Complex Numbers in Trigonometry

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The final version- with better LaTeX, more contest problems, and some new topics. Credit to Binomial-Theorem and djmathman for the LaTeX template. Note: This article describes what Franklyn Wang might call "Vincent Huang bashing". Since this method has always been somewhat known, and the name makes it sound like one is trying to bash a person, please do not refer to it by this name.

We take for granted that the reader knows a decent amount of algebra/manipulations, the basics about complex numbers, their polar forms, operations between complex numbers, and the definitions of trigonometric functions as well as their basic properties.

1

Introducing The Method

Note that if an uppercase letter denotes an angle, the lowercase letter is the complex number corresponding to the angle. The format of this chapter is slightly different from others.

1.1 Formulas for sine and cosine

Theorem 1.1. *Let θ be an angle and let $z = e^{i\theta}$. Then*

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

.

Proof. We begin by establishing that along with $z = e^{i\theta} = \cos\theta + i\sin\theta$ we have the second equation

$$\frac{1}{z} = \overline{z} = \cos\theta - i\sin\theta$$

Thus we may solve the system in terms of z to obtain the equations listed above for $\cos, \sin\theta$ □

Theorem 1.2. *With the same notations, we have*

$$\cos k\theta = \frac{1}{2}\left(z^k + \frac{1}{z^k}\right)$$

and

$$\sin k\theta = \frac{1}{2i}\left(z^k - \frac{1}{z^k}\right)$$

Proof. Just as z is the number corresponding to the angle θ, we have by DeMoivre that z^k is the number corresponding to the angle $k\theta$, which by theorem 1.1 immediately gives the result. □

These two formulas are very useful and immediately allow us to do a ton of things.

Example 1.1. *Verify that* $\sin^2\theta + \cos^2\theta = 1$

Solution. Just expand and things cancel. □

Example 1.2. *Suppose* A, B, C *exist with*

$$\sin A + \sin B + \sin C = \cos A + \cos B + \cos C = 0$$

Show that $3\cos(A+B+C) = \cos 3A + \cos 3B + \cos 3C$ *and* $3\sin(a+b+c) = \sin 3A + \sin 3B + \sin 3C$.

Solution. The condition tells us that $\sum_{cyc} a + \frac{1}{a} = \sum_{cyc} a - \frac{1}{a} = 0$, or that $\sum_{cyc} a = \sum_{cyc} \frac{1}{a} = 0$. The first equation we wish to show is equivalent with

$$3abc + \frac{3}{abc} = \sum_{cyc} a^3 + \frac{1}{a^3}$$

But note that $a+b+c=0 \implies a^3+b^3+c^3 = 3abc, \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 0 \implies \frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3} = \frac{3}{abc}$, so we're done. The other equation follows similarly. (Our two equations just now follow from a well-known algebraic identity.) □

There really isn't much theory to cover at this point, so I'll just give some exercises. The solutions are pretty much all to expand, so feel free to skip as needed.

1.2 Exercises/Problems

1. Derive formulas for $\tan, \cot x$.
2. Verify $\sin(x+y) = \sin x \cos y + \sin y \cos x$.
3. Show the identity $\sin 3x(2\cos 2x - 1) = \sin x(2\cos 4x + 1)$.
4. Solve for x: $\cos x + \cos 2x + \cos 3x = \sin x + \sin 2x + \sin 3x$.
5. Find a closed form for $\cos x \cos 2x \cos 4x .. \cos 2^k x$.
6. Find a closed form for $\cos x + \cos 2x + \cos 3x + .. + \cos nx$. (Hint: use the geometric series formula!)

2

Roots of Unity and Applications

When given an angle like $\theta = \dfrac{2\pi}{7}$, it is quite easy to apply complex numbers: This is because $e^{i\theta} = z$ is a root of unity, i.e. $z^7 = 1$, so we obtain an extra condition. This is easily demonstrated on examples.

2.1 Examples

Example 2.1. *Find* $2\cos 72°$.

Solution. We let $z = e^{\frac{2i\pi}{5}}$. We know that $z^5 = 1$, but since $z \neq 1$ we obtain that

$$z^4 + z^3 + z^2 + z + 1 = 0$$

Meanwhile, the expression we want to find is $t = 2\cos 72° = z + \dfrac{1}{z}$. Divide the preveious equation by z^2, and then it becomes

$$\left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) + 1 = 0$$

, or $(t^2 - 2) + t + 1 = 0 \implies t^2 + t - 1 = 0 \implies t = \dfrac{-1+\sqrt{5}}{2}$. $\square$

Example 2.2. *Let* $a = \cos\dfrac{2\pi}{7}, b = \cos\dfrac{4\pi}{7}, c = \cos\dfrac{8\pi}{7}$. *Evaluate* $ab + bc + ca$.

Solution. We present two solutions. Let $z = e^{\frac{2i\pi}{7}}$.

Solution 1: We do a straightforward bash. Factoring out the 0.25 factor, we would like to evaluate

$$\left(z + \frac{1}{z}\right)\left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right)\left(z^4 + \frac{1}{z^4}\right) + \left(z^2 + \frac{1}{z^2}\right)\left(z^4 + \frac{1}{z^4}\right)$$

Miraculously, using $z^7 = 1$, this expands and reduces as $2(z + z^2 + z^3 + z^4 + z^5 + z^6) = 2(0-1) = -2$, so our desired answer is $-\dfrac{2}{4} = -0.5$.

Solution 2: This is the 2nd symmetric sum of a, b, c. We will use Vieta's formulas on a cubic whose roots are a, b, c. Following example 2.1, we know that
$z^6+z^5+z^4+z^3+z^2+z+1=0$. Let $t = 0.5\left(z+\dfrac{1}{z}\right)$ and note that we can divide the equation by z^3:

$$\left(z^3+\frac{1}{z^3}\right)+\left(z^2+\frac{1}{z^2}\right)+\left(z+\frac{1}{z}\right)+1=0$$

But this, after numerous manipulations, is equivalent with $8t^3+8t^2-4t-1=0$. Since a, b, c are all distinct, they must be the only roots of this cubic, so by Vieta's Formulas we obtain the desired answer. $\square$

Example 2.3. *(HMMT 2014) Compute* $\displaystyle\sum_{k=1}^{1007}\left(\cos\left(\frac{\pi k}{1007}\right)\right)^{2014}$.

Solution. Let $z = \dfrac{2\pi}{2014}$ and $w = e^{iz}$, so $w^{2014} = 1$.

Using complex numbers we see the desired equals
$\displaystyle\frac{1}{2^{2014}}\sum_{k=1}^{1007}\left(w^k+\frac{1}{w^k}\right)^{2014}$. Ignore the $\dfrac{1}{2^{2014}}$. We find the sigma equals, by Binomial Theorem,
$\displaystyle\sum_{k=1}^{1007} w^{2014k}+w^{2012k}\binom{2014}{1}+\ldots$, which can be written more concisely as

$$\sum_{k=1}^{1007}\sum_{j=0}^{2014} w^{2014k-2jk}\binom{2014}{j}=\sum_{j=0}^{2014}\binom{2014}{j}\left(\sum_{k=1}^{1007} w^{2014k-2jk}\right)$$

Thus we should try to find the smaller sums $\displaystyle\sum_{k=1}^{1007} w^{2014k-2jk}$.

This is a geometric series with 1007 terms, first term $w^{2014-2j}$, and common ratio $w^{2014-2j}$. Thus it equals $w^{2014-2j}\dfrac{(w^{(2014-2j)1007}-1)}{w^{2014-2j}-1}$. This clearly equals 0 as the term in the numerator can be rewritten as $w^{2014\cdot(1007-j)}-1=1-1=0$.

However, here we have glossed over a point, namely when $w^{2014-2j}=1$. Since j varies from 0 to 2014 this occurs at $j=0,1007,2014$, giving us an extra $2014+1007\binom{2014}{1007}$ in our answer. (This happens when you evaluate the series for $j=0,1007,2014$). Thus, the desired quantity is
$\dfrac{2014+1007\binom{2014}{1007}}{2^{2014}}$. Done! Our next example is a useful lemma involving tangents... $\square$

Example 2.4. *Let m be an odd integer, and let $\theta = \dfrac{2\pi}{2m+1}$. Then*

$$\tan(x-m\theta)\tan(x-(m-1)\theta)\ldots\tan(x+m\theta)$$

and $\tan(2m+1)x$ *have the same magnitude.*

Solution. Note: This means we can compare the two quantities directly, as long as we know the signs they possess. Now let $w = e^{i\theta}, z = e^{ix}$. We now go into the computations for the left hand side. First of all,

$$\tan(x+k\theta) = \frac{1}{i}\frac{zw^k - \frac{1}{zw^k}}{zw^k + \frac{1}{zw^k}} = \frac{1}{i}\frac{z^2w^{2k}-1}{z^2w^{2k}+1}$$

As the left hand side multiplies these terms for $k = -m, -m+1, ...m$, we see the left side must equal, up to a sign difference

$$i\prod_{-m\le k\le m}\frac{z^2w^{2k}-1}{z^2w^{2k}+1} = i\prod_{-m\le k\le m}\frac{z^2 - w^{2m+1-2k}}{-z^2-w^{2m+1-2k}} = i\prod_{0\le j\le 2m}\frac{z^2-w^j}{-z^2-w^j}$$

The last equality follows since the set $[2m+1-2k]$, taken mod $2m+1$ because $z^{2m+1} = 1$, is equivalent to the set $[k]$. Now we introduce a useful fact:

For any a, the quantity $\prod_{0\le j\le 2m}(a-w^j) = a^{2m+1}-1$. The proof is very simple, consider both sides as a polynomial in a and compare their roots and leading coefficients. Then applying the lemma twice, the final product equals

$$i\frac{z^{2(2m+1)}-1}{-z^{2(2m+1)}-1} = \frac{1}{i}\frac{z^{2m+1}-\frac{1}{z^{2m+1}}}{z^{2m+1}+\frac{1}{z^{2m+1}}} = \tan(2m+1)x$$

, as desired. □

2.2 Exercises/Problems

1. Let $x = \frac{180}{7}$ degrees. Find $\tan x \tan 2x \tan 3x$.

2. (USAMO 1996) Show that the average of the numbers $n \sin n^\circ$ for $n = 2, 4, 6, ..180$ is $\cot 1^\circ$.

3. (Basically 2015 AIME I 13) Evalute $\sin^2 1^\circ \sin^2 3^\circ ... \sin^2 89^\circ$.

4. (ISL) Prove that for all positive integers n the following is true:

$$2^n\prod_{k=1}^{n}\sin\frac{k\pi}{2n+1} = \sqrt{2n+1}$$

(Hint: Square to kill the square-root. Then we have some identical terms on the left side, so replace them with equivalent but non-identical terms.)

3

Triangles I: Pure Trig

In this section (rather large) and the next, we focus on a very olympiad-useful aspect of complex numbers in trig- their effectiveness in dealing with triangles and triangle identities. Indeed, our methods will be able to prove identities not just for triangles, but for any three angles summing to 180 degrees. Because of the vast range of these problems, we split them into two classes- "pure" trig, and "non-pure" trig. Non-pure trig has strange terms relating to a triangle ABC such as $R, r, [ABC], a, b, c, s$, and so on.

We will assume ABC always stands for a triangle, and that a, b, c are the complex numbers associated with angles A, B, C. So how do we deal with the $A + B + C = 180$ condition?

In short, $a * b * c = e^{i(A+B+C)} = -1$! This simple condition is incredibly useful.

Note: This section will require a bit of expanding, and I will not explain how I explain in a detailed way. As long as one gets the same final result it doesn't matter. Also I am lazy so I use cyclic sums occasionally.

3.1 Demonstrating on Victims

Example 3.1. *Let ABC be a triangle. Show that* $4\sin A\sin B\sin C = \sin 2A + \sin 2B + \sin 2C$.

Solution. So after you clear the silly $2i$ terms and the $a^2b^2c^2$, what we're trying to show is $(a^3 - a)(b^3 - b)(c^3 - c) = -\sum(a^4 - b^2c^2)$. Since $abc = -1$ this is equivalent with

$$(a^2 - 1)(b^2 - 1)(c^2 - 1) = a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2$$

But expanding the LHS, this is equivalent with showing $a^2b^2c^2 - 1 = 0$, which is obvious. □

Example 3.2. *Let ABC be a triangle and suppose that* $\sin^2 A + \sin^2 B + \sin^2 C = 2$. *Show that ABC is right.*

Solution. We bash, of course. First multiply both sides by $-4a^2b^2c^2 = -4$ and we need

$$-8 = (a^2bc - bc)^2 + (b^2ca - ca)^2 + (c^2ab - ab)^2$$

Expanding yields

$$-2 = a^4b^2c^2 = b^4c^2a^2 + c^4a^2b^2 + a^2b^2 + b^2c^2 + c^2a^2$$

and using $abc = -1$ lets us rewrite this as $(a^2+1)(b^2+1)(c^2+1) = 0$.

But then one of $a, b, c = i$, so ABC is right as desired. $\square$

Example 3.3. *Let ABC be a triangle. Show* $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$.

Solution. The problem is equivalent with

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 + \left(c+\frac{1}{c}\right)^2 + \left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) = 4$$

or

$$4 = \sum_{cyc}\left(a^2 + \frac{1}{a^2} + 2\right) + abc + \frac{1}{abc} + \sum_{cyc}\left(\frac{ab}{c} + \frac{c}{ab}\right)$$

Using $abc = -1$ a ton, this reduces to

$$0 = \sum_{cyc}\left(a^2 + \frac{1}{a^2}\right) + \sum_{cyc} - \left(\frac{1}{c^2} + c^2\right)$$

which is trivial, so we're done.

Basically the plan here is- expand everything systematically (it's really not bad with cyclic sums and stuff), then use $abc = -1$ to "homogenize" or create like terms which cancel.

3.2 Exercises/Problems

1. Show in a triangle ABC that $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$.
2. (AIME) If ABC is a triangle with $\cos 3A + \cos 3B + \cos 3C = 1$, show that one of the angles of the triangle is $120°$
3. Show in an acute triangle ABC that $2\cos A\cos B\cos C + \cos 2A + \cos 2B + \cos 2C = -1$

$\square$

Note: It's probably instructive to derive some of the identities which we bashed in this chapter through other methods. I'll include a short proof of example 3.1 "synthetically" to give an idea of what I'm saying: Let ABC be a triangle with circumcenter O. We compute

$$2[ABC] = ab\sin C = 4R^2\sin A\sin B\sin C$$

Meanwhile,

$$2[ABC] = 2[OAB] + 2[OBC] + 2[OCA] = R^2\sin 2A + R^2\sin 2B + R^2\sin 2C$$

so canceling the R^2s gives the desired.

4

Triangles II: Non-pure (and harder) Trig

As stated earlier, as you move into olympiad-level problems, trigonometric identities will not be so simple- they may involve terms like $r, R, [ABC]$, and so on. In this section we give some insight on how to deal with these terms effectively.

1. First, note that all identities or equations are homogeneous- this means, if we have a triangle ABC, and replace it with a triangle $A'B'C'$ similar to ABC, the scale factor k should cancel from both sides of the equation. Last chapter we dealt with identities of "degree 0".
2. If a, b, c (sides of a triangle) are in the equation, write (by Law of Sines) $a = 2R \sin A$, and so on.
3. If $[ABC]$ appears, write it as $0.5ab \sin C = 2R^2 \sin A \sin B \sin C$.
4. If r appears, either write it as $\dfrac{2[ABC]}{a+b+c}$ and use steps 2,3 or write it as $4R \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$. It depends on the circumstance. We will prove the latter formula in this chapter.
5. After steps 2-4, since the identity was homogeneous (see step 1 which isn't a step), all the R terms should cancel from both sides and leave you with a nice simple equation in terms of only sines and cosines. Now, we do what we did last chapter.

4.1 Examples

Example 4.1. *As promised, we will show* $r = 4R \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$. *This will be very instructive, for reasons you will see, namely dealing with half-angles...*

Solution. Before we can even start, we have to ask ourselves how to deal with half-angles. The easiest way to explain this is to use a modified version of square-root for complex numbers: We know that radicals are not well-defined, so define $\sqrt{w}$ for a complex number $w = e^{i\theta}$ as $e^{0.5i\theta}$, but **only when** $0 \le \theta < 2\pi$**.**

Now, we may continue smoothly. First multiply both sides by two and rewrite the equation-

$$2r = iR\left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)\left(\sqrt{b} - \frac{1}{\sqrt{b}}\right)\left(\sqrt{c} - \frac{1}{\sqrt{c}}\right)$$

But this is not so bad- we have

$$\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)\left(\sqrt{b}-\frac{1}{\sqrt{b}}\right)\left(\sqrt{c}-\frac{1}{\sqrt{c}}\right)=\frac{(a-1)(b-1)(c-1)}{\sqrt{abc}}$$

and we know $\sqrt{abc}=\sqrt{-1}=i$. This is perfectly consistent with our definition of square-root. Thus the i's also cancel and we want to show

$$2r = R(a-1)(b-1)(c-1)$$

But now $2r=\dfrac{4[ABC]}{a+b+c}=\dfrac{8R^2\sin A\sin B\sin C}{2R(\sin A+\sin B+\sin C)}=\dfrac{4R\sin A\sin B\sin C}{\sin A+\sin B+\sin C}$ so it suffices to show

$$4\sin A\sin B\sin C=(a-1)(b-1)(c-1)(\sin A+\sin B+\sin C)$$

Now multiply both sides by two and we would like to show

$$\left(a-\frac{1}{a}\right)\left(b-\frac{1}{b}\right)\left(c-\frac{1}{c}\right)=-(a-1)(b-1)(c-1)\left(a+b+c-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}\right)$$

Some manipulation of the left and right sides (like $a^2-1=(a-1)(a+1)$) allows us to cancel things and we are left with $(a+1)(b+1)(c+1)=a+b+c+ab+bc+ca$. Now this is not bad at all to expand! We have finished establishing a useful formula for r in terms of R. If you didn't like the fact that $\sqrt{abc}=i$, note that if $\sqrt{abc}=-i$ it would only have resulted in a sign change and we would get a senseless identity. □

Example 4.2. *Show that* $\sum_{cyc} a^3\cos(B-C)=3abc$ *(Here lowercase letters mean side lengths.)*

Solution. First note that R^3 cancels from both sides nicely. Now multiply both sides by $-2i$ and we would like to show

$$\sum_{cyc}\left(a-\frac{1}{a}\right)^3\left(\frac{b}{c}+\frac{c}{b}\right)=6\left(a-\frac{1}{a}\right)\left(b-\frac{1}{b}\right)\left(c-\frac{1}{c}\right)$$

Now we multiply both sides by $a^3b^3c^3$ and hope for the best-

$$\sum_{cyc}\left(a^2-1\right)^3\left(b^4c^2+b^2c^4\right)=6(a^2-1)(b^2-1)(c^2-1)$$

This may sound insane, but now we simply expand. Honestly, the left side only has 8 terms within the cyclic sum, so it shouldn't be bad at all. And the best part is, since $b^4c^2+b^2c^4$ is symmetric in b, c, we can turn this into a SYMMETRIC sum of only 4 terms! (**When bashing, use small tricks to hugely reduce work.**):

$$\sum_{cyc}(a^6-3a^4+3a^2-1)(b^4c^2+b^2c^4)=\sum_{sym}a^6b^4c^2-3a^4b^4c^2+3a^2b^4c^2-b^4c^2$$

Now of course, since $abc=-1$ stuff cancels and we're left with

$\sum_{sym} 3b^2 - 3a^2b^2 = 6\sum_{cyc} b^2 - a^2b^2$. Note that symmetric sums count things like a^2 twice.
And now, it's easy to verify that this equals $6(a^2-1)(b^2-1)(c^2-1)$ by expansion. □

I've only included two examples in this section, but it should be quite clear how this method operates now. Since we're discussing triangles and there are tons of triangle inequalities, I may as well include a note- **DO NOT try to bash trigonometric inequalities** with complex numbers. You will get stupid results. You can use complex numbers to show an intermediate equality, but never an inequality.

4.2 Exercises/Problems

1. Show that $a\cos A + b\cos B + c\cos C = \frac{abc}{2R^2}$ (There is a nice "synthetic" proof of this as well.)

2. Show that acute triangle ABC is isosceles if and only if

$$a\cos B + b\cos C + c\cos A = \frac{a+b+c}{2}$$

   (Hint: To show ABC is isosceles, it is enough to show $(a-b)(b-c)(c-a) = 0$.)

3. Prove that in any triangle ABC,

$$\frac{a-b}{a+b} = \tan\frac{A-B}{2}\sin\frac{C}{2}$$

4. In triangle ABC, $2\angle A = 3\angle B$. Show that

$$(a^2-b^2)(a^2+ac-b^2) = b^2c^2$$

Up next... a short conclusion and even more problems, some of them quite non-standard!

5

Conclusion and Problems

These are the basics of mastering trig through complex numbers. Although trigonometry is somewhat dead on olympiads recently, nobody knows when it will reappear. Meanwhile, this method is still very helpful on the AIME, as shown in problem 13 of the 2015 AIME I, and is especially good if you want a short derivation of an identity. Enjoy the problems, in no particular order! We have mixed in a few gems, not *completely* relevant to complex numbers, but involving trigonometry and some nice concepts. **Feel free to post solutions on the AoPS thread.**

1. (Iran) Let α be an angle such that $\cos\alpha = \frac{p}{q}$, where p and q are two integers. Prove that the number $q^n \cos n\alpha$ is an integer for positive integers n.

2. (Mathematical Reflections) Let a be a real number. Prove that

$$5(\sin^3 a + \cos^3 a) + 3\sin a \cos a = 0.04$$

   if and only if

$$5(\sin a + \cos a) + 2\sin a \cos a = 0.04$$

3. Suppose x, y, z, p satisfy $p(\cos(x + y + z)) = \cos x + \cos y + \cos z$ and similarly for sin. Prove that $\cos(x + y) + \cos(y + z) + \cos(z + x) = p$.

4. This is a very common equation which has been mysteriously absent from this article... Prove the Law of Cosines using the Law of Sines: That is, in a triangle we have

$$c^2 = a^2 + b^2 - 2ab\cos C$$

5. Let r_a, r_b, r_c be the lengths of the exradii of a triangle (google it if you don't know). Show that $r_a + r_b + r_c = 4R + r$.

6. (Mathematical Reflections) Show that triangle ABC is right if and only if

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 0.5$$

7. Triangle ABC has the following property- there is an interior point P with $\angle PAB = 10°, \angle PBA = 20°, \angle PCA = 30°, \angle PAC = 40°$. Show that triangle ABC is isosceles.

   (Hint: How do we turn this into trig? Use trig Ceva!)

8. If ABC is a scalene triangle, show that angle C is right iff

$$(\cos^2 a + \cos^2 b)(1 + \tan a \tan b) = 2$$

9. (ISL) Given real a and integer $m > 0$, and $P(x) = x^{2m} - 2|a|^m x^m \cos\theta + a^{2m}$, factorize $P(x)$ as a product of m real quadratic polynomials

10. (USAMO) Let $F_r = x^r \sin rA + y^r \sin rB + z^r \sin rC$, where x, y, z, A, B, C are real and $A+B+C$ is an integral multiple of π. Prove that if $F_1 = F_2 = 0$, then $F_r = 0$ for all positive integral r.

11. (IMO) Solve the equation in $(0, 2\pi)$ of $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$

12. Show the identity $\sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y)$- almost like difference of squares!

13. (IMO) Show that triangle ABC is isosceles if

$$a + b = \tan\frac{C}{2}(a \tan A + b \tan B)$$

14. Show for ARBITRARY angles a, b, c that

$$\sin A + \sin B + \sin C - \sin(A+B+C) = 4 \sin\frac{A+B}{2} \sin\frac{B+C}{2} \sin\frac{C+A}{2}$$

and find a similar identity for cosine. (I'm not telling you what it is- this will really test your algebraic and trigonometric intuition. Let's just say it's very similar, maybe with sign differences :))

15. Let ABC be a triangle. Show that

$$\sum_{cyc} \cos^3\frac{A}{2} \sin\frac{B-C}{2} = 0$$

Note: A lot of these problems are harder than the examples and slightly or maybe a lot harder than the exercises and problems. This is absolutely intentional, and hopefully it gives you better training. Feel free to discuss solutions on the AoPS thread for the article. Have fun bashing!