only if there is at least one object whose neighbour on the right is different in the two arrangements.

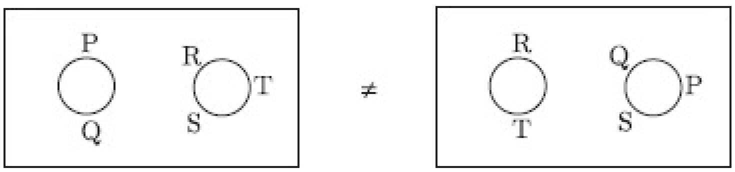
Thus, for instance, we agree that



while



and



With this clarification, we are now ready to solve the problem. There are two ways to split 5 distinct objects into 2 nonempty groups; namely,

$$\text{(i) } 4+1, \qquad \text{(ii) } 3+2.$$

Case (i) There are 4 objects around a circle and 1 object around another circle.

In this case, there are $\binom{5}{4}$ ways to select 4 objects from 5 distinct ones to put them around one circle. By (17.6), there are $(4 - 1)!$ ways to arrange the selected 4 objects around the circle. (Of course, the remaining object is on the other circle.) Thus the number of ways of arrangement is, by (MP),

$$\binom{5}{4}(4-1)! = 5 \cdot 3! = 30.$$

Case (ii) There are 3 objects around a circle and 2 objects around another circle.

In this case, there are $\binom{5}{3}$ ways to select 3 objects from 5 distinct ones to put them around one circle. By (17.6), there are $(3 - 1)!$ ways to arrange the selected 3 objects around the circle. By (17.6) again, the remaining 2 objects can be arranged around the other circle in $(2 - 1)!$ ways. Thus the number of ways of arrangement is, by (MP),

$$\binom{5}{3}(3-1)!(2-1)! = 20.$$

Finally, by (AP), the required number of arrangements is given by $30 + 20 = 50$.

We thus conclude that the number of ways of arranging 5 distinct objects around 2 identical circles with at least one object at each circle is 50.

Note that 50 is related to a Stirling number of the first kind. Indeed, $s(5,2) = -50$ ($m = 5$ corresponds to 5 objects and $k = 2$ corresponds to 2 circles).

For convenience, let us denote by $s^*(m, k)$, with $k \le m$, the number of ways of arranging m distinct objects around k identical circles with at least one object at each circle. Thus, as shown above, $s^*(5, 2) = 50 = |s(5,2)|$, where $|x|$ denotes the absolute value of the real number x.

By comparing the answers of Problems 17.1 and 17.3, we have $s^*(6, 3) = 225 = |s(6, 3)|$.

We define $s^*(0,0) = 1$. Clearly, $s^*(m, 0) = 0$ and $s^*(m, 1) = (m - 1)!$ by (17.6).

Our aim is to show that, indeed,

$$s^*(m,k) = |s(m,k)|.$$

The result (17.5) provides us with a recurrence relation for the numbers $s(m, k)$. In what follows, we shall establish a corresponding recurrence relation for $s^*(m, k)$.

$$\boxed{\begin{array}{l}\text{For } m,k \in \mathbb{N} \text{ with } k \le m,\\ s^*(m,k) = s^*(m-1,k-1) + (m-1)s^*(m-1,k).\end{array}} \qquad (17.7)$$

Let us give a combinatorial argument to see why (17.7) holds. The number $s^*(m, k)$ counts the number of ways of arranging m distinct objects, say $X_1, X_2, \ldots, X_m$ around k identical circles with at least one object at each circle. Let us fix one of the objects, say X_m. Clearly, in any such arrangement, either (i) X_m is the only object at a circle, or (ii) X_m is with others at a circle. We now count $s^*(m, k)$ by splitting our consideration into the above two cases.

Case (i) X_m is the only object at a circle.

In this case, the remaining $(m - 1)$ objects $X_1, X_2, \ldots, X_{m-1}$ are arranged around $k - 1$ circles with at least one object at each circle. By definition, there are $s^*(m - 1, k - 1)$ ways to do this.

Case (ii) X_m is mixed with others at a circle.

In this case, we can accomplish the task by first arranging the objects $X_1, X_2, \ldots, X_{m-1}$ around k circles with at least one object at each circle and then place X_m in one of the circles. By definition, there are $s^*(m - 1, k)$ ways to perform the first step. How many ways are there for the second step? After arranging $m - 1$ objects around the circles, X_m can be placed at any of the $m - 1$ spaces to the right of each object, and so there are $m - 1$ ways to do so. Thus, by (MP), there are $(m - 1)s^*(m - 1, k)$ ways in this case.

Finally, by (AP), we arrive at the result (17.7).

With the help of (17.5) and (17.7), we shall now see why $s^*(m,k) = |s(m,k)|$ for all $m,k \in \mathbb{N}$ with $k \le m$. Note that:

$$\text{For all} \quad m \ge 1, s^*(m,m) = 1 = s(m,m)$$
$$\text{and} \quad s^*(m,1) = (m-1)! = |s(m,1)|.$$

Also, as $s(m - 1, k - 1)$ and $s(m - 1, k)$ are *different* in sign or one of them is zero, we have

$$\boxed{\begin{aligned}&|s(m-1,k-1) - (m-1)s(m-1,k)|\\ &\quad = |s(m-1,k-1)| + (m-1)|s(m-1,k)|.\end{aligned}} \qquad (17.8)$$

Consider $(m,k) = (3,2)$. Observe that

$$\begin{aligned}|s(3,2)| &= |s(2,1) - 2s(2,2)| && \text{by (17.5)}\\ &= |s(2,1)| + 2|s(2,2)| && \text{by (17.8)}\\ &= s^*(2,1) + 2s^*(2,2) && \text{(see above)}\\ &= s^*(3,2) && \text{by (17.7).}\end{aligned}$$

When $(m,k) = (4,2)$, we have

$$\begin{aligned}|s(4,2)| &= |s(3,1) - 3s(3,2)| && \text{by (17.5)}\\ &= |s(3,1)| + 3|s(3,2)| && \text{by (17.8)}\\ &= s^*(3,1) + 2s^*(3,2) && \text{(see above)}\\ &= s^*(4,2) && \text{by (17.7).}\end{aligned}$$

When $(m,k) = (4,3)$, we have

$$\begin{aligned}|s(4,3)| &= |s(3,2) - 3s(3,3)| && \text{by (17.5)}\\ &= |s(3,2)| + 3|s(3,3)| && \text{by (17.8)}\\ &= s^*(3,2) + 3s^*(3,3) && \text{(see above)}\\ &= s^*(4,3) && \text{by (17.7).}\end{aligned}$$

If we proceed in this manner by following the ordering, say, $(m,k) = (3,2), (4, 2), (4, 3), (5, 2), (5, 3), (5, 4), (6, 2), (6, 3), (6, 4), (6, 5), (7, 2), (7, 3), \ldots$, we shall always find that

$$\boxed{|s(m,k)| = s^*(m,k).} \qquad (17.9)$$

Let us explain why (17.9) holds for all (m,k), where $1 \le k \le m$, with the help of Figure 17.1. We have already verified that (17.8) holds when $k = 1$ and $k = m$. This is indicated in Figure 17.1 at the entries (m, k) enclosed by rectangles. The key tools in the process are the recurrence relations (17.5) and (17.7) (and, of course, (17.8) also). We now start with $(m, k) = (3,2)$. Using the verified results for (2,1) and (2, 2), and applying (17.5) and (17.7), we show that (17.9) holds for (3, 2). This fact is indicated in Figure 17.1 by the two arrows pointing to entry (3, 2) from entries (2, 1) and (2, 2). We then proceed to $(m, k) = (4, 2)$. Using the verified results for (3, 1) and (3, 2), and applying (17.5) and (17.7), we show that (17.9) holds for (4, 2). Again, this is indicated in Figure 17.1 by the two arrows pointing to entry (4, 2) from entries (3, 1) and (3, 2). Thus, following the ordering of (m, k) as fixed above and the arrows pointing to the corresponding entries in Figure 17.1, we see that the result (17.9) is indeed valid for each (m, k) with $1 \le k \le m$.

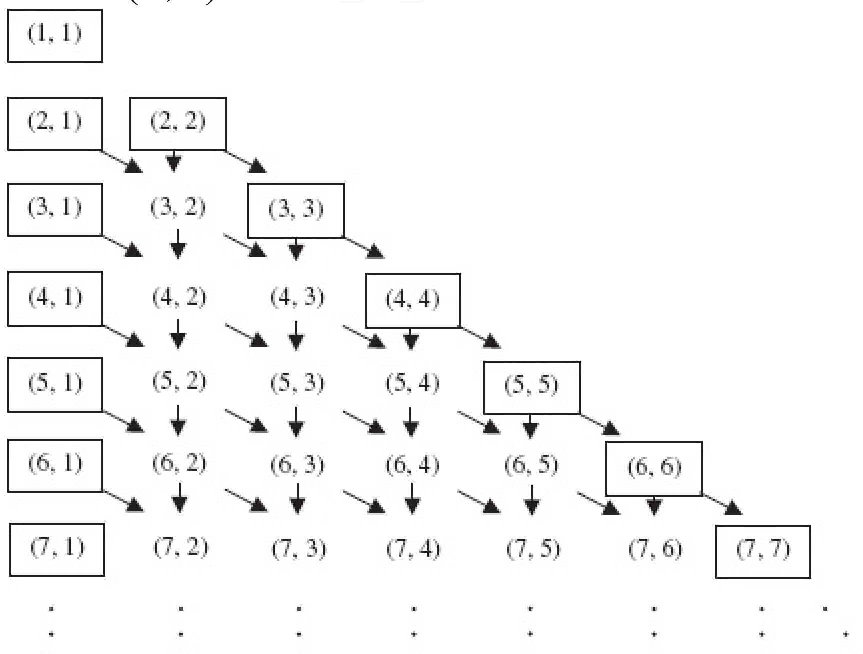


Figure 17.1

Following this, the reader may want to attempt a rigorous proof of (17.9) by mathematical induction (see Problem 17.4) using the inductive steps suggested by Figure 17.1.

Finally, we shall give a direct combinatorial argument for particular values of (17.9).

Let us consider the case when $m = 8$ and $k = 3$. By the algebraic definition, the Stirling number $s(8, 3)$ is the coefficient of x^3 in the expansion of $x(x - 1)(x - 2) \ldots (x - 7)$. Then $|s(8,3)|$ is the coefficient of x^3 in the expansion of $x(x + 1)(x + 2) \ldots (x + 7)$. The combinatorial definition says that $s^*(8,3)$ counts the number of ways that elements 0, 1,2,… ,7 can be arranged around 3 identical circles with at least one object at each circle. We shall show that the coefficient of x^3 the expansion of $x(x+1)(x+2)\cdots(x+7)$ is the same as the number of ways of arranging the 8 distinct elements around 3 identical circles with at least one object at each circle.

Note that the sum of the terms with x^3 in the expansion of

$$x(x+1)(x+2)\cdots(x+7)$$

is given by

$$3 \cdot 4 \cdot 5 \cdot 6 \cdot 7x^3 + 2 \cdot 4 \cdot 5 \cdot 6 \cdot 7x^3$$
$$+ 2 \cdot 3 \cdot 5 \cdot 6 \cdot 7x^3 + \cdots + 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5x^3.$$

Each term is a product of x^3 and five numbers chosen from among 1 through 7. This can be seen as taking three x's from three factors and the "numbers" from the remaining five factors. Thus

Coefficient of x^3 in the expansion of $x(x+1)(x+2)\cdots(x+7)$
$$= 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 + 2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 + 2 \cdot 3 \cdot 5 \cdot 6 \cdot 7 + \cdots$$
$$+ 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5.$$

Now, let us see how this expression $3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 + 2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 + 2 \cdot 3 \cdot 5 \cdot 6 \cdot 7 + \ldots + 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$ counts the number of ways elements 0 through 7 can be arranged around 3 circles with at least one object at each circle.

Let us take a term, say $1 \cdot 3 \cdot 4 \cdot 5 \cdot 7$. We place the "missing" elements 0, 2 and 6 on three different circles and make the rule that these "missing" elements will be the least elements of the circles. Next we place the remaining elements one at a time in increasing order. The number 1 has just one option, which is on the right of 0 (it cannot be placed on either of the other circles for then it would violate the least property of the incumbent "missing" elements 2 and 6, respectively). The element 3 then has three options: the right of 0, the right of 1 or the right of 2 on the other circle. The element 4 now has four options: the right of 0, the right of 1, the right of 2 or the right of 3. We can similarly see that element i has i options. Thus, by (MP), the number of ways of placing the eight elements 0, 1,2, … ,7 around three circles such that elements 0, 2 and 6 are the least elements in each of their circles is $1 \cdot 3 \cdot 4 \cdot 5 \cdot 7$.

Every term in the expression $3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 + 2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 + 2 \cdot 3 \cdot 5 \cdot 6 \cdot 7 + \ldots + 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$ can be combinatorially interpreted as above and every arrangement of the eight elements 0,1,2,…, 7 around three circles with at least one object at each circle can be matched uniquely to one term in the expression. For example, the arrangement with 3, 4, 0 in clockwise order around one circle, 1, 2, 7 in clockwise order around another circle and 5, 6 around a third circle, is matched uniquely to the term with "missing" elements 0, 1, 5, i.e. the term $2 \cdot 3 \cdot 4 \cdot 6 \cdot 7$. Hence, we have shown that $s^*(8,3) = |s(8,3)|$.

The reader is invited to prove the general result using the combinatorial argument above in Problem 17.5.

Exercise

17.1 Find the values of $s(6, k)$, where $1 \le k \le 6$.

17.2 Show that for $m \ge 2$,

(i) $s(m,1) = (-1)^{m+1}(m-1)!$

(ii) $s(m,m-1) = -\binom{m}{2}$.

17.3 Show, from "first principles", that the number of ways of arranging 6 distinct objects around 3 identical circles with at least one object at each circle is given by 225.

17.4 Use mathematical induction to show that $s(m,1) = (-1)^{m+1}(m-1)!$

17.5 A permutation $(a_1,a_2,\ldots, a_n)$ of the integers 1,2,… ,n is used to distribute these n integers into indistinguishable circles as follows. Locate $a_{i1} = 1$ and arrange $a_{i1},a_{i1}+1,\ldots,a_n$ in a clockwise direction around a circle. Next, locate $a_{i2} = \min\{a_1,a_2,\ldots,a_{i1-1}\}$ and arrange $a_{i2},a_{i2}+1,\ldots,a_{i1}-1$ in a clockwise direction around a second circle. Continue doing this until all the integers are distributed around k tables. Use this correspondence to show that

$$\sum_{k=1}^{n} s^*(n,k) = n!.$$

Chapter 18

The Stirling Numbers of the Second Kind

In the previous chapter, we introduced the Stirling number of the first kind $s(m, k)$ which is defined as the coefficient of x^k in the expansion of

$$[x]_m = x(x-1)\cdots(x-m+1);$$

namely,

$$[x]_m = \sum_{i=0}^{m} s(m,i)x^i.$$

The sequence of numbers $s(m, 1)$,$s(m, 2)$,..., $s(m, m)$ alternate in sign with $s(m, 1)$ positive when and only when m is odd.

We also gave a combinatorial interpretation of $s(m, k)$, i.e. the absolute value of $s(m, k)$ is the number of ways of arranging m distinct objects around k identical circles with at least one object at each circle.

In this chapter, we shall introduce the other sequence of Stirling numbers, called the *Stirling numbers of the second kind.*

Let us begin with a simple example. Consider four distinct objects: *a, b, c* and *d.* Clearly, there is one and only one way to group them into one group, i.e. $\{a, b, c, d\}$; and there is one and only one way to divide them into four groups, i.e.

$$\{a\} \cup \{b\} \cup \{c\} \cup \{d\}.$$

Now, (i) how many ways are there to divide them into two groups? There are 7 ways as shown below:

$$\begin{array}{lll} \{a,b,c\} \cup \{d\}, & \{a,b,d\} \cup \{c\}, & \{a,c,d\} \cup \{b\}, \\ \{b,c,d\} \cup \{a\}, & \{a,b\} \cup \{c,d\}, & \{a,c\} \cup \{b,d\}, \\ \{a,d\} \cup \{b,c\}. & & \end{array}$$

(ii) How many ways are there to divide them into three groups?

There are 6 ways as shown below:

$$\begin{array}{lll} \{a,b\} \cup \{c\} \cup \{d\}, & \{a,c\} \cup \{b\} \cup \{d\}, & \{a,d\} \cup \{b\} \cup \{c\}, \\ \{b,c\} \cup \{a\} \cup \{d\}, & \{b,d\} \cup \{a\} \cup \{c\}, & \{c,d\} \cup \{a\} \cup \{b\}. \end{array}$$

Given two positive integers n and k with $k \le n$, the *Stirling number of the second kind,* denoted by *S(n,k),* is defined as the number of ways of dividing n *distinct* objects into k (nonempty) groups; i.e. the number of ways of partitioning an n-element set into k nonempty subsets. Thus, as shown in the above example, we have

$$S(4,1) = 1, \quad S(4,2) = 7, \quad S(4,3) = 6, \quad S(4,4) = 1.$$

Example 18.1 *Find the number of ways to express* 2730 as a product ab of two numbers a and b, where a > b > 2.

Solution Observe that 2730 = 2 · 3 · 5 · 7 · 13, and such a pair *a,b* of factors is obtained by dividing {2,3, 5, 7,13} into two groups (and then taking the product of all the elements within each group). Thus, the desired number of ways is given by $S(5, 2)$ (= 15 (see Table 18.1)).

It is clear that

(i) $S(n,1) = S(n,n) = 1$,

(ii) $S(n,k) = 0$ if $k > n \ge 1$, and

(iii) $S(n,0) = S(0,k) = 0$ if $n \ge 1$ and $k \ge 1$.

We define

(iv) $S(0, 0) = 1$.

It can also be proved (see Problem 18.3) that for $n \geq 1$,

(v) $S(n, 2) = 2^{n-1} - 1$,

(vi) $S(n, n-1) = \binom{n}{2}$.

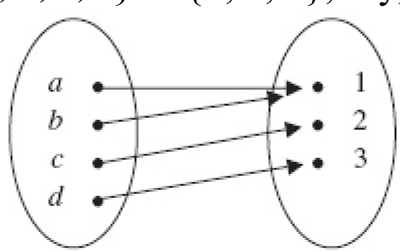
Table 18.1 The values of $S(n, k)$, $0 \leq k \leq n \leq 9$.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	1	1							
3	0	1	3	1						
4	0	1	7	6	1					
5	0	1	15	25	10	1				
6	0	1	31	90	65	15	1			
7	0	1	63	301	350	140	21	1		
8	0	1	127	966	1701	1050	266	28	1	
9	0	1	255	3025	7770	6951	2646	462	36	1

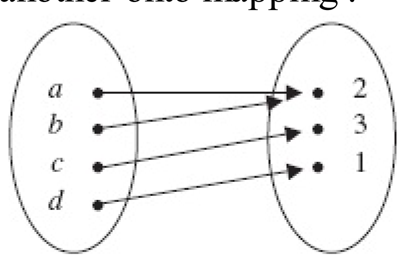
The Number of Onto Mappings

We pointed out in Chapter 13 that the problem of counting the number of onto mappings from a finite set to another finite set is not straightforward, and we showed by an example how to tackle this problem by applying (PIE). Here, we shall point out that this counting problem is actually closely related to the problem of evaluating the $S(n, k)$'s.

Consider an onto mapping from $\{a, b, c, d\}$ to $\{1, 2, 3\}$, say,



This onto mapping can be regarded as first dividing the 4 elements a, b, c, d into 3 groups {a, b}, {c}, {d}, and then naming the groups as "1 , "2" and "3" respectively. If we rename the groups as "2, "3" and "1" respectively, then we get another onto mapping :



Since there are 3! ways to name the 3 groups, we see that a way of dividing 4 distinct objects into 3 groups gives rise to 3! onto mappings from $\{a, b, c, d\}$ to $\{1, 2, 3\}$. It thus follows that the number of onto mappings from {a, b, c, $d\}$ to {1, 2, 3} is given by $3!S(4, 3)$ (= 36).

In general, we have:

The number of *onto* mappings from an n-element set to a k-element set is given by
$$k!S(n, k). \qquad (18.1)$$

Using the general statement of (PIE), as shown in Chapter 14, one can show that (see Problem 14.5) the number of *onto* mappings from an n-element set to a k-element set is given by

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r} (k-r)^n.$$

Combining this with (18.1), we have:

$$\boxed{S(n,k) = \frac{1}{k!}\sum_{r=0}^{k} (-1)^r \binom{k}{r} (k-r)^n.} \qquad (18.2)$$

The formula (18.2) provides us with a way to evaluate $S(n, k)$'s. There is another way to do so. As shown in Chapter 17, the Stirling numbers of the first kind $s(m,k)$*'s* satisfy the following recurrence relation:

$$s(m,k) = s(m-1,k-1) - (m-1)s(m-1,k).$$

For the Stirling numbers of the second kind, likewise, we have the following recurrence relation :

$$\boxed{\begin{array}{l}\text{For positive integers } n,k \text{ with } n \geq k,\\ S(n,k) = S(n-1,k-1) + kS(n-1,k).\end{array}} \qquad (18.3)$$

To see why (18.3) holds, suppose a_1, a_2,...,a_n are the n distinct objects which are divided into k groups. Consider a particular object, say $a1$.

Case (1) a_1 forms a group by itself.

In this case, the $n-1$ objects $a_2,a_3,\ldots,a_n$ are then divided into $k-1$ groups. By definition, there are $S(n-1,k-1)$ ways of grouping.

Case (2) a_1 is in a group with at least one other object.

In this case, the $n-1$ objects a_2, $a_3,\ldots,a_n$ are then divided into k groups and by definition, there are $S(n-1,k)$ ways of grouping. In any such grouping, $a1$ has k choices to be in one of the k groups. Thus there are $kS(n-1,k)$ ways in this case.

The relation (18.3) now follows by (AP).

Using the initial values shown earlier as (i)-(iv) and to be worked out in Problem 18.3 as (v) and (vi), and applying (18.3), one can find out the values of other $S(n,k)$'s. For instance,

$$S(3,2) = S(2,1) + 2S(2,2) = 1 + 2\cdot 1 = 3;$$
$$S(4,2) = S(3,1) + 2S(3,2) = 1 + 2\cdot 3 = 7;$$
$$S(4,3) = S(3,2) + 3S(3,3) = 3 + 3\cdot 1 = 6; \quad \text{etc.}$$

It is in this way that one can easily construct Table 18.1 for the values of the $S(n, k)$'s.

Expressing x^n in terms of $[x]_i$'s

As shown in Chapter 17, when $[x]_m$ is expressed in terms of x^i's, the Stirling numbers of the first kind are the coefficients. Suppose, conversely, we wish to express x^n in terms of [x]j' s. What can be said about the coefficients? To answer this question, let us consider the following counting problem:

Let $\mathbb{N}_n = \{1,2,3,\ldots, n\}$. Determine a, the number of mappings from $\mathbb{N}_n$ to $\mathbb{N}_k$.

We shall now use two different methods to count a. The first method is the "natural" one (see (13.7)):

$$\alpha = \underbrace{k\cdot k\cdot \cdots \cdot k}_{n} = k^n. \qquad (18.4)$$

The second method is a "stupid" one. According to the size $|f(\mathbb{N}_n)|$ of the range of a mapping $f:\mathbb{N}_n \to \mathbb{N}_k$, the set of mappings from $\mathbb{N}_n$ to $\mathbb{N}_k$ can be partitioned into k groups $A_i, i = 1,2,\ldots,k,$ where Ai

consists of those mappings whose ranges have exactly i elements, i.e.

$$A_i = \{f : \mathbb{N}_n \to \mathbb{N}_k \mid |f(\mathbb{N}_n)| = i\}.$$

What is the value of $|A_i|$? Well, $|A_i|$ counts the number of onto mappings from $\mathbb{N}_n$ to an i-element subset of $\mathbb{N}_k$. There are $\binom{k}{i}$ number of ways to choose an i-element subset of $\mathbb{N}_k$, and the number of onto mappings from $\mathbb{N}_n$ to this chosen i-element subset of $\mathbb{N}_k$ is $i!S(n,i)$ by (18.1). Thus

$$\begin{aligned}|A_i| &= \binom{k}{i} i!S(n,i) \\ &= k(k-1)(k-2)\cdots(k-i+1)S(n,i) \\ &= [k]_i S(n,i).\end{aligned}$$

Now, by (AP), we have

$$\begin{aligned}\alpha &= \sum_{i=1}^{k} |A_i| \\ &= \sum_{i=1}^{k} [k]_i S(n,i) \\ &= \sum_{i=1}^{n} [k]_i S(n,i) \quad ([k]_i = 0 \text{ if } i \geq k+1).\end{aligned}$$

Comparing this result with (18.4) and noting that both count for a, we have

$$k^n = \sum_{i=1}^{n} [k]_i S(n,i).$$

If we replace k by a real variable x, we then obtain:

$$\boxed{x^n = \sum_{i=1}^{n} S(n,i)[x]_i.} \qquad (18.5)$$

Thus, we see that when x^n is expressed in terms of $[x]_i$'s, the Stirling numbers of the second kind are the coefficients.

For instance, when $n = 4$,

$$\begin{aligned}\sum_{i=1}^{4} S(4,i)[x]_i &= S(4,1)[x]_1 + S(4,2)[x]_2 + S(4,3)[x]_3 + S(4,4)[x]_4 \\ &= 1 \cdot x + 7x(x-1) + 6x(x-1)(x-2) \\ &\quad + 1 \cdot x(x-1)(x-2)(x-3) \\ &= x + 7x^2 - 7x + 6x^3 - 18x^2 + 12x + x^4 - 6x^3 \\ &\quad + 11x^2 - 6x \\ &= x^4.\end{aligned}$$

Exercise

18.1 Find the value of $S(10, k)$, where k = 1,2,3,4, 5. You may want to write a simple computer program to generate the values.

18.2 Find, in terms of $S(n, k)$, the number of ways to express 35310 as a product abc of three integers $a, b,$ and c, where $a > b > c > 2$.

18.3 Show that for $n > 1$,

(i) $S(n,2) = 2^{n-1} - 1$;

(ii) $S(n,n-1) = \binom{n}{2}$.

18.4 Show that for $n > 3$,

(i) $S(n,3) = \frac{1}{2}(3^{n-1}+1) - 2^{n-1}$;

(ii) $S(n,n-2) = \binom{n}{3} + 3\binom{n}{4}$.

18.5 In Example 13.4, we applied (PIE) to compute the number of onto mappings from a 5-element set to a 3-element set, and found it to be “150”. Verify this result by applying (18.1) and the value for $S(5, 3)$ in Table 18.1.

Chapter 19
The Catalan Numbers

In Chapter 5, we learnt that the number of shortest P–Q routes in the 2×4 rectangular grid of Figure 19.1 is, by (BP), equal to the number of 6-digit binary sequences with four 0's and two 1's. This is $\binom{6}{2}$.

> In general, in the rectangular coordinate system of Figure 19.2, the number of shortest routes from $P(a,b)$ to $Q(c,d)$, where a, b, c and d are integers with $a \le c$ and $b \le d$, is given by
> $$\binom{(c+d)-(a+b)}{c-a} \text{ or } \binom{(c+d)-(a+b)}{d-b}.$$

(19.1)

Consider the case when $O = (0,0)$ and $A = (n,n)$, where n is a positive integer. By (19.1), the number of shortest O–A routes is given by $\binom{2n}{n}$. As shown in Figure 19.3 (where $n = 4$), we observe that the $\binom{2n}{n}$ shortest O-A routes can be divided into two groups: those that cross the diagonal $y = x$ (see (i)) and those that do not (see (ii) and (iii)).

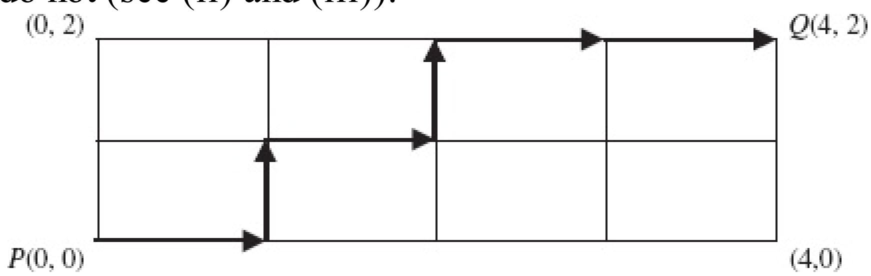


Figure 19.1

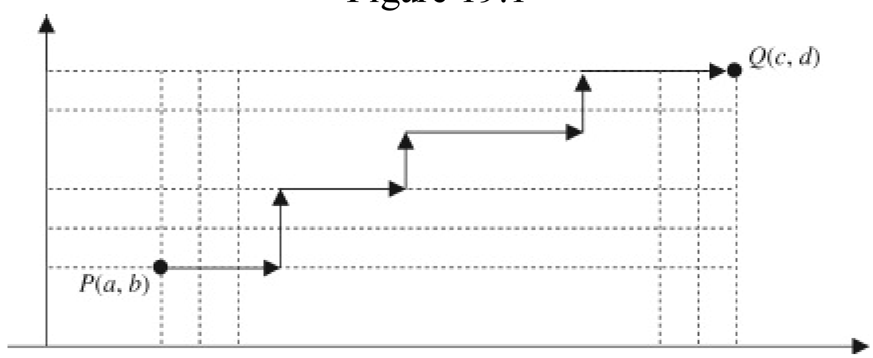
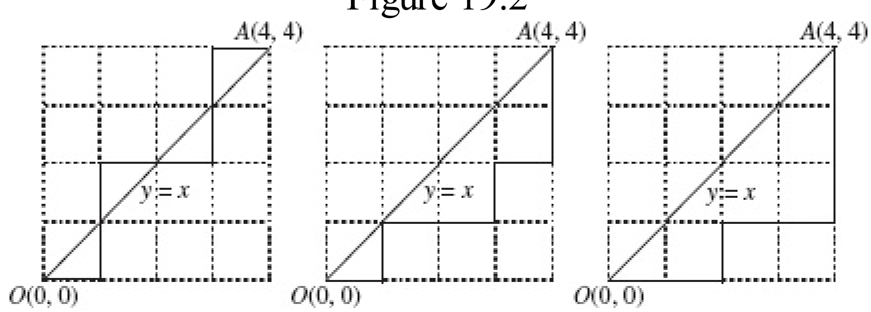


Figure 19.2



(i) crossing $y = x$

(ii) not crossing $y = x$ but meeting $y = x$ internally

(iii) not meeting $y = x$ internally and so not crossing $y = x$

Figure 19.3

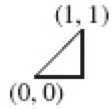
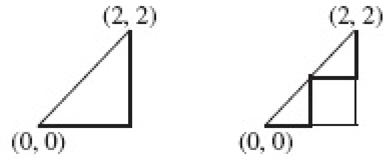
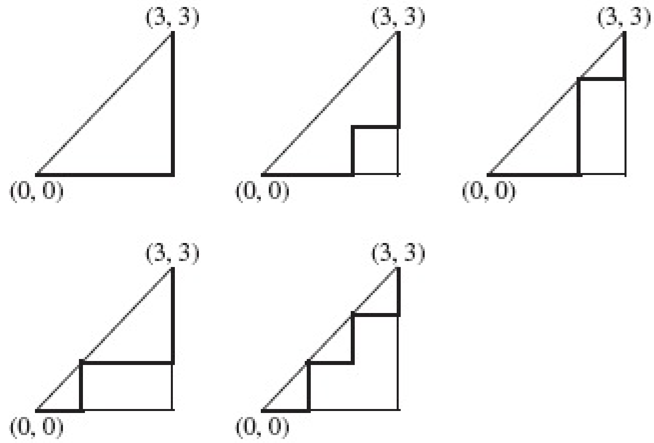
Around 1887, the French combinatorist Désiré André (1840-1917) studied the following problem.

> **A** How many shortest routes from $O(0,0)$ to $A(n,n)$ are there which do not cross the diagonal $y = x$ in the rectangular coordinate system?

(19.2)

Table 19.1

Table 19.1

n	The desired routes	$f(n)$
1		1
2		2
3		5

For convenience, let us denote by *f*(n) the number of such nondiagonal-crossing shortest routes from O(0,0) to A(n, n). For n = 1,2,3, all such routes and the values of *f*(n) are shown in Table 19.1.

In what follows, we shall present André's elegant idea in solving the problem.

By translating a route in the coordinate system one unit to the right as shown in Figure 19.4, we see that there is a 1-1 correspondence between the family of shortest routes from $O(0, 0)$ to $A(n, n)$ that do *not cross* $y = x$ and the family of shortest routes from $P(1,0)$ to $Q(n + 1$, n) that do *not meet* $y =$ x.

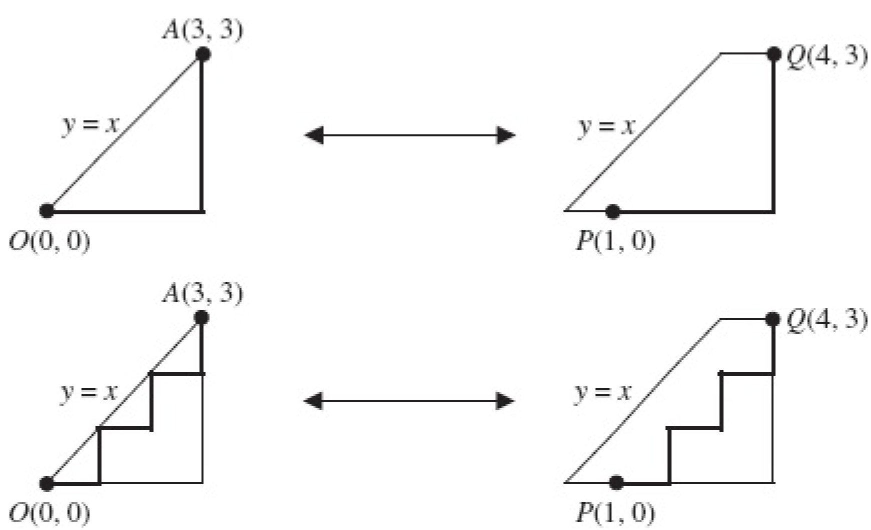


Figure 19.4

Thus, by (BP), we have:

$$\boxed{\begin{array}{l} f(n) \text{ is equal to the number of shortest routes from } P(1,0) \text{ to} \\ Q(n+1,n) \text{ that do not meet } y = x \text{ in the coordinate system.} \end{array}} \tag{19.3}$$

Now, let *g(n)* denote the number of shortest routes from $P(1,0)$ to $Q(n + 1$, n) that *meet* y = x. Clearly, *f* (n)+ *g(n)* is the number of shortest routes from $P(1, 0)$ to $Q(n + 1$,n). Thus, by (19.1), we have:

$$\boxed{f(n)+g(n)=\binom{2n}{n}.} \qquad (19.4)$$

Accordingly, to evaluate f (n), we may instead evaluate g(n).

But how to evaluate g(n)? Is it a less difficult problem? Let us first of all consider an example and make some observations.

Figure 19.5 shows a shortest route from $P(1,0)$ to Q(8, 7) (here, $n = 7$) that meets $y = x$.

Imagine that we are now traversing the route from P to Q. Let X be the point where the route meets $y = x$ for the first time (in Figure 19.5, $X = (2,2)$); note that such an X always exists. Consider the reflection of this part of the route from P to X with respect to $y = x$ as shown in Figure 19.6. Beginning with this image of reflection and following the rest of the original shortest route from X to Q, we obtain a shortest route from $P'(0,1)$ to Q(8, 7).

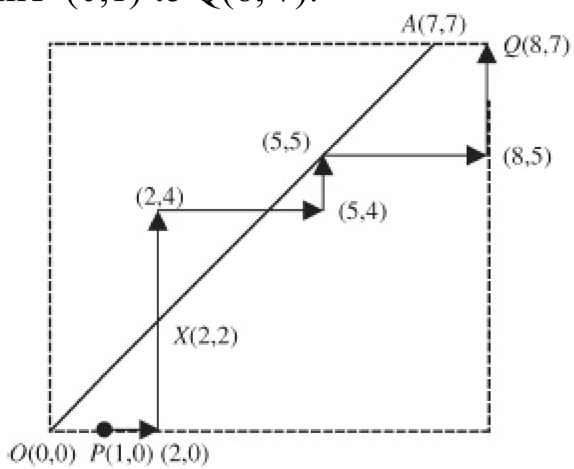


Figure 19.5

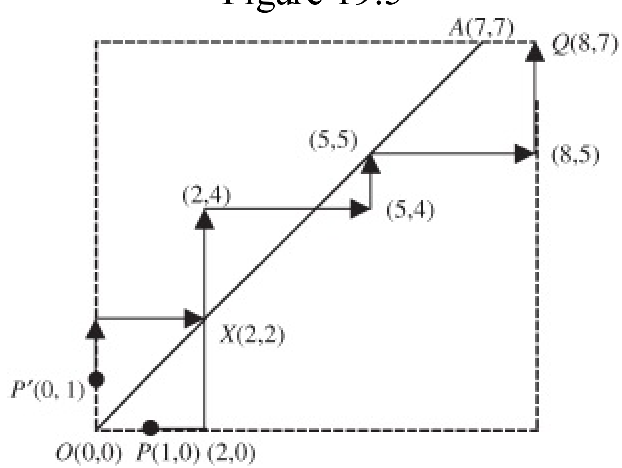


Figure 19.6

The reader may check that this reflection does provide a 1-1 correspondence between the family of shortest routes from $P(1, 0)$ to $Q(8, 7)$ that *meet* $y = x$ and the family of shortest routes from $P'(0,1)$ to Q(8, 7). Thus, by (BP) and (19.1),

$$g(7)=\binom{8+7-1}{8}=\binom{14}{6}.$$

In general, we have

$$g(n)=\binom{n+1+n-1}{n+1}=\binom{2n}{n+1}=\binom{2n}{n-1}.$$

Combining this with (19.4), we see that

$$\begin{aligned} f(n) &= \binom{2n}{n} - g(n) \\ &= \binom{2n}{n} - \binom{2n}{n-1} \\ &= \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

That is:

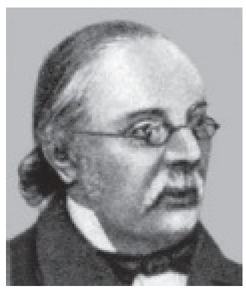
$$1, 2, 5, 14, 42, 132, 429, \ldots, \frac{1}{n+1}\binom{2n}{n}, \ldots$$

In particular, $f(1) = 1$, $f(2) = 2$ and $f(3) = 5$, which agree with Figure 19.1.

The numbers

$$1, 2, 5, 14, 42, 132, 429, \ldots, \frac{1}{n+1}\binom{2n}{n}, \ldots$$

that have just been obtained above are called *Catalan* numbers (denoted by $C(n)$) after the Belgium mathematician Eugene Charles Catalan.



Catalan (1814-1894)

Indeed, around 1838, Catalan studied the following problem:

> **B** Consider the product of n (≥ 2) numbers:
>
> $$x_1 \cdot x_2 \cdot \ldots \cdot x_n.$$
>
> How many ways are there to put $n-1$ pairs of parentheses "(" and ")" on it so that there are terms $a \cdot b$ bracketed by each pair of parentheses?

The ways of parenthesising $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ for $n = 2,3,4$ are shown in Table 19.2.

It turns out that the number of ways obtained are 1,2 and 5, and these are the first three Catalan numbers.

Let us proceed to present another problem which is equivalent to the one introduced by E. Just in *The American Mathematics Monthly* (76).

Table 19.2

n	$x_1 \cdot x_2 \cdot \cdots \cdot x_n$	Number of ways
2	$(x_1 \cdot x_2)$	1
3	$((x_1 \cdot x_2) \cdot x_3), (x_1 \cdot (x_2 \cdot x_3))$	2
4	$(((x_1 \cdot x_2) \cdot x_3) \cdot x_4)$, $((x_1 \cdot (x_2 \cdot x_3)) \cdot x_4)$, $((x_1 \cdot x_2) \cdot (x_3 \cdot x_4))$, $(x_1 \cdot ((x_2 \cdot x_3) \cdot x_4))$, $(x_1 \cdot (x_2 \cdot (x_3 \cdot x_4)))$	5

> **C** For each positive integer n, how many $2n$-digit binary sequences $b_1 b_2 \cdots b_{2n}$ with n "0"s and n "1"s are there such that for each $i = 1, 2, \ldots, 2n$, the number of "0"s is larger than or equal to the number of "1"s in the subsequence $b_1 b_2 \cdots b_i$?

Table 19.3 shows all such binary sequences for $n = 1,2,3$. Notice that the numbers of such sequences are again the first three Catalan numbers.

Table 19.3

n	$b_1 b_2 \cdots b_{2n}$	Number of ways
1	01	1
2	0011, 0101	2
3	000111, 001011, 001101, 010011, 010101	5

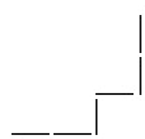
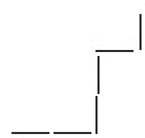
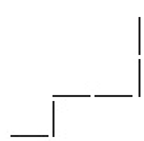
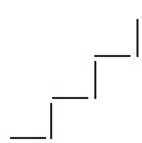
The solution of Problem (A) given by André gives rise to the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. The answers for the first three initial cases of Problems (B) and (C) are 1,2 and 5, which are Catalan numbers. Is it true that the answers for Problems (B) and (C) for general cases are also Catalan numbers?

Yes, they are! In Table 19.4, we exhibit, by examples, 1-1 correspondences among the routes for Problem (A), the ways of parenthesising $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ for Problem (B) and the binary sequences for Problem (C); and we leave it to the reader to figure out the rules for the correspondences.

One of the more general problems of this type, known as the Ballot Problem, is stated below.

The Ballot Problem X and Y are the two candidates taking part in an election. Assume that at the end, X receives x votes and Y receives y votes with $x > y$ (and so X wins). What is the probability that X always stays ahead of Y throughout the counting of the votes?

Table 19.4

Problem (C) ($n = 3$)	Problem (A) ($n = 3$)	Problem (B) ($n = 4$)
000111		$(((x_1 \cdot x_2) \cdot x_3) \cdot x_4)$
001011		$((x_1 \cdot (x_2 \cdot x_3)) \cdot x_4)$
001101		$((x_1 \cdot x_2) \cdot (x_3 \cdot x_4))$
010011		$(x_1 \cdot ((x_2 \cdot x_3) \cdot x_4))$
010101		$(x_1 \cdot (x_2 \cdot (x_3 \cdot x_4)))$

To find out the desired probability, the essential part of the solution is to find out the *number of ways* that X always stays ahead of Y throughout the counting of the votes. The problem is clearly an extension of Problems (A) and (C). Employing the ideas and techniques used to solve Problem (A), Andre solved in 1887 this more general problem. Indeed, the Ballot Problem was first posed and solved by Joseph Louis Francois Bertrand (1822-1900) in the same year 1887. The reader may refer to *An Introduction to Probability Theory and Its Applications* by W. Feller for more details.

It was said that in 1751, the Swiss mathematician Leonhard Euler (1707-1783) proposed to Christian Goldbach (1690-1764) the following problem which later became quite famous. The problem was solved by Johann Andreas von Segner (1704-1777) in 1758 and by Catalan in 1838 using different methods.

> (D) Euler's Polygon Division Problem A *triangulation* of an n-sided polygon, where $n \geq 3$, is a subdivision of the polygon into triangles by means of its nonintersecting diagonals. How many different triangulations of an n-sided polygon are there?

All the triangulations of an n-sided polygon, where n = 3,4, 5, are shown in Table 19.5. The reader may notice that the respective numbers of triangulations are the first three Catalan numbers.

Finally, let us introduce another interesting problem.

> E There are $2n$ ($n \geq 1$) distinct fixed points on the circumference of a circle. How many ways are there to pair them off using n nonintersecting chords?

Table 19.5

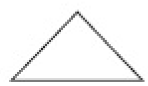
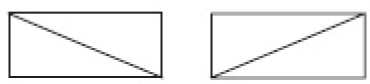
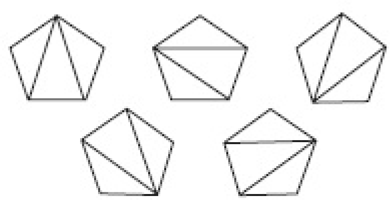
n	The triangulations	Number of ways
3		1
4		2
5		5

Table 19.6

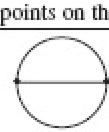
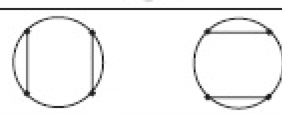
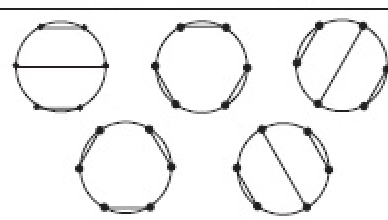
n	Pairing off $2n$ points on the circumference	Number of ways
1		1
2		2
3		5

Table 19.6 shows all the ways for n = 1,2,3. Again, the numbers of ways are the first three Catalan numbers.

The reader is invited in the Exercise to show that the numbers of ways for Problems (D) and (E) are indeed Catalan numbers by establishing 1-1 correspondences between Problem (D) (respectively, (E)) and any of Problems (A)-(C).

Exercise

19.1 Show that the numbers of ways for Problem (D) are Catalan numbers by establishing a 1-1 correspondence between Problem (D) and any of Problems (A)-(C).

19.2 Show that the numbers of ways for Problem (E) are Catalan numbers by establishing a 1-1 correspondence between Problem (E) and any of Problems (A)-(C).

19.3 Using Problem (D) (triangulation of polygons), prove geometrically the following recurrence relation:

$$C(n) = \sum_{r=0}^{n-1} C(r)C(n-r-1).$$

19.4 A valid grouping of *n* pairs of parentheses is one where each open parenthesis has a matching closed parenthesis. For example, ()(()) is valid, but ()())(is not. Let *P*(n) be the number of valid groupings of *n* pairs of parentheses. Show that *P*(n) = *C*(n) by proving the recurrence relation:

$$P(n) = \sum_{r=0}^{n-1} P(r)P(n-r-1).$$

19.5 A *stack* is a data structure where insertions and deletions take place at the *top* of the stack. The integers 1,2,...,*n* are in sequence in stack *X* with 1 on top. All the integers are to be moved to stack *Z* via stack *Y*. Let the stacks *X, Y, Z* be placed in order from left to right. A move from one stack to the stack on its immediate right is the deletion of the integer at the top of the left stack and the insertion of the same integer to the top of the right stack. After exactly 2*n* such moves, stacks *X* and Y will be empty and stack *Z* will contain a permutation of the integers 1,2,...,*n*.

(i) Show that if $n = 3$, the permutation (2,1,3) cannot be generated.

(ii) Find P_n, the number of permutations of 1,2,... ,*n* possible in stack *Z*.

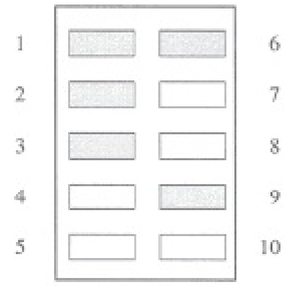
19.6 Use the formula $C(n) = \frac{1}{n+1}\binom{2n}{n}$ to obtain the following recurrence relation:

$$C(n+1) = \frac{2(2n+1)}{n+2}C(n).$$

Chapter 20

Miscellaneous Problems

20.1 One commercially available ten-button lock may be opened by depressing — in any order — the correct five buttons. The sample shown below has {1,2,3,6,9} as its combination. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve as combinations. How many additional combinations would this allow?



(AIME)

20.2 Calculate in how many ways each of the following choices can be made.

(i) 4 movies are to be downloaded from a list of 10 movies to be enjoyed during a holiday.

(ii) 200 essays have been shortlisted for a competition, and three are to be chosen so as to receive the 1st, 2nd and 3rd prizes.

(iii) Eight children are to be chosen from a group of 20 children; the chosen children are then to pair up and line up a pair behind the other, but order within each pair does not matter.

20.3 A society is planning a ballot for the office of president. There are 5 candidates for the office. In order to eliminate the order of the candidates on the ballot as a possible influence on the election, there is a rule that on the ballot slips, each candidate must appear in each position the same number of times as any other candidate. What is the smallest number of different ballot slips necessary?

20.4 In the waiting area of a specialist clinic, patients sit on chairs arranged 10 to a row with an aisle on either side. Ten patients are sitting in the second row. How many ways are there for all the patients in the second row to see the doctor if at least one patient has to pass over one or more other patients in order to reach an aisle?

20.5 In how many ways can 4 a's, 4 *b's,* 4 c's and 4 d's be arranged in a 4×4 array so that exactly one letter occurs in each row and in each column? (Such an arrangement is called a Latin square.)

20.6 A card is drawn from a full pack of 52 playing cards. If the card is a King, Queen or Jack, two dice are thrown and the total T is taken to be the sum of the scores on the dice. If any other card is drawn, only one die is thrown and T is taken to be the sum of the scores on the card (an Ace is considered as 1) and the die. Find the number of ways for each of the following:

(i) $T \leq 2$;

(ii) $T \geq 13$;

(iii) T is odd.

20.7 In each of the following 5-digit numbers

$$25225, 33333, 70007, 11888, \ldots$$

every digit appears more than once. Find the number of such 5-digit numbers.

20.8 The following list contains some permutations of $\mathbb{N}_9$ in which each of the digits 2, 3, 4 appears in between 1 and 9:

$$8\underline{1}\underline{4}7\underline{3}6\underline{2}5\underline{9},\ 56\underline{9}\underline{3}\underline{2}\underline{4}\underline{1}78,\ 7\underline{9}\underline{3}5\underline{4}8\underline{2}\underline{1}6, \ldots.$$

Find the number of such permutations of $\mathbb{N}_9$.

20.9 The following list contains some permutations of $\mathbb{N}_9$ in which each of the digits 1, 2, 3 appears to the right of 9:

458971263, 695438172, 854796123, ...

Find the number of such permutations of $\mathbb{N}_9$.

20.10 Find the number of '0's at the end of $1 \times 2 \times 3 \times \ldots \times 2012$.

20.11 Find the number of 15-digit ternary sequences (formed by 0, 1 and 2) in each of the following cases:

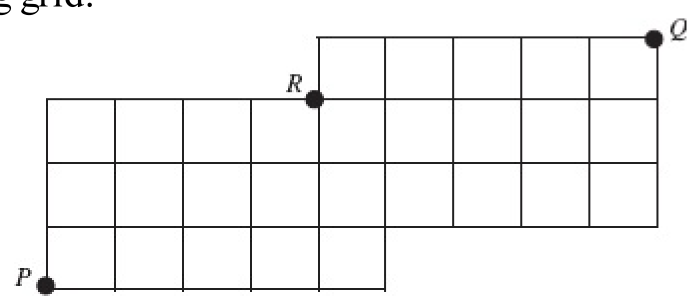
(i) there is no restriction;
(ii) there are exactly three '0's;
(iii) there are exactly four '0's and five '1's;
(iv) there are at most two '0's;
(v) there is at least one pair of consecutive digits that are the same;
(vi) there are exactly one '00', three '11', three '22', three '02', two '21' and two '10' (for instance, 002211102221102).

20.12 Find the number of (i) positive divisors, (ii) even positive divisors of 2160.

20.13 It can be checked that '12' and '18' are the only two positive integers that are divisible by 6 and have exactly 6 different positive divisors.

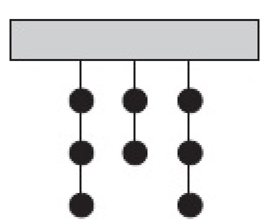
(i) Find all natural numbers which are divisible by 30 and have exactly 30 different divisors.
(ii) How many positive integers are there which are divisible by 105 and have exactly 105 different positive divisors?

20.14 Consider the following grid:



Find in the grid
(i) the number of shortest P-R routes;
(ii) the number of shortest P-Q routes.

20.15 In a shooting match, eight clay targets are arranged in two hanging columns of three each and one column of two, as pictured.



A marksman is to break all eight targets according to the following rules:
(1) The marksman first chooses a column for which a target is to be broken.
(2) The marksman must break the lowest remaining unbroken target in the chosen column.
If these rules are followed, in how many different orders can the eight targets be broken?

(AIME)

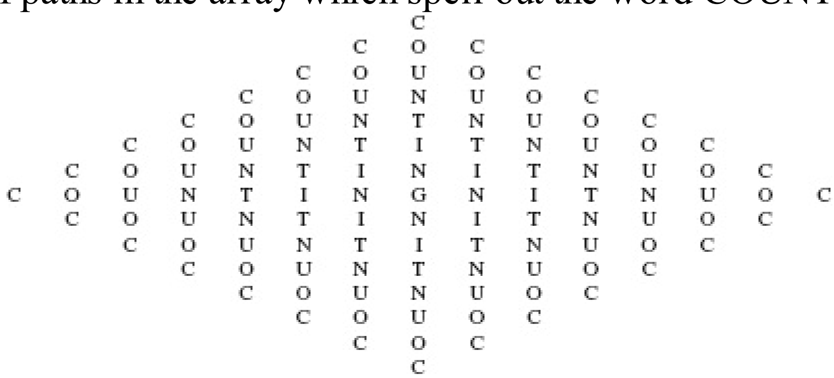
20.16 Six scientists are working on a secret project. They wish to lock up the documents in a cabinet such that the cabinet can be opened when and only when 3 or more of the scientists are present. What is the smallest number of locks needed? What is the smallest number of keys each scientist must carry?

20.17 A team for a boxing competition consists of a heavyweight, a middleweight and a lightweight. There are 5 teams in the competition.

(i) If each person fights with each person of a similar weight class, how many fights take place?

(ii) At the end of the competition, everyone shakes hands exactly once with every other person, except his teammates (they have to tend to each other's wounds later). How many handshakes take place?

20.18 Find the number of paths in the array which spell out the word COUNTING.



20.19 Let $A = \{1,2,..., 500\}$. Find

(i) the number of 2-element subsets of A;

(ii) the number of 2-element subsets $\{a, b\}$ of A such that $a \cdot b$ is a multiple of 3;

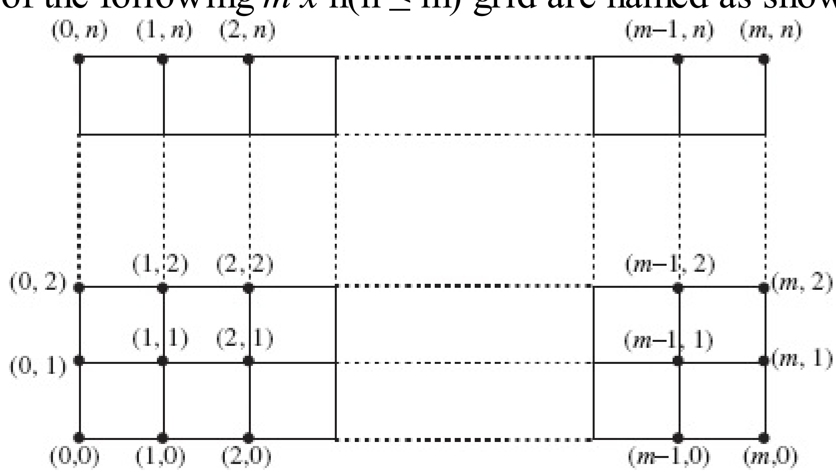
(iii) the number of 2-element subsets $\{a, b\}$ of A such that $a+b$ is a multiple of 3.

20.20 Two integers p and q, with $p \geq 2$ and $q \geq 2$, are said to be *coprime* if p and q have no common prime factor. Thus 8 and 9 are coprime while 4 and 6 are not.

(i) Find the number of ways to express 360 as a product of two coprime numbers (the order of these two numbers is unimportant).

(ii) In general, given an integer $n \geq 2$, how do you find the number of ways to express n as a product of two coprime numbers where the order is immaterial?

20.21 The lattice points of the following m x n($n \leq m$) grid are named as shown:



For $k \in \{1, 2,..., n\}$, let p be the number of shortest $(k,k-1)$–(m, n) routes and q be the number of shortest $(k-1, k)$–(m, n) routes. Show that $p(n+1-k) = q(m+1-k)$.

20.22 The face cards (Kings, Queens and Jacks) are removed from a pack of playing cards. Six cards are drawn one at a time from this pack of cards such that they are in increasing order of magnitude. How many ways are there to do this?

20.23 There are 12 coins on a table. I pick up a number (non-zero) of coins each time. Find the number of ways of picking up all the 12 coins in the following cases:

(i) I pick up all the 12 coins in an even number of picks.

(ii) I pick up an even number of coins each time.

20.24 Find the number of 4-tuples of integers

(i) *(a, b, c, d)* satisfying $1 \le a \le b \le c \le d \le 30$;

(ii) (p, q, r, s) satisfying $1 \le p \le q \le r \le s \le 30$.

20.25 Consider the following two 15-digit ternary sequences (formed by 0, 1 and 2):

0 0 0 1 1 1 2 2 0 0 1 1 2 2 2

0 1 2 2 2 0 0 0 0 1 1 1 1 2 2

Observe that each of the sequences contains exactly three '00',three '11', three '22', two '01', two '12' and one '20'. Find the number of such ternary sequences.

20.26 There are n upright cups in a row. At each step, I turn over n-1 of them. Show that I can end up with all the cups upside down if and only if n is even. Find the number of ways this can be done in a minimum number of steps.

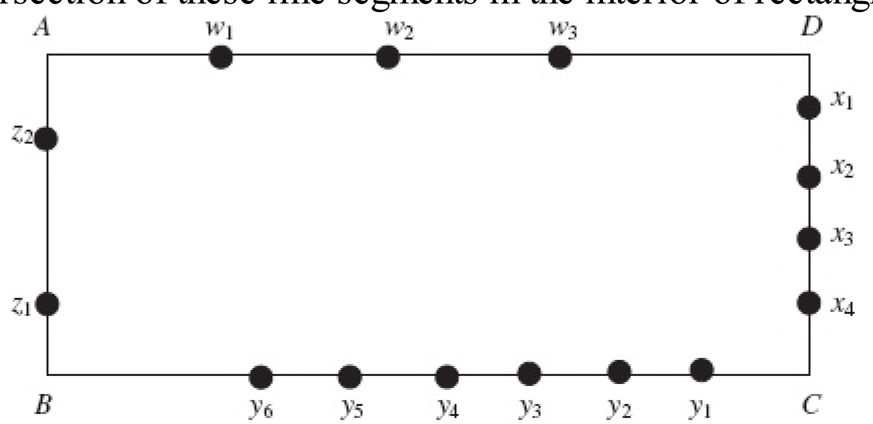
20.27 The following diagram shows 15 distinct points: w1, w2, w3, x1, . . . , x4, y1, . . . , y6, z1, z2 chosen from the sides of rectangle ABCD.

(i) How many line segments are there joining any two points each on different sides?

(ii) How many triangles can be formed from these points?

(iii)How many quadrilaterals can be formed from these points?

(iv) If no three line segments are concurrent in the interior of the rectangle, find the number of points of intersection of these line segments in the interior of rectangle ABCD.



20.28 A ternary sequence is a sequence formed by 0, 1 and 2. Let *n* be a positive integer. Find the number of *n*-digit ternary sequences

(i) which contain at least one '0';

(ii) which contain one '0' and one '1';

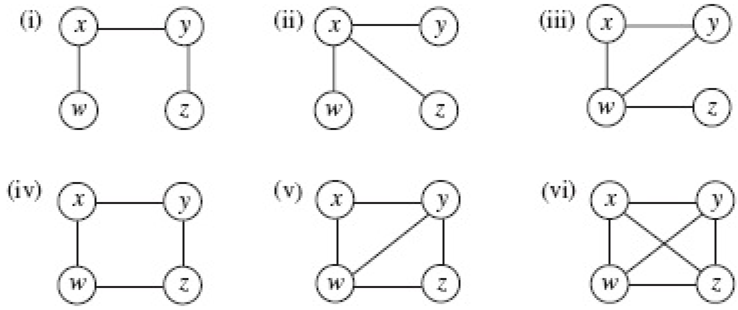
(iii) which contain three '2's.

20.29 Each of the following six configurations consists of 4 vertices *w, x, y, z* with some pairs of vertices joined by lines. We are now given five colours 1, 2, 3, 4, 5 to colour the 4 vertices such that

(1) each vertex is coloured by one colour and

(2) any two vertices which are joined by a line must be coloured by different colours.

How many different ways are there to colour each configuration?



20.30 If repetitions are not allowed, find the number of different 5-digit numbers which can be formed from 0,1, 2,..., 9 and are

(i) divisible by 25;
(ii) odd and divisible by 25;
(iii) even and divisible by 25;
(iv) greater than 75000;
(v) less than 75000;
(vi) in the interval [30000, 75000] and divisible by 5.

20.31 There are 12 boys and 8 girls, including a particular boy B and two particular girls G_1 and G_2, in a class. A class debating team of 4 speakers and a reserve is to be formed for the interclass games. Find the number of ways this can be done if the team is to contain

(i) exactly one girl;
(ii) exactly two girls;
(iii) at least one girl;
(iv) at most two girls;
(v) Gi;
(vi) no B;
(vii) B and G_1;
(viii) neither B nor G_1;
(ix) exactly one from G_1 and G_2;
(x) an odd number of girls.

20.32 A group of 6 people is to be chosen from 7 couples. Find the number of ways this can be done if the group is to contain

(i) three couples;
(ii) no couples;
(iii) exactly one couple;
(iv) exactly two couples;
(v) at least one couple.

20.33 Find the number of ways in which 6 people can be divided into

(i) 3 groups consisting of 3, 2, and 1 persons;
(ii) 3 groups with 2 persons in each group;
(iii) 4 groups consisting of 2, 2, 1 and 1 persons;
(iv) 3 groups with 2 persons in each group, and the groups are put in 3 distinct rooms.

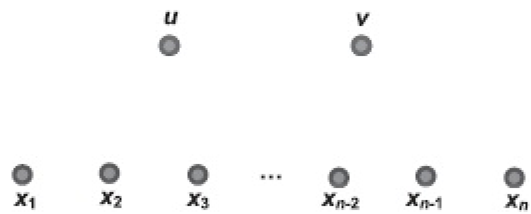
20.34 Show that the number of r-combinations of N_n which contain no consecutive integers is given by $\binom{n-r+1}{r}$,, where $0 \le r \le n - r + 1$.

20.35 Suppose the n integers in N_n are arranged consecutively round a table so that '1' is also adjacent to 'n'. For $0 \le r \le \lfloor n \rfloor$, let $T(r)$ denote the number of r-combinations of N_n in which no two integers are adjacent around the table. Show that

$$T(r) = \frac{n}{r}\binom{n-r-1}{r-1}.$$

20.36 Three girls and five boys are to line up in a row. Find the number of ways if each boy is adjacent to at most one girl.

20.37 There are $n + 2$ vertices, denoted by $u, v, x_1, x_2, \ldots, x_n$, in a graph (see Chapter 15 for the concept of a 'graph') as shown below:

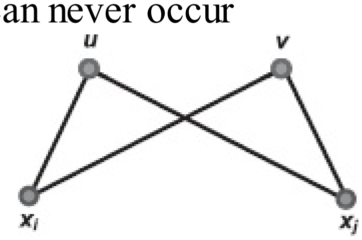


You are allowed to draw exactly $n + 1$ edges to link the vertices in such a way that

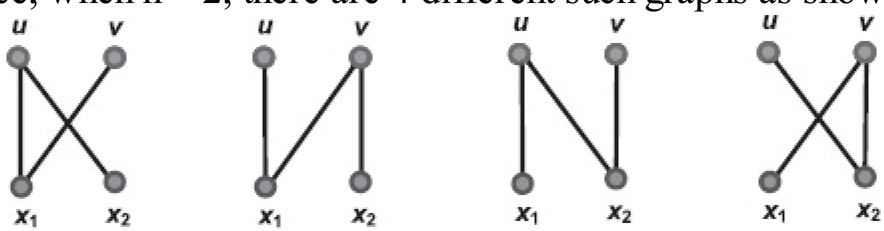
(1) each edge joins u or v to some xi,

(2) no two edges join the same pair of vertices, and

(3) the following configuration can never occur



Thus, for instance, when n = 2, there are 4 different such graphs as shown below:



(i) Show all different such graphs for $n = 3$.

(ii) For all $n \ge 4$, evaluate the number of different such graphs.

20.38 Find the coefficient of x^9 in the expansion of $(1 + x + x^3 + x^5)^5$.

20.39 Find the number of 4-element subsets $\{w, x, y, z\}$ of N_{25} in each of the following cases:

(i) there is no restriction;

(ii) the product $w \cdot x \cdot y \cdot z$ is odd;

(iii) the product $w \cdot x \cdot y \cdot z$ is even;

(iv) the sum $w + x + y + z$ is odd;

(v) the sum w ? $x + y$? z is odd.

20.40 Find the number of positive divisors of 67500 in each of the following cases:

(i) there is no restriction;

(ii) the divisors are even;

(iii) the divisors are odd;

(iv) the divisors are multiples of 5;

(v) the divisors are multiples of 6.

20.41 Six distinct numbers a, b, c, d, e and f are chosen from the 8-element set $\{2,3,...,9\}$ and are arranged in a row in the order a, b, c, d, e, f. Find the number of ways this can be done if
(i) there is no restriction;
(ii) the product $a \cdot d$ is even;
(iii) the product $a \cdot b \cdot c$ is odd;
(iv) the product $a \cdot b \cdot c$ is odd and the product $d \cdot e \cdot f$ is a multiple of 10;
(v) a, b and c are divisible by d, e and f respectively.

20.42 Every day, an inter-city train arrives at a station early (E), on time (T) or late (L), or the trip is cancelled (C). Let S_n denote the number of possible sequences of arrival for the train in a series of n days. Given that the train is never cancelled on two successive days, find a recurrence relation for S_n. Prove also that S_{2n} and S_{2n+1} are both divisible by 3^n.

20.43 A special deck of 52 playing cards consists of cards without the usual numbers or pictures but just bearing the suit (i.e. there are 13 cards of each of the four suits, Heart, Diamond, Spade and Club, and the cards in each suit are indistinguishable from each other).
(i) Eight cards are selected from the deck. Find the number of different selections if there is at least one card of each suit.
(ii) Eight cards are arranged in a row. Find the number of distinct sequences if there must be at least one card of each suit.
(iii) Two cards of each suit are selected. Find the number of ways they can be arranged in a row if no two cards of the same suit are together.

20.44 Let n and k be positive integers with $n \geq k$. A *partition* of n into *exactly* k parts is a way of expressing n as a sum of k positive integers in which the ordering is immaterial; that is,
$$n = n_1 + n_2 + \cdots + n_k,$$
and we may assume that $n_1 \geq n_2 \geq \cdots \geq n_k$.
For instance, three partitions of '8' into exactly 4 parts are shown below:
$$8 = 5+1+1+1 = 3+3+1+1 = 2+2+2+2.$$
Note that a partition of n into exactly k parts can also be considered as a way of distributing n identical objects into k identical boxes so that no box is empty.
Let $p(n, k)$ denote the number of partitions of n into exactly k parts.
(i) Given that $n \geq k+2$ and $k \geq 2$, prove that $p(n,k) \geq 2$. Hence determine all values of n and k such that $p(n, k) = 1$.
(ii) Prove that, for $n \geq 2$, $p(n,2) = \lfloor \frac{n}{2} \rfloor$.
(iii) Find the values of $p(6,1)$, $p(6,2)$, $p(6,3)$, $p(6,4)$ and $p(10,4)$, and verify that $p(6,1) + p(6,2) + p(6,3) + p(6,4) = p(10,4)$.
(iv) Prove that, for $1 \leq m \leq n$,
$$\sum_{k=1}^{m} p(n,k) = p(n+m, m).$$
(v) Find the value of $p(9,3)$ and verify that $p(10,4) = p(9, 3) + p(\mathbf{6}, 4)$.
(vi) Show that, for $2 \leq k \leq n$,
$$p(n,k) = p(n-1, k-1) + p(n-k, k).$$
(vii) Prove that for $m \geq 1$, $p(\mathbf{6}m, 3) = 3m^2$.

20.45(i) Ice-cream sticks are tied in bundles of 1, 2, 3 or 4. Let a_n be the number of ways of bundling and arranging n icecream sticks in a line. Thus, $a_1 = 1$, $a_2 = 2$ and $a_3 = 4$. For example, the arrangements (1,1,1), (1,2), (2,1), and (3) show that $a_3 = 4$. Find a_4 and a recurrence relation for a_n.

(ii) Ice-cream sticks and satay sticks are tied in bundles of 1, 2 or 3. Let b_n be the number of ways of bundling and arranging n sticks in a line. For example, if we label the two types of sticks as c and s, the arrangements ({c}, {c}), ({c}, {s}), ({s}, {c}), ({s}, {s}), ({c,c}), *({c,s}),* and ({s,s}) show that $b_2 = 7$. Find a recurrence relation for b_n.

:0.46 (i) There are 20 seats in a row labelled from left to right with the numbers 1, 2,..., 20. Find the number of ways of choosing 5 disjoint pairs of adjacent seats (for instance, {{3, 4}, {9,10}, {11,12}, {15,16}, {19, 20}} is a way) from them.

(ii) There are n seats in a row. Let r be a positive integer such that $2r \le n$. Find the number of ways of choosing r disjoint pairs of adjacent seats from them.

20.47 In the game of Mastermind, a code is constructed by arranging 4 out of 6 distinct coloured counters in a row. This code is hidden from a player who attempts to break the code by making a series of guesses. Each guess consists of placing counters (not necessarily distinct), which can be chosen from the 6 available colours, into 4 slots in a row. Any of the 4 slots may also be left 'empty' instead of placing a counter in it. After each guess, the codemaker will check the guess against the code. For each colour correctly placed, he will allocate a black token, and for each colour present in the code but incorrectly placed, he will allocate a white token. For example, suppose the code is (Blue, Green, Red, Yellow). A guess of (Red, Red, Empty, Empty) will be allocated 1 white token (for the incorrectly placed Red counter). A guess of (Blue, Red, Red, Green) will be allocated 2 black tokens (for the correctly placed Blue and Red counters) and 1 white token (for the incorrectly placed Green counter).

(i) How many possible guesses are there altogether?

(ii) How many guesses will result in no tokens?

(iii) How many guesses will result in 3 white and no black tokens?

(iv) How many guesses will result in 1 white and 3 black tokens?

20.48 Show that if a and b are integers, then the decimal representation of $\frac{a}{b}$ either terminates or eventually has a block of digits that repeats itself to infinity. (For example, $\frac{6}{5} = 1.2$ and $\frac{19}{14} = 1.3\underline{571428}571428571428\ldots$.)

In what follows, let S be an n-element universal set, and let $P_1, P_2, \ldots, P_q$ be q properties for the elements of S, where $q > 1$. For integer m with $0 \le m \le q$, let $E(m)$ denote the number of elements of S that possess *exactly* m of the q properties and for $1 \le m \le q$, let $\omega(P_{i_1} P_{i_2} \cdots P_{i_m})$ denote the number of elements of S that possess the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$, and let

$$\omega(m) = \sum \omega(P_{i_1} P_{i_2} \cdots P_{i_m}),$$

where the summation is taken over all m-combinations $\{i_1, i_2, \ldots, i_m\}$ of $\{1,2,\ldots, q\}$. We also define w(0) to be $|S|$, i. e. $w(0) = |S| = n$.

In the following problems, we shall establish the following generalised principle of inclusion and exclusion (GPIE) and show two applications of it.

> (GPIE)
>
> Let S be an n-element set and let $\{P_1, P_2, \ldots, P_q\}$ be a set of q properties for elements of S. Then for each $m = 0, 1, 2, \ldots, q$,
>
> $$E(m) = \sum_{k=m}^{q} (-1)^{k-m} \binom{k}{m} \omega(k).$$

20.49 Prove the following identities.

(i) $\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}$. (See Problem 10.1(ii).)

(ii) $\sum_{r=0}^{n}(-1)^r\binom{n}{r} = 0$.

20.50 Prove (GPIE).

Hint: Let $x \in S$ and assume that x possesses exactly t properties. Consider the different possible values of t and count the contribution of x to each side of the inequality.

20.51 Find the number of integers from the set {1, 2,... , 1000} which are divisible by exactly two of 2, 5 and 7.

20.52 Using (GPIE) on a particular value of m, prove that the number of onto mappings from N_m to N_n, where $m \geq n \geq 1$, is given by $\sum_{r=0}^{n}(-1)^r\binom{n}{r}(n-r)^m$ (see Problem 14.5).

20.53 Use a direct combinatorial argument to show that $\backslash s(m,k)\backslash = s^*(m, k)$.

20.54 For integers $m,n \geq 0$, prove the following identities:

(i) $s^*(n+1, m+1) = \sum_k s^*(n,k)\binom{k}{m}$,

(ii) $s^*(m+n+1, m) = \sum_{k=0}^{m}(n+k)s^*(n+k,k)$.

20.55 For integers m, $n > 0$, prove the following identities.

(i) $S(n+1, m+1) = \sum_{k=1}^{n} S(k,m)\binom{n}{k}$,

(ii) $S(m+n+1, m) = \sum_{k=0}^{m} kS(n+k,k)$.

20.56 Let n, k and d be positive integers with $n \geq k \geq d$. Denote by $S^d(n, k)$ the number of ways of partitioning the n-element set N_n into k nonempty subsets such that for any two elements i and j in each same subset, $|i - j| \geq d$. For instance, {{4}, {1,3}, {2,5}} is a partition counted in $S^2(5,3)$ and *{{1}, {4}, {2, 5}, {3, 6}}* is one counted in $S^2(6, 4)$ as well as $S^3(6,4)$. Clearly, $S^1(n,k) = S(n,k)$.

(i) Verify that $S^2(5, 3) = 7$ by listing all possible partitions.

(ii) Verify that $S^3(6, 4) = 7$ by listing all possible partitions.

(iii) Show that $S^d(n,k) = S^d(n-1,k-1) + (k-d+1)S^d(n-1,k)$, where $n \geq k \geq d \geq 2$.

(iv) Show that $S^d(n,k) = S(n-d+1,k-d+1)$ where $n \geq k \geq d \geq 2$.

(Mohr, A. and Porter, T. D., Applications of Chromatic Polynomials Involving Stirling Numbers, *Journal of Combinatorial Mathematics and Combinatorial Computing* 70 (2009), 5764.)

20.57 For any positive integer n, the nth *Bell number* (named after E.T. Bell (1883–1960)), denoted by B_n, is defined as the number of ways of dividing n distinct objects into (nonempty) groups i.e. the number of ways of partitioning an n-element set into nonempty subsets. For instance, $B_3 = 5$ and the 5 ways of partitioning {1, 2, 3} into nonempty subsets are shown below: {{1, 2, 3}}, {{1, 2}, {3}}, {{1,3}, {2}},{{2,3},{1}}, {{1},{2},{3}}.

The first 10 Bell numbers are shown below:

B1 = 1, B2 = 2, B3 = 5,B4 = 15, B5 = 52, Be = 203,
B7 = 877, Bg = 4140, B9 = 21147, B10 = 115975.

(i) Explain why $B_n = S(n, 1) + S(n, 2) + \ldots + S(n, n)$.

(ii) Show that

$$B_{n+1} = \sum_{k=0}^{n}\binom{n}{k}B_k.$$

20.58 A "depth" sequence of non-negative integers $d_1,d_2,...,d_n$, satisfies

(i) $d_1 = 0$

(ii) for $i \geq 1$, $d_{i+1} \leq di + 1$.

Show that the number of "depth" sequences of length n is C (n).

20.59 A standard Young tableau consists of two rows of boxes with n boxes in each row in which the integers 1,2<...,2n are placed such that the numbers increase from left to right and each number in the bottom row is larger than the number in the box above it.

Show that the number of standard Young tableau with $2n$ boxes is $C(n)$.

20.60 For $r = 1,2,\ldots,2012$, let A_r be a set such that $|A_r| = 44$. Assume that $|A_i \cap A_j| = 1$ for all i, $j \in \{1,2,\ldots, 2012\}$ with $i \neq j$. Evaluate $\left|\bigcup_{r=1}^{2012} A_r\right|$.

Books Recommended for Further Reading

1. K. P. Bogart, *Introductory Combinatorics* (3rd ed.), S. I. Harcourt Brace College Publishers, 1998.
2. R. A. Brualdi, *Introductory Combinatorics* (5th ed.), Prentice Hall, 2009.
3. C. C. Chen and K. M. Koh, *Principles and Techniques in Combinatorics,* World Scientific, 1992.
4. D. I. A. Cohen, *Basic Techniques of Combinatorial Theory,* John Wiley & Sons, 1978.
5. R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics* (2nd ed.), Addison-Wesley, 1994.
6. B. W. Jackson and T. Dmitri, *Applied Combinatorics with Problem Solving,* Addison-Wesley, 1990.
7. C. L. Liu, *Introduction to Combinatorial Mathematics*, McGraw-Hill, 1968.
8. F. Roberts and B. Tesman, *Applied Combinatorics* (2nd ed.), Prentice Hall, 2002.
9. A. Tucker, *Applied Combinatorics* (4th ed.), John Wiley & Sons, 2002.

Answers to Exercises

1.1 6

1.2 1, 5, 14, 55, $\sum_{r=1}^{n} r^2$

1.3 (i) 20 (ii) $6n - 4$

1.4 29

1.5 27

1.6 60

1.7 31

1.8 29

1.9 14

2.1 90

2.2 (i) 6 (ii) 15

2.3 (i) 20 (ii) mn (iii) mnt

2.4 30

2.5 (i) 3^n (ii) 2^n (iii) $2^{n-1}(2+n)$ (iv) n^2+n+1

2.6 (i) 47 (ii) 205 (iii) 378

3.4 $(n+1)! - 1$

4.1 (v) $2 \cdot 9!$ (vi) $2 \cdot 8!$ (vii) $9!3!$ (viii) $7!5!$ (ix) $2 \cdot 5!6!$ (x) $8!9 \cdot 8 \cdot 7$ (xi) $5!5!2$ (xii) $2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11$

4.2 (iii) 728 (iv) 280 (v) 1319

4.3 $9 \cdot 7 \cdot 5 \cdot 3$

4.4 (i) $10!$ (ii) $8!3!$ (iii) $7! \cdot 8 \cdot 7 \cdot 6$ (iv) $7!3!70$

4.5 $7 \cdot 5 \cdot 3$

4.6 (i) $5!$ (ii) $4!2!$ (iii) $4!3$

4.7 $\binom{4}{2}\binom{7}{3}$ (ii) $\binom{4}{3}\binom{7}{2} + \binom{4}{4}\binom{7}{1}$ (iii) $\binom{11}{5} - \binom{9}{3}$

4.8 $9 \cdot 10^{\lfloor \frac{n-1}{2} \rfloor}$

4.9 (i) $\binom{10}{6}$ (ii) P_6^{10} (iii) $\binom{10}{2}\binom{8}{2}\binom{6}{2}$

4.10 162

4.11 $\binom{9}{6}$ (ii) $\frac{1}{2}\binom{9}{3}\binom{6}{3}\binom{3}{2}$

4.12 (i)324 (ii) 72

4.13 $\binom{7}{2}$ (i) $\binom{7}{2}\binom{7}{2}$ (ii) $2\binom{7}{1}\binom{7}{2}$

4.14 (a)$\binom{5}{3}\binom{10}{3}$ (b) $\binom{2}{2}\binom{8}{1}\binom{5}{3} + \binom{2}{0}\binom{8}{3}\binom{5}{3}$, $640 \cdot 3 \cdot 2$

5.1 (a) (i) 60 (ii) 36

(b) By FTA, express n as $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$. Then number of positive divisors is $\prod_{i=1}^{k}(m_i+1)$.

5.2 $3^2 \cdot 5^4 \cdot 7^6, 3^2 \cdot 5^6 \cdot 7^4, 3^4 \cdot 5^2 \cdot 7^6, 3^4 \cdot 5^6 \cdot 7^2, 3^6 \cdot 5^2 \cdot 7^4, 3^6 \cdot 5^4 \cdot 7^2$

5.3 (i) $\binom{5}{2}\binom{7}{2}$ (ii) $\binom{5}{2}\binom{6}{2}$ (iii) $\binom{5}{2}\binom{4}{1}\binom{3}{1}$ (iv) $\binom{12}{4} - \binom{5}{2}\binom{6}{2}$

5.7 $\binom{7}{2}\binom{8}{2}$

5.8 $\binom{8}{4}$

5.9 $\binom{n}{6}$ (ii) $5\binom{n}{5}$ (iii) $4\binom{n}{4}$ (iv) $\binom{n}{6} + 5\binom{n}{5} + 4\binom{n}{4} + \binom{n}{3}$

6.1 $\binom{6}{3}$

6.2 $5! \cdot 6 \cdot 5 \cdot 4 \cdot 3$

6.3 $2 \cdot \frac{10!}{3!}$

6.4 $6!\binom{12}{4}$

7.1 (i) $\binom{55}{4}$ (ii) $\binom{47}{4}$ (iii) $\binom{55}{4}-\binom{46}{4}$ (iv) $\binom{11}{1}\binom{43}{2}$ (v) $\binom{27}{4}$, 0

7.2 (i) $\binom{23}{7}-\binom{19}{7}$ (ii) $\binom{19}{7}$

7.3 (i) $\binom{7}{2}\binom{6}{2}$ (ii) $\binom{5}{2}^3$

7.4 (i) $\binom{33}{3}$ (ii) $\binom{31}{3}-\binom{25}{3}$ (iii) $\binom{36}{3}$

7.5 $\binom{2006}{4}$

7.6 $\binom{16}{2}+\binom{11}{2}+\binom{6}{2}$

7.7 $\binom{n+r-1}{r}$

7.8 $\binom{r-1}{n-1}$

7.9 $\binom{14}{4}$

7.10 $\binom{5}{3}\binom{8}{3}$

7.11 4!7!

7.12 (i) 4!5!(ii) 4!2 (iii) $4!2^5$ (iv) 5!5!

7.13 (i) 10!(ii) 4!6!10

7.14 (i) 10!(ii) 4!6!10

7.15 (i) 4!2!(ii) 4 3 4!

8.1 (i) 9^6(ii) P_6^9

8.3 (i) 8^4(ii) 7^5

8.4 $\binom{8}{4}\binom{4}{2}\binom{2}{2}3+\binom{8}{2}\binom{6}{3}\binom{3}{3}3$

8.5 $m \geq n$(i) n! $\binom{n+1}{2}\ n!$ (iii) $\binom{n+2}{3}n!+\frac{1}{2}\binom{n+2}{2}\binom{n}{2}n!$

9.1 $\frac{10!}{5!2!}$

9.2 $\frac{9!}{3!2!}, \frac{6!}{2!}$

9.3 (i)$\frac{15!}{4!5!6!}$ (ii) $\binom{10}{4}\binom{11}{5}$ (iii) $\binom{11}{5}\binom{9}{4}$

9.4 (i) n^m(ii)P_m^n (iii) $\binom{n}{m}$ (iv) n^{m-1}

9.5 (i) $n!$ (ii) $\binom{n+1}{2}\ n!$ (iii) $\binom{n+2}{3}n!+\frac{1}{2}\binom{n+2}{2}\binom{n}{2}n!$

9.6 $\binom{13}{3}10!$

9.7 $n!, n(n+1)!, n(n+2)!+\binom{n}{2}(n+2)!$

10.2 25

10.3 $k=3; (1+x)^{14}; \binom{14}{4}x^4, \binom{14}{5}x^5, \binom{14}{6}x^6$

10.4 $\binom{23}{17}(3x)^{17}, \binom{23}{18}(3x)^{18}$

12.1 $\binom{101}{6}$

12.2 $\binom{n+1}{4}-1$

12.6 $\frac{(3n+1)!}{(n+1)(2n)!}$

12.7 (i) $\sum_{k=3}^{33}\binom{k}{3}$ (ii) $\binom{34}{4}$

12.8 $k = 44, n = 98$

13.1 56

13.2 1171

13.3 2

13.4 (i) 734(ii) 266

13.5 (i) 3^7(ii) 1806

13.7 356
13.8 10
13.9 150
13.10 250848

14.1 *Hint:* For $i = 1,2,...,n$, let A_i be the set of ways such that Couple i are seated together.

14.2 $\frac{23}{1260}$
Hint: Divide into cases according to the number of matchings and find the corresponding numbers of derangements.

14.3 *Hint:* For $i = 1,2,...,11$, let A_i be the set of integer solutions with $x_i \geq 10$.

14.4 *Hint:* For $i = 1,2,...,10$, let A_i be the set of ways such that Lady i gets back her hat and B_i be the set of ways such that Lady i gets back her umbrella.

14.5 *Hint:* For $i = 1,2,...,n$, let A_i be the set of mappings where i is not an image.

14.6 (i) (a) 5^8 (b) 5×4^7 (c) 60505
(ii) *Hint:* Let A_i be the set of ways without jersey i, for $i = 1, 2,3,4,5$.
(iii) The number of ways the team can choose a jersey from 5 jerseys for n matches if the team uses each jersey at least once; 0

15.1 *Hint:* Objects — sums along the rows, columns and diagonals; Boxes — all possible sums.

15.2 *Hint:* Objects — coordinates of 5 lattice points; Boxes — all permutations of parities (odd or even) of the x and y coordinates.

15.3 *Hint:* Objects — 19 points; Boxes — (i) 16 squares of side 1 unit within bigger square; (ii) 9 squares of side $\frac{4}{3}$ units within bigger square.

15.4 $\lceil\sqrt{n}\rceil$. *Hint:* Follow the proof of Example 15.5. Show also a sequence of n distinct numbers where there is no increasing or decreasing subsequences of $k + 1$ numbers, for $k = \lceil\sqrt{n}\rceil$.

15.5 *Hint:* Suppose none of the boxes contain at least $\lceil\frac{m}{n}\rceil$ objects.

15.6 *Hint:* Suppose for all $i = 1,2,...,n$, the ith box contains less than k_i objects.

15.7 *Hint:* Objects — pairs formed from the 16 objects; Boxes — possible absolute differences. Watch out for a twist!

15.8 Hint: Objects — the kings; Boxes — squares of side 2 units.

15.9 (PP) is wrongly used.

16.1 (i) $b_n = 1.05b_{n-1} - X$ for $n \geq 3$, $b_1 = 303000 - X, b_2 = 309060 - 2.02X$.
(ii) $X = 36432$ (to the nearest integer).

16.2 $s_n = 3s_{n-1} + 3s_{n-2}$ for $n \geq 3$, $s_1 = 4, s_2 = 15$.

16.3 $s_n = s_{n-1} + n$ for $n \geq 2$, $s_1 = 2$.

16.4 $s_n = s_{n-1} + 2(n - 1)$ for $n \geq 2$, $s_1 = 2$.

16.5 *Hint:* Observe that F_{n-i}, $0 \leq i \leq n - 1$, is also the number of rabbits that are at least i months old at the beginning of the nth month.

16.6 $s_n = s_{n-1} + s_{n-2}$ for $n \geq 3$, $s_1 = 2, s_2 = 3$.

16.7 $s_n = s_{n-1} + s_{n-2} + \ldots + s_{n-k}$ for $n \geq k + 1$, $s_i = 2^i$ for $i \leq k - 1, s_k = 2^k - 1$.

16.8 $s_n = 2s_{n-1} - s_{n-2} + s_{n-3}$ for $n \geq 4$, $s_1 = 2, s_2 = 4$, $s_3 = 7$.

16.9 *Hint:* Prove by induction on n.

16.10 *Hint:* Suppose $a_1 = i, i = 2,3,...,n$. Consider the cases $a_i = 1$ and $a_i = 1$.

17.1

k	1	2	3	4	5	6
$s(6, k)$	−120	274	−225	85	−15	1

17.2 (i) *Hint:* Collect all the x terms in the expansion of $[x]_m$.

(ii) *Hint:* Collect all the x^{m-1} terms in the expansion of $[x]_m$.

17.3 *Hint:* Consider the cases (4, 1, 1), (3, 2, 1) and (2, 2, 2).

17.5 *Hint:* Remember to show that every arrangement of n integers around k indistinguishable circles corresponds to a unique permutation of 1,2,…,n.

18.1

k	1	2	3	4	5
$S(10, k)$	1	511	9330	34105	42525

18.2 5(5, 3)

18.3 (i) *Hint:* Suppose first that the two groups are distinct.

(ii) *Hint:* First choose 2 items to put in one group.

18.4 (i) *Hint:* Use induction on n. (ii) *Hint:* Use induction on n.

18.5 *Hint:* First divide the 5 elements in the domain into 3 nonempty indistinguishable groups.

(vi) *Hint*: Split the counting according to whether there is a box with exactly one object.

20.45 (i) 8; $a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}$, $a_2 = 2$, $a_3 = 4$, $a_4 = 8$.

(ii) $b_n = 2b_{n-1} + 3b_{n-2} + 4b_{n-3}$, for $n \geq 4$, $b_1 = 2$, $b_2 = 7$, $b_3 = 24$.

20.46 (i) $\binom{15}{5}$ (ii) $\binom{n-r}{r}$

20.47 **(i)** 7^4 (ii) 3^4 (iii) 92 (iv) 0

20.48 *Hint:* Objects — "remainders" in the "long division" of a by b; Boxes — remainders when divided by b.

20.49 **(i)** *Hint:* Consider two methods of dividing a group of n persons into three groups containing $n - m$, $m - r$ and r persons respectively.

(ii) *Hint:* Consider the binomial expansion of $(1 + x)^n$ for a particular value of x.

20.50 *Further hint:* Consider the cases $t < m$, $t = m$ and $t > m$.

20.51 157

Hint: Let P_1 be the property "divisible by 2", P_2 be the property "divisible by 5", and P_3 be the property "divisible by 7".

20.52 *Hint:* Let $m = 0$.

20.53 *Hint:* Generalise the argument at the end of Chapter 17.

20.54(i) *Hint:* Use mathematical induction. Alternatively, use the idea and result of Problem 17.6.

(ii) *Hint:* Unfold the recurrence $s^*(n + m + 1, m) = s^*(n + m, m - 1) + (n + m)s^*(n + m, m)$.

20.55 **(i)** *Hint:* Divide into cases according to the number of persons who are not in the same group with a particular person A.

(ii) *Hint:* Unfold the recurrence relation $S(n + m + 1, m) = S(n + m, m - 1) + mS(n + m, m)$.

20.56 **(i)** {{1, 3}, {2, 4}, {5}}, {{1, 3}, {2, 5}, {4}}, {{1, 4}, {2, 5}, {3}},{{1,4},{3,5}, {2}}, {{1,5}, {2, 4}, {3}},{{2,4},{3,5}, {1}}, {{1, 3, 5}, {2}, {4}}.

(ii) {{1, 4}, {2, 5}, {3}, {6}}, {{1, 4}, {2, 6}, {3}, {5}}, {{1, 4}, {3,6},{2},{5}}, {{1, 5}, {2, 6}, {3}, {4}}, {{1,5}, {3, 6}, {2}, {4}},{{1,6},{2, 5}, {3}, {4}},{{2,5}, {3, 6}, {1}, {4}}.

(iii) *Hint*: Divide into cases according to whether the subset {1} exists or not.

(iv) *Hint*: Use induction on $n + k$.

20.57 **(i)** *Hint:* Consider the different number of nonempty subsets that an n-element set can be partitioned into.

(ii) *Hint*: Divide into cases according to the number of elements in the subset of which "n + 1" is an element.

20.58 *Hint:* Obtain the sequence $0 - d_1$, $1 - d_2$, ..., $i\text{-}1 - d_i$, ..., $n - 1 - d_n$. Start with the 2n-digit binary sequence 00 … 011 … 1. Now move the ith "0" to a position right of ($i - 1 - d_i$) "1"s. Show that the resulting sequence fulfils the conditions of Problem (C).

20.59 *Hint:* Obtain a correspondence with Problem (C) as follows. Given a 2n-digit binary sequence, the integers *i*, $i = 1,2,\ldots,2n$, are placed in order according to the following rules:

- On the leftmost empty cell in row *A* if the ith digit is 0.
- On the leftmost empty cell in row *B* if the ith digit is 1.

20.60 86517

(vi) *Hint*: Split the counting according to whether there is a box with exactly one object.

20.45 (i) 8; $a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}$, $a_2 = 2$, $a_3 = 4$, $a_4 = 8$.

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20.51 157

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20.60 86517

Index