Snake oil method: evaluation of sums

In this note we discuss a method for evaluating sums.

Before we start it, recall the following generating functions:

$$\sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n,$$

$$\sum_{n=0}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}},$$

and

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}.$$

Warning: it's easy to confuse the first two identities, be careful whether n or k is the running parameter.

The snake oil method is a very simple method, let's see an example.

Problem 1.1. Find a closed form for the sum

$$\sum_k \binom{n}{2k}\binom{k}{m}.$$

Proof. Let

$$A_n = \sum_k \binom{n}{2k}\binom{k}{m}.$$

Then

$$\sum_{n=0}^{\infty} A_n x^n = \sum_{n=0}^{\infty}\left(\sum_k \binom{n}{2k}\binom{k}{m}\right) x^n =$$

$$= \sum_{k=0}^{\infty} \binom{k}{m}\left(\sum_n \binom{n}{2k} x^n\right) = \sum_{k=0}^{\infty} \binom{k}{m} \frac{x^{2k}}{(1-x)^{2k+1}} =$$

$$= \frac{1}{1-x}\sum_{k=0}^{\infty}\binom{k}{m}\left(\frac{x^2}{(1-x)^2}\right)^k = \frac{1}{1-x}\frac{\left(\frac{x^2}{(1-x)^2}\right)^m}{\left(1-\frac{x^2}{(1-x)^2}\right)^{m+1}} =$$

$$= (1-x)\frac{x^{2m}}{(1-2x)^{m+1}} = (1-x)(x/2)^m \frac{(2x)^m}{(1-2x)^{m+1}} =$$

$$= (1-x)(x/2)^m \sum_{r=m}^{\infty}\binom{r}{m}(2x)^r.$$

By comparing the coefficients of x^n we get that

$$A_n = \frac{1}{2^m}\left(\binom{n-m}{m}2^{n-m} - \binom{n-m-1}{m}2^{n-m-1}\right) =$$

$$= 2^{n-2m-1}\left(2\binom{n-m}{m} - \frac{n-2m}{n-m}\binom{n-m}{m}\right) = 2^{n-2m-1}\frac{n}{n-m}\binom{n-m}{m}.$$

□

So the solution consisted of the steps

(i) call A_n the sum, and write up $\sum A_n x^n$,
(ii) interchange the two sums,
(iii) used the above identities to find a closed form for the generating function,
(iv) find a closed form for the sum.

Problem 1.2. Find a closed form for the sum

$$\sum_{k=0}^{n}\binom{n+k}{2k}2^{n-k}.$$

Proof. Again let

$$A_n = \sum_{k=0}^{n}\binom{n+k}{2k}2^{n-k}.$$

Then

$$\sum_{n=0}^{\infty} A_n x^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n+k}{2k}2^{n-k}\right)x^n =$$

$$\sum_{k=0}^{\infty}\frac{1}{2^k}\left(\sum_{n}\binom{n+k}{2k}(2x)^n\right) = \sum_{k=0}^{\infty}\frac{1}{2^k}\frac{(2x)^k}{(1-2x)^{2k+1}} =$$

$$= \frac{1}{1-2x}\sum_{k=0}^{\infty}\left(\frac{x}{(1-2x)^2}\right)^k = \frac{1}{1-2x}\frac{1}{1-\frac{x}{(1-2x)^2}} =$$

$$\frac{1-2x}{1-5x+4x^2} = \frac{1-2x}{(1-x)(1-4x)} = \frac{2}{3}\frac{1}{1-4x} + \frac{1}{3}\frac{1}{1-x} = \frac{2}{3}\sum_{n}(4x)^n + \frac{1}{3}\sum_{n}x^n.$$

Hence

$$\sum_{k=0}^{n}\binom{n+k}{2k}2^{n-k} = \frac{1}{3}(2\cdot 4^n + 1).$$

□

Problem 1.3. Show that

$$\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{m}{k}\binom{n}{k}2^k.$$

Proof. Let

$$A_n = \sum_k \binom{m}{k}\binom{n+k}{m},$$

and

$$B_n = \sum_k \binom{m}{k}\binom{n}{k}2^k.$$

Then

$$\sum_{n=0}^{\infty} A_n x^n = \sum_{n=0}^{\infty}\left(\sum_k \binom{m}{k}\binom{n+k}{m}\right)x^n =$$

$$= \sum_{k=0}^{\infty}\binom{m}{k}\left(\sum_n \binom{n+k}{m}x^n\right) = \sum_{k=0}^{\infty}\binom{m}{k}\frac{x^{m-k}}{(1-x)^{m+1}} =$$

$$= \frac{x^m}{(1-x)^{m+1}}\sum_k \binom{m}{k}x^{-k} = \frac{x^m}{(1-x)^{m+1}}\left(1+\frac{1}{x}\right)^m = \frac{(1+x)^m}{(1-x)^{m+1}}.$$

On the other hand,

$$\sum_{n=0}^{\infty} B_n x^n = \sum_{n=0}^{\infty}\left(\sum_k \binom{m}{k}\binom{n}{k}2^k\right)x^n =$$

$$= \sum_{k=0}^{\infty}\binom{m}{k}2^k\left(\sum_n \binom{n}{m}x^n\right) = \sum_{k=0}^{\infty}\binom{m}{k}2^k\frac{x^k}{(1-x)^{k+1}} =$$

$$= \frac{1}{1-x}\sum_{k=0}^{\infty}\binom{m}{k}\left(\frac{2x}{1-x}\right)^k = \frac{1}{1-x}\left(1+\frac{2x}{1-x}\right)^m = \frac{(1+x)^m}{(1-x)^{m+1}}.$$

Hence $A_n = B_n$ for all n. □

Problem 1.4. Find a closed form for the sum

$$A_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{2k}{k}\binom{k}{n-k}.$$

Proof. Let

$$A_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{2k}{k} \binom{k}{n-k}.$$

Then

$$\sum_{n=0}^{\infty} A_n x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^{n-k} \binom{2k}{k} \binom{k}{n-k} \right) x^n =$$

$$= \sum_{k=0}^{\infty} \binom{2k}{k} x^k \left(\sum_{n} \binom{k}{n-k} (-x)^{n-k} \right) = \sum_{k=0}^{\infty} \binom{2k}{k} x^k (1-x)^k = \sum_{k=0}^{\infty} \binom{2k}{k} (x(1-x))^k =$$

$$= \frac{1}{\sqrt{1-4x(1-x)}} = \frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n.$$

Hence

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{2k}{k} \binom{k}{n-k} = 2^n.$$

□

Problem 1.5. Find a closed form for the sum

$$\sum_{k=0} \binom{m}{k} \binom{2m-2k}{m-k} (-2)^k.$$

Proof. First we carry out a change of variable: $m - k = r$. Then

$$A_m = \sum_{k=0} \binom{m}{k} \binom{2m-2k}{m-k} (-2)^k = \sum_{r=0} \binom{m}{r} \binom{2r}{r} (-2)^{m-r}.$$

Then

$$\sum_{m=0}^{\infty} A_m x^m = \sum_{m=0}^{\infty} \left(\sum_{r=0} \binom{m}{r} \binom{2r}{r} (-2)^{m-r} \right) x^m = \sum_{r=0}^{\infty} \binom{2r}{r} (-2)^{-r} \left(\sum_{m=r}^{\infty} \binom{m}{r} (-2x)^m \right) =$$

$$= \sum_{r=0}^{\infty} \binom{2r}{r} (-2)^{-r} \frac{(-2x)^r}{(1+2x)^{r+1}} = \frac{1}{1+2x} \sum_{r=0}^{\infty} \binom{2r}{r} \left(\frac{x}{1+2x} \right)^r =$$

$$= \frac{1}{1+2x} \frac{1}{\sqrt{1-4\frac{x}{1+2x}}} = \frac{1}{\sqrt{1-4x^2}} =$$

$$= \sum_{n=0}^{\infty} \binom{2n}{n} x^{2n}.$$

Hence $A_m = 0$ if m is odd, and $A_m = \binom{m}{m/2}$ if m is even. □