Functional Equations

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17 Chapters and 199 Problems With Solution

Content:

Enjoy!

Constructive Problems

This problems involve explicit construction of functions, or inductive arguments.

Problem 1. Let k be an even positive integer. Find the number of all functions $f\colon N_0 \to N_0$ such that

$$f(f(n)) = n + k$$

for any $n \in N_0$.

Solution. We have

$$f(n + k) = f(f(f(n))) = f(n) + k$$

and it follows by induction on m that

$$f(n + km) = f(n) + km$$

for all $n, m \in N_0$.

Now take an arbitrary integer p, $0 \le p \le k-1$, and let $f(p) = kq+r$, where $q \in N_0$ and $0 \le r \le k - 1$. Then

$$p + k = f(f(p)) = f(kq + r) = f(r) + kq.$$

Hence either $q = 0$ or $q = 1$ and therefore

$$\text{either} \quad f(p) = r, \;\; f(r) = p + k \quad \text{or} \quad f(p) = r + k, \;\; f(r) = p.$$

In both cases we have $p \ne r$ which shows that f defines a pairing of the set $A = \{0, 1, \ldots, k\}$. Note that different functions define different pairings of A.

Conversely, any pairing of A defines a function $f\colon N_0 \to N_0$ with the given property in the following way. We define f on A by setting $f(p) = r$, $f(r) = p + k$ for any pair (p, r) of the given pairing and $f(n) = f(q) + ks$ for $n \ge k + 1$, where q and s are respectively the quotient and the remainder of n in the division by k.

Thus the number of the functions with the given property is equal to that of all pairings of the set A. It is easy to see that this number is equal to $\dfrac{k!}{(k/2)!}$.

Remark. The above solution shows that if k is an odd positive integer then there are no functions $f\colon N_0 \to N_0$ such that

$$f(f(n)) = n + k$$

for all $n \in N_0$. For $k = 1987$ this problem was given at IMO '1987.

Problem 2. (IMO '1996). Find all functions $f\colon N_0 \to N_0$ such that

$$f(m + f(n)) = f(f(m)) + f(n)$$

for all $m, n \in N_0$.

Solution. Setting $m = n = 0$ gives $f(0) = 0$ and therefore $f(f(n)) = f(n)$, i.e. $f(n)$ is a fixed point of f for any n. Hence the given identity is equivalent to

$$f(0) = 0 \text{ and } f(m + f(n)) = f(m) + f(n).$$

It is obvious that the zero function is a solution of the problem.

Now suppose that $f(a) \neq 0$ for some $a \in N$ and denote by b the least fixed point of f. Then

$$2f(b) = f(b + f(b)) = f(2b)$$

and it follows by induction that $f(nb) = nb$ for any $n \in N_0$. If $b = 1$ then $f(n) = n$ for any $n \in N_0$ and this function is also a solution of the problem. Hence we may assume that $b \geq 2$. Let c be an arbitrary fixed point of f. Then $c = kb + r$, where $k \in N_0$, $0 \leq r < b$, and we get

$$kb + r = c = f(c) = f(kb + r) = f(f(kb)) + f(r) = kb + f(r).$$

Thus $f(r) = r$ and therefore $r = 0$ since $r < b$. Hence any fixed point of f has the form kb. Now the identity $f(f(i)) = f(i)$ implies that $f(i) = bn_i$ for all i, $0 \leq i < b$, where $n_i \in N_0$ and $n_0 = 0$. Thus if $n = kb + i$ then $f(n) = (k + n_i)b$. Conversely, it is easily checked that for any fixed integers $b \geq 2$, $n_0 = 0$ and $n_1, n_2, \ldots, n_b - 1 \in N_0$ the function $f(n) = \left(\left[\dfrac{n}{b}\right] + n_i\right) b$ has the given property.

Problem 3. Find all functions $f\colon N \to R\backslash\{0\}$ which satisfy

$$f(1) + f(2) + \ldots + f(n) = f(n)f(n + 1)$$

Solution. If we try to set $f(x) = cx$ we compute that $c = \frac{1}{2}$. However the condition of the problem provides a clear recurrent relation for f, therefore there are as many solutions as possible values for $f(1)$. So set $f(1) = a$. Then setting $n = 1$ in the condition we get $a = af(2)$ and as $a \neq 0$ we get $f(2) = 1$. Then setting $n = 2$ we get $f(3) = a + 1$. Setting $n = 3$ we get $f(4)(a + 1) = a + 1 + (a + 1)$ so $f(4) = 2$ as $a + 1 = f(3) \neq 0$. Now we see a pattern: for even numbers k $f(k) = \frac{k}{2}$ as desired, whereas for odd numbers k we have an additional a, and we can suppose that $f(k) = [\frac{k}{2}] + (k \ mod\ 2)a = \frac{k}{2} + (k \ mod\ 2)(a - \frac{1}{2})$. Let's now prove by induction on k this hypothesis. Clearly we have to consider two cases according to the parity of k.

a) $k = 2n$. Then we have $f(1) + f(2) + \ldots + f(k) = f(k)f(k+1)$ or $\frac{1}{2} + \frac{2}{2} + \ldots + \frac{2n}{2} + n(a - \frac{1}{2}) = nf(2n+1)$ so $\frac{2n(2n+1)}{4} + na - \frac{n}{2} = nf(2n+1)$ which gives us $f(2n+1) = n + a$, as desired.

b)$k = 2n + 1$. This case is absolutely analogous.

Hence all desired functions are of form $f(k) = [\frac{k}{2}] + (k \ mod \ 2)a$ for some a. They clearly satisfy the conditions of the problem provided that a is not a negative integer (in which $f(-2a+1) = 0$).

Problem 5. Find all functions $f\colon N \to N$ for which

$$f^3(1) + f^3(2) + \ldots + f^3(n) = (f(1) + f(2) + \ldots + f(n))^2$$

Solution. The function $f(x) = x$ comes to the mind of everyone who knows the identity $1^3 + 2^3 + \ldots + n^3 = (\frac{n(n+1)}{2})^2 = (1 + 2 + \ldots + n)^2$. We shall prove this is the only solution, proving in the meantime the identity, too. By setting $n = 1$ we get $f(1) = 1$. If we subtract the identity for n from the identity for $n+1$ we get $f^3(n+1) = f(n+1)(2f(1) + 2f(2) + \ldots + 2f(n) + f(n+1))$ so we get an identity of a smaller degree: $f^2(n+1) = 2f(1) + 2f(2) + \ldots + 2f(n) + f(n+1)$ (*). Doing the same procedure (subtracting (*) for n from (*) for $n+1$) we get $f^2(n+2) - f^2(n+1) = f(n+2) + f(n+1)$ and reducing we get $f(n+2) - f(n+1) = 1$. It's thus clear by induction that $f(n) = n$. The verification is clear by the same induction, as we actually worked by equivalence.

Problem 6. Find all non-decreasing functions $f\colon Z \to Z$ which satisfy

$$f(k) + f(k+1) + \ldots + f(k+n-1) = k$$

, for any $k \in Z$, and fixed n.

Solution. Again subtracting the condition for k from the condition for $k+1$ we get $f(k+n) = f(k) + 1$. Therefore f is determined by its' values on $\{0, 1, \ldots, n-1\}$ and the relation $f(k) = [\frac{k}{n}] + f(kmodn)$. As f is non-decreasing and $f(n) = f(0) + 1$, we see that there is a $0 \leq m \leq n-1$ such that $f(0) = f(1) = \ldots = f(m)$, $f(0) + 1 = f(m+1) = \ldots = f(n)$. Now by writing the condition for $k = 0$ we get $nf(0) + (n - m - 1) = 0$ which implies $m = n - 1$ thus $f(0) = f(1) = \ldots = f(n-1) = 0$. It's now clear that $f(k) = [\frac{k}{n}]$. This value clearly satisfies the condition, as it is a consequence of Hermite's Identity $[x] + [x + \frac{1}{n}] + \ldots + [x + \frac{n-1}{n}] = [nx]$. Note that we have also proven the identity by induction during the proof.

Remark In this problem and in preceding ones, we could replace the function f by the sequence a_n, so transforming a functional equation into a sequence problem. It can be therefore asked if this kind of problems are functional problems or problems on sequences? While the answer is insignificant and is left at the mercy of the reader, sequences in general play a very important role in many functional equations, as we shall see in a lot of examples.

Problem 7. Find all functions $f\colon N \to R$ for which we have $f(1) = 1$ and

$$\sum_{d|n} f(d) = n$$

whenever $n \in N$.

Solution. Again a little mathematical culture helps us: an example of such a function is Euler's totient function ϕ. So let's try to prove that $f = \phi$. As ϕ is multiplicative, let's firstly show that f is multiplicative, i.e. $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. We do this by induction on $m + n$. Note that when one of m, n is 1 this is clearly true. Now assume that $m, n > 1$, $(m, n) = 1$. Then the condition written for mn gives us $\sum_{d|mn} f(d) = mn$. But any $d|mn$ can be written uniquely as $d = d_1 d_2$ where $d_1|m.d_2|n$. If $d < mn$ then $d_1 + d_2 < m + n$ and by the induction hypothesis we get $f(d) = f(d_1 d_2) = f(d_1)f(d_2)$ for $d < mn$. Therefore $mn = \sum_{d|mn} f(d) = \sum_{d|mn,d<mn} f(d) + f(mn) = \sum_{d_1|m,d_2|n} f(d_1)f(d_2) - f(m)f(n) + f(mn) = (\sum_{d|m} f(d))(\sum_{d|n} f(d)) = mn - f(m)f(n) + f(mn)$, so $f(mn) = f(m)f(n)$, as desired. So it suffices to compute f for powers of primes. Let p be a prime. Then writing the condition for $n = p^k$ we get $f(1) + f(2) + \ldots + f(p^k) = p^k$. Subtracting this for the analogous condition for $n = p^{k+1}$ we get $f(p^{k+1}) = p^{k+1} - p^k = \phi(p^{k+1})$, and now the relation $f = \phi$ follows from the multiplicativity. It remains to verify that $\sum_{d|n} \phi(d) = n$. There are many proofs of this. One of the shortest is evaluating the numbers of subunitary (and unitary) non-zero fractions with denominator n. On one hand, this number is clearly n. On the other hand, if we write each fraction as $\frac{k}{l}$ in lowest terms, then $l|n$ and the number of fractions with denominator l is $\phi(l)$-the number of numbers not exceeding l which are coprime with l. So this number is also $\sum_{d|n} \phi(d)$.

Problem 8. Find all functions $f\colon N \to N$ for which we have $f(1) = 1$ and

$$f(n+1) = [f(n) + \sqrt{f(n)} + \frac{1}{2}]$$

$n \in N$.

Solution. $f(n + 1)$ depends on $[\sqrt{f(n)} + \frac{1}{2}]$. So suppose that $[\sqrt{f(n)} + \frac{1}{2}] = m$, thus $(m - \frac{1}{2})^2 \le f(n) < (m + \frac{1}{2})^2$ or $m^2 - m \le f(n) \le$

m^2+m, then $f(n+1)=f(n)+m$ so $m^2\le f(n+1)\le m^2+2m<(m+1)(m+2)$. Then $[\sqrt{f(n+1)}+\frac{1}{2}]$ is either m or $m+1$ hence $f(n+2)$ is either $f(n)+2m$ or $f(n)+2m+1$, so $m^2+m\le f(n+2)\le m^2+3m+1$ so $m(m+1)\le f(n+2)\le(m+1)(m+2)$. Thus if we denote $g(x)=[\sqrt{x+\frac{1}{2}}]$ then $g(f(n+2))=g(f(n))+1$. As $g(f(1))=1, g(f(2))=1$, we deduce that $g(f(n))=[\frac{n}{2}]$ by induction. Hence $f(n+1)=f(n)+[\frac{n}{2}]$. Then $f(n+2)=f(n)+[\frac{n}{2}]+\frac{[}{n+1}2]=f(n)+n$ (Hermite). So $f(2k+2)=f(2k)+2k, f(2k+1)=f(2k-1)+2k-1$ thus $f(2k)=(2k-2)+(2k-4)+\ldots+2+f(2)=k(k-1)+1$ and $f(2k+1)=(2k-1)+(2k-3)+\ldots+1+f(1)=k^2+1$. This can be summed up to $f(n)=[\frac{n}{2}][\frac{n+1}{2}]+1$.

Problem 9. Find all functions $f\colon N\to N$ that satisfy $f(1)=2$ and

$$f(n+1)=[1+f(n)+\sqrt{1+f(n)}]-[\sqrt{f(n)}]$$

Solution. As $[\sqrt{1+f(n)}]=[\sqrt{f(n)}]$ unless $1+f(n)$ is a perfect square, we deduce that $f(n+1)=f(n)+1$ unless $f(n)+1$ is a perfect square, in which case $f(n)+1$ is a perfect square. Thus f jumps over the perfect squares, and $f(n)$ is the n-th number in the list of numbers not perfect squares. To find an explicit expression for f, assume that $f(n)=k$. Then there are $[\sqrt{k}]$ perfect squares less than k so $k-[\sqrt{k}]$ numbers which are not perfect squares. As k is the n-th number we get $k-[\sqrt{k}]=n$ so $k-\sqrt{k}<n<k-\sqrt{k}+1$. We claim that $k=n+[\sqrt{n}+\frac{1}{2}]$. Indeed $n+[\sqrt{n}+\frac{1}{2}]$ is not a perfect square, for if $n+[\sqrt{n}+\frac{1}{2}]=m^2$ then we deduce $n<m^2$ so $[\sqrt{n+\frac{1}{2}}]\le m-1$ so $n\ge m^2-m+1$ and then $[\sqrt{n}+\frac{1}{2}]\ge m$ which implies $n+[\sqrt{n}+\frac{1}{2}]\ge m^2+1$. Next we have to prove that $[\sqrt{n+[\sqrt{n}+\frac{1}{2}]}]=[\sqrt{n}+\frac{1}{2}]$. Indeed, if $m(m-1)\le n\le m(m+1)$ then $[\sqrt{n}+\frac{1}{2}]=m$ so $n+[\sqrt{n}+\frac{1}{2}]=n+m$ hence $m^2\le n+[\sqrt{n}+\frac{1}{2}]\le m^2+2m$ so $[\sqrt{n+[\sqrt{n}+\frac{1}{2}]}]=m$ and we are finished.

Problem 10. Find all functions $f\colon N_0\to N_0$ that satisfy $f(0)=1$ and

$$f(n)=f([\frac{n}{a}])+f([\frac{n}{a^2}])$$

.

Solution. Partition N into sets $S_k=\{a^k,a^k+1,\ldots,a^{k+1}-1\}$. We see that if $n\in S_k$ then $[\frac{n}{a}]\in S_{k-1},[\frac{n}{a^2}]\in S_{k-2}$ (for $k\ge 2$). Next we see that if $k\in S_0$ then $f(k)=2$ and if $k\in S_1$ then $f(k)=3$. So we can easily prove by induction that f is constant on each S_k. If we let $g(k)$ be the value of f on S_k, then $g(k) = g(k-1) + g(k-2)$ for $k \geq 2$. It's clear now that $g(k) = F_{k+2}$ where $(F_n)_{n \in N_0}$ is the Fibonacci sequence. So $f(n) = F_{[log_a n]+2}$ for $n \geq 1$.

Problem 11. Let $f \colon N_0 \to N_0$ be a function such that $f(0) = 1$ and

$$f(n) = f([\frac{n}{2}]) + f([\frac{n}{3}])$$

whenever $n \in N$. Show that $f(n-1) < f(n)$ if and only if $n = 2^k 3^h$ for some $k, h \in N_0$.

Solution. The solution if by induction (recall f is non-decreasing by the same induction). The basis for $n \leq 6$ is easy to check. Now let's perform the induction step. For $[\frac{n}{2}]$ and $[\frac{n}{3}]$, the residue of n modulo 6 matters. So we distinguish 6 cases:

a) $n = 6k$. Then $f(n) = f(2k) + f(3k)$ while $f(n-1) = f(2k-1) + f(3k-1)$. So $f(n-1) < f(n)$ if and only if $f(2k-1) < f(2k)$ or $f(3k-1) < f(3k)$ thus $2k$ or $3k$ is of form $2^i 3^j$, which is equivalent to $n = 6k$ being of the same form.

b)$n = 6k+1$. In this case n is not of form $2^i 3^j$, and $f(n-1) = f(n) = f(2k) + f(3k)$.

c)$n = 6k+2$. Then $f(n) = f(3k+1) + f(2k)$ while $f(n-1) = f(3k) + f(2k)$ and $f(n-1) < f(n)$ if and only if $3k+1$ is of form $2^i 3^j$, which is equivalent to $n = 6k+2$ being of the same form.

d)$n = 6k+3$. $f(n) = f(3k+1) + f(2k+1)$ and $f(n-1) = f(3k+1) + f(2k)$, so $f(n-1) < f(n)$ if and only if $f(2k) < f(2k+1)$ or $2k+1 = 2^i 3^j$, which is equivalent to $6k+3 = 2^i 3^{j+1}$.

e)$n = 6k+4$. We have $f(n) - f(n-1) = (f(3k+2) + f(2k+1)) - (f(3k+1) + f(2k+1)) = f(3k+2) - f(3k+1)$ which is possible if and only if $3k+2$ is of form $2^i 3^j$, or the same condition for $n = 2(3k+2)$.

f)$n = 6k+5$. Like in case b) we have $f(n) = f(n-1)$ and n is not of the desired form, since it's neither even nor divisible by 3.

The induction is finished.

Problem 12. Find all functions $f \colon N \to [1; \infty)$ for which we have $f(2) = 4$,

$$f(mn) = f(m)f(n)$$
$$\frac{f(n)}{n} \leq \frac{f(n+1)}{n+1}$$

.

Solution. It's clear that g defined by $g(n) = \frac{f(n)}{(}n)$ is increasing and multiplicative. Therefore $g(1) = 1$. Also $g(2) = 2$ and we try to prove that g is the identity function. Indeed, assume that $l = g(k) \neq k$. We are done if we find such x, y that satisfy $(2^x - k^y)(2^x - l^y) < 0$ because then the monotonicity is broken. Now as k and l are distinct we can find a positive integer such that the ratio between the largest of k^y, l^y and the smallest of them is greater than 2. Then there exists a power of two between them, and taking 2^x to be that power we get the desired conclusion.

Problem 13. Find all functions $f :: Z \to Z$ that obey

$$f(m+n) + f(mn-1) = f(m)f(n) + 2$$

Solution. If $f = c$ is a constant function we get $2c = c^2 + 2$ so $(c-1)^2 + 1 = 0$ impossible. Now set $m = 0$ to get $f(n) + f(-1) = f(0)f(n) + 2$ so $f(n)(1 - f(0)) = 2 - f(-1)$. As f is not constant we get $f(0) = 1, f(-1) = 2$. Next set $m = -1$ to get $f(n-1) + f(-n-1) = 2f(n) + 2$. If we replace n by $-n$ the left-hand side does not change therefore right-hand side does not change too so f is even. So $f(n-1) + f(n+1) = 2f(n) + 2$. Now we can easily prove by induction on $n \geq 0$ that $f(n) = n^2 + 1$ and the evenness of f implies $f(n) = n^2 + 1$ for all n.

Problem 14. Find all functions $f :: Z \to Z$ that obey

$$f(m+n) + f(mn-1) = f(m)f(n)$$

.

Solution. If $f = c$ is constant we have $2c = c^2$ so $c = 0, 2$. If f is not constant setting $m = -1$ gives us $f(n)(1 - f(0)) = -f(-1)$ possible only for $f(-1) = 0, f(0) = 1$. Then set $m = -1$ to get $f(n-1) + f(-n-1) = 0$. Now set $m = 1$ to get $f(n+1) + f(n-1) = f(1)f(n)$. This is a quadratic recurrence in $f(n)$ with associated equation $x^2 - f(1)x + 1 = 0$. If $f(1) = 0$ we get $f(n-1) + f(n+1) = 0$ which implies $f(n+2) = -f(n)$ so $f(2k) = (-1)^{2k} f(0) = (-1)^k, f(2k+1) = (-1)^k f(1) = 0$. This function does satisfy the equation. Indeed, if m, n are both odd then $mn - m - n - 1 = (m-1)(n-1) - 2$ and so $m+n, mn-1$ are even integers which give different residues mod 4 hence $f(m+n) + f(mn-1) = 0$ while $f(m)f(n) = 0$, too. If one of m, n is odd and the other even then $m+n, mn-1$ are both odd hence $f(m+n) + f(mn-1) = f(m)f(n) = 0$. Finally if m, n are even then $f(mn-1) = 0$ and we have $f(m+n) = 1$ if $4|m-n$ and -1 otherwise, and the same for $f(m)f(n)$. If $f(1) = -1$ then we get $f(n) = (n-1) mod 3 - 1$ for all n by induction on n. It also satisfies the condition as we can check by looking at m, n modulo 3. If

$f(1)=2$ then $f(n+1)-2f(n)+f(n-1)=0$ and $f(n)=n+1$ by induction on $|n|$. It also satisfies the condition as $(m+n+1)+mn=(m+1)(n+1)$. If $f(1)=1$ then we have $f(n+1)+f(n-1)=f(n)$. So $f(-2)+f(0)=f(-1)$ so $f(-2)=-1$. Then $f(-3)+f(-1)=f(-2)$ so $f(-3)=-1$. $f(-4)+f(-2)=f(-3)$ so $f(-4)=-2$. Also $f(0)+f(2)=f(1)$ so $f(2)=0$. But then $f(2)+f(-4)\neq 0$, contradiction. If $f(1)=-2$ then $f(n+1)+f(n-1)+2f(n)=0$ and $f(-2)+2f(-1)+f(0)=0$ so $f(-2)=-1$ and then $f(-3)+2f(-2)+f(-1)=0$ so $f(-3)=3$ and we have $f(-3)+f(1)\neq 0$, again contradiction. Finally if $f(1)\neq 0,1,-1,2,-2$ then the equation $x^2-f(1)x+1=0$ has two solutions $\frac{f(1)\pm\sqrt{f^2(1)-4}}{2}$, one of which is greater than one in absolute value and one is smaller. If we solve the recurrence we find that $f(n)=cr^n+ds^n$ where $c,d\neq 0$ and without loss of generality $|r|>1,|s|<1$. In this case we have $f(n)\sim cr^n$ for $n\to\infty$. Then $f(m+n)+f(mn-1)=f(m)f(n)$ cannot hold because the left-hand side is asymptotically equivalent to cr^{mn-1} for $m=n\to\infty$ while the right-hand side is asymptotically equivalent to c^2r^{m+n} and $mn-1$ is much bigger than $m+n$.

Problem 15. Find all functions $f\colon Z\to Z$ that verify

$$f(f(k+1)+3)=k$$

.

Solution. Let us start by noting that f is injective, as if $f(m)=f(n)$ then plugging $k=m-1,n-1$ we get $m=n$. Therefore if we set $k=f(n)$ we get $f(f(f(n)+1)+3)=f(n)$ and the injectivity implies $f(f(n)+1)+3=n$ or $f(f(n)+1)=n-3$. By plugging $k=n-3$ in the condition we get $f(f(n-2)+3)=n-3$ and the injectivity gives us $f(n-2)+3=f(n)+1$ so $f(n)=f(n-2)+2$. From here we deduce that if $f(0)=a,f(1)=b$ then $f(2n)=2n+a,f(2n+1)=2n+b$. Also from the given condition f is surjective so a and b have distinct parities and we encounter two cases:

a) a is even and b is odd. Plugging $k=2n$ we get $f(f(2n+1)+3)=2n$ or $f(2n+b+3)=2n$ so $2n+b+3+a=2n$ hence $a+b+3=0$. Plugging $k=2n-1$ we get $f(f(2n)+3)=2n-1$ or $f(2n+a+3)=2n-1$ so $2n+a+2+b=2n-1$ and again $a+b+3=0$. Conversely, if $a+b+3=0$ the f defined by $f(2n)=2n+a,f(2n+1)=2n+b$ satisfies the condition.

b) a is odd and b is even. Then plugging $k=2n$ we deduce $f(2n+b+3)=2n$ so $2n+2b+2=2n$ hence $b=-1$ which contradicts the evenness of b.

To conclude, all solutions are given by $f(2n) = 2n + a, f(2n+1) = 2n + b$ where a us even, b is odd and $a + b + 3 = 0$.

Problem 16. Find all functions $f\colon N \to Z$ verifying $f(mn) = f(m) + f(n) - f(gcd(m,n))$

Solution. Again we smell some dependency between f and the prime decomposition of n. Let's set $m = p^k, n = p^l, k \leq l$ where p is some prime. Then $f(p^{k+l}) = f(p^k) + f(p^l) - f(p^l) = f(p^k)$, so whenever $t \leq 2k$ we deduce $f(p^t) = f(p^k)$, and from this relation we immediately get $f(p^k) = f(p)$ for any $k > 0$. Next consider two coprime numbers m, n. Then $f(mn) = f(m) + f(n) - f(1)$ and we easily deduce the more general version $f(m_1 m_2 \dots m_k) = f(m_1) + f(m_2) + \dots + f(m_k) - (k-1)f(1)$ when $m_1, m_2, \dots, m_k$ are pairwise coprime. Particularly if $n = \prod_{i=1}^m p_i^{k_i}$ (p_i are distinct primes) then $f(n) = \sum_{i=1}^m f(p_i^{k_i}) - (m-1)f(1) = \sum_{i=1}^m f(p_i) - (m-1)f(1)$. A more comfortable setting is obtained if we work with $f(x) - f(1)$. Indeed, if we denote $g(x) = f(x) - f(1)$ then we get $f(x) = \sum_{p|x} g(p) + f(1)$ where the sum is taken over all prime divisors of n. It's straightforward to check that this function satisfies the condition: If $m = \prod_{i=1}^k p_i^{k_i} \prod_{i=1}^l q_i^{m_i}$, $n = \prod_{i=1}^j r_i^{j_i} \prod_{i=1}^l q_i^{n_i}$ where p_i, q_i, r_i are distinct primes and $k_i, m_i, j_i, n_i > 0$ then $gcd(m,n) = \prod_{i=1}^l q_i^{min\{m_i, n_i\}}$ and we have $f(mn) = \sum_{i=1}^k f(p_i) + \sum_{i=1}^l f(q_i) + \sum_{i=1}^j f(r_i) + f(1) = \sum_{i=1}^k f(p_i) + \sum_{i=1}^l + f(q_i) f(1) + \sum_{i=1}^j f(r_i) + \sum_{i=1}^l f(q_i) + f(1) - (\sum_{i=1}^l f(q_i) + f(1)) = f(m) + f(n) - f(\gcd(m,n))$.

Problem 17. Find all surjective functions $f\colon N \to N$ such that $m|n$ if and only if $f(m)|f(n)$ for any $m, n \in N$.

Solution. f is actually a bijection, as if $f(k) = f(l)$ for $k < l$ then $f(l)|f(k)$ so $l|k$ impossible. Since 1 divides every number, $f(1) = 1$. Now we also see that $f(n)$ has as many divisors as n because f provides a bijection between the set of divisors of n and the set of divisors of $f(n)$. Next, let's prove that f is multiplicative. If $(m,n) = 1$ then $(f(m), f(n)) = 1$ because if $f(e) = d|(f(m), f(n))$ then $e|m, e|n$ so $e = 1$. Hence if $(m,n) = 1$ then $f(m)|f(mn), f(n)|f(mn)$ so $f(m)f(n)|f(mn)$. As $f(m)$ has as many divisors as m, $f(n)$ as many divisors as n and $f(m), f(n)$ are coprime, $f(m)f(n)$ has as many divisors as mn, hence $f(mn) = f(m)f(n)$ thus f is multiplicative. Now if p is prime then $f(p)$ must also be a prime, and the converse is also true, so f is a bijection on the set of all prime numbers. We prove that if n is a prime power then $f(p)$ is a prime power of the same exponent. Indeed, we've just proven the basis. Now if we have proven that $f(p^k) = q^k$, then $q^k|f(p^{k+1})$ so $f(p^{k+1}) = q^{k+r}M$ where M is a number coprime to q. As $f(p^{k+1})$ has $k+2$ divisors, we must have $(k+r+1)t = k+2$ where t is the number of divisors of M. If $t \geq 2$ then this is impossible. So $t = 1, M = 1$ and $k+r+1 = k+2$ hence $f(p^{k+1}) = q^{k+1}$ as desired. Now using the multiplicativity of f we deduce $f(\prod p_i^{k_i}) = \prod q_i^{k_i}$ where $q_i = f(p_i)$. Any f defined by this relation and by a bijection on the set of primes clearly satisfies the relation, and so this is the form of all solutions.

Problem 18. (ISL 2001)Find all functions $f\colon N_0^3 \to R$ that satisfy

$$f(p, q, r) = 0$$

if $pqr = 0$ and

$$f(p, q, r) = 1 + \frac{1}{6}(f(p+1, q-1, r) + f(p-1, q+1, r) + f(p-1, q, r+1) +$$
$$+ f(p+1, q-1, r) + f(p, q+1, r-1) + f(p, q-1, r+1))$$

otherwise.

Solution. It's clear that the second most important condition computes $f(p, q, r)$ in terms of $f(p+1, q-1, r)$ etc. and the sum of coordinates remains the same: if $p+q+r = s$ then $(p+1)+(q-1)+r = s$ etc. This implies that it suffices to compute f on each of the planes $p+q+r = s$, as the conditions take place within this planes. Also the fact that $f(p, q, r) = 0$ when $pqr = 0$ may suggest that $f(p, q, r) = kpqr$. So let's try to set $f(p, q, r) = kpqr$. Then $\frac{1}{6}(f(p+1, q-1, r) + f(p-1, q+1, r) + f(p-1, q, r+1) + f(p+1, q-1, r) + f(p, q+1, r-1) + f(p, q-1, r+1)) - f(p, q, r) = \frac{k}{6}((p+1)(q-1)r + (p-1)(q+1)r + (p+1)q(r-1) + (p-1)q(r+1) + p(q-1)(r+1) + p(q+1)(r-1) - 6pqr) = \frac{k}{6}(6pqr + 2p + 2q + 2r - 6pqr) = \frac{k}{3}(p+q+r)$ so $k = \frac{3}{p+q+r}$. So we have found a solution $f(p, q, r) = \frac{3pqr}{p+q+r}$ and we try to prove now it's unique. As it satisfies the condition, it suffices to show that the values of f may be deduced inductively on each (x, y, z). Indeed, we can perform an induction on $s^2 - (x+y+z)^2$. When it's zero, the condition follows from the condition as two of the numbers must be zero. Now assume we have proven the claim when the minimal number is $s^2 - (x+y+z)^2 \leq k$ and let's prove it when the minimal number is $s^2 - (x+y+z)^2 = k+1$. Without loss of generality $x \leq y \leq z$. If $x = 0$ we are done otherwise set $p = x-1, q = y, r = z+1$. We see $(p^2+q^2+r^2) - (x^2+y^2+z^2) = 2(z-x+1) > 0$ and so the induction assumption applies to (p, q, r). Moreover it also applies for $(p-1, q+1, r), (p-1, q, r+1), (p, q-1, r+1), (p, q+1, r-1), (p+1, q+1, r)$ as it's easy to check, because the sum of squares of coordinates of each of these numbers equals $p^2+q^2+r^2$ plus one of $2(p-q+1), 2(q-p+1), 2(q-r+1), 2(r-q+1), 2(p-r+1)$, whereas the sum of squares of coordinates of x, y, z is $p^2+q^2+r^2+2(r-p+1)$ and is clearly less than all the others, because $p<q<r$. So the induction assumption allows us to say f is computed on all points $(p,q,r), (p-1,q,r+1), (p,q+1,r-1), (p,q-1,r+1), (p-1,q,r+1), (p+1,q-1,r)$ and writing the condition for (p,q,r) allows us to compute the value of $f(p+1,q,r-1)=f(x,y,z)$. Therefore f is unique and it equals $\frac{3pqr}{p+q+r}$ when $(p,q,r)\neq(0,0,0)$ and 0 when $p=q=r=0$.

Problem 19. Find all functions $f\colon Z\to Z$ that satisfy the relation

$$f(m+n)+f(m)f(n)=f(mn+1)$$

whenever m, n are integers.

Solution. By setting $m=n=11$ we get $f(2)+f^2(1)=f(2)$ so $f(1)=0$. Now let $m=0$ to get $f(n)+f(0)f(n)=0$. Therefore either $f(0)=-1$ or f is identically zero. Excluding this trivial case we get $f(0)=-1$. Now take $m=-1$ to get $f(n-1)+f(-1)f(n)=f(1-n)$ so $f(n-1)-f(1-n)=-f(-1)f(n)$ and we have two cases:

a) $f(-1)=0$. In this case we have $f(-x)=f(x)$. Firstly let's try to settle the unicity of f. Set $f(2)=f(-2)=a$. Then by setting $m=2, n=-2$ we compute $f(3)=f(-3)=a^2-1$. Set $m=2, n=-3$ to get $f(5)=f(2)f(3)=a(a^2-1)$. Set $m=n=2$ to compute $f(4)=a^3-a^2-a$. Set $m=3, n=-3$ to get $f(8)=a^4-2a^2$. Next by setting $m=4, n=-4$ and $m=6, n=-2$ we get $f(15)=f^2(4)-1=f(6)+f(8)f(2)$ which allows us to deduce that $f(6)=a^3+a^2-1$. Next by setting $m=2, n=3$ we get $f(7)=f(5)+f(2)f(3)=2a(a^2-1)$. Also by setting $m=4, n=-2$ we get $f(7)=f(2)+f(2)f(4)=a^4-a^3-a^2+a$. Thus we get $2a^2(a^2-1)=a^4-a^3-a^2-a=0$ or $a(a-1)(a+1)(a-3)=0$ so we have four values for a. Next we feel that all values of f can be determined from the value of a only. Indeed, let's prove by induction on $|n|$ that $f(n)$ is uniquely determined by $f(2)=a$ for each n. We see this holds true for any n with $|n|\leq 8$. Now assume this holds whenever $|n|\leq k-1$ and let's prove it for $n=k\geq 9$ ($n=-k$ is the same). We seek numbers $0<x<u<v<y$ with $x+y=k, xy=uv$. Then applying the condition for $m=x, n=y$ and $m=u, n=v$ we get $f(x+y)+f(x)f(y)=f(xy+1)=f(uv+1)=f(u+v)+f(u)f(v)$ so $f(k)=f(x+y)=f(u+v)+f(u)f(v)-f(x)f(y)$ and we can easily prove that $u, v, x, y, u+v<k$ so $f(k)$ is determined and the induction is finished $u+v<k=x+y$ because $u+v-x-y=(u-x)+(v-y)=u-x+(\frac{uv}{u}-\frac{uv}{x})=(u-x)-\frac{uv(u-x)}{ux}=\frac{(u-x)(x-v)}{vx}<0$. Now if $k=(2a+1)2^b$

for $a > 0$ then we can set $x = 2^b, y = 2^{b+1}a, u = 2^{b+1}, v = 2^b a$. If k is a power of 2 then either $k = 3p+1$ and $x = 1, y = 3p, u = 3, v = p$ or $k = 3p+2$ with $x = 2, y = 3p, u = 3, v = 2p$.

So at most one f stems from each value of a. Surprisingly, each of them gives one value of f, so the general set of solutions is quite unexpected. Let's analyze them one by one:

i) $a = 0$. By substituting we get $f(0) = f(3) = f(6) = -1, f(1) = f(2) = f(4) = f(5) = f(7) = f(8) = 0$. We guess the solution $f(x) = -1$ when x is divisible by 3 and $f(x) = 0$ otherwise. Indeed it satisfies the conditions: If both m, n are divisible by 3 we get the identity $-1 + 1 = 0$ as $3|m + n$ but $mn + 1$ is not divisible by 3; if one of m, n divisible by 3 and the other is not we get $0 + 0 = 0$ as neither $mn + 1$ nor $m + n$ are divisible; finally if both neither of m, n is divisible by 3 the either they give the same residue mod 3, in which $m + n$ and $mn + 1$ are not divisible by 3 and we get $0 + 0 = 0$ or they give different residues mod 3 which means $m + n, mn + 1$ are divisible by 3 so $1 + 0 = 1$.

ii) $a = 1$. By substituting we get $f(2) = f(6) = 1, f(4) = f(8) = -1, f(1) = f(3) = f(5) = f(7) = -1$ so we suppose $f(4k + 2) = 1, f(4k) = -1, f(2k + 1) = 0$. Indeed, if m, n are both odd then $m + n$ and $mn + 1$ give the same residue modulo 4 as $mn + 1 - m - n = (m - 1)(n - 1)$ is a product of two even integers, so the equality holds. If one of m, n is odd and the other is even then $mn + 1, m + n$ are both odd and again the equality holds. If both m, n are even then either they give the same residue mod 4 in which we have the identity $-1 + 1 = 0$ or they give different residues which implies $1 + (-1) = 0$, again true.

iii) $a = -1$. By substituting we get $f(2) = f(4) = f(6) = f(8) = -1, f(1) = f(3) = f(5) = f(7) = 0$ so we guess $f(2k) = -1, f(2k + 1) = 0$. Indeed, if at least one of m, n is odd then $(mn+1)-(m+n) = (m - 1)(n - 1)$ is even so the identity holds true, while when they are both even we get the true $-1 + 1 = 0$.

iv) $a = 3$. We compute $f(3) = 8, f(4) = 15$ and so on, guessing $f(x) = x^2 - 1$. Indeed, $(m + n)^2 - 1 + (m^2 - 1)(n^2 - 1) = m^2 + n^2 + 2mn - 1 + m^2n^2 - m^2 - n^2 + 1 = m^2n^2 + 2mn = (mn + 1)^2$.

This case is exhausted.

b) $f(-1) \neq 0$. Then we have $f(n) = -\frac{f(n-1)-f(-(n-1))}{f(-1)}$. Therefore we get $f(-n) = \frac{f(n+1)-f(-n-1)}{f(-1)}$. If we set $a_n = f(n) - f(-n)$ we observe by subtracting the previous two results the nice recursive identity $a_n = a(a_{n-1} + a_{n+1})$ where $a = -\frac{1}{f(-1)}$. We can write it as $a_{n+1} = ba_n - a_{n-1}$ where $b = \frac{1}{a} = -f(-1)$. Also $a_0 = 0, a_1 = b$. Then $a_2 = b^2, a_3 =$

$b^3-b, a_4 = b^4-2b^2, a_5 = b^5-3b^3+b$ and so on. Next $f(n) = \frac{a_{n-1}}{b}$. Hence $f(2) = 1, f(3) = b, f(4) = b^2 - 1, f(5) = b^3 - 2b, f(6) = b^4 - 3b^2 + 1$. Setting $m = n = 2$ we get $f(4) + f^2(2) = f(5)$ or $b^2 - 1 + 1 = b^3 - 2b$ so $b^2 = b^3 - 2b$ and as $b \neq 0$ we get $b^2 = b + 2$ so $b = 2$ or $b = -1$.

i) $b = 2$. In this case we can guess the identity $f(x) = x - 1$. It holds by induction as we prove that $a_n = 2n$ by induction (another proofs can be obtained by using the same unicity idea that we deduced in case a)). Indeed $f(x) = x - 1$ satisfies the condition as $(m + n - 1) + (m - 1)(n - 1) = mn = (mn + 1) - 1$.

ii) $b = -1$. We compute that $f(-1) = 1, f(0) = -1, f(1) = 0, f(2) = 1, f(3) = -1, f(4) = 0, f(5) = 1, f(6) = -1$. We conjecture that $f(3k) = -1, f(3k + 1) = 0, f(3k + 2) = 1$. Indeed, this can be either proven by showing inductively that $b_{3k} = 1, b_{3k+1} = 0, b_{3k+2} = 1$, or by using again the unicity idea after we verify that f satisfies our equation:

$3|m, 3|n$ then we get $-1 + 1 = 0$. $3|m, 3|n - 1$ or $3|n, 3|m - 1$ then we get $0 + 0 = 0$. $3|m, 3|n - 2$ or $3|n, 3|m - 2$ then we get $1 + (-1) = 0$. $3|m - 1, 3|n - 1$ then $1 + 0 = 1$. $3|m - 1, 3|n - 2$ or $3|m - 2, 3|n - 1$ then we get $0 - 1 = -1$. $3|m - 2, 3|n - 2$ then $0 + 1 = 1$. All the identities are true so this function is indeed a solution.

Concluding, we have plenty of different solutions: $f(x) = x^2 - 1, f(x) = x-1, f(x) = (x+1) \mathit{mod}\, 2, f(x) = x \mathit{mod}\, 3-1, f(x) = ((x+1) \mathit{mod}\ 2)(x \mathit{mod}\ 4 + 1), f(x) = (x \mathit{mod}\ 3)^2 - 1$.

Problem 20. Find all functions $f\colon R\backslash\{0; 1\} \to R$ satisfying

$$f(x) + f(\frac{1}{1-x}) = \frac{2(1-2x)}{x(1-x)}$$

for all x in the domain of f.

Solution. The condition links $f(x)$ to $f(\frac{1}{1-x})$ and only. Set $g(x) = \frac{1}{1-x}, h(x) = g^{-1}(x) = 1 - \frac{1}{x}$. Using the condition we can establish a dependance only between $f(x), f(g(x)), f(h(x)), f(g(g(x))), f(h(h(x))), \ldots$. So we have to look at properties of g or h. We see that $g(g(x)) = \frac{x-1}{x} = 1 - \frac{1}{x} = h(x)$ so $g(g(g(x))) = x$ and we know $f(x) + f(g(x)), f(g(x)) + f(g(g(x))), f(g(g(x))) + f(x)$. From here we can find $f(x)$ by solving the linear system. We can do this manually by substituting into the condition, and we get $f(x) = \frac{x+1}{x-1}$. It satisfies the conditions because we worked by equivalency.

Problem 21. Find all functions $f\colon R \to R$ that satisfy

$$f(-x) = -f(x)$$

for all real x;

$$f(x+1) = f(x) + 1$$

for all real x and

$$f(\frac{1}{x}) = \frac{f(x)}{x^2}$$

for all non-zero x.

Solution. All the conditions are in one variable: x. In this case, some graph theory helps us understand the path to the solution. Consider the reals as vertices of a graph, and connect x with $x+1, -x, \frac{1}{x}$. The conditions link two values of the function in two vertices joined by an edge. So if we pick up a x_0, we can deduce from $f(x_0)$ the values of f on C where C is the set of numbers connected to x_0 by some chain of edges. Now we can get a contradiction if and only if there is a cycle somewhere. So finding a cycle would impose a condition on f(x_0) and maybe would exactly find the value of $f(x_0)$. So let's try to construct such a cycle for any x. After some tries we see $x \to x+1 \to \frac{1}{x+1} \to -\frac{1}{x+1} \to 1 - \frac{1}{x+1} = \frac{x}{x+1} \to \frac{x+1}{x} = 1 + \frac{1}{x} \to \frac{1}{x} \to x$. Set $f(x) = y$. Then $f(x+1) = y+1, f(\frac{1}{x+1}) = \frac{y+1}{(x+1)^2}, f(-\frac{1}{x+1}) = -\frac{y+1}{(x+1)^2}, f(\frac{x}{x+1}) = \frac{x^2+2x-y}{(x+1)^2}, f(\frac{x+1}{x}) = \frac{x^2+2x-y}{x^2}, f(\frac{1}{x}) = \frac{2x-y}{x^2}, f(x) = 2x - y$. So $y = 2x - y$ thus $y = x$. Note that we need to have $x \neq 0, -1$ in order not to divide by zero. This is no problem for us, as $f(0) + 1 = f(1)$ and we know that $f(1) = 1$ so $f(0) = 0$, also $f(-1) = -f(1) = 1$ hence $f(x) = x$ for all x, and it satisfies the condition.

Problem 22. Let f be an increasing function on R such that $f(x+1) = f(x) + 1$. Show that the limit $\lim_{n\to\infty} \frac{f_n(x)}{n}$ exists and is independent of x, where f_n is f iterated n times.

Solution. The clear properties of the function are that if $x < y$ there is a $k = [y - x]$ such that $x + k \leq y < x + k + 1$ hence $f(x) + k \leq f(y) < f(x) + k + 1$ so $k \leq f(y) - f(x) \leq k + 1$ hence $y - x - 1 \leq f(y) - f(x) \leq y - x + 1$ .It's easier to set the independence. Indeed if for a fixed x_0 we have $lim_{n\to\infty} \frac{f_n(x_0)}{n} = a$ then if $[x - x_0] = k$ we have $[f(x) - f(x_0)] = k$ from the above result and by induction $[f_n(x) - f_n(x_0)] = k$ thus $|f_n(x) - f_n(x_0)| < |k| + 1$ is bounded, hence $\lim_{n\to\infty} \frac{f_n(x) - f_n(x_0)}{n} = 0$ so $\lim_{n\to\infty} \frac{f_n(x)}{n} = \lim_{n\to\infty} \frac{f_n(x_0)}{n} = a$.

Now let's prove that the limit exists for some fixed x, say $x = 0$. Let $a_i = f_i(0)$,. Then we have proven above that $[f_k(x) - f_k(y)] = [x - y]$, particularly $[f_k(a_i) - f_k(0)] = [a_i - 0]$ hence $[a_{i+k} - a_k] = [a_i]$ so $a_k + a_i - 1 \leq a_{i+k} \leq a_k + a_i + 1$. Now we prove that $\frac{a_n}{n}$ converges. Indeed, if $n = km + r$ then we can deduce $ma_k + a_r - m - 1 \leq a_n \leq$

$ma_k + a_r + m + 1$ so $\frac{n-r}{k}a_k + a_r - \frac{n}{k} - 1 \leq a_n \leq \frac{n-r}{k}a_k + a_r + \frac{n}{k} + 1$ hence $\frac{n-r}{n}\frac{a_k}{k} + \frac{a_r}{n} - \frac{1}{k} - \frac{1}{k^2} \leq \frac{a_n}{n} \leq \frac{n-r}{n}\frac{a_k}{k} + \frac{a_r}{n} + \frac{1}{k} + \frac{1}{k^2}$. Now choosing n sufficiently big such that $|a_r| < \frac{n}{k}, \frac{r}{n}\frac{|a_k|}{k} < \frac{1}{k}$ for all $r = 0, 1, 2, \ldots, k-1$ we ensure that $|\frac{a_n}{n} - \frac{a_k}{k}| < \frac{4}{k}$ hence all $\frac{a_n}{n}$ starting from some position on belong to a closed interval I_k of length less than $\frac{8}{k}$. The intervals $J_k = I_1 \bigcap I_2 \bigcap \ldots \bigcap I_k$ have non-empty intersection for all k because infinitely many $\frac{a_n}{n}$ belong to them, however their length tends to 0 and $J_k \subset J_{k-1}$ hence their intersection is a single point, which then should be the limit of our sequence.

Problem 23. Find all functions $f\colon Q^+ \to Q^+$ that satisfy

$$f(x) + f(\frac{1}{x}) = 1$$

and $f(1+2x) = \frac{f(x)}{2}$ for all x in the domain of f.

Solution. Let's firstly try to guess the function. It's natural to suppose $f(x) = \frac{ax+b}{cx+d}$. The condition $f(x) + f(\frac{1}{x}) = 1$ now translates as $\frac{(ax+b)}{(cx+d)} + \frac{bx+a}{dx+c} = 1$ and we conclude that $c = d$. We can assume $c = d = 1$ otherwise divide a, b by c to see $\frac{ax+b}{cx+c} = \frac{\frac{a}{c}x + \frac{b}{c}x}{x+1}$. Next we need to have $\frac{ax+b}{x+1} + \frac{bx+a}{x+1} = 1$ which is possible for $a + b = 1$. Now the condition $f(1+2x) = \frac{f(x)}{2}$ means $\frac{a(1+2x)+b}{2x+2} = \frac{1}{2}\frac{ax+b}{x+1}$ or $a(1+2x) + b = ax + b$ $a = 0$ hence $f(x) = \frac{1}{x+1}$ which satisfies the conditions. Indeed, if we set $x = 1$ we get $2f(1) = 1$ so $f(x) = \frac{1}{2}$. The hint here is that all the values of f are positive. So we try to prove that if $f(x) \neq \frac{1}{1+x}$ then some of values of f should be negative. Indeed, set $g(x) = f(x) - \frac{1}{x+1}$. Then $g(\frac{1}{x}) = -g(x), g(1+2x) = \frac{g(x)}{2}$. As $g(x) + 1 > g(x) + \frac{1}{x+1} = f(x) > 0$ we see that $g(x) > -1$ and as $g(x) = -g(\frac{1}{x})$ we deduce $g(x) < -1$ so $|g(x)| < 1$. Now the second condition can be rewritten as $g(\frac{x-1}{2}) = 2g(x)$ for $x > 1$. Now if $g(x) = a \neq 0$ we find by induction the numbers x_n such that $|g(x_n)| = 2^n a$. We set $x_0 = x$. Now assume we found x_k. Then $x_k \neq 1$ as $g(1) = 0$. If $x_k > 1$ then $g(\frac{x_k-1}{2}) = 2g(x_k)$ and we set $x_{k+1} = \frac{x_k-1}{2}$. If $x_k < 1$ then $\frac{1}{x_k} = 1, g(\frac{1}{x_k}) = -g(x_k)$ therefore $g(\frac{\frac{1}{x_k}-1}{2}) = -2g(x_k)$ and we set $x_{k+1} = \frac{\frac{1}{x_k}-1}{2}$. Then if we find k such that $2^k|a| > 1$ we obtain a contradiction. Thus $g(x) = 0$ for all x and $f(x) = \frac{1}{x+1}$.

Problem 24.(China)Find all functions $f\colon [1,\infty) \to [1,\infty)$ given that

$$f(x) \leq 2(1+x)$$

and

$$xf(x+1) = f^2(x) - 1$$

for all x in the range of f.

Solution. We can guess the solution $f(x) = x+1$ and now we try to prove this is the only. As in many other situations we assume that $f(x_0) \neq x_0 + 1$ and try to obtain a x such that $f(x) < 1$ or $f(x) > 2(1+x)$. Indeed, we observe that $xf(x+1) = f^2(x) - 1$ can be interpreted as a recurrence on $a_n = f(n+x_0)$ by $a_{n+1} = \frac{a_n^2 - 1}{n+x_0}$. Consider now $b_n = \frac{a_n}{n+1+x_0}$. Then $b_{n+1} = \frac{(n+1+x_0)^2 b_n^2 - 1}{(n+x_0)(n+2+x_0)} = b_n^2 + \frac{b_n^2 - 1}{(n+2)(n+2+x_0)}$. If now $b_0 > 1$ then we prove by induction that $b_n > 1$ and then $b_{n+1} > b_n^2$ which implies that $b_n > 2$ for some n hence $f(n+x_0) > 2(1+n+x_0)$, contradiction. If $b_0 < 1$ then we prove by induction that $b_n < 1$ and therefore $b_{n+1} < b_n^2$ so $b_n < b_0^{2^n}$ so $\frac{1}{b_n} > (\frac{1}{b_0})^{2^n}$. However $\frac{1}{b_n} = \frac{n+1+x_0}{f(n+x_0)} < n+1+x+0$ and as $b_0 < 1$, $(\frac{1}{b_0})^{2^n} > n+1+x_0$ contradiction. So $b_0 = 1$ hence $f(x_0) = x_0 + 1$. As x_0 was picked up at random, $f(x) = x+1$.

Another solution is as follows:

We have

$$f(x) = \sqrt{xf(x+1)+1} \leq \sqrt{2x(x+2)+1} < \sqrt{2}(x+1)$$

and it follows by induction that

$$f(x) < 2^{\frac{1}{2^n}}(x+1)$$

for all $n \in N$. Hence $f(x) \geq x+1$ which shows that $f(x) = x+1$. It is easy to check that this function satisfies the conditions of the problem.

Problem 25. Find all functions $f :: N_0 \to R$ that satisfy $f(4) = f(2) + 2f(1)$

$$f(\binom{n}{2} - \binom{m}{2}) = f(\binom{n}{2}) - f(\binom{m}{2})$$

for $n > m$

Solution. $f(x) = cx$ clearly satisfy the condition. Set $f(1) = a$ and $f(2) = b$. It's natural to try to prove that $b = 2a$. To do this, we have to compute some values of f. If $S = \{\binom{n}{2} | n \in N\} = \{0, 1, 3, 6, 10, 15, 21, 28, \ldots, \}$ then $f(x-y) = f(x) - f(y)$ for $x, y \in S$. Now we can find $f(3) = f(2) + f(1) = a + b$, $f(4) = 2a + b, f(6) = 2f(3) = 3(a+b), f(5) = f(6) - f(1) = a + 2b$, $f(10) = f(6) + f(4) = 4a + 3b, f(7) = f(10) - f(3) = 3a + 2b$. If we continue like this, we will not find a contradiction, a fact that leads us to the conclusion that there are more functions that satisfy the condition. If we look attentively at the list of computed value, we see that $f(3k) = k(a+b), f(3k+1) = (k+1)a + kb, f(3k+2) = ka + (k+1)b$ so we are led to looking at the residues of n modulo 3, and find another example $g(x) = x \ mod\ 3$ where we set $2\ mod\ 3 = -1$. It satisfies the conclusion because S does not contain numbers congruent to 2 modulo 3. So $f(x) = cx + dg(x)$ also satisfies the conclusion. Now we can try to prove that this are the only solutions. Let $c = \frac{a+b}{3}, d = \frac{2(a-b)}{3}$, and let $h(x) = f(x) - cx - dg(x)$. It satisfies the conclusion and also $h(1) = h(2) = h(3) = \ldots = h(7) = 0$. We need to prove that $h(x) = 0$ for all x. We do this by induction on x. Suppose that we have $f(x) = 0$ for all $x = 1, 2, \ldots, n$ and let's prove that $f(n+1) = 0$. Firstly, as $\binom{m+1}{2} - \binom{m}{2} = m$ we deduce that $f(\binom{k}{2}) = 0$ for $k \leq n+1$, and we need to prove that $f(\binom{n+2}{2}) = 0$. It could be done if we would find $k, x, y \leq n+1$ such that $\binom{n+2}{2} - \binom{n+2-k}{2} = \binom{x}{2} - \binom{y}{2}$ then we are done. This relation is equivalent to $k(2n+3-k) = (x-y)(x+y-1)$. Now let's find a $k \leq 3$ such that $3|2n+3-k$. Then $k(2n+3-k) = 3k(\frac{2n+3-k}{3})$ and $3k, \frac{2n+3-k}{3}$ have opposite parities, hence we can set $2x = \frac{2n+3-k}{3} + 3k, 2y = \frac{2n+3-k}{-}3k$. We only need to verify that $x, y \leq n+1$ (we could also have $y < 0$ but then $\binom{y}{2} = \binom{-y+1}{2}$ so there is no problem here) which is equivalent to $2n+4 > \frac{2n+3-k}{3} + 3k$ or $6n+12 > 2n+8k+3$ or $4n+9 > 8k$. As $k \leq 3$ this is satisfied when $n \geq 4$. Since we have already $f(1), f(2), f(3), f(4)$ zero, we are done.

Problem 26. Find all continuous functions $f :: R \to R$ that satisfy

$$f(1+x^2) = f(x)$$

Solution. Set $g(x) = 1 + x^2$. As g is even we see that $f(x) = f(g(x)) = f(-x)$ so f is even thus we need to find f only on $[0;\infty)$. Now we know that $f(g_k(x)) = f(x)$. Also $g(x)$ is increasing and $g(x) > x$ as $1 + x^2 > x$ for $x \geq 0$. Set $x = 0$ to get $f(0) = f(1)$. Nest set $x_k = g_k(0)$ so that $x_0 = 0, x_1 = 1$. Then g maps $[x_{k-1}; x_k]$ into $[x_k; x_{k+1}]$ so g_k maps $[0;1]$ into $[x_k; x_{k+1}]$. As $g(x) > 0$ is increasing and $g(x) > 1$ we cannot establish any condition between $f(x)$ and $f(y)$ for $0 < x < y < 1$ because we cannot link x and y by operating with g: if $g_k(x) = g_l(y)$ then as $g_k(x) \in (x_k; x_{k+1}); g_l(y) \in (x_l; x_{l+1})$ we conclude $k = l$ and by injectivity $x = y$. Thus we may construct f as follows: define f a continuous function on $[0;1]$ with $f(0) = f(1)$ and extend f to R^+ by setting $f(g_k(x)) = f(x)$ and $f(-x) = -f(x)$. Indeed f satisfies $f(1+x^2) = f(x)$. Moreover it's continuous: the graphs of f on $[x_k; x_{k+1}]$ are continuous as they are the composition of the continuous functions f on $[0;1]$ and g_k^{-1} on $[x_k;x_{k+1}]$. As $f(x_k) = f(x_{k+1})$ the continuous graphs of f on intervals $[x_k;x_{k+1}]$ unite to form a continuous curve, and reflecting it with respect to the y axis we get the continuous graph of f.

Exercises

Problem 27. Find all functions $f\colon N \to R$ which satisfy $f(1) \neq 0$ and

$$f^2(1) + f^2(2) + \ldots + f^2(n) = f(n)f(n+1)$$

Problem 28. Find all functions $f\colon N \to R$ for which $f(1) = 1$ and

$$\sum_{d|n} f(d) = 0$$

whenever $n \geq 2$.

Problem 29. Find all functions $f\colon N \to N$ that satisfy $f(0) = 0$ and

$$f(n) = 1 + f([\frac{n}{k}])$$

for all $n \in N$.

Problem 30. Let $k \in Z$. Find all functions $f\colon Z \to Z$ that satisfy

$$f(m+n) + f(mn-1) = f(m)f(n) + k$$

Problem 31. Find all functions $f\colon Z \to Z$ that satisfy

$$f(m+n) + f(mn) = f(m)f(n) + 1$$

Problem 32. Find all functions $f\colon Z \to R$ satisfying

$$f(a^3 + b^3 + c^3) = f(a^3) + f(b^3) + f(c^3)$$

whenever $a, b, c \in Z$.

Problem 33. Let f be a strictly increasing function on N with the property that $f(f(n)) = 3n$. Find $f(2007)$.

Problem 34. Find all functions $f\colon N \to N$ satisfying

$$f(m + f(n)) = n + f(m+k)$$

for $m, n \in N$ where $k \in N$ is fixed.

Problem 35. Let $f, g \colon N_0 \to N_0$ that satisfy the following three conditions:
i) $f(1) > 0, g(1) > 0$;
ii) $f(g(n)) = g(f(n))$
iii) $f(m^2 + g(n)) = f^2(m) + g(n)$;
iv) $g(m^2 + f(n)) = g^2(m) + f(n)$.
Prove that $f(n) = g(n) = n$.

Problem 36. Find all functions $f :: Q^+ \to Q^+$ that satisfy $f(x) + f(\frac{1}{x}) = 1$ and $f(f(x)) = \frac{f(x+1)}{f(x)}$.

Binary (and other) bases

Problem 37. Find all functions $f \colon N_0 \to N_0$ such that $f(0) = 0$ and

$$f(2n+1) = f(2n) + 1 = f(n) + 1$$

for any $n \in N_0$.

Solution. The statement suggests that we look at the binary expansion of f. As $f(2n+1) = f(n)+1$ and $f(2n) = f(n)$ it's straightforward to observe and check that $f(n)$ is the number of ones (or the sum of digits) of the binary representation of n.

Problem 38. (China)Find all functions $f \colon N \to N$ for which $f(1) = 1, f(2n) < 6f(n)$ and

$$3f(n)f(2n+1) = f(2n)(3f(n)+1)$$

.

Solution. Rewrite the main condition as $\frac{f(2n+1)}{f(2n)} = \frac{3f(n)+1}{3f(n)}$, or $\frac{f(2n+1)-f(2n)}{f(2n)} = \frac{1}{3f(n)}$. Thus $f(2n+1)-f(2n) > 0$ and $3f(n)(f(2n+1)-f(2n)) = f(2n)$. As $f(2n) < 6f(n)$ we deduce $f(2n+1) - f(2n) < 2$ thus the only possibility is $f(2n+1) - f(2n) = 1$ and $f(2n) = 3f(n)$. We have already encountered this problem, whose solution is: $f(n)$ is the number obtained by writing n in base 2 and reading the result in base 3.

Problem 39. (ISL 2000) The function f on the non-negative integers takes non-negative integer values and satisfies $f(4n) = f(2n) + f(n), f(4n+2) = f(4n) + 1, f(2n+1) = f(2n) + 1$ for all n. Show that the number of non-negative integers n such that $f(4n) = f(3n)$ and $n < 2^m$ is $f(2^{m+1})$.

Solution. The condition suggests us look at the binary decomposition of n. Firstly as $f(4n) = f(2n) + f(n)$ we can easily deduce that $f(2^k) = F_{k+1}$, where $(F_n)_{n\in N}$ is the Fibonacci sequence. Indeed, setting $n = 0$ we get $f(0) = 0$ thus $f(1) = 1, f(2) = 1$. Now the conditions $f(4n+2) = f(4n)+1 = f(4n)+2, f(2n+1) = f(2n)+1$ may suggest some sort of additivity for f, at least $f(a + b) = f(a) + f(b)$ when a do not share digits in base 2. And this is indeed the case if we look at some small particular values of f. So we conjecture this assertion, which would mean that $f(n)$ is actually n transferred from base 2 into "Fibonacci base", i.e. $f(b_k 2^k + \ldots + b_0) = b_k F_{k+1} + \ldots + b_0$. This is easily accomplished by induction on n: if $n = 4k$ then $f(n) = f(2k) + f(k)$, if $n = 2k+1$ then $f(n) = f(2k)+1$ and if $n = 4k+2$ then $f(n) = f(4k)+1$ and the verification is direct.

Now as we found f, let's turn to the final question. It asks when $f(4n) = f(3n)$. Actually f should be some sort of increasing function, so we could suppose $f(3n) \leq f(4n)$. Indeed this holds true if we check some particular cases, with equality sometimes. Now what connects $4n$ and $3n$? The condition says us that $f(4n) = f(2n) + f(n)$ but we have $3n = 2n + n$. So we can suppose that $f(a + b) \leq f(a) + f(b)$ and look for equality cases.

We work of course in binary. The addition of two binary numbers can be thought of as adding their corresponding digits pairwise, and then repeating a number of times the following operation: if we have reached a 2 in some position, replace it by a zero and add a 1 to the next position. (Note that we will never have digits greater than two if we eliminate the 2 at the highest level at each step). For example $3+9 = 11_2+1001_2 = 1012_2$ and then we remove the 2 to get 1020_2 and again to get $1100_2 = 10$ so $3 + 7 = 10$. We can extend f to sequences of 0's, 1's and 2;s by setting $f(b_k, \ldots, b_0) = b_k F_{k+1} + b_{k-1} F_k + \ldots + b_0$. Then we can see that if S is the sequence obtained by adding a and b componentwise (as vectors), then $f(s) = f(a) + f(b)$. And we need to prove that the operation of removing a 2 does not increase f. Indeed, if we remove a 2 from position k and add a 1 to position $k + 1$ the f changes by $F_{k+2} - 2F_{k+1}$. This value is never positive and is actually zero only for $k = 0$. So f indeed is not increased by this operation (which guarantees the claim that $f(a+b) \leq f(a)+f(b)$), and moreover it is not decreased by it only if the operation consists of removing the 2 at the units position. So $f(a+b) = f(a)+f(b)$ if and only if by adding them componentwise we either reach no transfer of unity, or have only one transfer at the lowest level. Hence $f(4n) = f(3n)$ if and only if adding $2n + n$ we can reach at most a transfer at the lowest level. But it cannot occur as the last digit of $2n$ is 0. So $f(4n) = f(3n)$ if and only if by adding $2n$ and n we have no transfer i.e. $2n$ and n don't share a unity digit in the same position. But as the digits of $2n$ are just the digits of n shifted one position, this is possible if and only if n has no two consecutive unities in its binary representation. So we need to prove that there are exactly $f(2^{m+1}) = F_{m+2}$ such numbers less than 2^m. Let $g(m)$ be this numbers. Then $g(0) = 1, g(1) = 2$. Now note that if n is such a number and $n \geq 2^{m-1}$ then $n = 2^{m-1} + n'$ where $n' < 2^{m-2}$ (as it cannot have a unity in position $m-1$ that would conflict with the leading unity), so we have $g(m-2)$ possibilities for this case. For $n < 2^{m-1}$ we have $g(m-1)$ possibilities. So $g(m) = g(m-1) + g(m-2)$ and an induction finishes the problem.

Exercises

Problem 40. (Iberoamerican)Find all functions $f\colon N \to R$ for which $f(1) = 1$ and

$$f(2n+1) = f(2n) + 1 = 3f(n) + 1$$

$n \in N$.

Problem 41. (IMO 1978)Find all functions $f\colon N \to N$ that satisfy $f(1) = 1, f(3) = 3$ and

$$f(2n) = f(n)$$
$$f(4n+1) = 2f(2n+1) - f(n)$$
$$f(4n+3) = 3f(2n+1) - 2f(n)$$

for any $n \in N$.

Constructing functions by iterations

There is a class of functional equations, most of them on N, like $f(f(x)) = g(x)$. They can be solved by constructing the "orbits" of x: $O(x) = (x, g(x), g(g(x), \ldots))$ and investigating the relations determined by f on this orbits. This type of equations will be exemplified here.

Problem 42. Show that there are infinitely many odd functions $g\colon Z \to Z$ for which $g(g(k)) = -k, k \in Z$.

Solution. We may set $g(0) = 0$. Then $Z \setminus \{0\}$ can be divided into an infinite number of pairs $(a_1, -a_1), (a_2, -a_2), \ldots$ where $a_1, a_2, \ldots,$ is some enumeration of N. We can then set $g(a_{2k}) = a_{2k+1}, g(a_{2k+1}) =$

$-a_{2k}, g(-a_{2k}) = -a_{2k+1}, g(-a_{2k+1}) = a_{2k}$ and check that the condition is verified.

Problem 43. Find all functions $f\colon N \to N$ that verify

$$f(f(n)) = an$$

for some fixed $a \in N$.

Solution. If $a = 1$ then $f(f(x))$ so f is an involution and is obtained by paring all the natural numbers into pairs and mapping one element of a pair into another. Next, suppose $a > 1$. If $f(x) = y$ then $f(y) = ax$, $f(ax) = f(f(y)) = ay$ and we prove by induction on k the following statement: (*) $f(a^k x) = a^k y, f(a^k y) = a^{k+1}x$. Let S be the set of all numbers not divisible by a. Every natural number can be represented uniquely as $a^k b$ where $b \in S$. Now let $s \in S$ and $f(s) = a^k t$ where $t \in S$. If we set $u = f(t)$ then using (*) we get $f(a^k t) = a^k u$. But $f(a^k t) = f(f(s)) = as$ therefore $a^k u = as$ so as s is divisible by u we get either $k = 1, u = s$ or $k = 0, u = as$. In the first case $f(t) = s, f(s) = at$ and in the second case $f(s) = t, f(t) = as$. In any case, f maps one of s, t into another. Therefore S separates into pairs (x, y) that satisfy $f(x) = y, f(y) = x$, hence by (*) $f(a^k x) = a^k y, f(a^k y) = a^{k+1}x$. It's clear that all such functions satisfy our requirements.

Problem 44. (Romanian TST 1991) Let $n \geq 2$ be a positive integer, $a, b \in Z$, $a \notin \{0, 1\}$. Show that there exist infinitely many functions $f\colon Z \to Z$ such that $f_n(x) = ax + b$ for all $x \in Z$, where f_n is the n-th iterate of f. Show that for $a = 1$ there exist b such that $f_n(x) = ax + b$ has no solutions.

Solution. The second part of the problem is already known for us when $n = 2$, n is odd, and the procedure is the same. If $b = n - 1$ and we let $a_i = f_i(0)$ $(a_{i+n} = a_i + b)$, then for some $0 \leq i < j \leq n$ we have $a_i \equiv a_j \ (mod\ n)$, thus $a_i = a_j + hb$, hence $a_{i+hn} = a_j$, and thus $a_{r+hn+i-j} = a_r$ for all sufficiently big r, which in turn as $hn + i - j \neq 0$ imply $a_{r+n(hn+i-j)} = a_r$ which contradicts the conclusion that $a_{r+n(hn+i-j)} = a_r + b(hn + i - j)$. Actually with one more effort one can prove that f exists if and only if $n|b$.

Now let's turn to the first part, which seems more challenging but bear also some similarity to the simpler case $a = 1, n = 2$. Let $g(x) = ax + b$. Firstly consider the case $a \neq -1$ (it is special because $g(g(x)) = x$ in this case, whereas in the general case $|g_n(x)|$ tends to infinity for almost all x). We see that $g_n(x) = a^n(x - \frac{b}{a-1}) + \frac{b}{a-1}$. Particularly this guarantees our claim that $|g_n(x)|$ tends to infinity for almost all x, particularly for all x except maybe $x = \frac{b}{a-1}$ (if it's an integer).

Now call $C(x) = \{x, g(x), g_2(x), \ldots, g_n(x), \ldots\}$ the chain generated by x and call a chain maximal if it is not a proper subchain of another chain (to put it otherwise, if $x \neq g(y)$ for some $y \in Z$). We claim that maximal chains constitute a partition of $N \setminus \{-\frac{b}{a-1}\}$. Indeed, firstly pick up a number $n \neq -\frac{b}{a-1}$. Then $n = g_k(m)$ is equivalent to $n = a^k(m + \frac{b}{a-1}) - \frac{b}{a-1}$, or $(a-1)n + b = a^k((a-1)m + b)$. So take k be the greatest power of a dividing $(a-1)n+b$ and let $s = \frac{(a-1)m+b}{a^k}$. Then s is not divisible by a and moreover $s - b$ is divisible by $a - 1$. Hence if we set $m = \frac{s-b}{a-1} + b$ then m is an integer and the equation $g(t) = m$ has no solutions in N (because otherwise $at = m - b = \frac{s-b}{a-1}$ so $s - b$ is divisible by a). So $C(m)$ is the desired maximal chain. Next, let's prove that two distinct maximal chains do not intersect. If $C(x)$ and $C(y)$ intersect for $x \neq y$ then $g_m(x) = g_n(y)$ for some $m \neq n$. Without loss of generality $m \geq n$. Then as g is invertible on R we deduce $g_{m-n}(x) = y$ hence $C(y) \subset C(x)$ contradicting the fact that $C(y)$ is a maximal chain. Now consider all the maximal chains (there are infinitely many of them since every element x such that the equation $g(y) = x$ has no solutions in N generates such a chain). We can group then into n-uples. Now we define f on each of the n-uples. Let $(C(x_1), C(x_2), \ldots, C(x_n))$ be such an n-uple. Then we define $f(g_k(x_i)) = g_k(x_{i+1})$ for $i = 1, 2, \ldots, n-1$ and $f(g_k(x_n)) = g_{k+1}(x_1)$. Define also $f(-\frac{b}{a-1}) = -\frac{b}{a-1}$. f is seen to satisfy our requirements.

Let's investigate now the case $a = 1$. In this case, $N \setminus \{\frac{b}{2}\}$ splits into infinitely many disjoint pairs (x, y) with $x + y = b$. Again we can group the pairs into n-uples and define f on each n-uple $(x_1, y_1), \ldots, (x_n, y_n)$ as $f(x_i) = x_{i+1}, f(y_i) = y_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $f(x_n) = y_1, f(y_n) = x_1$. Define $f(\frac{b}{2}) = \frac{b}{2}$ if necessary. Again we see that f satisfies the conditions.

Finally in both cases as we can group the chains or the pairs into n-uples in infinitely many ways, we have infinitely many such functions. It can be also proven that all functions with the desired property are of form we found.

Exercises

Problem 45. Let $n \in N$. Find all continuous $f\colon R \to R$ that satisfy $f_n(x) = -x$ where f_n is the n-th iterate of f.

Problem 46. Show that there exist functions $f\colon N \to N$ such that

$$f(f(n)) = n^2, n \in N$$

.

Problem 47. Let $f\colon N\to N$ ne a function satisfying

$$f(f(n))=4n-3$$

and

$$f(2^n)=2^{n+1}-1$$

Find $f(1993)$. Can we find explicitly the value of $f(2007)$? What values can $f(1997)$ take?

Approximating with linear functions

There are some weird functional equations on N that seem untouchable. But sometimes we can prove that they are unique. In this case guessing the function would be very helpful, and very often, the solutions are linear, thus it's natural to try $f(x)=cx$. But sometimes c can be rational or even irrational, and we can have formulae like $f(x)=[cx]$. To overpass this difficulty, we write $f(x)\sim cx$ meaning that $|f(x)-cx|$ is bounded. Now we can guess c from the condition and then look at some initial case to guess the exact formula. The examples are given below.

Problem 48. Find all increasing functions $f\colon N\to N$ such that the only natural numbers who are not in the image of f are those of form $f(n)+f(n+1), n\in N$.

Solution. Firstly, let's assume $f(x)\sim cx$. Let's compute the effective value of c. If $f(n)=m$ then there are exactly $m-n$ natural numbers up to m that are not values of f. Therefore we conclude that they are exactly $f(1)+f(2),\dots,f(m-n)+f(m-n+1)$. Hence $f(m-n)+f(m-n+1)<m<f(m-n+1)+f(m-n+2)$. Now as $f(x)\sim cx$ we conclude $m\sim cn$ so we get $2c(m-n)\sim m$ or $2c(c-1)n\sim cn$ which means $2c-2=1$ so $c=\frac{3}{2}$. Hence we make the assumption that $f(x)=[\frac{3}{2}x+a]$ for some a. Let's search for a. Clearly $f(1)=1, f(2)=2$ as $1,2$ must necessarily belong to Imf. Then 3 does not belong to Imf hence $f(3)\geq 4$ so $f(2)+f(3)\geq 6$. Thus 4 belongs to Imf and $f(3)=4$. We continue to $f(4)=5, f(5)=7$ and so on. So $[\frac{3}{2}+a]=1, [3+a]=2$ which implies $a\in[-\frac{1}{2};0)$. And we see that for any a,b in this interval, $[\frac{3}{2}x+a]=[\frac{3}{2}x+b]$. So we can assume $a=-\frac{1}{2}$ and infer $f(n)=[\frac{3n-1}{2}]$, and try to prove it. Firstly we wish to show that $[\frac{3n-1}{2}]$ satisfies the conditions. Indeed, $[\frac{3n-1}{2}]+[\frac{3(n+1)}{2}]=[\frac{3n-1}{2}]+1+[\frac{3n}{2}]=3n+1$ by Hermite's Identity, and we need to prove that the only numbers that are not of form $\frac{3n-1}{2}$

are those that give residue 1 to division by 3. Indeed, if $n = 2k$ then $[\frac{3n-1}{2}] = 3k - 1$ and if $n = 2k + 1$ then $[\frac{3n-1}{2}] = 3k$, and the conclusion is straightforward.

The fact that $f(n) = [\frac{3n-1}{2}]$ stems now from the inductive assertion that f is unique. Indeed, if we have determined $f(1), f(2), \ldots, f(n-1)$ then we have determined all $f(1)+f(2), f(2)+f(3), \ldots, f(n-2)+f(n-1)$. Then $f(n)$ must be the least number which is greater than $f(n-1)$ and not among $f(1)+f(2), f(2)+f(3), \ldots, f(n-2)+f(n-1)$. This is because if m is this number and $f(n) \neq m$ then $f(n) > m$ and then m does not belong neither to $Im(f)$ nor to the set $\{f(n)+f(n+1)|n \in N\}$, contradiction. Hence $f(n)$ is computed uniquely from the previous values of f and thus f is unique.

Problem 49.(IMO 1979)Find all increasing functions $f\colon N \to N$ with the property that all natural numbers which are not in the image of f are those of form $f(f(n)) + 1, n \in N$.

Solution. Again f is unique. If $f(x) \sim cx$ then we conclude that $m \sim c^2(m - n)$ where $m = f(n)$ so $c = c^2(c - 1)$ or $c^2 - c - 1 = 0$ so $c = \frac{1+\sqrt{5}}{2} \sim 1.618$, the positive root of the quadratic equation. So we try to set $f(x) = [cx + d]$ for some constant d. Now we compute $f(1) = 1,\ f(2) = 3,\ f(3) = 4, f(4) = 6, f(5) = 8$, and we can try to put $d = 0$, so $f(n) = [cn]$. Let's prove that it satisfies the hypothesis. If $f(n) = m$ then $m < cn < m + 1$ so $\frac{m}{c} < n < \frac{m+1}{c}$. As $\frac{1}{c} = c - 1$ we get $(c - 1)m < n < (c - 1)(m + 1)$ and so m is in $Im(f)$ if an only if the interval $(cm; cm + c - 1)$ contains an integer which is equivalent to the fact that $\{cm\} > 2 - c$. And if $f(f(n)) + 1 = m$ then $[c[cn]] = m - 1$ so $[cn] \in ((m - 1)(c - 1); m(c - 1))$ so $n \in ((m-1)(c-1)^2; m(c-1)^2+(c-1)) = ((2-c)m+c-2; (2-c)m+c-1)$, so $n = [(2-c)m+c-1] = 2m-[c(m-1)]-2$. Therefore $m = f(f(n))+1$ if and only if the number $n = 2m-[c(m-1)]-2$ satisfies the condition $f(f(n))+1 = m$. Set $u = \{c(m-1)\}$. Then $n = (2-c)m+c-2+u$ so $f(n) = [c(2-c)m+cu-2c+c^2] = [(c-1)m+cu-c+1] = [(c-1)(m-1)+cu] = [c(m-1)-m+1+cu] = c(m-1)-m+1+cu-\{u(c+1)\}$. Set $s = \{u(c+1)\}$. Then $f(f(n)) = [c(c-1)(m-1)+c^2u-cs] = [m-1+(c+1)u-cs]$. So $f(f(n))+1 = m$ if and only if $0 < (c+1)u-cs < 1$. If $t = u(c+1) \in (0; 1+c)$ this is equivalent to $t - c\{t\} \in (0; 1)$. When $t < 1$ this is false as the requested value is negative. When $1 < t < 2$ we have $t-c\{t\} = t-c(t-1) = c-(c-1)t \in (0; 1)$. When $t > 2$ $t-c\{t\} = t-c(t-2) = 2c-(c-1)t > 2c-(c-1)(c+1) = 2c-c^2+1 = c > 1$. So our condition is equivalent to $t \in (1; 2)$ or $u \in (\frac{1}{c+1}; \frac{2}{c+1}) = (2-c; 4-2c)$

so $\{cm-c\} \in (2-c;4-2c)$ or $\{cm\} \in \{0;2-c\}$ So this condition is equivalent to $\{cm\} < 2-c$.

Thus we see that the condition $m = f(n)$ is equivalent to $\{cm\} > 2-c$ and the condition $m = f(f(n))+1$ is equivalent to $\{cm\} < 2-c$. So these two conditions are complementary and the proof is finished.

Exercises

Problem 50. Find all increasing functions $f\colon N \to N$ such that the only natural numbers who are not in the image of f are those of form $2n + f(n), n \in N$.

Problem 51. Find all functions $f\colon N \to N$ such that

$$f(f(n)) + f(n+1) = n+2$$

for $n \in N$.

Problem 52. Find all functions $f :: N \to N$ that satisfy $f(1) = 1$ and $f(n+1) = f(n) + 2$ if $f(f(n) - n + 1) = n$, $f(n+1) = f(n) + 1$ otherwise.

Extremal element method

Problem 53. Find all bijections

$$f, g, h\colon N \to N$$

for which

$$f^3(n) + g^3(n) + h^3(n) = 3ng(n)h(n)$$

whenever $n \in N$.

Solution. $f = g = h$ satisfies the condition. Next, by AM-GM we have $f^3(n) + g^3(n) + h^3(n) =\geq 3f(n)g(n)h(n)$ with equality if and only if $f(n) = g(n) = h(n)$. Therefore $f(n) \geq n$ with equality if and only if $f(n) = g(n) = h(n) = n$. The problem now follows clearly from the fact that f is a bijection: if $f(a) = 1$ then $1 \geq a$ so $a = 1$ and then equality holds so $f(1) = g(1) = h(1) = 1$. We then proceed by induction: if we have shown that $f(k) = g(k) = h(k) = k$ for $k \leq n$ then if $f(m) = n+1$ then $n+1 \leq m$ (all numbers less than m are already occupied) but from the other side $n+1 \geq m$ hence equality holds and then we have $f(n+1) = g(n+1) = h(n+1) = n+1$, as desired.

Problem 54. Find all functions $f\colon N \to N$ such that

$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$

for all $n \in N$.

Solution. We shall show by induction that $f(n) = n$ for all $n \in N$. We have

$$f(f(f(1))) + f(f(1)) + f(1) = 3$$

and therefore

$$f(f(f(1))) = f(f(1)) = f(1) = 1.$$

Suppose that $f(k) = k$ for all $k \le n$. It follows by the given condition that the function f is injective. This implies that $f(m) > n$ for $m > n$. Hence

$$f(n+1) \ge n+1, \;\; f(f(n+1)) \ge f(n+1) \ge n+1$$

and

$$f(f(f(n+1))) \ge f(n+1) \ge n+1.$$

Summing up gives

$$f(f(f(n+1))) + f(f(n+1)) + f(n+1) \ge 3(n+1).$$

On the other hand we have that

$$f(f(f(n+1))) + f(f(n+1)) + f(n+1) = 3(n+1)$$

and

$$f(f(f(n+1))) = f(f(n+1) = f(n+1) = n+1.$$

Hence it follows by induction that

$$f(n) = n$$

for all $n \in N$.

Problem 55.(Ukraine) Find all functions $f : \mathbb{Z} \to \mathbb{N}_{\not{\vdash}}$ for which we have

$$6f(k+3) - 3f(k+2) - 2f(k+1) - f(k) = 0$$

.

Solution. This looks like a recurrence relation, but it's not! Indeed, the first inconvenience is that we have f defined on Z, not on N, so computing f inductively would lead to induction in both directions, and the second is the 6 before $f(k+3)$ which implies that if we would try to compute f inductively, we could obtain non-integer numbers. That's why we must use another idea. As surprisingly the range of f is in N (not in Z like usually when functions are defined on Z), we might try to use some of the properties of N that distinguish it from Z. A first that comes to our mind is that N contains a minimal element, and indeed as $6 = 1 + 2 + 3$ this is the best idea. So let $a = min\{Imf\}, f(x) = a$. Then writing the conditions for $k = x-3$ we get $6a = 3f(x-1)+2f(x-2)+f(x) \geq 3a+2a+a = 6a$. So equalities hold everywhere thus $f(x-1) = f(x-2) = f(x-3) = a$. A straightforward induction now shows that $f(y) = a$ whenever $y \leq x$. Another simple induction using the recurrence relation shows that $f(y) = a$ for $y > x$ (the conditions written for $k = x - 2$ gives us $f(x+1) = a$ and so on). It's clear that constant functions satisfy our claim.

Exercises

Problem 56. (IMO '1977). Let $f\colon N \to N$ be a function such that $f(n+1) > f(f(n))$ for all $n \in N$. Show that $f(n) = n$ for all $n \in N$.
Problem 57. (BMO '2002) Find all functions $f\colon N \to N$ such that

$$2n + 2001 \leq f(f(n)) + f(n) \leq 2n + 2002$$

for all $n \in N$.

Multiplicative Cauchy Equation

Problem 58. (IMO '1990) Construct a function $f\colon Q^+ \to Q^+$ such that

$$yf(xf(y)) = f(x)$$

for all $x, y \in Q^+$.

Solution. Let f be a function with the given properties. Then setting $x = 1$ gives $yf(f(y)) = f(1)$. Hence the function f is injective and from $f(f(1)) = f(1)$ we get $f(1) = 1$. Thus

$$yf(f(y)) = 1. \tag{1}$$

Now replacing x by xy and y by $f(y)$ in the given identity gives

$$f(xy) = f(x)f(y). \tag{2}$$

Conversely, it is easy to check that any function $f\colon Q^+ \to Q^+$ satisfying (1) and (2) satisfies the given condition as well.

It follows by (2) that $1 = f(1) = f(y)f\left(\dfrac{1}{y}\right)$ and using (2) again we get that $f\left(\dfrac{x}{y}\right) = \dfrac{f(x)}{f(y)}$. This shows that if the function f is defined on the set P of the prime numbers then it has a unique continuation on Q^+ given by: $f(1) = 1$; if $n \geq 2$ is an integer and $n = {p_1}^{\alpha_1} \dots {p_k}^{\alpha_k}$ is its canonical representation as a product of primes then $f(n) = (f(p_1))^{\alpha_1}(f(p_2))^{\alpha_2} \dots (f(p_k))^{\alpha_k}$; if $r = \dfrac{m}{n} \in Q^+$ then $f(r) = \dfrac{f(m)}{f(n)}$. If moreover f satisfies (1) for all $p \in P$ then one checks easily that (1) and (2) are satisfied for all $x, y \in Q^+$.

Hence we have to construct a function $f\colon P \to Q^+$ which satisfies (1). To this end let $p_1 < p_2 < \dots$ be the sequence of prime numbers. Set $f(p_{2n-1}) = p_{2n}$ and $f(p_{2n}) = \dfrac{1}{p_{2n-1}}$ for all $n \geq 1$. Then it is obvious that $f(f(p)) = \dfrac{1}{p}$ for all $p \in P$, i.e. the identity (1) is satisfied.

Problem 59. (IMO '1998) Consider all functions $f\colon N \to N$ such that

$$f(n^2 f(m)) = m f^2(n)$$

for all $m, n \in N$. Find the least possible value of $f(1998)$.

Solution. Let f be an arbitrary function satisfying the given condition. Set $f(1) = a$. Then setting $n = 1$ and $m = 1$ gives $f(f(m)) = a^2 m$ and $f(an^2) = f^2(n)$. Then $f^2(m)f^2(n) = f^2(m)f(an^2) = f(m^2 f(f(an^2))) = f(m^2 a^3 n^2) = f^2(amn)$, i.e. $f(m)f(n) = f(amn)$. In particular, $f(am) = af(m)$ and therefore

$$af(mn) = f(m)f(n). \tag{1}$$

We shall prove that a divides $f(n)$ for all $n \in N$. Let p be a prime number and let $\alpha \geq 0$, $\beta \geq 0$ be the degrees of p in the canonical representations of a and $f(n)$, respectively. It follows by induction from (1) that $f^k(n) = a^{k-1} f(n^k)$ for all $k \in N$. Hence $k\beta \geq (k-1)\alpha$, which implies that $\beta \geq \alpha$. This shows that a divides $f(n)$.

Set $g(n) = \dfrac{f(n)}{a}$. Then $g\colon N \to N$ satisfies the conditions

$$g(mn) = g(m)g(n) \tag{2}$$

$$g(g(m)) = m \tag{3}$$

for all $m, n \in N$. Conversely, given a function g with the above properties then for any $a \in N$ the function $f(n) = ag(n)$ satisfies the given condition.

Setting $m = n = 1$ in (2) gives

$$g(1) = 1. \tag{4}$$

No we shall prove that $g(p) \in P$ for all $p \in P$. Indeed, let $p \in P$ and $g(p) = uv$. Then it follows from (3) and (2) that $p = g(g(p)) = g(uv) = g(u)g(v)$. We may assume that $g(u) = 1$. Hence $u = g(g(u)) = g(1) = 1$ and therefore $g(p) \in P$. Now let $n \geq 2$ and $n = {p_1}^{\alpha_1} \dots {p_k}^{\alpha_k}$ be the canonical representation of n as a product of prime numbers. Then it follows from (2) that

$$g(n) = g^{\alpha_1}(p_1) \dots g^{\alpha_k}(p_k). \tag{5}$$

Hence we have proved that any function $f \colon N \to N$ with the given property is uniquely determined by its value $f(1)$ and a function $g \colon P \to P$ such that $g(g(p)) = p$, i.e. by a pairing of the set of prime numbers.

We shall show now that the least possible value of $f(1998)$ is 120. We have that

$$f(1998) = f(2.3^3.37) = f(1)g(2)g^3(3)g(37).$$

Since $g(2), g(3)$ and $g(37)$ are different prime numbers it follows that

$$g(2)g^3(3)g(37) \geq 3.2^3.5 = 120,$$

i.e. $f(1998) \geq 120$. To construct a function f satisfying the given condition and $f(1998) = 120$ set $a = f(1) = 1$, $g(2) = 3$, $g(3) = 2$, $g(5) = 37$, $g(37) = 5$ and $g(p) = p$ for all prime numbers $p \neq 2, 3, 5, 37$. Then $g(g(p)) = p$ for all $p \in P$ and as we said above these data determine uniquely a function $f \colon N \to N$ with desired properties.

Substitutions

Problem 60. Find all functions $f :: R \to R$ such that

$$f(x+y) - f(x-y) = f(x)f(y)$$

Solution. Set $x = y = 0$ to get $f^2(0) = 0$ so $f(0) = 0$. If we set $y \to -y$ we get $f(x-y) - f(x+y) = f(x)f(-y) = -f(x)f(y)$. Particularly $f(y)f(-y) = -f(y)^2, f(-y)^2 = -f(y)f(y)$ so $f(y)(f(y) + f(-y)) = f(-y)(f(y) + f(-y)) = 0$, so either $f(y) + f(-y) = 0$ or $f(y) = f(-y) = 0$ and again $f(y) + f(-y) = 0$ thus f is odd. Now set

$y = x$ so get $f(2x) = f(x)^2$. Then $f(-2x) = f(-x)^2 = f(x)^2 = f(2x)$. As $f(-2x) = -f(2x)$ we deduce $f(2x) = 0$ and f is identically zero.

Problem 61.(Vietnam 1991)Find all functions $f :: R \to R$ for which

$$\frac{1}{2}f(xy) + \frac{1}{2}f(xz) - f(x)f(yz) \geq \frac{1}{4}$$

Solution. If we set $x = y = 1$ the condition is equivalent to $-(f(1)-\frac{1}{2})^2 \geq 0$ possible only for $f(1) = \frac{1}{2}$. Next set $y = z = 1$ to get $f(x) - \frac{1}{2}f(x) \geq \frac{1}{4}$ so $f(x) \geq \frac{1}{2}$. Set $y = z = \frac{1}{x}, x \neq 0$ so get $f(1) - f(x)f(\frac{1}{x^2}) \geq \frac{1}{4}$. But $f(x) \geq \frac{1}{2}, f(\frac{1}{x^2}) \geq \frac{1}{2}, f(1) = \frac{1}{2}$ so $f(1) - f(x)f(\frac{1}{x^2}) \geq \frac{1}{4}$. Thus equalities hold and $f(x) = \frac{1}{2}$ for $x \neq 0$. Next set $y = z = 0, x \neq 0$ to get $f(0) - \frac{1}{2}f(0) \geq \frac{1}{4}$ so $f(0) \leq \frac{1}{2}$. As we have proven $f(x) \geq \frac{1}{2}$ we get $f(0) = \frac{1}{2}$. So $f(x) = \frac{1}{2}$ which satisfies the condition.

Problem 62. Find all functions $f \colon R \to R$ that are continuous in zero and satisfy $f(x+y) - f(x) - f(y) = xy(x+y)$

Solution. We can guess the solution $\frac{x^3}{3}$, thus $g(x) = f(x) - \frac{x^3}{3}$ is additive. Now we claim $f(x) = cx$ for $c = f(1)$. Indeed, assume that $d = f(t) \neq ct$. If t is irrational then we can find $m, n \in Z$ with $|m + nt| < \epsilon$ for any $\epsilon > 0$. Then $f(m + nt) = mc + nd = c(m+nt) + n(d-ct)$. But now if we take ϵ small enough we force n to be as big as we wish and thus $|f(m+nt)| > n|d-ct| - c\epsilon$ increases to infinity which contradicts the continuity of f in 0. So $f(x) = \frac{x^3}{3} + cx$.

Problem 63. Find all functions $f :: R^2 \to R$ which satisfy the following conditions:

i)$f(x+u, y+u) = f(x,y) + u$

ii) $f(xu, yu) = f(x,y)u$

Solution. Firstly according to ii) $f(x,0) = f(1,0)x$ so $f(x,0) = cx$ for some c. Next according to i) $f(x,y) = f(x-y,0)+y = c(x-y)+y = cx + (1-c)y$. Conversely, any function of form $cx + (1-c)y$ obviously satisfies the conditions.

Problem 64. (Belarus 1995)Find all functions $f :: R \to R$ for which

$$f(f(x+y)) = f(x+y) + f(x)f(y) - xy$$

Solution. Set $y = 0$ to get $f(f(x)) = f(x)(1 + f(0))$. So $f(x) = (1+f(0))x$ on $Im(f)$. Substituting we get $f(x)f(y) - xy = f(x+y)f(0)$. Set $a = f(0)$. Now set $x = -a, y = a$ to get $a^2 f(-a) + a^2 = a^2$ so $f(-a) = 0$. Therefore $0 \in A$ hence $f(0) = (1+a)0 = 0$ so $a = 0$.

Hence we get $f(x)f(y) = xy$. Hence for $x \neq 0$ we have $f(x) \neq 0$ and if we set $g(x) = \frac{f(x)}{x}$ we get $g(x)g(y) = 1$ for all x, y. Keeping y fixed we get that $g(x) = c$ is constant. Then $c^2 = 1$ so $c = \pm 1$ hence $f(x) = x$ or $f(x) = -x$. As $f(x) = x$ on $Im(f)$, $f(x) = x$ is the only solution.

Problem 65. Find all continuous functions $f\colon R \to R$ that satisfy

$$f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$$

Solution. We know that $tan\ x$ satisfies this equation. Therefore if we set $g(x) = arctan f(x)$ then $g(x+y) = g(x) + g(y) \pm 2k\pi$. As a multiple of 2π does not matter to us we can show in the usual way that $g(x) = cx\ mod\ 2\pi$ and hence $f(x) = tan\ cx$.

Problem 66. Let f, g be two continuous functions on R that satisfy

$$f(x-y) = f(x)f(y) + g(x)g(y)$$

Find f, g.

Solution. Set $y = x$ to conclude that $f(x)^2 + g(x)^2 = c = f(0)$. Particularly for $x = 0$ we get $c^2 + g^2(0) = c$ so $c \leq 1$. If $c < 1$ we set $y = 0$ to get $f(x) = cf(x) + g(0)g(x)$ so $g(x) = \frac{1-c}{g(0)}f(x)$. Then $f^2 + g^2(x) = f^2(x)(1 + \frac{(1-c)^2}{g^2(0)}) = f^2(x)(1 + \frac{(1-c)^2}{c-c^2}) = \frac{f^2(x)}{c}$ so $f^2(x) = c^2$ thus $f(x) = \pm c$ and by continuity $f(x) = c, g(x) = \sqrt{c-c^2}$ or $g(x) = -\sqrt{c-c^2}$. We have left to investigate the case $c = 1$ in which case we get $f(0) = 1, g(0) = 0, f^2(x) + g^2(x) = 1$. This already suggests the sine-cosine formula. Indeed, let's prove the "sister" identity $g(x-y) = g(x)f(y) - g(y)f(x)$. If holds for $y = 0$. Note that $f^2(x-y) + g^2(x-y) = 1$ so $g^2(x-y) = (f^2(x)+g^2(x))(f^2(y)+g^2(y)) - (f(x)f(y)+g(x)g(y))^2 = (g(x)f(y) - g(y)f(x))^2$. Thus $g(x-y) = \pm(g(x)f(y) - g(y)f(x))$. Suppose $g(x-y) = -(g(x)f(y) - g(y)f(x))$ for some x, y. Let A be the set of such (x, y) and B be the set of (x, y) for which $g(x-y) = g(x)f(y) - g(y)f(x)$. Then A, B are closed and $A \bigcap B = R^2$ thus there is a point (u, v) which belongs to both A, B. Then $g(u-v) = (g(u)f(v) - g(v)f(u)) = -(g(u)f(v) - g(v)f(u))$ possible only when $g(u-v) \neq 0$. If we interchange x with y we get that f is even. If we set $y \to -y$ we get $f(x+y) = f(x)f(y) + g(x)g(-y)$ and analogously $f(x+y) = f(x)f(y) + g(-x)g(y)$. Thus $g(x)g(-y) = g(y)g(-x)$so $g(-y) = g(y)\frac{g(-x)}{g(x)}$ if $g(x) \neq 0$. If $g(x) = 0$ for all x we immediately conclude that $f(x) = 1$. If $g(x_0) \neq 0$ then as $f^2(x_0) + g(x_0^2) = 1 = f(-x_0)^2 + g(-x_0)^2$ we get $\frac{g(-x_0)}{g(x_0)} = \pm 1$ thus g is either even or odd. If g is even by setting $y \to -y$ we get $f(x+y) = f(x)f(y) + g(x)g(y) = f(x+y)$ and so f is constant. If f is not constant then g is odd. We can now set $y \to -y$ to get $f(x+y)+f(x-y)=2f(x)f(y)$ and solve it by the familiar D'Alembert's Equation (see the chapter "Polynomial recurrences and continuity"), or we can proceed by another way. Set $z=x-y$. Then $f(y)=f(x-z)=f(x)f(z)+g(x)g(z)=f(x)(f(x)f(y)+g(x)g(y))+g(x)g(z)=f^2(x)f(y)+f(x)g(x)g(y)$. So $g(x)g(z)=f(y)-f^2(x)f(y)-f(x)g(x)g(y)=g^2(x)f(y)-f(x)g(x)g(y)$ thus if $g(x)\neq 0$ we get $g(x-y)=g(x)f(y)-f(x)g(y)$. Now if $g(x)=0$ but there is a sequence $x_n \to x$ such that $g(x_n)\neq 0$ we can conclude $g(x-y)=g(x)f(y)-f(x)g(y)$ by continuity. Otherwise there is a non-empty open interval (a,b) on which g vanishes. Then if $x\in(a,b)$ we get $f(x+x)=f^2(x)-g^2(x)=f^2(x)+g^2(x)=1$ hence $f(2x)=0$ thus on $(2a;2b)$ we have f identically 1 and g identically zero. Setting now $y\in(2a;2b)$ we deduce $f(x-y)=f(x)$ for all x thus if $|u-v|<2(b-a)$ we can find x and $y_1,y_2\in(2a,2b)$ with $u=x-y_1, v=x-y_2$ hence $f(u)=f(v)=f(x)$. We deduce from here that f is constant so f is identically 1 and g identically zero. If not, then we have proven $g(x-y)=g(x)f(y)-f(x)g(y)$ for all x,y. Now set $u(x)=f(x)+ig(x)$. Then $u(x+y)=f(x)f(y)-g(x)g(y)+i(g(x)f(y)-f(x)g(y))=u(x)u(y)$. So u is multiplicative. Since it is also continuous we get $u(x)=e^{kx}$ by using the lemma. Now as $|u(x)|=\sqrt{f^2(x)+g^2(x)}=1$ we must have $k\in iR$ thus $u(x)=e^{\alpha ix}$ and we conclude $f(x)=\cos\alpha x, g(x)=\sin\alpha x$.

Problem 67.(Ukraine 1997)Find all functions $f :: Q^+ \to Q^+$ such that $f(x+1)=f(x)+1$ and $f(x^2)=f(x^2)$

Solution. From the first condition we conclude that $f(x+k)=f(x)+k$ for $k\in N$ by induction on k. Assume that $f(\frac{p}{q})=r$. Then $f(\frac{p}{q}+kq)=a+kq$ so $f((\frac{p}{q}+kq)^2)=a^2+2kaq+k^2q^2$ for $k\in N$. But $(\frac{p}{q}+kq)^2=\frac{p^2}{q^2}+2pk+k^2q^2$ hence $f((\frac{p}{q}+kq)^2)=f(\frac{p^2}{q^2})+2pk+k^2q^2=q^2+2pq+k^2q^2$. So $2kaq=2pk$ thus $a=\frac{p}{q}$. The identity function clearly satisfies the condition.

Problem 68. Find all functions $f :: R\to R$ that obey

$$(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2)$$

Solution. The substitution $u=x+y, v=x-y$ is immediately seen and it simplifies our condition to $uf(v)-vf(u)=uv(u^2-v^2)$. Now if we set $v=0$ we get $uf(0)=0$ for all u so $f(0)=0$. Now if $uv\neq 0$ we divide by uv to get $\frac{f(v)}{v}-\frac{f(u)}{u}=u^2-v^2$ hence $\frac{f(v)}{v}+v^2=\frac{f(u)}{u^2}$ for all $u,v\neq 0$. Therefore $\frac{f(x)}{x}+x^2=c$ for some fixed c and all $x\neq 0$. Then

$f(x) = cx - x^3$ and this relation holds even for $x = 0$. We substitute to have $u(cv - v^3) - v(cu - u^3) = u^3v - v^3u$ which is true. All solutions are therefore of form $f(x) = cx - x^3$.

Problem 69. Find all pairs of functions $f, g :: R \to R$ such that g is a one-one function and $f(g(x) + y) = g(x + f(y))$.

Solution. As g is injective, $g(x) + y = g(z) + t$ implies $x + f(y) = z + f(t)$. This can be written as $f(y + g(x) - g(z)) = f(y) + x - z$. Leaving y and x constant we deduce that f is surjective. Then so is g, as seen from the main condition. Then picking up z_0 with $g(z_0) = 0$ we get $f(y + g(x)) = f(y) + x - z_0$. But $f(y + g(x)) = g(x + f(y))$. So $g(x+f(y)) = x+f(y)-z_0$ and then $g(x) = x-z_0$. Hence $f(x+y-z_0) = x + f(y) - z_0$ and if we set $y = z_0$ we get $f(x) = x + f(z_0) - z_0$. We conclude that both f, g are of form $f(x) + c$. Indeed all such pairs of functions satisfy the problem.

Problem 70. Let $g \colon C \to C$ be a given function, $a \in C$ and w the primitive cubic root of unity. Find all functions $f :: C \to C$ such that

$$f(z) + f(wz + a) = g(z)$$

Solution. Let $h(z) = wz + a$. Then $f(z) + f(h(z)) = g(z)$. Now $h(h(z)) = w(wz + a) + a = w^2z + (w + 1)a, h(h(h(z))) = w(w^2z + (w + 1)a) + a = w^3z + (w^2 + w + 1)a = z$. Therefore $h(h(h(z))) = z$. So $f(z) + f(h(z)) = g(z), f(h(z)) + f(h(h(z))) = g(h(z)), f(h(h(z))) + f(z) = f(h(h(z))) + f(h(h(h(z)))) = g(h(h(z)))$. If we solve the obtained non-singular linear system in $f(z), f(f(z)), f(f(f(z)))$ we get $f(z) = \frac{g(z)+g(w^2z+(w+1)a)-g(wz+a)}{2}$, and similar expressions for $f(h(z)), f(h(h(z)))$. As the system is non-singular, this f satisfies the condition.

Problem 71. Find all functions $f :: R_0 \to R_0$ that satisfy the conditions

i)$f(xf(y))f(y) = f(x + y)$ for $x, y \geq 0$

ii) $f(2) = 0$

iii) $f(x) > 0$ for $x \in [0; 2)$

Solution. Setting $y = 2$ into i) we get $f(x+2) = 0$ so f is identically zero on $[2; \infty)$. Thus $f(x) = 0$ if and only if $x \geq 2$. Now consider a fixed $y < 2$. From condition i) we deduce that $f(xf(y)) = 0$ if and only if $f(x + y) = 0$, or $xf(y) \geq 2$ if and only if $x + y \geq 2$. Hence if $x < 2 - y$ then $xf(y) < 2$ so $f(y) < \frac{2}{x}$. Taking $x \to (2 - y)_-$ we deduce $f(y) \leq \frac{2}{2-y}$. But setting $x = 2 - y$ we get $x + y = 2$ so $xf(y) \geq 2$ and $f(y) \geq \frac{2}{x} = \frac{2}{2-y}$. We conclude that $f(y) = \frac{2}{2-y}$. Indeed the function f with $f(x) = 0$ for $x \geq 2$ and $f(x) = \frac{2}{2-x}$ otherwise satisfies the conditions: if $y \geq 2$ this is trivial and if $y < 2$ we need to check $f(\frac{2x}{2-y})\frac{2}{2-y} = f(x+y)$. If $x \geq 2-y$ then $\frac{2x}{2-y} \geq 2$ and $x+y \geq 2$ and we get the true $0 = 0$. If $x < 2-y$ then $\frac{2x}{2-y} < 2$ so $f(\frac{2x}{2-y}) = \frac{2-y}{2-x-y}$ and $x+y < 2$ so $f(x+y) = \frac{2}{2-x-y}$ and so we get the identity $\frac{2}{2-y}\frac{2-y}{2-x-y} = \frac{2}{2-x-y}$, which is true.

Problem 72. Find all functions $f\colon R \to R$ that satisfy

$$f(f(x-y)) = f(x) - f(y) + f(x)f(y) - xy$$

.

Solution. By setting $y = x$ we get $f(f(0)) = f^2(x) - x^2$ thus $f(x) = \pm\sqrt{x^2+d}$ for any x where $d = f(f(0))$. $d \geq 0$ otherwise we couldn't extract square root for $x = 0$. Now for $x = 0$ we get $f(0) = \pm\sqrt{d}$ and for $x = f(0)$ we have $f(f(0)) = \sqrt{2d}$. So $d = \sqrt{2d}$ which is possible for $d = 0, d = 2$. Now let's consider $y = x-1$. We get $f(f(1)) = f(x) - f(x-1) + f(x)f(x-1) - x(x-1)$. Consider x be sufficiently big. Then $f(x)$ and $f(x-1)$ have the same sign otherwise $f(f(1)) = (f(x)-1)(f(x-1)-1) - x(x-1) < -x(x-1)$ as $|f(x)|, |f(x-1)| > 1$ thus $f(x)-1, f(x-1)-1$ would have the same signs as $f(x), f(x-1)$ so $(f(x)-1)(f(x-1)-1)$ would be negative. However $|x(x-1)| > |f(f(1))| = \sqrt{1+2d}$ for sufficiently big x, contradiction. Thus $f(f(1)) = f(x) - f(x-1) + f(x)f(x-1) - x(x-1) = \pm(\sqrt{x^2+d} - \sqrt{(x-1)^2+d}) + \sqrt{(x^2+d)((x-1)^2+d))} - x(x-1)$. Now we use the identity $a - b = \frac{a^2-b^2}{a+b}$ to get $\sqrt{x^2+d} - \sqrt{(x-1)^2+d} = \frac{2x-1}{\sqrt{x^2+d}+\sqrt{(x-1)^2+d}} \sim 1$, $\sqrt{(x^2+d)((x-1)^2+d))} - x(x-1) = \frac{d(2x^2-2x+1)}{\sqrt{(x^2+d)((x-1)^2)+d}+x(x-1)} \sim d$. Therefore when x is sufficiently big $right - handside \sim \pm 1 + d$ so actually $f(f(1)) = \pm 1 + d$. But $f(f(1)) = \pm\sqrt{1+2d}$, which holds only for $d = 0$, as $\sqrt{5} \neq \pm 1 + 2$ which would hold for $d = 2$. So we have $f(x) = \pm x$. Now we prove $f(x) = x$. Indeed assume $f(x_0) = -x_0 . x_0 \neq 0$. Then $f(f(x-x_0)) = f(x) - f(x_0) + f(x)f(x_0) - xx_0 = (f(x)-1)(f(x_0)+1) - xx_0 + 1 = (x-1)(-x_0+1) - xx_0 + 1$ or $(-x-1)(-x_0+1) - xx_0 + 1$. But $f(f(x-x_0)) \in \{x-x_0, x_0-x\}$. Thus $\{x-x_0, x_0-x\} \bigcap \{(x-1)(-x_0+1) - xx_0+1, (-x-1)(-x_0+1) - xx_0+1\} = \{-2xx_0 + x + x_0, x_0 - x\}$, as the common value $f(f(x-x_0)) = f(x) - f(x_0) + f(x)f(x_0) - xx_0$ belongs to both sets. Now the equalities $x - x_0 = -2xx_0 + x + x_0, x - x_0 = x_0 - x, x_0 - x = -2xx_0 + x + x_0$ are equivalent to $x = 1, x = x_0, x(x_0-1) = 0$ which have at most one solution each for $x_0 \neq 1$, therefore for $x_0 \neq 1$ we have $f(f(x-x_0)) = x_0 - x, f(x) = -x$

for all x except three values. However this is impossible: if we take x such that neither x nor $x_0 - x$ is among these three values then $f(f(x-x_0)) = f(x_0-x) = x-x_0$ while we have $f(f(x-x_0)) = x_0-x$. Contradiction. So $f(x) = x$ for all $x \neq 1$. Finally if we set $x = 3, y = 1$ we get $2 = f(2) = f(3) - f(1) + f(3)f(1) - 3 = 3 - f(1) + 3f(1) - 3 = 2f(1)$ hence $f(1) = 1$. Thus f is the identity function. The identity function satisfies the condition.

Problem 73. Find all functions $f\colon Q \to Q$ such that

$$f(\frac{x+y}{3}) = \frac{f(x)+f(y)}{2}$$

for all non-zero x, y in the domain of f.

Solution. If we set $2x = y$ we get $f(x) = \frac{f(x)+f(2x)}{2}$ for any x therefore $f(2x) = f(x)$. This suggest the fact that f is a constant function. Indeed, let S be the set of all integers k for which $f(kx) = f(x)$ for all x. Then we know that $1, 2 \in S$. Also if $a, b \in S$ then $3a-b \in S$ (just set $x \to (3a-b)x, y \to bx$), and clearly if $a, b \in S$ then $ab \in S$. Now we can prove by induction on $|k|$ that $k \in S$ for all $k \in Z^*$. Indeed, $4 = 2 \cdot 2 \in S$, $-1 = 3 \cdot 1 - 4 \in S, -2 = 3 \cdot (-1) - (-1) \in S$ and the basis in proven. To prove the induction step, we have to show that every k with $|k| \geq 3$ can be written as $3a-b$ where $|a|, |b| < k$. Indeed, if we denote $a = [\frac{k}{3}], b = -3\{\frac{k}{3}\}$ then $k = 3a-b$ and $|a| \leq |\frac{k}{3}| < k, |b| \leq 2 < k$. Thus $Z^* \subset S$. Therefore if $a, b \in Q \setminus \{0\}$ and $\frac{a}{b} = \frac{p}{q}, p, q \in Z$ then $qa = pb$ $f(a) = f(qa) = f(pb) = f(b)$ so f is constant on $Q \setminus \{0\}$. Also if $x \neq 0$ then setting $y = 0$ we get $f(\frac{x}{3}) = \frac{f(x)+f(0)}{2}$ which as $f(\frac{x}{3}) = f(x)$ implies that $f(0) = f(x)$. So f in constant on Q and all constant functions satisfy the requirements.

Problem 74. Find all polynomials $P(x, y) \in R^2[x, y]$ that satisfy $P(x+a, y+b) = P(x, y)$ where a, b are some reals, not both zero.

Solution. Assume that $b \neq 0$. Consider the polynomial $R \in R^2[x, y]$ defined by $R(x, y) = P(x + \frac{a}{b}y, y)$. Then we can tell $P(x, y) = R(x - \frac{a}{b}y, y)$. So $P(x+a, y+b) = P(x, y)$ can be rewritten in terms of R as $R((x+a) - \frac{a}{b}(y+b), y+b) = R(x - \frac{a}{b}y, y)$ or $R(x - \frac{a}{b}y, y+b) = R(x - \frac{a}{b}y, y)$. If we set $x \to x - \frac{a}{b}y$ we get $R(x, y) = R(x, y+b)$. Then by induction on n $R(x, y) = R(x, y+b) = \ldots = R(x, y+nb)$. Set $Q_x(y) = R(x, y)$. Then $Q_x(y) = Q_x(y+b) = \ldots = Q_x(y+nb)$ and taking $n > degQ_x$ we get Q_x is constant so $Q_x(y) = Q_x(0) = R(x, 0)$. As $R(x, 0)$ is a polynomial in x, R is a polynomial in x so $R(x, y) = Q(x)$ for some polynomial $Q \in R[x]$. Then $P(x, y) = R(x - \frac{a}{b}y, y) = Q(x - \frac{a}{b}y) = Q(bx - ay)$. Any such polynomial satisfies the condition, as $P(x+a, y+b) = Q(b(x+a) - a(y+b)) = Q(bx - ay) = P(x, y)$. If $b = 0$ then $a \neq 0$ and we repeat the reasoning by replacing b with a and y with x to get again $P(x, y) = Q(bx - ay)$.

Problem 75. (ISL 1996) Find a bijection $f\colon N_0 \to N_0$ that satisfies $f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n)$.

Solution. If we denote $g(3k+1) = f(k)$ then the condition becomes $g((3m+1)(3n+1)) = 4g(3m+1)g(3n+1) + g(3m+1) + g(3n+1)$. Next if we denote $4g(x) + 1 = h(x)$ the condition is rewritten as $h((3m+1)(3n+1)) = h(3m+1)h(3n+1)$. Now we understand that we need to construct a multiplicative bijection of A into B where $A = \{3k+1 | k \in N\}, B = \{4k+1 \in N\}$. We can set $h(1) = 1$. Now consider U the set of all primes of form $3k-1$, V the set of all primes of form $3k+1$, X the set of all primes of form $4k-1$, Y the set of all primes of form $4k+1$. All these four sets are infinite. So we can provide a bijection h between U and X and between V and Y. Now we extend it by multiplicativity to the whole A. We prove this is the required bijection. Indeed, assume that $3k+1 = \prod p_i^{a_i} \prod q_i^{b_i}$ where $p_i \in U, q_i \in B$. Then p_i are $1-$ mod 3, q_i are 1 mod 3 so $\sum a_i$ must be even. Then $h(3k+1) = \prod h(p_i)^{a_i} \prod h(q_i)^{b_i}$ where $h(p_i) \in X, h(q_i) \in Y$. As $h(p_i)$ is -1 mod 4 and $h(q_i)$ is 1 mod 4 but $\sum a_i$ is even we conclude that $h(3k+1)$ is 1 mod 4 so $h(3k+1) \in B$. We can analogously prove the reverse implication: assume that $4k+1 = \prod p_i^{a_i} \prod q_i^{b_i}$ where $p_i \in X, q_i \in Y$. As p_i are -1 mod 4 but q_i are 1 mod 4 $\sum a_i$ must be even. Then $x = \prod h^{-1}(p_i)^{a_i} \prod h^{-1}(q_i)^{b_i}$ satisfies $h(x) = 4k+1$. Moreover as $h^{-1}(p_i)$ are -1 mod 3, $h^{-1}(q_i)$ are 1 mod 3 and $\sum a_i$ is even, we conclude that x is 1 mod 3 so $x \in A$. Finally h is injective because of the uniqueness of the decomposition of a number into product of primes.

Problem 76. (IMO '1999) Find all functions $f\colon R \to R$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in R$.

Solution. Setting $x = f(y)$ in the given equation gives

$$f(x) = \frac{c + 1 - x^2}{2}, \tag{1}$$

where $c = f(0)$. On the other hand if $y = 0$ we get $f(x - c) - f(x) = f(c) + cx - 1$. Hence $f(-c) - c = f(c) - 1$ which shows that $c \neq 0$.

Thus for any $x \in R$ there is $t \in R$ such that $x = y_1 - y_2$, where $y_1 = f(t-c)$ $y_2 = f(t)$. Now using the given equation we get

$$\begin{aligned} f(x) = f(y_1 - y_2) &= f(y_2) + y_1 y_2 + f(y_1) - 1 = \\ &= \frac{c+1-{y_2}^2}{2} + y_1 y_2 + \frac{c+1-{y_1}^2}{2} - 1 = \\ &= c - \frac{(y_1-y_2)^2}{2} = c - \frac{x^2}{2}. \end{aligned}$$

This together with (1) gives $c = 1$ and $f(x) = 1 - \dfrac{x^2}{2}$. Conversely, it is easy to check that this function satisfies the given equation.

Problem 77. Find all functions $f\colon {R_0}^+ \to {R_0}^+$ such that

$$f(y)f(xf(y)) = f(x+y)$$

for all $x, y \in {R_0}^+$.

Solution. Setting $x = 0$ gives $f(0)f(y) = f(y)$. Hence $f^2(0) = f(0)$. If $f(0) = 0$ then $f(y) = 0$ for all $y \in {R_0}^+$. Hence we may assume that $f(0) = 1$. We shall consider two cases.

1. Let $f(y) = 0$ for some $y > 0$. Set $a = inf\{y > 0 : f(y) = 0\}$. For any $x > a$ there is y such that $a < y < x$ and $f(y) = 0$. Then

$$f(x) = f(y)f((x-y)f(y)) = 0.$$

So, if $a = 0$ then $f(x) = 0$ for all $x > 0$.

Suppose now that $a > 0$ and let $0 < y < a$. For any $\varepsilon > 0$ we have

$$f(y)f((a+\varepsilon-y)f(y)) = f(a+\varepsilon) = 0.$$

Hence $(a+\varepsilon-y)f(y) \geq a$ since $f(y) > 0$. On the other hand

$$f\left(\frac{a+\varepsilon}{f(y)} + y\right) = f(a+\varepsilon)f(y) = 0$$

and therefore $\dfrac{a+\varepsilon}{f(y)} + y \geq a$. Now letting $\varepsilon \to 0$ in the above two inequalities gives $f(y) = \dfrac{a}{a-y}$ for any $0 < y < a$. In particular $f\left(\dfrac{a}{2}\right) = 2$ and setting $x = y = \dfrac{a}{2}$ in the given equation gives $2f(a) = f(a)$, i.e. $f(a) = 0$.

2. Let $f(y) > 0$ for any $y > 0$. First we shall show that $f(y) \leq 1$ for any $y \geq 0$. Indeed, assume that $f(y) > 1$ for some $y > 0$. Then we set $x = \dfrac{y}{f(y)}$ in the given equation and get $f(y) = 1$, a contradiction.

The inequality $f(y) \le 1$ together with the given equation shows that the function f is decreasing.

Suppose that $f(y) = 1$ for some $y > 0$. Then $f(x+y) = f(x)$ for any $x \ge 0$ and using the fact that the function f is decreasing we get that $f(x) = 1$ for all $x \ge 0$.

It remains to consider the case when $f(y) < 1$ for any $y > 0$. Then the function f is strictly decreasing and hence injective. Now the identity

$$f(y)f(xf(y)) = f(x+y) = f(xf(y) + y + x(1-f(y))) =$$

$$= f(xf(y))f((y + x(1-f(y)))f(f(xf(y)))$$

implies that

$$y = (y + x(1-f(y)))f(xf(y)).$$

Setting $y = 1$, $xf(1) = z$ and $\dfrac{f(1)}{1-f(1)} = a$ we get that $f(z) = \dfrac{a}{a+z}$ for any $z > 0$.

Thus all the functions we have found are the following:

$$f(x) = 0,\ \ f(x) = 1,\ \ f(x) = \frac{a}{x+a},$$

$$f(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x > 0, \end{cases}$$

$$f(x) = \begin{cases} \dfrac{a}{a-x} & \text{for } 0 \le x < a \\ 0 & \text{for } x \ge a, \end{cases}$$

where $a > 0$ is an arbitrary constant. It is easy to check that the first four functions are solutions of the problem. Now we shall show that the fifth function also gives a solution.

If $x, y \ge 0$ and $x + y < a$ then

$$f(y)f(xf(y)) = \frac{a}{a-y} f\left(\frac{ax}{a-y}\right) = \frac{a}{a-y} . \frac{a-y}{a-(x+y)} = f(x+y).$$

Let $x, y \ge 0$ and $x + y \ge a$. Then $f(x+y) = 0$. If $y \ge 0$ then $f(y) = 0$. If $y < a$ then $\dfrac{ax}{a-y} \ge a$ and $xf(y) \ge a$. Hence $f(xf(y)) = 0$. Therefore in both cases we have

$$f(y)f(xf(y)) = 0 = f(x+y).$$

Remark. The above problem under the additional conditions $f(2) = 0$ and $f(x) \neq 0$ for $0 \leq x < 2$ is a problem from IMO '1986 and was solved before, the only solution being the function

$$f(x) = \begin{cases} \dfrac{2}{2-x} & \text{for } 0 \leq x < 2 \\ 0 & \text{for } x \geq 2. \end{cases}$$

Exercises

Problem 78. Find all continuous functions $f\colon R \to R$ that satisfy

$$f(x)y + f(y)x = (x+y)f(x)f(y)$$

Problem 79. Find all continuous functions $f\colon R \to R$ for which

$$f(x+y) - f(x-y) = 2f(xy+1) - f(x)f(y) - 4$$

Problem 80.(ISL 2000)Find all pairs of functions $f :: R \to R$ that obey the identity

$$f(x + g(y)) = xf(y) - yf(x) + g(x)$$

Problem 81. If $a > 0$ find all continuous functions f for which

$$f(x+y) = a^{xy} f(x)f(y)$$

Problem 82. Find all continuous functions $f\colon R \to R$ that satisfy

$$f(x+y)\frac{f(x) + f(y) + 2f(x)f(y)}{1 - f(x)f(y)}$$

Problem 83. Find all continuous function $f \cdot (a;b) \to R$ that satisfy $f(xyz) = f(x) + f(y) + f(z)$ whenever $xyz, x, y, z \in (a;b)$, where $1 < a^3 < b$.

Problem 84. Find all continuous functions $f\colon R \to R$ that satisfy

$$f(xy) = xf(y) + yf(x)$$

Problem 85. Find all functions $f\colon Q^+ \to Q^+$ that obey the relations

$$f(x+1) = f(x) + 1$$

if $x \in Q^+$ and

$$f(x^3) = f(x)^3$$

if $x \in Q^+$.

Problem 86. Show that if $f\colon R \to R$ satisfies

$$f(xy) = xf(x) + yf(y)$$

then f is identically zero.

Problem 87. Find all functions $f\colon R \to R$ that obey the condition

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

.

Problem 88. Let $k \in R^+$. Find all functions $f\colon [0,1]^2 \to R$ such that the following four conditions hold for all $x, y, z \in [0;1]$:

i)

$$f(f(x,y),z) = f(x, f(y,z))$$

ii)

$$f(x,y) = f(y,x)$$

iii)

$$f(x,1) = x$$

iv)

$$f(zx, zy) = z^k f(x,y)$$

Problem 89. Find all continuous functions $f\colon R \to R$ that satisfy $3f(2x+1) = f(x) + 5x$.

Problem 90. (Shortlisted problems for IMO '2002) Find all functions $f\colon R \to R$ such that

$$f(f(x) + y) = 2x + f(f(y) - x)$$

for all $x, y \in R$. **Problem 91.** (Bulgaria, 1996) Find all functions

$f\colon R \to R$ such that

$$f(f(x) + xf(y)) = xf(y+1)$$

for all $x, y \in R$.

Problem 92. (BMO '1997 and BMO '2000) Find all functions $f\colon R \to R$ such that

$$f(xf(x) + f(y)) = f^2(x) + y$$

for all $x, y \in R$.

Problem 93. (USA, 2002) Find all functions $f\colon R \to R$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all $x, y \in R$.

Fixed Points

Problem 94. (IMO '1983) Find all functions $f\colon R^+ \to R^+$ such that:

(*i*) $f(xf(y)) = yf(x)$ for all $x, y \in R^+$;

(*ii*) $\lim\limits_{x\to+\infty} f(x) = 0$.

Solution. It follows from (*i*) that $f(xf(x)) = xf(x)$ for all $x > 0$. Then it follows by induction on n that if $f(a) = a$ for some $a > 0$ then $f(a^n) = a^n$ for any $n \in N$. Note also that $a \le 1$ since otherwise

$$\lim_{n\to\infty} f(a^n) = \lim_{n\to\infty} a^n = +\infty,$$

a contradiction to (*ii*).

On the other hand $a = f(1.a) = f(1.f(a)) = af(1)$. Hence

$$1 = f(1) = f(a^{-1}a) = f(a^{-1}f(a)) = af(a^{-1}),$$

i.e. $f(a^{-1}) = a^{-1}$. Thus we have (as above) $f(a^{-n}) = a^{-n}$ for all $n \in N$ and therefore $a^{-1} \le 1$.

In conclusion, the only $a > 0$ such that $f(a) = a$ is $a = 1$. Hence the identity $f(xf(x)) = xf(x)$ implies that $f(x) = \dfrac{1}{x}$ for any $x > 0$. It is easy to check that this function satisfies the conditions (*i*) and (*ii*) of the problem.

Problem 95. (IMO '1994) Let S be the set of all real numbers greater than -1. Find all functions $f\colon S \to S$ such that

(*i*) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all $x, y \in S$;

(*ii*) $\dfrac{f(x)}{x}$ is strictly increasing in the intervals $(-1, 0)$ and $(0, +\infty)$.

Solution. If $x = y > -1$ we have from (*i*) that

$$f(x + (1 + x)f(x)) = x + (1 + x)f(x). \tag{1}$$

On the other hand (*ii*) implies that the equation $f(x) = x$ has at most one solution in each of the intervals $(-1, 0)$ and $(0, +\infty)$.

Suppose that $f(a) = a$ for some $a \in (-1, 0)$. Then (1) implies that $f(a^2 + 2a) = a^2 + 2a$ and therefore $a^2 + 2a = a$ since $a^2 + 2a = (a+1)^2 - 1 \in (-1, 0)$. Hence $a = -1$ or $a = 0$, contradiction. The same arguments show that the equation $f(x) = x$ has no solutions in the interval $(0, +\infty)$.

Then we conclude from (1) that $x + (1+x)f(x) = 0$, i.e. $f(x) = -\dfrac{x}{1+x}$ for any $x > -1$. It is easy to check that this function satisfies the conditions (*i*) and (*ii*) of the problem.

Problem 96. (Tournament of the towns '1996) Prove that there is no function $f\colon R \to R$ such that

$$f(f(x)) = x^2 - 1996$$

for any $x \in R$.

Solution. We shall prove the following more general result.

Proposition. Let $g(x)$ be a quadratic function such that the equation $g(g(x)) = x$ has at least three different real roots. Then there is no function $f\colon R \to R$ such that

$$f(f(x)) = g(x) \tag{1}$$

for all $x \in R$.

Proof. The fixed points of $g(x)$ are also fixed points of the forth degree polynomial $h(x) = g(g(x))$. Hence it follows by the given conditions that $g(x)$ has one or two real fixed points. Denote them by x_1 and x_2. Then $h(x)$ has one or two real fixed points, different from x_1 and x_2. Denote them by x_3 and x_4. The identity

$$f(g(x)) = f(f(f(x))) = g(f(x))$$

implies that $\{f(x_1), f(x_2)\} = \{x_1, x_2\}$. On the other hand we have

$$f(f(g(x))) = f(g(f(x))) \text{ and } f(f(f(g(x)))) = f(f(g(f(x)))),$$

i.e. $f(h(x)) = h(f(x))$. Hence $\{f(x_3), f(x_4)\} \in \{x_1, x_2, x_3, x_4\}$. Suppose that $f(x_l) = x_k$ for some $k \in \{1, 2\}$ and $l \in \{3, 4\}$. Then

$$x_l = h(x_l) = f(f(f(f(x_l)))) = f(g(x_k)) = f(x_k) \in \{x_1, x_2\},$$

a contradiction. Hence $f(x_3) = x_3$ if $x_3 = x_4$ and $\{f(x_3), f(x_4)\} = \{x_3, x_4\}$ if $x_3 \neq x_4$. In both cases we have $g(x_3) = f(f(x_3)) = x_3$, a contradiction. Thus the proposition is proved. Turning back to the problem we note that the equation $g(g(x)) = (x^2 - 1996)^2 - 1996 = x$ has four different real roots since $(x^2 - 1996)^2 - 1996 - x = (x^2 - 1996 - x)(x^2 + x - 1996)$.

Remark. Set $g(x) = ax^2 + bx + c$. Then

$$g(g(x)) - x = (ax^2 + (b-1)x + c)(a^2x^2 + a(b+1)x + ac + b + 1).$$

Therefore the four roots of the equation $g(g(x)) = x$ are equal to:

$$\frac{1-b+\sqrt{D}}{2a},\quad \frac{1-b-\sqrt{D}}{2a},\quad \frac{-1-b+\sqrt{D-4}}{2a} \quad \text{and} \quad \frac{-1-b-\sqrt{D-4}}{2a},$$

where $D = (b-1)^2 - 4ac$. All these roots are real if and only if $D \geq 4$. If $D > 4$ then all the roots are different whereas for $D = 4$ one of them is equal to $\dfrac{3-b}{2a}$ and the other three are equal to $-\dfrac{1+b}{2a}$.

The proposition proved above says that if $D > 4$, then there are no functions $f\colon R \to R$ such that $f(f(x)) = g(x)$ for all $x \in R$. On the other hand if $D = 4$ then there are infinitely many continuous functions $f\colon R \to R$ satisfying the above equation.

Additive Cauchy Equation

Problem 97. (AMM 2001) Find all functions $f\colon R \to R$ if

$$f(x^2 + y + f(y)) = 2y + f^2(x)$$

for all reals x, y.

Solution. If we fix x then the right-hand side is surjective on R therefore f is surjective. Also if $y_1 + f(y_1) = y_2 + f(y_2)$ then writing the condition for some x and $y = y_1, y_2$ the get $2y_1 = 2y_2$ thus $x + f(x)$ is injective. Thus for some c we have $f(c) = 0$. This implies $f(c^2 + y + f(y)) = 2y$. Pick up now two fixed a, b. Set $c = b + f(b) - a - f(a) > 0$. If $x > a + f(a)$ then there is an u such that $x = u^2 + a + f(a), x + d = u^2 + b + f(b)$. Then $f(x) = 2a + f^2(x), f(x+c) = 2b + f^2(x)$. Therefore we conclude that $f(x+c) - f(x) = 2(a-b)$ for all sufficiently big x. This means that $f(x+c) = f(x) + d$ for $d = 2(b-a)$ and $c + d = 3(b-a) + f(b) - f(a)$ and from here $f(x+nc) = f(x) + nd$ for all sufficiently big x and any fixed natural n. If $d < 0$ then $f(x+nc) < 0$ for all sufficiently big n. However if $x+nc > f(0)$ then $x+nc = u^2+f(0)$ and applying the condition for u and 0 we get $f(x+nc) = f^2(u) > 0$ contradiction. So $d > 0$. Then $f(x+nc)+x+nc = f(x)+x+n(c+d)$.

This means $f(x)+x$ takes arbitrarily large values. Take now a y. We then get $f((x+c)^2+y+f(y)) = 2y+(f(x)+d)^2$ for $x \geq x_0$. Assume now that we have $f(x+u) = f(x)$ for all sufficiently big x when $u > 0$. Then we get $f((x+u)^2+y+f(y)) = f(x^2+y+f(y))$. However we can choose x such that $2xu+u^2 = lc, l > 0$ and then choosing y such that $y+f(y)$ is big enough we have $f((x+u)^2+y+f(y)) = f(x^2+2xu+u^2+y+f(y)) = f(x^2+y+f(y))+ld$ contradiction. Consider now $g(x) = x+f(x)$. Take a fixed s and $x \neq y$ with $g(x+s)-g(x) \leq g(y+s)-g(y)$. Then we have $f(t+g(x+s)-g(x)) = f(t)+2s, f(t+g(y+s)-g(y)) = f(t)+2s$ for all $t \geq t_0$ and if we set $z = t+g(x+s)-g(s)), v = (g(y+s)-g(y))-(g(x+s)-g(x))$ then the identity turns to $f(z+v) = f(z)$ for all $z \geq 0$. As $v \geq 0$ we must have $v = 0$ as we have proven such a relation is impossible for $v \neq 0$. Hence $g(x+s)-g(x) = g(y+s)-g(y)$. As x, y were chose arbitrarily we conclude that $g(x+s)-g(x)$ is independent of x thus $g(x+s)-g(x) = g(s)-g(0)$ so $g(x+s)+g(0) = g(x)+g(s)$ thus $h(x) = g(x)-g(0)$ is an additive function, hence so is $f(x)-g(0) = f(x)-f(0)$. We then get $f(x^2+g(y)) = f(x^2)+f(g(y))-f(0) = 2y+f^2(x)$ hence $f(x^2)-f^2(x) = 2y-f(g(y))$. If we now fix y then we get $f(x^2) = f^2(x)+e$ for fixed e hence $f(x) \geq -e$ for $x \geq 0$. Hence $f(x)-f(0)$ is additive and bounded below, and then using an already known problem we deduce $f(x)-f(0) = rx$ for some r hence $f = rx+s$ is linear. The condition $f(x^2)-f^2(x) = 2y-f(g(y))$ now implies $rx^2+s-(rx+s)^2 = 2y-r((r+1)y+s)-s$ or $r(1-r)x^2-2rsx-s^2 = (2-r(r+1))y-(r+1)s$ which is possible only when $r(1-r) = 2rs = 2-r(r+1) = 0$. Then $r = 0$ or $r = 1$. If $r = 0$ then $2-r(r+1) \neq 0$. So $r = 1$ and then $s = 0$. So f is the identity function. It can be easily verified that the identity function satisfies the condition.

Problem 98. Find all functions $f, g, h \colon R \to R$ such that

$$f(x+y) = f(x)g(y)+h(y)$$

Solution. Set $y = 0$ to get $f(x) = f(x)g(0)+h(0)$ so $f(x)(1-g(0)) = h(0)$ and either $1-g(0) = 0$ or $f(x) = \frac{h(0)}{1-g(0)}$. In the second case f is a constant then if we set $f(x) = c$ we get $c = cg(y)+h(y)$ and any functions g, h with $h(x) = c-cg(x)$ satisfy the condition. So assume that f is not constant thus $g(0) = 1$ and $h(0) = 0$. Now set $x = 0$ to get $f(y) = f(0)g(y)+h(y)$. Set $f(0) = c$. Then $f(x) = cg(x)+h(x)$ so we substitute to get $cg(x+y)+h(x+y) = cg(x)g(y)+h(x)g(y)+h(y)$. Symmetrize the condition to get $cg(x+y)+h(x+y) = cg(x)g(y)+h(x)g(y)+h(y) = cg(x)g(y)+h(y)g(x)+h(x)$ therefore

$h(x)g(y)+h(y) = h(y)g(x)+h(x)$ hence $h(x)(g(y)-1) = h(y)(g(x)-1)$. Now we distinguish some cases:

a) $g(x) = 1$ for all x. Then the condition transforms to $h(x+y) = h(x)+h(y)$ hence h is an additive function, $f = h+c, g = 1$ and we check that this functions indeed satisfy the conditions.

b) $h(x) = 0$ for all x. Then the condition becomes $cg(x+y) = cg(x)g(y)$ which implies that either c is zero so $f = h = 0, g$ is any function, which satisfy the condition or $c \neq 0$ and g is a multiplicative function while $f = cg, h = 0$, again satisfy the condition.

c) There are x_0, y_0 for which $g(x_0) \neq 1, h(y_0) \neq 0$. If we set $x = x_0, y = y_0$ we deduce that $h(x_0)(g(y_0)-1) \neq 0$ so $h(x_0), g(y_0)-1$ are not zero. Now set $y = y_0$ and we get $h(x)(g(y_0)-1) = h(y_0)(g(x)-1)$ thus $h(x) = (g(x)-1)\frac{h(y_0)}{g(y_0)-1}$. If we set $\frac{h(y_0)}{g(y_0)-1} = d$ then $h(x) = dg(x)-d$. Thus our condition becomes $(c+d)g(x+y)-d = cg(x)g(y)+(dg(x)-d)g(y)+dg(y)-d = (c+d)g(x)g(y)-d$ hence either $c+d = 0$ or g is additive multiplicative. In the first case we get $d = -c$ so $f = cg(x)-cg(x)+c = c, h(x) = -cg(x)+c$ which satisfy the condition. In the second case we get g a multiplicative function, $h = dg-d, f = cg+h = (c+d)g-d$ and again they satisfy the condition.

We have exhausted all solutions. Note that we have used the terms "additive function" and "additive multiplicative" function for the solutions because these function cannot be otherwise defined. In fact there are many different additive functions, even non-continuous, and the same with additive multiplicative functions ($g(x)$ is additive multiplicative then $g(x) = g^2(\frac{x}{2})$ so $g(x) \geq 0$. If $g(x_0) = 0$ then $g(x) = g(x-x_0)g(x_0) = 0$, otherwise $g(x) > 0$ and then $ln(g)$ makes sense and is an additive function).

Problem 99. Prove that any additive function f on R^+ which is bounded from below (above) on an interval of R^+ has the form $f(x) = f(1)x$ for all $x \in R^+$.

Solution. Set $g(x) - f(1)x$. Then $g(x)$ is an additive function with $g(1) = 0$. It follows by induction that $g(nx) = ng(x)$ for any $x \in R^+, n \in N$. In particular $g(n) = 0$. Moreover, for any $k, l \in N$ we have $lg\left(\frac{k}{l}\right) = g(k) = 0$, i.e. $g\left(\frac{k}{l}\right) = 0$. Without loss of generality we may assume that there are constants c and d such that $f(x) > c$ and $f(1)x < d$ in an interval of R^+. Then the identity $g(x+r) = g(x)$ for all $x \in R^+,\ r \in Q^+$ shows that $g(x) > c-d$ for all $x \in R^+$. Hence $g(x) = \dfrac{g(nx)}{n} > \dfrac{c-d}{n}$ for any $n \in N$, i.e. $g(x) \geq 0$ for any $x > 0$.

This implies that the function $g(x)$ is increasing since for $x > y > 0$ we have $g(x) = g(x-y) + g(y) \geq g(y)$. Now for any $x \in R^+$ take $r, s \in Q^+$ such that $r > x > s$. Then $0 = g(r) \geq g(x) \geq g(s) = 0$ which shows that $g(x) = 0$.

Remark. Note that the statement of Problem 99 holds true if we replace R^+ by R. Note also that any function which is continuous at a part or monotone in an interval of $R^+(R)$ satisfies the conditions of the problem. On the other hand there are additive functions that are unbounded in any interval of $R(R^+)$.

Problem 100. (Tuymaada 2003) Find all continuous functions $f\colon R^+ \to R$ that satisfy

$$f(x+\frac{1}{x}) + f(y+\frac{1}{y}) = f(x+\frac{1}{y}) + f(y+\frac{1}{x})$$

for any $x, y \in R^+$.

Solution. Linear functions satisfy our condition, and we prove that only these do. If we replace y by $\frac{1}{y}$ we can enhance the condition:

$$f(x+\frac{1}{x}) + f(y+\frac{1}{y}) = f(x+\frac{1}{y}) + f(y+\frac{1}{x}) = f(x+y) + f(\frac{1}{x}+\frac{1}{y})$$

Now pick up a fixed $1 < y < C$ for some $C > 1$ and let x be sufficiently big. Rewriting the second two parts of the condition as

$$f(x+y) - f(x+\frac{1}{y}) = f(\frac{1}{x}+y) - f(\frac{1}{x}+\frac{1}{y})$$

Now as f is continuous then f is uniformly continuous on $[\frac{1}{C}; 2C]$ hence for any desired $\epsilon > 0$ there exists an $a > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in [\frac{1}{C}; 2C], |x - y| < a$. Thus taking $x > max\{C, \frac{1}{a}\}$ we deduce that $|f(\frac{1}{x}+\frac{1}{y}) - f(\frac{1}{y})|, |f(\frac{1}{x}+y) - f(y)| < \epsilon$. This means $(f(\frac{1}{x}+y) - f(\frac{1}{x}+\frac{1}{y})) - (f(y) - f\frac{1}{(}y)) < 2a$ and we have proven that $\lim_{x\to\infty} f(x+y) - f(x+\frac{1}{y}) = f(y) - f(\frac{1}{y})$ and the convergence is uniform on any interval $[1; C]$ for y. Now any $b > 0$ can be written as $y - \frac{1}{y}$ in a unique way for $y > 1$. Set $g(b) = f(y) - f(\frac{1}{y})$. Then we see that $g(b) = \lim_{x\to\infty} f(x+b) - f(x)$. It is from here pretty clear that $g(a+b) = g(a) + g(b)$ and since g is continuous we find that $g(x) = cx$. Now we can suppose that $c = 0$ otherwise take $f(x) - cx$ instead of f. So we then have $f(x) = f(\frac{1}{x})$ and also $\lim_{x\to\infty}(f(x+a) - f(x)) = 0$ uniformly for $a \in [0; C]$. Now take y be fixed and let $x \to \infty$. The condition $f(x+\frac{1}{x}) + f(y+\frac{1}{y}) = f(x+\frac{1}{y}) + f(y+\frac{1}{x})$ is rewritten as

$f(x+\frac{1}{x})-f(x+\frac{1}{y})=f(y+\frac{1}{x})-f(y+\frac{1}{y})$. The right-hand side now tends to zero according to the limit result obtained just before, and the left-hand side tends to $f(x+\frac{1}{x})-f(x)$. Thus $f(x+\frac{1}{x})=f(x)$. This is enough to prove that f is constant. Indeed, let $1\leq a<b$. Consider $x_0=a, y_0=b, x_{i+1}=x_i+\frac{1}{x_i}, y_{i+1}=y_i+\frac{1}{y_i}$. As $x+\frac{1}{x}$ is increasing for $x\geq 1$ and also $(x+\frac{1}{x})^2\geq x^2+2$ we have $x_i<y_i$ and x_i, y_i grow infinitely large. Finally we can note that $|x+\frac{1}{x}-y-\frac{1}{y}|<|x-y|$ for $x,y\geq 1$ therefore $|x_i-y_i|\leq|a-b|$. Therefore $\lim_{n\to\infty}(f(x_n)-f(y_n))=0$. But $f(x_n)=f(a), f(y_n)=f(b)$. We conclude that $f(a)=f(b)$ and this finishes the proof.

Problem 101. (Sankt-Petersburg) Find all continuous functions $f\colon R\to R$ that satisfy

$$f(f(x+y))=f(x)+f(y)$$

Solution. We observe that f satisfies the equation if and only if $f+c$ satisfies it, where c is any real. Therefore we can suppose that $f(0)=0$. Then by letting $x=0$ we get $f(f(x))=f(x)$. Therefore f is the identity function on $Im(f)$. We now try to prove f is the identity function, or $f=0$. To do this, it suffices to prove that $Im(f)=R$ or $Im(f)=0$. Indeed, as f is continuous, we see that if $f(t)\neq 0$ then $Im(f)$ contains the image of $[0;t]$ under f which is an interval which contains zero. Without loss of generality $f(t)>0$ otherwise assume work with $-f$. Let A be the set of all a for which $[0;a]$ belongs to Imf. Let $b=supA$. If $b<\infty$ then we may find $c=f(x)$ such that $c?\frac{b}{2}$. Then $f(f(2x))=2c>b$ therefore from the continuity of f we deduce $[0;2c]$ which contradicts the maximality of b. Therefore $b=\infty$ and R^+ belongs to Imf, and we are done after noting that $0=f(f(0))=f(f(x-x))=f(x)+f(-x)$ so $f(-x)=-f(x)$. To conclude, f can be a constant function or a function of the form $f(x)=x+a$. A generalization of this equation comes next.

Problem 102. Find all pairs of continuous functions $f,g\colon R\to R$ that satisfy

$$f(x)+f(y)=g(x+y)$$

Solution. We remark that $f(x+y)+f(0)=f(x)+f(y)=g(x+y)$ therefore $f(x)-f(0)+f(y)-f(0)=f(x+y)-f(0)$ so $f(x)-f(0)$ is an additive function, therefore $f(x)=x+c$ for some c and hence $g(x)=x+2c$.

Problem 103. Find all functions $f :: N \to N$ that satisfy

$$f(m^2 + f(n)) = f(m)^2 + n$$

Solution. It's clear that f is injective as if $f(n_1) = f(n_2)$ set $n = n_1, n_2$ to get $n_1 = n_2$. Therefore applying the condition and the injectivity of f we find that $f(m_1)^2 + n_1 = f(m_2)^2 + n_2$ if and only if $m_1^2 + f(n_1) = m_2^2 + f(n_2)$. This can be restated as: $n_1 - n_2 = f(m_2)^2 - f(m_1)^2$ if and only if $f(n_1) - f(n_2) = m_2^2 - m_1^2$. Particularly if $x - y \in S$ where $S = \{a^2 - b^2 | a, b \in N\}$ then $f(x) - f(y)$ depends only on $x - y$ so we can write $f(x) - f(y) = g(x - y)$. But S consists precisely of those integer numbers that are not 2 mod 4 or ± 1 or ± 4 (in the latter two cases this would force a or b equal zero). If $x - y = 1$ then $f(x) - f(y) = f(x) - f(x-1) = f(x) - f(x+7) + f(x+7) - f(x-1) = g(-7) + g(8)$. If we set $g(-7) + g(8) = a$ then $f(x) - f(x-1) = a$ so $f(x) = ax + b$. We substitute to find $a(m^2 + an + b) + b = (am + b)^2 + n$ or $am^2 + a^2 n + a(b+1) = a^2 m^2 + 2abm + n + b^2$ and comparing the corresponding coefficients we get $a^2 = a$ so $a = 1$ (a cannot be zero because the function cannot be constant as easily seen), $2ab = 0$ so $b = 0$. Hence $f(x) = x$ and it satisfies the condition.

Problem 104. Find all functions $f\colon R \to R$ that satisfy

$$f(f(x) + yz) = x + f(y)f(z)$$

Solution. f is injective as setting $x = x_1, x_2$ for $f(x_1) = f(x_2)$ would immediately imply $x_1 = x_2$. f is also surjective because if we fix y, z right-hand side runs over the whole R, hence so does left-hand side. Now pick up z_1 with $f(z_1) = 1$ and set $x = x_1, z = x_1$ to get $f(f(0) + yz_1) = f(y)$ so the injectivity implies $f(0) + yz_1 = y$ for any y, possible only for $f(0) = 0, z_1 = 1$. So $f(0) = 0, f(1) = 1$. Next set $y = 0$ to get $f(f(x)) = x$. Also set $z = 1, x = f(u), y = v$ to get $f(u + v) = f(u) + f(v)$. Hence f is additive. Now if we set $x = 0$ we get $f(yz) = f(y)f(z)$ so $f(y^2) = f(y)^2$ hence f is positive on R^+ thus as f is additive, f is increasing. So $f(x) = cx$ and as $f(1) = 1$ we conclude that f is the identity function.

Problem 105. Find all functions $f :: R \to R$ such that

$$f(f(x)^2 + y) = x^2 + f(y)$$

Solution. If $f(x_1) = f(x_2)$ then setting $x = x_1, x_2$ we get $x_1^2 = x_2^2$ so $x_2 = \pm x_1$. Now consider the function $h :: R^+ \to R^+$ defined by $h(x) = f^2(\sqrt{x})$. We can rewrite the condition as $f(h(x) + y) = x + f(y)$ for $x > 0$. Then $f(h(u) + h(v) + y) = u + f(h(v) + y) = u + v +$

$f(y) = f(h(u+v)+y)$. Therefore $h(u)+h(v)+y = \pm(h(u+v)+y)$. $h(u)+h(v)+y = -(h(u+v)+y)$ cannot hold for all y, so for at least one y we have $h(u)+h(v)+y = h(u+v)+y$. Thus h is additive. As h is non-negative by definition, $h(x) = cx$ where $c \geq 0$. Therefore we can deduce that $f(x) = \pm\sqrt{c}x$. Hence $f(cx^2+y) = x^2+f(y)$. If $f(y) = -\sqrt{c}y, y \neq 0$ then $f(cx^2+y) = x^2-\sqrt{c}y$ so $f^2(cx^2+y) = (x^2-\sqrt{c}y)^2 \neq (cx^2+y)^2$ for at least some x, because $(x^2-\sqrt{c}y)^2 = (cx^2+y)^2$ is equivalent to the not identically zero polynomial equation $(c^1-1)x^4+2(c+\sqrt{c})yx+(1-c^2)y^2 = 0$. So $f(y) = \sqrt{c}y$. In this case we get analogously $(x^2+\sqrt{c}y)^2 = (cx^2+y)^2$ which for $x=0$ becomes $cy^2 = y^2$ so $c = 1$. Hence $f(x) = x$ for all x. The identity function satisfies our requirements.

Problem 106. (generalization of Problem 92) Find all functions $f\colon R^+ \to R^+$ such that

$$f(xf(x)+f(y)) = f^2(x)+y$$

for all $x, y \in R^+$.

Solution. This problem differs from Problem 92 only by the fact that the set R is replaced by R^+ but this increases it difficulty considerably. For example we can not use the value $f(0)$ (as we did in the solution of Problem 92) as well as to compute directly the value of f at some particular positive number. That is why we shall first reduce the given equation to the additive Cauchy equation.

To this end we set $f(1) = a$. Then

$$f(f(y)+a) = a^2+y \tag{1}$$

and

$$f(xf(x)+a) = f^2(x)+1. \tag{2}$$

It follows from (1) that

$$f(y)+a+a^2 = f(f(f(y)+a)+a) = f(y+a+a^2).$$

Now induction on n gives

$$f(y+n(a+a^2)) = f(y)+n(a+a^2) \tag{3}$$

for any $n \in N$. On the other hand (1) and (2) imply that $xf(x)+a+a^2 = f(f(xf(x)+a)+a) = f(f^2(x)+1+a)$. This together with the given equation gives

$$\begin{aligned} f(f(f^2(x)+1+a)+f(y)) &= f(xf(x)+a+a^2+f(y)) = \\ &= f^2(x)+y+a+a^2. \end{aligned} \tag{4}$$

It follows from (1) that the function $f(x)$ attains any value greater than a^2 and therefore the function $f^2(x)+1+a$ attains any value greater than a^n+a+1. Hence (4) shows that

$$f(f(x)+f(y)) = x+y+a^2-1 \tag{5}$$

for any $x > a^4+a+1,\ y>0$. Now it follows from (3) that (5) is fulfilled for any $x,y>0$. Indeed, given $x,y>0$ choose an $n \in N$ such that $x+n(a+a^2) > a^4+a+1$. Then

$$f(f(x)+f(y)) = f(f(x+n(a+a^2))+f(y)) - n(a+a^2) =$$

$$= x+n(a+a^2)+y+a^2-1-n(a+a^2) = x+y+a^2-1.$$

Replacing x and y respectively with $f(x)$ and $f(y)$ in (5) we get

$$f(x+y+2a^2) = f(x)+f(y)+a^2-1. \tag{6}$$

Hence

$$f(x+2a^2)+f(y-2a^2) = f(x+y+2a^2)+1-a-a^2 = f(x)+f(y).$$

This shows that $f(x+2a^2)-f(x) = b$ for any $x \in R^+$ where b is a constant. Set $g(x) = f(x)+c$ where $c = a^2+a-1-b$. Then (6) can be rewritten as $g(x+y) = g(x)+g(y)$ for all $x,y \in R^+$. Taking into account that $f(x)>0$ we see that $g(x)>c$ for any $x \in R^+$. Hence it follows from Problem 99 that $g(x) = g(1)x$, i.e. $f(x) = g(1).x - c$ for any $x \in R^+$. Now it is easy to check that this function satisfies the given conditions if and only $f(x) = x$ for any $x \in R^+$.

Problem 107. (Bulgaria '2004) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(f(x)-f(y))f\Big(\frac{x+y}{x-y}\Big) = f(x)+f(y) \tag{2}$$

for any $x \neq y$.

Solution. It follows by (2) that if $x \neq y$ and $f(x) = f(y)$, then $f(x) = f(y) = 0$. Assume now that $f(a) = 0$ for some $a \neq 0$. Then either $f(x) = 0$, or $f\Big(\frac{x+a}{x-a}\Big) = 1$. So, if $f(x) \neq 0$ for some x, then $f\Big(\frac{x+a}{x-a}\Big) = f\Big(\frac{1+a}{1-a}\Big) = 1$. Then the proved above implies that $\frac{x+a}{x-a} = \frac{1+a}{1-a}$, that is, $x = 1$. Hence $f(x) = 0$ for $x \neq 1$. Now (2) shows that $f(1) = 0$. Thus, either $f \equiv 0$, or $f(x) \neq 0$ for $x \neq 0$. The zero function obviously satisfies (2).

Let now $f(x) \neq 0$ for $x \neq 0$. Then f is an injection. For $y = 0$ we get that $f(x)(f(1)-1) = f(0)(f(1)+1)$. Since f is not constant, then $f(1) = 1$ and $f(0) = 0$. Replacing y by xy in (2), we obtain that

$$f\Big(\frac{1+y}{1+y}\Big) = \frac{f(x)+f(xy)}{f(x)-f(xy)}.$$

In particular,

$$f\Big(\frac{1+y}{1+y}\Big) = \frac{f(1)+f(y)}{f(1)-f(y)}.$$

Since $f(1) = 1$, it follows that

$$\frac{f(x)+f(xy)}{f(x)-f(xy)} = \frac{f(1)+f(y)}{f(1)-f(y)}.$$

It is easy to see that this equality is equivalent to

$$f(xy) = f(x)f(y),$$

that is f is an multiplicative function. Then $f(x^2) = f^2(x) = f^2(-x)$ and the injectivity of f implies that $f(x) = -f(-x) > 0$ for $x > 0$. Now (2) shows that $f(x) > f(y)$ for $x > y > 0$. Then $\lg f(e^x)$ is an additive strictly increasing function and hence $f(x) = x^\alpha$ for $x > 0$. Substituting this function in (2) shows that $\alpha = 1$, that is $f(x) = x$ for any x.

Problem 108. (India '2003) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + f(x)f(y) = f(x) + f(y) + f(xy) \tag{3}$$

for any x, y.

Solution. Using (3) several times, we obtain that

$$\begin{gathered}
f(x+y+z) = f(x) + f(y+z) + f(xy+xz) - f(x)f(y+z) = \\
f(x) + (1-f(x))(f(y)+f(z)+f(yz)-f(y)f(z)) \\
+f(xy) + f(xz) + f(x^2yz) - f(xy)f(xz) = \\
f(x)+f(y)+f(z)+f(xy)+f(yz)+f(zx)+f(x)f(y)f(z) \\
-f(x)f(y) - f(y)f(z) - f(z)f(x) \\
+f(x^2yz) - f(xy)f(xz) - f(x)f(yz).
\end{gathered}$$

Hence the term in the last line is a symmetric function of x, y z, which implies that

$$f(x^2yz) - f(xy)f(xz) - f(x)f(yz) = f(xy^2z) - f(xy)f(yz) - f(y)f(xz).$$

For $y = 1$ we get that

$$f(x^2z) = (a-1)f(xz) + f(x)f(xz),$$

where $a = 2 - f(1)$. On the other hand, again (3) shows that

$$f(x^2 z) = f(x + xz) + f(x)f(xz) - f(x) - f(xz).$$

Therefore,

$$f(x + xz) = af(xz) + f(x).$$

For $z = 0$ we obtain that $af(0) = 0$.

If $a = 0$, then $f(1 + z) = f(1)$, that is, $f \equiv 2$.

Let $f(0) = 0$. Then $f(x) = -af(-x) = a^2 f(x)$ and hence either $f \equiv 0$, or $a^2 = 1$.

If $a = -1$, then $-3 = f(1) = f\left(\frac{1}{2}\right) - f\left(\frac{1}{2}\right) = 0$, a contradiction.

Let $a = 1$. Setting $z = \frac{y}{x}$ leads to

$$f(x + y) = f(x) + f(y)$$

for any $x \neq 0$ and any y. The same remains true holds if $x = 0$. It follows now by (3)that

$$f(xy) = f(x)f(y).$$

Then $f(x + y) = f(x) + (f(\sqrt{y}))^2 \geq f(x)$ for $y \geq 0$. Hence f is an additive increasing function and therefore $f(x) = f(1)x = x$.

So, $f \equiv 2$, $f \equiv 0$ or $f(x) \equiv x$. It clear that all the three functions satisfy (3).

Exercises

Problem 109. (Bulgaria, 1994) Find all functions $f\colon R \to R$ such that

$$xf(x) - yf(y) = (x - y)f(x + y)$$

for all $x, y \in R$.

Problem 110. Find all functions $f :: R \to R$ that satisfy

$$f(x + y) + f(xy) = f(x)f(y) + 1$$

Problem 111. Find all functions $f\colon N \to N$ such that

$$f(f(m) + f(n)) = m + n$$

for all $m, n \in N$.

Problem 112. Denote by T the set of real numbers greater than 1. Given on $n \in N$ find all functions $f\colon T \to R$ such that

$$f(x^{n+1} + y^{n+1}) = x^n f(x) + y^n f(y)$$

for all $x, y \in T$.

Problem 113. (Russia '1993). Find all functions $f\colon R^+ \to R^+$ such that
$$f(x^y) = f(x)^{f(y)}$$
for all $x, y \in R^+$.

Problem 114. (generalization of Problem 94) Find all functions $f\colon R^+ \to R^+$ which are bounded from above on an interval and such that
$$f(xf(y)) = yf(x)$$
for all $x, y \in R^+$.

Problem 115. (generalization of Problem 95) Let S be the set of all real numbers greater than -1. Find all functions $f\colon S \to S$ which are bounded from above on an interval and such that
$$f(x + f(y) + xf(y)) = y + f(x) + yf(x)$$
for all $x, y \in S$.

Problem 116. (IMO '2002) Find all functions $f\colon R \to R$ such that
$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$
for all $x, y, z, t \in R$.

Problem 117. (Korea 1998) Find all functions $f\colon N_0 \to N_0$ that satisfy
$$2f(m^2 + n^2) = f(m)^2 + f(n)^2$$
for all $m, n \in N_0$.

Problem 118. Find all functions $f\colon R \to [0; \infty)$ that satisfy
$$f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$$

Problem 119. Find all functions $f :: R \to R$ that satisfy
$$f(y + zf(x)) = f(y) + xf(z)$$

Problem 120. Find all functions $f :: R \to R$ that satisfy
$$f(xf(z) + y) = zf(x) + y$$

Problem 121. Find all continuous functions $f\colon R^n \to R$ that satisfy

$$f(x_1, x_2, \ldots, x_n) + f(y_1, y_2, \ldots, y_n) = f(x_1 + y_1, \ldots, x_n + y_n)$$

Problem 123. Given an integer $n \geq 2$ find all functions $f\colon R \to R$ such that

$$f(x^n + f(y)) = f^n(x) + y$$

for all $x, y \in R$.

Problem 124. Let $n \geq 3$ be a positive integer. Find all continuous functions $f\colon [0;1] \to R$ for which $f(x_1) + f(x_2) + \ldots + f(x_n) = 1$ whenever $x_1, x_2, \ldots, x_n \in [0;1]$ and $x_1 + x_2 + \ldots + x_n = 1$.

Functional Equations for Polynomials

Problem 125. (Romania '2001) Find all polynomials $P \in R[x]$ such that

$$P(x)P(2x^2 - 1) = P(x^2)P(2x - 1)$$

for all $x \in R$.

First Solution. It is obvious that the constant polynomials are solutions of the problem. Suppose now that $degP = n \geq 1$. Then $P(2x - 1) = 2^n P(x) + R(x)$ where either $R \equiv 0$ or $degR = m < n$. Assume that $R \not\equiv 0$. It follows from the given identity that $P(x)(2^n P(x^2) + R(x^2)) = P(x^2)(2^n P(x) + R(x))$, i.e. $P(x)R(x^2) = P(x^2)R(x)$ for all $x \in R$. Hence $n + 2m = 2n + m$, i.e. $n = m$, a contradiction. Thus $R \equiv 0$ and $P(2x-1) = 2^n P(x)$. Set $Q(x) = P(x+1)$. Then

$$Q(2x) = 2^n Q(x) \tag{1}$$

for any $x \in R$. Set

$$Q(x) = \sum_{k=0}^{n} a_k x^{n-k}.$$

Then comparing the coefficients of $x^n - k$ on both sides of (1) gives $a_k 2^{n-k} = 2^n a_k$, i.e. $a_k = 0$ for $k \geq 1$. Hence $Q(x) = a_0 x^n$ and therefore $P(x) = a_0(x-1)^n$.

Second Solution. Suppose that $P \not\equiv 0$ and set

$$P(x) = \sum_{k=0}^{n} a_k x^{n-k}$$

where $n = degP$ and $a_0 \neq 0$. Then

$$\sum_{k=0}^{n} a_k x^{n-k} \sum_{k=0}^{n} a_k (2x^2-1)^{n-k} = \sum_{k=0}^{n} a_k x^{2(n-k)} \sum_{k=0}^{n} a_k (2x-1)^{n-k}.$$

Comparing the coefficients of $x^{3n-k}, k \geq 1$, on both sides gives

$$a_k a_0 + R_1(a_0, \ldots, a_{k-1}) = a_0 a_k 2^{n-k} + R_2(a_0, \ldots, a_{k-1})$$

where R_1 and R_2 are polynomials of $k-1$ variables. Hence a_k is determined uniquely by $a_0, \ldots, a_{k-1}$. This shows that for given a_0 and n there is at most one polynomial satisfying the given condition. On the other hand it is easy to check that the polynomials $P(x) = a_0(x-1)^n$ are solutions and therefore they give all the solutions of the problem.

Third Solution. Suppose that the polynomial $P(x)$ has a complex root $\alpha \neq 1$. Of all these roots take that for which the number $|\alpha - 1| \neq 0$ is the least possible. Let β be a complex number such that $\alpha = 2\beta^2 - 1$. Setting $x = \pm\beta$ in the given equation we see that either $P\left(\frac{\alpha+1}{2}\right) = 0$ or $P(2\beta - 1) = P(-2\beta - 1) = 0$. The inequality $\left|\frac{\alpha+1}{2} - 1\right| < |\alpha - 1|$ shows that $P\left(\frac{\alpha+1}{2}\right) \neq 0$, i.e. $P(2\beta - 1) = P(-2\beta - 1) = 0$. Then

$$2|(\beta-1)(\beta+1)| = |\alpha - 1| \leq \min(|(2\beta-1)-1|, |(-2\beta-1)-1|)$$

and $\beta \neq \pm 1$ imply that $\max(|\beta - 1|, |\beta + 1|) \leq 1$, i.e. $\beta = 0$. Hence $\alpha = -1$ and therefore $P(x) = (x+1)^k Q(x)$ where $k \geq 1$ and $Q(-1) \neq 0$. Substituting in the given identity gives

$$(x+1)^k x^k Q(x) Q(2x^2-1) = (x^2+1) Q(x^2) Q(2x-1).$$

Setting $x = 0$ in this identity gives $Q(0) = 0$ since $Q(-1) \neq 0$. Thus $P(0) = 0$ which contradicts the choice of $\alpha = -1$ since $|-1-1| > |0-1|$.

Hence all the roots of the polynomial $P(x)$ are equal to 1 and therefore $P(x) = a_0(x-1)^n$for some real constant a_0.

Problem 126. (Bulgaria '2001) Find all polynomials $P \in R[x]$ such that

$$P(x)P(2x^2+1) = P(x^2)(P(2x+1)-4)$$

for all $x \in R$.

First Solution. This solution is similar to the first solution of Problem 26. It is obvious that $P \equiv 0$ is a solution. Suppose now that $P \not\equiv 0$. Then $P(2x+1) = 2^n P(x) + R(x)$ where $n = degP$ and either $R \equiv 0$ or $degR = m < n$. It follows from the given identity that

$$P(x)R(x^2) = P(x^2)(R(x) - 4x).$$

Hence $R \not\equiv 0$ since otherwise $P \equiv 0$. Suppose that $m \geq 2$. Then comparing the degrees of both sides gives $n+2m = 2n+m$, i.e. $n = m$, a contradiction. Thus $m \leq 1$ and $1 \geq k = deg(R(x) - 4x)$. Now the equality $n+2m = 2n+k$ shows that $n = 2, m = 1, k = 0$ and therefore $P(x)$ is a quadratic function such that

$$P(2x+1) = 4P(x) + 4x + c. \tag{1}$$

On the other hand setting $x = 1$ on the given identity gives $P(1) = 0$, i.e. $P(x) = a(x-1)(x-b)$. Substituting this in (1) implies that $P(x) = x^2 - 1$ and one checks easily that this polynomial is a solution of the problem. Thus $P \equiv 0$ or $P(x) = x^2 - 1$.

Second Solution. (based on an idea of M. Manea) First we shall show that if P is a nonconstant solution then all the roots of the polynomial $P(x)$ are real. Assume the contrary and let $\alpha \in C$ be a root of $P(x)$ with argument $\varphi \in (0, 2\pi)$. Since the coefficients of $P(x)$ are real it follows that $\overline{\alpha}$ is a root of $P(x)$. Hence we may assume that $\varphi \in (0, \pi)$ and that φ is the least possible argument of the complex roots of $P(x)$. It follows from the given identity that at least one of the complex numbers $\sqrt{|\alpha|}\left(\cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}\right)$ and $2\alpha+1$ is a root of $P(x)$. This is a contradiction since the arguments of both numbers belong to the interval $(0, \varphi)$.

Suppose now that $degP = n > 2$ and set

$$P(x) = \sum_{k=0}^{n} a_k x^{n-k}.$$

Then comparing the coefficients of x^3n-1 on both sides of the given identity we get $2^n a_0 a_1 = a_0(n.2^{n-1}a_0 + 2^{n-1}a_1)$, i.e. $\dfrac{a_1}{a_n} = n$. Hence the sum of all roots of $P(x)$ is equal to $-n$. Since 1 is a root of $P(x)$ (set $x = 1$ in the given identity) we conclude that the least root α of $P(x)$ is less than -1. On the other hand it is easily seen that at least one of the numbers $i\sqrt{-\alpha}$ and $2\alpha+1$ is a root of $P(x)$. This is a contradiction since $i\sqrt{-\alpha}$ is not a real number and $2\alpha + 1 < \alpha$.

Finally, we see as in the first solution that if $degP \le 2$ then $P \equiv 0$ or $P(x) = x^2 - 1$.

Problem 127. Let $\{P_n\}_{n=1}^{\infty}$ be the sequence of polynomials defined by:

$$P_1(x) = x, P_{n+1}(x) = P_n^2(x) + 1, \quad n \ge 1.$$

Prove that a polynomial P satisfies the identity

$$P(x^2 + 1) = P^2(x) + 1$$

for all $x \in R$ if and only if P belongs to the above sequence.

Solution. Let P satisfies the given identity. Then $P^2(x) = P^2(-x)$, hence for any x either $P(x) = P(-x)$ or $P(x) = -P(-x)$. It follows that $P(x) \equiv P(-x)$ or $P(x) \equiv -P(-x)$. In the second case we get $P(0) = 0$ and an easy induction shows that $P(n) = n$ for any $n \in N$. Hence $P(x) = x$ for all $x \in R$ and this polynomial belongs to the given sequence. In the first case it follows easily that $P(x) = Q(x^2)$ where Q is a polynomial. Then

$$Q((x^2 + 1)^2) = P(x^2 + 1) = P^2(x) + 1 = Q^2(x^2) + 1$$

and setting $R(x) = Q(x - 1)$ we see that $R(y^2 + 1) = R^2(y) + 1$ for $y = x^2 + 1$. Hence $R(y^2 + 1) = R^2(y) + 1$ for all $y \in R$. Thus

$$P(x) = R(x^2 + 1) = R^2(x) + 1$$

where $degR = \dfrac{degR}{2}$ and the polynomial R satisfies the given condition. Conversely, if R is a polynomial satisfying the given identity then the same is true for the polynomial $P(x) = R(x^2 + 1)$. Now the statement of the problem follows by induction on the degree of P.

Problem 128. (Bulgaria 2003) Assume that $P \in Z[X]$ is a polynomial such that $P(x) = 2^n$ has at least one integer root for all natural n. Prove that P is linear.

Solution. We use the following lemma, approximating polynomials by powers of linear functions:

Let $P(x) = ax^n + bx^{n-1} + \dots, a > 0,\ n \geq 2$. Set $u = \sqrt[n]{a}, v = \frac{b}{na^{n-1}}$. Then

$$\lim_{|x|\to\infty} |\sqrt[n]{P(x)} - ux - v| = 0$$

(note that if n is even then $P(x) > 0$ for all sufficiently big $|x|$ so $\sqrt[n]{P(x)}$ makes sense. We also have two values of $\sqrt[n]{P(x)}$ for n even, we pick up the logical one)

Proof: consider $(ux+v)^n$. It's leading two coefficients coincide with those of P hence $(ux+v)^n - P(x)$ has degree at most $n-2$. Let $\sqrt[n]{P}(x) = uf(x)$. If $|f(x) - ux - v| > \epsilon$ then $|(ux+v)^n - P(x)| = |(ux+v)^n - f^n(x)| = |(ux+v) - f(x)|(f^{n-1}(x) + \ldots + (ux+v)^{n-1}) > \epsilon(ux+v)^{n-1}$ is $|x|$ is big enough so that $f(x)$ and $ux+v$ have the same sign (positive if $x > 0$ and negative if $x < 0$. For n even we have two complementary possible values for f, we choose the one which has the same sign with $ux+v$). But $\epsilon(ux+v)^{n-1}$ has degree bigger that $(ux+v)^n - P(x)$ so our inequality can hold only $x < C$ for some C thus for $x > C$ we get $|f(x) - ux - v| < \epsilon$ and taking $\epsilon \to 0$ we get the result. Note that we might have $f(x) = -\sqrt[n]{P(x)}$ if n is even and $x < 0$, it's still a n-th root of P.

Now assume $deg(P) \geq 2$ and approximate P by $(ux+v)^m$ as in the lemma. Let $k_n \in Z$ such that $P(k_n) = 2^n$. Clearly $|k_n| \to \infty$ because (k_n) is a sequence of distinct numbers. Thus if $f(x) = \sqrt[m]{(P(x))}$ (if n is even consider only $|x|$ sufficiently big such that $P(x) > 0$) then we have $\lim_{n\to\infty} |f(k_n) - uk_n - v| = 0$. Now $f(k_n) = \sqrt[m]{2^n}$. If u is rational, pick up $q \in N$ suck that $uq \in Z$ then we have $\lim_{n\to\infty} |qf(k_n) - quk_n - qv| = 0$ so $\lim_{n\to\infty} |q\sqrt[m]{2^n} - quk_n - qv = 0|$ thus $\lim_{n\to\infty}\{q\sqrt[m]{2^n}\} = \{qv\}$. If we take $m|n$ we see that the Left-Hand Side is zero so $\{qv\} = 0$. Now take $m = ln+1$ to deduce that $\lim_{l\to\infty}\{q2^l\sqrt[m]{2}\} = \{qv\}$. Now $q\sqrt[m]{2}$ is irrational hence its representation in base 2 has infinitely many digits of 1 and 0 so infinitely many blocks 10. If the digit on position $l+1$ after zero is 1 and the digit on position $l+2$ after zero is 0 then $\frac{1}{2} < \{q2^l\sqrt[m]{2}\} < \frac{3}{4}$ and it cannot tend to zero, contradiction. If u is irrational then we can take $n = lm$ to get $\lim_{l\to\infty} |2^l - uk_{lm} - v| = 0$ so $\lim l \to \infty |2^l\frac{1}{u} - \frac{v}{u} - k_{lm}| = 0$ which implies $\lim_{l\to\infty}\{2^l\frac{1}{u}\} = \{\frac{v}{u}\}$. Now let's look at the irrational number $\frac{1}{u}$ in base 2. It must have infinitely many ones and zeroes. Now if the block 11 or 00 would meet in the binary representation a finite number of times, then starting from some point the digits of $\frac{1}{u}$ would be $101010\ldots$ so $\frac{1}{u}$ would be rational, impossible. So one of them, say 11 meets an infinite number of times. As we have infinitely many zeroes and ones, the blocks 01 must also occur an infinite number of times. So if the $l+1^{th}, l+2^{th}$ digits of $\frac{1}{u}$ are 1 we get $\frac{3}{4} < \{2^l \frac{1}{u}\} < 1$ but if $l+1^{th}, l+2^{th}$ digits of $\frac{1}{u}$ are 0 and 1 we have $\frac{1}{4} < \{2^l \frac{1}{u}\} < \frac{1}{2}$. So $\{2^l \frac{1}{u}\}$ has infinitely many members in the disjoint intervals $[\frac{1}{4}; \frac{1}{2}]$ and $[\frac{3}{4}; 1]$ thus cannot converge. We derive an analogous contradiction if 00 meets infinitely many times.

[Remark: The conclusion $lim_{n\to\infty}\{x_n\} = x$ if $lim_{n\to\infty} x_n = x$ is inaccurate as seen by the example $x_n = 1 - \frac{1}{n}, x = 1$ so $\{x_n\} \to 1, \{x\} = 0$. This is however the only possible kind of counter-example, and it doesn't affect our reasonings above, as you can check.]

Problem 129. (Belarus '1996) Prove that if $P, Q \in R[x]$ and $P(P(x)) = Q(Q(1-x))$ for all $x \in R$ then there exists $R \in R[x]$ such that $P(x) = Q(x) = R(x(1-x))$ for all $x \in R$.

Solution. Set $F(x) = P\left(x + \frac{1}{2}\right) - \frac{1}{2}$ and $G(x) = Q\left(x - \frac{1}{2}\right) - \frac{1}{2}$. Then

$$P(P(x)) = F\left(P(x) - \frac{1}{2}\right) + \frac{1}{2} = F\left(F\left(x - \frac{1}{2}\right)\right) + \frac{1}{2}.$$

Analogously $Q(Q(x)) = G\left(G\left(x - \frac{1}{2}\right)\right) + \frac{1}{2}$. Hence

$$F\left(F\left(x - \frac{1}{2}\right)\right) = G\left(G\left(\frac{1}{2} - x\right)\right),$$

i.e.

$$F(F(x)) = G(G(-x)) \qquad (1)$$

for all $x \in R$.

We shall show that $F(x) = G(x) = G(-x)$. It is obvious that $degF = degG$. Set

$$F(x) = \sum_{k=0}^{n} a_k x^{n-k}, G(x) = \sum_{k=0}^{n} b_k x^{n-k}$$

where $a_0, b_0 \neq 0$. Then comparing the coefficients of x^{n^2} on both sides of (1) gives $a_0^{n+1} = b_0^{n+1}(-1)^{n^2}$. Hence n is an even number and $a_0 = b_0$. Now rewrite (1) as

$$a_0(F^n(x) - G^n(-x)) = b_1 G^{n-1}(-x) + \cdots + b_n - a_1 F^{n-1}(x) - \cdots - a_n.$$

It is clear that the degree of the polynomial on the right hand side is less or equal to $n(n-1)$. On the other hand

$$F^n(x) - G^n(-x) =$$

$(F(x)-G(-x))(F^{n-1}(x)+F^{n-2}(x)G(-x)+\cdots+F(x)G^{n-2}(-x)+G^{n-1}(-x))$ and the coefficient of $x^{n(n-1)}$ in the second factor on the right hand side is equal to na_0^{n-1} since n is even and $a_0 = b_0$. This shows that $G(-x) = F(x) + c$, where c is a constant, If F is a constant, then G is the same constant by (1). If F is not a constant, then neither is G, so $F(y) = G(y + c)$ for infinitely many y by (1), hence for any y. Thus, $G(-x) - c = G(x + c)$ and for $x = -\dfrac{c}{2}$ we get that $c = 0$. Hence $G(-x) = F(x) = G(x)$. It follows that $G(x)$ is an even function hence there is a polynomial $H(x)$ such that $F(x) = G(x) = H(x^2)$. Then $P(x) = Q(x) = R(x(1-x))$ where $R(x) = H\left(\dfrac{1}{4} - x\right) + \dfrac{1}{2}$.

Another solution is also possible, based on the lemma established while solving the previous problem. Let $n = deg(f) = deg(g) > 0$ (if f, g are constants the problem is trivial). Let a be the leading coefficient of P, b the leading coefficient of Q. As the leading coefficient of $P(P(x))$ is a^{n+1} while the leading coefficient of $Q(Q(1-x))$ is $(-1)^n b^{n+1}$ we get $a = \pm b$. If $a = -b$ then we get $a^{n+1} = (-1)^{2n+1}a^{n+1}$ so $a = 0$ impossible. Thus $a = b$ and n is even. If $a > 0$ approximate $P(x)$ by $(ux+v)^n$ and $Q(x)$ by $(ux+w)^n$. We get $lim_{x\to\infty}|\sqrt[n]{P(P(x))}-uP(x)-v| = \lim_{x\to\infty}|\sqrt[n]{Q(Q(1-x))} - uQ(1-x) - w| = 0$. As $\sqrt[n]{P(P(x))} = \sqrt[n]{Q(Q(1-x))}$ we conclude that $lim_{x\to\infty}|uP(x)+v-uQ(1-x)-w| = 0$ possible only when $Q(x) = P(1-x) + c$ for $c = \frac{v-w}{u}$. Thus $P(P(x)) = Q(Q(1-x))$ can be rewritten as $P(P(x)) = Q(P(x)+c)$ and we deduce from here $P(x) = Q(x + c)$. Particularly if $1 - x - c = x$ which holds for $x = \frac{1-c}{2}$ we have $P(x) = Q(x + c) = P(1 - x - c) + c$ so $c = 0$. Thus we get $P(x) = Q(x)$ and $Q(x) = P(1 - x)$. As $P(x) = P(1 - x)$ if r is a root of P then so is $1 - r$ and $1 - r \neq r$ for $r \neq \frac{1}{2}$. So all roots not equal to $\frac{1}{2}$ group into pairs $(r, 1 - r)$. Hence $P(x) = a(x-\frac{1}{2})^m \prod(x-r)(x-1+r)$. As n is even, m is also even so we get $P(x) = Q(x) = a((x - \frac{1}{2})^2)^{\frac{m}{2}} \prod(x^2 - x + r(1 - r)) = aR(x(1 - x))$ where $R(x) = (-1)^{\frac{n}{2}}(-\frac{1}{4} - x)^{\frac{m}{2}} \prod(x - r(1 - r))$.

Problem 130. Find all polynomials P with rational coefficients that satisfy

$$P(x) = P(\frac{-x + \sqrt{3(1 - x^2)}}{2})$$

whenever $|x| \leq 1$.

Solution. We have (*) $P(\frac{-x+\sqrt{3(1-x^2)}}{2}) = Q(x) + \sqrt{3(1-x^2)}R(x)$ (this can be proven by Newton's Binomial Formula) hence $R = 0$ and $Q = P$ and then $P(x) = P(\frac{-x+\sqrt{3(1-x^2)}}{2}) = P(\frac{-x-\sqrt{3(1-x^2)}}{2})$. The condition holds for all x if we extend it to complex number. Let $r(x) = \frac{-x+\sqrt{3(1-x^2)}}{2}$. We have $r^3(x) = 1$. This can be checked manually but can also be prove if we note that $P(\cos t) = \cos(t + \frac{2\pi}{3})$. If w is a root of p then so is $P(r(w)) = 0$ from (*) and then $P(r(r(w))) = 0$. Then $(x-w)(x-r(w))(x-r(r(w)))|P$. But $Q_w(x) = (x-w)(x-r(w))(x-r(r(w))) = (x^3 - \frac{3}{4}x - w^3)$ satisfies our conditions and hence so does $\frac{P}{Q_w}$. Continuing this operation we shall reach a constant at some point hence $P = \prod Q_w = R(x^3 - \frac{3}{4}x)$ where $R = \prod(x - w^3)$. We must have $R \in Q[x]$ otherwise if a_k is the irrational coefficient at the smallest power k then the coefficient of x^k in $R(x^3 - \frac{3}{4}x)$ would be irrational. It's clear from the proof that $R(x^3 - \frac{3}{4}x)$ satisfies our hypothesis.

Problem 131. Find all polynomials P with only real zeroes that satisfy

$$P(x)P(-x) = P(x^2 - 1)$$

Solution. If r is a root of P the by setting $x = r$ we conclude that $g(r) = r^2 - 1$ is also a root of P. Then $g(g(r))$ is also a root of P and so on. As we may have a finite number of roots, we may encounter a root for a second time, so $g(g(\ldots(s))) = s$ for some s in the sequence. Now let's find r. $g(r) - r = (r-u)(r-v)$ where $u = \frac{-1-\sqrt{5}}{2}, v = \frac{-1+\sqrt{5}}{2}$. $g(g(r)) - r = r(r+1)(r-u)(r-v)$. If $r < -1$ then set $x = \sqrt{1+r}$ to obtain that $\pm\sqrt{1+r}$ is a root of P but it is not real so this case is not possible. If $r = -1$ then $g(r) = 0, g(g(r)) = -1$ so $x(x+1)|P$. If $r \in (-1;u)$ then $g(r) \in (u;0)$ and $g(g(r)) \in (-1;u)$ but $g(g(r)) - r = r(r+1)(r-u)(r-v) < 0$ so $g(g(r)) < r$. We repeat the reasoning with $g(g(r))$ and so on to obtain and infinite decreasing sequence of roots of P in $(-1;u)$ contradiction. If $r = u$ then $u - x|P$. If $r \in (u;0)$ then $g(r) \in (-1;u)$ and we have shown no root can occur in $(-1;u)$. If $r = 0$ then $g(r) = -1$ and $x(x+1)|P$. If $0 < r < v$ then $\pm sqrt1 + r$ is a root of P. As P has no roots less than -1, $\sqrt{1+r}$ is a root of P. Also $r < \sqrt{1+r}, \sqrt{1+r} < v$ and we can build an increasing sequence of roots of P in $(0;v)$. If $r = v$ then $v - x|P$. If $r > v$ then $g(r) > v$ is a root of P and continuing this operation we get an infinite increasing set of roots of P greater than v. So all roots can be $-1, 0, u, v$. As $x(x+1), u-x, v-x$ all satisfy the condition, we can divide P by any of them and repeat of reasoning to get that

$P(x) = x^m(x+1)^m(u-x)^q(v-x)^q$. If P is a constant then $P = 0$ or $P = 1$.

Problem 132. Suppose that f is a rational function in x that satisfies $f(x) = f(\frac{1}{x})$. Prove that f is a rational function in $x + \frac{1}{x}$.

Solution. For a polynomial P with $P(x) \neq 0$ let P^* be $x^{deg(P)}P(\frac{1}{x})$. We directly prove that $(PQ)^* = P^*Q^*$ and if $P(x) = a_nx^n + \ldots + a_0$ then $P^*(x) = a_0x^n + \ldots + a_n$ where $a_0a_n \neq 0$. Next if $P = a_{2n}x^{2n} + \ldots + a_0$ is a polynomial of degree $2n$ that satisfies $P = P^*$ we conclude $a_{n+k} = a_{n-k}$ thus $\frac{P(x)}{x^n} = a_n + \sum_{i=1}^{n} a_{n-i}(x^i + \frac{1}{x^i})$. Now for any k $x^k + \frac{1}{x^k}$ is a polynomial in $x + \frac{1}{x}$. This is proven by induction on n: if we set $q_n = x^n + \frac{1}{x^n}$ then $q_{n+1} = q_nq_1 - q_{n-1}$ and from here it's clear how to show that q_n is a polynomial in q_1. Therefore $\frac{P(x)}{x^n}$ is a polynomial in $x + \frac{1}{x}$. Finally let $f = \frac{g}{h}$ where g, h are coprime polynomials. Set $deg(g) = k, deg(h) = l$. We can assume g, h are monic. We distinguish three cases:

a) $h(0) \neq 0$. Then let $g = x^m g_1(x)$ where $g_1(0) \neq 0$. We have $\frac{x^m g_1(x)}{h(x)} = \frac{g_1(\frac{1}{x})}{x^m h(\frac{1}{x})}$ thus $x^{2m}g_1(x)h^*(x)\frac{1}{x^k} = h(x)g_1^*(x)\frac{1}{x^{l-m}}$ so $x^{l+m-k}g_1(x)h^*(x) = g_1^*(x)h(x)$. We get $l + m = k$ and $g_1^*(x)h^*(x) = g_1^*(x)h(x)$. Now as $(g_1, h) = 1$ we conclude $g_1 | g_1^*$ so $g_1^* = kg_1$ then $h^* = kh$. Then the roots of $g_1$group into pairs $w, \frac{1}{w}$ which consist of different numbers unless $w = \pm 1$ hence the free coefficient of g_1 is ± 1 depending on whether 1 is a root of g_1. So $k = \pm 1$. If $k = -1$ then 1 is a root of g_1 and analogously a root of h contradicting the coprimality of g, h. So $k = 1$. Also note that $deg(g_1) = deg(h_1) - 2m$. We can suppose $deg(g_1)$ and $deg(h)$ are even because otherwise we can multiply g_1, h by $x + 1$ and still have $g_1^* = g_1, h^* = h$ because of the multiplicativity of * and since $(x+1)^* = x + 1$. Therefore $\frac{g_1}{x^{\frac{1}{2}deg(g_1)}}, \frac{h}{x^{\frac{1}{2}degh}}$ are polynomials in $x + \frac{1}{x}$ as we have proven above. Thus $\frac{g_1}{h}x^{\frac{1}{2}(deg(h)-deg(g_1))} = x^m\frac{g_1(x)}{h(x)} = f(x)$ is a rational function in $x + \frac{1}{x}$, as desired.

b) $h(0) = 0$. Then $g(0) \neq 0$ as $(g, h) = 1$. We repeat the argument of a) for $\frac{1}{f}$.

Problem 133. (Bulgaria '2006) Find all polynomials P and Q with real coefficients such that for infinitely many $x \in \mathbb{R}$ one has that

$$\frac{P(x)}{Q(x)} - \frac{P(x+1)}{Q(x+1)} = \frac{1}{x(x+2)}.$$

Solution. Set $R(x) = \dfrac{P(x)}{Q(x)}$. Then

$$R(x)-R(x+n) = \sum_{i=0}^{n-1}(R(x+i)-R(x+i+1)) = \sum_{i=0}^{n-1}\frac{1}{(x+i)(x+i+2)}$$

$$= \frac{1}{2}\sum_{i=0}^{n-1}\left(\frac{1}{x+i}-\frac{1}{x+i+2}\right) = \frac{1}{2}\left(\frac{1}{x}+\frac{1}{x+1}-\frac{1}{x+n}+\frac{1}{x+n+1}\right).$$

Therefore,

$$\lim_{n\to\infty} R(x+n) = R(x) - \frac{1}{2x} - \frac{1}{2(x+1)}.$$

Since this limit does not depend of x (why?), we conclude that $R(x) = c + \dfrac{1}{2x} + \dfrac{1}{2(x+1)}$. Thus, $\dfrac{P(x)}{Q(x)} = \dfrac{P_0(x)}{Q_0(x)}$, where

$$P_0(x_0) = x + \frac{1}{2} + cx(x+1),\ Q_0(x) = x(x+1).$$

Since P_0 and Q_0 are relatively prime, then for infinitely many x, hence for any x, one has that $P(x) = R(x)P_0(x)$ and $Q(x) = R(x)Q_0(x)$, where R is an arbitrary nonzero polynomial and $c \in \mathbb{R}$. Conversely, the polynomials of these forms satisfy the given condition.

Remark. One can show the following:

Let $a \in \mathbb{R}$ and R be a rational function with real coefficients such that $R(x) - R(x+1) = \dfrac{1}{x(x+a)}$ for infinitely many $x \in \mathbb{R}$. Then $a \in \mathbb{Z}$, $a \neq 0$. Moreover, if $a > 0$, then $R(x) = c + \dfrac{1}{a}\displaystyle\sum_{i=0}^{a-1}\frac{1}{x+i}$, and if $a < 0$, then $R(x) = c - \dfrac{1}{a}\displaystyle\sum_{i=-1}^{a}\frac{1}{x+i}$

Exercises

Problem 134. (Bulgaria '2001) Find all polynomials $P \in R[x]$ such that

$$P(x)P(x+1) = P(x^2)$$

for all $x \in R$.

Problem 135. (IMO '1979, Shortlisted Problem) Find all polynomials $P \in R[x]$ such that

$$P(x)P(2x^2) = P(2x^3+x)$$

for all $x \in R$.

Problem 136. (Romania '1990) Find all polynomials $P \in R[x]$ such that

$$2P(2x^2-1) = P^2(x) - 2$$

for all $x \in R$.

Problem 137. Let $k, l \in N$ be integers. Find all polynomials P for which $xP(x-k) = (x-l)P(x)$

Problem 138. Find all nonconstant polynomials P that satisfy $P(x)P(x+1) = P(x^2+x+1)$.

Problem 139. Find all polynomials $P \in C[X]$ that satisfy $P(x)P(-x) = P(x^2)$

Problem 140. Find all polynomials $P(x)$ which are solutions of the equation $P(x^2-y^2) = P(x-y)P(x+y)$

Problem 141. Find all polynomials $P \in C[X]$ that satisfy $P(2x) = P'(x)P"(x)$

Iterations and Recurrence Relations

Problem 142.(Nordic Contest 1999)A function $f\colon N \to R$ satisfies for some positive integer m the conditions $f(m) = f(1995), f(m+1) = 1996, f(m+2) = 1997$ and $f(n+m) = \frac{f(n)-1}{f(n)+1}$. Prove that $f(n+4m) = f(n)$ and find the least m for which this function exists.

Solution. If $h(x) = \frac{x-1}{x+1}$ then $f(n+m) = h(f(n))$ so $f(n+4m) = h_4(f(n))$. We need to check that $h_4(x) = x$. Indeed $h_2(x) = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \frac{-1}{x}$ and therefore $h_4(x) = h_2(h_2(x)) = x$. We've solved the first part of the problem. The least possible value of m is 1. Then $f(n+4) = f(n)$ so $f(1997) = f(5) = f(1)$. But we know that $f(1997) = f(3) = h_2(f(1)) = \frac{-1}{f(1)}$. Thus $m = 1$ gives us $f(1) = \frac{-1}{f(1)}$ so $f(1)^2 = -1$ impossible. similarly $m = 2$ gives $f(1995) = f(3) = f(2)$, $f(1996) = f(4) = f(3), f(1997) = f(5) = f(4)$ thus $f(2) = f(3) = f(4) = f(5)$. Then $h(f(2)) = f(2)$. But the equation $\frac{x-1}{x+1} = x$ gives us $x-1 = x^2+x$ so again $x^2 = -1$ impossible. Finally if $m = 3$ then for any value of

$f(1), f(2), f(3)$ we can compute f inductively. Because $12|1992$, we get $f(1995) = f(3) = f(m)$ and so on so $m = 3$ is the answer.

Problem 143. Find all $f\colon N \to N$ that satisfy $f(n) + f(n+1) = f(n+2)f(n+3) - k$ where $k+1$ is a prime number.

Solution. This is mainly a sequence problem. Set $a_n = f(n)$ to get $a_n + a_{n+1} = a_{n+2}a_{n+3} - k$. Writing this condition for $n-1$ we get $a_n + a_{n-1} = a_{n+2}a_{n+1} - k$. Subtracting them we get $(a_{n+1} - a_{n-1}) = a_{n+2}(a_{n+3} - a_{n+1})$. So if we set $b_k = a_{k+2} - a_k$ we get $b_{n-1} = a_{n+2}b_{n+1}$ so $b_{n+1} = \frac{b_{n-1}}{a_{n+2}}$ so $|b_{n+1}| = \frac{|b_{n-1}|}{|a_{n+2}|}$. So $|b_{n+1}| \leq |b_n|$ and if $b_{n-1} \neq 0, a_{n+2} \neq 1$ then $|b_{n+1}| < |b_{n-1}|$ and $b_{n+1} \neq 0$. Then again if $a_{n+4} \neq 1$ we will get $|b_{n+3}| < |b_{n+1}|, b_{n+3} \neq 0$ and so on. This would produce an infinite sequence of decreasing positive integers which is impossible. Therefore either $b_{n-1} = 0$ or the sequence $a_{n+2}, a_{n+4}, \ldots,$ becomes eventually 1. Set $n = 3$ to see that (*) either $a_2 = a_4$ or $a_{2m+1} = 1$ for all $m \geq m_0$. Likewise if we set $n = 2$ we see that (**) either $a_3 = a_1$ or $a_{2m} = 1$ for all $m \geq m_0$.

a) Assume that $a_2 = a_4$. Then $b_2 = 0$ and hence by induction $b_{2m} = 0$ so $a_{2m+2} = a_{2m}$ hence $a_{2m} = a_2$. If $a_2 \neq 1$ then we must have $a_1 = a_3$ by (**) and hence by induction $a_{2m+1} = a_1$. Thus the condition written for $n = 1$ becomes $a_2 + a_1 = a_2a_1 - k$ or $(a_2 - 1)(a_1 - 1) = k+1$. As $k+1$ is prime, one of $a_2 - 1$ is $k+1$ and the other is 1. So either $f(n) = k+2$ for even n and $f(n) = 2$ for odd n or viceversa: $f(n) = 2$ for odd n and $f(n) = 2$ for even n. Both functions satisfy the condition. Now if $a_2 = 1$ then $a_{2m} = 1$ and we have $1 + f(2m+1) = f(2m+3) - k$ so $f(2m+3) = f(2m+1) + k + 1$. We conclude that $f(2m) = 1, f(2m+3) = m(k+1) + a$ where $a = f(1)$. This function also satisfies the condition.

b) Assume that $a_{2m+1} = 1$ for all $m \geq m_0$. By (**) either $a_{2p} = 1$ for all $p \geq p_0$ or $a_3 = a_1$. If a_{2p} becomes eventually 1 set $n \geq 2m_0 + 1, 2p_0$ to get $1 + 1 = 1 - k$ impossible. Hence $a_3 = a_1$ and like in a) we conclude $a_{2m+1} = a_1$. As a_{2m+1} is eventually 1 we have $a_1 = 1$. Then like in a) we conclude that $f(2m) = (m-1)(k-1) + a$ where $a = f(2)$. It also satisfies the condition.

Problem 144. Find all functions $f\colon N \to N$ such that

$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$

for all $n \in N$.

Second Solution. In this case we get the recurrence relation

$$a_{k+3} + a_{k+2} + a_{k+1} = 3a_k$$

with characteristic equation $x^3 + x^2 + x = 3$. Its roots are equal to 1 and $-1 \pm \sqrt{2}$, i.e.

$$a_k = c_0 + c_1(-1+\sqrt{2})^k + c_2(-1-\sqrt{2})^k, \quad k \geq 0.$$

Since $a_k > 0$ and $|-1-\sqrt{2}| > 1 > |-1+\sqrt{2}|$ we conclude as in the solutions of the previous two problems that $c_2 = 0$ from where $c_1 = 0$. Hence $a_1 = a_0$, i.e. $f(n) = n$ for all $n \in N$.

Problem 145. (BMO '2002) Find all functions $f\colon N \to N$ such that

$$2n + 2000 \leq f(f(n)) + f(n) \leq 2n + 2002$$

for all $n \in N$.

Second Solution. Fix an n and set

$$a_0 = n, \ \ a_{k+1} = f(a_k), \ \ c_k = a_{k+1} - a_k - 667, \quad k \geq 0.$$

Then

$$2a_k + 2001 \leq a_{k+2} + a_{k+1} \leq 2a_k + 2002,$$
$$0 \leq c_{k+1} + 2c_k \leq 1, \quad k \geq 0.$$

We shall prove that $c_0 = 0$. Assume the contrary. Then we may assume that $c_0 \geq 1$ since otherwise $c_1 \geq -2c_0 \geq 2$ and we consider the sequence $c_1, c_2, \ldots$ We have

$$c_{2k+2} \geq -2c2k+1 \geq 4c_{2k} - 2 \geq 2c_{2k}$$

and it follows by induction that $c_{2k} \geq 2^k$, $k \geq 0$. Hence

$$a_{2k+2} = a_{2k} + c_{2k} + c2k+1 + 1334 \leq a_{2k} + 1335 - c_{2k} \leq$$
$$\leq a_{2k} + 1335 - 2^k, \quad k \geq 0.$$

Summing up these inequalities gives

$$a_{2k} \leq a_0 + 1335k - 2^k, \quad k \geq 0.$$

This inequality shows that $a_{2k} \leq 0$ for all sufficiently large k, a contradiction. Thus $c_0 = 0$ and $f(n) = n + 667$ for all n. It is easy to check that this function satisfies the given conditions.

Problem 146. (IMO '1997, shortlisted problem) Prove that if the function $f\colon R \to R$ is such that $|f(x)| \leq 1$ and

$$f(x) + f\left(x + \frac{13}{42}\right) = f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right)$$

for all $x \in R$ then it is periodic.

Solution. We have

$$f\left(x+\frac{1}{6}+\frac{1}{7}\right)-f\left(x+\frac{1}{7}\right)=f\left(x+\frac{1}{6}\right)-f(x)$$

which implies

$$f\left(x+\frac{k}{6}+\frac{1}{7}\right)-f\left(x+\frac{k-1}{6}+\frac{1}{7}\right)=f\left(x+\frac{k}{6}\right)-f\left(x+\frac{k-1}{6}\right)$$

for $1\le k\le 6$. Summing up these inequalities gives

$$f\left(x+1+\frac{1}{7}\right)-f\left(x+\frac{1}{7}\right)=f(x+1)-f(x).$$

Set $g(x)=f(x+1)-f(x)$. Then $g\left(x+\frac{1}{7}\right)=g(x)$ which implies $g(x)=g\left(x+\frac{1}{7}\right)=g\left(x+\frac{2}{7}\right)=\cdots=g(x+1)$. Hence $g(x)=g(x+n)$ for any $n\in N$. Then

$$f(x+n)-f(x)=(f(x+n)-f(x+n-1))+\cdots+(f(x+1)-f(x))=$$
$$=g(x+n-1)+\cdots+g(x)=ng(x),$$

i.e.

$$f(x+n)-f(x)=ng(x)$$

for any $x\in R$ and $n\in N$. Hence

$$n|g(x)|=|f(x+n)-f(x)|\le|f(x+n)|+|f(x)|\le 2,$$

i.e. $n|g(x)|\le 2$ for any $x\in R$ and $n\in N$. This shows that $g(x)=0$ for any $x\in R$, i.e. $f(x+1)=f(x)$.

Problem 147 Let $0<a_1<a_2<\ldots<a_k$ be integer numbers, $b_0,b_2,b_3,\ldots,b_k$ be reals such that $b_k=\pm1$ and $b_0+b_1x^{a_1}+\ldots+b_kx^{a_k}$ has all roots of absolute value 1. Let f be a bounded function such that

$$b_0f(x)+b_1f(x+a_1)+\ldots+b_kf(x+a_k)=0$$

Show that f is periodic.

Solution This is a generalization of the previous problem. (Set $g(x)=f(\frac{x}{42})$ to obtain $g(x+13)+g(x)=g(x+6)+g(x+7)$ and the polynomial $x^{12}-x^6-x^7+1=(x^6-1)(x^7-1)$ has all roots of absolute value 1). However the method is hard to generalize as here we have a very vague and complex relation. The fact that a_i are rational can help us to reduce the problem to a polynomial recurrence. Now we employ two lemmas which will help us. Both are well-known.

Lemma 1: If $w_1, w_2, \ldots, w_k$ have absolute value 1 and $a_n = w_1^n + w_2^n + \ldots + w_k^n$ is not identically zero then there exists an $\epsilon > 0$ such that $|a_n| > \epsilon$ for infinitely many n.

Proof: Let $w_i = e^{2\pi i a_i}$ where $a_i \in R$. For any n, consider the k-uple $(\{na_1\}, \{na_2\}, \ldots, \{na_k\})$. If we divide $[0;1)^k$ into N^k boxes $[\frac{i}{n}; \frac{i+1}{n}) \times [\frac{j}{n}; \frac{j+1}{n}) \times \ldots$ then taking $n > N^k$ we deduce that for some $i, j < n$ the k-uples $(\{ia_1\}, \{ia_2\}, \ldots, \{ia_k\})$ and $(\{ja_1\}, \{ja_2\}, \ldots, \{ja_k\})$ will fall into the same box, and this means that $|\{ia_m\} - \{ja_m\}| < \frac{1}{N}$ therefore $\langle (i-j)a_m \rangle < \frac{1}{N}$ where we denote $\langle x \rangle = \min(\{x\}, 1 - \{x\})$. We thus conclude that $|w_m^i - w_m^j| = |1 - w_m^{i-j}| < |1 - e^{\frac{2\pi i}{N}}| = 2\sin\frac{\pi}{N} < \frac{2\pi}{N}$ hence if we denote by $r = i - j$ we get $|a_i - a_{i+r}| < \frac{2\pi k}{N}$. Now if we take a_i such that $a_i \neq 0$ we can denote $\epsilon = \frac{|a_i|}{2}$. Now taking N_1 such that $\frac{2\pi k}{N_1} < \frac{\epsilon}{2}$ we find r_1 such that $|a_{i+r_1} - a_i| < \frac{\epsilon}{2}$ so $|a_{i+r_1}| > (1 + \frac{1}{2})\epsilon$. Now taking N_2 such that $\frac{2\pi k}{N_2}$ we analogously find r_2 such that $|a_{i+r_1+r_2}| > (1 + \frac{1}{4})\epsilon$. Reasoning by induction we find $r_1, r_2, \ldots, r_l$ such that $|a_{i+r_1+\ldots+r_l}| > (1 + \frac{1}{2^l})\epsilon$ and this guarantees the claim.

Lemma 2: If $P \in Z[X]$ is monic and has all roots of absolute value 1 then this roots are roots of unity.

Proof: Let $P(X) = (x - w_1)(x - w_2)\ldots(x - w_n)$. Let $P_k(X) = (x - w_1^k)(x - w_2^k)\ldots(w - w_n^k)$. As P_k is symmetric in $w_1, w_2, \ldots, w_n$ its coefficients express as integer polynomials in the symmetric sums of $w_1, w_2, \ldots, w_n$. These sums are integers as $P \in Z[X]$ thus $P_k \in Z[X]$. However $[x^m]P_k(x) = |\sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} w_{i_1}^k w_{i_2}^k \ldots w_{i_m}^k| \le \binom{n}{m}$ as $|w_i| = 1$. So the coefficients of $P_k(X)$ are bounded and therefore for some $k < l$ we have $P_k(X) = P_l(X)$. This means that $(w_1^k, w_2^k, \ldots, w_n^k)$ is a permutation of $(w_1^l, w_2^l, \ldots, w_n^l)$. So $w_i^k = w_{i_1}^l$. Then $w_{i_1}^k = w_{i_2}^l$ so $w_i^{k^2} = w_{i_2}^{l^2}$. Reasoning inductively we get $w_i^{k^j} = w_{i_j}^{l^j}$. Eventually we return to i $(i_j = i)$ so we get $w_i^{k^j} = w_i^{l^j}$ so $w_i^{l^j - k^j} = 1$ so w_i is a root of unity.

Now we return to the problem. If we set $c_n = f(x+n)$ then this is a polynomial recurrence with associated polynomial $b_0 + b_1 x^{a_1} + \ldots + b_k x^{a_k}$. Then $c_n = \sum_{i=1}^{l} p_i(n) w_i^n$ for w_i the roots of the equation. Now we claim that p_i are constants. Indeed, assume not. Then $c_n = (d_0(n)n^m + d_1(n)n^{m-1} + \ldots + d_m(n))$ where $d_0, d_1, \ldots, d_m$ are simple polynomial recurrences in $w_1, w_2, \ldots$. Now if k is the smallest such that d_k is not identically zero, then applying lemma 1 we get infinitely many n for which $|d_k(n)| > \epsilon$. Then $\frac{c_n}{r^n n^k} = d_k(n) + \frac{d_{k+1}(n)}{n} + \ldots$. Also $|d_i(n)|$ is bounded because w_i have absolute value 1. Now it's clear that for sufficiently big n we have $|\frac{c_n}{n^k} - d_k()| < \frac{\epsilon}{2}$ thus for infinitely many n we have $\frac{c_n}{n^k} > \frac{\epsilon}{2}$ which contradicts the boundedness of f unless $k = 0$.

Thus $c_n = d_0(n)$ and this guarantees the claim. Now as w_i are roots of unity according to Lemma 2 we have N such that $w_i^N = 1$ hence $c_n = c_{n+N}$. As N does not depend on x we get $f(x) = f(x+N)$ as desired.

Problem 148. (Belarus '1997) Let $f\colon R^+ \to R^+$ be a function such that

$$f(2x) \geq x + f(f(x))$$

for all $x \in R^+$. Prove that $f(x) \geq x$ for any $x \in R^+$.

Solution. First note that $f(x) > \dfrac{x}{2}$ and let $f(x) > a_n x$ for all $x \in R^+$ where a_n is a constant. Then

$$f(x) \geq \frac{x}{2} + f\left(f\left(\frac{x}{2}\right)\right) > \frac{x}{2} + a_n f\left(\frac{x}{2}\right) > \frac{1+a_n^2}{2}x.$$

Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by: $a_1 = \dfrac{1}{2},\ \ a_{n+1} = \dfrac{1+a_n^2}{2},\ \ n \geq 1$. Then $a_{n+1} - a_n = \dfrac{(1-a_n)^2}{2} \geq 0$, i.e. the sequence is monotone increasing. Moreover, it follows by induction on n that $a_n < 1$ for any $n \in N$. Hence the sequence is convergent and denote by a its limit. Then $a = \dfrac{1+a^2}{2}$, i.e. $a = 1$. Now letting $n \to \infty$ in the inequality $f(x) > \dfrac{1+a_n^2}{2}x$ gives $f(x) \geq x$.

Problem 149. (China '1998) Let $f\colon R \to R$ be a function such that

$$f^2(x) \leq 2x^2 f\left(\frac{x}{2}\right)$$

for all $x \in R$ and $f(x) \leq 1$ for $x \in (-1,1)$. Prove that $f(x) \leq \dfrac{x^2}{2}$ for all $x \in R$.

Solution. It is obvious that $f(0) = 0$. Hence we have to prove the desired inequality for $x \neq 0$. Set $g(x) = \dfrac{2f(x)}{x^2}$ for $x \neq 0$. Then

$$g^2(x) \leq g\left(\frac{x}{2}\right)$$

and it follows by induction that

$$g^{2^n}(x) \leq g\left(\frac{x}{2^n}\right)$$

for all $x \neq 0$ and $n \in N$. Note that $g(x) \geq 0$. Hence

$$g(x) \leq \sqrt[2^n]{g\left(\frac{x}{2^n}\right)} \leq \sqrt[2^n]{\frac{2^{2n+1}}{x^2}}$$

if $\dfrac{x}{2^n} \in (-1,1)$. Now letting $n \to \infty$ and using the fact that $\lim\limits_{n\to\infty} \dfrac{n}{2^n} = 0$ we get $g(x) \leq 1$. Thus $f(x) \leq \dfrac{x^2}{2}$ for all $x \in R$.

Problem 150. (Bulgaria '1996) Find all strictly monotone functions $f\colon R^+ \to R^+$ such that

$$f\left(\frac{x^2}{f(x)}\right) = x$$

for all $x \in R^+$.

Solution. We shall show that the function $g(x) = \dfrac{f(x)}{x}$ is a constant. We have $g\left(\dfrac{x}{g(x)}\right) = g(x)$ and it follows by induction that $g\left(\dfrac{x}{g^n(x)}\right) = g(x)$, i.e. $f\left(\dfrac{x}{g^n(x)}\right) = \dfrac{x}{g^{n-1}(x)}$ for any $n \in N$. On the other hand the given condition gives $f\left(\dfrac{f^2(x)}{f(f(x))}\right) = f(x)$. Since the function f is injective we get $\dfrac{f^2(x)}{f(f(x))} = x$, i.e. $g(xg(x)) = g(x)$. Now it follows by induction that $g(xg^n(x)) = g(x)$, i.e. $f(xg^n(x)) = xg^{n+1}(x)$ for any $n \in N$. Denote $f^{(m)}(x) = \underbrace{f(f\ldots f(x)\ldots)}_{m-times}$. Then

$$f^{(m)}(xg^{-k}(x)) = xg^{m-k}(x) \tag{1}$$

for any $k, m \in N$.

Now suppose that the function $g(x)$is not constant. Then $g(x_1) < g(x_2)$ for some $x_1 \neq x_2$. Now choose a k such that $\left(\dfrac{g(x_2)}{g(x_1)}\right)^k \leq \dfrac{x_2}{x_1}$. Since the function f is monotone it follows that $f^{(2m)}$ is a strongly increasing function and (1) implies that $\left(\dfrac{g(x_1)}{g(x_2)}\right)^{2m-k} \geq \dfrac{x_2}{x_1}$ for all $m \in N$. On the other hand for m large enough the converse inequality holds, a contradiction. Thus the function $g(x)$ is a constant and therefore $f(x) = Cx$ where $C > 0$. It is easy to check that these functions satisfy the given conditions.

Exercises

Problem 151. (Bulgaria '1996) Find all functions $f\colon Z \to Z$ such that

$$3f(n) - 2f(f(n)) = n$$

for all $n \in Z$. **Problem 152.** Find all functions $f\colon R^+ \to R^+$ such

that

$$f(f(x)) + f(x) = 6x$$

for all $x \in R^+$. **Problem 153.** Find all functions $f\colon R^+ \to R^+$ such

that

$$f(f(f(x))) + f(f(x)) = 2x + 5$$

for all $x \in R^+$. **Problem 154.** Find all continuous functions $f\colon R \to$

R that satisfy

$$f(f(x)) = f(x) + 2x$$

for any $x \in R$.

Problem 155. Find all increasing bijections f of R onto itself that satisfy

$$f(x) + f^{-1}(x) = 2x$$

where f^{-1} is the inverse of f.

Problem 156. (M^+209) Find all functions $f\colon R \to R$ such that

$$f(x+1) \geq x + 1 \text{ and } f(x+y) \geq f(x)f(y)$$

for all $x, y \in R$.

Problem 157. (Belarus '1998) Prove that:

a) if $a \leq 1$ then there is no function $f\colon R^+ \to R^+$ such that

$$f\left(f(x) + \frac{1}{f(x)}\right) = x + a \tag{1}$$

for all $x \in R^+$;

b) if $a > 1$ then there are infinitely many functions $f\colon R^+ \to R^+$ satisfying (1).

Problem 158. (Bulgaria '2003) Find all $a > 0$ for which there exists a function $f\colon R \to R$ having the following two properties:
(i) $f(x) = ax + 1 - a$ for any $x \in [2, 3)$;
(ii) $f(f(x)) = 3 - 2x$ for any $x \in R$.

Polynomial recurrences and continuity

We have already seen that sequences help solving functional equations. Up to now, this was only for functional equations on N. However they can be also used for functions on R, provided continuity. We already saw a simple example in Cauchy's equation, as we have established that $f(nx) = nf(x)$ which is a basically a sequence relation if we denote $a_n = f(nx)$. Now we shall see how much more complicated sequences apply to difficult functional equations.

Problem 159. (D'Alembert's functional equation)Find all continuous $f\colon R \to R$ if

$$f(x + y) + f(x - y) = 2f(x)f(y)$$

Solution. We rely heavily on sequences for this problem. Set $y = 0$ to get $f(0) = 1$ unless f is identically zero, case we disregard as trivial. Firstly let's settle f on N. Set $a_n = f(n), a_1 = a$. We then get $a_{n+1} + a_{n-1} = 2a_n a$ hence $a_{n+1} = 2a_n a - a_{n-1}$. If $a = 1$ then we get $a_n = 1$ by induction, if $a = -1$ we get $a_n = (-1)^n$. Otherwise consider the equation $x^2 - 2ax + 1 = 0$ with different roots $w, \frac{1}{w}$. Then $a_n = cw^n + d\frac{1}{w^n}$ for some c, d. As $a_0 = 0, a_1 = a$ we get $c + d = 1, cw + \frac{d}{w} = a$. This is a linear equation in c, d which has a unique solution for $w \neq 1, -1$, which holds as $a \neq 1, -1$. The solutions are $c = \frac{1}{2}, d = \frac{1}{2}$. Therefore $a_n = f(n) = \frac{w^n + \frac{1}{w^n}}{2}$

Analogously we can show that $f(nx) = \frac{w_x^n + \frac{1}{w_x^n}}{2}$ for some w_x, this formula holds even in the case $w_x = \pm 1$. If we set $x = \frac{1}{k}$ we get $f(n) = \frac{w_k^{nk} + \frac{1}{w_k^{nk}}}{2}$. It is clear then that $w = w_k^k$ or $w = \frac{1}{w_k^k}$. We can assume it's the former as there is symmetry between w_k and $\frac{1}{w_k}$. We now want to show that there exists a number a such that $w_k = e^{\frac{a}{k}}$. Let $a_k = ln(w_k)$. Then $ka_k - a_1 = r_k 2\pi i$ where r_k is an integer, because $w_k^k = w_1$. We may assume $-\frac{k}{2} \leq r_k \leq \frac{k}{2}$ otherwise subtract from a_k a suitable multiple of $2\pi i$. Then $a_k = \frac{a_1}{k} + \frac{r_k}{k} 2\pi i$. Also as $\frac{1}{k} \to 0$, $f(\frac{1}{k}) \to f(0) = 1$ which implies that $e^{\frac{a_1}{k}} e^{\frac{r_k}{k} 2\pi i} \to 1$. As $\frac{a_k}{k} \to 0, e^{\frac{a_k}{k}} \to 1$ hence $e^{\frac{r_k}{k}} \to 1$ therefore $\frac{r_k}{k} \to 0$. Analogously as for the case $k = 1$ we conclude that

$la_{lk} - a_k$ is a multiple of $2\pi i$ which implies that $r_{kl} - r_k$ is a multiple of k hence $r_k - r_l$ is a multiple of $gcd(k,l)$. Particularly $r_{2k} - r_k$ is a multiple of k. But for all sufficiently big k_0 $\frac{|r_n|}{n} < \frac{1}{3}$ for $n \geq k_0$ therefore $\frac{|r_{2k}-r_k|}{k} \leq 2\frac{|r_{2k}|}{2k} + \frac{|r_k|}{k} < 2\frac{1}{3} + \frac{1}{3} = 1$ hence $r_{2k} - r_k = 0$. We then conclude $r_{2^m k} = r_k$ for all m when $k \geq k_0$. Now pick up $k \geq k_0, m > k$. Then $r_{2^m k} = r_k$ but $2^m | r_{2^{mk}} - r_{2^m}$. As $|r_{2^{mk}}| = |r_k| < |\frac{k}{3}|, |r_{2^m}| < \frac{2^m}{3}$ we conclude that $|r_{2^{mk}} - r_{2^m}| < \frac{k}{2} + \frac{2^m}{3} < 2^m$ so $r_k = r_{2^m}$. Analogously we conclude $r_{k+1} = r_{2^m}$ so $r_k = r_{k+1}$ and thus r_k is eventually constant. Now we claim that r_k can be made in fact constant by changing a_k by an irrelevant $2\pi i$ multiple. Indeed, let a be the eventual value of (the initial) r_k and let m be the smallest integer with $a_m \neq a$. Then $m | r_m - r_{2m}$ so we can change a_m by $2\pi i \frac{r_{2m} - r_m}{m}$ to have $r_m = a$. Keeping doing this operation we will have $r_k = a$ for all $k \geq 1$. Then by setting $t = a_1 + a$ we ensure our claim.

We then deduce $f(r) = \frac{e^{rt} + e^{-rt}}{2}$ for some t and rational r.

Then using the continuity of f and the continuity of the function $\frac{e^{xt} + e^{xt}}{2}$ since Q is dense in R we conclude that $f(x) = \frac{e^{xt} + e^{-xt}}{2}$. Now $e^t = w$ is the solution to the equation $x^2 - 2ax + 1 = 0$. If $a > 1$ w is real and if $a < 1$ then w is a complex number of absolute value 1. This means t is either real or completely imaginary. So either $f = \frac{e^{ixt} + e^{-ixt}}{2} = cos(xt)$ for real t or $f = \frac{e^{xt} + e^{-xt}}{2} = cosh(xt)$ again for real t. Both this functions satisfy the equation.

Remark: D'Alembert's Equation is considered only for real-valued functions. However our method works well for complex-valued functions too: the solutions will be $f(x) = \frac{e^{ax} + e^{-ax}}{2}$ for any complex number a.

Problem 160. Find all continuous functions $f\colon R \to R$ if

$$f(x+y)f(x-y) = f^2(x) - f^2(y)$$

for all $x, y \in R$.

Solution. Disregard the trivial solution $f = 0$. If we interchange x and y we deduce f is an odd function thus $f(0) = 0$. Without loss of generality $f(1) \neq 0$. Next let $a = f(1), a_n = \frac{f(n)}{a}$ for $n \in Z$. The condition written for $x = n, y = 1$ turns to the recurrence $a_{n-1}a_{n+1} = a_n^2 - 1$. If $a_2 = 2$ then $a_n = n$ by induction otherwise set $a_2 = x + \frac{1}{x}$ and we prove by induction $a_n = \frac{x^n - \frac{1}{x^n}}{x - \frac{1}{x}}$. We then conclude that $f(n) = an$ or $f(n) = a\sin(nu)$ or $f(n) = asinh(nu)$, like in D'Alembert's equation. And if $f(1) = 0$ then the relation $f(3)f(1) = f(2)^2 - f(1)^2$ tells us that $f(2) = 0$ and then using the relation $f(n+1)f(n-1) = f^2(n) - f^2(1)$

we show by induction that $f(n) = 0$. We can proceed analogously to show that $f(nx) = a\sin(nux), asinh(nux), anx$.

Now we distinguish two cases:

a) There is some $x_0 \neq 0$ for which $f(nx_0) = anx_0, a \neq 0$. Without loss of generality $x_0 = 1$. Then if $f(\frac{n}{k}) = bsin(nu)$ we conclude $bsin(nu) = n$ for all n, impossible. The same if $f(\frac{n}{k}) = bsinh(nu)$. So $f(\frac{n}{k}) = bn$ and we conclude $b = \frac{a}{k}$. Hence $f(x) = ax$ for all rational x and by continuity $f(x) = ax$.

b) There are no such x_0. Without loss of generality $f(1) \neq 0$ so $f(n) = a\sin(nu)$ or $f(n) = asinh(nu)$.

i) $f(n) = a\sin(nu)$. For $x = \frac{1}{k}$ we conclude $f(\frac{n}{k}) = a_k \sin(nu_k)$ or $f(\frac{n}{k}) = a_k sinh(nu_k)$. The latter is impossible, as $a_k sinh(nu_k)$ would be unbounded for $k|n$ whereas it must be bounded as it equals $a\sin(nu)$. So $f(\frac{n}{k}) = a_k \sin(nu_k)$. Now we prove $a_k = a$. Indeed we have $f(n) = a\sin(nu) = a_k \sin(nku_k)$. We use the property $lim_{n\to\infty} \frac{\sum_{i=1}^n \sin(ia)}{n} = \frac{1}{2}$ for $a \neq 0$ to conclude that $lim_{n\to\infty} \frac{\sum_{i=1}^n f^2(i)}{n} = a = a_k$ so $a = a_k$. Therefore $f(\frac{n}{k}) = a\sin(nu_k)$. Therefore $g(x) = \frac{a^2 - 2f^2(x)}{a^2}$ satisfies $f(\frac{n}{k}) = cos(nu_k)$. Like in the proof of D'Alembert's equation we find a t such that $g(x) = \cos(xt)$ for rational x. Then $f(x) = a \pm sin(\frac{xt}{2})$ for rational x. But as we have $f(x+y)f(x-y) = f^2(x) - f^2(y)$ we conclude that we either have $f(x) = a\sin(\frac{xt}{2})$ for all rational x or $f(x) = -a\sin(\frac{xt}{2})$ for all rational x. Then $f(x) = asinux$ for some a, u and all rational x and by continuity this holds for all x.

ii) $f(n) = asinhu$. Like in i) we prove $f(\frac{n}{k}) = a_k sinh(nu_k)$. We then have $f(n) = a_k sinh(nku_k)$ which is asymptotically equivalent to $a_k e^{nku_k}$. As $f(n) = a\sinh(nu)$ is asymptotically equivalent to ae^{ku} we conclude $u_k = \frac{u}{k}, a_k = a$. We then get $f(x) = nsinh(ux)$ for all rational x and this holds by continuity for all x.

To conclude, the solutions are given by $f(x) = ax, f(x) = a\sin ux), f(x) = asinh(ux)$.

After solving these problems we felt a strong connection between them, namely that their solutions were very similar to each other. Based on this, we deduce the main result of this chapter which helps us solve a lot of functional equations, including those mentioned above.

Problem 161. Prove the following general Lemma: Assume that $f \colon R \to C$ is a continuous function that satisfies the following condition: for any x, there is a number w and $p_1, p_2, p_3 \in C[X]$ polynomials, such that $f(nx) = p_1(x)w^{nx} + p_2(x)w^{-nx} + p_3$. Prove then that $f(x) = p_1(x)e^{tx} + p_2(x)e^{-tx} + p_3$ for some fixed t, p_2, p_2, p_3 and all x.

Solution. We use the following helpful result: if $\sum_{i=1}^{k} p(n)w_i^n = \sum_{i=1}^{m} q(n)r_i^n$ for all n where r_i and w_i are two sequences of distinct numbers then for (q_i, r_i) are some permutation of (p_i, w_i).

Proof: assume not. Then we can write $\sum_{i=1}^{k} p(n)w_i^n - 0\sum_{i=1}^{m} q(n)r_i^n = \sum_{i=1}^{l} s(n)u_i^n$ where s_i are not zero. Then the generating function of this recurrent sequence is also zero. However we know that the generating function of it can be written as $\sum_{i=1}^{l} \frac{f_i(x)}{(x-u_i)^{deg(s_i+1)}}$ where $f_i \neq 0, deg(f_i) \leq deg(s_i)$ and is not zero (if we multiply it by $(x-u_i)^{deg(s_i+1)}$ then we get all terms divisible by $(x-u_i)^{deg(s_i)+1}$ except the term $f_i(x)$, so the sum cannot be zero).

Let's return to the problem. We shall consider only $x = \frac{1}{k}$. If for all rational x we have $p_1 = p_3 = 0$ then $f(nx) = p_3(nx)$ for all n. Moreover p_3 does not depend on x, because if x, x' are rational, p_3 is defined for x and p_3' for x' then if $\frac{x}{x'} = \frac{p}{q}$ then $qnx = pnx'$ so $p_3(qnx) = p_3'(qnx')$ so p_3 coincides with p_3' for infinitely many n so $p_3 = p_3'$. So assume for some x we have p_1 or p_2 non-zero. Without loss of generality $x = 1$. Let w_k be the value of w defined for $x = \frac{1}{k}$. We claim we can pick up w_i in such a way that $w_{kl}^l = w_k$ by induction on i. Assume we can proven this for $i < k$ and let's prove it for $i = k$. We have $f(\frac{n}{k}) = p_1(\frac{n}{k})w_k^n + p_2(\frac{n}{k})w_k^{-n} + p_3(\frac{n}{k})$ and if $d|n$ then $f(\frac{n}{d}) = q_1(\frac{n}{d})w_d^n + q_2(\frac{n}{d})w_d^{-n} + q_3(\frac{n}{d})$. But $f(\frac{n}{d}) = p_1(\frac{n}{d})w_k^{\frac{k}{d}n} + p_2(\frac{n}{d})w_k^{-\frac{k}{d}n} + p_3(\frac{n}{d})$. This applying the helpful result we deduce $w_k^{\frac{k}{d}} = w_d$ or $w_k^{\frac{k}{d}} = \frac{1}{w_d}$. In the second case we can replace w_k by $\frac{1}{w_k}$ to ensure $w_k^{\frac{k}{d}} = w_d$, so we conclude that $w_{dl}^l = w_d$ if we set $l = \frac{k}{d}$. So we deduce $p_1 = q_1, p_2 = q_2, p_3 = q_3$. Thus p_1, p_2, p_3 also do not depend on k thus we have $f(\frac{n}{k}) = p_1(\frac{n}{k})w_k^n + p_2(\frac{n}{k})w_k^{-n} + p_3(\frac{n}{k})$. We can continue the proof just like in the proof of D'Alembert's Equation to conclude that $w_k = e^{\frac{t}{k}}$ for some w and we are done for $x \in Q$. As f is continuous and Q is dense in R, we are done for all x.

Problem 162. Find all continuous functions $f, h, k \colon R \to R$ that satisfy $f(x+y) + f(x-y) = 2h(x)k(y)$

Solution. The problem is a generalization of the already difficult D'Alembert Equation. However as we shall soon see, the solution is not very difficult and quite analogous to D'Alembert's Equation's one. Pick up an x and set $a_n = f(nx)$. The condition written for nx instead of x and x instead of y gives us $a_{n+1} + a_{n-1} = 2h(nx)k(x)$. However if we write the condition for nx instead of x and 0 instead of y we get $2f(nx) = 2h(nx)k(0)$. If $k(0) = 0$ then f is identically zero and it's clear from here that either h or k is identically zero (otherwise pick up x

with $h(x) \neq 0$ and y with $k(y) \neq 0$ to get a contradiction). Otherwise we get the linear recurrence $a_{n+1} + a_{n-1} = ba_n$ where $b = \frac{k(x)}{k(0)}$. It's associated polynomial is $x^2 - bx + 1$ and thus either $a_n = \alpha w^n + \beta \frac{1}{w^n}$ where $w \neq \frac{1}{w}$ are the solutions of the equation, or if $x^2 - bx + 1$ has a double root -1 or 1 then $a_n = (cn + d)$ or $a_n = (cn + d)(-1)^n$. Now apply the previous result to conclude that $f(x) = \alpha e^{ax} + \beta e^{-ax}$ for some constants α, a, β or $f(x) = ax + b$. In the first case we get $f(x + y) + f(x - y) = (\alpha e^{ax} + \beta e^{-ax})(e^{ay} + e^{-ay}) = h(x)k(y)$. If we set $y = 0$ we get $h(x) = c(\alpha e^{ax} + \beta e^{-ax})$ where $c = \frac{2}{k(0)}$ and from here we deduce $k(y) = \frac{1}{c}(e^{ay} + e^{-ay})$. It's clear such functions satisfy the condition. If $f(x) = ax+b$ then $f(x+y)+f(x-y) = 2ax+2b$ possible only when $h(x) = c(ax + b)$ and k is identically $\frac{2}{c}$.

Problem 163. Find all continuous functions $f\colon R \to R$ that satisfy

$$f(x + y) + f(y + z) + f(z + x) = f(x + y + z) + f(x) + f(y) + f(z)$$

Solution. If $x = y = z = 0$ we get $3f(0) = 4f(0)$ so $f(0) = 0$. Set $z = -y$ to deduce $f(x+y)+f(x-y)+f(0) = f(x)+f(x)+f(y)+f(-y)$. So $f(x+y) + f(x-y) - 2f(x) = f(y) + f(-y)$. Then if we set $a_n = nx$ and $x \to nx, y \to x$ we get $a_{n+1}-2a_n+a_{n-1} = b$ where $b = f(x)+f(-y)$. If we set $b_n = a_n - \frac{b}{2}n^2$ then we check that $b_{n+1} - 2b_n + b_{n-1} = 0$ so $b_n = cn + d$ because the quadratic recurrence $b_{n+1} - 2b_n + b_{n-1}$ has associated polynomial $(x - 1)^2$. Thus $f(nx) = \frac{b}{2}n^2 + cn + d$ and applying the lemma we get $f(x) = ax^2 + bx + c$. As $f(0) = 0$ $c = 0$ so $f(x) = ax^2 + bx$ which satisfies the condition.

Problem 164. Find all differentiable functions $f\colon R \to R$ that satisfy

$$f(x + y) - f(x - y) = y(f'(x + y) + f'(x - y))$$

Solution. Set $g(t) = f(x + t) - f(x - t)$. Then $g'(t) = f'(x + t) + f'(x - t)$ so the condition tells us that $g(y) = yg'(y)$ thus $(\frac{g(y)}{y})' = 0$ so $g(y) = cy$. Thus $f(x+y) - f(x-y) = cy$ for a fixed x. Next by taking $y \to 0$ we find $c = 2f'(x)$. So $f(x + y) - f(x - y) = 2f'(x)y$. As we know $f(x + y) - f(x - y) = y(f'(x + y) + f'(x - y))$ we conclude that $f'(x + y) + f'(x - y) = 2f'(x)$. If we denote $a_n = f(nx), b_n = f'(nx)$ we get $b_{n+1} + b_{n-1} = 2b_n$ so $b_n = an + b$ thus $a_{n+1} - a_{n-1} = x(an + b)$ and from here we deduce $a_{2n} = ux^2 + vx + w$ for some u, v, w. Now by applying the lemma we get $f(x) = ux^2 + vx + w$ and it satisfies the condition.

Exercises

Problem 165. Find all continuous functions $f, g\colon R \to R$ that satisfy
$$f(x+y) + f(x-y) = 2f(x)g(y)$$

Problem 166. Find all continuous functions $f, g, h\colon R \to R$ that satisfy
$$f(x+y) + g(x-y) = 2(h(x) + h(y))$$

Problem 167. Find all continuous functions $f, g, h, k\colon R \to R$ that satisfy $f(x+y) + g(x-y) = 2h(x)k(y)$

Problem 168. Find all continuous functions $f, g, h\colon R \to R$ that satisfy
$$f(x+y) + f(y+z) + f(z+x) = g(x) + g(y) + g(z) + h(x+y+z)$$

Problem 169. Find all continuous functions $f \cdot R \to R$ that satisfy
$$f(x+y)f(x-y) = f^2(x)f^2(y)$$

The odd and even parts of functions

This chapter exemplifies using the even part of a function ($f_e(x) = \frac{f(x)+f(-x)}{2}$) and the odd part of a function ($f_o(x) = \frac{f(x)-f(-x)}{2}$) to find the function. The main advantages are that f_e and f_o are even, respectively odd, so they might be easier to find.

Problem 170. Find all continuous functions $f\colon R \to R$ for which
$$f(x+y) + f(x)f(y) = f(xy+1)$$

Solution. If we replace x, y by $-x, -y$ and compare with the initial condition we get $f(x+y)+f(x)f(y) = f(-x-y)+f(-x)f(-y)$. Now write $f = g + h$ where $g(x) = \frac{f(x)+f(-x)}{2}, h(x) = \frac{f(x)-f(-x)}{2}$ are the even and odd parts of f.So $g(x+y) + h(x+y) + (g(x) + h(x))(g(y) + h(y)) = g(x+y) - h(x+y) + (g(x) - h(x))(g(y) - h(y))$ so we get $2h(x+y) + 2g(x)h(y) + 2h(x)g(y) = 0$. Next we replace y by $-y$ to get $2h(x-y) - 2g(x)h(y) + 2h(x)g(y) = 0$ and from here $h(x+y) + h(x-y) = -2h(x)g(y)$. We have solved this problem, with h being $c\cos ax + d\sin ax$ or $ccoshax + dsinhax$ and $g = coshx$ which does satisfy the original condition, or $h(x)$ linear and $g(x) = 1$ or $h(x) = 0$ and any g. If $h = a + bx$ is linear then $a = 0$ as h is odd. As

$f(x) = g(x) + h(x)$ $f(x) = 1 + bx$ and substituting into the original condition we see that only $1+x$ satisfies the condition. If $h(x) = 0$ then f is an even function. We then deduce $f(x+y)+f(x)f(y) = f(xy+1)$ and if we replace y by $-y$ we get $f(x-y) + f(x)f(y) = f(xy-1)$ so $f(x+y) - f(x-y) = f(xy+1) - f(xy-1) = t(4xy)$ where $t(x) = f(\frac{x}{4}+1) - f(\frac{x}{4}-1)$. Then we set $r(t) = f(\sqrt{t})$ for $t > 0$ we conclude that $r(x) - r(y) = t(x-y)$ for all $x, y \geq 0$ due to the identity $(x+y)^2 - (x-y)^2 = 4xy$ hence $r(x) - r(y) = r(x-y) - r(0)$ and thus r is a linear function on R. We conclude that $f(x) = a + bx^2$ and replacing into the original condition we get $f(x) = x^2 - 1$ or $f(x) = 0$. Hence there are three solution: $f(x) = 1 - x, f(x) = x^2 - 1$ and $f(x) = 0$.

Exercises

Problem 171. Find all continuous functions $f, g, h \colon R \to R$ that obey

$$f(x+y) + g(xy) = h(x)h(y) + 1$$

Problem 172. Find all continuous functions $f, g, h \colon R \to R$ that obey

$$f(x+y) + h(x)h(y) = g(xy+1)$$

Solution. This problem is very similar to the previous. Set $y = 0$ to get $f(x) + h(x)h(0) = g(1)$. From here $f(x) = g(1) - h(x)h(0)$. Again we can suppose $g(1) = 1$ because otherwise we an subtract $g(1)$ from both g and f and the condition will still hold. So we get $h(x)h(y) - h(0)h(x+y) = g(xy+1)$. If $h(0) = 0$ we deduce $h(x)h(y) = h(xy)h(1) = g(xy+1)$ so $h(x) = ax^b, g(x) = a^2(x-1)^b$. Otherwise we can suppose $h(0) = 1$. Then $h(x)h(y) - h(x+y) = g(xy+1)$. If we set $y = 1$ we get $g(x+1) = ah(x) - h(x+1)$ so $g(x) = ah(x-1) - h(x)$ where $a = h(1)$. Exactly like in the previous problem we conclude that either $h(x) = 1 + cx$ or h is even. For $h(x) = 1 + cx$ we get $(1+cx)(1+cy) - 1 - c(x+y) = (1+c)(1+cxy) - 1 - c(xy+1)$ and by looking at the coefficient of xy we get $c = 1$ so $h(x) = 1, f(x) = -1, g(x) = 0$. If h is even then we get $h(x)h(y) - h(x+y) = ah(xy) - h(xy+1)$ and $h(x)h(y) - h(x-y) = ah(xy) - h(xy-1)$ thus $h(x+y) - h(x-y) = h(xy+1) - h(xy-1)$ again so $h(x) = cx^2 + 1$. We easily draw the conclusions from here.

Symmetrization and additional variables

Sometimes we have a condition in x, y, say $u(x, y) = v(x, y)$ such that one side of it is symmetric in x, y but the other is not (or we can obtain such a condition by an appropriate substitution). Then swapping x with y we get a new condition, which might prove helpful. For example if $u(x,y) = u(y,x)$ then as $u(x,y) = v(x,y)$ and $u(y,x) = v(y,x)$ thus $v(x,y) = v(y,x)$. In other cases we might need to add one additional variable to get one side of the equation symmetric. See the examples below.

Problem 173. Find all continuous functions $f, g, h \colon R \to R$ that satisfy

$$f(x+y) + g(xy) = h(x) + h(y)$$

Solution. Set $y = 0$ to get $f(x) = h(x) + h(0) - g(0)$. So the condition rewrites as $h(x+y) - h(x) - h(y) = g(xy)$ where replace g by $g - g(0) - h(0)$ for simplicity. Thus $h(x+y+z) = h(x) + h(y+z) + g(xy+xz) = h(x) + h(y) + h(z) + g(yz) + g(xy+xz)$. Symmetrizing this we conclude that $g(yz) + g(xy+xz) = g(xz) + g(xy+yz) = g(xy) + g(xz+yz)$. As for $a, b, c > 0$ we can find x, y, z with $yz = a, xz = b, xy = c$ we get $g(a) + g(b+c) = g(b) + g(a+c) + g(c) + g(a+b)$ and taking $c \to 0^+$ we get $g(a+b) + g(0) = g(a) + g(b)$. Next if we take $a > 0, b < 0, c < 0$ we can also find x, y, z with $yz = a, xz = b, xy = c$ so $g(a) + g(b+c) = g(b) + g(a+c) + g(c) + g(a+b)$. Now taking $c \to 0^-$ we get $g(a) + g(b) = g(0) + g(a+b)$. Finally if we take $a < 0, b < 0, c > 0$ and take $c \to 0^+$ we get $g(a) + g(b) = g(a+b)$ in this case too. So $g(a+b) + g(0) = g(a) + g(b)$ holds for all non-zero a, b by continuity and then $f(x) = ax + b$ is linear. So $h(x+y) - h(y) - h(z) = axy + b$. If we consider $H(x) = h(x) - \frac{a}{2}x^2 + b$ then we see that $H(x) + H(y) = H(x+y)$ so $H(x) = cx$. Therefore we find a representation $h(x) = ux^2 + vx + w, g(x) = 2ux - w$. The problem is now solved.

Problem 174. Find all continuous $f \colon R \to R$, solutions of the equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(xy+1)$$

Solution. Set $g(x) = f(x+1) - f(x)$. Then $f(x+y) - f(x) - f(y) = g(xy)$. Then $f(x+y+z) - f(x+y) - f(z) = g(xz+yz)$ so $f(x+y+z) - f(x) - f(y) - f(z) = g(xz+yz) + g(xy)$. Due to the symmetry among x, y, z we conclude that $f(x+y+z) - f(x) - f(y) - f(z) = g(xz+yz) + g(xy) = g(xz+xy) + g(yz) = g(xy+yz) + g(xz)$. Now if we set $a = xy, b = yz, c = xz$ we get $g(a+b) + g(c) = g(a+c) + g(b) = g(b+c) + g(a)$. The condition that $a = xy, b = xz, c = yz$ can be satisfied if $abc > 0$ by setting $x = \frac{\sqrt{abc}}{c}, y = \frac{\sqrt{abc}}{b}, z = \frac{\sqrt{abc}}{a}$. Thus we get $g(a+b) + g(c) = g(a+c) + g(b) = g(b+c) + g(a)$ for $abc > 0$. Now we claim $g(x+y) + g(0) = g(x) + g(y)$ for $xy \neq 0$. Indeed either for $z < 0$

or for $z > 0$ we have $xyz > 0$ thus $g(x+y)+g(z) = g(x+z)+f(y)$. Now taking $z \to 0$ we obtain $g(x+y)+g(0) = g(x)+g(y)$. This also holds by continuity even when $xy = 0$. Hence $g(x) - g(0)$ is additive thus $g(x) = ax + b$ is linear. Hence $f(x+y) - f(x) - f(y) = axy + b$. Now if we set $h = f(x) - \frac{a}{2}x^2 + b$ then we see that $h(x+y) - h(x) - h(y) = f(x+y) - \frac{a}{2}(x+y)^2 + b - f(x) + \frac{a}{2}x^2 - b - f(y) + \frac{a}{2}y^2 - b = (f(x+y) - f(x) - f(y)) + \frac{a}{2}(x^2+y^2-(x+y)^2) - b = axy+b-axy-b = 0$. Hence h is additive so $h(x) = cx$. We conclude that f is a polynomial of degree at most 2. Let $f(x) = ax^2+bx+c$. We have $f(x+y)+f(xy) = a(x+y)^2 + b(x+y)+c+ax^2y^2+bxy+c = a(x^2+y^2)+ax^2y^2+(2a+b)xy+b(x+y)+2c$ while $f(x)+f(y)+f(xy+1) = ax^2+bx+c+ay^2+y^2+c+a(xy+1)^2 + b(xy+1)+c = a(x^2+y^2)+ax^2y^2+(2a+b)xy+b(x+y)+a+b+3c$. Therefore by comparing the two expressions we get $a+b+c = 0$ hence $f(x) = ax^2+bx-a-b$. These functions clearly satisfy the condition.

Problem 175. Find all functions $f\colon R \to R$ obeying

$$f((x-y)^2) = f^2(x) - 2xf(y) + y^2$$

Solution. Symmetrize the condition to get $f((x-y)^2) = f^2(x) - 2xf(y) + y^2 = x^2 - 2f(x)y + f^2(y)$ and the equality of the last two expressions can be written as $(f(x)+y)^2 = (f(y)+x)^2$. One can guess that only the function $f(x) = x+a, f(x) = -x$ satisfy the condition. Indeed, assume that $f(a) \neq -a$. Let $f(a) = b$. Pick up another c and let $f(c) = d$. We wish to prove that $d = c+b-a$. Indeed, we have $(a+d)^2 = (b+c)^2$ so either $d = c+b-a$ or $d = -a-b-c$. If it is the latter, pick up any x. We have $(f(x)+a)^2 = (x+b)^2$ so either $f(x) = x+b-a$ or $f(x) = -x-b-a$. We also have $(f(x)+c)^2 = (x-a-b-c)^2$ so either $f(x) = x-a-b-2c$ or $f(x) = a+b-x$. It follows that the sets $\{x+b-a, -x-a-b\}$ and $\{x-a-b-2c, a+b-x\}$ must intersect. We can pick up such an x that satisfies $x+b-a \neq a+b-x$ and also $-x-a-b \neq x+a-b-2c$. Then either $x+b-a = x-a-b-2c$ or $-x-a-b = a+b-x$ thus either $b+c = 0$ or $a+b = 0$. $a+b \neq 0$ as $f(a) \neq a$. Hence $b+c = 0$ and in this case $d = -a-b-c = c+b-a$. Hence $d = c+b-a$ so $f(c) = c+b-a$. As c is arbitrary, we get $f(x) = x+b-a$. This guarantees our claim, so $f(x) = -x$ or $f(x) = x+a$. It remains only to check which of them satisfies the condition. $f(x) = -x$ then $f(x-y)^2 = -(x-y)^2$ while $f^2(x) - 2xf(y) + y^2 = x^2 + 2xy + y^2 = (x+y)^2$ and the condition is not satisfied. $f(x) = x+a$ then $f((x-y)^2) = x^2 - 2xy + y^2 + a$ while $f^2(x) - 2xf(y) + y^2 = (x+a)^2 - 2x(y+a) + y^2 = x^2 - 2xy + y^2 + a^2$ so the identity hold if and only if $a^2 = a$ or $a = 0, 1$. So $f(x) = x, f(x) = x+1$ are the solutions of the problem.

Exercises

Problem 176. Find all functions $f :: R \to R$ for which

$$f(x+y) = f(x)f(y)f(xy)$$

Problem 177. Find all continuous functions $f\colon R \to R$ such that

$$f(x+y) + f(xy-1) = f(x) + f(y) + f(xy)$$

Problem 178. (Hosszu's functional equation) Show that a function $f :: R \to R$ which satisfies

$$f(x+y-xy) + f(xy) = f(x) + f(y)$$

is an additive function plus some constant.

Problem 179. Find all functions $f :: R \to R$ for which

$$xf(x) - yf(y) = (x-y)f(x+y)$$

holds.

Functional Inequalities without Solutions

Problem 180. (Bulgaria '1998) Prove that there is no function $f\colon R^+ \to R^+$ such that

$$f^2(x) \geq f(x+y)(f(x)+y)$$

for all $x, y \in R^+$.

Solution. Suppose that there is a function f with the given properties. Then

$$f(x) - f(x+y) \geq \frac{f(x)y}{f(x)+y} \tag{1}$$

which shows that f is a strictly increasing function. Given an $x \in R^+$ we choose an $n \in N$ such that $nf(x+1) \geq 1$. Then

$$f\left(x+\frac{k}{n}\right) - f\left(x+\frac{k+1}{n}\right) \geq \frac{f(x+\frac{k}{n}).\frac{1}{n}}{f(x+\frac{k}{n})+\frac{1}{n}} > \frac{1}{2n}$$

for any $k \in N$. (Note that $nf\left(x+\dfrac{k}{n}\right) > nf(x+1) > 1$.) Summing up these inequalities for $k = 0, 1, \ldots, n-1$ we get

$$f(x) - f(x+1) > \frac{1}{2}.$$

Now take an $m \in N$ such that $m \geq 2f(x)$. Then

$$f(x)-f(x+m) = (f(x)-f(x+1))+\cdots+(f(x+m-1)-f(x+m)) > \frac{m}{2} \geq f(x).$$

Hence $f(x+m) < 0$, a contradiction.

Problem 181. Prove that there is no function $f\colon R \to R$ such that $f(0) > 0$ and

$$f(x+y) \geq f(x) + yf(f(x)) \qquad (1)$$

for all $x, y \in R$.

Solution. Suppose that there is a function f with the given properties. If $f(f(x)) \leq 0$ for any $x \in R$ then

$$f(x+y) \geq f(x) + yf(f(x)) \geq f(x)$$

for any $y \leq 0$ and the function f is decreasing. Now the inequalities $f(0) > 0 \geq f(f(x))$ imply $f(x) > 0$ for any x, a contradiction to $f(f(x)) \leq 0$. Hence there exists z such that $f(f(z)) > 0$. Then the inequality

$$f(z+x) \geq f(z) + xf(f(z))$$

shows that $\lim\limits_{x\to+\infty} f(x) = +\infty$ and therefore $\lim\limits_{x\to\infty} f(f(x)) = +\infty$. In particular, there exist $x, y > 0$ such that

$$f(x) \geq f(f(x)) > 1,\ \ y \geq \frac{x+1}{f(f(x))-1},\ \ f(f(x+y+1)) \geq 0.$$

Then

$$f(x+y) \geq f(x) + yf(f(x)) \geq x+y+1$$

and therefore

$$f(f(x+y)) \geq f(x+y+1) + (f(x+y) - (x+y+1))f(f(x+y+1)) \geq$$
$$\geq f(x+y+1) \geq f(x+y)+f(f(x+y)) \geq f(x)+yf(f(x))+f(f(x+y) > f(f(x+y)),$$

a contradiction.

Remark. Note that the only function $f\colon R \to R$ with $f(0) = 0$ and satisfying the inequality (1) is the constant 0. Indeed, as in the second part of the above solution we conclude that $f(f(x)) \leq 0$ for all $x \in R$. On the other hand setting $x = 0$ in (1) gives $f(y) \geq 0$ for all x. Hence

$f(x+y) \geq f(x)$ for any $x, y \in R$ which easily implies that $f(x) = 0$ for all x.

It is not known to the authors if there is a function $f\colon R \to R$ with $f(0) < 0$ and satisfying the inequality (1).

Exercise

Problem 182. (Romania '2001) Prove that there is no function $f\colon R^+ \to R^+$ such that

$$f(x+y) \geq f(x) + yf(f(x))$$

for all $x, y \in R^+$.

Miscellaneous

Problem 183. (Iran 1998)Let $f\colon R^+ \to R^+$ be a decreasing function that satisfies

$$f(x+y) + f(f(x)+f(y)) = f(f(x+f(y)) + f(y+f(x)))$$

Show that $f(f(x)) = x$.

Solution. Set $y = x$ so get $E = f(2x) + f(2a) = f(2f(a+x))$ where $a = f(x)$. Now replace x by $f(x)$ to get $F = f(2b) + f(2a) = f(2f(a+b))$ where $b = f(f(x))$. If $b < x$ then $f(a+b) > f(a+x)$ so $f(2f(a+b)) < f(2f(a+x))$ as f decreasing. Also $f(2b) > f(2x)$ hence we get $f(2x) + f(2a) < f(2b) + f(2a)$. So we get $F > E$ from the first relation and $F < E$ from the second, contradiction. If $b > x$ then we change the signs to get $F < E$ and $F > E$ again contradiction. So $b = x$.

Problem 184. Find all continuous functions $f\colon R \to R$ that satisfy the equation

$$f(x+yf(x)) = f(x)f(y)$$

Solution. $f = 0$ or $f = 1$ satisfy the condition. Next set $y = 0$ to get $f(x) = f(x)f(0)$ thus if f is not identically zero we get $f(0) = 1$. If $f(x) = 1 + ax$ is linear then we get $f(x+yf(x)) = f(x+y(1+ax)) = f(x+y+ax) = 1 + ax + ay + a^2x = (1+ax)(1+ay) = f(x)f(y)$. If $f(x) \neq 1$ and $y = \frac{x}{1-f(x)}$ then we get $x + yf(x) = y$ hence the condition says $f(x)f(y) = f(y)$. As $f(x) \neq 1$ we get $f(\frac{x}{1-f(x)}) = 0$. So if f is not identically 1 then the set A of t for which $f(t) = 0$ is not empty. Now if $t \in A, x \notin A$ then set $y = \frac{t-x}{f(x)}$ to get $x + yf(y) = t$ and from here we get $0 = f(t) = f(x)f(y)$ so $f(y) = 0$ so $\frac{t-x}{f(x)} \in A$ If $\frac{t-x}{f(x)}$ is constant then f

is linear but linear functions were already investigated by us, otherwise $\frac{t-x}{f(x)}$ is a continuous non-constant function so A contains infinitely many numbers. Without loss of generality A contains infinitely many positive positive numbers (the second case is analogous). Let $b = infA \bigcap R^+$. As A is closed $b \in A$ but $[0; b)$ does not intersect A. As $f(0) = 1$ we deduce f is positive on $[0; b)$. Thus if x in $[0; b)$ then $g(x) = \frac{c-x}{f(x)} \in A$ where $c \in A, c > b$. But $g(0) = b, \lim_{x \to b} g(x) = \infty$ hence we conclude that $[b; \infty) \subset A$. Then if $x \notin A$ we get $h_x(y) = x + yf(x) \in A$ if and only if y in A. Therefore $h_x(y) \notin A$ as $y < b$ but $h_x(b) \in A$ which is possible only when $h_x(b)$ is a bordering point of A as A is closed. But $h_x(b)$ is continuous in x. As b cannot be written as a limit of bordering points of A except b itself, we conclude that $h_x(b) = b$ for all $x \in [0; b]$. So $x + bf(x) = b$ hence $f(x) = \frac{b-x}{b}$ for x in $[0; b]$. If A contains negative points let $-c \in A, c > 0$. Then set $y = -c$ to get $f(x - \frac{(b-x)c}{b}) = 0$. But is x is sufficiently close to b and less than b, $0 < x - \frac{(b-x)c}{b} =< b$. Contradiction. Hence A contains no point in R^- and by continuity of h on $(-\infty; b)$ we conclude h is constant so $f(x) = \frac{b-x}{b}$ for $x \leq b$, $f(x) = 0$ for $x \geq b$. If A contains negative numbers, we get analogously that $f(x) = \frac{b-x}{b}$ for $x \geq b$, $f(x) = 0$ for $x \leq b$, where b is negative. These two functions can be checked to verify the problem. Together with $f = 0$ and $f = 1$ they form the answer set.

Problem 185. Suppose $f\colon Q \to \{0, 1\}$ is such that $f(1) = 1, f(0) = 0$ and if $f(x) = f(y)$ then $f(\frac{x+y}{2}) = f(x) = f(y)$. Prove that $f(x) = 1$ whenever $x \geq 1$.

Solution. Let $A = \{x|f(x) = 1\}, B = \{x|f(x) = 0\}$. We have that if x, y belong to a set, then $\frac{x+y}{2}$ also belongs to the same set. If $2 \in B$ then $1 = \frac{2+0}{2} \in B$, contradiction, so $2 \in A$. Next we prove by induction on n that $n \in A$. If $n = 2k$ then as $0 \in B$, $2k \in B$ would imply $\frac{2k+0}{2} = k \in B$ contradicting the induction step. If $n = 2k + 1$ then we prove like above that $2k, 2k + 2 \in A$ so $2k + 1 = \frac{2k+2k+2}{2} \in A$. Now assume that $f(1 + a) = 0$ for $a > 0$. We prove by induction on n that $1 + na \in B$ like above: if $n = 2k$ then $1 + 2ka \in A$ together with $1 \in A$ would imply $1 + ka \in A$ contradicting the induction hypothesis, and if $n = 2k + 1$ then we show that $1 + 2ka, 1 + (2k + 2)a \in B$ hence their mean $1 + (2k + 1)a$ is also in B. Finally if n is such that $na \in N$ then $1 + na$ which contradicts our conclusion above that all natural numbers are in A. QED.

Problem 186. Find for which a there exist increasing multiplicative functions on N (i.e. $f(n) < f(n+1), f(mn) = f(m)f(n)$ if $(m,n) = 1$) with $f(2) = a$.

Solution. We claim the function must be $f(n) = n^k$ for some k. Assume $f(x) = x^u, f(y) = y^v$. If $x^k < y^l$ then $x^{uk} < y^{vl}$ and if $x^k > y^l$ then $x^{uk} > y^{vl}$. Pick up now k and let l be the biggest for which $y^l < x^k$. Then $x^k \leq y^{l+1}$ so we get $y^{vl+l} > x^{uk} > y^{vl}$. Now if $v > u$ we cannot have $x^{uk} > y^{vl}$ for sufficiently big k, as $x^{uk} > y^{vl} > y^{ul+l(v-u)} > y^{u(l+1)}$ for $l \geq \frac{u}{v-u}$. If $u > v$ then $y^{vl+l} > x^{uk}$ so $y^{vl} > x^{uk-1} > x^{vk}$ for $k \geq \frac{1}{u-v}$ again contradiction. Thus taking $k > \frac{1}{u-v}$ if $u > v$ or k such that $x^k > y^{\frac{u}{v-u}+1}$ if $u < v$ we would obtain contradiction. So $u = v$ and hence $f(x) = x^u$ for all x. As f is from N to N, we must have u integer. So a must be a power of 2, and conversely if $a = 2^k$ then $f(x) = x^k$ is good.

Problem 187. (Russia '2005; a slight generalization) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function such that

$$f^2(x+y) \geq f^2(x) + 2f(xy) + f^2(y)$$

for any x, y. Prove that $-2 \leq f(x) \leq 0$ for any x.

Solution. First, we shall prove that $f(x) \leq 0$ for any x. Let $M = \sup\limits_{x\neq 0} |f(x)|$. Then there is a sequence $x_1, x_2, \ldots$ of non-zero real numbers such that $|f(x_n)| \to M$. Fixing an x, it follows that

$$M^2 \geq f^2\left(x_n + \frac{x}{x_n}\right) \geq f^2(x_n) + 2f(x) + f^2\left(\frac{x}{x_n}\right)$$
$$\geq f^2(x_n) + 2f(x) \to M^2 + 2f(x).$$

Thus, $f(x) \leq 0$.

Then $M = -\inf\limits_{x\neq 0} |f(x)|$. Now the inequalities

$$M^2 \geq f^2(2x_n) \geq 2f^2(x_n) + 2f(x_n^2) \geq 2f^2(x_n) + 2M \to 2M^2 + 2M$$

imply that $M^2 \geq 2M^2 - 2M$, that is $M(M-2) \leq 0$. Since $M \geq 0$, then $M \leq 2$, which means that $f(x) \geq -2$ for any $x \neq 0$. It remains to observe that the inequalities $f^2(0) \geq f^2(0)+2f(0)+f^2(0)$ and $f(0) \leq 0$ implies that $f(0) \geq -2$, too.

Remark. Obviously the constant function 0 and -2 satisfy the given inequality. We claim that the unbounded functions x and $-x$ also satisfy it.

Problem 188. (IMO '2005, shortlisted problem) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + f(x)f(y) = f(xy) + 2xy + 1 \tag{1}$$

for any x, y.

Solution. It is easy to check that the functions $f(x) = 2x - 1$, $f(x) = -x - 1$ and $f(x) = x^2 - 1$ satisfy (1). We shall prove that there are the only solutions of the problem.

Setting $y = 1$ gives

$$f(x+1) = af(x) + 2x + 1, \tag{2}$$

where $a = 1 - f(1)$. Then we change y to $y + 1$ in (1) and use (2) to expand $f(x + y + 1)$ and $f(y + 1)$. The result is

$$a(f(x+y) + f(x)f(y)) + (2y+1)(1 + f(x)) = f(x(y+1)) + 2xy + 1,$$

or, using (1) again,

$$a(f(xy) + 2xy + 1) + (2y+1)(1 + f(x)) = f(x(y+1)) + 2xy + 1.$$

Set now $x = 2t$ and $y = -\dfrac{1}{2}$ to obtain

$$a(f(-t) - 2t + 1) = f(t) - 2t + 1.$$

Replacing t by $-t$ gives also

$$a(f(t) + 2t + 1) = f(-t) + 2t + 1.$$

We now eliminate $f(-t)$ from the last two equations. Then

$$(1-a)^2 f(t) = 2(1-a)^2 t + a^2 - 1. \tag{3}$$

Note that $a \neq 1$ (or else $8t = 0$ for any t, which is false). If additionally $a \neq 1$, then $1 - a^2 \neq 0$; therefore

$$f(t) = 2\frac{1-a}{1+a}t - 1.$$

Setting $t = 1$ and recalling that $f(1) = 1 - a$, we get $a = 0$ or $a = 3$, which gives the first two solutions.

Let $a = 1$. Then (1) implies that f is an even function. Set now $y = x$ and $y = -x$ in the original equation. It follows that

$$f(2x) + f^2(x) = f(x^2) + 2x^2 + 1, \;\; f(0) + f^2(x) = f(x^2) - 2x^2 + 1,$$

respectively. Subtracting gives $f(2x) = 4x^2 + f(0)$. Set $x = 0$ in (2). Since $f(1) = 1 - a = 0$, this yields $f(0) = -1$. Hence $f(2x) = (2x)^2 - 1$, i.e., $f(x) = x^2 - 1$. This completes the solutions.

Problem 189. (IMO '2005, shortlisted problem) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x)f(y) = 2f(x + yf(x))$$

for any $x, y > 0$.

Solution. First, we shall show that f is increasing. Indeed, suppose that $f(x) < f(z)$ for some $x > z > 0$. Setting $y = (x-z)(f(x)-f(z)) > 0$, it follows that $x + yf(x) = z + yf(z)$. Then

$$f(x)f(y) = 2f(x + yf(x)) = 2(f(z + yf(z) = f(z)f(x),$$

therefore $f(x) = f(z)$, a contradiction.

Assume now that f is not strictly increasing, i.e., $f(x) = f(z)$ for some $x > z > 0$. If $y \in (0, (x-z)/f(x)]$, then $z < z+yf(z) \geq x$. Hence

$$f(z) \geq f(z + yf(z)) \geq f(x) = f(z)$$

and therefore $f(z + yf(x)) = f(x)$. Thus,

$$f(z)f(y) = 2f(z + yf(z)) = 2f(x) = 2f(z)$$

which implies that $f(y) = 2$ for all y in the above interval.

But if $f(y_0) = 2$ for some $y_0 > 0$, then

$$4 = f^2(y_0) = 2f(y_0 + y_0 f(y_0)) = 2f(3y_0).$$

So, $f(3y_0) = 2$ and by induction $f(3^n y_0) = 2$ for any $n \in \mathbb{N}$. Since f is increasing, it follows that $f \equiv 2$. Obviously, this function satisfies the given equation.

Assume now that f is a strictly increasing function. Then

$$f(x)f(y) = 2f(x + yf(x)) > 2f(x)$$

implies that $f(y) > 2$ for any $y > 0$. On the other hand,

$$2f(x + f(x)) = f(x)f(1) = f(1)f(x) = 2f(1 + xf(1))$$

and since f is injective, we obtain $x + f(x) = 1 + xf(1)$. Thus, $f(x) = x(f(1)-1)+1$ for any $x > 0$. Taking a small x, we get the contradiction $f(x) < 2$.

Remark. Similar arguments if $k > 0$, $k \neq 1$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the equation $f(x)f(y) = kf(x + yf(x))$ for any $x, y > 0$, then $f \equiv k$. On the other, the case $k = 1$ is the Gołab-Schinzel equation and its solutions are $f \equiv 1$ and $f(x) = x + 1$.

Problem 190. (Romania '1998) Find all functions $f : \mathbb{R} \to \mathbb{R}_0^+$ such that $f(x^2 + y^2) \equiv f(x^2 - y^2) + f(2xy)$ for any x, y.

Solution. It is easy that for any a, b there are x, y such that $x^2 - y^2 = a$ and $2xy = b$. Since $x^2 + y^2 = \sqrt{a^2 + b^2}$, the given equations becomes

$$f(a) + f(b) = f(\sqrt{a^2 + b^2}).$$

In particular, $f(a) = f(-a)$, i.e., f is an even function. For $a \geq 0$ set $g(a) = f(\sqrt{a})$. Then $g(a^2) + g(b^2) = g(a^2 + b^2)$, i.e., g is a non-negative additive function on $\mathbb{R}_0^+$. Therefore $g(x) = cx$ and hence $f(x) = cx^2$, where $c \geq 0$ is a constant.

Exercises

Problem 191. (IMO '2003, shortlisted problem) Find all function $f : \mathbb{R}^+ \to \mathbb{R}^+$, which are increasing in the segment $[1, \infty)$ and such that

$$f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$$

for any $x, y, z > 0$.

Problem 192. (AMM 1998) Find all functions $f\colon N^2 \to N$ that satisfy:
a) $f(n, n) = n$;
b)$f(m, n) = f(n, m)$;
c)$\frac{f(m,n+m)}{f(m,n)} = \frac{n+m}{n}$.

Problem 193. Find for which a there exist increasing multiplicative functions on N (i.e. $f(n) < f(n+1), f(mn) = f(m)f(n)$ if $(m, n) = 1$) with $f(2) = a$.

Problem 194. Find all functions $f\colon Z \to Z$ that satisfy:
a) if $p|m - n$ then $f(m) = f(n)$.
b) $f(mn) = f(m)f(n)$

Problem 195. Find all $f\colon N_0 \to N_0$ that satisfy

$$f(f^2(m) + f^2(n)) = m^2 + n^2$$

Problem 196. (Bulgaria '2003) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y + f(y)) = 2y + f^2(x) \tag{1}$$

for any x, y

Problem 197. (Bulgaria '2006) Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be such a function that

$$f(x+y) - f(x-y) = 4\sqrt{f(x)f(y)}$$

for any $x > y > 0$.

a) Prove that $f(2x) = 4f(x)$ for any $x > 0$.

b) Find all such functions f.

Problem 198. (Ukraine '2003) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x) + f(y)) \equiv x^2 + y$$

for any x, y (compare with Problem 92).

Problem 199. (Bulgaria '2004) Find all non-constant polynomials P and Q with real coefficients such that

$$P(x)Q(x+1) = P(x+2004)Q(x) \tag{1}$$

for any $x \in \mathbb{R}$.

Solutions to exercises
Constructive Problems

Problem 3. Find all functions $f\colon N \to R$ which satisfy $f(1) \neq 0$ and

$$f^2(1) + f^2(2) + \ldots + f^2(n) = f(n)f(n+1)$$

Solution. It's clear that $f(n) \neq 0$ as the RHS of the condition is positive. Now by subtracting the condition written for n from the condition written for $n+1$ we get $f(n+1)^2 = f(n+1)(f(n+2) - f(n))$ thus reducing by $f(n+1) \neq 0$ we get the Fibonacci-type relation $f(n+2) = f(n+1) + f(n)$. If we set $a = f(1)$ then taking $n = 1$ we get $f(2) = a$, thus $f(n) = aF_n$ by induction using the recurrent relation. It's clear that this function satisfies the condition by induction, as this is clear for $n = 1$ and the induction step was proven to be equivalent to the true relation $f(n+2) = f(n+1) + f(n)$.

Problem 7. Find all functions $f\colon N \to R$ for which $f(1) = 1$ and

$$\sum_{d|n} f(d) = 0$$

whenever $n \geq 2$.

Solution. A good example is the Mobius function, which is multiplicative. The fact that f is multiplicative too i.e. $f(mn) = f(m)f(n)$ for $(m,n) = 1$ can be proven by induction on $m+n$ just like in the previous problem. Indeed, if $m,n > 1$ we get $0 = \sum_{d|mn} f(d) = \sum_{d_1|m, d_2|n} f(d_1d_2) = f(mn) - f(m)f(n) + (\sum_{d|m} f(d))(\sum_{d|n} f(d)) = 0$ so $f(m)f(n) = f(mn)$. Next writing the condition for $n = p^k$ (p-prime) for $k \geq 1$ and subtracting it from the condition for p^{k+1} we get $f(p^{k+1}) = 0$. Therefore $f(p^k) = 0$ for $k \geq 2$. And writing the condition for p prime we get $f(p) = -1$. Therefore if $n = \prod {p_i}^{k_i}$ then $f(n) = \prod f(p_i^{k_i})$ so $f(n) = 0$ if some of k_i is greater than 1 and $f(n) = (-1)^k$ where k is the number of prime divisors of n otherwise. This is Moebius function.

Problem 8. Find all functions $f \colon N \to N$ that satisfy $f(0) = 0$ and

$$f(n) = 1 + f([\frac{n}{k}])$$

for all $n \in N$.

Solution. This is seen as a recurrence. To compute $f(n)$, we need to compute $f([\frac{n}{k}])$ first. To compute $f([\frac{n}{k}])$ we need to compute $f([\frac{[\frac{n}{k}]}{k}]) = f([\frac{n}{k^2}])$ and so on. Thus if $n \geq k^{r-1}$, by repeating this argument we get $f(n) = r + f([\frac{n}{k^r}])$. Thus if $k^r \leq n < f^{r+1}$ then $f(n) = r + 1 + f(0) = r+1$, so $f(n) = 1 + [log_k n]$ for $n > 0$. It clearly satisfies the condition.

Note We have used in the proof the identity $[\frac{[\frac{x}{m}]}{n}] = [\frac{x}{mn}]$ where $m,n \in N$. Indeed, if $a = [\frac{x}{mn}], b = [\frac{x}{m}]$ then $mna \leq x < mna + mn$, thus $na \leq \frac{x}{m} < na + n$ hence $na \leq b < n(a+1)$ so $[\frac{b}{n}] = a$, and we are done. The problem could be solved without this identity, if we would look at the representation of x in base k.

Problem 87. Let $k \in Z$. Find all functions $f \colon Z \to Z$ that satisfy

$$f(m+n) + f(mn-1) = f(m)f(n) + k$$

Solution. This is a generalization of the previous problems. We have already investigated the cases $k = 2, k = 0$ so we shall not deal with them any more. If $f = c$ is constant then $2c = c^2 + k$ so $(c-1)^2 = 1-k$ hence $f(x) = 1 \pm \sqrt{1-k}$ if $\sqrt{1-k}$ is an integer. So assume f is not constant. Set $m = 0$ to get $f(n)(1 - f(0)) = k - f(-1)$, possible only for $f(0) = 1, f(-1) = k$. Next set $m = -1$ to get $f(n-1) + f(-n-1) = kf(n) + k$. If we replace n by $-n$ the left-hand side does not change hence neither does the right-hand side so f is even (recall that $k \neq 1$). Therefore we can write $f(n-1) + f(n+1) =$

$kf(n)+k$. If $k=-1$ we get $f(n-1)+f(n)+f(n+1)=-1$ and we deduce $f(3k)=1, f(3k\pm 1)=-1$ which satisfies the equation. If $k=1$ we get $f(n-1)+f(n-1)=f(n)+1$. In this case we prove by induction on $|n|$ that $f(n)=1$ so f is not constant. If $k=-2$ then we have $f(n-1)+2f(n)+f(n+1)=-2$ so $a_n=f(n)+\frac{1}{2}$ satisfies the equation $a_{n-1}+2a_n+a_{n+1}=0$ whose polynomial equation x^2+2x+1 has double root -1 thus $a_n=(an+b)(-1)^n$. As f is even $a_n=a_{-n}$ which is possible only for $a=b=0$ so $a_n=0$ hence f is identically $-\frac{1}{2}$, which is not possible. Finally if $|k|>2$ then set $a_n=f(n)+\frac{k}{k-2}$. It satisfies the condition $a_{n-1}-ka_n+a_{n+1}=0$ with associated equation $x^2-kx+1=0$. This equation has two roots $r,\frac{1}{r}$ where $|r|>1$. Then we find a_n to be $cr^n+d(\frac{1}{r})^n$. As f is even, $a_n=a_{-n}$ so $cr^n+d\frac{1}{r^n}=c\frac{1}{r^n}+dr^n$ or $(c-d)(r^n-\frac{1}{r^n})$ so $c=d\neq 0$ as the function is not constant. Then $f(n)\sim cr^n$ for $n\to\infty$. We set $m=n\to\infty$ to get $f(2n)+f(n^2-1)=f^2(n)+k$. But $f(2n)+f(n^2-1)\ cr^{n^2-1}, f^2(n)+k\ c^2r^{2n}$, and for $n\to\infty$ we get contradiction. So are there are no solutions in this case.

Problem 88. Find all functions $f\colon Z\to Z$ that satisfy

$$f(m+n)+f(mn)=f(m)f(n)+1$$

Solution. Set $m=0,n=0$ to get $2f(0)=f^2(0)+1$ so $f(0)=1$. Next set $m=-1,n=1$ to get $f(0)+f(-1)=f(1)f(-1)+1$ so $f(-1)(f(1)-1)=0$. If $f(1)=1$ then set $m=1$ to get $f(n+1)+f(n)=f(n)+1$ hence f is identically 1 which satisfies our condition. If $f(-1)=0$ set $m=-1$ to get $f(n-1)+f(-n)=1$. Also set $m=1$ to get $f(n+1)=f(n)(f(1)-1)+1$. If $a=f(1)-1$ we get $f(n+1)=af(n)+1$. From here $(a-1)f(n+1)-1=a((a-1)f(n)-1)$. We conclude that $(a-1)f(n-k)-1=\frac{a^k}{(}(a-1)f(n)-1)$. So $a^k|(a-1)f(n)-1$ for any k which is possible only when $f(n)=\frac{1}{a-1}$ or $a=\pm 1$. If $f(n)=\frac{1}{a-1}$ then $f(1)=a+1=\frac{1}{a-1}$ so $a^2=2$ impossible. Hence $a=\pm 1$. If $a=1$ we get $f(n+1)=f(n)+1$ so $f(n)=n+1$ by induction on $|n|$, which obeys the equation as $(m+n+1)+(mn+1)=(m+1)(n+1)+1$. If $a=-1$ we get $f(n+1)=1-f(n)$. We conclude by induction on $|n|$ that $f(n)=1$ if n is even and $f(n)=0$ if n is odd. Indeed, if m,n have the same parity then $m+n$ is even so $f(m+n)=1$ and also $f(m)f(n)=f(mn)$ so the condition holds, while if one of m,n is even and the other odd we get $f(m+n)=0, f(mn)=1, f(m)f(n)$ so the condition holds again. Hence all solutions are given by $f(n)=1, f(n)=n+1$ and $f(n)=1-n\ mod\ 2$.

Problem 28. Find all functions $f\colon Z \to R$ satisfying

$$f(a^3+b^3+c^3) = f(a^3)+f(b^3)+f(c^3)$$

whenever $a, b, c \in Z$.

Solution. If we try to set $f(x) = cx$ we get $c = 0, 1, -1$. If we try to set $f(x) = c$ we get $c = 0, c = \pm\frac{1}{\sqrt{3}}$. We try to prove that this are the only solutions. Firstly we want to find a way of computing $f(n)$ by induction on $|n|$ (we note that $f(0)^3 = f(n^3+(-n)^3+0^3) = f(n)^3 + f(-n)^3 + f(0)^3$ so we can easily compute $f(-n)$ from $f(n)$. To do this, note that if $a^3 + b^3 + c^3 = m^3 + n^3 + p^3$ then $f(a)^3 + f(b)^3 + f(c)^3 = f(m)^3 + f(n)^3 + f(p)^3$. So if we can write n^3 as a sum of five cubes of numbers having absolute value less than n, or we can write n as sum of three cubes of numbers having absolute value less than n, then we are done. So let's find such representations. We note $5^3 = 4^3 + 4^3 - 1^3 - 1^3 - 1^3, 7^3 = 6^3 + 4^3 + 4^3 - 1^3 + 0^3, 6^3 = 3^3+4^3+5^3$, hence for all numbers multiples of $5, 7, 6$ we are done. Also $32 = 3^3+2^3-1^3-1^3-1^3$, $64 = 3^3+3^3+2^3+1^3+1^3$, $16 = 2^3+2^3, 8 = 2^3 + 0^3 + 0^3$, therefore this also holds for $n = 2^k$ for $k \geq 4$. Finally if we have $n = 2^m(2a + 1)$, then if a is $1, 2, 3$ then n is a multiple of $5, 7$ or 6 and we have proven this case, otherwise if $a \geq 4$ then $(2a + 1)^3 = (2a - 1)^3 + (a + 4)^3 - (a - 4)^3 - 5^3 - 1^3$ and we get the result by multiplying by 2^m. So f is uniquely determined by $f(0), f(1)$ (since then we compute $f(2) = f(1^3 + 1^3), f(3) = f(1^3 + 1^3 + 1^3)$ and use the method above to deduce f). So let's look at $f(0)$ and $f(1)$. By setting $a = b = c = 0$ then we get $f(0) = 3f(0)^3$ thus $f(0) = 0$ or $f(0) = \pm\frac{1}{\sqrt{3}}$. If $f(0) = 0$ then setting $a = 1, b = c = 0$ then $f(1) = f(1)^3$ hence $f(1) = 0, 1, -1$ in which case we get the solutions $f(x) = x, f(x) = -x, f(x) = 0$. Now assume $f(0) \neq 0$. Without loss of generality let $f(0) = \frac{1}{\sqrt{3}}$ (the second case is analogous since we could look at $-f$ instead). Then by setting $a = 1, b = c = 0$ we deduce $f(1) = f(1)^3 + 2f(0)^3$, which is a polynomial equation in $f(1)$ with solutions $\frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}$. If $f(1) = \frac{1}{\sqrt{3}}$ then we get the solution $f(x) = \frac{1}{\sqrt{3}}$. We are left with the case $f(1) = -\frac{2}{\sqrt{3}}$. Also $f(0) = f(-x^3+x^3+0^3) = f(-x)^3 + f(x)^3 + f(0)^3$ therefore $f(-x)^3 = -f(x)^3 + \frac{2}{3\sqrt{3}}$. Thus $f(-1)^3 = \frac{2}{\sqrt{3}} + \frac{2}{3\sqrt{3}} = \frac{8}{3\sqrt{3}}$ so $f(-1) = \frac{2}{\sqrt{3}} = -f(1)$. Then $f(x^3) = f(x^3+1^3+(-1)^3) = f(x)^3+f(1)^3+f(-1)^3 = f(x)^3$. From the other side, $f(x^3) = f(x^3+0^3+0^3) = f(x)^3+f(0)^3+f(0)^3 = f(x)^3+\frac{2}{3\sqrt{3}}$, contradiction. All the cases are now investigated.

Problem 31. Let f be a strictly increasing function on N with the property that $f(f(n)) = 3n$. Find $f(2007)$.

Solution. We have $f(f(1)) = 3$. If $f(1) = 1$ this is impossible. Hence $f(1) > 1$ and then $f(f(1)) > f(1)$ so $f(1) < 3$ and $f(1) = 2$. So $f(2) = 3$. Then $f(3) = f(f(2)) = 6$ and we get by induction on k $f(3^k) = 2 \cdot 3^k, f(2 \cdot 3^k) = 3^{k+1}$. Then $f(3) = 6, f(6) = 9$ and as f is increasing we get $f(4) = 7, f(5) = 8$. Thus $f(7) = 12, f(8) = 15$. We then come to the following hypothesis: If $3^k \leq n < 3^{k+1}$ then $f(n) = n + 3^k$ for $n \leq 2 \cdot 3^k$, $f(n) = 3(n - 3^k)$ for $n \geq 2 \cdot 3^k$. Indeed this function is increasing and the assumption holds for $k = 1$. We now reason by induction: Assume that it holds for $k = m-1$ and let's prove it for $k = m$. Assume $3^m \leq n < 2 \cdot 3^m$. If $n = 3s$ then $f(n) = f(3s) = f(f(f(s))) = 3f(s) = 3(s+3^{m-1}) = n+3^m$ by the induction hypothesis. If $3s < n < 3s + 3$ then $f(3s) < f(3s + 1) < f(3s + 2) < f(3s + 3)$. But we have proven $f(3s) = 3^m + 3s, f(3s + 3) = 3^m + 3s + 3$. As f is increasing, we can only have $f(3s + 1) = 3^m + 3s + 1, f(3s + 2) = 3^m + 3s + 2$ and so $f(n) = n + 3^m$. If now $2 \cdot 3^m \leq n < 3^{m+1}$ then $3^m \leq n - 3^m < 2 \cdot 3^{m+1}$ thus $f(n - 3^m) = n$ so $f(n) = f(f(n - 3^m)) = 3(n - 3^m)$. The induction step is proven and so we have found f. It remains to see that $2 \cdot 3^6 = 1458 < 2007 < 3^7 = 2187$ so $f(2007) = 3(2007 - 3^6) = 3(2007 - 729) = 3834$.

Problem 36. Find all functions $f\colon N \to N$ satisfying

$$f(m + f(n)) = n + f(m + k)$$

for $m, n \in N$ where $k \in N$ is fixed.

Solution. It's clear that f is injective (if $f(n_1) = f(n_2)$ taking $n = n_1, n_2$, the left-hand side of the condition stays the same, while right-hand side is $n_1+f(m+k)$, respectively $n_2+f(m+k)$, so $n_1 = n_2$). Now let's symmetrize the condition, by taking $n \to f(n + k)$. We get $f(m + f(f(n + k))) = f(n + k) + f(m + k)$. Tossing m and n we get $f(n+f(f(m+k))) = f(n+k)+f(m+k)$ and the injectivity of f gives us $n+f(f(m+k)) = m+f(f(n+k))$, or if we take $x = m+k, y = n+k$ we deduce $f(f(x))-x = f(f(y))-y = a$ so $f(f(x)) = x+a$. Then $a \geq 0$. Now set $n \to f(n)$ to get $f(m+n+a) = f(n)+f(m+k)$. Interchanging m, n we get $f(m + n + a) = f(m) + f(n + k)$ so $f(n + k) - f(n) = f(m + k) = b$. So $f(n + k^2) = f(n) + bk$ thus $f(f(n + k^2)) = f(f(n) + bk) = f(f(n)) + b^2$. But $f(f(n + k^2)) = n + k^2 + a, f(f(n)) = n + a$ which implies $b^2 = a^2$ so $b = a$ as b is clearly non-negative. Therefore $f(m + n + a) = f(m) + f(n) + a$. This Cauchy-type equation can be solved in a usual way: $f(m + n + a) = f(m + n - 1) + f(1) + a =$

$f(m) + f(n) + a$ so $f(m + n - 1) + f(1) = f(m) + f(n)$. Now set $m = 2$ to get $f(n+1)+f(1) = f(2)+f(n)$, and therefore f is a linear function. If $f(x) = cx + d$ then substituting into the condition we get $c(m + cn + d) + d = n + c(m + k) + d$ so $c(cn + d - k) = n$ which is possible only for $c = 1$ (as $c \geq 0$) and $d = k$. So $f(x) = x + k$ and it satisfies the requirements.

Problem 77. Let $f, g\colon N_0 \to N_0$ that satisfy the following three conditions:
i) $f(1) > 0, g(1) > 0$;
ii) $f(g(n)) = g(f(n))$
iii) $f(m^2 + g(n)) = f^2(m) + g(n)$;
iv) $g(m^2 + f(n)) = g^2(m) + f(n)$.
Prove that $f(n) = g(n) = n$.

Solution. If we set $m = 0$ into iii) and iv) and compare them we get $f^2(0) + g(n) = f(g(n)) = g(f(n)) = g^2(0) + f(n)$ so $f(n) - g(n) = f^2(0) - g^2(0)$. Particularly if we set $n = 0$ we get $f(0) - g(0) = f^2(0)-g^2(0)$ or $(f(0)-g(0))(f(0)+g(0)-1) = 0$ so either $f(0) = g(0)$ or $f(0) + g(0) = 1$. If it is the latter the one of $f(0), g(0)$ is 1 and the other is 0. As the conditions are symmetric in f, g we can suppose $f(0) = 1, g(0) = 0$. Then set $m = 1, n = 0$ into iii) to get $f(1) = f^2(1)$ so $f(1) = 1$. Now we set $m = 1, n = 1$ into iv) to get $g(1) = 1$. So $g(1) = f(1)$. But this contradicts the fact that $f(n) - g(n) = f^2(0) - g^2(0) = 1$. Hence we have $f(0) = g(0)$ and thus $f(n) = g(n)$. So the conditions iii) and iv) merge into $f(m^2+f(n)) = f^2(m)+f(n)$. Next let $A = \{f(x) - f(y)\}$. If $u = f(x) - f(y) \in A$ then $u + f^2(1) = f(1 + f(x)) - f(y) \in A$ hence A contains all multiples of $k = f^2(1)$ as it contains 0. Now let $f(a) = b$. Assume that $b \neq a$. Pick up $n > a^2, b^2$ and pick up x, y with $f(x) - f(y) = (nk + a)^2 - a^2$, so $f(x) + a^2 = f(y)+(nk+a)^2$. Then $f(f(x)+a^2) = f(f(y)+(nk+a)^2)$ so $f(x)+b^2 = f(y) + f(nk+a)^2$ hence $(nk+a)^2 - a^2 = f(x) - f(y) = f(nk+a)^2 - b^2$ or $b^2 - a^2 = f(nk+a)^2 - (nk+a)^2$. As $b \neq a$, $f(nk+a) \neq nk+a$. But then $|f(nk+a)^2 - (nk+a)^2| \geq 2(nk+a) - 1 > max\{a^2, b^2\} > |b^2 - a^2|$ contradiction. This shows that $b = a$ so f is the identity function, and so is g.

Problem 95. Find all functions $f :: Q^+ \to Q^+$ that satisfy $f(x) + f(\frac{1}{x}) = 1$ and $f(f(x)) = \frac{f(x+1)}{f(x)}$.

Solution. We have already encountered a function that satisfies $f(x)+f(\frac{1}{x}) = 1$, namely $f(x) = \frac{1}{x+1}$. There are of course many of them, but this seems more comfortable, as it indeed satisfies the condition.

So let's prove $f(x) = \frac{1}{x+1}$. If we set $x = 1$ we get $f(1) + f(1) = 1$ so $f(1) = \frac{1}{2}$. Let A be the set of all x that satisfy $f(x) = \frac{1}{x+1}$. If $x \in A$ then $f(\frac{1}{x}) = 1 - f(x) = 1 - \frac{1}{x+1} = \frac{x}{x+1} = \frac{1}{1+\frac{1}{x}}$. Also writing the second condition for x yields $f(\frac{1}{x+1}) = \frac{f(x+1)}{\frac{1}{x+1}}$ so $f(\frac{1}{x+1}) = (x+1)f(x+1)$. As $f(\frac{1}{x+1}) = 1 - f(x+1)$ we get $(x+2)f(x+1) = 1$ so $f(x+1) = \frac{1}{x+2}$. Thus if $x \in A$ then $\frac{1}{x}$ and $x+1$ also belong to A. It remains to show that every positive rational number $\frac{p}{q}$ can be obtained from 1 by means of operations $x \to \frac{1}{x}, x \to x+1$. This can be proven by induction on $p + 2q \geq 3$. For $p + 2q = 3, p = q = 1$ and the basis holds true. Next if $p > q$ then $\frac{p-q}{q}$ obeys the induction step and the operation $x \to x+1$ turns $\frac{p-q}{q}$ to $\frac{p}{q}$. If $p < q$ then $\frac{q}{p}$ obeys the induction step and the operation $x \to \frac{1}{x}$ turns $\frac{q}{p}$ to $\frac{p}{q}$. The verification is easy.

Binary (and other) bases

Problem 14. (Iberoamerican)Find all functions $f\colon N \to R$ for which $f(1) = 1$ and

$$f(2n+1) = f(2n) + 1 = 3f(n) + 1$$

$n \in N$.

Solution. Again the problem is pretty directly solved if we look at the binary representation of n. $f(n)$ is obtained by writing n in base 2 and reading the result in base 3.

Problem 15. (IMO 1978)Find all functions $f\colon N \to N$ that satisfy $f(1) = 1, f(3) = 3$ and

$$f(2n) = f(n)$$
$$f(4n+1) = 2f(2n+1) - f(n)$$
$$f(4n+3) = 3f(2n+1) - 2f(n)$$

for any $n \in N$.

Solution. Again the function is uniquely determined, and the key to finding it should be the binary representation of n. By direct computation we get that $f(1) = 1, f(2) = 1, f(3) = 3, f(4) = 1, f(5) = 5, f(6) = 3, f(7) = 7, f(8) = 1, f(9) = 9, f(10) = 5, f(11) = 13$. Up to $f(11)$ we could conjecture that $f(n)$ is obtained from n by deleting the zeroes at the end. However if we write 11 in base 2 we get $11 = 1011_2$ and $f(11) = 13 = 1101_2$. So it's natural to suppose that $f(n)$ is obtained by reversing the digits of n. Indeed, this is confirmed by all the previously computed values of f, since all numbers less than 11, if we delete their last zeroes, become palindromes. This claim is easy to verify by strong induction. Indeed assume that it's true for all numbers less than $k-1$ and let's prove it for k. Of course we need to consider three cases according to the three cases of the condition:

a) k is even. In this case $f(k) = f(\frac{k}{2})$ and the claim follows from the induction step.

b) k is of form $4n+1$. Assume that n has r digits, and let $m = f(n)$ be obtained from n by reversing its binary digits. Then the number obtained from k by reversing its digits is $2^{r+1}+m$. Also $f(2n+1) = 2^r + m$ so $f(k) = 2f(2n+1) - f(n) = 2^{r+1} + 2m - m = 2^{r+1} + m$ and the claim holds in this case.

c) k is of form $4k+3$. Again assuming that n has k digits, and $m = f(n)$ then the inverse of k is $2^{r+1} + 2^r + m$, while $f(k) = 3f(2n+1) - 2f(n) = 3(2^r + m) - 2m = 2^{r+1} + 2^r + m$, and this case is true, too.

Constructing functions by iterations

Problem 69. Let $n \in N$. Find all continuous $f\colon R \to R$ that satisfy $f_n(x) = -x$ where f_n is the n-th iterate of f.

Solution. f is clearly injective, so is monotonic. As it cannot be increasing (otherwise $f_n(x) = -x$ would be increasing too) we conclude that f is decreasing. Set $a_m = f_m(x)$. Then $f_{2n}(x) = -f_n(x) = x$. Hence the set $A = \{a_k | k \in N\}$ is periodic of period $2n$ so is finite. Let a_k be the smallest number of a. The as f is decreasing we conclude that $a_{k+1} = f(a_k)$ is the largest in A. However as a_k is the smallest, we have $a_{i+n} \geq a_k$ hence $a_i = -a_{i+n} \leq -a_k$ so $a_{k+n} = -a_k$ is the largest. Hence $f(a_k) = -a_k$. Also $f(a_{k+1})$ should be the smallest in A hence $f(-a_k) = a_k$ and by induction on p that $a_{k+p} = (-1)^p a_k$. Since A is periodic $a_{k+p} = a_0 = x$ for some p and therefore $f(x) = -x$. This function satisfies the conditions if and only if n is odd.

Problem 26. Show that there exist functions $f\colon N \to N$ such that

$$f(f(n)) = n^2, n \in N$$

.

Solution. The problem presents no difficulty to us once we have enough experience in solving this kind of problems. If m is not a perfect square, define a "chain" $C(m)$ as the sequence $(x_n)_{n\in N}$ with $x_i = m^{2^i}$. The chains partition all $N \setminus \{1\}$ (1 presents no problem as we can merely set $f(1) = 1$ and ignore this number). Then by pairing the infinitely many chains: $(C(x_1), C(y_1))$, $(C(x_2), C(y_2))$, …and setting $f(x_i^{2^k}) = y_i^{2^k}, f(y_i^{2^k}) = x_i^{2^{k+1}}$ we construct the desired function. It presents no difficulty proving that all solutions can be written in such form by merely copying the reasoning from the previous problems.

Problem 23. Let $f\colon N \to N$ ne a function satisfying

$$f(f(n)) = 4n - 3$$

and

$$f(2^n) = 2^{n+1} - 1$$

. Find $f(993)$. Can we find explicitly the value of $f(2007)$? What values can $f(1997)$ take?

Solution. It's clear that $f(x) = 2x - 1$ satisfies the condition, so we try to prove that $f(993) = 1995$. Now if we can prove $f(t) = 2t - 1$ by using the first condition we prove that $f(2t-1) = 4t-3, f(4t-3) = 8t-7$ and so on, so $f(t_n) = 2t_n - 1$ for t_n being recurrently defined as $t_0 = t, t_{k+1} = 2t_k - 1$. We know $f(2^n) = 2^{n+1} - 1$, so it seems natural to try to write $993 = t_n$ where $t_0 = 2^k$ for some k. Let's try to go backwards: $993 = 2 \cdot 497 - 1, 497 = 2 \cdot 249 - 1, 249 = 2 \cdot 125 - 1, 125 = 2 \cdot 63 - 1, 63 = 2 \cdot 32 - 1$ and 32 is a power of two. This procedure works for any number of form $2^m(2^n - 1) + 1$. But if we try to obtain $f(2007)$ by this procedure we fail because $2007 = 2 \cdot 1004 - 1$ and at 1004 we stop. Now we try to construct a different function than $2x - 1$ to answer negatively to the second question. Consider a number $n > 1$ which is not 1 mod 4 then we can define a sequence $S(x) = (x_n)_{n \in N}$ for $x_0 = x$, $x_{k+1} = 4x_k - 3$, and call x the ancestor of the sequence. It's easy to prove that g is injective (thus so is f) and these sequences partition $N \setminus \{1\}$. Let $g(x) = 4x - 3$, $g_n(x)$ be g iterated n times. Now we can prove that if $f(n) = m$ then $f(m) = g(n)$ then $f(g(n)) = g(m)$ and so on, to get $f(g_k(n)) = g_k(m)$. Hence f provides a pretty comfortable intuitive mapping between these sequences, which we shall define now. Let x, y be ancestors of two sequences and assume $f(x) = y_m$. Let $f(y) = u$ then $f(g_m(y)) = g_m(f(y)) = g_m(u)$, as proven just before. However $f(g_m(y)) = f(y_m) = g(x)$. So $g(x) = g_m(u)$ hence $x = g_{m-1}(u)$ (if $m > 0$). As x is not 1 modulo 4 this is only possible for $m = 1$ or $m = 0$. In the first case we have $f(y) = x$ and in the second $f(x) = y$. In any case, we see that f maps one ancestor into another ancestor, and then f can be defined recursively without contradiction on the sequences of these ancestors. In the condition we have defined f for the powers of two and for powers of two minus one. However we have infinitely many ancestors left which we can pair up as we want, constructing many different functions. Particularly pairing up 2007 with different ancestors we can obtain different values for $f(2007)$, so $f(2007)$ can not be uniquely determined. Now if 2007 is paired with another ancestor u then either $f(2007) = u$ or $f(u) = 2007$ and $f(2007) = 4u - 3$. Hence all possible values for $f(2007)$ are either u or $4u - 3$ where u is a number not 1 modulo 4 and not a power of two or one less a power of two.

Approximating by linear functions

Problem 19. Find all increasing functions $f\colon N \to N$ such that the only natural numbers who are not in the image of f are those of form $2n + f(n), n \in N$.

Solution. Like in the previous problem we prove that f is unique. It remains now to find it. Again setting $f(x) \sim cx$ we get $m \sim 2(m - n) + c(m - n)$ where $m = f(n)$ so $c = 2(c - 1) + c(c - 1)$ so $c^2 = 2$. Therefore $f(n) \sim \sqrt{2}n$ so we can suppose that $f(n) = [\sqrt{2}n + a]$. By computing some values of f we can even infer $a = 0$ so $f(n) = [\sqrt{2}]$. To prove that this function satisfies the condition we must prove that the sets $\{[\sqrt{2}n] | n \in N\}$ and $\{[(2 + \sqrt{2})n] | n \in N\}$ partition N. This follows from the more general Beatty Theorem: If $\alpha, \beta \in R^+ \setminus Q$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ then the sets $A = \{[n\alpha] | n \in N\}$ and $B = \{[n\beta] | n \in N\}$ partition N, as $\frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{2-\sqrt{2}}{2} = 1$. To prove Beatty Theorem, note that $|A \bigcap \{1, 2, \ldots, n\}| = [\frac{n+1}{\alpha}]$ (the number of numbers m that satisfy $m\alpha < n + 1$)and $|B \bigcap \{1, 2 \ldots, n\}| = [\frac{n+1}{\beta}]$. Therefore $|A \bigcap \{1, 2, \ldots, n\}| + |B \bigcap \{1, 2, \ldots, n\}| = [\frac{n+1}{\alpha}] + [\frac{n+1}{\beta}]$. And as $[\frac{n+1}{\alpha}] \in (\frac{n+1}{alpha} - 1; \frac{n+1}{\alpha})$ and $\frac{n+1}{\beta} \in (\frac{n+1}{\beta} - 1; \frac{n+1}{beta} - 1)$ thus $[\frac{n+1}{\alpha}] + [\frac{n+1}{\beta}] \in (\frac{n+1}{\alpha} + \frac{n+1}{\beta} - 2; \frac{n+1}{\alpha} + \frac{n+1}{\beta}) = (n-1; n+1)$. As this number is an integer, we finally conclude that $|A \bigcap \{1, 2, \ldots, n\}| + |B \bigcap \{1, 2, \ldots, n\}| = n$. By writing the condition for n and for $n+1$ and subtracting them we deduce that $|A \bigcap \{n\}| + |B \bigcap \{n\}| = 1$ which implies that A, B partition N.

Problem 34. Find all functions $f\colon N \to N$ such that

$$f(f(n)) + f(n + 1) = n + 2$$

for $n \in N$.

Solution. We firstly note that $f(n+1) \leq n+1$ and $f(f(n)) \leq n+1$ from this relation. Hence $f(k) \leq k$ for $k > 1$. If we try to set $f(x) \sim cx$ then we get $f(f(n)) \sim c^2 n$ thus $c^2 n + cn \sim n$ so $c^2 + c = 1$ hence $c = \frac{\sqrt{5}-1}{2}$. Let's compute now $f(1)$. Assume that $f(1) = a$. Then $f(a) + f(2) = 3$. We have two cases:

a) $f(2) = 1, f(a) = 2$. Immediately $a \geq 3$. Set $n = a - 1$ to get $f(f(a - 1)) + 2 = a$ so $f(f(a - 1)) = a - 2$. This is possible only when $f(a - 1) = a - 2, f(a - 2) = a - 2$. Then set $n = a - 2$ to get

$a-2+a-2=a$ so $a=4$. But this contradicts the fact that $f(2)=1$ as we got $f(a-2)=2$. So this case is impossible.

b) $f(2)=2, f(a)=1$. We claim $a=1$. Indeed assume for sake of contradiction that $a \geq 3$. Set $n=a-1$. Then $f(f(a-1))+a=a+1$ so $f(f(a-1))=1$. Let $f(a-1)=b$. Then $f(b)=1$ so $b \geq 3$. Set $n=b-1$ to get $f(f(b-1))=b$. As $f(k) \leq k$ for $k \geq 2$ this is possible only when $f(b-1)=1$ so $b=a$. But then $f(a-1)=a$, impossible. So $f(1)=1$.

Thus we can conclude that $f(k) \leq k$ for any k and we can determine $f(n)$ by strong induction on n: if we have found $f(1), f(2), \ldots, f(k)$ then by setting $n=k+1$ we compute $f(k+1)$. So f is uniquely determined and it remains to find an example. We seek $f(x) \sim cx$ where $c=\frac{\sqrt{5}-1}{2}$. As $f(1)=1, f(2)=2$ we get $f(3)=2$ then $f(4)=3$, $f(5)=4$, $f(6)=4$. We can conjecture that $f(x)=[cx]+1$, as this holds for $x=1,2,3,4,5,6$. Indeed, we prove $f(x)=[cx]+1$ satisfies the condition. $f(f(n))+f(n+1)=[c([cn]+1)]+1+[cn+c]+1$ so we need to show that $[c[cn]+c]+[cn+c]=n$. We prove this by induction on n by showing that either $[c(n+1)+c]=[cn+c]+1$ and $[c[c(n+1)]+c]=[c[cn]+c]$ or $[c(n+1)+c]=[cn+c]$ and $[c[c(n+1)]+c]=[c[cn]+c]+1$. Indeed, set $x=cn$. Then $[c(n+1)+c]-[cn+c]=[x+2c]-[x+c]$. It equals 0 when $\{x+c\}<1-c$ and 1 otherwise, thus 0 when $1-c<\{x\}<2-2c$ and 1 otherwise. $[c[c(n+1)]+c]-[c[cn]+c]=[c[x+c]+c]-[c[x]+c]$. Let $\{x\}=t$. If $t<1-c$ then $t<c$ $[c[x+c]+c]-[c[x]+c]=[c[x]+c]-[c[x]+c]=0$. If $2-2c>t>1-c$ then $[c[x+c]+c]-[c[x]+c]=[c(x-t+1)+c]-[c(x-t)+c]=[cx+2c-ct]-[cx+c-ct]$. Now $cx=c^2n=n-cn=n-x$ so we substitute to get $[n-x+2c-ct]-[n-x+c-ct]=[2c-ct-x]-[c-ct-x]$. As $\{x\}=t$ it equals $[2c-(c+1)t]-[c-(c+1)t]$. Now $t \in(1-c;2-2c)$ so $2c-(c+1)t \in(2c-2(1-c^2);2c-(1-c^2))=(0;c)$ while $c-(c+1)t \in (-c;0)$ and the difference is 1. Finally if $t>2-2c$ then we get $c-1<2c-(c+1)t<0$ and $-1<c-(c+1)t<-c$ and the difference is zero. The condition we seek therefore holds and $f(x)=[cx]+1$.

Problem 78. Find all functions $f :: N \rightarrow N$ that satisfy $f(1)=1$ and $f(n+1)=f(n)+2$ if $f(f(n)-n+1)=n$, $f(n+1)=f(n)+1$ otherwise.

Solution. Let's put $f(n) \sim cn$. Then $1<c<2$. Also $f(f(n)-n+1)$ must equal n an infinite number of times so $c(cn-n+1) \sim n$ or $c(c-1)=1$ hence $c=\frac{1+\sqrt{5}}{2}$. Next we compute $f(2)=3, f(3)=4$ so we suppose that $f(n)=[cn]$. Firstly we prove that $[cn]$ satisfies the condition. Indeed, let $t=\{cn\}$. Then $[c(n+1)]-[cn]=[cn+c]-[cn]$

is 2 when $t \geq 2 - c$ and 1 otherwise. However $[c([cn] - n + 1)] = [c(cn - t - n + 1)] = [(c^2 - c)n + c - ct] = [n + c(1 - t)]$. It equals n when $1 - t < \frac{1}{c}$ or $t > \frac{1}{-}\frac{1}{c} = 1 - (c - 1) = 2 - c$ and $n + 1$ otherwise. Hence we see that both conditions $[c(n+1)] - [cn] = 2$ and $[c([cn] - n + 1)] = n$ are equivalent to the same condition $\{cn\} > 2 - c$ and thus $[cn]$ satisfies the condition.

Now we prove $f(n) = [cn]$ by induction on n. It's true for $n \leq 3$. Next assume it holds for all $n \leq k$. We prove it for $k+1$. We have $f(k+1) = f(k)+2 = [ck]+2$ if $f(f(k)-k+1) = k$ and $f(k+1) = f(k)+1 = [ck]+1$ otherwise. But $f(k)-k+1 < ck-k+1 = k+1-(2-c)k < k+1$ hence $f(f(k) - k + 1) = [c(f(k) - k + 1)] = [c([ck] - k + 1)]$ using the induction step. However we have proven that $[c(n + 1)] = [cn] + 2$ if $[c([ck] - k + 1)] = k$ and $[c(n + 1)] = [cn] + 1$ otherwise. Therefore $f(k + 1) = [c(k + 1)]$ and the induction is finished.

The Extremal Principle

Problem 4. (IMO '1977). Let $f\colon N \to N$ be a function such that $f(n+1) > f(f(n))$ for all $n \in N$. Show that $f(n) = n$ for all $n \in N$.

Solution. First note that if $f(n) \geq k$ for all $n \geq k$, then

$$f(n + 1) > f(f(n)) \geq f(k) \geq k.$$

Hence it follows by induction on k that$f(n) \geq k$ for all $n \geq k$. In particular, $f(k) \geq k$ for all $k \in N$. Suppose that $f(k) > k$ for some $k \in N$. Denote by $f(m)$ the least element of the set $A = \{f(n)\colon n \geq k\}$. If $m - 1 > k$ then $f(m - 1) \geq m - 1 > k$ and if $m - 1 = k$ then $f(m - 1) = f(k) > k$. Hence $f(m - 1) \in A$. On the other hand $f(m) > f(f(m - 1))$ which is a contradiction. Thus $f(k) = k$ for all $k \in N$.

Problem 5. (BMO '2002) Find all functions $f\colon N \to N$ such that

$$2n + 2001 \leq f(f(n)) + f(n) \leq 2n + 2002$$

for all $n \in N$.

Solution. First we shall show that $f(n) > n$ for all $n \in N$. Assume the contrary and let $m \in N$ be such that $f(m) \leq m$ and $k = f(m)$ be the least possible. Then for $l = f(k)$ we have

$$k + l \geq 2m + 2001 \text{ and } f(l) + l \leq 2k + 2002.$$

Hence

$$2k + 2002 \geq f(l) + 2m + 2001 - k$$

and we get

$$f(l) \leq 3k - 2m + 1 < k < m < l,$$

a contradiction. Hence the function $g(n) = f(n) - n$ is positive. Let $g(p)$ be its least value and set $q = f(p)$. Then it follows by the given condition that

$$2g(p) + g(q) \geq 2001 \text{ and } 2g(q) + g(f(q)) \leq 2002.$$

These inequalities imply that

$$4g(p) \geq 4002 - 2g(q) \geq 2000 + g(f(q)) \geq 2000 + g(p),$$

i.e. $g(p) \geq 667$. Now it follows by the inequality

$$2g(n) + g(f(n)) \leq 2002$$

that $g(n) = 667$, i.e. $f(n) = n + 667$ for all $n \in N$. Conversely, it is easy to check that this function satisfies the given conditions.

Substitutions

Problem 123. Find all continuous functions $f\colon R \to R$ that satisfy

$$f(x)y + f(y)x = (x + y)f(x)f(y)$$

Solution. If $f(y) = 0$ then $yf(x) = 0$ so if $y \neq 0$ then f is identically zero. Otherwise f is non-zero on $R \setminus \{0\}$. Then set $y = x$ to get $2xf(x) = 2xf^2(x)$ so $f(x) = 1$. As f is continuous, in this case f is identically 1.

Problem 119. Find all continuous functions $f\colon R \to R$ for which

$$f(x + y) - f(x - y) = 2f(xy + 1) - f(x)f(y) - 4$$

Solution. If we set $x = 0$ we get $f(y) - f(-y) = 2f(1) - f(0)f(y) - 4$. If we set $y = 0$ we get $2f(1) - f(x)f(0) - 4 = 0$, so if f is not constant then $f(0) = 0$ hence $f(1) = 2$ and from the first relation we get $f(y) - f(-y) = 0$ so f is even. (Note that if $f = c$ then $2c - c^2 - 4 = 0$ which has no real roots so f cannot be constant). Next set $y = 1$ to get $f(x + 1) - f(x - 1) = 2f(x + 1) - 2f(x) - 4$ so $f(x + 1) - 2f(x) + f(x - 1) = 4$. From here we deduce by induction that $f(n) = 2n^2$ for all integers n and that $f(x + n) = 2n^2 + bn + c$ for some b, c depending only on x and not on $N \in N$. For $x = \frac{1}{2}$ we have $b = 0$ because $f(\frac{1}{2}) = f(-\frac{1}{2})$. Next set $y = \frac{1}{2}$ to get $f(x + \frac{1}{2}) - f(x - \frac{1}{2}) = 2f(\frac{x}{2} + 1) - f(x)f(\frac{1}{2}) - 4$. If now $x = 2k$ where $k \in N$ we get $(2(2k+\frac{1}{2})^2+c)-(2(2k-\frac{1}{2})^2+c)=2(k+1)^2-8k^2(\frac{1}{2}+c)-4$ and we deduce $c=0$ so $f(k+\frac{1}{2})=2(k+\frac{1}{2})^2$. Now we prove that $f(t)=2t^2$ for all $t=\frac{a}{2^k}$ where $a,k\in N$ by induction on k. The basis is proven. For the induction step, write the previously obtained relation $f(x+\frac{1}{2})-f(x-\frac{1}{2})=2f(\frac{x}{2}+1)-f(x)f(\frac{1}{2})-4$. Now if $t=\frac{a}{2^k}$ where $k>1$ and a is odd then taking $x=2(t-1)$ and plotting it into the relation we get $f(t)=2t^2$ as desired. As the numbers $\frac{a}{2^k}$ are dense in R $f(x)=2x^2$.

Problem 104.(ISL 2000)Find all pairs of functions $f::R\to R$ that obey the identity

$$f(x+g(y))=xf(y)-yf(x)+g(x)$$

Solution. If f is identically zero then so is g. Now assume f is not identically zero. Let's prove g take the value zero. Set $x=0$ to get $f(g(y))=-yf(0)+g(0)$. Particularly f takes the value 0 because if $f(0)\neq 0$ then $-yf(0)+g(0)$ is surjective. Next $setx\to g(x)$ to obtain $f(g(x)+g(y))=g(x)f(y)-yf(0)+xyf(0)+g(g(x))$. Swapping x and y we get $f(g(x)+g(y))=f(x)g(y)-xf(0)+xyf(0)+g(g(y))$. Therefore $g(x)f(y)-yf(0)+f(f(x))=g(y)f(x)-xf(0)+g(g(x))$ (1). Now set $y=t$ with $f(t)=0$ to get $-tf(0)+g(g(x))=g(t)f(x)-xf(0)+g(g(t))$ so $g(g(x))=c-ax+uf(x)$ where $c=tf(0)+g(g(t)),a=f(0),u=g(t)$. If we substitute into (1) we get $g(x)f(y)+uf(x)=g(y)f(x)+uf(y)$ hence $(g(x)-u)=\frac{g(y)-u}{f(y)}f(x)$ if $f(y)\neq 0$. Taking a fixed value of y in which f does not vanish we get $g(x)-u=kf(x)$ so $g(x)=kf(x)+u$ or g depends linearly on f. If $k=0$ then g is constants and $f(x+u)=xf(y)-yf(x)+u$ so setting $y=x$ gives us $f(x+u)=u$ so $f=g=u$. Now assume $k\neq 0$. We have $g(g(x))=c-ax+uf(x)$ and also $f(g(x))=-xf(0)+g(0)$. But $g(g(x))=kf(g(x))+u=k(-xf(0)+g(0))+u=-kf(0)x+kg(0)+u$. So $c-ax+uf(x)=-kf(0)x+kg(0)+u$. If $u\neq 0$ then we express f as a linear function hence so is g. If $u=0$ then $g(t)=0$ and $-kf(0)x+g(0)=c-ax$ hence $f(0)=a=-kf(0),c=g(0)$. Then set $y=t$ to get $f(x)=-tf(x)+kf(x)=(k-t)f(x)$ so $k-t=1$. Now the original condition can be rewritten as $f(x+kf(y))=xf(y)-yf(x)+kf(x)$. Particularly we deduce f is injective as if $f(y_1)=f(y_2)$ then by setting $y=y_1,y_2$ we conclude $y_1f(x)=y_2f(x)$ for all x hence $y_1=y_2$. If $f(0)=0$ then $f(g(y))=g(0)$ so the injectivity of f implies g is constant hence so is f. Otherwise we get $f(kf(y))=-yf(0)+g(0)=(k-y)f(0)$. Now set $x\to kf(x)$ to get $f(kf(x)+kf(y))=kf(x)f(y)+(k-y)f(kf(x))=kf(x)f(y)+(k-y)^2f(0)$. By symmetry we deduce also $f(kf(x)+$

$kf(y)) = kf(x)f(y) + (k-x)^2 f(0)$ so $(k-x)^2 f(0) = (k-y)^2 f(0)$ for all x, y which is impossible.

We have thus proven that f, g are linear functions. Set $f(x) = ax+b, g(x) = cx+d$. We substitute to get $f(x+cy+d) = x(ay+b) - y(ax+b) + cx + d$ or $ax + acy + ad + b = (b-d)x - by + d + cx$ so $(a+d-b-c)x + (ac+b)y + ad + b - d = 0$. So $b = -ac, d = b+c-a = c - ac - a$. Then $ad + b - d = 0$ so $a(c - ac - a) - ac - c + ac + a = 0$ or $ac - a^2c - a^2 - c + a = 0$ so $c(a^2 - a + 1) = a(1-a)$ hence $c = \frac{a(1-a)}{a^2-a+1}$.

Problem 136. If $a > 0$ find all continuous functions f for which

$$f(x+y) = a^{xy} f(x) f(y)$$

Solution. The function $g(x) = \frac{f(x)}{a^{\frac{x^2}{2}}}$ will satisfy $g(x+y) = g(x)g(y)$ hence $g(x) = a^{bx}$ for some b so $f(x) = a^{\frac{x^2}{2}+bx}$.

Problem 138. Find all continuous functions $f\colon R \to R$ that satisfy

$$f(x+y)\frac{f(x) + f(y) + 2f(x)f(y)}{1 - f(x)f(y)}$$

Solution. If we let $t(x) = \frac{x}{x+1}$ then we observe that $t(f(x+y)) = t(f(x)) + t(f(y))$. Thus $t(f(x)) = ax$ hence $f(x) = \frac{t(x)}{1-t(x)} = \frac{ax}{1-ax}$.

Problem 130. Find all continuous function $f \cdot (a;b) \to R$ that satisfy $f(xyz) = f(x) + f(y) + f(z)$ whenever $xyz, x, y, z \in (a;b)$, where $1 < a^3 < b$.

Solution. Let $a = e^k, b = e^l$ with $0 < 3k < l$. Consider the function $g\colon (0; l-k) \to R, g(t) = f(e^{k+t})$. The condition rewrites as $g(u+v+w+2k) = g(u) + g(v) + g(w)$ whenever $u+v+w < l-3k$. Particularly $g(u+v+2k) = g(u) + g(v) + g(0) = g(u+v) + 2g(0)$ for $u+v < k-2l$. (Note that $g(0)$ is actually not defined but we can define $g(0) = \frac{1}{3}g(2k)$ and observe that when $c \to 0$ we have $g(3c+2k) = 3g(c)$ so by continuity $g(x) \to \frac{g(2k)}{3} = g(0)$. The condition $g(u)+g(v)+g(0) = g(u+v)+2g(0)$ is then obtained by taking $c \to 0$ in $g(u)+g(v)+g(2c) = g(u+v)+2g(c))$). Then $g - g(0)$ is additive and continuous on $[0; l-3k)$ which implies $g(x) = cx + d$ on $[0; l-3k]$. Then if $t < l-3k$ then we have $g(2k+t) = g(\frac{t}{3}) + g(\frac{t}{3}) + g(\frac{t}{3}) = ct + 3d$. So $f(x) = c\ln x + d$ for $x \in (a; \frac{b}{a^2})$ and $f(x) = x\ln x + 3d$ for $x \in (a^3; b)$. Note that if $b < a^5$ then the condition says nothing about f on $(\frac{b}{a^2}; a^3)$ except that it's continuous: if $xyz \in (a;b)$ then $xyz > a^3$ $x, y, z < \frac{b}{a^2}$ because $b > xyz > xa^2, ya^2, za^2$. So in this case every continuous function that satisfies $f(x) = c \ln x + d$ for $x \in (a; \frac{b}{a^2})$ and $f(x) = c \ln x + 3d$ for $x \in (a^3; b)$ is a solution. If $b \geq a^5$ then $\frac{b}{a^2} > a^3$ hence $f(\frac{b}{a^2}) = c \ln(\frac{a}{b^2}) + d$ from one side and $c \ln(\frac{a}{b^2}) + 3d$ from the other side. So $d = 0$ and since the intervals $(a; \frac{b}{a^2}]$ and $[a^3; b)$ cover (a, b) we have $f(x) = c \ln x$ the only solutions.

Problem 131. Find all continuous functions $f\colon R \to R$ that satisfy

$$f(xy) = xf(y) + yf(x)$$

Solution. If $y = 0$ we get $f(0) = xf(0)$ so $f(0) = 0$. If $x, y \neq 0$ then divide the condition by xy to get $\frac{f(xy)}{xy} = \frac{f(y)}{y} + \frac{f(x)}{x}$. Hence if we denote $g(u) = \frac{f(e^u)}{e^u}$ we get $g(u+v) = g(u) + g(v)$. As g is additive and continuous we get $g(x) = cx$ thus $f(x) = cx \ln x$ for $x > 0$. For $x < 0$ set $y = x$ to get $f(x^2) = 2xf(x)$ so $f(x) = \frac{f(x^2)}{2x} = \frac{cx^2 \ln -x^2}{2x} = cx \ln |x|$. So $f(x) = cx \ln |x|$ for all $x \neq 0$, $f(0) = 0$. We only need to check the continuity in zero which is equivalent to $\lim_{t \to 0^+} t \ln t = 0$. But if $t = \frac{1}{e^x}$ this turns to $\lim_{x \to \infty} \frac{x}{e^x} = 0$ which is true.

Problem 46. Find all functions $f\colon Q^+ \to Q^+$ that obey the relations

$$f(x+1) = f(x) + 1$$

if $x \in Q^+$ and

$$f(x^3) = f(x)^3$$

if $x \in Q^+$.

Solution. It's clear that the identity function satisfies the condition. Now pick up any rational number $r = \frac{p}{q}$. Then we must have $f(r+k) = f(r) + k$ for $k \in Z$ and then $f((r+k)^3) = f(r+k)^3 = (f(r)+k)^3 = f^3(r) + 3kf^2(r) + 3k^2 f(r) + k^3 = f(r^3) + 3kf^2(r) + 3k^2 f(r) + k^3$. But $f((r+k)^3) = f(r^3 + 3r^2k + 3rk^2 + k^3)$ and if $q^2 | k$ then r^2k, rk^2, k^3 are integers hence $f((r+k)^3) = f(r)^3 + 3r^2k + 3rk^2 + k^3$. So the identity $3kf^2(r) + 3k^2 f(r) + k^3 = 3r^2k + 3rk^2 + k^3$ holds for all k divisible by q^2. The identity can be rewritten as $(f(r) - r)(3k^2 + 3k(f(r) + r)) = 0$ and $3k^2 + 3k(f(r) + r)$ as a quadratic in k will not vanish for all k divisible by q^2 (there are infinitely many of them). Therefore $f(r) - r = 0$ so f is the identity function.

Problem 89. Show that if $f\colon R \to R$ satisfies

$$f(xy) = xf(x) + yf(y)$$

then f is identically zero.

Solution. Set $y = 1$ to get $f(x) = xf(x) + f(1)$ so $(1 - x)f(x) = f(1)$. For $x = 1$ we get $f(1) = 0$ so $(1 - x)f(x) = 0$ hence $f(x) = 0$ for $x \neq 1$. Hence f is identically zero.

Problem 44. Find all functions $f\colon R \to R$ that obey the condition

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

.

Solution. We can guess the solution $f(x) = x^2$ together with the trivial $f = 0$. By setting $y = 0$ we get $f(f(x)) = x^2$. If f would be injective, we would immediately conclude that $f(x) = x^2$. Assume not that $f(x_1) = f(x_2)$. Then substituting $x = x_1, x = x_2$ into the condition we get $f(x_1^2 - y) = f(x_2^2 - y)$ so $f(t) = f(t + b)$ where $b = x_1^2 - x_2^2$. If $b \neq 0$ then we can find t_0 such that $(t_0 + b)^2 - t_0^2 = a$ where $0 \leq a \leq \frac{b^2}{2}$ is a fixed number, thus setting $x_1 = t_0, x_2 = t_0 + b$ we get $f(t+a) = f(t)$. Hence if $|x-y| \leq \frac{b^2}{2}$ we can assert that $f(x) = f(y)$ hence f is a constant function c and substituting into the condition we get $c = 0$ which contradicts the fact that we've picked up f not identically zero. The contradiction arose from the fact that we assumed that $f(x_1) = f(x_2)$ and $x_1^2 \neq x_2^2$. Thus $f(x) = f(y)$ can hold only if $y = \pm x$. Returning to $f(f(x)) = f(x^2)$ we deduce $f(x) = \pm x^2$ for any x. Now assume for some $x_0 \neq 0$ we have $f(x_0) = -x_0^2$. Then setting $x = x_0$ we get $f(-x_0^2 + y) = f(x_0^2 - y) - 4x_0^2 y$ therefore $f(-x_0^2 + y) - f(x_0^2 - y) = -4x_0^2 y$. As $f(-x_0^2 + y) = \pm(x_0^2 - y)^2, f(x_0^2 - y) = \pm(x_0^2 - y)^2$, we have $f(-x_0^2 + y) - f(x_0^2 - y) = 0, 2(x_0^2 - y)^2$ or $-2(x_0 - y)^2$. So for any y, $-4x_0^2 y \in \{0, 2(x_0^2 - y)^2$ or $-2(x_0 - y)^2\}$. However for $x_0 \neq 0$ each of $-4x_0^2 y = 0, -4x_0^2 y = 2(x_0^2 - y)^2, -4x_0^2 y = -2(x_0^2 - y)^2$ is an equation in y which is either linear or quadratic hence has at most two solutions as it's leading coefficient is not zero. Therefore we have at most 6 values of y for which the assertion holds, contradiction. So $f(x) = x^2$.

Problem 52. Let $k \in R^+$. Find all functions $f\colon [0,1]^2 \to R$ such that the following four conditions hold for all $x, y, z \in [0;1]$:

i)

$$f(f(x,y),z) = f(x, f(y,z))$$

ii)

$$f(x,y) = f(y,x)$$

iii)

$$f(x,1) = x$$

iv)

$$f(zx, zy) = z^k f(x,y)$$

Solution. Assume that $x \leq y$ (the second case is reduced to the first by ii)). Then set $z = y$ and we have $f(x,y) = f(z\frac{x}{y}, z) = z^k f(\frac{x}{y}, 1) = y^k \frac{x}{y} = xy^{k-1}$. Hence we have $f(x,y) = min\{x,y\}max\{x,y\}^{k-1}$. This function clearly satisfies the last three conditions. We've got just the first one to check. Assume that $x < y = z$. Then $f(f(x,y),z) = f(xy^{k-1},z) = f(xy^{k-1},y)$. If $x < (\frac{1}{y})^{2-k}$ then $xy^{k-1} < y$ hence $f(f(x,y),z) = xy^{k-1}y^{k-1} = xy^{2k-2}$. From the other side $f(x, f(y,z))) = f(x, y^k)$ and if $x < y^k$ then $f(x, f(y,z)) = xy^{k(k-1)}$. So $xy^{2k-2} = xy^{k(k-1)}$ for all $x, y \in [0;1]^2$ that satisfy $x < y^l, x < y^{2-k}$. This is possible only when $k(k-1) = 2k-2$ so $k = 1$ or $k = 2$. If $k = 1$ then $f(x,y) = min\{x,y\}$ and if $k = 2$ then $f(x,y) = xy$. It's obvious that both functions satisfy the first condition.

Problem 68. Find all continuous functions $f \colon R \to R$ that satisfy $3f(2x+1) = f(x) + 5x$.

Solution. If we seek $f(x) = ax+b$ then we must have $3(2ax+a+b) = (a+5)x+b$ hence $3b+3a = b, 6a = a+5$ hence $a = 1, b = -\frac{3}{2}$. Then if we set $g(x) = f(x) + \frac{3}{2} - x$ then we get $3g(x) = g(2x+1)$. If we let $h(x) = g(x-1)$ we get $3h(x+1) = h(2x+2)$ or $3h(2x) = h(x)$ or $h(x) = \frac{1}{3}h(2x)$ thus $h(x) = \frac{1}{3^n}h(\frac{x}{2^n})$. As $\frac{x}{2^n}$ tends to zero, $f(\frac{x}{2^n})$ tends to $f(0)$ so $\frac{1}{3^n}f(\frac{x}{2^n})$ tends to zero. Hence $h = 0$ thus $f(x) = x - \frac{3}{2}$.

Problem 8. (Shortlisted problems for IMO '2002) Find all functions $f \colon R \to R$ such that

$$f(f(x)+y) = 2x + f(f(y)-x)$$

for all $x, y \in R$.

Solution. Setting $y = -f(x)$ in the given equation gives

$$f(f(-f(x)) - x) = -2x + f(0)$$

for all $x \in R$. Hence the function $f(x)$ is surjective since the function $-2x + f(y)$ takes all real values when x runs over R. Hence there is $a \in R$ such that $f(a) = 0$. Setting $x = a$ in the given equation gives

$$f(y) = 2a + f(f(y) - a)$$

for all $y \in R$. Set $z = f(y) - a$. Then $f(z) = z - a$ for all $z \in R$ since the function $f(y) - a$ takes all real values. Conversely, it is easy to check that for any $a \in R$ the function $f(x) = x - a$ satisfies the given condition.

Problem 9. (Bulgaria, 1996) Find all functions $f\colon R \to R$ such that

$$f(f(x) + xf(y)) = xf(y+1)$$

for all $x, y \in R$.

Solution. Set $f(0) = a$. Then the given equation for $x = 0$ gives $f(a) = 0$ and setting $y = a$ gives

$$f(f(x)) = xf(a+1). \tag{1}$$

We shall consider two cases.

1. Let $f(a+1) = 0$. Then $f(f(x)) = 0$ for all $x \in R$. Suppose that $f(y+1) \neq 0$ for some $y \in R$. Then the given equation shows that the function $f(x)$ is surjective. Then the function $f(f(x))$ is also surjective which contradicts to $f(f(x)) \equiv 0$. Thus $f(x) \equiv 0$.

2. Let $f(a+1) \neq 0$. Then it follows from (1) that the function $f(x)$ is injective (if $f(x_1) = f(x_2)$ then $x_1 f(a+1) = f(f(x_1)) = f(f(x_2)) = x_2 f(a+1)$, i.e. $x_1 = x_2$). Setting $x = 1$ in the given condition gives

$$f(f(1) + f(y)) = f(y+1),$$

i.e. $f(1)+f(y) = y+1$. In particular, $f(1)+f(1) = 1+1$, i.e. $f(1) = 1$. Hence $f(y) = y$ for all $y \in R$.

Thus there are two functions satisfying the given functional equation - $f(x) = 0$ and $f(x) = x$.

Problem 10. (BMO '1997 and BMO '2000) Find all functions $f\colon R \to R$ such that

$$f(xf(x) + f(y)) = f^2(x) + y$$

for all $x, y \in R$.

Solution. Setting $x = 0$ in the given equation gives $f(f(y)) = f^2(0) + y$. In particular, $f(b) = 0$ for $b = -f^2(0)$. Then

$$f(f(y)) = f(bf(b) + f(y)) = f^2(b) + y = y$$

which shows that $f(0) = 0$ and $f(f(y)) = y$ for all $y \in R$. Now setting $y = 0$ in the given equation we get $f(xf(x)) = f^2(x)$. Hence

$$f^2(x) = f(xf(x)) = f(f(x)f(f(x))) = f^2(f(x)) = x^2,$$

i.e. $f(x) = \pm x$ for all $x \in R$. Suppose that $f(x) = x$ and $f(y) = -y$ for some x and y. Then we get from the given equation that $\pm(x^2 - y) = x^2 + y$ which implies $x = 0$ or $y = 0$.

Thus the only solution of the problem are the functions $f(x) = x$ and $f(x) = -x$.

Problem 13. (USA, 2002) Find all functions $f\colon R \to R$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all $x, y \in R$.

Solution. Setting $y = 0$ gives $f(x^2) = xf(x)$ which implies $f(0) = 0$. If $x \neq 0$ then $xf(x) = f(x^2) = -xf(-x)$, i.e. $f(-x) = -f(x)$. Hence $f(-x) = -f(x)$ and $f(x^2 - y^2) = f(x^2) - f(y^2)$ for all $x, y \in R$. These two equations imply that

$$f(x) + f(y) = f(x + y)$$

for all $x, y \in R$. Note that the above equation together with $f(x^2) = xf(x)$ is equivalent to the given equation. For any x and $y = 1 - x$ we get

$$f(x) - f(y) = f(x - y) = f(x^2 - y^2) = xf(x) - yf(y) =$$

$$= xf(x) - (1 - x)f(y) = x(f(x) + f(y)) - f(y) = xf(1) - f(y),$$

i.e. $f(x) = xf(1)$. Hence $f(x) = cx$, where $c \in R$ is a constant.

Additive Cauchy Equation

Problem 12. (Bulgaria, 1994) Find all functions $f\colon R \to R$ such that

$$xf(x) - yf(y) = (x - y)f(x + y)$$

for all $x, y \in R$.

Solution. It follows from the given equation that

$$(x + y)f(x + y) - yf(y) = xf(x + 2y).$$

Subtracting this from the given equation gives

$$f(x) + f(x + 2y) = 2f(x + y)$$

which is equivalent to

$$f(x) + f(y) = 2f\left(\frac{x + y}{2}\right) \tag{1}$$

for all $x, y \in R$. Set $b = f(0)$. Then

$$f(x) + b = 2f(\frac{x}{2})$$

which together with (1) gives

$$f(x) + f(y) = f(x + y) + b.$$

Now using the given equation we get

$$x(f(y) - b) = y(f(x) - b).$$

Hence $f(x) - b = x(f(1) - b$, i.e. $f(x) = ax + b$, where a and b are constants.

Conversely, it is easily checked that any linear function satisfies the given equation.

Problem 109. Find all functions $f :: R \to R$ that satisfy

$$f(x + y) + f(xy) = f(x)f(y) + 1$$

Solution. We can guess $f(x) = x + 1$ as a solution. If $x = y = 0$ we get $2f(0) = f(0)^2 + 1$ so $f(0) = 1$. If we set $y = 1$ we get $f(x+1)+f(x) = f(x)f(1)+1$. Set $f(1) = a$. Then we get $f(x+1) = (a - 1)f(x) + 1$. If $a = 1$ we get $f(x + 1) = 1$ so f is identically 1. Otherwise we get $1 = f(-1 + 1) = (a - 1)f(-1) + 1$ so $f(-1) = 0$. Also $f(2) = a(a-1)+1 = a^2 - a+1$. $f(3) = (a-1)(a^2 - a+1)+1 = a^3 - 2a^2 + 2a$, $f(4) = (a-1)(a^3 - 2a^2 + 2a) + 1 = a^4 - 3a^3 + 4a^2 - 2a + 1$. Now set $x = y = 2$ into the condition to get $2f(4) = f(2)^2 + 1$ so $2(a^4 - 3a^3 + 4a^2 - 2a + 1) = a^4 - 2a^3 + 3a^2 - 2a + 2$ so $a^4 - 4a^3 + 5a^2 - 2a = 0$ so $a(a-2)(a-1)^2 = 0$. As $a \neq 1$ we either have $a = 2$ or $a = 0$. If $a = 0$ we get $f(x+1) = 1-f(x)$ hence $f(x+2) = f(x)$. Then set $y = 2$ to get $f(x+2)+f(2x) = f(2)f(x)+1 = f(x)+1$. As $f(x+2) = f(x)$ we have $f(2x) = 1$ and then for $x = \frac{1}{2}$ we get $f(1) = 1$ contradiction. So $a = 2$ and then $f(x+1) = f(x)+1$ so $f(x) = x+1$ for $x \in Z$. Now as $x+1$ is a solution, we may suppose $f(x) = g(x) + 1$ and try to prove $g(x) = x$. The condition transforms to $g(x+y) + g(xy) = g(x)g(y) + g(x) + g(y)$. Then $g(x) = x$ for $x \in N$ and $g(x + 1) = g(x) + 1$. If we set $y = k \in N$ then $g(x + k) + g(kx) = kg(x) + g(x) + k$ so $g(kx) = kg(x)$. Then $y = x$ gives $g(x^2) + g(2x) = g(x)^2 + 2g(x)$ and as $g(2x) = 2g(x)$ we get $g(x^2) = g(x)^2$ so g is nonnegative on R^+. Now write $y \to y + 1$ to get $g(x+y+1)+g(xy+x) = g(x)g(y+1)+g(x)+g(y+1)$. As $g(x+y+1) = g(x+y)+1, g(y+1) = g(y)+1$ and $g(x+y)+g(xy) = g(x)g(y)+g(y)$, subtracting these two relations we deduce $g(xy + x) = g(xy) + g(x)$. Now if $u, v \neq 0$ we set $y = \frac{v}{u}, x = u$ to get $g(u + v) = g(u + v)$ for $u, v \neq 0$. As $g(0) = 0$ we conclude that g is additive. And since g is nonnegative on R^+ $g(x) = cx$. Since $g(x) = x$ for $x \in N$, $c = 1$. So $f(x) = x + 1$ and $f(x) = 1$ are the solutions.

Problem 30. Find all functions $f\colon N \to N$ such that

$$f(f(m) + f(n)) = m + n$$

for all $m, n \in N$.

Solution. If $f(m_1) = f(m_2)$ the by setting $m = m_1, m_2$ we deduce $m_1+n = m_2+n$ so $m_1 = m_2$. Thus f is injective. Hence if $m+n = k+l$ then $f(f(m) + f(n)) = f(f(k) + f(l))$ so $f(m) + f(n) = f(k) + f(l)$. Therefore $f(m+n-1)+f(1) = f(m)+f(n)$. Hence by setting $m = 2$ we get $f(n+1) = f(n)+f(2)-f(1)$. If we set $f(2)-f(1) = a, f(1) = b$ we deduce by induction on n that $f(n) = a(n-1)+b = an+b-a$. Hence $f(f(m)+f(n)) = f(am+b-a+an+b-a) = f(a(m+n)+2(b-a)) = a(a(m+n) + 2(b-a)) + (b-a) = a^2(m+n) + (2a+1)(b-a)$. As $f(f(m) + f(n)) = m + n$ we get $a^2 = 1$ hence $a = 1$ ($a = -1$ yields f negative for sufficiently big n) and $(2a+1)(b-a) = 3(b-a) = 0$ so $b = a = 1$ and f is the identity function. It satisfies our condition.

Problem 18. Denote by T the set of real numbers greater than 1. Given on $n \in N$ find all functions $f\colon T \to R$ such that

$$f(x^{n+1} + y^{n+1}) = x^n f(x) + y^n f(y)$$

for all $x, y \in T$.

Solution. The solution is similar to that of Problem 13. Setting $x = y$ gives $f(2x^{n+1}) = 2x^n f(x)$. Hence

$$2f(x^{n+1} + y^{n+1}) = f(2x^{n+1}) + f(2y^{n+1}),$$

i.e.

$$2f(x+y) = f(2x) + f(2y)$$

for all $x, y > 1$. Then

$$f(x+y)+f(z) = f\left(2.\frac{x+y}{2}\right)+f\left(2.\frac{z}{2}\right) = 2f\left(\frac{x+y+z}{2}\right) = f(x)+f(z+y)$$

for all $x, z > 2$ and $y > 0$. This shows that the functions $f(x+y)-f(x)$, where $x > 2,\ y > 0$ depends only on y. Set $g(y) = f(x+y) - f(x)$. Then

$$\frac{f(2x)}{2} + \frac{f(2y)}{2} = f(x+y) = f(x) + g(y),$$

$x > 2,\ y > 1$. Hence $f(2x) = 2f(x) + a$, where a is a constant. Then it follows by induction that

$$f(2^k) = 2^{k-2}f(4) + a(2^{k-2} - 1)$$

for any $k \geq 2$. Hence

$$f(2.4^{n+1}) = f(2^{2n+3}) = 2^{2n+1}f(4)+a(2^{2n+1}-1) = 2.4^n f(4)+a(2^{2n+1}-1).$$

On the other hand we have

$$f(2.4^{n+1}) = 2.4^n f(4)$$

and therefore $a = 0$. Then

$$f(x+y) = f(x) + f(y), f(x^{n+1}) = x^n f(x),$$

$x, y > 2$. It follows by induction that $f(kx) = kf(x)$ for any $k \in N$. In particular for any $s \in N$ we have $f(3^s) = 3^{s-1}f(3) = 3^s.c$ where $c = \dfrac{f(3)}{3}$. Thus for any $x > 2$ we have

$$f(x+3^s) = f(x) + f(3^s) = f(x) + 3^s.c \ . \tag{1}$$

Set $k = 3^s$ where $s \in N$. Then

$$\sum_{j=0}^{n+1} \binom{x}{y} f(x^j)k^{n+1-j} = f((x+k)^{n+1}) =$$

$$(x+k)^n f(x+k) = \sum_{j=0}^{n} \binom{n}{j} x^j k^{n-j}(f(x)+kc),$$

where in the last identity we have used (1). Comparing the coefficients of k^n on both sides we get $(n+1)f(x) = f(x) + nxc$, i.e. $f(x) = xc$ for all $x > 2$. Now let $x > 1$ and take $y > 2$. Then

$$c(x^{n+1} + y^{n+1}) = x^n f(x) + y^n f(y) = x^n f(x) + cy^{n+1}$$

which shows that $f(x) = cx$. Thus the solutions of the problem are all functions $f(x) = cx$ where c is a constant.

Problem 20. (Russia '1993). Find all functions $f\colon R^+ \to R^+$ such that

$$f(x^y) = f(x)^{f(y)}$$

for all $x, y \in R^+$.

Solution. We shall show that the function $f(x) = x$ is the only solution of the problem. Suppose that $f(a) \neq 1$ for some $a > 0$. Then

$$f(a)^{f(xy)} = f(a^{xy}) = f(a^x)^{f(y)} = f(a)^{f(x)f(y)},$$

i.e. $f(xy) = f(x)f(y)$. Hence

$$f(a)^{f(x+y)} = f(a^{x+y}) = f(a^x)f(a^y) = f(a)^{f(x)+f(y)},$$

i.e. $f(x+y) = f(x) + f(y)$. Now it follows from Problem 19 ($f(x)$ is bounded from below since $f(x) > 0$) that $f(x) = cx$ where $c = f(1) >$

0. Hence $cx^y = (cx)^{cy}$. In particular $c = c^{cy}$ for all $y > 0$ which shows that $c = 1$.

Problem 21. (generalization of Problem 15) Find all functions $f\colon R^+ \to R^+$ which are bounded from above on an interval and such that

$$f(xf(y)) = yf(x)$$

for all $x, y \in R^+$.

Solution. For any $z > 0$ set $x = \dfrac{z}{f(1)}$ and $y = 1$. Then

$$f(f(z)) = f(f(xf(1))) = f(1.f(x)) = xf(1) = z,$$

i.e. $f(f(z)) = z$. Hence $f(xy) = f(xf(f(y))) = f(y)f(x)$. Set $g(x) = \lg f(10^x), x \in R$. Then $g(x) + g(y) = \lg f(10^x) + \lg f(10^y) = \lg f(10^x)f(10^y) = \lg f(10^x.10^y) = \lg f(10^{x+y}) = g(x+y)$, i.e. $g(x+y) = g(x) + g(y)$ for all $x, y \in R$. Moreover the function $g(x)$ is bounded from above on an interval since $f(x)$ is so. Then $g(x) = cx$ for all $x \in R$, where $c = g(1)$ (see the Remark after Problem 19). Hence $\lg f(10^x) = cx$, i.e. $f(10^x) = (10^x)^c$ showing that $f(x) = x^c$ for all $x \in R^+$. Now the given equation gives $(xf(y))^c = yx^c$, i.e. $x^c.y^{c^2} = yx^c$. Hence $c^2 = 1$, i.e. $c = \pm 1$. Thus $f(x) = x$ for any $x \in R^+$ or $f(x) = \dfrac{1}{x}$ for any $x \in R^+$.

Problem 22. (generalization of Problem 16) Let S be the set of all real numbers greater than -1. Find all functions $f\colon S \to S$ which are bounded from above on an interval and such that

$$f(x + f(y) + xf(y)) = y + f(x) + yf(x)$$

for all $x, y \in S$.

Solution. Replacing x and y by $x - 1$ and $y - 1$, respectively we get $f(x(f(y-1)+1)-1) = y(f(x-1)+1) - 1$ for any $x, y > 0$. Hence the function $g(x) = f(x-1)+1$ satisfies the functional equation $g(xg(y)) = yg(x)$ for $x, y \in R^+$. It follows from Problem 21 that either $g(x) = x$ or $g(x) = \dfrac{1}{x}$, $x \in R^+$. Hence the solutions of the problem are the functions $f(x) = x$ and $f(x) = -\dfrac{x}{1+x}$.

Problem 23. (IMO '2002) Find all functions $f\colon R \to R$ such that

$$(f(x)+f(z))(f(y)+f(t)) = f(xy-zt)+f(xt+yz)$$

for all $x, y, z, t \in R$.

Solution. Setting $y = z = t = 0$ gives $2(f(x)+f(0))f(0) = 2f(0)$. Hence $2(f(0))^2 = f(0)$ and it follows that either $f(0) = \dfrac{1}{2}$ or $f(0) = 0$. If $f(0) = \dfrac{1}{2}$ then $f(x) = \dfrac{1}{2}$ for all $x \in R$ and this function is a solution of the problem.

Suppose now that $f(0) = 0$. Setting $z = t = 0$ gives $f(x)f(y) = f(xy)$. In particular, $f^2(1) = f(1)$, i.e. either $f(1) = 0$ or $f(1) = 1$. If $f(1) = 0$ then $f(x) = f(x)f(1) = 0$ for all $x \in R$ and this function is also a solution of the problem.

So we may assume that $f(1) = 1$. Setting $x = 0, y = t = 1$ gives $f(z) = f(-z)$, i.e. f is an even function. Then the identity $f(x^2) = f^2(x)$ shows that $f(x) \geq 0$ for any x. Now setting $x = t, y = z$ we get

$$(f(x)+f(y))^2 = f(x^2+y^2).$$

Consider the function $g(x) = \sqrt{f(x)}$. Then $g(x^2) = \sqrt{f(x^2)} = f(x) = g^2(x)$ and the above identity implies

$$g(x^2)+g(y^2) = g(x^2+y^2).$$

Thus g is a non-negative even function which is additive on R^+ and $g(1) = 1$ (this function is also multiplicative on R^+). Now it follows from Problem 19 that $g(x) = x$ for $x \in R^+$ and the fact that g is even shows that $g(x) = |x|$ for any $x \in R$. Thus $f(x) = x^2$ and it is easy to check that this function satisfies the given condition (in fact we have to check the Lagrange identity).

Problem 79. (Korea 1998) Find all functions $f\colon N_0 \to N_0$ that satisfy

$$2f(m^2+n^2) = f(m)^2 + f(n)^2$$

for all $m, n \in N_0$.

Solution. Again the clue to the problem is not the condition itself but an observation that follows directly from it: if $m^2+n^2 = x^2+y^2$ then $f(m)^2+f(n)^2 = f(x)^2+f(y)^2$. Using this we can compute $f(n)$ from $f(0)$ and $f(1)$: set $m = n = 1$ to compute $f(2)$, then $m = 0.n = 2$ to compute $f(4)$, $m = 1, n = 2$ to compute $f(5)$, then $m = 3, n = 4$ to compute $f(3)$, $m = n = 2$ to find $f(8)$, $m = 1, n = 3$ to get $f(10)$

then $m = 6, n = 8$ to compute $f(6)$. If $n > 6$ we can compute $f(n)$ inductively on n as follows: we only need to find $x, y, z < n$ with $x^2 + n^2 = y^2 + z^2$ or equivalently $n^2 - y^2 = z^2 - x^2$ or $(n+y)(n-y) = (z-x)(z+x)$. Set $y = n - 2k$ then $(n+y)(n-y) = (2n-2k)2k$ so if we have $z + x = n - k, z - x = 4k$ we satisfy the condition. Now this implies $z = \frac{n-k+4k}{2} = \frac{n+3k}{2}, x = \frac{n-5k}{2}$. So this is possible if $n - k$ is even, $n \geq 5k, \frac{n+3k}{2}$. Now take $k = 1 + (n - 1(\ mod\ 2 \leq 1$. Then $n > 5 \geq 5k, \frac{n+3k}{2} \leq \frac{n+3}{2} < n$ so we are done.

It remains therefore to investigate $f(0)$ and $f(1)$. If we set $m = n = 0$ we get $2f^2(0) = 2f(0)$ so $f(0) = 1$ or $f(0) = 0$. If $f(0) = 1$ set $m = 1, n = 0$ to get $2f(1) = f^2(1) + 1$ so $(f(1) - 1)^2 = 0$ hence $f(1) = 1$. Now $f(x) = 1$ satisfies the conditions. As f can be uniquely computed from $f(0)$ and $f(1)$ we conclude $f(x) = 1$ is the only solution in this case. Now if $f(0) = 0$ then set $m = 1, n = 0$ to get $2f(1) = f^2(1)$ so $f(1) = 0$ or $f(1) = 2$. $f(x) = 0$ satisfies the conditions and $f(0) = f(1) = 0$ so it's the only solution with $f(0) = f(1) = 0$. $f(x) = 2x$ also satisfies the conditions and $f(0) = 0, f(1) = 2$, so it's the solution with $f(0) = 0, f(1) = 2$.

$f(x) = 2x, f(x) = 1, f(x) = 0$ are therefore the solutions to our equation.

Problem 91. Find all functions $f\colon R \to [0; \infty)$ that satisfy

$$f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$$

Solution. If we replace y by $-y$ we see that $f(2xy) = f(-2xy)$ so f is even. Now set $g(x) = f(x^2)$. We claim $g(u) + g(v) = g(u + v)$ for $u, v \geq 0$. Indeed, to prove this we need to find x, y such that $(x^2 + y^2)^2 = u + v, (x^2 - y^2)^2 = u, 4x^2y^2 = v$. Indeed, we get $2x^2 = \sqrt{u+v} + \sqrt{u}, 2y^2 = \sqrt{u+v} - \sqrt{u}$ thus $x = \sqrt{\frac{\sqrt{u+v}+\sqrt{u}}{2}}, y = \sqrt{\frac{\sqrt{u+v}-\sqrt{u}}{2}}$ satisfy our claim. Therefore g is additive on R^+. Moreover by definition g is non-negative hence if $x > y$ the $g(x) = g(y) + g(x - y) \geq g(y)$ so g is non-decreasing. Therefore $g(x) = cx$ for $c \geq 0$ hence $f(x) = cx^2$ for $c \geq 0$. This function satisfies the condition as $(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2$. Note that the key to the proof was exactly this well-known identity which suggested us to substitute $g(x) = f(x^2)$ in order to get additivity.

Problem 92. Find all functions $f :: R \to R$ that satisfy

$$f(y + zf(x)) = f(y) + xf(z)$$

Solution. If f is not identically zero then for $f(z) \neq 0$ $f(x_1) = f(x_2)$ implies $x_1 = x_2$ if we set $x = y_1, y_2$, so f is injective. Also f is surjective as if we fix y, z with $f(z) \neq 0$ then the right-hand side of the condition written for x, y, z will span the whole real line, hence so will the left-hand side. Thus f is a bijection. Now if $x = 0$ then $f(y + zf(0)) = f(y)$ and the injectivity of f implies $y + zf(0) = y$ for all z, possible only for $f(0) = 0$. Next if we set $y = 0, x = 1$ we get $f(zf(1)) = f(z)$ hence $zf(1) = z$ for all z so $f(1) = 1$. Next set $y = 0, z = x$ to get $f(f(x)) = x$. Hence if we set $y = 0, x \to f(x)$ we get $f(zx) = f(z)f(x)$ so f is multiplicative. If we set $z = 1, x \to f(x)$ we get $f(y+x) = f(y)+f(x)$ so f is additive. As we know the only additive and multiplicative function is the identity function. Thus $f(x) = 0$ and $f(x) = x$ are the only solutions

Problem 93. Find all functions $f :: R \to R$ that satisfy

$$f(xf(z) + y) = zf(x) + y$$

Solution. f is obviously surjective because the right-hand side runs through all real numbers when we fix x, z. If $f(y_1) = f(y_2)$ setting $z = 0, y = y_1, y_2$ and comparing the conditions we deduce $y_1 = y_2$. So f is a bijection. Now set $y = 0$ to get $f(xf(z)) = zf(x)$. Particularly for $z = 0$ we get $f(xf(0)) = 0$. As f is injective we deduce $xf(0)$ is constant hence $f(0) = 0$. Now set $x = 0$ to get $f(y) = y$. The identity function satisfies the condition.

Problem 114. Find all continuous functions $f\colon R^n \to R$ that satisfy

$$f(x_1, x_2, \ldots, x_n) + f(y_1, y_2, \ldots, y_n) = f(x_1 + y_1, \ldots, x_n + y_n)$$

Solution. If we set $f_i(x) = f(0, 0, \ldots, 0, x, 0, \ldots, 0)$ where x in the i-th position then $f(x_1, x_2, \ldots, x_n) = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n)$. Now as f is additive so are f_i (just set $x_k = 0, y_k = 0$ for $k \neq i$ to get $f_i(x_i) + f_i(y_i) = f_i(x_i + y_i)$). Moreover since f is continuous so are f_i therefore $f_i(x) = c_i x$. Hence $f(x_1, x_2, \ldots, x_n) = c_1x_1 + \ldots + c_nx_n$, and this function clearly satisfies the condition.

Problem 24. Given an integer $n \geq 2$ find all functions $f\colon R \to R$ such that

$$f(x^n + f(y)) = f^n(x) + y$$

for all $x, y \in R$.

Solution. Set $f(0)=a$. Then

$$f(f(y))=y+a^n \qquad (1).$$

Applying twice this identity we get

$$f(f(x^n+f(f(y))))=x^n+f(f(y))+a^n=x^n+y+2a^n.$$

On the other hand using the given identity and (1) gives

$$f(f(x^n+f(f(y))))=f(f^n(x)+f(y))=f^n(f(x))+y=(x+a^n)^n+y.$$

Hence $x^n+2a^n=(x+a^n)^n$ for any $x\in R$. Comparing the coefficients of x^{n-1} on both sides we conclude that $a=0$. Thus (1) takes the form $f(f(y))=y$ and the given condition for $y=0$ gives

$$f(x^n)=f^n(x) \qquad (2).$$

Then for any $x\in R_0^+$ and $y\in R$ we get

$$f(x+y)=f((\sqrt[n]{x})^n+f(f(y)))=f^n(\sqrt[n]{x})+f(y)=f(x)+f(y),$$

i.e. $f(x+y)=f(x)+f(y)$. Now it follows easily that the function f is additive on the whole R.

Now we shall show that either $f(x)\geq 0$ for any $x\in R^+$ or $f(x)\leq 0$ for any $x\in R^+$. Set $f(1)=b$. Then for any $x\in R$ and $r\in Q^+$ we have

$$\sum_{k=0}^{n}\binom{n}{k}f(x^k)r^{n-k}=f((x+r)^n)=f^n(x+r)=$$

$$=(f(x)+br)^n=\sum_{k=0}^{n}\binom{n}{k}f^k(x)(br)^{n-k}.$$

Comparing the coefficients of r^{n-2} on both sides gives $f(x^2)=b^{n-2}f^2(x)$. This shows that for any $x\in R^+$ the sign of $f(x)$ is that of b^{n-2}.

Thus the additive function f is bounded from below or from above on R^+ and therefore (see the Remark after Problem 19) $f(x)=bx$ for all $x\in R$. Going back to the given condition we get $b(x^n+by)=b^nx^n+y$ for all $x,y\in R$. Hence $b=b^n$ and $b^2=1$ which shows that $b=1$ for n even and $b=\pm 1$ for n odd. Thus the solutions of the problem are the following functions: $f(x)=x$ if n is even; $f(x)=x$ and $f(x)=-x$ if n is odd.

Remarks.

1. For $n=2$ this is Problem 2 from IMO '1992.

2. If $n=1$ the above equation is equivalent to $f(x)+f(y)=0$ and $f(f(x))=0$ for all $x,y\in R$. We should note that there exist functions which satisfy these conditions and are unbounded on any interval.

Problem 134. Let $n \geq 3$ be a positive integer. Find all continuous functions $f\colon [0;1] \to R$ for which $f(x_1) + f(x_2) + \ldots + f(x_n) = 1$ whenever $x_1, x_2, \ldots, x_n \in [0;1]$ and $x_1 + x_2 + \ldots + x_n = 1$.

Solution. If $x, y \in [0;1]$ and $x+y \leq 1$ then we conclude $f(x)+f(y)+f(1-x-y)+f(0)+\ldots+f(0) = 1$ and $f(x+y)+f(0)+f(1-x-y)+f(0)+\ldots+f(0) = 1$ thus $f(x)+f(y) = f(x+y)+f(0)$ so $f(x) = ax+b$. Then $f(x_1)+f(x_2)+\ldots+f(x_n) = a(x_1+x_2+\ldots+x_n)+nb = a+nb$ for $x_1 + x_2 + \ldots + x_n = 1$ hence $a = 1-nb$ so $f(x) = (1-nb)x+b$.

Equations for Polynomials

Problem 28. (Bulgaria '2001) Find all polynomials $P \in R[x]$ such that

$$P(x)P(x+1) = P(x^2)$$

for all $x \in R$.

Solution. If $P \equiv const$ then $P \equiv 0$ or $P \equiv 1$. Suppose now that $P \not\equiv const$. Note that the given identity is satisfied for any $x \in C$. If $\alpha \in C$ is a root of P then $P(\alpha^2) = 0$ and it follows by induction that $P(\alpha^{2^n}) = 0$. Hence either $\alpha = 0$ or $|\alpha| = 1$ since otherwise P would have infinitely many roots.

On the other hand it follows from the given identity that $(\alpha-1)^2$ is a root of P and therefore $\beta((\alpha-1)^2-1)^2$ is also a root of P. Then either $\beta = 0$ or $\beta = 1$, i.e. $\alpha = 0, \alpha = 2$ or $|\alpha(\alpha-2)| = 2$. If $\alpha \neq 0$ then $|\alpha| = 1$ and $|\alpha - 2| = 1$, i.e. $\alpha = 1$.

Hence all the roots of the polynomial $P(x)$ are equal to 0 or 1. Hence $P(x) = cx^n(x-1)^m$ and the given condition implies that $P(x) = x^n(x-1)^n$ where $n \in N$.

Problem 29. (IMO '1979, Shortlisted Problem) Find all polynomials $P \in R[x]$ such that

$$P(x)P(2x^2) = P(2x^3+x)$$

for all $x \in R$.

Solution. If $P \equiv const$ then $P \equiv 0$ or $P \equiv 1$. Now set $P(x) = \sum_{k=0}^{n} a_k x^{n-k}$ where $n = degP \geq 1$ and $a_0 \neq 0$. Comparing the coefficients of x^3n on both sides of the given identity we get $a_0^2 = a_0$, i.e. $a_0 = 1$.

Let $P(x) = x^k P_1(x)$ where $k \geq 0$ and $P_1(0) \neq 0$. Then the given identity can be rewritten as

$$2^k x^2 k P_1(x) P_1(2x^2) = (2x^2+1)^k P_1(2x^3+x).$$

It follows that $k = 0$ since otherwise $P_1(0) = 0$, a contradiction. Now the above identity for $x = 0$ gives $a_n = P(0) = 1$. Hence by the Vieta theorem we see that the product of all roots of P is equal to 1.

Now let $\alpha \in C$ be the root of P whose modulus is a maximum. Then $P(2x^3+\alpha) = 0$ and therefore $|\alpha| \leq 1$ since otherwise $|2\alpha^3+\alpha| \geq |\alpha|(2|\alpha|^2 - 1) > |\alpha|$, a contradiction. Hence $|\alpha| = 1$ and $|2\alpha^2+1| = 1$. Set $\alpha = \cos\varphi + i\sin\varphi$. Then $2\alpha^2 + 1 = (2\cos 2\varphi + 1) - i.2\sin 2\varphi$, i.e. $(2\cos 2\varphi + 1)^2 + (2\sin 2\varphi)^2 = 1$ It follows that $\cos 2\varphi = -1$. Hence $\alpha = \pm i$ and since the coefficients of $P(x)$ are real we conclude that i and $-i$ are roots of $P(x)$. Set $P(x) = (x^2+1)^m Q(x)$ where $m \geq 1$ and $Q(i)Q(-i) \neq 0$. Then using the identity $(x^2+1)((2x^2)^2+1) = ((2x^3+x)^2+1)$ we see that the polynomial $Q(x)$ satisfies the given conditions. Now the same arguments as above show that $Q \equiv 0$ or $Q \equiv 1$ (recall that $Q(i)Q(-i) \neq 0$). Hence the solutions of the problems are the polynomials $Q \equiv 0, Q \equiv 1$ and $Q(x) = (x^2+1)^n$ where $n \in N$.

Another solution is also possible:

If u is a root of P then by setting $x = u$ we get that $2u^3 + u$ is also a root of P. Let r be the root of largest absolute value. Then $|2r^3 + r| \leq |r|$ hence $2|r|^3 - |r| \leq |r|$ so $|r| \leq 1$. So all roots of P have absolute value at most 1. Now set $x \to -x$ to get $P(-x)P(2x^2) = P(-2x^3-x)$. Comparing it with the condition we get $\frac{P(x)}{P(-x)} = \frac{P(2x^3+x)}{P(-2x^3-x)}$. Now if we set $P(x) = Q(x^2)R(x)$ where R has no two roots that add up to zero (thus $R(x), R(-x)$ have no common root) then we have $\frac{P(x)}{P(-x)} = \frac{R(x)}{R(-x)}$. So $\frac{R(x)}{R(-x)} = \frac{R(2x^3+x)}{R(-2x^3-x)}$. Now $R(2x^3+x)$ and $R(-2x^3-x)$ also have no common root: if w is a common root of both $R(2x^3+x)$ and $R(-2x^3-x)$ then $2w^3+w, -2w^3-w$ are two solutions of $R(x)$ which add up to zero. Hence we get $R(2x^3+x)R(-x) = R(-2x^3-x)R(x)$ and thus $R(2x^3+x)|R(x)$. This is possible only when R is constant because otherwise the degree of $R(2x^3+x)$ is higher that the degree of R. So $P(x) = Q(x^2)$ and we get $Q(x^2)Q(4x^4) = Q(4x^6+4x^4+x^2)$. Now if we replace x^2 by x we get $Q(x)Q(4x^2) = Q(4x^3+4x+1)$. All roots of Q have absolute value at most 1, as we have shown. Now let $Q(x) = a\prod_{i=1}^{n}(x - r_i)$ then $Q(4x^2) = 4^n a \prod_{i=1}^{n}(x - \sqrt{\frac{r_i}{4}})(x + \sqrt{\frac{r_i}{4}})$ and $Q(4x^3+4x^2+x) = 4^n a \prod_{i=1}^{n}(x^3 + x^2 + \frac{x}{4} - r_i)$. Consider now the sum of roots of $Q(x)Q(4x^2)$ and the sum of roots of $Q(4x^3+4x^2+1)$. The sum of roots of $Q(x)$ is $\sum_{i=1}^{n} r_i$, the sum of roots of $Q(4x^2)$ is zero, and the sum of roots of $Q(4x^3+4x^2+1)$ is $-n$ because we have n factors $x^3 + x^2 + \frac{x}{4} - r_i$ the sum of whose roots is -1. So $\sum_{i=1}^{n} r_i = -n$ and as $|r_i| \leq 1$ this is possible only when $r_i = -1$. So $Q(x) = a(x+1)^n$. Now if $Q \neq 0$ then we get $a^2 = a$ if looking at the leading coefficient of $Q(x)Q(4x^2) = Q(4x^3 + 4x + x)$ so $a = 1$. Finally as $(x+1)(4x^2+1) = (4x^3+4x^2+x+1)$ we deduce that $Q(x) = (x+1)^n$ satisfies the conditions.

So all solutions to our original problem are $P(x) = 0$ and $P(x) = (x^2+1)^n$.

Problem 30. (Romania '1990) Find all polynomials $P \in R[x]$ such that

$$2P(2x^2 - 1) = P^2(x) - 2$$

for all $x \in R$.

Solution. Suppose that $P(x) \neq P(1)$ and set $P(x) = (x-1)^n Q(x) + P(1)$ where $n \in N$ and $Q(1) \neq 0$. Then

$$4(x-1)^n(x+1)^n Q(2x^2-1) + 2P(1) =$$

$$(x-1)^2 n Q(x) + 2(x-1)^n Q(x)P(1) + P^2(1) - 2$$

and using $2P(1) = P^2(1) - 2$ we get

$$4(x+1)^n Q(2x^2-1) = (x-1)^n Q(x) + 2Q(x)P(1).$$

This identity for $x = 1$ gives $Q(1)(2^{n+1} - P(1)) = 0$. Taking into account that $P(1) = 1 \pm \sqrt{3}$ we get $Q(1) = 0$, a contradiction. Thus $P(x) \equiv P(1)$ and the only solutions are the constant polynomials $P(x) \equiv 1 + \sqrt{3}$ and $P(x) \equiv 1 - \sqrt{3}$.

Problem 60. Let $k, l \in N$ be integers. Find all polynomials P for which $xP(x-k) = (x-l)P(x)$

Solution. If the degree of P is $n, P(x) = ax^n + bx^{n-1} + \dots$ then $P(x-k) = ax^n + (b - nak)x^{n-1} + \dots$ hence $xP(x-k) = ax^{n+1} + (b-nak)x^n, (x-l)P(x) = ax^{n+1} + (b-la)x^n + \dots$ so $l = nk$. So $xP(x-k) = (x-nk)P(x)$. Then $x|P(x)$, k is a root of $P(x)$ hence $2k$ is a root of $P(x-k)$ and as $P(x-k)|(x-nk)P(x)$ we conclude that if $n > 2$ then $2k$ is a root of x and so on, obtaining $x(x-k)\dots(x-(n-1)k)|P(x)$. If $P(x) = x(x-k)\dots(x-(n-1)k)Q(x)$ the condition transforms to $Q(x-k) = Q(x)$ which is possible only for constant Q, so $P = cx(x-k)\dots(x-(n-1)k)$, which is clearly a solution.

Problem 61. Find all nonconstant polynomials P that satisfy $P(x)P(x+1) = P(x^2+x+1)$.

Solution. If P is non-constant, let w be its root of maximal absolute value. Set $x = w$ to conclude that $w_1 = w^2+w+1$ is a root of P and $x = w-1$ to conclude that $w_2 = w^2-w+1$ is a root of P. Then $|w_1-w_2| = 2|w|$ but $|w_1-w_2| \le |w_1|+|w_2| = 2|w|$. The equality can hold only if $w_1+w_2 = 0$ so $w^2+1 = 0$ hence $w = \pm i$. In this case $(w_1, w_2) = (i, -i)$. Thus $x^2+1|P(x)$. However $Q(x) = x^2+1$ satisfies $Q(x)Q(x+1) = Q(x^2+x+1)$, therefore $\frac{P}{Q}$ satisfies the same condition. We can repeat the same operation until we reach a constant polynomial, so $P(x) = c(x^2+1)^n$, which satisfies the condition.

Problem 126. Find all polynomials $P \in C[X]$ that satisfy $P(x)P(-x) = P(x^2)$

Solution. If $P = c$ is constant then $c^2 = c$ so $c = 0, 1$. Assume now P is not constant. If r is a non-zero root of P then r^2 is also a root of P (just set $x = r$). Then r^{2^k} is also a root of P by induction. As P has finitely many roots, $r^{2^k} = r^{2^m}$ for some k, m hence $|r| = 1$. We have found thus a root w for which $w^{2^n} = w$ thus $w^{2^n-1} = 1$. If n is minimal with this property, then $w, w^2, \dots, w^{2^{n-1}}$ are different roots of P hence $Q(x) = (x-w)(x-w^2)\dots(x-w^{2^{n-1}})$ divides P. Now $Q(x)Q(-x) = (x-w)(-x-w)\dots(x-w^{2^{n-1}})(-x-w^{2^{n-1}}) = (-1)^n(x^2-w^2)\dots(x^2-w^{2^n}) = (-1)^nQ(x^2)$ because $w^{2^n} = w$. Thus $(-1)^nQ$ satisfies the original condition, so we can divide P by Q and the condition holds for $(-1)^n\frac{P}{Q}$. Now repeating this argument as many times as necessary we shall reach a polynomial with no non-zero roots. Then if $P(x) = cx^n$ we have $cx^ncx^n(-1)^n = cx^{2n}$ which gives $c = (-1)^n$. Concluding, all set of solutions is given by $P(x) = (-x)^n \prod Q_i(x)$ where each $Q_i = (w-x)(w^2-x)(w^4-x)\dots(w^{2^{m-1}}-x)$ where $w^{2^m} = w$.

Problem 127. Find all polynomials $P(x)$ which are solutions of the equation $P(x^2-y^2) = P(x-y)P(x+y)$

Solution. Suppose w is a root of P. Then $x-y = w$ implies $x^2-y^2 = w(x+y) = w(2x-w)$ is also a root of w. Now if $w \neq 0$ x^2-y^2 can take any value being a non-constant linear function in x so P is identically zero. Thus P is either identically zero or has all roots zero. If $P = cx^n$ we get $c(x^2-y^2)^n = c(x-y)^nc(x+y)^n = c^2(x^2-y^2)^n$ so $c = 1$. So all solutions are $P(x) = 0, P(x) = x^n$ for $n \ge 0$.

Problem 129. Find all polynomials $P \in C[X]$ that satisfy $P(2x) = P'(x)P''(x)$

Solution. If $deg(P) = k > 0$ then the degree of $P'P$" is $2k-3$ (unless $k = 1$ when it is zero) so $k = 3$. Now if the leading coefficient of P is a then the leading coefficients of $P(2x), P'(x), P"(x)$ are $8a, 3a, 6a$ respectively so $8a = 18a^2$ hence $a = \frac{4}{9}$. Now let $P'(x) = \frac{4}{3}(x-4a)(x-4b)$. Then $P"(x) = \frac{8}{3}(x-2a-2b)$ and hence $P(2x) = \frac{4}{9}(x-4a)(x-4b)(x-2a-2b)$ so $P(x) = \frac{4}{9}(x-2a)(x-2b)(x-a-b)$, $P'(x) = \frac{4}{9}(3x^2 - 6(a+b)x + 2a^2 + 2b^2 + 8ab)$. As $P'(x) = \frac{4}{3}(x-4a)(x-4b) = \frac{4}{9}(3x^2 - 12(a+b)x + 48ab)$ we conclude $b = -a$ and $-4a^2 = -48a^2$ hence $a = b = 0$ and $P(x) = \frac{4}{9}x^3$.

Iterations

Problem 33. (Bulgaria '1996) Find all functions $f\colon Z \to Z$ such that

$$3f(n) - 2f(f(n)) = n$$

for all $n \in Z$.

Solution. For a given n set

$$a_0 = n, a_{k+1} = f(a_k), \quad k \geq 0.$$

Then the given identity gives

$$3a_{k+1} - 2a_{k+2} = a_k, \quad k \geq 0.$$

The characteristic equation of this recurrence is $3x - 2x^2 = 1$ with roots 1 and $\dfrac{1}{2}$. Hence

$$a_k = c_0.1^k + c_1\left(\frac{1}{2}\right)^k, \tag{1}$$

where $c_0 = 2a_1 - a_0$ and $c_1 = 2(a_0 - a_1)$. It follows from (1) that 2^k divides c_1 for any $k \geq 0$. Hence $c_1 = 0$ and $a_k = c_0 = 2a_1 - a_0$ for any $k \geq 0$. In particular $a_1 = a_0$, i.e. $f(n) = n$.

Problem 34. Find all functions $f\colon R^+ \to R^+$ such that

$$f(f(x)) + f(x) = 6x$$

for all $x \in R^+$.

Solution. For a given $x \in R^+$ set

$$a_0 = x, \;\; a_{k+1} = f(a_k), \quad k \geq 0.$$

Then we obtain the recurrence relation

$$a_{k+2} + a_{k+1} = 6a_k, \quad k \geq 0.$$

Its characteristic equation

$$x^2 + x = 6$$

has roots 2 and -3. Hence

$$a_k = c_0 2^k + c_1(-3)^k$$

where $c_0 = \dfrac{3a_0 + a1}{5}$ and $c_1 = \dfrac{2a_0 - a_1}{5}$. Note that $\lim\limits_{k\to\infty} \dfrac{3^k}{2^k} = +\infty$. Hence $\lim\limits_{k\to\infty} a_{2k} = -\infty$ if $a < 0$ and $\lim\limits_{k\to\infty} a2k+1 = -\infty$ if $a > 0$. Hence $c_1 = 0$, i.e. $a_{k+1} = 2a_k$ and therefore $f(x) = 2x$ for any $x \in R^+$.

Problem 35. Find all functions $f\colon R^+ \to R^+$ such that

$$f(f(f(x))) + f(f(x)) = 2x + 5$$

for all $x \in R^+$.

Solution. Using the same notation as in the solution of the previous problem we have

$$a_{k+s} + a_{k+2} = 2a_k + 5, \quad k \geq 0. \tag{1}$$

Subtracting this equality from

$$a_{k+4} + a_{k+3} = 2a_{k+1} + 5$$

we get

$$a_{k+4} = a_{k+2} + 2a_{k+1} + 2a_k, \quad k \geq 0.$$

The characteristic equation $x^4 = x^2 + 2x + 2$ can be written as $(x-1)^2(x^2+2x+2) = 0$, i.e. it has a double root 1 and two complex roots $\sqrt{2}\left(\cos\dfrac{\pi}{4} \pm i\sin\dfrac{\pi}{4}\right)$. Hence

$$a_k = c_0 + c_1 k + 2^{\frac{k}{2}}\left(c_2\cos\frac{k\pi}{4} + c_3\sin\frac{k\pi}{4}\right)$$

where the constants c_n, $0 \leq n \leq 3$ are real and depend only on the first four terms of the sequence. Considering the subsequences with indexes congruent respectively to 0, 2, 4, 6 modulo 8 we conclude, as in the solution of the previous problem, that respectively $c_2 \geq 0$, $c_3 \geq 0$, $c_2 \leq 0$, $c_3 \leq 0$, *i.e.* $c_2 = c_3 = 0$. Thus $a_{k+1} = a_k + c_1$ and using (1) we get $c_1 = 1$. Hence $f(x) = x + 1$ for all $x \in R^+$.

Problem 66. Find all continuous functions $f\colon R \to R$ that satisfy

$$f(f(x)) = f(x) + 2x$$

for any $x \in R$.

Solution. It is clear that f is injective. Now we prove it's surjective. Assume for contradiction that $f(x) \neq a$. Then as f is continuous we either have $f(x) > a$ or $f(x) < a$. If $f(x) > a$ then we deduce $f(f(x)) - f(x) = 2x$ so $f(x) > a - 2x$. Now f is injective and continuous, so either increasing or decreasing. If f is increasing then $f(-n) \geq 2n + a$ for $n \in N$ hence $f(0) > f(-n) \geq 2n + a$ for any $n \in N$ which is impossible. If f is decreasing then we get $f(f(x)) \leq f(a)$ hence $f(x) \leq f(a) - 2x < a$ when x is sufficiently large contradiction. Assume now that $f(x) < a$. We then deduce $f(x) < a - 2x, f(f(x)) < 2x - a$. If f is increasing then as $f(x) < x$ for $x > a$ we get $2x = f(f(x)) - f(x) < 0$ for $x > a$ impossible if $x > 0$. If f is decreasing then $f(n) < a - 2n$ for $n \in N$ hence $f(0) < a - 2n$ for all $n \in N$, impossible. So we have proven that f is injective and surjective, so has an inverse g. If we try the usual pattern: setting $a_n = f_n(x)$ and using the recurrence $a_{n+2} = a_{n+1} + 2a_n$, this will lead us nowhere because we cannot link the recurrence with the continuity of f, as the roots of the equation are -1 and 2 so the members of the sequence will get larger and sparser with increasing n. However we may use the backwards formula: if $y = g(x)$ then $2y + x = f(x)$ so $y = \frac{f(x)-x}{2}$. So if we set $a_n = g_n(x)$ then a_n satisfies the recurrence $a_{n+2} + \frac{a_{n+1}}{2} - \frac{a_n}{2}$ and the associated equation $x^2 + \frac{x}{2} - \frac{1}{2} = 0$ has roots $-1, \frac{1}{2}$. The general term has formula $a_n = u(-1)^n + v(\frac{1}{2})^n$ hence the odd terms of the sequence converge to $-u$ and the even to u. As $f(a_{n+1}) = a_n$ using the continuity we deduce $f(u) = -u, f(-u) = u$. Now u can be computed from the initial values: if $a_0 = f(x), a_1 = x$ we have $u + v = f(x), -u + \frac{v}{2} = x$ so $u = \frac{f(x)-2x}{3}$. Let $h(x) = \frac{f(x)-2x}{3}$, $A = Imh$. As h is continuous, A must be connected. Also A is symmetric with respect to the origin, because if $t \in A$ then $f(t) = -t$ hence $\frac{f(t)-2t}{3} = -t$ so $-t \in A$. A is also not empty by the definition. Hence either A consists of a single point 0 so $f(x) = 2x$ or A contains an interval $(-a; a)$. We prove that if A contains an interval then $A = R$. Without loss of generality assume $A = [-a; a]$ or $A = (-a; a)$. Suppose $h(x) = t \neq 0$, $|t| < a$. As $a_{2n} \to t, a_{2n-1} \to -t$, for some n we have $a_{2n} \in A$ so $a_{2n-1} \in A$ thus we have $a_{2n} = -a_{2n-1}$ and thus we can conclude by induction on k $a_{2n-2k} = a_{2n}, a_{2n-2k-1} = a_{2n-1}$ so $f(x) = -x$ and x is in A. We have only one trouble: if A is $[-a; a]$ and $t = \pm a$. Then we cannot say $a_{2n} \in t$ for some n as t is in the boundary of A. We can handle it like this: Let

B be the set of all x for which $f(x) \neq -x$. $B = (-\infty; -a) \bigcup (a; +\infty)$. B can be split in two sets: C of x which satisfy $h(x) = a$ and D of x which satisfy $h(x) = -a$. Both sets are connected and closed therefore either $C = B$, $D = B$, $C = (-\infty; -a), D = (a; +\infty)$ or $D = (-\infty; -a), C = (a; +\infty)$. However $f(x) = 2x + 3a$ if $x \in C$ and $f(x) = 2x - 3a$ if $x \in D$. Thus if $(a; +\infty) \subset C$ we get $f(x_0) = 2x_0 + 3a$ for $x_0 > a$ and taking the limit in a we get $f(a) = 5a$ impossible. Hence $(a; +\infty) \subset D$. Analogously $(-\infty; -a) \subset C$. Thus we conclude $f(x) = -x$ if $|x \leq a|$, $f(x) = 2x - 3a$ if $x > a$, $f(x) = 2x + 3a$ is $x < -a$. Now if $x > 2a$ then $f(x) = 2x - 3a, f(f(x)) = 2f(x) - 3a = 4f(x) - 9a$ but $2x + f(x) = 4x - 3a \neq f(f(x))$ for $a > 0$. So $A = R$ and the solutions are $f(x) = 2x, f(x) = -x$.

Problem 67. Find all increasing bijections f of R onto itself that satisfy

$$f(x) + f^{-1}(x) = 2x$$

where f^{-1} is the inverse of f.

Solution. Set $x \to f(x)$ into the condition to get $f(f(x)) = 2x - f(x)$. Hence if we set $a_n = f_n(x)$ we get $a_{n+2} = 2a_{n+1} - a_n$ so $a_{n+2} - a_{n+1} = a_{n+1} - an$ and from here we get $a_n = a_1 + (n-1)(a_2 - a_1)$. So $f(x + k(f(x) - x)) = x + (k+1)(f(x) - x)$. Now we prove $f(x) - x$ is constant. Indeed assume that $a = f(u) - u < b = f(v) - v$. We shall find such m that satisfy $u + ka \leq v + lb$ and $u + (k+1)a < v + (l+1)b$. If $\frac{a}{b}$ is rational then there is an $x > 0$ such that $a = px, v = qx$ where $(p, q) = 1$ and $p < q$. Also set $[\frac{u-v}{x}] = r$. Then we take k, l with $lq - kp = r$ hence $lb - ka = rx$. So $(u + ka) - (v + lb) = (u - v) - (lb - ka) = (u - v) - [\frac{u-v}{x}]x \geq 0$ but $(u + (k+1)a) - (v + (l+1)b) = (u - v) - [\frac{u-v}{x}] + a - b < x + a - b = x + px - qx = (p + 1 - q)x \leq 0$. If $\frac{a}{b}$ is irrational then by Kronecker's Theorem we can find k, l with $u - v + a - b < lb - ka < u - v$ hence $u + ka > v + lb$ while $u + (k+1)a < v + (l+1)b$.

But $f(u + ka) = u + (k+1)a, f(v + lb) = v + (l+1)b$ and this contradicts the fact that f is increasing, contradiction. Hence $f(x) = x + c$ and all such functions satisfy the conditions.

Problem 40. (M^+209) Find all functions $f\colon R \to R$ such that

$$f(x+1) \geq x + 1 \text{ and } f(x+y) \geq f(x)f(y)$$

for all $x, y \in R$.

Solution. The second condition implies that $f(x_1+x_2+\cdots+x_n) \geq f(x_1)f(x^2)\dots f(x_n)$ for any $x_i \in R,\ 1 \leq i \leq n$. In particular

$$f(x) \geq f^n\left(\frac{x}{n}\right)$$

for any $x \in R$ and $n \in N$. Hence

$$f(x) \geq f^n\left(\frac{x}{n}\right) \geq \left(1+\frac{x}{n}\right)^n$$

and letting $n \to \infty$ gives $f(x) \geq e^x$. In particular $f(0) = 1$. Now

$$1 = f(0) \geq f(x)f(-x) \geq e^x.e^{-x} = 1$$

which shows that $f(x) = e^x$. It is easy to check that this function satisfies the conditions of the problem.

Problem 41. (Belarus '1998) Prove that:

a) if $a \leq 1$ then there is no function $f\colon R^+ \to R^+$ such that

$$f\left(f(x)+\frac{1}{f(x)}\right) = x + a \tag{1}$$

for all $x \in R^+$;

b) if $a > 1$ then there are infinitely many functions $f\colon R^+ \to R^+$ satisfying (1).

Solution. a) Suppose that $f\colon R^+ \to R^+$ satisfies (1). Set

$$g(x) = f(x) + \frac{1}{f(x)}.$$

If $f(x) > 1$ then $y = f(x) - a > 0$ and $f(g(y)) = f(x)$. But (1) implies that the function f is injective and therefore $x = g(y) \geq 2$. Hence $f(x) < 1$ for any $x \in (0,2)$ with at most one exception. For any $x \in (0,2)$ we have

$$y = \frac{1}{f(x)} - a > 0,\ \ z = f(g(y)) \geq 2,\ \ f(z) = \frac{1}{f(x)},$$

$$z + a = f(g(z)) = f(g(x)) = x + a.$$

Thus $2 \leq z = x < 2$, a contradiction.

b) Let $a > 1$ and f be an arbitrary strongly increasing continuous function on the interval [0,2] such that $f(0) = 1$ and $f(2) = a$. Then (1) defines a unique strongly increasing continuous function f on R_0^+. Indeed, let $g(x) = f(x) + \dfrac{1}{f(x)}$ and define the sequence $\{a_n\}_{n=0}^{\infty}$ by: $a_0 = 0,\ \ a_n = g(a_{n-1}),\ \ n \in N$. Suppose that f has been defined as a strongly increasing and continuous function on the interval $(a_{n-1}, a_n]$ greater than 1. Then it is easy to check that $g(x)$ is strongly increasing and continuous function on this interval with $g(a_{n-1}) = a_n$, $g(a_n) = a_{n+1}$. Hence for any $x \in (a_n, a_{n+1}]$ there is a unique $y \in (a_{n-1}, a_n]$ such that $x = g(y)$ and set $f(x) = y + a$. It follows by induction that the function $f(x)$ is defined on R_0^+ and it satisfies (1) on R^+.

Problem 42. (Bulgaria '2003) Find all $a > 0$ for which there exists a function $f\colon R \to R$ having the following two properties:
(i) $f(x) = ax + 1 - a$ for any $x \in [2, 3)$;
(ii) $f(f(x)) = 3 - 2x$ for any $x \in R$.

Solution. Set $h(x) = f(x+1) - 1$. Then the conditions (i) and (ii) can be written as $h(x) = ax$ for $x \in [1, 2)$ and $h(h(x)) = 2x$ for any $x \in R$. Then $h(-2x) = h(h(h(x))) = -2h(x)$; in particular $h(0) = 0$. It follows by induction that $h(4^n x) = 4^n h(x)$ for any integer n and hence $h(x) > 0$ for $x \in [4^n, 2.4^n)$. On the other hand $0 > -2x = h(h(x)) = h(ax)$ for $x \in [1, 2)$ and therefore $[a, 2a) \subset [2.4^k, 4^{k+1})$ for an integer k. Thus $a = 2.4^k$. Conversely, let $a = 2.4^k$ for some integer k. Then one checks easily that the function

$$h(x) = \begin{cases} ax & \text{for} \quad x \in [4^n, 2.4^n), \\ -\dfrac{2x}{a} & \text{for} \quad x \in [2.4^n, 4^{n+1}), \\ 0 & \text{for} \quad x = 0, \\ ax & \text{for} \quad x \in (-4^{n+1}, -2.4^n], \\ -\dfrac{2x}{a} & \text{for} \quad x \in (-2.4^n, -4^n] \end{cases}$$

where n runs over the integers has the desired properties. One can easily prove that this is the only function with the given properties.

Polynomial Recurrences and Continuity

Problem 56. Find all continuous functions $f, g\colon R \to R$ that satisfy

$$f(x+y) + f(x-y) = 2f(x)g(y)$$

Solution. This resembles D'Alembert's equation, but it's a generalized version of it. We try to reduce it to D'Alembert's equation. If f is identically zero, then g could be any function. Otherwise, if $f(x_0) = 0$ we get $f(x_0 + y) + f(x_0 - y) = 2f(x_0)g(y) = f(x_0 - y) + f(x_0 + y) = 2f(x_0)g(-y)$ so g is an even function. Now set $y = 0$ to get $g(0) = 1$. We now distinguish two cases:

a) $f(0) = 0$. In this case set $x = 0$ to get $f(y) + f(-y) = 0$ so f is an odd function. Therefore $f(x + y) + f(x - y) = 2f(x)g(y)$ and $f(x + y) - f(x - y) = f(y + x) + f(y - x) = 2g(x)f(y)$. Hence $f(x+y) = f(x)g(y) + g(x)f(y), f(x-y) = f(x)g(y) - g(x)f(y)$. This resembles the formulae for the sine of sum, with f for sine and g for cosine (this is confirmed by the oddness of f and evenness of g). We only lack the cosine of the sum formula, so let's set $x \to x+y$ into our second formula to get $f(x) = f(x+y)g(y)-g(x+y)f(y) = (f(x)g(y)+g(x)f(y))g(y) - g(x+y)f(y)$ hence $g(x+y) = g(x)g(y) - \frac{f(x)(1-g^2(y))}{f(y)}$. This is not really the formula we expected, but if we replace y by $-y$ we get $g(x-y) = g(x)g(y)+\frac{f(x)(1-g^2(y))}{f(y)}$ so $g(x+y)+g(x-y) = 2g(x)g(y)$ for $f(y) \neq 0$. So g can be found from D'Alembert's equation but we have a problem: if $f(y) = 0$ then the condition $g(x + y) + g(x - y) = 2g(x)g(y)$ may not hold. If there is a sequence y_n that tends to y and $f(y_n) \neq 0$ then it holds by continuity. Otherwise there is an interval I containing y s.t. f is identically zero on I. Let l be the length of I. If we denote by $X - Y$ the set $\{x - y|x \in X, y \in Y\}$ then the formula $f(x - y) = f(x)g(y) - f(y)g(x)$ says that if f is zero on X, Y then it's zero on $X - Y$. Hence f is zero on $I - I = (-l, l)$. Then f is zero on $(l, -l) - (l, -l) = (2l, -2l)$ and so on, f being zero on $(2^k l, -2^k l)$ for any k, so f is identically zero, contradiction. Therefore D'Alembert's equation is satisfied by g for all x and y hence $g(x) = \cos(ax)$ or $g(x) = cosh(ax)$ for some a. Without loss of generality $g(x) = cos(ax)$, the second case being handled analogously. For $a = 0$ $g = 1$ and we get $f(x + y) = f(x) + f(y)$ hence f is linear. Now suppose $a \neq 0$. If now $f(x) = b_x sin(ax)$ for $f(x) \neq 0, \sin ax \neq 0$ then we prove by induction on n that $f(nx) = b_x sin(anx)$, because $c\sin(ax), \cos(ax)$ satisfy our initial condition. Now we claim $b_x = b_y$ for all x, y. If $\frac{x}{y} \in Q$ then there is z such that $x = mz, y = nz$ for $m, n \in N$ thus $b_x = b_z = b_y$. If $\frac{x}{y} \notin Q$ then there is a sequence x_n such that $\frac{x_n}{x} \in Q, x_n \to y$. Then $b_{x_n} = x$ and using the continuity of f we get $f(y) = b_y sin(ay)$ so $b_x = b_y$. We thus conclude that $f(x) = b_{x_0} \sin(ax)$ when $f(x) \neq 0$. If $f(x) = 0$ or $\sin(ax)$ then there is a sequence $x_n \to x$ such that $f(x_n) \neq 0, \sin(ax_n) \neq 0$ and by continuity we get $f(x) = b_{x_0} \sin(ax)$ therefore $f(x) = b\sin(ax), g(x) = \cos(ax)$. It's clear they satisfy our condition. For the case $g(x) = cosh(ax)$ analogously we get $f(x) = bsinh(ax)$.

b) $f(0) \neq 0$. We employ case a. Set $f^+(x) = f(x) + f(-x), f^-(x) = f(x) - f(-x)$. If f satisfies the condition then $-f$ also satisfies the condition, hence so do f^+, f^- (the condition is linear in f). But f^- falls into case a) Therefore $g(x) = \cos(ax), f^-(x) = b\sin(ax)$ or $g(x) = cosh(ax), f^-(x) = bsinh(ax)$. We can also suppose $f^+(0) = 1$ because we can multiply or divide f by any constant we want. Now we have $f^+(x+y)+f^+(x-y)=2f^+(x)g(y)$ and $f^+(x+y)+f^+(y-x)=2g(x)f^+(y)$ if x,y change places. As f^+ is even $f^+(y-x)=f^+(x-y)$, thus $f^+(x)g(y)=f^+(y)g(x)$. Particularly $f^+(x)g(0)=f^+(0)g(x)$. As $f^+(0)=g(0)=1$ we have $f^+=g$.

If we combine all the cases, we get $f(x)=bx+c, g(x)=1$, $f(x)=c\sin(ax)+d\cos(ax), g(x)=\cos(ax)$, $f(x)=csinh(ax)+dcosh(ax), g(x)=cosh(ax)$. All of these satisfy the condition.

Problem 58. Find all continuous functions $f,g,h\colon R\to R$ that satisfy

$$f(x+y)+g(x-y)=2(h(x)+h(y))$$

Solution. If we interchange x with y we obtain that g is even. Now set $y=0$ to get $f(x)+g(x)=2(h(x)+h(0))$. Without loss of generality $h(0)=0$ because otherwise we can work with $f(x)-2h(0), g(x)-2h(0), h(x)-h(0)$ instead. So $f(x)+g(x)=2h(x)$. Now replace y by $-y$ to get $f(x-y)+g(x+y)=2(h(x)+h(-y))$. Adding it with the original condition we get $f(x-y)+g(x-y)+f(x+y)+g(x+y)=4h(x)+2h(y)+2h(-y)$ or $2h(x-y)+2h(x+y)=4h(x)+2h(y)+2h(-y)$. Let's solve this equation. We settle it for h even and for h odd, because h can be written as h^++h^-, where $h^+(x)=\frac{h(x)+h(-x)}{2}$ is even, $h^-(x)=\frac{h(x)-h(-x)}{2}$ is odd.

If h is even we get $h(x-y)+h(x+y)=2h(x)+2h(y)$. Set $y=x$ to get $h(2x)=4h(x)$. Then we can prove by induction on n that $h(nx)=n^2x$, the induction step following directly from the condition written for $y=nx$. If $h(1)=a$ then $q^2h(\frac{1}{q})=a$ so $h(\frac{1}{q})=\frac{a}{q^2}$ hence $h(\frac{p}{q})=a\frac{p^2}{q^2}$. As Q is dense using the continuity we get $h(x)=ax^2$.

If h is odd we get $h(x-y)+h(x+y)=2h(x)$. Set $y=0$ to get $h(2x)=2h(x)$ hence $h(x-y)+h(x+y)=h(2x)$ and as $2x=x-y+x+y$ we conclude that h is additive so $h(x)=bx$.

Combining these two results we deduce that $h(x)=ax^2+bx$. Then $f(x+y)+g(x-y)=2a(x^2+y^2)+b(x+y)$ so $f_1(x+y)+g_1(x-y)=0$ where $f_1(x)=f(x)-ax^2-bx, g_1(x)=g(x)-ax^2$. As $x+y, x-y$ are independent, we conclude $f_1(u)=-g_1(v)$ for all u,v so $f_1(x)=d, g_1(x)=-d$ for some d. We conclude that $f(x)=ax^2+bx+d, g(x)=ax^2-d, h(x)=ax^2+bx$.

As we have supposed that $h(0)=0$ in the general case if $h(0)=c$ we get $f(x)=ax^2+bx+2c+d, g(x)=ax^2+2c-d, h(x)=ax^2+bx+c$. These functions satisfy the condition.

Problem 113. Find all continuous functions $f, g, h, k\colon R \to R$ that satisfy $f(x+y) + g(x-y) = 2h(x)k(y)$

Solution. This is a generalization of the previous problem. In order to connect it to the previous one we try to eliminate g. Replace y by $-y$ to get $f(x-y) + g(x+y) = 2h(x)k(y)$. Now subtracting these conditions and setting $u = f - g, l(x) = k(x) - k(-x)$ we get $u(x+y) - u(x-y) = 2h(x)l(y)(*)$. From here find u explicitly by using the same helpful lemma. Indeed if we set $a_n = f(nx)$ then setting nx instead of x, x instead of y he get $a_{n+1} - a_{n-1} = 2h(nx)l(x)$. If $l(x) = 0$ then $a_{2n} = a_0$ thus $f(2nx)$ is constant. If $l(x) \neq 0$ then taking $y = 2x$ we deduce $a_{n+2} - a_{n-2} = 2h(nx)l(2x)$ thus $a_{n+2} - a_{n-2} = b(a_{n+1} - a_{n-1})$ where $b = \frac{l(2x)}{l(x)}$. The associated polynomial of this quadratic recurrence is $x^4 - b(x^3 - x) - 1 = (x+1)(x-1)(x^2 - bx + 1)$. If $b \neq \pm 2$ this polynomial has four different roots $1, -1, w, \frac{1}{w}$. Thus $a_n = \alpha + \beta(-1)^n + \gamma w^n + \theta \frac{1}{w^n}$. so $a_{2n} = \alpha + \beta + \gamma w^n + \theta \frac{1}{w^n}$. If $b = 2$ the polynomial has a triple roots 1 and we have $a_n = p(n) + c(-1)^n$ where p is a polynomial of degree at most 2. So $u(2nx) = p(2nx)$ where p is some polynomial of degree at most 2. Finally if $b = -2$ we get a triple root -1 and a root -1 and similarly $u(nx) = p(2nx)$ where p is some polynomial of degree at most 2. Thus we can apply the lemma to establish that either $u(x)$ is a polynomial of degree at most 2 or $u(x) = \alpha e^{ax} + \beta e^{-ax} + \gamma$ (1). Now if we add the equalities in the beginning instead of subtracting them we get $v(x+y) + v(x-y) = 2h(x)j(y)(**)$ where $j(y) = k(y) + k(-y)$. We proceed exactly like in the previous problem to establish that v is either linear or $\alpha' e^{a'x} + \beta' e^{-a'x}$ (2). If v is linear then so is h. Therefore u cannot be of form $\alpha e^{ax} + \beta e^{-ax}$ for $a \neq 0$ because then $l(y) = \frac{u(x+y) - u(x-y)}{2h(x)} = \frac{\alpha e^{ax} - \beta e^{ax}}{2h(x)}(e^{ay} - e^{-ay})$ and does depend on x. So u is a polynomial of degree 2 and hence so are $f = \frac{u+v}{2}, g = \frac{v-u}{x}$. If $f(x) = ax^2 + bx + c, g(x) = -ax^2 + dx + e$ (the coefficients of x^2 are complementary because they come only from u) then we have $f(x+y) - g(x-y) = a((x+y)^2 - (x-y)^2) + b(x+y) + d(x-y) + c + e = 4axy + (b+d)x + (b-d)y + c + e$. As h is linear, $h(x) = kx + l$ then $k(y)$ is also linear. Thus $k(x)h(y) = (kx + l)(my + n) = mnxy + knx + lmy + ln$. Therefore $a = \frac{mn}{4}, b = \frac{kn+lm}{2}, c = \frac{kn-lm}{2}, c + e = ln$. Next suppose that v is of form $\alpha' e^{a'x} + \beta' e^{-a'x}$ then we conclude $h(x) = c(\alpha' e^{a'x} + \beta' e^{-a'x})$. In this case $a = a'$ or $a = -a'$ because then looking at (*) we get $k(y) = \frac{u(x+y) - u(x-y)}{h(x)}$. As k does not depend on x but h depends exponentially on $a'x$ we readily conclude that u must be of form $\alpha e^{ax} + \beta e^{-ax} + \gamma$ and then $a = \pm a'x$ in order for $h(x)$ to cancel in the expression for $k(y)$. We substitute u, v into

f,g to get $f(x) = ke^{ax} + le^{-ax} + m, g(x) = re^{ax} + se^{-ax} - m$. Then $f(x+y)+g(x-y) = (ke^{ay}+re^{-ay})e^{ax}+(le^{-ay}+se^{ay})e^{-ax} = h(x)k(y)$. If we avoid the trivial case $k = r = l = s = 0$ then h, k cannot be identically zero thus selecting $x = x_0$ with $h(x_0) \neq 0$ we can express $k(y)$ as $k_1e^{ay} + k_2e^{-ay}$, and selecting $y = y_0$ with $k(y_0) \neq 0$ we can select $h(x) = h_1e^{ax} + h_2e^{-ax}$. Thus substituting we get $k = h_1k_1, l = k_2h_2, r = h_1k_2, s = h_2k_1$ to obtain the parametric representation for f, g, h, k.

Problem 116. Find all continuous functions $f, g, h\colon R \to R$ that satisfy

$$f(x + y) + f(y + z) + f(z + x) = g(x) + g(y) + g(z) + h(x + y + z)$$

Solution. Set $x = y = 0$ to get $3f(0) = 3g(0) + h(0)$. Thus we can assume $f(0) = g(0) = h(0)$ otherwise work with $f - f(0), g - g(0), h - h(0)$ which satisfy the original condition. Next like in the previous problem set $z = -x$ to get $f(x + y) + f(x - y) = g(x) + g(-y) + g(y) + h(x)$. Particularly $y = 0$ gives us $2f(x) = g(x) + h(x)$ hence $f(x + y) + f(x - y) = 2f(x) + g(y) + g(-y)$. Now we proceed exactly like in the previous problem to conclude that $f(x) = ax^2 + bx$. Now set $u(x) = g(x) - f(x) = f(x) - h(x)$. As f satisfies the condition $f(x + y) + f(y + z) + f(z + x) = f(x + y + z) + f(x) + f(y) + f(z)$ of the previous problem, comparing it with the condition of this problem yields $u(x)+u(y)+u(z) = u(x+y+z)$. So u is additive and continuous. From here we establish f, g, h completely: $f(x) = ax^2 + bx + k, g(x) = ax^2 + (b + c)x + l, h(x) = ax^2 + (b - c)x + m$ where $k = f(0), l = g(0).m = h(0)$ obey $3k = 3l + m$.

Problem 133. Find all continuous functions $f \cdot R \to R$ that satisfy

$$f(x + y)f(x - y) = f^2(x)f^2(y)$$

Solution. If $f(0) = 0$ then we conclude $f^2(x) = 0$ so f is identically zero. If $f(0) \neq 0$ the for some $z > 0$ we get $f(x) \neq 0$ for all $x \in (-z; z)$. Now by setting $y = x$ we get $f(2x)f(0) = f^4(x)$ hence if $f(x) \neq 0$ then $f(2x) \neq 0$ and since f is non-zero on $(-z; z)$ on the whole real line. Also f has constant sign because $f(x + y)f(x - y)$ is positive for all x, y. Without loss of generality let $f > 0$ (the second case is analogous by working with $-f$ instead of f). Let $g(x) = \ln f(x)$. Then we get $g(x + y) + g(x - y) = 2g(x) + 2g(y)$ For $x = y = 0$ we get $g(0) = 0$. Thus if we set $a_n = f(nx)$ we deduce $a_{n+1} - 2a_n + a_{n-1} = 2g(x)$ thus $a_n = n^2g(x)$. We conclude using the lemma that $g(x) = ax^2$. So the solutions are $f(x) = 0, f(x) = e^{ax^2}, f(x) = -e^{ax^2}$.

The Odd and Even Parts of a Function

Problem 120. Find all continuous functions $f, g, h\colon R \to R$ that obey
$$f(x+y)+g(xy)=h(x)h(y)+1$$

Solution. If $y=0$ then $f(x)=h(x)h(0)+1-g(0)$. We can assume $g(0)=0$ otherwise replace f by $f+g(0), g-g(0)$ to get the same condition. We get $f(x)=h(x)h(0)+1$ so $h(x+y)h(0)+g(xy)=h(x)h(y)$. If $h(0)=0$ we get $h(x)h(y)=g(xy)=h(xy)h(1)$ so from were $h(x)=ax^k$ for some k, $g(x)=a^2x^k$. If $h(0)\neq 0$ we can assume $h(0)=1$ otherwise replace h by $\frac{h}{h(0)}$, g by $\frac{g}{g_0}$ to maintain the condition. So we have $h(x+y)+g(xy)=h(x)h(y)$. If we set $y=1$ we get $g(x)=ah(x)-h(x+1)$ where $a=h(1)$. So we get $h(x+y)+ah(xy)=h(x)h(y)+h(xy+1)$. Now let u, v be the even and odd parts of h. We have $u(x+y)+v(x+y)+au(xy)+av(xy)=(u(x)+v(x))(u(y)+v(y))+u(xy+1)+v(xy+1)$. Now if we replace x, y by $-x, -y$ we get $u(x+y)-v(x+y)+au(xy)+av(xy)=(u(x)-v(x))(u(y)-v(y))+u(xy+1)+v(xy+1)$. Subtracting these two relations we get $2v(x+y)=2v(x)u(y)+2u(x)v(y)$. Now replacing y by $-y$ the relation turns into $2v(x-y)=2v(x)u(y)-2u(x)v(y)$ so $v(x+y)-v(x-y)=2u(x)v(y)$. We have met this equation before, and have shown that the only solutions u, v with u even and v odd are $v=c\sin ax, u=\cos ax$ or $v=csinhax, u=coshax$ or $v=cx, u=1$ or $v=0$, u any even function. We can eliminate the first two solutions immediately, as an expression of h as sine-cosine surely doesn't satisfy the condition (just look that $h(xy+1)-ah(xy)$ has nothing to share with $h(x+y)$ or $h(x)h(y)$). If $v=cx, u=1$ then $h(x)=1+cx, a=1+c$ so we get $1+c(x+y)+(1+c)(1+cxy)=(1+cx)(1+cy)+1+c(xy+1)$ which satisfies the condition, and $g(x)=(1+c)(1+cx)-c(x+1)=c^2x+1$. If $v=0$ then h is even. Then writing the condition on h for x, y and $x, -y$ then subtracting them we get $h(x+y)-h(x-y)=h(1+xy)-h(xy-1)$. This function was solved previously by us with solutions $h(x)=ax^2+1$. In this case $h(1)=a+1$ and $g(x)=(a+1)(ax^2+1)-(a(x+1)^2+1)=a^2x^2-2ax$.

Problem 121. Find all continuous functions $f, g, h\colon R \to R$ that obey
$$f(x+y)+h(x)h(y)=g(xy+1)$$

Solution. This problem is very similar to the previous. Set $y=0$ to get $f(x)+h(x)h(0)=g(1)$. From here $f(x)=g(1)-h(x)h(0)$. Again we can suppose $g(1)=1$ because otherwise we an subtract

$g(1)$ from both g and f and the condition will still hold. So we get $h(x)h(y)-h(0)h(x+y)=g(xy+1)$. If $h(0)=0$ we deduce $h(x)h(y)=h(xy)h(1)=g(xy+1)$ so $h(x)=ax^b, g(x)=a^2(x-1)^b$. Otherwise we can suppose $h(0)=1$. Then $h(x)h(y)-h(x+y)=g(xy+1)$. If we set $y=1$ we get $g(x+1)=ah(x)-h(x+1)$ so $g(x)=ah(x-1)-h(x)$ where $a=h(1)$. Exactly like in the previous problem we conclude that either $h(x)=1+cx$ or h is even. For $h(x)=1+cx$ we get $(1+cx)(1+cy)-1-c(x+y)=(1+c)(1+cxy)-1-c(xy+1)$ and by looking at the coefficient of xy we get $c=1$ so $h(x)=1, f(x)=-1, g(x)=0$. If h is even then we get $h(x)h(y)-h(x+y)=ah(xy)-h(xy+1)$ and $h(x)h(y)-h(x-y)=ah(xy)-h(xy-1)$ thus $h(x+y)-h(x-y)=h(xy+1)-h(xy-1)$ again so $h(x)=cx^2+1$. We easily draw the conclusions from here.

Symmetrization and Additional Variables

Problem 100. Find all functions $f :: R \to R$ for which

$$f(x+y)=f(x)f(y)f(xy)$$

Solution. If $f(u)=0$ then setting $y=x-u$ we deduce $f(x)=0$. If f is not identically zero, then f is non-zero on R. We add again a new variable z: $f(x+y+z)=f(x)f(y+z)f(xy+yz)=f(x)f(y)f(z)f(yz)f(xy)f(xz)f(x^2yz)$. Now if as the left-hand side is symmetric by swapping x and y we get the relation $f(x+y+z)=f(x)f(y+z)f(xy+yz)=f(x)f(y)f(z)f(yz)f(xy)f(xz)f(xy^2z)$. Hence $f(x^2yz)=f(xy^2z)$. Then picking up $u,v\neq 0$ and setting $x=u, y=v$ we get $x^2yz=u^2z, y^2xz=uv^2z$ so $z=\frac{1}{uv}$ implies $x^2yz=u^2z$ thus $f(u)=f(v)$. So f is constant on $R\setminus\{0\}$. If $f=c$ then $c=c^3$ so $c=0$ or $c=\pm 1$. In any case by setting $y=-x\neq 0$ we get $f(0)=c^3=c$. Thus f is either identically zero, or identically 1, or identically -1. All of them satisfy the condition.

Problem 122. Find all continuous functions $f\colon R\to R$ such that

$$f(x+y)+f(xy-1)=f(x)+f(y)+f(xy)$$

Solution. If $g(x)=f(x)-f(x-1)$ then we have $f(x+y)-f(x)-f(y)=g(xy)$. From here $f(x+y+z)=f(x)+f(y+z)+g(xy+zz)=f(x)+f(y)+f(z)+g(yz)+g(xy+xz)$. We have encountered this sort of equation before, establishing that g is linear. Hence $f(x+1)-f(x)=cx+d$. Setting $y=1$ we get $f(x+1)-f(x)-f(1)=g(x)=cx+d$ so $f(0)=0$. Then setting $y=0$ we get $g(0)=0$ so $g(x)=cx$. We then have $f(x+y)-f(x)-f(y)=cxy$ thus $f(x+y)-\frac{c}{2}(x+y)^2-$

$f(x)+\frac{c}{2}x^2-f(y)+\frac{c}{2}y^2=0$ and so $f(x)=\frac{c}{2}x^2+dx$. Then we have $f(x)-f(x-1)=\frac{c}{2}(2x-1)+d$ hence $\frac{c}{2}=d$ so $f(x)=dx^2+dx$.

Problem 75. (Hosszu's functional equation) Show that a function $f::R\to R$ which satisfies

$$f(x+y-xy)+f(xy)=f(x)+f(y)$$

is an additive function plus some constant.

Solution. This problem is of form $f(a)+f(b)=f(c)+f(d)$ where $a+b=c+d$. The inconvenience if that (a,b) and (c,d) are linked to each other, so we cannot state that $f(a)+f(b)=f(c)+f(d)$ whenever $a+b=c+d$. We try to eliminate the inconvenience by adding a new variable and symmetrizing:

$f(x)+f(y)+f(z)=f(x+y-xy)+f(xy)+f(z)=f(x+y-xy)+f(xy+z-xyz)+f(xyz)$. By symmetry it also equals $f(x+z-xz)+f(xz+y-xyz)+f(xyz)$ and $f(y+z-yz)+f(yz+x-xyz)+f(xyz)$. We thus deduce that $f(x+y-xy)+f(xy+z-xyz)=f(x+z-xz)+f(xz+y-xyz)=f(y+z-yz)+f(yz+x-xyz)$. This is again an equation of form $f(a)+f(b)=f(c)+f(d)$ where $a+b=c+d$, but this time the restraints are milder. Indeed, let's find for which a,b,c,d with $a+b=c+d$ we can find x,y,z with $(1-x)(1-y)=a,(1-z)(1-xy)=b,(1-x)(1-z)=c$. For comfort we set $u=1-x,v=1-y,w=1-z$ to get $uv=a,w(u+v-uv)=b,uw=c$. Hence $w=\frac{c}{u},v=\frac{a}{u}$ and $\frac{c}{u}(u+\frac{a}{u}-a)=b$ or $(c-b)u^2-acu+ac=0$. For this equation to have a non-zero solution we need to have a positive discriminant so $a^2c^2-4ac(c-b)\geq 0$ or $(ac-2c)^2+4(abc-c^2)\geq 0$. When $abc>c^2$ this is certainly true. Now consider (a,b) and (c,d) with $a+b=c+d$. If ab,cd have the same sign then we can find such an e sufficiently small in absolute value such that $abe>e^2,cde>e^2$. Setting $e_1=a+b-e=c+d-e$ we get $f(a)+f(b)=f(e)+f(e_1)=f(c)+f(d)$ so $f(a)+f(b)=f(c)+f(d)$. So $f(a)+f(b)=f(c)+f(d)$ when $a+b=c+d$ and $abcd>0$. Next if we have $a+b=c+d=s\notin[0;4]$ and $abcd<0$ then there are u,v such that $u+v=s$ $u+v-uv<0$ (just take $u=v=\frac{s}{2}$). Then $u+v=uv+(u+v-uv)$ by $uvuv(u+v-uv)=u^2v^2(u+v-uv)<0$. Without loss of generality $ab>0,cd<0$. Then if $s>0$ we have $f(a)+f(b)=f(u)+f(v)=f(uv)+f(u+v-uv)=f(c)+f(d)$ and if $s<0$ we have $f(c)+f(d)=f(u)+f(v)=f(uv)+f(u+v-uv)=f(a)+f(b)$. Finally, even if $s\in[0;4]$ (we actually cannot have $s=0$ and $abcd<0$ as $abcd=a^2c^2$), we can find a sufficiently big e such that $a+c+e,b+d+e,a+b+2e,c+d+2e>4$ and state $f(a+e)+f(b+e)=f(c+e)+f(d+e)$ while $f(a)+f(c+e)=f(c)+f(a+e)$ and $f(b)+f(d+e)=f(d)+f(b+e)$

hence $f(c+e)-f(c)=f(a+e)-f(a), f(b+e)-f(b)=f(d+e)-f(d)$ and $f(a+e)+f(b+e)=f(c+e)+f(d+e)$ implies thus $f(a)+f(b)=f(c)+f(d)$ in this last case.

So $f(a)+f(b)=f(c)+f(d)$ whenever $a+b=c+d$ and $abcd\neq 0$. Now what if $abcd=0$? If one of the is zero, say d we must prove $f(a)+f(b)=f(a+b)+f(0)$ when $ab\neq 0$. If $a+b\notin[0;4]$ we deduce the existence of $x,y\neq 0$ s.t. $xy=x+y=a+b$ thus $f(a)+f(b)=f(x)+f(y)=f(x+y-xy)+f(xy)=f(a+b)+f(0)$. If $a+b\in[0;4]$ again we find a sufficiently big e to have $a+b+e, 2a+b+e>4$ and conclude $f(a+b+e)+f(0)=f(a+e)+f(b)$ while $f(a+e)+f(a+b)=f(a)+f(a+b+e)$ hence adding these expressions and cancelling common terms we have $f(a+b)+f(0)=f(a)+f(b)$, as desired. If two of them are zero, then either we get $(a,b)=(c,d)$ or one of $(a,b),(c,d)$ is $(0,0)$. The first case is clear. In the second case we need to prove $f(a)+f(-a)=2f(0)$ for $a\neq 0$. In this case $f(2a)+f(0)=2f(a)$ while $f(2a)+f(-a)=f(a)+f(0)$. Subtracting this two relations we get $f(0)-f(-a)=f(a)-f(0)$ so $f(a)+f(-a)=2f(0)$. If three of a,b,c,d are zero, the fourth is also zero and the conclusion is clear.

We have established that $f(a)+f(b)=f(c)+f(d)$ whenever $a+b=c+d$. Then $f(a+b)+f(0)=f(a)+f(b)$ thus $f(x)-f(0)$ is an additive function and the conclusion follows.

Problem 98. Find all functions $f :: R\to R$ for which

$$xf(x)-yf(y)=(x-y)f(x+y)$$

holds.

Solution. We add a new variable to get a relation: $xf(x)-zf(z)=(x-z)f(x+z)$ but also $xf(x)-zf(z)=xf(x)-yf(y)+yf(y)-zf(z)=(x-y)f(x+y)+(y-z)f(x+z)$. Therefore $(x-z)f(x+z)=(x-y)f(x+y)+(y-z)f(y+z)$. If we want $x+z=u, x+y=1, y+z=0$ by solving the system of equations we get $x=\frac{u+1}{2}, y=\frac{1-u}{2}, z=\frac{u-1}{2}$ and thus our condition becomes $f(u)=uf(1)+(1-u)f(0)$. Thus $f(x)=ax+b$ is a linear function, and linear functions satisfy the condition.

Functional Equations without Solution

Problem 46. (Romania '2001) Prove that there is no function $f\colon R^+\to R^+$ such that

$$f(x+y)\geq f(x)+yf(f(x))$$

for all $x,y\in R^+$.

Solution. Use the same arguments as in the second part of the solution of Problem 45.

Miscellaneous

Problem 1. (IMO '2003, shortlisted problem) Find all function $f : \mathbb{R}^+ \to \mathbb{R}^+$, which are increasing in the segment $[1, \infty)$ and such that

$$f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$$

for any $x, y, z > 0$.

Solution. Replacing x, y and z with $\frac{x}{y}$, $\frac{y}{x}$ and xy, respectively, we get that

$$2f(xy) + f\left(\frac{x}{y}\right) + f\left(\frac{y}{x}\right) = f(1)f(x)f(y). \tag{1}$$

For $y = 1$ it follows that $3f(x) + f\left(\frac{1}{x}\right) = f^2(1)f(x)$. In particular, $4f(1) = (f(1))^3$. Since $f(1) > 0$, then $f(1) = 2$. Thus, $f(x) = f\left(\frac{1}{x}\right)$ and (1) can be written as

$$f(xy) = f(x)f(y) - f\left(\frac{x}{y}\right).$$

Further, since $e > 1$, then $f(e) \geq f(1) = 2$ and hence $f(e) = e^\alpha + e^{-\alpha}$ for some $\alpha \geq 0$. Using that $f(x^2) = f^2(x) - 2$, it follows by induction that

$$f(e^{2^{-n}}) = e^{\alpha 2^{-n}} + e^{-\alpha 2^{-n}}$$

for any $n \in \mathbb{N}_0$. Having in mind the equality

$$f(e^{(m+1)2^{-n}}) = f(e^{2^{-n}})f(e^{m)2^{-n}}) - f(e^{(m-1)2^{-n}}),$$

we get again by induction that

$$f(e^{m2^{-n}}) = e^{\alpha m2^{-n}} + e^{-\alpha m2^{-n}}$$

for any $m, n \in \mathbb{N}_0$.

Since the set of numbers of the form $m2^{-n}$, $m, n \in \mathbb{N}_0$, is dense in $\mathbb{R}^+$ (use the binary representation of the positive real numbers) and f is a monotone function in the segment $[1, \infty)$, we conclude that $f(x) = x^\alpha + x^{-\alpha}$ in this segment. The same is true in the segment $(0, 1)$ in virtue of the equality $f(x) = f\left(\frac{1}{x}\right)$.

Conversely, it is easy to see that any function of this form satisfies the condition of the problem

Problem 81. (AMM 1998) Find all functions $f\colon N^2 \to N$ that satisfy:
a) $f(n,n) = n$;
b)$f(m,n) = f(n,m)$;
c)$\frac{f(m,n+m)}{f(m,n)} = \frac{n+m}{n}$.

Solution. We can immediately see from a) the algorithm of finding f, because the three conditions follow exactly the three possible steps of the Euclidean Algorithm. Indeed, following the Euclidean Algorithm by means of steps b),c) we shall reach from (m,n) the pair (d,d) where $d = gcd(m,n)$. Also we note that all steps a),b), c) preserve the quantity $\frac{f(m,n)}{mn}$. Therefore $\frac{f(m,n)}{mn} = \frac{f(d,d)}{d^2} = \frac{d}{d^2} = \frac{1}{d}$. So $f(m,n) = \frac{mn}{d} = LCM(m,n)$. Indeed, LCM satisfies trivially the conditions a), b). For c) we use the fact that $gcd(m,n+m) = gcd(m,n)$ or $\frac{m(n+m)}{LCM(m,n+m)} = \frac{mn}{LCM(m,n)}$ which can be rewritten as $\frac{LCM(m,n+m)}{LCM(m,n)} = \frac{n+m}{n}$

Problem 29. Find for which a there exist increasing multiplicative functions on N (i.e. $f(n) < f(n+1), f(mn) = f(m)f(n)$ if $(m,n) = 1$) with $f(2) = a$.

Solution. We claim the function must be $f(n) = n^k$ for some k. Assume $f(x) = x^u, f(y) = y^v$. If $x^k < y^l$ then $x^{uk} < y^{vl}$ and if $x^k > y^l$ then $x^{uk} > y^{vl}$. Pick up now k and let l be the biggest for which $y^l < x^k$. Then $x^k \leq y^{l+1}$ so we get $y^{vl+l} > x^{uk} > y^{vl}$. Now if $v > u$ we cannot have $x^{uk} > y^{vl}$ for sufficiently big k, as $x^{uk} > y^{vl} > y^{ul+l(v-u)} > y^{u(l+1)}$ for $l \geq \frac{u}{v-u}$. If $u > v$ then $y^{vl+l} > x^{uk}$ so $y^{vl} > x^{uk-1} > x^{vk}$ for $k \geq \frac{1}{u-v}$ again contradiction. Thus taking $k > \frac{1}{u-v}$ if $u > v$ or k such that $x^k > y^{\frac{u}{v-u}+1}$ if $u < v$ we would obtain contradiction. So $u = v$ and hence $f(x) = x^u$ for all x. As f is from N to N, we must have u integer. So a must be a power of 2, and conversely if $a = 2^k$ then $f(x) = x^k$ is good.

Problem 83. Find all functions $f\colon Z \to Z$ that satisfy:
a) if $p|m-n$ then $f(m) = f(n)$.
b) $f(mn) = f(m)f(n)$

Solution. We see that $f(n)$ takes one of the p values $f(0), f(1), \dots, f(p-1)$. Therefore we can pick up an a for which $|f(a)|$ has the maximal value. Then $|f(a^2)| = |f(a)|^2 \leq |f(a)|$ thus $|f(a)| \leq 1$ hence $f(n) \in \{-1,0,1\}$. Now if we set $m = 0$ into b) we get $f(0)(f(n)-1) = 0$ so either f is identically 1 (which is a solution) or $f(0) = 0$. If f is not identically 1 then $f(0) = 0$ hence $f(n) = 0$ whenever f is divisible by p. If $f(k) = 0$ for k not divisible by p then for any n we can find l

such that $p|kl-n$ so $f(n)=f(kl)=f(k)f(l)=0$, so f is identically zero, a solution. Assume f is not identically zero, so $f(n)\neq 0$ if n is not divisible by p. Now if n is a quadratic residue modulo p then there is a b such that $p|n-b^2$ so $f(n)=f(b^2)=f(b)^2=1$. Next pick up a quadratic non-residue r. Then for every n which is a non-quadratic residue modulo p we can find b such that $p|n-rb^2$ hence $f(n)=f(rb^2)=f(r)f(b)^2=f(r)$. If $f(r)=1$ then $f(n)=1$ for all n not divisible by p. If $f(r)=-1$ then f is 1 for quadratic residues and -1 for quadratic non-residues.

Concluding, $f(n)=1; f(n)=0$; $f(n)=1$ for n not divisible by p and $f(n)=0$ for n divisible by p; and Legendre's symbol are all solutions to our problem.

Problem 84. Find all $f\colon N_0\to N_0$ that satisfy

$$f(f^2(m)+f^2(n))=m^2+n^2$$

Solution. If $f(n_1)=f(n_2)$ then setting $n=n_1, n=n_2$ we conclude that $n_1=n_2$ so f is injective. If $f(0)=a$ then $f(2a^2)=0$ and then setting $m=n=2a^2$ we get $f(0)=8a^4$ so $a=8a^4$ thus $a=0$. Now is $m^2+n^2=x^2+y^2$ then $f(f^2(m)+f^2(n))=m^2+n^2=x^2+y^2=f(f^2(x)+f^2(y))$. So the injectivity of f implies that $f^2(m)+f^2(n)$ if $m^2+n^2=x^2+y^2$. Set $f(1)=b$. If we set $m=0, n=1$ we get $f(b^2)=1$. Then set $m=0, n=b^2$ to get $f(1)=b^4$ so $b^4=b$. Hence $f(1)=1$ or $f(1)=0$. As $f(0)=0$ we conclude $f(1)=1$. Next set $m=n=1$ to get $f(2)=2$. Set $m=2, n=0$ to get $f(4)=4$, $m=2, n=1$ to get $f(5)=5$, $m=n=2$ to compute $f(8)$. Next as $3^2+4^2=5^2+0^2$ we get $f(3)=3$. Set $m=1, n=3$ to get $f(10)=10$. As $6^2+8^2=10^2+0^2$ we conclude $f(6)=6$. Now we have proven during the proof of a previous problem that the condition $f(m)^2+f(n)^2=f(x)^2+f(y)^2$ for $m^2+n^2=x^2+y^2$ helps us compute $f(n)$ inductively on n for $n>6$. Therefore f is unique and as the identity function is a solution, we conclude that $f(x)=x$.

Problem 3. (Bulgaria '2003) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^2+y+f(y))=2y+f^2(x) \tag{1}$$

for any x, y

Solution. It follows by (1) that f is a surjective function. Moreover, $(f(x))^2=(f(-x))^2$. Let a be such that $f(a)=0$. Then $f(-a)=0$. Setting $x=0, y=\pm a$ in (1) gives $0=f(\pm a)=(f(0))^2\pm 2a$, i.e., $a=0$.

Substitute $y = -\dfrac{(f(x))^2}{2}$ again in (1). It follows that $f(x^2+y+f(y)) = 0$ and hence $y+f(y) = -x^2$. Thus, $y+f(y)$ runs over all the non-positive real numbers. Since $f(0) = 0$, (1) implies that

$$f(x^2) = ((f(x))^2 \geq 0 \quad \text{and} \quad f(y + f(y)) = 2y.$$

Setting $z = x^2$, $t = y+f(y)$ and using (1), we conclude that $f(z+t) = f(z) + f(t)$ for any $z \geq 0 \geq t$. Then for $z = -t$ we obtain that $f(-t) = -f(t)$. The it is easy to see that $f(z + t) = f(z) + f(t)$ for arbitrary z and t. Since $f(t) \geq 0$ for $t \geq 0$, it follows that f is an increasing function. Assuming now that $f(y) > y$, then $f(f(y)) \geq f(y)$ and we get the contradiction

$$2y = f(y + f(y)) = f(y) + f(f(y)) > 2f(y).$$

Similar arguments show that the inequality $f(y) < y$ is impossible. Therefore, $f(x) \equiv x$ and this function obviously satisfies (1).

Problem 4. (Bulgaria '2006) Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be such a function that

$$f(x + y) - f(x - y) = 4\sqrt{f(x)f(y)}$$

for any $x > y > 0$.

a) Prove that $f(2x) = 4f(x)$ for any $x > 0$.

b) Find all such functions f.

Solution. a) Since $f(x + y) - f(x - y) > 0$, then f is a (strictly) increasing function. Hence $f(x)$ has a limit $l \geq 0$ as $x \to 0$, $x > 0$ (prove). Letting $x, y \to 0$, $x > y > 0$ gives $l - l = 4\sqrt{l^2}$, i.e., $l = 0$. Fix now x. Letting $y \to 0$, $y > 0$, implies that $f(x + y) - f(x - y) \to 0$. Since f is a monotonic function, we conclude that it is continuous at x. Finally, tending $y \to x$, $y < x$, we get that $f(2x) = 4f(x)$.

b) Put $x = ny > 0$ and $n \geq 2$. Then

$$f((n + 1)y) = f((n - 1)y) + 4\sqrt{f(ny)f(y)}.$$

Using that $f(2y) = 4f(y)$, we conclude by induction that $f(ny) = n^2f(y)$. Set $f(1) = c > 0$. Then $f(n) = n^2c$. For $p, q \in \mathbb{N}$ one has that $cp^2 = f(q.p/q) = q^2f(p/q)$, i.e. $f(p/q) = c(p/q)^2$. Since f is a continuous function, it follows that $f(x) = cx^2$ for any $x > 0$. Conversely, the functions of this form satisfy the given condition.

Problem 6. (Ukraine '2003) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x) + f(y)) \equiv x^2 + y$$

for any x, y (compare with Problem 10).

Solution. Setting $x = 0$ in the given equation gives $f(f(y)) =$. Then

$$x^2 + y = f(xf(x) + f(y)) = f(f(x)f(f(x)) + f(y)) = f^2(x) + y,$$

i.e. $|f(x)| = |x|$ for any x. Suppose that $f(x) = x$ and $f(y) = -y$ for some x and y. Then we get from the given equation that $\pm(x^2 - y) = x^2 + y$ which implies $x = 0$ or $y = 0$.

Thus the only solution of the problem are the functions $f(x) = x$ and $f(x) = -x$.

Problem 1. (Bulgaria '2004) Find all non-constant polynomials P and Q with real coefficients such that

$$P(x)Q(x+1) = P(x+2004)Q(x) \tag{1}$$

for any $x \in \mathbb{R}$.

Solution. Set $R(x) = P(x)P(x+1)\dots P(x+2003)$. It follows by (1) that if x is greater than the real zeros of P, then

$$\frac{Q(x)}{R(x)} = \frac{Q(x+1)}{R(x+1)} \tag{2}.$$

We conclude by induction that $\dfrac{Q(x)}{R(x)} = \dfrac{Q(x+n)}{R(x+n)}$ for any $n \in \mathbb{N}$. Note that $\lim\limits_{n\to\infty} \dfrac{Q(x+n)}{R(x+n)}$ is a number independent of x, or ∞. On the other hand, this limit equals $\dfrac{Q(x)}{R(x)}$. Hence $Q(x) = cR(x)$ for any x, where $c \neq 0$ is a constant. Conversely, if $Q(x) = cP(x)P(x+1)\dots P(x+2003)$, then (1) is satisfied.

Remark. Using the equality (2), the solution can be also done by comparing the coefficients of the respective polynomials.