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On the sine functional equation

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ON THE SINE FUNCTIONAL EQUATION

A THESIS

SUBMITTED TO THE FACULTY OF ATLANTA UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF MASTER OF SCIENCE

BY

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CHAPTER I

INTRODUCTION

Functional equations for familiar elementary functions have been studied by Cauchy and others. In this paper the writer can only praise these renowned and worthy scholars of previous generations who have provided such a noble heritage for students of today. However, this paper is concerned with the theory of more recent writers on certain functional equations — such as Kestelman, Rosenbaum, Segal, and Wilson.

It is well known and has been rediscovered several times that if f is a real-valued finite-valued function of a real variable, continuous at a point, and satisfying the functional equation

$$(1) \quad f(x+y) f(x-y) = f^2(x) - f^2(y)$$

for all real x and y , then f must be continuous everywhere and be one of the functions defined by the following three relations:

$f(x) = k_1 x$, $f(x) = k_2 \sin k_3 x$, $f(x) = k_4 \sinh k_5 x$, where the k_i are arbitrary real constants.

The functional equation (1) is closely connected with Cauchy functional equation

$$(2) \quad \phi(x+y) = \phi(x) + \phi(x)$$

for real-valued functions of a real variable. In fact, clearly every solution of (2) is a solution of (1).

In chapter two we will establish various theorems on approximating measurable functions by means of continuous functions. These theorems will lead us to the basic structural property for chapter three. We will show in chapter three that the only solution of (1) measurable on some interval are continuous.

Besides weakening the continuity condition of f one might also consider the functional equation (1) with $=$ replaced by \leq . The analogous substitution in (2) gives the important class of subadditive functions; in (1), however nothing new is obtained.

CHAPTER II

MATHEMATICAL TOOLS

We need the following definitions:

Definition 2.1

A function f defined on U , is said to be measurable if the set $E[f(x) \leq k]$ is measurable for every real k , where $E = \{x \mid f(x) \leq k\}$.

Definition 2.2

Let the function f be defined on the set E , let $x_0 \in E$, and let $F(x_0)$ be finite. We say the function f is continuous at the point x_0 in two cases: (1) if x_0 is an isolated point of E ; (2) if $x_0 \in E'$, where E' is a subset of E , and the relations

$$x_n \longrightarrow x_0, x_n \in E$$

implies that

$$f(x) \longrightarrow f(x_0).$$

If f is continuous at every point on the set E , it is said to be continuous on the set E .¹

Definition 2.3

If G is a bounded open set, then the sum of the lengths

¹I. P. Natanson, Theory of Functions of a Real Variable (New York : Fredrick Ungar Publishing Company, 1961), I, 102.

of the component intervals of G is called the measure of G and is denoted by $m(G)$.

Definition 2.4

A function f is subadditive according as

$$f(x_1 + x_2) \leq f(x_1) + f(x_2)$$

or

$$f(x_1 + x_2) \geq f(x_1) + f(x_2)$$

for all x_1, x_2 and $x_1 + x_2$ in the domain of definition of f (this domain is usually taken to be an interval of the form $0 \leq x \leq a$).¹

In addition, we need the following lemmas, which we state without proof:

Lemma 2.1

Let the sets F_1, F_2, \dots, F_n be closed and pairwise disjoint. If the function ϕ , defined on the set

$$F = \sum_{k=1}^n F_k,$$

is a constant on each of the sets F_k , it is continuous on the set F .²

Lemma 2.2

Let F be a closed set contained in the closed interval $[a, b]$. If the function ϕ is defined and continuous on the set F , then it is possible to define a function ψ on $[a, b]$ with the following properties: (1) ψ is continuous; (2) if

¹ Glenn James and Robert C. James, Mathematics Dictionary (Princeton, New Jersey: D. Van Nostrand Company, Inc., 1956), p. 5.

² Natanson, loc. cit.

$x \in F$, then $\psi(x) = \phi(x)$; and (3) $\max |\psi(x)| = \max |\phi(x)|$.¹

For many purposes of applied analysis it is convenient to separate the sine part and cosine part of certain functional equations.

Consider the system of equations

$$(3) \quad S(x - y) = S(x) C(y) - C(x) S(y)$$

and

$$(4) \quad C(x - y) = C(x) S(y) - k^2 S(x) S(y),$$

where S and C represent the sine and cosine functions respectively. We will show that equations (3) and (4) have their solutions in common if C is an even function and $S(x) \neq 0$.

Theorem 2.1

If S and C satisfy equation (3) and $S(x) \neq 0$, then the odd component of C is a constant multiple of S , and S and the even component $E(x)$ of C satisfy equations (3) and (4) simultaneously, where $k^2 = \frac{E(a) - E(0)}{S^2(a)}$ and $S(a) \neq 0$, where

a is some fixed real number.

Proof:

Interchange x and y . Then

$$S(y - x) = -S(x - y),$$

that is S is an odd function and $S(0) = 0$. Discarding the trivial solution $S(x) \equiv 0$, there is some value a of x such that $S(a) \neq 0$. Let

$$C(x) = E(x) + O(x),$$

¹Ibid.

where E is even and O is odd. Equation (3) becomes

$$(3') \quad S(x - y) = S(x) E(y) + S(x) O(y) - E(x) S(y) - O(x) S(y).$$

Replace x by $-x$ and y by $-y$ in (3') and add the equation thus obtained to equation (3'). Then

$$S(x) O(y) - O(x) S(y) = 0.$$

If $y = a$,

$$O(x) = \frac{O(a) S(x)}{S(a)}.$$

Equation (3') now becomes

$$S(x - y) = S(x) E(y) - E(x) S(y),$$

whence

$$E(x) S(y) = S(x) E(y) - S(x - y).$$

Now replace x by $x - y$ and y by a .

$$\begin{aligned} E(x - y) S(a) &= S(x - y) E(a) - S(x - y - a) \\ &= [E(a) S(x) - S(x - a)] E(y) - \\ &\quad [E(a) E(x) - E(x - a)] S(y) \\ &= S(a) E(x) E(y) - [E(a) E(x) - \\ &\quad E(x - a)] S(y). \end{aligned}$$

Interchange x and y and compare the equation thus obtained with the last equation. Then

$$[E(a) E(x) - E(x - a)] S(y) = [E(a) E(y) - E(y - a)] S(x).$$

Let $y = a$. Then

$$E(x) E(a) - E(x - a) = k^2 S(a) S(x),$$

where

$$k^2 = \frac{E^2(a) - E(0)}{S^2(a)}.$$

Therefore,

$$E(x - y) = E(x) E(y) - k^2 S(x) S(y).$$

Q. E. D.

Theorem 2.2

If S and C satisfy (4) and if $C(x) \not\equiv \text{constant}$, then $k \neq 0$, $S(x) \not\equiv 0$, and S and C satisfy equation (3) and (4) simultaneously.

Proof:

Interchange x and y . Then

$$C(y - x) = C(x - y),$$

that is, C is even. If $S(x) \equiv 0$, or $k = 0$, then

$$C(x + y) = C(x) C(-y) = C(x) C(y) = C(x - y)$$

and if $y = x$

$$C(2x) = C(0),$$

that is, C is identically constant (0 or 1).

Setting aside this trivial case, $k \neq 0$, there is some value a of x such that $C(a) \neq 0$.

Set

$$S(x) = E_1(x) + O_1(x),$$

where E_1 is even and O_1 is odd.

Equation (4) becomes

$$C(x - y) = C(x) C(y) - k^2 E_1(x) E_1(y) + E_1(x) O_1(y) + E_1(y) O_1(x) + O_1(x) O_1(y).$$

Replace x by $-x$ and y by $-y$ and subtract the equation so obtained from the last equation. Then

$$E_1(x) O_1(y) + E_1(y) O_1(x) = 0.$$

If $O_1(x) \equiv 0$, then as in Theorem 2.1 $C(x) \equiv \text{constant}$. Discarding this trivial case, there is some value \bar{a} of x such

that $O_1(\bar{a}) \neq 0$. Let $x = y = \bar{a}$. Then

$$2E_1(\bar{a}) O_1(\bar{a}) = 0,$$

whence $E_1(\bar{a}) = 0$.

Now let $y = \bar{a}$ and let x vary;

$$O_1(\bar{a}) E_1(x) = 0,$$

whence $E_1(x) \equiv 0$ and S is odd and $S(0) = 0$. In (4) let $y = 0$

and $x = b$. Then $C(0) = 1$ and hence if $y = x$ in (4),

$$C^2(x) - k^2 S^2(x) = 1.$$

Replace y by $-y$ in equation (4). Then

$$C(x + y) = C(x) C(y) + k^2 S(x) S(y).$$

Now replace x by $x + a$ and y by a in equation (4);

$$\begin{aligned} k^2 S(a) S(x + a) &= C(a) C(x + a) - C(x) \\ &= C^2(a) C(x) + k^2 C(a) S(a) S(x) - C(x) \\ &= [C^2(a) - 1] C(x) + k^2 C(a) S(a) S(x) \\ &= k^2 S^2(a) C(x) + k^2 C(a) S(a) S(x), \end{aligned}$$

whence

$$S(x + a) = S(a) C(x) + C(a) S(x).$$

Finally, replace x by $x - y$ and y by a in equation (4).

It follows that

$$\begin{aligned} k^2 S(a) S(x - y) &= C(a) C(x - y) - C(x - y - a) \\ &= C(a) C(x) C(y) - k^2 C(a) S(x) S(y) - \\ &\quad C(x) C(y + a) + k^2 S(x) S(y + a) \\ &= k^2 S(x) [S(y + a) - C(a) S(y)] - \\ &\quad C(x) [C(y + a) - C(a) C(y)] \\ &= k^2 S(a) S(x) C(y) - k^2 S(a) C(x) S(y). \end{aligned}$$

Therefore,

$$S(x - y) = S(x) C(y) - C(x) S(y).$$

Q. E. D.

Theorem 2.3

Suppose that S and C satisfy equation (3) and (4) simultaneously. If $k \neq 0$, $S(x) = \frac{F(x) - F(-x)}{2k}$ and $C(x) = \frac{F(x) + F(-x)}{2}$, where $F(x+y) = F(x)F(y)$. If $k = 0$, and $S(x) \neq 0$, $S(x+y) = S(x) + S(y)$ and $C(x) \equiv 1$.

Proof:

By interchanging x and y , we see that S is odd and C is even. Replace y by $-y$. Then

$$S(x+y) = S(x)C(y) + C(x)S(y)$$

$$C(x+y) = C(x)C(y) + k^2 S(x)S(y).$$

The function $F(x) = C(x) = k S(x)$ satisfies the equation

$$F(x+y) = F(x)F(y).$$

Now

$$F(-x) = C(x) - k S(x)$$

and, therefore,

$$C(x) = \frac{1}{2} [F(x) + F(-x)]$$

and

$$S(x) = \frac{1}{2} [F(x) - F(-x)]$$

if $k \neq 0$.

If $k = 0$, we have seen from equation (4) that $C(x) \equiv C(0)$.

If $S(x) \neq 0$, then by equation (3) $C(x) \neq 0$. But if $y = 0$ in equation (4),

$$C(x) = C(x)C(0),$$

whence if $C(b) \neq 0$, $C(0) = 1$ and equation (3) becomes

$$S(x-y) = S(x) - S(y).$$

Replace y by $-y$. Thus,

$$S(x + y) = S(x) + S(y).$$

Discarding the trivial solution $S(x) \equiv 0$, suppose a is a value of x for which $S(x) \neq 0$. Since $\frac{S(x)}{m}$ (where m is any constant different from zero) satisfies equation (1) we may suppose that $S(a) = 1$.

Q. E. D.

Theorem 2.4

If S satisfies equation (1), then $S(x)$ and $C(x) = \frac{1}{2} [S(x+a) - S(x-a)]$ satisfy equations (3) and (4) simultaneously, where $k^2 = C^2(a) - 1$.

Proof:

Interchange x and y . It follows that

$$S(x+y) S(y-x) = -S(x+y) S(x-y).$$

Replace $x+y$ by a and $x-y$ by x . Then

$$S(-x) = -S(x).$$

Now

$$\begin{aligned} 2C(-x) &= S(-x+a) - S(-x-a) \\ &= -S(x-a) + S(x+a) \\ &= 2C(x). \end{aligned}$$

Moreover, $C(0) = 1$. From equation (1) it readily follows that

$$S(x+y) = S(x+y) S(a) = S^2\left(\frac{x+y+a}{2}\right) - S^2\left(\frac{x+y-a}{2}\right)$$

and

$$S(x-y) = S(x-y) S(a) = S^2\left(\frac{x-y+a}{2}\right) - S^2\left(\frac{x-y-a}{2}\right).$$

Therefore,

$$S(x+y) + S(x-y) = S(x) S(y+a) - S(x) S(y-a) = 2S(x) C(y).$$

Interchange x and y , and subtract the equation thus obtained from the last equation. Then

$$S(x - y) = S(x) C(y) - C(x) S(y).$$

The hypothesis of theorem 2.1 is fulfilled. Since C is even the functions S and C satisfy equations (3) and (4) simultaneously.

To prove the converse theorem when $S(x) \not\equiv 0$, let us observe that if $y = x$ in (4),

$$C^2(x) - k^2 S^2(x) = 1,$$

whence

$$\begin{aligned} S(x + y) S(x - y) &= S^2(x) C^2(y) - C^2(x) S^2(y) \\ &= S^2(x) - S^2(y). \end{aligned}$$

Q. E. D.

That is, S satisfies equation (1).

Since the sine functional equation is in the realm of analysis, some of the powerful mathematical tools of analysis are applied. These tools include Hamel's Basis, E. Borel Theorem, and Lusin's Theorem.

Hamel's Basis

The German mathematician, Hamel, conceived the notion of basis for all real numbers. It is as follows: Let H be a set of real numbers with properties;

- 1) If $\{x_1, \dots, x_n\}$ is any finite subset of H and if r_1, \dots, r_n are rational numbers for which $r_1 x_1 + \dots + r_n x_n = 0$, then $r_1 = \dots = r_n = 0$.
- 2) Every real number x can be expressed as a finite linear combination of elements of H , with rational coefficients.

In terms of such basis, Hamel then discussed real function f satisfying the equation

$$f(x + y) = f(x) + f(y)$$

for all real x and y . If H is a basis in the foregoing sense, let a real function f be defined as follows: If $x \in H$, assign the values of f arbitrarily. Any real x has a unique representation $x = r_1 x_1 + \dots + r_n x_n$, where x_1, \dots, x_n are in H and r_1, \dots, r_n are rational (n may vary with x , of course). We then define $f(x) = r_1 f(x_1) + \dots + r_n f(x_n)$, the values $f(x_1), \dots, f(x_n)$ having already been assigned. With this definition f turns out to satisfy the condition $f(x + y) = f(x) + f(y)$ for every x and y .¹

E. Borel Theorem

Let a measurable function f be defined and be finite almost everywhere on the closed interval $[a, b]$. For all numbers $\sigma > 0$ and $\epsilon > 0$, there exists a function ψ continuous on $|a, b|$ for which

$$m \left\{ E \left(|f - \psi| \geq \sigma \right) \right\} < \epsilon.$$

If $|f(x)| \leq k$, we can choose ψ so that $|\psi(x)| \leq k$.

Proof:

Suppose first that

$$|f(x)| \leq k,$$

That is to say, that the function f is bounded. Fixing arbitrarily $\sigma > 0$ and $\epsilon > 0$, we can choose a natural number

¹Angus E. Taylor, Introduction to Functional Analysis (New York: John Wiley & Sons, Inc., 1958), p. 44.

m so large that

$$\frac{K}{m} < \epsilon,$$

and construct the sets

$$E_i = E \left(\frac{i-1}{m} K \leq f < \frac{i}{m} K \right) \quad (i = 1 - m, 2 - m, \dots, m - 1),$$

$$E_m = E \left(\frac{m-1}{m} K \leq f \leq K \right).$$

These sets are measurable, pairwise disjoint, and have the property that

$$[a, b] \supset \bigcup_{i=1-m}^m E_i.$$

For every i , we will choose a closed set $F_i \subset E_i$ such that

$$m(F_i) > m(E_i) - \frac{\epsilon}{2m}$$

and set

$$F = \bigcup_{i=1-m}^m F_i.$$

It is clear that

$$[a, b] - F = \bigcup (E_i - F_i),$$

and therefore

$$m[a, b] - m(F) < \epsilon.$$

Now we define a function ϕ in the set F by setting

$$\phi(x) = \frac{i}{m} K \text{ for } x \in F_i \quad (i = 1 - m, \dots, m).$$

By Lemma 2.1, the function is continuous on the set F , we also have

$$|\phi(x)| \leq K,$$

finally for $x \in F$,

Now applying Lemma 2.2, we find a continuous function ψ which coincides with the function ϕ on the set F , and which has

the property that $|\psi(x)| \leq K$.

Inasmuch as $E(|f - \psi| \geq \delta) \subset [a, b] - F$,

it is clear the function ϕ satisfies the requirements of the theorem. The theorem is thus proved for a bounded function f .

Suppose now that f is unbounded. Then using the following theorem:

Let a measurable function $f(x)$ be defined and finite almost everywhere on the set E . For any $\epsilon > 0$, there exists a measurable bounded function $g(x)$ such that

$$m\{E(f \neq g)\} < \epsilon^1,$$

we can find a bounded function g such that

$$m\{E(f \neq g)\} < \frac{\epsilon}{2}.$$

Applying the present theorem to the bounded function g , we can find a continuous function ψ such that

$$m\{E(|g - \psi| \geq \delta)\} < \frac{\epsilon}{2}.$$

It is easy to see that

$$E(|f - \psi| \geq \delta) \subset E(f \neq g) + E(|g - \psi| \geq \delta).$$

Hence the function ψ satisfies all requirements. Q. E. D.

Lusin's Theorem

Every measurable function f defined in a measurable set

¹Natanson, loc. cit., p. 101.

E can be made continuous by removing from E the points contained in suitably chosen open intervals whose total length is arbitrarily small.

Proof:

It suffices to consider the case where E can be enclosed in a finite interval which we can assume closed, say the interval $[a, b]$. We extend the function f to the entire interval by obtaining a function which is measurable in $[a, b]$ and is consequently the limit almost everywhere of a sequence of step functions ϕ_n . The points where the sequence does not converge to f and the points of discontinuity of the functions ϕ_n form a set of measure zero; they can therefore be covered by open intervals of total length less than $\frac{\epsilon}{2}$. In the closed set which remains after having removed these intervals from $[a, b]$ all the functions ϕ_n are continuous and tend everywhere to f . We can, therefore, remove from this closed set the points contained in a system of suitably chosen open intervals, of total length less than $\frac{\epsilon}{2}$, so that in the set which remains the sequence ϕ_n tends uniformly to f , and thus applying the theorem which states that the limit of a uniformly convergent sequence of continuous functions itself is continuous, the proof is complete.

CHAPTER III

APPLICATION OF THE SINE FUNCTIONAL EQUATION

We will now show that the only solutions of (1) measurable on some interval are continuous.

Theorem 3.1

If f is a real-valued (finite-valued) function of a real variable, measurable on some interval, and satisfying the functional inequality

$$(5) \quad f(x+y) f(x-y) \leq f^2(x) - f^2(y)$$

for all real x and y , then f is one of the functions defined by the following three relations:

$$f(x) = k_1 x, \quad f(x) = k_2 \sin k_3 x, \quad f(x) = k_4 \sinh k_5 x,$$

where the k_i are arbitrary real constants.

Proof:

First, we will prove that the functional inequality (5) is valid for all x and y implies that (1) is valid for all real x and y .

Putting $x = y = 0$ in (5), we have

$$f(y+y) f(y-y) \leq f^2(y) - f^2(y),$$

so that

$$(6) \quad f(0) = 0.$$

Putting $x = -y$ in (5), we have

$$f(-y + y) f(-y - y) \leq f^2(-y) - f^2(y),$$

that is to say,

$$f(0) \leq f^2(-y) - f^2(y).$$

Now using (5), we have

$$0 \leq f^2(-y) - f^2(y),$$

that is

$$f^2(y) \leq f^2(-y).$$

Hence by (5) and (6)

$$0 \geq f^2(y) - f^2(-y) \geq f(0) f^2(y) = 0$$

and so

$$f^2(y) = f^2(-y).$$

It follows that for each real number y , $f(-y)$ equal either $-f(y)$ or $f(y)$. Suppose for some real y_0 , $f(y_0) = f(-y_0)$; then putting $x = 0$ and $y = y_0$ in (5) and using (6), we have

$$\begin{aligned} f(0 - y_0) f(0 - y_0) &\leq f^2(0) - f^2(y_0), \\ f(-y_0) f(-y_0) &\leq -f^2(y_0). \end{aligned}$$

But by our hypothesis $f(-y_0) = f(y_0)$.

Hence

$$f^2(y_0) \leq -f^2(y_0)$$

or

$$2f^2(y_0) = 0.$$

Hence $f(y_0) = f(-y_0) = 0$ and so for all real y ,

$$(8) \quad f(-y) = -f(y).$$

Hence for (5), for all real x and y

$$f^2(y) \leq f^2(x) - f(x + y) f(x - y)$$

$$\begin{aligned}
&= f^2(x) + f(x+y) f(y-x) \\
&\leq f^2(x) + f^2(y) - f^2(x) \\
&= f^2(y)
\end{aligned}$$

and hence equation (1) holds for all real x and y .

Now suppose f is measurable on the interval (a, b) . We will show that f is continuous at zero¹. By Lusin's theorem, given $\sigma > 0$, there is a function F such that F is continuous on (a, b) and $f(x) = F(x)$ for all $x \in (a, b)$ except for a set of measure $< \sigma$. Let $\delta = \frac{(b-a)}{6}$.

Since F is continuous, given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ (where δ clearly may be taken $< \delta$) such that for all h with $|h| < \delta$ and for $x \in (a, b)$,

$$(10) \quad |F(x+h) F(x-h) - F^2(x)| < \epsilon.$$

Since

$$(11) \quad F(x) = f(x)$$

for all $x \in (a, b)$ except for the set of measure $< \sigma$,

$$(12) \quad F(x+h) = f(x+h)$$

for all $x \in (a, b)$ except for the set of measure $< \sigma + |h| < \sigma + \delta$.

Similarly,

$$(13) \quad F(x-h) = f(x-h)$$

for all $x \in (a, b)$ except for a set of measure $< \sigma + \delta$.

Hence by (11), (12), (13)

$$(14) \quad F(x+h) F(x-h) - F^2(x) = f(x+h) f(x-h) - f^2(x)$$

¹The essential idea of our method is due to Banach and would seem to be adaptable to many functional equations to prove that if a solution is measurable on some interval, it is continuous at some point.

for all $x \in (a, b)$ except for a set of measure $< (\sigma + \delta) + (\sigma + \delta) + \sigma$
 $= 3\sigma + 2\delta < 5\sigma = \frac{5(b-a)}{6} < b-a.$

Hence given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that given
 h with $|h| < \delta$, there exists $x = x(h) \in (a, b)$ such that

$$|f(x+h) f(x-h) - f^2(x)| < \epsilon.$$

But by equation (1)

$$f(x+h) f(x-h) - f^2(x) = -f^2(h^2)$$

for all real x and h .

Hence given ϵ , there exists $\delta = \delta(\epsilon) > 0$ such that if
 $|h| < \delta$, then $|f^2(h)| < \epsilon$. Hence $\lim_{h \rightarrow 0} f^2(h) = 0$.

So by (6) f is continuous at zero.

Q. E. D.

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