Olympiad Combinatorics

Pranav A. Sriram

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About the Author

Pranav Sriram graduated from high school at The International School Bangalore, India, and will be a Freshman at Stanford University this Fall.

5. COMBINATORIAL GAMES

I'm afraid that sometimes,
you'll play lonely games too.
Games you can't win
'cause you'll play against you.

-Doctor Seuss

Introduction

In this chapter, we study combinatorial games involving two players. Typical problems describe a game, and ask us to find a winning strategy for one of the players or determine whether one exists.

As in problems in the chapters on algorithms and processes, games are specified by a starting position, allowed move(s) and the final or winning position. These and other attributes make problems in this chapter seem superficially similar to those in the first three chapters, but a fundamental difference is that our algorithms or strategies now need to compete against those of an opponent. Some of the techniques we will develop in this chapter are hence significantly different from earlier ones. One simple yet surprisingly powerful technique we use is exploiting *symmetry*.

Others include coloring and invariants, recursion, induction, parity and a very important technique we will introduce known as positional analysis.

The term "combinatorial games" is generally used to describe games with the following characteristics:

(i) There are no elements of luck or chance (so no dice rolling or coin flipping involved).
(ii) There is usually perfect information, unlike in card games, where I cannot see your hand (unless I peak).
(iii) Players typically move alternately. In this chapter, all our games are played by Alice and Bob, whom we refer to as A and B respectively. In general A starts.
(iv) There is no cheating (unfortunately).

One more thing we will use frequently is that if a game is finite and does not have ties, then someone has to eventually lose. Because of the perfect information property and the absence of probabilistic elements, one player will hence have a winning strategy. In particular, one concept we will use frequently to develop strategies is that if one player can ensure that he always "stays alive", the other player will eventually lose.

Symmetry, Pairing and Copying

They said no cheating, but we can still copy!

This section is devoted to the very important technique of "copycat strategies" and pairing techniques. The next few examples should illustrate what these techniques are about.

Example 1
A and **B** each get an unlimited supply of identical circular coins. **A**

and **B** take turns placing the coins on a finite square table, in such a way that no two coins overlap and each coin is completely on the table (that is, it doesn't stick out). The person who cannot legally place a coin loses. Assuming at least one coin can fit on the table, prove that **A** has a winning strategy.

Answer:
A first places a coin such that its center is at the center of the table. Then whenever **B** places a coin with center at a point X, **A** places a coin with center at the point X', where X' is the reflection of X in the center of the table. This ensures that after each of **A**'s moves, the board is completely symmetrical. Thus if **B** can make a legal move, then by symmetry, **A**'s next move is also legal. Since the area of the table is finite, eventually the game must terminate and someone must lose. Since **A** can always "stay alive", **B** loses. ■

Example 2 [Saint Petersburg 1997]
The number N is the product of k different primes ($k \geq 3$). **A** and **B** take turns writing composite divisors of N on a board, according to the following rules. One may not write N. Also, there may never appear two coprime numbers or two numbers, one of which divides the other. The first player unable to move loses. If **A** starts, who has the winning strategy?

Answer:
A has a winning strategy. **A** first writes pq for some primes p and q dividing N. Then all the subsequent numbers written must be of the form pm or qm for some m dividing N, by the conditions of the problem. Whenever **B** writes qm, **A** writes qn. This "copying strategy" ensures that **A** always has a move. Since the game is finite (N has a finite number of divisors), **A** will eventually win. ■

Example 3 [USAMO 2004 – 4]
Alice and Bob play a game on a 6×6 grid. On his or her turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all of the squares have numbers

written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if she can then draw a path from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if she can't. (A path is a sequence of squares such that any two consecutive squares in the path share a vertex). Find, with proof, a winning strategy for one of the players.

Answer:

X	X	X			
X	X	X			
X	X			X	
			X	X	X
			X	X	X
			X	X	X

Figure 5.1

B has a winning strategy. The idea is to ensure that the largest number in each row is in one of the squares marked X in Figure 5.1. Then clearly there will be no path and **B** will win.

To do this, **B** pairs each square marked with an X with a square not marked with an X in the same row. Whenever **A** plays in a square marked with an X, **B** writes a smaller number in the paired square. Whenever **A** writes a number in a square not having an X, **B** writes a larger number in the paired square. This ensures that after each of **B**'s moves, the largest number in each row is in one of the squares marked with an X. ■

Example 4 [Tic Tac Toe for mathematicians]
On a 5 × 5 board, **A** and **B** take turns marking squares. **A** always writes an X in a square and **B** always writes O. No square can be marked twice. **A** wins if she can make one full row, column or diagonal contain only Xs. Can **B** prevent **A** from winning?

Answer:
Yes. Mark the board as shown.

5	9	11	9	6
8	1	1	2	7
12	4		2	12
8	4	3	3	7
6	10	11	10	5

Figure 5.2: Squares with equal numbers are paired

Since each number from 1 to 12 appears in 2 squares, **B** can ensure that he always marks at least one square of each number (if **A** marks an X in a square with number i, **B** puts an O in the other square marked i). Observe that each row, column and diagonal contains a pair of squares having equal numbers. Since **B** ensures that at least one number in each pair is marked, each row, column and diagonal would have been marked by **B** at least once. ■

Remark 1: This is one of those solutions that initially appears to come out of nowhere, but actually has a very intuitive explanation. Here's one way of thinking about it. There are 5 columns, 5 rows and 2 diagonals, for a total of 12 "winning lines". We can only make 12 pairs of numbers (since there are 25 squares and 12 x 2 = 24). Thus our idea is to construct a pairing strategy such that each row, column and diagonal is covered by exactly one pair. In this example, a natural way to construct the pairing is to start with the inner 3 x 3 square, covering its border, and then fill the outer layer, ensuring that the remaining winning lines are taken care of.

Remark 2: The game of tic-tac-toe can be generalized to getting k squares in a row on an $m \times n$ board (regular tic-tac-toe has $k = m = n = 3$ and this problem has $k = m = n = 5$). For most values of k, m and n, the question of whether there exists a winning strategy for the starting player remains an open problem. In addition, the game of tic-tac-toe in higher dimensions (such as on 3-D cubes or in general, n-dimensional hyperspaces) has strong connections to a branch of extremal combinatorics known as Ramsey theory.

Example 5 [IMO Shortlist 1994, C1]
A and **B** play alternately on a 5 × 5 board. **A** always enters a 1 into an empty square, and **B** always enters a 0 into an empty square. When the board is full, the sum of the numbers in each of the nine 3 × 3 squares is calculated and **A**'s score S is the largest such sum. What is the largest score **A** can make, regardless of the responses of **B**?

Answer:

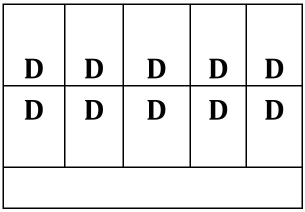

Figure 5.3: A pairing strategy for B

Tile the board with 10 dominoes (marked **D** in the figure), leaving the bottom row untiled. **B** can ensure that each domino contains at least one 0, since whenever **A** plays in a domino **B** writes a 0 in the other square. Each 3 × 3 square has at least 3 full dominoes, and hence will have at least three 0s. Thus **B** can ensure that S is at most 6. Now we leave it to the reader to show that **A** can ensure that S is at least 6. ■

Parity Based Pairing

We solved the last few examples by explicitly constructing pairing strategies. But all we actually need to do sometimes is prove the *existence* of a pairing strategy. This is often remarkably easy- observe that if there are an even number of objects, there exists a way to pair them up.

Example 6 [Based on Italy TST 2009, problem 6]
A and **B** play the following game. First **A** writes a permutation of the numbers from 1 to n, where n is a fixed positive integer greater than 1. In each player's turn, he or she must write a sequence of numbers that has not been written yet such that either:

a) The sequence is a permutation of the sequence the previous player wrote, OR
b) The sequence is obtained by deleting one number from the previous player's sequence

For example, if **A** first writes 4123, **B** could write 3124 or 413. The player who cannot write down a sequence loses. Determine who has the winning strategy.

Answer:
If $n = 2$, **B** wins: after **A**'s first move, **B** deletes the number 2 and is left with the sequence {1}. Then **A** has no move.

The idea now is to construct an inductive strategy for **B**. Suppose **B** wins for $n = k$; we now want a strategy for $n = k+1$. **B**'s aim is to make **A** be the first player to delete a number from the sequence. Then from this point the game is reduced to a game with k numbers, and **B** will win this by induction. But this is very easy to do – whenever **A** writes a sequence of $k+1$ numbers, there will always exist at least one permutation of the $k+1$ numbers that has not been written yet, simply because the total number of permutations is even ($(k+1)!$ is even). ■

Remark: You can also use an explicit pairing strategy. One example is that whenever **A** writes a sequence, **B** writes the same sequence backwards.

Example 7 [USAMO 1999-5]
The *Y2K Game* is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

Answer:
Call an empty square *bad* if playing in that square will let the other player form SOS in the next turn. We claim that an empty square is bad **if and only if** it is in a block of 4 squares of the form S_ _ S.

It is easy to see that both these empty squares are bad, as playing either S or O will allow the other player to form SOS. Conversely, if a square is bad, then playing an O in it will allow the other player to win, so it must have an S next to it and an empty square on the other side. Also, playing an S in a bad square allows the other player to win, so there must be another S beyond the empty square. This forces the configuration to be S_ _S, proving our claim.

Now after **A**'s first move, **B** writes an S at least 4 squares away from either end of the grid and **A**'s first move. On **B**'s second move, he writes S three squares away from his first S so that the two squares in between are empty. These two squares are bad. Note that at any point in the game there will always be an even number of bad squares (since they come in pairs, by our claim above). So whenever it is **B**'s turn, an odd number of moves would have been made, so an odd number of squares would be empty, of which an even number would be bad. Hence there will always be at least one square that is not bad on **B**'s turn, so **B** won't lose. Eventually the game will end since there are at least 2 bad squares, so **B** must win. ■

Positional Analysis

Heads I win, tails you lose.

For the rest chapter, we will use the following convention: a position in a game is a ***P-position*** if the player who has just played can force a win (that is, if he has a winning strategy). A position is called an ***N-position*** if the player whose turn it is can force a win. P and N respectively stand for Player and Next player.

This means that the starting player has a winning strategy if and only if the initial position of a game is an N-position (he is the "next player" as it is his turn at the start of the game). The second player has a winning strategy if and only if the initial position is a P-position (even though the game has not yet begun, by convention, he is the player who has "just played", since it is not his turn).

Our definition of P- and N-positions also implies the following: From an N-position, the player whose turn it is can move into a P-position. In other words, the player who has a winning strategy can move to a position in which he still has a winning strategy. From a P-position, the player whose turn it is must move into an N-position. **The winning strategy for a player is to always move to P-positions.**

Example 8

A and **B** play a game as follows. First **A** says 1, 2 or 3. Then **B** can add 1, 2 or 3 to the number the first player said. The game continues with the players playing alternately, in each turn adding 1, 2 or 3 to the previous number. For example, **A** can say 2, then **B** can say 5, then **A** could say 6, and so on. The player who says 100 wins. Who has the winning strategy?

Answer:
Observe that **B** can always say a multiple of 4 in his turn. For example, consider the following sequence of moves: **A**-1; **B**-4; **A**-6; **B**-8 and so on. Regardless of what **A** says, **B can always say a multiple of 4**. This is fairly obvious, but if you want to be more rigorous you can prove it by induction – if **B** says $4k$ then **A** says $4k+1$, $4k+2$ or $4k+3$, and in all cases **B** can say $4k+4$. Hence **B** will say 100 and will win. ■

Remark 1: Let's analyze this proof. First of all, how would one come up with it? One idea is to work backwards. Clearly the player who says 100 wins. Hence 100 is a P-position – the person who has just played wins. Then 99, 98 and 97 are N-positions as the next player can reach 100 from these positions. But 96 is a P-position, since from 96 only N-positions can be reached (97, 98 and 99). Continuing in this manner, we see that every multiple of 4 is a P-position, so the winning strategy is to always play multiples of 4. This type of analysis of P- and N-positions will be the central idea in the rest of this chapter.

Remark 2: This game can be generalized- instead of 1, 2, or 3 we can allow a player to increase the number by 1, 2, …, or k for any positive integer k and we can replace 100 by n. If $k+1$ divides n, then all multiples of $k+1$ are P-positions, so 0 is a P-position and **B** has a winning strategy. Otherwise, suppose $n \equiv r \bmod (k+1)$. Then all numbers that are congruent to r mod $k+1$ are P-positions, and **A** has a winning strategy by saying r in her first move.

Note: In several problems, we will use arguments of the form "If one player does this, then the other player does that, then the first player does this…". In order to avoid referring to the players as "one" or "the other" or the particularly ambiguous "the first player", we will use X and Y in these situations. X could refer to either of the players A and B, and Y refers to the other one. (On the other hand this precludes the usage of personal pronouns… writing a book is just frustrating sometimes.)

Example 9 [Lithuania 2010]
In an $m \times n$ rectangular chessboard there is a stone in the lower leftmost square. **A** and **B** move the stone alternately, starting with **A**. In each step one can move the stone upward or rightward any number of squares. The player who moves it into the upper rightmost square wins. Find all (m, n) such that **A** has a winning strategy.

Answer:
For convenience, lets flip the board upside down so that the stone starts in the upper rightmost square and the player who moves it to the lower leftmost square wins. All moves are now down or left. Now we label the squares by their coordinates, so that the lower leftmost square is $(1, 1)$ and the upper rightmost square is (m, n). Note that the game must terminate – this is a key ingredient for positional analysis to work.

						P		Start
					P			
				P				
			P					
		P						
	P							
P								

Figure 5.4: The P-positions

The crucial observation is that the P-positions are squares with **equal coordinates**, that is, squares with coordinates (k, k) for some k. These are marked **P** in the figure. This is because if X moves to a square marked P, Y cannot reach any other P square in Y's turn. Then X can move back to a P-position. This continues – X keeps moving to P-positions, and eventually reaches $(1, 1)$ and wins.

Thus if $m \neq n$, the initial position is N, so **A** wins by always moving to a square marked P. If $m = n$, the initial position is P, so **B** wins. Hence **A** wins if and only if $m \neq n$. ■

The previous two examples illustrate the general approach to problems in this section: characterize the N- and P-positions. If the starting position is an N-position, **A** wins; otherwise **B** wins. The next example is similar to example 8, but the inductive proof for characterizing N-positions is slightly trickier. There is also an important difference between this example and the previous two, discussed in the remark at the end.

Example 10 [Saint Petersburg 2001]
The number 1,000,000 is written on a board. **A** and **B** take turns, each turn consisting of replacing the number n on the board with $n - 1$ or $\lfloor (n+1)/2 \rfloor$. The player who writes the number 1 wins. Who has the winning strategy?

Answer:
Note that the game eventually ends, so one of the players must have a winning strategy. After experimenting with 1,000,000 replaced with smaller values, we see that **A** wins when the starting number is 2, 4, 6 and 8 and conjecture that **A** wins for any even starting number. In other words, we claim that all even numbers are N-positions. We prove this by induction.

2 is an N-position, so now suppose that all even numbers less than $2k$ are N-positions. We show that $2k$ is also an N-position. If k is a P-position, **A** can write k in her first move and win. Otherwise **A** first writes $2k - 1$. Then **B** must write $2k - 2$ or k, both of which are N-positions by assumption, so **A** wins in this case as well. Thus all even numbers are N-positions, so 1,000,000 is an N-position and **A** has a winning strategy. ■

Remark: Note that unlike the solutions in examples 8 and 9, this solution does not characterize *all* positions as P or N. We proved all even numbers are N-positions, but didn't prove anything about odd numbers since we didn't need to. All we really cared about was the number 1,000,000, so characterizing the positions of even numbers was sufficient. Another interesting thing to observe is the following: suppose N = 50. **A**'s strategy says to write 25 **if 25 is a P-position**, and otherwise write 49. But **we don't know whether 25 is a P-position or an N-position!** Does that mean our solution is incomplete or wrong? No. The problem only asked *who* has the winning strategy, not *what* the winning strategy is. We have guaranteed *the existence* of a winning strategy for **A**, without explicitly finding all its details – that's Alice's problem! (Well, actually, it's kind of your problem too- exercise 6 asks you to characterize the odd positions of this problem.)

An idea that has started to recur in this book is to take the techniques used in constructive proofs and use them for existential proofs. The next proof is a beautiful example of this idea, in which we use positional analysis to prove the existence of a winning strategy for one player by contradiction.

Example 11 [Russia 2011, adapted]

There are $N > n^2$ stones on a table. **A** and **B** play a game. **A** begins, and then they alternate. In each turn a player can remove k stones, where k is a positive integer that is either less than n or a multiple of n. The player who takes the last stone wins. Prove that **A** has a winning strategy.

Answer:

The game is finite and deterministic so some player must have a winning strategy. Suppose to the contrary that **B** can win. **A** first removes kn stones. If **B** removes jn stones for some j, then that means $N - (k + j)\, n$ is a P-position (since we are assuming **B** is playing his winning strategy). In this case **A** could have removed $(k + j)\, n$ stones in her first move and won, meaning that **A** has a winning strategy, which is a contradiction.

Thus if **A** removes kn stones in the first move, where $1 \le k \le n$, let $f(k)$ denote the number of stones **B** takes in response according to his winning strategy. By the first paragraph, $1 \le f(k) \le n - 1$. Hence by the pigeonhole principle, for some distinct k and j, $f(k) = f(j)$. This means that both $N - kn - f(k)$ and $N - jn - f(k)$ are P-positions, since these are the positions that arise after **B**'s move.

Now WLOG let $k < j$. **A** first removes kn stones, then **B** removes $f(k)$ stones. Now **A** removes $(k - j)n$ stones. There are now $N - kn - f(k)$ stones remaining, which is a P-position by the second paragraph. Hence **A** will win, contradicting our assumption that **B** has a winning strategy. ■

In slightly harder problems recursion often proves to be a useful technique. This means that we relate the outcome of a game of size N to games of smaller size.

Example 12 [IMO Shortlist 2004, C5]
A and **B** take turns writing a number as follows. Let N be a fixed positive integer. First **A** writes the number 1, and then **B** writes 2. Hereafter, in each move, if the current number is k, then the player whose turn it is can either write $k + 1$ or $2k$, but no player can write a number larger than N. The player who writes N wins. For each N, determine who has a winning strategy.

Answer:
Step 1: We quickly observe that if N is odd, **A** can win. **A** can always ensure that she writes an odd number, after which **B** would have to write an even number. Hence **B** cannot say N so **A** wins. Now suppose N is even.

Step 2: The key observation is that all **even numbers greater than $N/2$ are P-positions**. This is because after this point **neither player can double the number** (otherwise it will exceed N). Hence they both must keep adding 1 in their turns, and one player will keep writing even numbers and the other will keep writing odd numbers. The player who wrote the even number greater than $N/2$ will hence write N since N is even.

Step 3: **If $N = 4k$ or $N = 4k+2$, then k is a P-position**. This is because if X writes k, Y must write $k+1$ or $2k$. Then X writes $2k+2$ if Y writes k and X writes $4k$ if Y writes $2k$. X has thus written an even number greater than $N/2$ and by step 2, X wins.

Steps 2 and 3 now give us the final critical lemma.

Step 4: **If X has a winning strategy for $N = k$, then X has a winning strategy for $N = 4k$ and $N = 4k+2$.**
Proof: Consider a game where $N = 4k$ or $4k + 2$. If X writes k at some point during the game, we are done by step 3. So X's aim is to write k, so X starts implementing the winning strategy for $N = k$. How can Y prevent X from writing k? By "jumping over" k at some point: after X says some number j with $k/2 < j < k$, Y doubles it, resulting in a number 2j with $(k + 1) \leq 2j \leq N/2$. But then X simply doubles this number, resulting in an even number at least equal to $2(k+1) > N/2$. So X wins by step 2.

Finally, we have a recursive method of determining the answer for even N. The answer for N is the same as that for $\lfloor N/4 \rfloor$. To convert this recursion into an explicit answer, write N in base 4. The function $\lfloor N/4 \rfloor$ is equivalent to removing the last digit of N in base 4. Starting from the base 4 representation of N, keep removing the rightmost digit. The resulting numbers will all be winning for the same player by our recursion. If at some point we obtain an odd number, then **A** wins for this number and hence **A** wins for N. Hence if N has an odd digit in base 4, then **A** wins. Otherwise, suppose **N** has only 0s and 2s in its base 4 representation. Then applying our procedure we eventually end up with the number 2, and since **B** wins for 2, **B** wins for N in this case. ∎

The final example of this chapter is different in a few ways. First, it is asymmetrical, in the sense that the two players have different (in fact, opposite) objectives. Such games are typically called "maker-breaker" games. We cannot define P- and N-positions the way we did before. However, similar ideas of analyzing positions based on outcome still apply, and winning strategies are still based on always sticking to some particular type of position. These positions are typically characterized by some specific property or invariant, as the next example shows.

Example 13 [IMO Shortlist 2009, C5]
Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then the next round begins. The Stepmother's goal is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

Answer:
Let the volume of water in the buckets be B_1, B_2, B_3, B_4 and B_5. Indices will be taken mod 5. Clearly if both B_i and B_{i+2} are greater than one before one Cinderella's moves, she cannot empty both of them and hence the stepmother will win enforce an overflow in her turn. Thus Cinderella's aim is to prevent this from happening. To do this, clearly it is sufficient to ensure that $B_i + B_{i+2}$ is at most one for each i after each of Cinderella's turns. Call such a situation *good*.

Suppose after some round we have a good situation: two buckets are empty, say $B_4 = B_5 = 0$. Then $B_1 + B_3 \leq 1$ and $B_2 \leq 1$ (since $B_2 + B_4 \leq 1$). After the stepmother's turn, we will have $B_1 + B_3 + B_4 + B_5 \leq 2$. Hence either $B_5 + B_3 \leq 1$ or $B_4 + B_1 \leq 1$. WLOG $B_5 + B_3 \leq 1$. Then Cinderella empties B_1 and B_2. Now observe that the new configuration is still good, since $B_4 \leq 1$ and $B_5 + B_3 \leq 1$.

Hence starting from a good configuration, Cinderella can ensure that at the end of the round the new configuration is still good. Initially the buckets are all empty, so this configuration is good. Hence Cinderella can prevent an overflow by always staying in a good position. ■

Exercises

1. **[Cram]**
   A and **B** take turns placing dominoes on an $m \times n$ rectangular grid, where mn is even. **A** must place dominoes vertically and **B** must place dominoes horizontally, and dominoes cannot overlap with each other or stick out of the board. The player who cannot make any legal move loses. Given m and n, determine who has a winning strategy, and find this strategy.

2. **[Double Chess]**
   The game of double chess is played like regular chess, except each player makes two moves in their turn (white plays twice, then black plays twice, and so on). Show that white can always win or draw.

3. **[Russia 1999]**
   There are 2000 devices in a circuit, every two of which were initially joined by a wire. The hooligans Vasya and Petya cut the wires one after another. Vasya, who starts, cuts one wire on his turn, while Petya cuts two or three. A device is said to be disconnected if all wires incident to it have been cut. The player who makes some device disconnected loses. Who has a winning strategy?

4. **[IMO Shortlist 2009, C1]**
   Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over,

so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

a) Does the game necessarily end?

b) Does there exist a winning strategy for the starting player?

5. [Russia 1999]
There are three empty jugs on a table. Winnie the Pooh, Rabbit, and Piglet put walnuts in the jugs one by one. They play successively, with the order chosen by Rabbit in the beginning. Thereby Winnie the Pooh plays either in the first or second jug, Rabbit in the second or third, and Piglet in the first or third. The player after whose move there are exactly 1999 walnuts in some jug loses. Show that Winnie the Pooh and Piglet can cooperate so as to make Rabbit lose.

6. Solve the problem in example 10 with 1,000,000 replaced by n, an arbitrary odd number. Use this complete characterization of positions to provide a complete description of the winning strategy. If you have some programming experience, you could also write a program to play this game against you.
Remark (for programmers): You could also write a program to solve this probem, that is, to determine for each n who has a winning strategy. A simple dynamic programming approach would run in $O(n)$ time. Using this as a subroutine, the program to play the game against you would take $O(n)$ time for each move. However, if you found the characterization of positions on your own first, the program to play the game would only take $O(\log n)$ time for each move.

7. [Bulgaria 2005]
For positive integers t, a, b, a *(t, a, b)-game* is a two player game defined by the following rules. Initially, the number t is written on a blackboard. In his first move, the first player replaces t with either $t - a$ or $t - b$. Then, the second player subtracts either a or b from this number, and writes the result on the blackboard, erasing the old number. After this, the first player once again erases either a or b from the number

written on the blackboard, and so on. The player who first reaches a negative number loses the game. Prove that there exist infinitely many values of t for which the first player has a winning strategy for all pairs (a, b) with $(a + b) = 2005$.

8. **[Rioplatense Math Olympiad 2010]**
Alice and Bob play the following game. To start, Alice arranges the numbers 1, 2, ..., n in some order in a row and then Bob chooses one of the numbers and places a pebble on it. A player's *turn* consists of picking up and placing the pebble on an adjacent number under the restriction that the pebble can be placed on the number k at most k times. The two players alternate taking turns beginning with Alice. The first player who cannot make a move loses. For each positive integer n, determine who has a winning strategy.

9. **[Russia 2007]**
Two players take turns drawing diagonals in a regular $(2n+1)$-gon $(n > 1)$. It is forbidden to draw a diagonal that has already been drawn or intersects an odd number of already drawn diagonals. The player who has no legal move loses. Who has a winning strategy?

10. **[Indian Practice TST 2013]**
A marker is placed at the origin of an integer lattice. Calvin and Hobbes play the following game. Calvin starts the game and each of them takes turns alternatively. At each turn, one can choose two (not necessarily distinct) integers a and b, neither of which was chosen earlier by any player and move the marker by a units in the horizontal direction and b units in the vertical direction. Hobbes wins if the marker is back at the origin any time after the first turn. Determine whether Calvin can prevent Hobbes from winning.
Note: A move in the horizontal direction by a positive quantity will be towards the right, and by a negative quantity will be towards the left (and similarly in the vertical case as well).

11. [Based on South Korea 2009, Problem 5]

Consider an $m \times (m+1)$ grid of points, where each point is joined by a line segment to is immediate neighbors (points immediately to the left, right, above or below). A stone is initially placed on one of the points in the bottom row. **A** and **B** alternately move the stone along line segments, according to the rule that no line segment may be used more than once. The player unable to make a legal move loses. Determine which player has a winning strategy.

12. [IMO Shortlist 1994, C6]

Two players play alternatively on an infinite square grid. The first player puts an X in an empty cell and the second player puts an O in an empty cell. The first player wins if he gets 11 adjacent X's in a line - horizontally, vertically or diagonally. Show that the second player can always prevent the first player from winning.

13. [Nim]

There are k heaps of stones, containing a_1, a_2, …, a_k stones respectively, where the a_i's are positive integers. Players **A** and **B** play alternately as follows: in each turn, a player chooses one non-empty heap and removes as many stones as he or she wants. The person who takes the last stone wins. Determine when each player has a winning strategy, and find this winning strategy.

14. [The name of this problem would give the answer away]

There is one pile of N counters. **A** and **B** play alternately as follows. In the first turn of the game, **A** may remove any positive number of counters, but not the whole pile. Thereafter, each player may remove at most twice the number of counters his opponent took on the previous move. The player who removes the last counter wins. Who has the winning strategy?