Problems of Vasc and Arqady

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1. Suppose that a, b, c are positive real numbers, prove that

$$1 < \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{b^2+c^2}} + \frac{c}{\sqrt{c^2+a^2}} \le \frac{3\sqrt{2}}{2}$$

2. If a, b, c are nonnegative real numbers, no two of which are zero, then

   - $2\left(\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b}\right) + (a+b+c)^2 \ge 4(a^2+b^2+c^2);$
   - $\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \le \frac{3(a^2+b^2+c^2)}{2(a+b+c)}.$

3. For all reals a, b and c prove that:

$$\sum_{cyc}(a-b)(a-c)\sum_{cyc}a^2(a-b)(a-c) \ge \left(\sum_{cyc}a(a-b)(a-c)\right)^2$$

4. Let a, b and c are non-negatives such that $a+b+c+ab+ac+bc=6$. Prove that:

$$4(a+b+c)+abc \ge 13$$

5. Let a, b and c are non-negatives. Prove that:

$$(a^2+b^2-2c^2)\sqrt{c^2+ab}+(a^2+c^2-2b^2)\sqrt{b^2+ac}+(b^2+c^2-2a^2)\sqrt{a^2+bc} \le 0$$

6. If a, b, c are nonnegative real numbers, then

   - $\sum_{cyc} a\sqrt{3a^2+5(ab+bc+ca)} \ge \sqrt{2}(a+b+c)^2;$
   - $\sum_{cyc} a\sqrt{2a(a+b+c)+3bc} \ge (a+b+c)^2;$
   - $\sum_{cyc} a\sqrt{5a^2+9bc+11a(b+c)} \ge 2(a+b+c)^2.$

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7. If a, b, c are nonnegative real numbers, then

- $$\sum_{cyc} a\sqrt{2(a^2+b^2+c^2)+3bc} \geq (a+b+c)^2;$$
- $$\sum_{cyc} a\sqrt{4a^2+5bc} \geq (a+b+c)^2.$$

8. If a, b, c are nonnegative real numbers, then

- $$\sum_{cyc} a\sqrt{ab+2bc+ca} \geq 2(ab+bc+ca);$$
- $$\sum_{cyc} a\sqrt{a^2+4b^2+4c^2} \geq (a+b+c)^2.$$

9. If a, b, c are nonnegative real numbers, then

$$\sum_{cyc} \sqrt{a(b+c)(a^2+bc)} \geq 2(ab+bc+ca)$$

10. If a, b, c are positive real numbers such that $ab+bc+ca=3$, then

- $$\sum_{cyc} \sqrt{a(a+b)(a+c)} \geq 6$$
- $$\sum_{cyc} \sqrt{a(4a+5b)(4a+5c)} \geq 27$$

11. If a, b, c are nonnegative real numbers, then

- $$\sum_{cyc} a\sqrt{(a+b)(a+c)} \geq 2(ab+bc+ca)$$
- $$\sum_{cyc} a\sqrt{(a+2b)(a+2c)} \geq 3(ab+bc+ca)$$
- $$\sum_{cyc} a\sqrt{(a+3b)(a+3c)} \geq 4(ab+bc+ca)$$

12. If a, b, c are nonnegative real numbers, then

- $$\sum_{cyc} a\sqrt{(2a+b)(2a+c)} \geq (a+b+c)^2$$
- $$\sum_{cyc} a\sqrt{(a+b)(a+c)} \geq \frac{2}{3}(a+b+c)^2$$
- $$\sum_{cyc} a\sqrt{(4a+5b)(4a+5c)} \geq 3(a+b+c)^2$$

13. If a, b, c are positive real numbers, then

- $a^3 + b^3 + c^3 + abc + 8 \geq 4(a + b + c)$;
- $a^3 + b^3 + c^3 + 3abc + 12 \geq 6(a + b + c)$;
- $4(a^3 + b^3 + c^3) + 15abc + 54 \geq 27(a + b + c)$.

14. If a, b, c are positive real numbers, then

- $\frac{a}{\sqrt{4a^2 + 3ab + 2b^2}} + \frac{b}{\sqrt{4b^2 + 3bc + 2c^2}} + \frac{c}{\sqrt{4c^2 + 3ca + 2a^2}} \leq 1$
- $\frac{a}{\sqrt{4a^2 + 2ab + 3b^2}} + \frac{b}{\sqrt{4b^2 + 2bc + 3c^2}} + \frac{c}{\sqrt{4c^2 + 2ca + 3a^2}} \leq 1$
- $\frac{a}{\sqrt{4a^2 + ab + 4b^2}} + \frac{b}{\sqrt{4b^2 + bc + 4c^2}} + \frac{c}{\sqrt{4c^2 + ca + 4a^2}} \leq 1$

The last is a known inequality.

15. If a, b, c are positive real numbers, then

$$1 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 2\sqrt{1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c}}$$

16. Let x, y, z be real numbers such that $x + y + z = 0$. Find the maximum value of

$$E = \frac{yz}{x^2} + \frac{zx}{y^2} + \frac{xy}{z^2}$$

17. If $a.b.c$ are distinct real numbers, then

$$\frac{ab}{(a-b)^2} + \frac{bc}{(b-c)^2} + \frac{ca}{(c-a)^2} + \frac{1}{4} \geq 0$$

18. If a and b are nonnegative real numbers such that $a + b = 2$, then

- $a^a b^b + ab \geq 2$;
- $a^a b^b + 3ab \leq 4$;
- $a^b b^a + 2 \geq 3ab$.

19. Let a, b, c, d and k be positive real numbers such that

$$(a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) = k$$

Find the range of k such that any three of a, b, c, d are triangle side-lengths.

20. If a, b, c, d, e are positive real numbers such that $a + b + c + d + e = 5$, then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} + 9 \geq \frac{14}{5}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right)$$

21. Let a, b and c are non-negatives such that $ab+ac+bc=3$. Prove that:

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{1}{2}abc\leq 2$$

22. Let a, b, c and d are positive numbers such that $a^4+b^4+c^4+d^4=4$. Prove that:

$$\frac{a^3}{bc}+\frac{b^3}{cd}+\frac{c^3}{da}+\frac{d^3}{ab}\geq 4$$

23. Let $a\geq b\geq c\geq 0$ Prove that:

- $(a-b)^5+(b-c)^5+(c-a)^5\leq 0$
- $\sum_{cyc}(5a^2+11ab+5b^2)(a-b)^5\leq 0$

24. Let a, b and c are positive numbers. Prove that:

$$\frac{a}{a^2+bc}+\frac{b}{b^2+ac}+\frac{c}{c^2+ab}\leq\frac{3}{2\sqrt[3]{abc}}$$

25. Let a, b and c are positive numbers. Prove that:

$$\sqrt{\frac{a+b}{c}}+\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}\geq\sqrt{\frac{11(a+b+c)}{\sqrt[3]{abc}}-15}$$

26. Let a, b and c are non-negative numbers. Prove that:

$$9a^2b^2c^2+a^2b^2+a^2c^2+b^2c^2-4(ab+ac+bc)+2(a+b+c)\geq 0$$

27. Let a,b,c,d be nonnegative real numbers such that $a\geq b\geq c\geq d$ and $3(a^2+b^2+c^2+d^2)=(a+b+c+d)^2$. Prove that

- $a\leq 3b$;
- $a\leq 4c$;
- $b\leq(2+\sqrt{3})c$.

28. If a,b,c are nonnegative real numbers, no two of which are zero, then

- $\dfrac{bc}{2a^2+bc}+\dfrac{ca}{2b^2+ca}+\dfrac{ab}{2c^2+ab}\leq\dfrac{a^2+b^2+c^2}{ab+bc+ca}$;
- $\dfrac{2bc}{a^2+2bc}+\dfrac{2ca}{b^2+2ca}+\dfrac{2ab}{c^2+2ab}+\dfrac{a^2+b^2+c^2}{ab+bc+ca}\geq 3.$

29. Let $a_1, a_2, ..., a_n$ be real numbers such that

$$a_1, a_2, ..., a_n \geq n - 1 - \sqrt{1 + (n-1)^2}, \quad a_1 + a_2 + ... + a_n = n.$$

Prove that

$$\frac{1}{a_1^2 + 1} + \frac{1}{a_2^2 + 1} + \cdots + \frac{1}{a_n^2 + 1} \geq \frac{n}{2}.$$

30. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. For given real $p \neq -2$, find q such that the inequality holds

$$\frac{1}{a^2 + pa + q} + \frac{1}{b^2 + pb + q} + \frac{1}{c^2 + pc + q} \leq \frac{3}{1 + p + q},$$

With two equality cases.
Some particular cases:

- $$\frac{1}{a^2 + 2a + 15} + \frac{1}{b^2 + 2b + 15} + \frac{1}{c^2 + 2c + 15} \leq \frac{1}{6},$$
  With equality for $a = 0$ and $b = c = \frac{3}{2}$;
- $$\frac{1}{8a^2 + 8a + 65} + \frac{1}{8b^2 + 8b + 65} + \frac{1}{8c^2 + 8c + 65} \leq \frac{1}{27},$$
  With equality for $a = \frac{5}{2}$ and $b = c = \frac{1}{4}$;
- $$\frac{1}{8a^2 - 8a + 9} + \frac{1}{8b^2 - 8b + 9} + \frac{1}{8c^2 - 8c + 9} \leq \frac{1}{3},$$
  With equality for $a = \frac{3}{2}$ and $b = c = \frac{3}{4}$;
- $$\frac{1}{8a^2 - 24a + 25} + \frac{1}{8b^2 - 24b + 25} + \frac{1}{8c^2 - 24c + 25} \leq \frac{1}{3},$$
  With equality for $a = \frac{1}{2}$ and $b = c = \frac{5}{4}$;
- $$\frac{1}{2a^2 - 8a + 15} + \frac{1}{2b^2 - 8b + 15} + \frac{1}{2c^2 - 8c + 15} \leq \frac{1}{3},$$
  With equality for $a = 3$ and $b = c = 0$.

31. If a, b, c are the side-lengths of a triangle, then

$$a^3(b + c) + bc(b^2 + c^2) \geq a(b^3 + c^3).$$

32. Find the minimum value of $k > 0$ such that

$$\frac{a}{a^2 + kbc} + \frac{b}{b^2 + kca} + \frac{c}{c^2 + kab} \geq \frac{9}{(1 + k)(a + b + c)},$$

for any positive a, b, c. See the nice case $k = 8$.
PS. Actually, this inequality, with a, b, c replaced by $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, is known.

33. If a, b, c, d are nonnegative real numbers such that

$$a + b + c + d = 4, \quad a^2 + b^2 + c^2 + d^2 = 7,$$

then

$$a^3 + b^3 + c^3 + d^3 \le 16.$$

34. If $a \ge b \ge c \ge 0$, then

- $$a + b + c - 3\sqrt[3]{abc} \ge \frac{64(a-b)^2}{7(11a+24b)};$$
- $$a + b + c - 3\sqrt[3]{abc} \ge \frac{25(b-c)^2}{7(3b+11c)}.$$

35. If $a \ge b \ge 0$, then

$$a^{b-a} \le 1 + \frac{a-b}{\sqrt{a}}.$$

36. If $a, b \in (0, 1]$, then

$$a^{b-a} + b^{a-b} \le 2.$$

37. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{24}{a^2b + b^2c + c^2a} + \frac{1}{abc} \ge 9.$$

38. Let x, y, z be positive real numbers belonging to the interval $[a, b]$. Find the best M (which does not depend on x, y, z) such that

$$x + y + z \le 3M\sqrt[3]{xyz}.$$

39. Let a and b be nonnegative real numbers.
(a) If $2a^2 + b^2 = 2a + b$, then $1 - ab \ge \frac{a-b}{3}$;
(b) If $a^3 + b^3 = 2$, then $3(a^4 + b^4) + 2a^4b^4 \le 8$.

40. Let a, b and c are non-negative numbers. Prove that:

$$\frac{a + b + c + \sqrt{ab} + \sqrt{ac} + \sqrt{bc} + \sqrt[3]{abc}}{7} \ge \sqrt[7]{\frac{(a+b+c)(a+b)(a+c)(b+c)abc}{24}}$$

41. Let a, b, c and d are non-negative numbers such that $abc+abd+acd+bcd = 4$. Prove that:

$$\frac{1}{a+b+c} + \frac{1}{a+b+d} + \frac{1}{a+c+d} + \frac{1}{b+c+d} - \frac{3}{a+b+c+d} \le \frac{7}{12}$$

42. Let a, b, c and d are positive numbers such that $ab+ac+ad+bc+bd+cd=6$. Prove that:

$$\frac{1}{a+b+c+1}+\frac{1}{a+b+d+1}+\frac{1}{a+c+d+1}+\frac{1}{b+c+d+1}\leq 1$$

43. Let $x\geq 0$. Prove without calculus: $(e^x-1)\ln(1+x)\geq x^2$.

44. Let a, b and c are positive numbers. Prove that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{24\sqrt[3]{abc}}{a+b+c}\geq 11$$

45. For all reals a, b and c such that $\sum\limits_{cyc}(a^2+5ab)\geq 0$ prove that:

$$(a+b+c)^6\geq 36(a+b)(a+c)(b+c)abc$$

The equality holds also when a, b and c are roots of the equation:

$$2x^3-6x^2-6x+9=0$$

46. Let a, b and c are non-negative numbers such that $ab+ac+bc\neq 0$. Prove that:

$$\frac{(a+b)^2}{a^2+3ab+4b^2}+\frac{(b+c)^2}{b^2+3bc+4c^2}+\frac{(c+a)^2}{c^2+3ca+4a^2}\geq\frac{3}{2}$$

47. a, b and c are real numbers such that $a+b+c=3$. Prove that:

$$\frac{1}{(a+b)^2+14}+\frac{1}{(b+c)^2+14}+\frac{1}{(c+a)^2+14}\leq\frac{1}{6}$$

48. Let a, b and c are real numbers such that $a+b+c=1$. Prove that:

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\leq\frac{9}{10}$$

49. Let a, b and c are positive numbers such that $4abc=a+b+c+1$. Prove that:

$$\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{b^2+a^2}{c}\geq 2(a^2+b^2+c^2)$$

50. Let a, b and c are positive numbers. Prove that:

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{b}\right)\geq 1+2\sqrt[3]{6(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+10}$$

51. Let a, b and c are positive numbers. Prove that:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{37(a^2+b^2+c^2)-19(ab+ac+bc)}{6(a+b+c)}$$

52. Let a, b and c are positive numbers such that $abc=1$. Prove that

$$a^3+b^3+c^3+4\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)+48\geq 7(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

53. Let a, b and c are non-negative numbers such that $ab+ac+bc=3$. Prove that:

- $\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}\geq\frac{3}{2}$
- $\frac{1}{2+3a^3}+\frac{1}{2+3b^3}+\frac{1}{2+3c^3}\geq\frac{3}{5}$
- $\frac{1}{3+5a^4}+\frac{1}{3+5b^4}+\frac{1}{3+5c^4}\geq\frac{3}{8}$

54. Let a, b and c are non-negative numbers such that $ab+ac+bc\neq 0$. Prove that

$$\frac{a+b+c}{ab+ac+bc}\leq\frac{a}{b^2+bc+c^2}+\frac{b}{a^2+ac+c^2}+\frac{c}{a^2+ab+b^2}\leq\frac{a^3+b^3+c^3}{a^2b^2+a^2c^2+b^2c^2}$$

55. Let a, b and c are non-negative numbers such that $ab+ac+bc=3$. Prove that

- $\frac{a+b+c}{3}\geq\sqrt[5]{\frac{a^2b+b^2c+c^2a}{3}}$
- $\frac{a+b+c}{3}\geq\sqrt[11]{\frac{a^3b+b^3c+c^3a}{3}}$

56. Let a, b and c are non-negative numbers. Prove that

$$(a^2+b^2+c^2)^2\geq 4(a-b)(b-c)(c-a)(a+b+c)$$

57. Let a, b and c are non-negative numbers. Prove that:

$$(a+b+c)^8\geq 128(a^5b^3+a^5c^3+b^5a^3+b^5c^3+c^5a^3+c^5b^3)$$

58. Let a, b and c are positive numbers. Prove that

$$\frac{a^2-bc}{3a+b+c}+\frac{b^2-ac}{3b+a+c}+\frac{c^2-ab}{3c+a+b}\geq 0$$

It seems that $\sum_{cyc}\frac{a^3-bcd}{7a+b+c+d}\geq 0$ is true too for positive a, b, c and d.

59. Let a, b and c are non-negative numbers such that $ab+ac+bc=3$. Prove that:

- $$a^2+b^2+c^2+3abc \geq 6$$
- $$a^4+b^4+c^4+15abc \geq 18$$

60. Let a, b and c are positive numbers such that $abc=1$. Prove that

$$a^2b+b^2c+c^2a \geq \sqrt{3(a^2+b^2+c^2)}$$

61. Let a, b and c are non-negative numbers such that $ab+ac+bc \neq 0$. Prove that:

$$\frac{1}{a^3+3abc+b^3}+\frac{1}{a^3+3abc+c^3}+\frac{1}{b^3+3abc+c^3} \geq \frac{81}{5(a+b+c)^3}$$

62. Let m_a, m_b and m_c are medians of triangle with sides lengths a, b and c. Prove that

$$m_a+m_b+m_c \geq \frac{3}{2}\sqrt{2(ab+ac+bc)-a^2-b^2-c^2}$$

63. Let a, b and c are positive numbers. Prove that:

$$\frac{a+b+c}{9\sqrt[3]{abc}} \geq \frac{a^2}{4a^2+5bc}+\frac{b^2}{4b^2+5ca}+\frac{c^2}{4c^2+5ab}$$

64. Let $\{a,b,c,d\} \subset [1,2]$. Prove that

$$16(a^2+b^2)(b^2+c^2)(c^2+d^2)(d^2+a^2) \leq 25(ac+bd)^4$$

65. Let a, b and c are positive numbers. Prove that

$$\sum_{cyc}\sqrt{a^2-ab+b^2} \leq \frac{10(a^2+b^2+c^2)-ab-ac-bc}{3(a+b+c)}$$

66. Let a, b and c are non-negative numbers. Prove that:

$$\sum_{cyc}\sqrt{2(a^2+b^2)} \geq \sqrt[3]{9\sum_{cyc}(a+b)^3}$$

67. Let a, b and c are positive numbers. Prove that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \sqrt{\frac{15(a^2+b^2+c^2)}{ab+bc+ca}-6}$$

68. Let a, b, c, d and e are non-negative numbers. Prove that

$$\left(\frac{(a+b)(b+c)(c+d)(d+e)(e+a)}{32}\right)^{128} \geq \left(\frac{a+b+c+d+e}{5}\right)^{125}(abcde)^{103}$$

69. Let a, b and c are positive numbers. Prove that

$$a^2b + a^2c + b^2a + b^2c + c^2a + c^2b \geq 6\left(\frac{a^2+b^2+c^2}{ab+ac+bc}\right)^{\frac{4}{5}} abc$$

70. Let a, b and c are non-negative numbers such that $a^3+b^3+c^3=3$. prove that

$$(a+b^2c^2)(b+a^2c^2)(c+a^2b^2) \geq 8a^2b^2c^2$$

71. Given real different numbers a, b and c. Prove that:

$$\frac{(a^2+b^2+c^2-ab-bc-ca)^3}{(a-b)(b-c)(c-a)}\left(\frac{1}{(a-b)^3}+\frac{1}{(b-c)^3}+\frac{1}{(c-a)^3}\right) \leq -\frac{405}{16}$$

72. Let $x \neq 1$, $y \neq 1$ and $x \neq 1$ such that $xyz = 1$. Prove that:

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2} \geq 1$$

When does the equality occur?

73. Let a, b, and c are non-negative numbers such that $a+b+c=3$. Prove that:

$$a^5+b^5+c^5+6 \geq 3(a^3+b^3+c^3)$$

74. $a > 1$, $b > 1$ and $c > 1$. Find the minimal value of the expression:

$$\frac{a^3}{a+b-2}+\frac{b^3}{b+c-2}+\frac{c^3}{c+a-2}$$

75. For all non-negative a, b and c prove that:

$$(ab-c^2)(a+b-c)^3+(ac-b^2)(a+c-b)^3+(bc-a^2)(b+c-a)^3 \geq 0$$

76. Let a, b, c and d are positive numbers such that $a^4+b^4+c^4+d^4=4$. Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a} \geq 4$$

REMARK. This inequality is not true for the condition $a^5+b^5+c^5+d^5=4$.

77. Let a, b and c are positive numbers such that $\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}=3$. Prove that

$$\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c} \leq \frac{3}{2}$$

78. Let a, b and c are positive numbers such that $abc=1$. Prove that:

$$(a+b+c)^3 \geq 63\left(\frac{1}{5a^3+2}+\frac{1}{5b^3+3}+\frac{1}{5c^3+2}\right)$$

79. Let a, b and c are positive numbers such that $\max\{ab, bc, ca\} \le \dfrac{ab+ac+bc}{2}$ and $a+b+c=3$. Prove that

$$a^2+b^2+c^2 \ge a^2b^2+b^2c^2+c^2a^2$$

80. Let a, b and c are positive numbers such that $a+b+c=3$. Prove that:

$$\frac{a^2}{3a+b^2}+\frac{b^2}{3b+c^2}+\frac{c^2}{3c+a^2} \ge \frac{3}{4}$$

81. Let a, b and c are non-negative numbers and $k \ge 2$. Prove that

- $\sqrt{2a^2+5ab+2b^2}+\sqrt{2a^2+5ac+2c^2}+\sqrt{2b^2+5bc+2c^2} \le 3(a+b+c)$;
- $\sum\limits_{cyc}\sqrt{a^2+kab+b^2} \le \sqrt{4(a^2+b^2+c^2)+(3k+2)(ab+ac+bc)}$.

82. Let x, y and z are non-negative numbers such that $x^2+y^2+z^2=3$. Prove that:

$$\frac{x}{\sqrt{x^2+y+z}}+\frac{y}{\sqrt{x+y^2+z}}+\frac{z}{\sqrt{x+y+z^2}} \le \sqrt{3}$$

83. Let a, b and c are non-negative numbers such that $a+b+c=3$. Prove that

$$\frac{a+b}{ab+9}+\frac{a+c}{ac+9}+\frac{b+c}{bc+9} \ge \frac{3}{5}$$

84. If x, y, z be positive reals, then

$$\frac{x}{\sqrt{x+y}}+\frac{y}{\sqrt{y+z}}+\frac{z}{\sqrt{z+x}} \ge \sqrt[4]{\frac{27(yz+zx+xy)}{4}}$$

85. For positive numbers a, b, c, d, e, f and g prove that:

$$\frac{a+b+c+d}{a+b+c+d+f+g}+\frac{c+d+e+f}{c+d+e+f+b+g} > \frac{e+f+a+b}{e+f+a+b+d+g}$$

86. Let a, b and c are non-negative numbers. Prove that:

$$a\sqrt{4a^2+5b^2}+b\sqrt{4b^2+5c^2}+c\sqrt{4c^2+5a^2} \ge (a+b+c)^2$$

87. Let a, b and c are positive numbers. Prove that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{(4\sqrt{2}-3)(ab+ac+bc)}{a^2+b^2+c^2} \ge 4\sqrt{2}$$

88. Let a, b and c are non-negative numbers such, that $a^4+b^4+c^4=3$. Prove that:

$$a^5b+b^5c+c^5a \le 3$$

89. Let a and b are positive numbers, $n \in \mathbb{N}$. Prove that:

$$(n+1)(a^{n+1}+b^{n+1}) \geq (a+b)(a^n + a^{n-1}b + \cdots + b^n)$$

90. Find the maximal α, for which the following inequality holds for all non-negative a, b and c such that $a+b+c=4$.

$$a^3b+b^3c+c^3a+\alpha abc \leq 27$$

91. Let a, b and c are non-negative numbers. Prove that

$$3\sqrt[9]{\frac{a^9+b^9+c^9}{3}} \geq \sqrt[10]{\frac{a^{10}+b^{10}}{2}} + \sqrt[10]{\frac{a^{10}+c^{10}}{2}} + \sqrt[10]{\frac{b^{10}+c^{10}}{2}}$$

92. Let a and b are positive numbers and $2-\sqrt{3} \leq k \leq 2+\sqrt{3}$. Prove that

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\frac{1}{\sqrt{a+kb}}+\frac{1}{\sqrt{b+ka}}\right) \leq \frac{4}{\sqrt{1+k}}$$

93. Let a, b and c are nonnegative numbers, no two of which are zeros. Prove that:

$$\frac{a}{b^2+c^2}+\frac{b}{a^2+c^2}+\frac{c}{a^2+b^2} \geq \frac{3(a+b+c)}{a^2+b^2+c^2+ab+ac+bc}.$$

94. Let x, y and z are positive numbers such that $xy+xz+yz=1$. Prove that

$$\frac{x^3}{1-4y^2xz}+\frac{y^3}{1-4z^2yx}+\frac{z^3}{1-4x^2yz} \geq \frac{(x+y+z)^3}{5}.$$

95. Let a, b and c are positive numbers such that $a^6+b^6+c^6=3$. Prove that:

$$(ab+ac+bc)\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right) \geq 9.$$

96. Let a, b and c are positive numbers. Prove that

$$\sqrt[3]{\frac{a}{2b+25c}}+\sqrt[3]{\frac{b}{2c+25a}}+\sqrt[3]{\frac{c}{2a+25b}} \geq 1.$$

97. Let a, b and c are sides lengths of triangle. Prove that

$$\frac{(a+b)(a+c)(b+c)}{8} \geq \frac{(2a+b)(2b+c)(2c+a)}{27}.$$

98. Let a, b and c are non-negative numbers. Prove that

$$\sqrt[3]{\frac{(2a+b)(2b+c)(2c+a)}{27}} \geq \sqrt{\frac{ab+ac+bc}{3}}.$$

99. Let a, b and c are positive numbers. Prove that

$$\sqrt{\frac{a^3}{b^3+(c+a)^3}}+\sqrt{\frac{b^3}{c^3+(a+b)^3}}+\sqrt{\frac{c^3}{a^3+(b+c)^3}}\geq 1.$$

100. Let x, y and z are non-negative numbers such that $xy+xz+yz=9$. Prove that

$$(1+x^2)(1+y^2)(1+z^2)\geq 64.$$