How to Use Directed Angles

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"WLOG, diagram as shown." – Everyone

This is a very brief note on what a directed angle is and how to use it to write olympiad solutions which are impervious to configuration issues. It is short because I don't actually need to teach you how to solve problems, but only to rewrite proofs of problems you've already essentially solved.

1 Introduction

Everyone hates configuration issues (both contestants and graders). Configuration issues come up because the most fundamental theorem in olympiad geometry actually has two cases.

Proposition 1.1 (Cyclic Quadrilaterals)
Let A, B, X, Y be any four points, no three collinear.

(i) If A and B lie on the same side of $\overline{XY}$, then the four points are concyclic if and only if $\angle XAY = \angle XBY$.

(ii) If A and B lie on different sides of $\overline{XY}$, then the four points are concyclic if and only if $\angle XAY + \angle XBY = 180^\circ$.

Doesn't that just look annoying? Any time you want to invoke a cyclic quadrilateral, you have to actually check the points lie on the correct side of some line.

In fact, you even have to worry about configuration issues for something as simple as adding two angles.

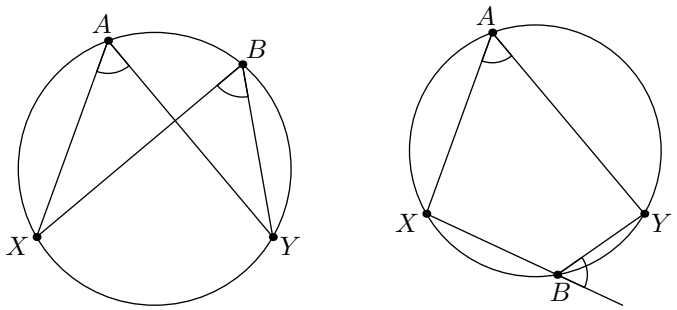


Figure 1: The two cases for cyclic quadrilaterals.

Question 1.2. When does the assertion

$$\angle AOP + \angle POB = \angle AOB$$

fail?

Thus any time you want to add two angles, you technically need to also check that point P lies "inside" $\angle AOB$.

Given all this disaster, you might wonder how we ever got any angle chasing done at all. The secret is that **configuration issues are the object of widespread scorn**: they are glossed over, swept under a carpet, or "left as an exercise". I've almost never seen them addressed seriously, except in the very rare circumstances in which they actually matter.

Let's fix this.

2 Directed Angles

In what follows I'm going to write $\measuredangle AOB$ for a directed angle to distinguish it from a "regular" angle $\angle AOB$. But I should warn that **this notation is absolutely not standard**. Thus if you wish to use directed angles on an olympiad, you should explicitly say so in your solution.

Here's the very general definition.

Definition 2.1. Given any two non-parallel lines ℓ and m, we define the directed angle

$$\measuredangle(\ell, m)$$

to be the measure of the angle *starting* from ℓ and *ending* at m, measured *counterclockwise*.

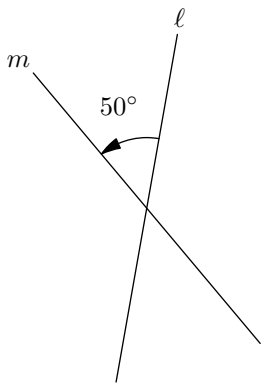


Figure 2: The directed angle $\measuredangle(\ell, m) = 50^\circ$.

Notice that

$$\measuredangle(\ell, m) + \measuredangle(m, \ell) = 180^\circ \tag{1}$$

holds universally. This is kind of nice, but it's a bit annoying to have that 180° lying around there, and so we will also **take all angle measures modulo** 180°. That means that $-70^\circ = 110^\circ = 290^\circ = \dots$. Once we take mod 180°, (1) becomes the following very important result.

Proposition 2.2

For any lines ℓ and m, $\measuredangle(\ell, m) = -\measuredangle(m, \ell)$. (In other words, measuring the angle clockwise instead of counterclockwise corresponds to negation.)

Observe why this is intuitively true in Figure 2.

You can verify now that with this identification, we have

$$\measuredangle(\ell, m) + \measuredangle(m, n) = \measuredangle(\ell, n)$$

for all concurrent lines ℓ, m, and n, regardless of configuration.

With this definition in place, we can define $\measuredangle AOB$ as the angle between the two lines AO and BO.

Definition 2.3. Given three points A, O, B we define

$$\measuredangle AOB \stackrel{\text{def}}{=} \measuredangle\left(\overline{AO}, \overline{BO}\right).$$

Equivalently, if ℓ and m are two lines which intersect at O, then $\measuredangle(\ell, m) = \measuredangle AOB$ for any point A on ℓ and B on m.

(Note that by $\overline{XY}$ I mean "line XY", not "segment XY".)

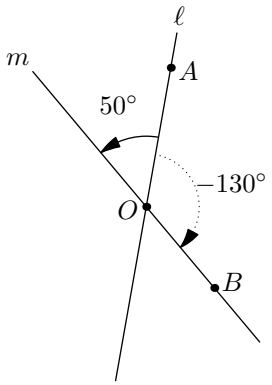


Figure 3: The directed angle $\measuredangle AOB = -130^\circ = 50^\circ$.

Most of the time we will be using the $\measuredangle AOB$ notation. But it is sometimes useful to use the $\measuredangle(\ell, m)$ notation in problems where the intersection point O of ℓ and m is not yet named.

3 Properties of Directed Angles

You might ask whether this strange convention is actually useful. Let me convince you it is with the following theorem.

Theorem 3.1 (Directed Cyclic Quadrilaterals)

Let A, B, X, Y be four points, no three collinear. Then they are concyclic if and only if

$$\measuredangle XAY = \measuredangle XBY.$$

Exercise 3.2. Go back to Figure 1 and verify that this theorem actually works for both configurations.

Now you should be paying attention, because I just eliminated one of the biggest pains in olympiad geometry for you. Unlike the Proposition 1.1 you grew up with, Theorem 3.1 has no case distinctions.

Many other things become vastly simpler.

Theorem 3.3 (Angle Addition Postulate)
We have $\measuredangle AOP + \measuredangle POB = \measuredangle AOB$.

Actually, this is a "special case" of the following result for three concurrent lines ℓ, m, n.

Theorem 3.4 (Triangles Sum to 180°)
For any lines ℓ, m, n we have

$$\measuredangle(\ell, m) + \measuredangle(m, n) + \measuredangle(n, \ell) = 0.$$

In particular, for any points A, B, C we have

$$\measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0.$$

Proof. Check it yourself. □

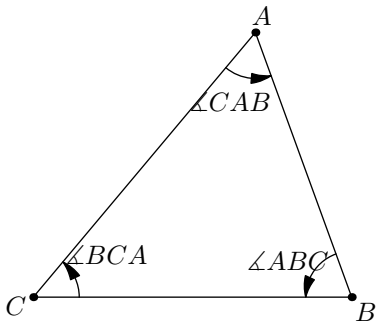


Figure 4: Triangle Sum

Theorem 3.5 (Collinearity Criteria)
Let X be any point. Points A, B, C are collinear if and only if

$$\measuredangle XBC = \measuredangle XBA.$$

Proof. Prove this yourself. (Show that the assertion is equivalent to $\measuredangle ABC = 0$.) □

Also, note that right angles have the very nice property that if $\ell \perp m$ we have

$$\measuredangle(\ell, m) = \measuredangle(m, \ell) = 90^\circ.$$

Hence any time you have perpendiculars, you can simply set the measure of the directed angle as 90° without thinking or worrying about counterclockwise versus clockwise.

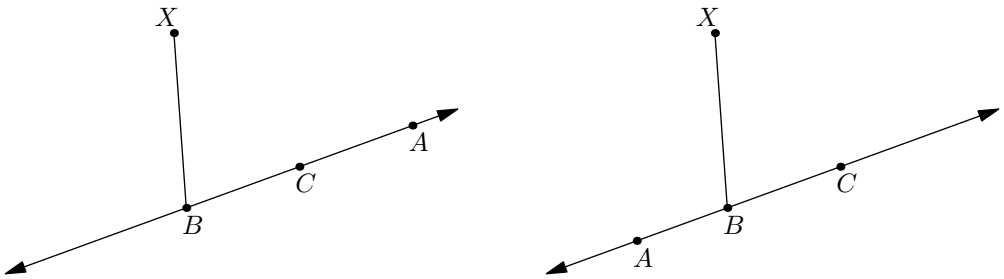


Figure 5: Collinearity with directed angles.

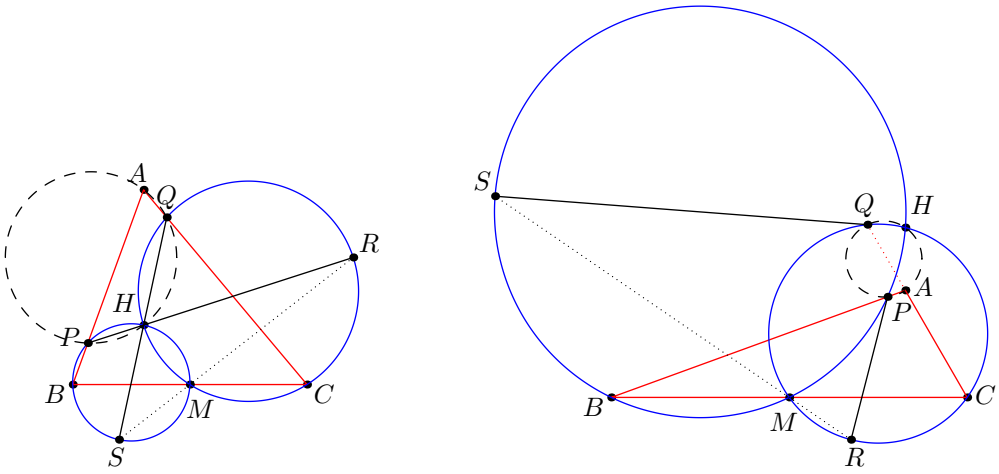
4 Examples

Writing a solution using directed angles should, in theory, not take any more additional effort. One simply solves the problem by looking at *one particular diagram*, but writes the solution up using the above rules so that they hold for *all diagrams*.

Let me actually give an example now.

Example 4.1 (NIMO Winter 2013, Adapted)
Let ABC be a triangle with orthocenter H and let M be the midpoint of $\overline{BC}$. Denote by ω_B the circle passing through B, H, and M, and denote by ω_C the circle passing through C, H, and M. Lines AB and AC meet ω_B and ω_C again at P and Q, respectively. Rays PH and QH meet ω_C and ω_B again at R and S, respectively. Prove that M, R, S are collinear.

This is just angle chasing. Unfortunately, the configurations when $\triangle ABC$ is acute and $\triangle ABC$ is obtuse look quite different from each other.



Follow along the proof in both diagrams, asking yourself why each equality is true. The beauty of directed angles is that to *write* this proof, I only had to look at the first diagram; it then works for the other diagram automatically.

Verbose Solution. Applying Miquel's Theorem (in the exercises later), we find that quadrilateral $APHQ$ is cyclic. Also from Triangle Sum we have

$$\measuredangle\left(\overline{HB},\overline{AB}\right)+\measuredangle\left(\overline{AB},\overline{AC}\right)+\measuredangle\left(\overline{AC},\overline{HB}\right)=0$$

we derive that $\measuredangle HBA + \measuredangle BAC = 90^\circ$.

Taking these for granted, we can compute

$$\begin{aligned}
\measuredangle BPS &= \measuredangle BHS && \text{Cyclic quads} \\
&= \measuredangle PHS + \measuredangle BHP && \text{Angle addition} \\
&= \measuredangle PHQ + \measuredangle BHP && S,\ H,\ Q \text{ collinear} \\
&= \measuredangle PAQ + \measuredangle BHP && \text{Cyclic quads} \\
&= \measuredangle PAQ - (\measuredangle HPB + \measuredangle PBH) && \text{Triangle Sum} \\
&= \measuredangle BAC - (\measuredangle HPB + \measuredangle ABH) && \text{Collinearity} \\
&= \measuredangle BAC + \measuredangle BPH + \measuredangle HBA && \text{Negation} \\
&= 90^\circ + \measuredangle BPH.
\end{aligned}$$

But

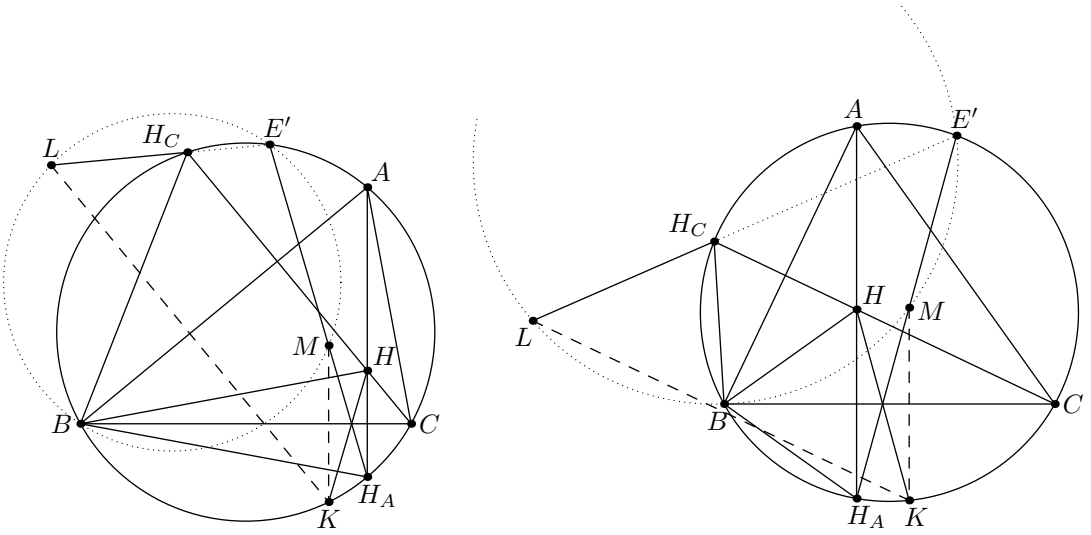
$$\measuredangle BPS - \measuredangle BPH = \measuredangle BPS + \measuredangle HPB = \measuredangle HPS.$$

so we deduce that $\measuredangle HPS = -90^\circ = 90^\circ$, hence $\measuredangle HMS = 90^\circ$ as well. In the same way, $\measuredangle HMR = 90^\circ$ as well. Hence $\measuredangle HMS = \measuredangle HMR$, so points M, R, S are collinear. □

In an actual olympiad one need not be so explicit. In particular, you can omit the part where I wrote out the explicit reasons for each step; I only provided this for reference. Here is a second example (much harder) in which I will be much more succinct. See if you can follow along the logic.

Example 4.2 (European Girl's MO 2012, Problem 7)

Let ABC be an acute-angled triangle with circumcircle Γ and orthocenter H. Let K be a point of Γ on the other side of BC from A. Let L be the reflection of K in the line AB, and let M be the reflection of K in the line BC. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM. Show that the lines KH, EM and BC are concurrent.



Solution. Let H_A and H_C be the reflections of H across $\overline{BC}$ and $\overline{BA}$; it is well-known that these lie on Γ. Let E' be the second intersection of line H_AM with Γ. By construction, lines $E'M$ and HK concur on $\overline{BC}$, and our goal is to show that B, L, E', M are concyclic.

First, we claim that L, H_C, and E' are collinear. Due to the reflections,

$$\measuredangle LH_CB = -\measuredangle KHB = \measuredangle MH_AB = \measuredangle E'H_AB = \measuredangle E'H_CB$$

which proves the claim. Then

$$\measuredangle LE'M = \measuredangle H_CE'H_A = \measuredangle H_CBH_A = 2\measuredangle ABC$$

(the last equality following from reflections; verify it yourself) and

$$\measuredangle LBM = \measuredangle LBK + \measuredangle KBM = 2\measuredangle ABK + 2\measuredangle KBC = 2\measuredangle ABC$$

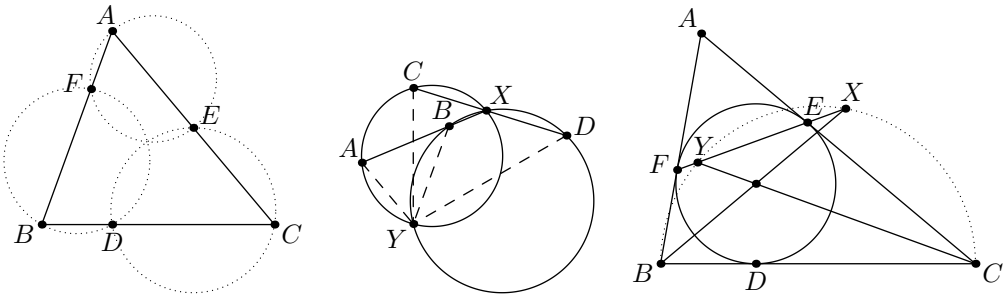
so B, L, E', M are concyclic. Hence $E = E'$ and we are done. □

5 A Word of Warning

Never take half of a directed angle – since we are working modulo 180°, taking half of an angle doesn't make sense.

6 Practice Problems

Here are some famous lemmas from olympiad geometry that you should know. They are not hard to prove (in fact, they are all direct angle chasing) but they do have configuration issues. See if you can solve them and then write a single proof which does not need to consider different cases.



Problem 6.1 (Miquel's Theorem)**.** Let ABC be a triangle. Consider any points D, E, F on lines BC, CA, AB (distinct from the vertices of ABC, but not necessarily in the interiors of the sides). Prove that the circumcircles of triangles AEF, BFD, and CDE intersect at a single point.

Problem 6.2 (Spiral Similarity Lemma)**.** Two circles ω_1 and ω_2 meet at points X and Y. A line through X intersects ω_1 and ω_2 again at A and B. A second line through X intersects ω_1 and ω_2 again at C and D. Show that $\triangle AYC \sim \triangle BYD$.

Problem 6.3 (Right Angles on Intouch Chord)**.** Let ABC be a triangle whose incircle touches the opposite sides at D, E, F. The angle bisectors of $\angle B$ and $\angle C$ meet line EF at points X and Y. Prove that X and Y lie on the circle with diameter $\overline{BC}$.

7 Contest Practice

Some of the following problems are much more nontrivial; the last one is infamously difficult. Directed angles will help clean up a solution but as stressed before will not actually help you find the solution in the first place.

Problem 7.1 (Folklore)**.** Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ be four distinct circles. For $i = 1, 2, 3, 4$, suppose that $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ intersect at two distinct points A_i and B_i (here $\mathcal{C}_5 = \mathcal{C}_1$). Prove that if $A_1A_2A_3A_4$ is cyclic then so is $B_1B_2B_3B_4$.

Problem 7.2 (Shortlist 2010/G1)**.** Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

Problem 7.3 (USAMO 2013/1)**.** In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

Problem 7.4 (Balkan 2009)**.** Let $\overline{MN}$ be a line parallel to the side $\overline{BC}$ of a triangle ABC, with M on side AB and N on side AC. The lines BN and CM intersect at point P. The circumcircles of $\triangle BMP$ and $\triangle CNP$ meet again at Q. Prove that $\angle BAQ = \angle CAP$.

Problem 7.5 (USA TST 2007/1)**.** Circles ω_1 and ω_2 meet at P and Q. Segments AC and BD are chords of ω_1 and ω_2 respectively, such that lines AB and CD meet at P. Lines BD and AC meet at X. Point Y lies on ω_1 such that $\overline{PY} \parallel \overline{BD}$. Point Z lies on ω_2 such that $\overline{PZ} \parallel \overline{AC}$. Prove that points Q, X, Y, Z are collinear.

Problem 7.6. Let ABC be a triangle with circumcircle Γ. Let ℓ be a line in the plane, and let ℓ_a, ℓ_b, ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA, and AB, respectively. Let $\triangle A'B'C'$ denote the triangle determined by the lines ℓ_a, ℓ_b, ℓ_c.

(a) (Iran 1995) Prove that the incenter of the $\triangle A'B'C'$ lies on Γ.

(b) (IMO 2011/6) Assume ℓ is tangent to Γ. Show that the circumcircle $\triangle A'B'C'$ is tangent to the circle Γ.