# Olympiad Combinatorics

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## About the Author

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# 3. PROCESSES

## Introduction

In this chapter we analyze combinatorial processes. In Chapter 1 on algorithms, we often encountered combinatorial processes from a different viewpoint. Problems from both chapters are similar in that they typically specify an initial configuration and allowed set of moves. In the chapter on algorithms, we were asked to prove that a certain final configuration could be reached using these moves, and we solved these problems by constructing procedures to reach the desired final configuration. In this chapter, our job is not to construct our own procedures, but rather to analyze given ones. Some questions ask us to determine whether a process terminates, and if it does, what the final configuration looks like. Others may ask us to bound the number of steps it takes for a certain configuration to arise.

Our main tools for this chapter are invariants, the extremal principle, induction and other clever ideas that we will develop as we go further, such as making transformations to a problem that simplify the problem but leave the end result invariant. It must also be stressed that **experimentation, trial and error, observation, intuition and conjectures** play a big role in solving problems related to processes (and combinatorics in general). We remind the reader that the ideas to solve combinatorial problems often arise from experimenting with small values.

# Invariants

Our first few examples use *invariants*, a technique we have already used in earlier chapters. The usefulness of invariants while analyzing combinatorial processes can hardly be overstated.

**Example 1 [Indian TST 2004]**
The game of pebbles is played as follows. Initially there is a pebble at (0, 0). In a move one can remove a pebble from $(i, j)$ and place one pebble each on $(i+1, j)$ and $(i, j+1)$, provided $(i, j)$ had a pebble to begin with and $(i+1, j)$ and $(i, j+1)$ did not have pebbles. Prove that at any point in the game there will be a pebble at some lattice point $(a, b)$ with $a+b \leq 3$.

**Answer:**
Clearly the pebbles will always be on lattice points in the first quadrant. How can we find an invariant? Just assign a weight of $2^{-(i+j)}$ to a pebble at $(i, j)$. Then in each move one pebble is replaced by two pebbles, each having half its weight. So the total weight of pebbles is invariant. Initially the weight is $2^0 = 1$. Suppose at some stage no pebble is on a point $(a, b)$ with $a+b \leq 3$. Then the maximum possible total weight of all pebbles is the weight of the whole first quadrant minus that of the squares $(a, b)$ with $a+b \leq 3$, which is

$$\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} 2^{-(i+j)}\right) - (1 + 2 \times \tfrac{1}{2} + 3 \times \tfrac{1}{4} + 4 \times \tfrac{1}{8})$$
$$= 4 - (1 + 1 + \tfrac{3}{4} + \tfrac{1}{2}) = \tfrac{3}{4} < 1.$$

This is a contradiction as the weight should always be 1. ■

**Remark**: The double summation was computed by noticing $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} 2^{-(i+j)} = \sum_{i=0}^{\infty} 2^{-i} \text{ x } \sum_{i=0}^{\infty} 2^{-j} = 2 \text{ x } 2 = 4$. The second parenthesis was the weight of all squares $a$, $b$ with $a+b \leq 3$.

**Example 2 [IMO shortlist 2005, C5]**
There are $n$ markers, each with one side white and the other side black, aligned in a row so that their white sides are up. In each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it and reverse the closest marker to the left and the closest marker to the right of it. Prove that one can achieve the state with only two markers remaining if and only if $n-1$ is not divisible by 3.

**Answer:**
If $n-1$ is not divisible by 3, it is easy to construct an inductive algorithm to make only two markers remain (chapter 2 FTW!). We leave this to the reader (just do it for $n$ = 5, $n$ = 6 and induct with step 3). Now we do the harder part: if $(n-1)$ is divisible by 3, we need to show that this cannot be done.

Call a marker *black* or $B$ if its black side is up and *white* or $W$ if its white side is up. One invariant we immediately find is that the number of black markers is always even since each move changes the number of black markers by 0, 2 or -2. Now we look for another invariant.

We assign numbers to each white marker. If a white marker has $t$ black markers to its left, we assign the number $(-1)^t$ to it. Let $S$ be the sum of all the labels. Initially all labels are $(-1)^0 = 1$, so $S$ = $n$ initially. The labels may keep changing, but we claim that $S$ stays invariant mod 3. For example, suppose we have the substring …*WWB*… and remove the middle white marker. Then it becomes …*BW*…. If the 2 white markers had $t$ black markers to their left initially, then the white marker now has $(t+1)$ black markers to its left. Thus the two white markers both had labels $(-1)^t$ initially, but now the white marker has label $(-1)^{t+1}$. The sum of the labels has changed by $(-1)^{t+1} - 2(-1)^t = 3(-1)^{t+1} \equiv 0 \bmod 3$. The reader can verify that in the other cases (*WWW*, *BWB*, *BWW*) as well the sum of labels $S$ doesn't change mod 3.

Now the rest is easy. If two markers remain, they are either both white or both black (number of black markers must be even). In the first case, both labels are 1 and $S = 2$. In the second case, $S = 0$ as no markers are labeled. So $S = 0$ or 2 at the end and $S = n$ in the beginning. Since $S$ stays invariant mod 3, $n \equiv 0$ or 2 mod 3 and we are done. ■

**Example 3 [IMO Shortlist 1998 C7]**
A solitaire game is played on an $m \times$ n board with markers having one white side and one black side. Each of the $mn$ cells contains a marker with its white side up, except for one corner square which has a marker with its black side up. The allowed move is to select a marker with black side up, remove it, and turn over all markers in squares sharing a side with the square of the chosen marker. Determine all pairs $(m, n)$ for which it is possible to remove all markers from the board.

**Answer**:
It is natural (but not essential) to rephrase the problem using graph theory. We take the markers as vertices. Each vertex is black or white. 2 vertices are connected by an edge if and only if the markers lie on adjacent squares. In each move, we are deleting one black vertex and all its incident edges, but all its white neighbors become black and all its white neighbors become black. Suppose in a move we delete a black vertex $v$ and $s$ edges, where $s$ is the degree of $v$. Suppose $w$ of $v$'s neighbors were white vertices, and $(s–w)$ were black vertices. Then these $w$ vertices become black and $(s–w)$ become white, so the number of white vertices increases by $s – 2w$.

This information alone does not immediately give us an invariant, since the quantity $s – 2w$ is quite random. However, suppose we consider the quantity $W + E$, where $W$ is the total number of remaining white vertices (don't confuse this with $w$) and $E$ is the number of edges. Then when $E$ reduces by $s$ and $W$

changes by $s - 2w$, $(W+E)$ decreases by $2w$, which is always even. Hence **the parity of $(W+E)$ remains the same**. So if $W+E$ is 0 at the end (when all markers are gone), we need $W+E$ to be even in the beginning. But initially

$$W = mn\text{-}1,\ E = m(n\text{-}1) + n(m\text{-}1), \text{ and}$$

$$W+E = 3mn - m - n - 1 \equiv mn - m - n + 1 \pmod 2 = (m\text{-}1)(n\text{-}1),$$

so at least one of $m$ and $n$ must be odd. In this case the task is indeed possible and we leave it to the reader to find an algorithm. (Assume $m$ is odd and use an inductive procedure that makes each column empty one by one). ■

## Good and Bad Objects

Another useful idea while analyzing processes is to distinguish between "good" and "bad" objects. For example, if at the end of a process we want to show that all objects satisfy a certain property, call objects with that property *good* and the other objects *bad*. We will use this idea in different forms several times throughout this chapter. The next example combines this idea with monovariants by showing that the number of "good" objects monotonically increases.

**Example 4 [Based on Canada 1994]**
There are $2n+1$ lamps placed in a circle. Each day, some of the lamps change state (from on to off or off to on), according to the following rules. On the $k^{\text{th}}$ day, if a lamp is in the same state as at least one of its neighbors, then it will not change state the next day. If a lamp is in a different state from both of its neighbors on the $k^{\text{th}}$ day, then it will change its state the next day. Show that regardless of the initial states of each lamp, after some point none of the lamps will change state.

**Answer**:
Call a lamp "good" if it is in the same state as at least one of its neighbors. Once a lamp is good, it will remain good forever (if two adjacent lamps are in the same state on the $k^{th}$ day, they will not change state the next day, and hence both remain good). Hence the number of good lamps never decreases and is a monovariant.

We show that in fact the number of good lamps strictly increases until it reaches $2n+1$. Initially there must be 2 adjacent lamps with the same state since the number of lamps is odd. Suppose at some point there are j good lamps and $2 \leq j < 2n+1$. Then there must exist 2 adjacent lamps such that one is bad and one is good. Then the bad lamp will switch states the next day and the good lamp will remain in the same state. Then the bad lamp will now be good, so the number of good lamps has increased (remember all good lamps remain good). So the number of good lamps increases until all lamps are good, and at this point there will be no more changes of state. ■

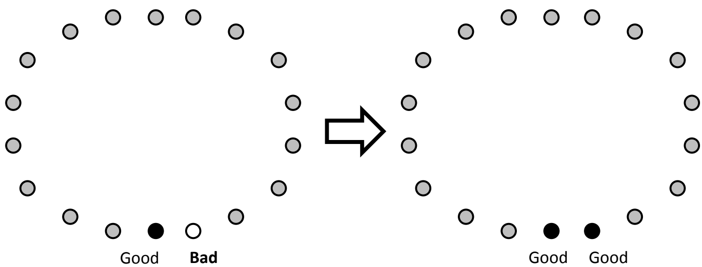


*Figure 3.1: Bad lamps next to good lamps become good*

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## Bounds on the number of steps

Now we look at another class of problems, which ask us to bound the number of steps or moves it takes for a process to terminate. To bound the total number of moves, it is often useful to bound the number of times a particular object is involved in a move.

**Example 5 [USAMO 2010-2]**
There are n students standing in a circle, one behind the other, all facing clockwise. The students have heights $h_1 < h_2 < \ldots < h_n$. If a student with height $h_k$ is standing directly behind a student with height $h_{k-2}$ or less, the two students are permitted to switch places. Prove that it is not possible to make more than ${}^nC_3$ such switches before reaching a position in which no further switches are possible.

**Answer:**
We bound the number of times an individual student can switch places with another student. Let $s_k$ denote the number of times the student with height $k$ switches with someone shorter. Obviously $s_1 = s_2 = 0$. Now consider the number of people between student $k$ and student $(k-1)$ (along the clockwise direction) of height less than $h_{k-1}$. This number is at most $(k-2)$. This quantity decreases by 1 each time student $k$ switches with someone shorter and increases by one each time student $k-1$ switches with someone shorter. This quantity doesn't change when students taller than student $k$ switch with either student $k$ or student $(k-1)$. Hence $s_k - s_{k-1}$ denotes the decrease from beginning to end in the number of students between student $k$ and student $(k-1)$, which cannot exceed $(k-2)$. Thus we get the bound $s_k - s_{k-1} \leq (k-2)$. Using this recursive bound and the initial values $s_1 = s_2 = 0$, we get $s_3 \leq 1$, $s_4 \leq 3$, etc. In general it is easy to show by induction $s_k \leq \binom{k-1}{2}$. Hence the total number of moves cannot exceed $\sum_{k=1}^{n}\binom{k-1}{2} = {}^nC_3$ and we are done. ■

**Note**: The last step uses a well-known binomial identity. More generally
$\sum_{k=1}^{n}\binom{k-1}{r} = \binom{n}{r+1}$, where by convention $\binom{j}{r} = 0$ for $j < r$.

**Example 6 [Based on Canada 2012, IMO Shortlist 1994 C4]**
A bookshelf contains n volumes, labeled 1 to $n$, in some order. The

librarian wishes to put them in the correct order as follows. The librarian selects a volume that is too far to the right, say the volume with label $k$, takes it out, and inserts it in the $k^{\text{th}}$ position. For example, if the bookshelf contains the volumes 3, 1, 4, 2 in that order, the librarian could take out volume 2 and place it in the second position. The books will then be in the order 3, 2, 1, 4. Show that the sequence $(1, 2, \ldots, n)$ is reached in fewer than $2^n$ moves.

**Answer:**

We bound the number of times book $k$ can be selected by the librarian. Clearly, book $n$ can never be selected since it will never be too far right. Book 1 can only be selected once, because once selected, it will move to the first position and never move again. Book 2 can be selected twice: it may be selected once and put in the correct position, but then it may move because of book 1.

More generally, let $f(k)$ denote the number of times book $k$ is selected for $1 \leq k \leq (n-1)$. We have

$$f(k) \leq 1 + f(k-1) + f(k-2) + \ldots + f(1).$$

This is because once $k$ is in the correct position, it can only be displaced $f(k-1) + f(k-2) + \ldots + f(1)$ times, because the only way in which book $k$ can be displaced is if one of the books with number less than $k$ "pushes" $k$.

For example: If we start from (4, 1, 3, 2, 5) and we choose book 2, it becomes (4, 2, 1, 3, 5). Book 3 was in the correct position, but has been "pushed out" because of book 2 being chosen.

Thus using this recursive bound on $f(k)$ and the fact that $f(1) = 1$, we obtain by a simple induction $f(k) \leq 2^{k-1}$. Hence the total number of moves required is at most

$f(1) + f(2) + \ldots + f(n-1) \leq 1 + 2 + 4 + \ldots + 2^{n-2} = 2^{n-1} - 1$. ■

**Remark**: A solution with monovariants is also possible.

# Induction

In the previous section, we essentially broke down the analysis of a process into the analysis of the individual entities involved. To find the total time for the process to terminate, we used recursive bounds to estimate the time a particular object could contribute to the total time. These essential elements of somehow "breaking down" a process and using induction and/or recursion will be central to this section as well. However, rather than the object-centric approach of the previous section, a structure-centric approach will be taken here: the inductive proofs will rely on exploiting the nice combinatorial structure of n x n boards.

**Example 7 [Belarus 2001]**
Let $n$ be a positive integer. Each unit square of a $(2n\text{-}1) \times (2n\text{-}1)$ square board contains an arrow, either pointing up, down left or right. A beetle sits in one of the squares. In one move, the beetle moves one unit in the direction of the arrow in the square it is sitting on, and either reaches an adjacent square or leaves the board. Then the arrow of the square the beetle left turns $90^{o}$ clockwise. Prove that the beetle leaves the board in at most $2^{3n-1}(n\text{-}1)! - 3$ moves.

**Answer:**
The base case $n = 1$ is trivial since the beetle leaves in the first move. Now suppose the result is true for $n = k$; we prove it for $n = k+1$. It is natural to distinguish between boundary squares (squares on the edge of the board) and interior squares, since the interior squares form a $(2k\text{-}1) \times (2k\text{-}1)$ board and we can use the induction hypothesis on this board. We further distinguish between corner squares and non-corner boundary squares.

Suppose the beetle is still on the board after $T$ moves. We want to show that $T < 2^{3(k+1)-1}k! - 3$. At this stage, if any non-corner boundary square has been visited 4 times, then one of the four times the arrow would have been pointing out of the board (since its direction changes each time). Similarly if a corner square has been visited 3 times, then at least once it would have pointed out of the board. Hence in each of the cases, the beetle would have left the board, contradiction. Hence the beetle has visited each corner square at most twice and each non-corner boundary square at most thrice. Moreover, the beetle can move at most once from a non-corner boundary square to an interior square. Thus:

i. The beetle has made at most $2 \times 4 + 3(8k - 4) = 24k - 4$ moves from boundary squares to other squares of the board (since there are 4 corner squares and $8k - 4$ non-corner boundary squares).

ii. The beetle has made at most $4(2k - 1) = 8k - 4$ moves from a boundary square to an interior square, since there are $8k - 4$ non-corner boundary squares.

iii. If a beetle is in the interior $(2k - 1) \times (2k - 1)$ square, it can make at most $M = 2^{3k-1}(k - 1)! - 3$ moves before reaching a boundary square, by the induction hypothesis.

iv. From (ii), the beetle can stay in the interior square for at most $8k - 3$ periods of time (once in the beginning, then once for each time it moves from a boundary square back to the interior). Each period lasts at most $2^{3k-1}(k - 1)! - 3$ moves by (iii). Hence the number of moves made from interior squares is at most

$$\begin{aligned} &(2^{3k-1}(k-1)! - 3) \times (8k - 3) \\ &< 8k \times (2^{3k-1}(k-1)! - 3) \\ &= 2^{3(k+1)-1}k! - 24k. \end{aligned}$$

From (i) and (iv), we see that

$T \le (24k - 4) + (2^{3(k+1)-1}k! - 24k) = 2^{3(k+1)-1}k! - 4$, as desired. ■

Now we look at another example using induction. This problem is different from the previous one in that we are not asked to bound the number of moves for a process to terminate. However, the idea of inducting by dividing an $n \times n$ board into an $(n-1)\times(n-1)$ sub board and an additional row and column is very similar to the idea in the previous example. This inductive technique is just one of many ways in which the structure of boards can be exploited.

**Example 8 [Russia 2010]**

On an $n \times n$ chart where $n \geq 4$, stand $n$ '+' signs in cells of one diagonal and a '–' sign in all the other cells. In a move, one can change all the signs in one row or in one column, (– changes to + and + changes to –). Prove that it is impossible to reach a stage where there are fewer than $n$ pluses on the board.

**Answer:**

Note that operating twice on a row is equivalent to not operating on it at all. So we can assume that each row and column has been operated upon 0 or 1 times. Now we use induction on $n$. The base case $n = 4$ is not entirely trivial, but is left to the reader in keeping with my general habit of dismissing base cases.

Now passing to the induction step, given an $n \times n$ board there are at least $(n - 1)$ pluses in the bottom right $(n - 1) \times (n - 1)$ square by the induction hypothesis. If we have a plus in the first row or column we are done. Suppose there is no plus in the first column or row. Then either the first row or the first column (but not both) has been operated upon (otherwise the top left square would have a plus). Suppose WLOG the first row has been operated upon. Then columns 2, 3, ..., $n$ have all been operated upon (otherwise row 1 would have a plus). Also no other row has been operated upon (otherwise the first column would have a plus). But in this case, the lower right $(n - 1) \times (n - 1)$ square has had all its columns and none of its rows operated upon, and hence each column has $(n - 2)$ pluses. In total it has $(n - 2)(n - 1) > n$ pluses, so in this case as well we are done. ■

## Problem Alteration: Don't play by the rules

Next we look at a very powerful technique of solving problems related to processes. In the next three examples, we will alter the problem statement slightly in such a way that the result we need to show doesn't change, but the process becomes much easier to analyze. In other words, we simplify the process to be analyzed while leaving the aspect of the process that we want to prove something about invariant. This may take some time to understand, so read through the next few examples slowly and carefully, and multiple times if necessary.

**Example 9 [warm up for example 11]**
There are $n$ ants on a stick of length one unit, each facing left or right. At time $t$ = 0, each ant starts moving with a speed of 1 unit per second in the direction it is facing. If an ant reaches the end of the stick, it falls off and doesn't reappear. When two ants moving in opposite directions collide, they both turn around and continue moving with the same speed (but in the opposite direction). Show that all ants will fall off the stick in at most 1 second. (We will use a very similar idea in example 11, so make sure you understand this trick.)

**Answer:**
The key observation is that the problem doesn't change if we alter it as follows: when two ants moving in opposite directions meet, they simply pass through each other and continue moving at the same speed. Thus instead of rebounding, if the ants pass through each other, the only difference from the original problem is that the identities of the ants get exchanged, which is inconsequential. Now the statement is obvious – each ant is unaffected by the others, and so each ant will fall of the stick of length one unit in at most 1 second. ■

**Example 10 [Russia 1993 generalized]**
The integers from 1 to $n$ are written in a line in some order. The following operation is performed with this line: if the first number is $k$ then the first $k$ numbers are rewritten in reverse order. Prove that after some finite number of these operations, the first number in the line of numbers will be 1.

**Answer:**
The base case $n = 1$ is trivial. Suppose the result is true for $(n - 1)$. First observe that if $n$ appears in the first position at some point, then in the next step $n$ will be in the last position and will remain there permanently. Then we can effectively ignore $n$ and we are done by induction. So suppose $n$ never appears in the first position. Let $j$ be the number in the last position. If we switch $n$ and $j$, it has absolutely no effect on the problem, as $j$ will never appear in the first position (since we assumed $n$ will never appear in the first position). Now $n$ is in the last position and as in the first case, we are done by induction. ■

**Remark**: Based on the above proof, it is not difficult to show that for $n > 1$ if the first number becomes 1 after at most $f(n)$ operations, we have the recursive bound $f(n+1) \leq 2f(n)+1$. I believe this bound can be further improved for most values of $n$.

As if "cheating" once isn't bad enough, we'll cheat twice in the next problem. Combining the insights obtained from these two instances of "cheating" will greatly restrict the possible positions of otherwise very chaotic ants.

**Example 11 [IMO Shortlist 2011, C5]**
Let $m$ be a positive integer and consider a checkerboard consisting of $m \times m$ unit squares. At the midpoints of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in opposite directions meet, they both turn $90^\circ$ clockwise and continue moving with speed 1. When more than two ants meet, or when two ants moving in perpendicular

directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not reappear. Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard or prove that such a moment does not necessarily exist.

**Answer:**
After experimenting with small values of $m$, we conjecture that the answer is $\frac{3m}{2}$ -1. Clearly this is attainable if initially there are only 2 ants, one in the bottom left square facing upwards and one in the top left square facing downward. Now we prove that it is the maximum possible. Let U, D, L, R represent the directions up, down, left and right respectively.

*Step 1*: We use a modified version of the trick in example 7. Using the same reasoning, we can change the rules so that each ant travels in only two directions- either U and R or D and L. So if an ant travelling R meets an ant travelling L, they now move U and D respectively (even though in the original problem they should now move D and U respectively). This doesn't affect the problem. Now based on their initial direction, each ant can be classified into two types: UR or DL. UR ants can only move up and right the whole time and DL ants only move down and left the whole time. Note that we can ignore collisions between two ants of the same type. From now on, "collision" only refers to collisions between two ants of opposite types.

*Step 2*: Choose a coordinate system such that the corners of the checkerboard are $(0, 0)$, $(m, 0)$, $(m, m)$ and $(0, m)$. At time $t$, there will be no UR ants in the region $\{(x, y): x + y < t + 1\}$ and no DL ants in the region $\{(x, y): x + y > 2m - t - 1\}$. So if a collision occurs at $(x, y)$ at time $t$, we have $t + 1 \le x + y \le 2m - t - 1$.

*Step 3*: In a similar manner, we can change the rules of the original problem (without affecting it) by assuming that each ant can only move U and L or D and R, so each ant is UL or DR. Using the same logic as in step 2, we get a bound $|x-y| \leq m-t-1$ for each collision at point $(x, y)$ and time $t$. Thus we have shown that all collisions at time $t$ are within the region bounded by the 4 lines represented by the equations $t + 1 \leq x + y \leq 2m-t-1$ and $|x-y| \leq m-t-1$.

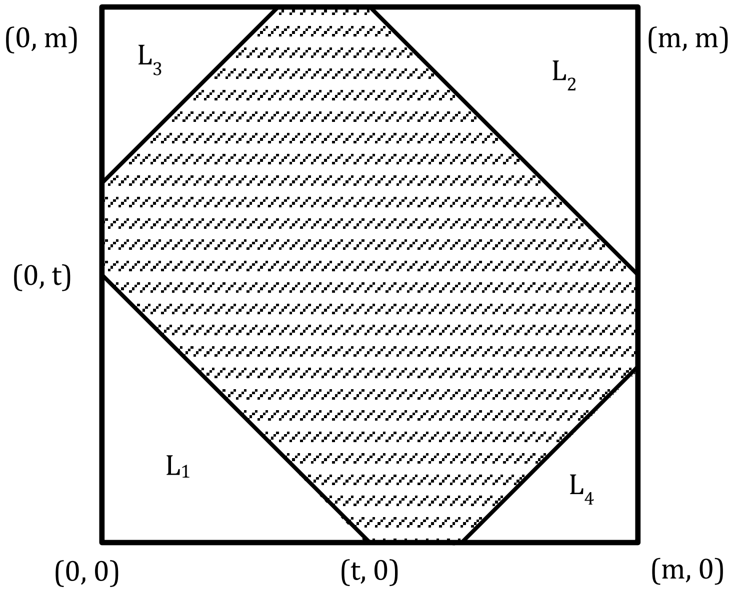


*Figure 3.1: All collisions at time t must lie within the shaded region*

*Step 4*: We finish the proof for a UR ant; by symmetry the same final bound will hold for DL ants. Take a UR ant and suppose its last collision is at $(x, y)$ at time $t$. Adding the bounds $x+y \geq t+1$ and $x - y \geq -(m - t - 1)$, we get $x \geq t + 1 - \frac{m}{2}$. Similarly, $y \geq t + 1 - \frac{m}{2}$. Since this is the last collision, the ant will now move straight to an edge and fall off. This takes at most $m-\min\{x, y\}$ units of time. The total amount of time the ant stays on the board is hence at most

$$t + m - \min\{x, y\} \leq t + m - \{t + 1 - m/2\} = \frac{3m}{2} - 1$$

units of time. ■

**Remark**: Let's reverse engineer the solution a little bit, to see how the main ideas fit together so nicely – did you notice how the parameter $t$ disappeared so conveniently in the last step? The basic goal in the above solution was to obtain tight bounds on the location of an ant after its last collision because after this the ant travels straight off the board. The intuition behind getting rid of t was that the longer an ant has been wandering around till its last collision, the closer it must be to an edge, and so the less time it will take to fall off now. But for this to work we need the ants to be "well behaved" - and hence the cheating!

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## Concluding Examples

Our final two examples lie at the heart of this chapter. Example 12 is a particular case of a more general and extensively studied process known as a "chip firing game", and Example 13 is a distant cousin of the chip firing game. Through these problems we introduce some important ideas such as using the extremal principle in different ways and obtaining contradictions, and combine these with ideas we have already seen like invariants and making assumptions that don't affect the problem. In example 12, we use the following idea: if a process never terminates, there must be some object that is moved or operated upon infinitely times. If we can find an object that is only operated upon finitely many times, we may be able to get a contradiction.

**Example 12 [IMO shortlist 1994, C5]**
1994 girls are seated in a circle. Initially one girl is given $n$ coins. In one move, each girl with at least 2 coins passes one coin to each of her two neighbors.

(a) Show that if $n < 1994$, the game must terminate.
(b) Show that if $n = 1994$, the game cannot terminate.

**Answer:**

(a) Label the girls $G_1$, $G_2$, ..., $G_{1994}$ and let $G_{1995} = G_1$, $G_0 = G_{1994}$. Suppose the game doesn't terminate. Then some girl must pass coins infinitely times. If some girl passes only finitely many times, there exist two adjacent girls, one of whom has passed finitely many times and one of whom has passed infinitely many times. The girl who has passed finitely many times will then indefinitely accumulate coins after her final pass, which is impossible. Hence every girl must pass coins infinitely many times.

Now the key idea is the following: For any two neighboring girls $G_i$ and $G_{i+1}$, let $c_i$ be the first coin ever passed between them. After this, we may assume that $c_i$ always stays stuck between $G_i$ and $G_{i+1}$, because whenever one of them has $c_i$ and makes a move, we can assume the coin passed to the other girl was $c_i$. Therefore, each coin is eventually stuck between two girls. Since there are fewer than 1994 coins, this means there exist two adjacent girls who have never passed a coin to each other. This contradicts the result of the first paragraph.

(b) This is simple using invariants. Let a coin with girl $i$ have weight $i$, and let $G_1$ have all coins initially. In each pass from $G_i$ to her neighbors, the total weight either doesn't change or changes by $\pm 1994$ (if $G_1$ passes to $G_{1994}$ or vice versa). So the total weight is invariant mod 1994. The initial weight is 1994, so the weight will always be divisible by 1994. If the game terminates, then each girl has one coin, so the final weight is $1+2+3+\ldots+ 1994 = (1994 \times 1995)/2$ which is not divisible by 1994. Contradiction. ■

Before reading the solution to the next problem, we recommend that the reader experiment with small values of $n$ and try to guess what the final configuration looks like. Several combinatorics problems require experimentation, observation and conjecturing before actually proving anything.

**Example 13 [IMO shortlist 2001, C7]**
A pile of $n$ pebbles is placed in a vertical column. This configuration is modified according to the following rules. A pebble can be moved if it is at the top of a column which contains at least two more pebbles than the column immediately to its right. (If there are no pebbles to the right, think of this as a column with 0 pebbles.) At each stage, choose a pebble from among those that can be moved (if there are any) and place it at the top of the column to its right. If no pebbles can be moved, the configuration is called a *final configuration*. For each $n$, show that, no matter what choices are made at each stage, the final configuration obtained is unique. Describe that configuration in terms of $n$.

**Answer**:
It is clear that if $s_i$ denotes the number of stones in column $i$, then in the final configuration $s_{i+1} = s_i$ or $s_i$ -1. After experimenting with small values of $n$, we are led to the following claim:

*Claim*: In the final configuration, there is at most one index $i$ such that $s_{i+1} = s_i$ (hence the remaining columns satisfy $s_{j+1} = s_j - 1$).

*Proof*: Call an index $j$ bad if $s_{j+1} = s_j$. Assume to the contrary that there exist (at least) 2 bad indices in the final configuration. Take $k$ and $m$ ($k > m$) to be consecutive bad indices. Then $s_{k+1} = s_k$, $s_{m+1} = s_m$ and $s_{i+1} = s_i - 1$ for $m < i < k$. Consider the *earliest* configuration, say $C$, with the 2 bad indices. Now look at the last move before $C$. Since $C$ is the earliest such configuration, the last move was either from the $k^{th}$ or $m^{th}$ column. But then in either case the configuration before $C$ also had 2 bad indices, contradicting our assumption. This proves the claim.

Now it is easy to see that the claim uniquely determines the final configuration. For example, for $n$ = 17 the final heights would be (5, 4, 3, 2, 2, 1). ■

# Exercises

1. **[Austrian-Polish Mathematical Competition 1997]**
The numbers $49/k$ for $k = 1, 2, \ldots, 97$ are written on a blackboard. A move consists of choosing two number $a$ and $b$, erasing them and writing the number $2ab - a - b + 1$ in their place. After 96 moves, only one number remains. Find all possible values of this number.

2. We have $n(n+1)/2$ stones in $k$ piles. In each move we take one stone from each pile and form a new pile with these stones (if a pile has only one stone, after that stone is removed the pile vanishes). Show that regardless of the initial configuration, we always end up with $n$ piles, having $1, 2, \ldots, n$ stones respectively.

3. **[ELMO Shortlist 2013, C9, generalized]**
There are $n$ people at a party. Each person holds some number of coins. Every minute, each person who has at least $(n - 1)$ coins simultaneously gives one coin to every other person at the party. (So, it is possible that A gives B a coin and B gives A a coin at the same time.) Suppose that this process continues indefinitely. That is, for any positive integer $m$, there exists a person who will give away coins during the $m^{\text{th}}$ minute. What is the smallest number of coins that could be at the party?

4. **[China TST 2003]**
There is a frog in every vertex of a regular $2n$-gon ($n \geq 2$). At a certain time, all frogs jump simultaneously jump to one of their neighboring vertices. (There can be more than one frog in one vertex). Suppose after this jump, no line connecting any two distinct vertices having frogs on it after the jump passes through the circumcentre of the $2n$-gon. Find all possible values of $n$ for which this can occur.

**5. [Chip firing lemma]**
Let $\boldsymbol{G}$ be a connected graph with $m$ edges. Consider $2m+1$ frogs, each placed on some vertex of $\boldsymbol{G}$. At each second, if a vertex $v$ contains at least $d_v$ frogs, then $d_v$ of the frogs on $v$ jump, one on each of the $d_v$ adjacent vertices. Show that every vertex will be visited by a frog at some point.

**6. [IMO 1986, Problem 3]**
An integer is assigned to each vertex of a regular pentagon, and the sum of all five integers is positive. If three consecutive vertices are assigned the numbers $x$, $y$, $z$ respectively, and $y < 0$, then the following operation is allowed: $x$, $y$, $z$ are replaced by $x+y$, $-y$, $z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

**7. [Russia 1997]**
There are some stones placed on an infinite (in both directions) row of squares labeled by integers. (There may be more than one stone on a given square). There are two types of moves:
(i) Remove one stone from each of the squares $n$ and $n - 1$ and place one stone on $n + 1$
(ii) Remove two stones from square $n$ and place one stone on each of the squares $n + 1$ and $n - 2$.
Show that at some point no more moves can be made, and this final configuration is independent of the choice of moves.

**8. [APMO 2007, Problem 5]**
A regular 5×5 array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all the possible positions of this light.

**9. [IMO Shortlist 2007, C4]**

Let $A_0 = \{a_1, a_2, \ldots, a_n\}$ be a finite sequence of real numbers. For each $k \geq 0$, from the sequence $A_k = \{x_1, x_2, \ldots, x_n\}$ we construct a new sequence $A_{k+1}$ in the following way:

(i) We choose a partition $\{1, 2, \ldots, n\} = I \cup J$, where $I$ and $J$ are two disjoint sets, such that the expression $|\sum_{i \in I} x_i - \sum_{j \in J} x_j|$ is minimized. (We allow $I$ or $J$ to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily.

(ii) We set $A_{k+1} = \{y_1, y_2, \ldots, y_n\}$, where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.

Prove that for some $k$, the sequence $A_k$ contains an element $x$ such that $|x| \geq n/2$.

**10. [Romanian TST 2002]**

After elections, every Member of Parliament (MP) has his own absolute rating. When the parliament is set up, he enters a group and gets a relative rating. The relative rating is the ratio of its own absolute rating to the sum of all absolute ratings of the MPs in the group. An MP can move from one group to another only if in his new group his relative rating is greater. In a given day, only one MP can change the group. Show that only a finite number of group moves is possible (that is, the process eventually terminates).

**11. [ELMO Shortlist 2013, C10]**

Let $N > 1$ be a fixed positive integer. There are $2N$ people, numbered $1, 2, \ldots, 2N$, participating in a tennis tournament. For any two positive integers $i, j$ with $1 \leq i < j \leq 2N$, player $i$ has a higher skill level than player $j$. Prior to the first round, the players are paired arbitrarily and each pair is assigned a unique court among $N$ courts, numbered $1, 2, \ldots, N$.

During a round, each player plays against the other person assigned to his court (so that exactly one match takes place per court), and the player with higher skill wins the match (in other words, there are no upsets). Afterwards, for $i = 2, 3, \ldots, N$, the winner of court $i$ moves to court $(i - 1)$ and the loser of court $i$ stays on court $i$; however, the winner of court 1 stays on court 1 and the loser of court 1 moves to court $N$.

Find all positive integers $M$ such that, regardless of the initial pairing, the players $2, 3, \ldots, N+1$ all change courts immediately after the $M^{\text{th}}$ round.

**12. [IMO 1993, Problem 3]**
On an infinite chessboard, a solitaire game is played as follows: at the start, we have $n^2$ pieces occupying a square of side $n$. The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which $n$ can the game end with only one piece remaining on the board?

**13. [South Korea TST 2009]**
2008 white stones and 1 black stone are in a row. A move consists of selecting one black stone and change the color of its neighboring stone(s). The goal is to make all stones black after a finite number of moves. Find all possible initial positions of the black stone for which this is possible.

**14. [IMO Shortlist 1996, C7]**
A finite number of coins are placed on an infinite (in both directions) row of squares. A sequence of moves is performed as follows: at each stage a square containing more than one coin is chosen. Two coins are taken from this square; one of them is placed on the square immediately to the left while the other is placed on the square immediately to the right of the chosen square. The sequence terminates if at some point there is at most one coin on each square. Given some initial

configuration, show that any legal sequence of moves will terminate after the same number of steps and with the same final configuration.

**15. [IMO Shortlist 2010, C6]**
Given a positive integer $k$ and other two integers $b > w > 1$. There are two strings of pearls, a string of $b$ black pearls and a string of $w$ white pearls. The length of a string is the number of pearls on it. One cuts these strings in some steps by the following rules. In each step:

i. The strings are ordered by their lengths in a non-increasing order. If there are some strings of equal lengths, then the white ones precede the black ones. Then $k$ first ones (if they consist of more than one pearl) are chosen; if there are less than $k$ strings longer than 1, then one chooses all of them.

ii. Next, one cuts each chosen string into two parts differing in length by at most one. The process stops immediately after the step when a first isolated white pearl appears.

Prove that at this stage, there will still exist a string of at least two black pearls.