Geometry Problems

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Note. Most of problems have solutions. Just click on the number beside the problem to open its page and see the solution! Problems posted by different authors, but all of them are nice! Happy Problem Solving!

1. Circles W_1, W_2 intersect at P, K. XY is common tangent of two circles which is nearer to P and X is on W_1 and Y is on W_2. XP intersects W_2 for the second time in C and YP intersects W_1 in B. Let A be intersection point of BX and CY. Prove that if Q is the second intersection point of circumcircles of ABC and AXY

$$\angle QXA = \angle QKP$$

2. Let M be an arbitrary point on side BC of triangle ABC. W is a circle which is tangent to AB and BM at T and K and is tangent to circumcircle of AMC at P. Prove that if $TK||AM$, circumcircles of APT and KPC are tangent together.

3. Let ABC an isosceles triangle and $BC > AB = AC$. D, M are respectively midpoints of BC, AB. X is a point such that $BX \perp AC$ and $XD||AB$. BX and AD meet at H. If P is intersection point of DX and circumcircle of AHX (other than X), prove that tangent from A to circumcircle of triangle AMP is parallel to BC.

4. Let O, H be the circumcenter and the orthogonal center of triangle $\triangle ABC$, respectively. Let M and N be the midpoints of BH and CH. Define

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B' on the circumcenter of $\triangle ABC$, such that B and B' are diametrically opposed. If $HONM$ is a cyclic quadrilateral, prove that $\overline{B'N} = \frac{1}{2}\overline{AC}$.

5. OX, OY are perpendicular. Assume that on OX we have wo fixed points P, P' on the same side of O. I is a variable point that $IP = IP'$. $PI, P'I$ intersect OY at A, A'.

a) If C, C' Prove that I, A, A', M are on a circle which is tangent to a fixed line and is tangent to a fixed circle.

b) Prove that IM passes through a fixed point.

6. Let A, B, C, Q be fixed points on plane. M, N, P are intersection points of AQ, BQ, CQ with BC, CA, AB. D', E', F' are tangency points of incircle of ABC with BC, CA, AB. Tangents drawn from M, N, P (not triangle sides) to incircle of ABC make triangle DEF. Prove that DD', EE', FF' intersect at Q.

7. Let ABC be a triangle. W_a is a circle with center on BC passing through A and perpendicular to circumcircle of ABC. W_b, W_c are defined similarly. Prove that center of W_a, W_b, W_c are collinear.

8. In tetrahedron $ABCD$, radius four circumcircles of four faces are equal. Prove that $AB = CD, AC = BD$ and $AD = BC$.

9. Suppose that M is an arbitrary point on side BC of triangle ABC. B_1, C_1 are points on AB, AC such that $MB = MB_1$ and $MC = MC_1$. Suppose that H, I are orthocenter of triangle ABC and incenter of triangle MB_1C_1. Prove that A, B_1, H, I, C_1 lie on a circle.

10. Incircle of triangle ABC touches AB, AC at P, Q. BI, CI intersect with PQ at K, L. Prove that circumcircle of ILK is tangent to incircle of ABC if and only if $AB + AC = 3BC$.

11. Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

12. Let $ABCD$ be an arbitrary quadrilateral. The bisectors of external angles A and C of the quadrilateral intersect at P; the bisectors of external angles B and D intersect at Q. The lines AB and CD intersect at E, and the lines BC and DA intersect at F. Now we have two new angles: E (this is the angle $\angle AED$) and F (this is the angle $\angle BFA$). We also consider a point R of intersection of the external bisectors of these angles. Prove that the points P, Q and R are collinear.

13. Let ABC be a triangle. Squares AB_cB_aC, CA_bA_cB and BC_aC_bA are outside the triangle. Square $B_cB_c'B_a'B_a$ with center P is outside square AB_cB_aC. Prove that BP, C_aB_a and A_cB_c are concurrent.

14. Triangle ABC is isosceles ($AB = AC$). From A, we draw a line ℓ parallel to BC. P, Q are on perpendicular bisectors of AB, AC such that $PQ \perp BC$. M, N are points on ℓ such that angles $\angle APM$ and $\angle AQN$ are $\frac{\pi}{2}$. Prove that

$$\frac{1}{AM} + \frac{1}{AN} \leq \frac{2}{AB}$$

15. In triangle ABC, M is midpoint of AC, and D is a point on BC such that $DB = DM$. We know that $2BC^2 - AC^2 = AB.AC$. Prove that

$$BD.DC = \frac{AC^2.AB}{2(AB + AC)}$$

16. H, I, O, N are orthogonal center, incenter, circumcenter, and Nagelian point of triangle ABC. I_a, I_b, I_c are excenters of ABC corresponding vertices A, B, C. S is point that O is midpoint of HS. Prove that centroid of triangles $I_aI_bI_c$ and SIN concide.

17. $ABCD$ is a convex quadrilateral. We draw its diagonals to divide the quadrilateral to four triangles. P is the intersection of diagonals. I_1, I_2, I_3, I_4 are excenters of PAD, PAB, PBC, PCD(excenters corresponding vertex P). Prove that I_1, I_2, I_3, I_4 lie on a circle iff $ABCD$ is a tangential quadrilateral.

18. In triangle ABC, if L, M, N are midpoints of AB, AC, BC. And H is orthogonal center of triangle ABC, then prove that

$$LH^2 + MH^2 + NH^2 \leq \frac{1}{4}(AB^2 + AC^2 + BC^2)$$

19. Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that, as A_1 and B_1 vary, the circumcentres of triangles A_1A_2C all lie on one fixed circle.

20. Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcentre of triangle BCD lies on the circumcircle of triangle ABC.

21. The bisectors of the angles A and B of the triangle ABC meet the sides BC and CA at the points D and E, respectively. Assuming that $AE + BD = AB$, determine the angle C.

22. Let A, B, C, P, Q, and R be six concyclic points. Show that if the Simson lines of P, Q, and R with respect to triangle ABC are concurrent, then the Simson lines of A, B, and C with respect to triangle PQR are concurrent. Furthermore, show that the points of concurrence are the same.

23. ABC is a triangle, and E and F are points on the segments BC and CA respectively, such that $\dfrac{CE}{CB} + \dfrac{CF}{CA} = 1$ and $\angle CEF = \angle CAB$. Suppose that M is the midpoint of EF and G is the point of intersection between CM and AB. Prove that triangle FEG is similar to triangle ABC.

24. Let ABC be a triangle with $\angle C = 90^\circ$ and $CA \neq CB$. Let CH be an altitude and CL be an interior angle bisector. Show that for $X \neq C$ on the line CL, we have $\angle XAC \neq \angle XBC$. Also show that for $Y \neq C$ on the line CH we have $\angle YAC \neq \angle YBC$.

25. Given four points A, B, C, D on a circle such that AB is a diameter and CD is not a diameter. Show that the line joining the point of intersection of the tangents to the circle at the points C and D with the point of intersection of the lines AC and BD is perpendicular to the line AB.

27. Given a triangle ABC and D be point on side AC such that $AB = DC$, $\angle BAC = 60 - 2X$, $\angle DBC = 5X$ and $\angle BCA = 3X$ prove that $X = 10$.

28. Prove that in any triangle ABC,

$$0 < \cot\left(\frac{A}{4}\right) - \tan\left(\frac{B}{4}\right) - \tan\left(\frac{C}{4}\right) - 1 < 2\cot\left(\frac{A}{2}\right).$$

29. Triangle $\triangle ABC$ is given. Points D i E are on line AB such that $D - A - B - E, AD = AC$ and $BE = BC$. Bisector of internal angles at A and B intersect BC, AC at P and Q, and circumcircle of ABC at M and N. Line which connects A with center of circumcircle of BME and line which connects B and center of circumcircle of AND intersect at X. Prove that $CX \perp PQ$.

30. Consider a circle with center O and points A, B on it such that AB is not a diameter. Let C be on the circle so that AC bisects OB. Let AB and OC intersect at D, BC and AO intersect at F. Prove that $AF = CD$.

31. Let ABC be a triangle.$X; Y$ are two points on $AC; AB$,respectively.CY cuts BX at Z and AZ cut XY at H ($AZ \perp XY$). $BHXC$ is a quadrilateral inscribed in a circle. Prove that $XB = XC$.

32. Let $ABCD$ be a cyclic quadrilatedral, and let L and N be the midpoints of its diagonals AC and BD, respectively. Suppose that the line BD bisects the angle ANC. Prove that the line AC bisects the angle BLD.

33. A triangle $\triangle ABC$ is given, and let the external angle bisector of the angle $\angle A$ intersect the lines perpendicular to BC and passing through B and C at the points D and E, respectively. Prove that the line segments BE, CD, AO are concurrent, where O is the circumcenter of $\triangle ABC$.

34. Let $ABCD$ be a convex quadrilateral. Denote $O \in AC \cap BD$. Ascertain and construct the positions of the points $M \in (AB)$ and $N \in (CD)$, $O \in MN$ so that the sum $\dfrac{MB}{MA} + \dfrac{NC}{ND}$ is minimum.

35. Let ABC be a triangle, the middlepoints M, N, P of the segments $[BC], [CA], [AM]$ respectively, the intersection $E \in AC \cap BP$ and the projection R of the point A on the line MN. Prove that $\widehat{ERN} \equiv \widehat{CRN}$.

36. Two circles intersect at two points, one of them X. Find Y on one circle and Z on the other, so that X, Y and Z are collinear and $XY \cdot XZ$ is as large as possible.

37. The points A, B, C, D lie in this order on a circle o. The point S lies inside o and has properties $\angle SAD = \angle SCB$ and $\angle SDA = \angle SBC$. Line which in which angle bisector of $\angle ASB$ in included cut the circle in points P and Q. Prove that $PS = QS$.

38. Given a triangle ABC. Let G, I, H be the centroid, the incenter and the orthocenter of triangle ABC, respectively. Prove that $\angle GIH > 90°$.

39. Let be given two parallel lines k and l, and a circle not intersecting k. Consider a variable point A on the line k. The two tangents from this point A to the circle intersect the line l at B and C. Let m be the line through the point A and the midpoint of the segment BC. Prove that all the lines m (as A varies) have a common point.

40. Let $ABCD$ be a convex quadrilateral with $AD \not\parallel BC$. Define the points $E = AD \cap BC$ and $I = AC \cap BD$. Prove that the triangles EDC and IAB have the same centroid if and only if $AB \parallel CD$ and $IC^2 = IA \cdot AC$.

41. Let $ABCD$ be a square. Denote the intersection $O \in AC \cap BD$. Exists a positive number k so that for any point $M \in [OC]$ there is a point $N \in [OD]$ so that $AM \cdot BN = k^2$. Ascertain the geometrical locus of the intersection $L \in AN \cap BM$.

42. Consider a right-angled triangle ABC with the hypothenuse $AB = 1$. The bisector of $\angle ACB$ cuts the medians BE and AF at P and M, respectively. If $AF \cap BE = \{P\}$, determine the maximum value of the area of $\triangle MNP$.

43. Let triangle ABC be an isosceles triangle with $AB = AC$. Suppose that the angle bisector of its angle $\angle B$ meets the side AC at a point D and that $BC = BD + AD$. Determine $\angle A$.

44. Given a triangle with the area S, and let a, b, c be the sidelengths of the triangle. Prove that $a^2 + 4b^2 + 12c^2 \geq 32 \cdot S$.

45. In a right triangle ABC with $\angle A = 90$ we draw the bisector AD . Let $DK \perp AC, DL \perp AB$. Lines BK, CL meet each other at point H . Prove that $AH \perp BC$.

46. Let H be the orthocenter of the acute triangle ABC. Let BB' and CC' be altitudes of the triangle ($BE \in AC$, $CE \in AB$). A variable line ℓ passing through H intersects the segments $[BC']$ and $[CB']$ in M and N. The perpendicular lines of ℓ from M and N intersect BB' and CC' in P and Q. Determine the locus of the midpoint of the segment $[PQ]$.

47. Let ABC be a triangle whit $AH \perp BC$ and BE the interior bisector of the angle ABC.If $m(\angle BEA) = 45$, find $m(\angle EHC)$.

48. Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.

49. Let D be inside the $\triangle ABC$ and E on AD different of D. Let ω_1 and ω_2 be the circumscribed circles of $\triangle BDE$ resp. $\triangle CDE$. ω_1 and ω_2 intersect BC in the interior points F resp. G. Let X be the intersection between DG and AB and Y the intersection between DF and AC. Show that XY is $\parallel$ to BC.

50. Let $\triangle ABC$ be a triangle, D the midpoint of BC, and M be the midpoint of AD. The line BM intersects the side AC on the point N. Show that AB is tangent to the circuncircle to the triangle $\triangle NBC$ if and only if the following equality is true:

$$\frac{BM}{MN} = \frac{(BC)^2}{(BN)^2}.$$

51. Let $\triangle ABC$ be a traingle with sides a, b, c, and area K. Prove that

$$27(b^2 + c^2 - a^2)^2(c^2 + a^2 - b^2)^2(a^2 + b^2 - c^2)^2 \leq (4K)^6$$

52. Given a triangle ABC satisfying $AC + BC = 3 \cdot AB$. The incircle of triangle ABC has center I and touches the sides BC and CA at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Prove that the points A, B, K, L lie on one circle.

53. In an acute-angled triangle ABC, we are given that $2 \cdot AB = AC + BC$. Show that the incenter of triangle ABC, the circumcenter of triangle ABC, the midpoint of AC and the midpoint of BC are concyclic.

54. Let ABC be a triangle, and M the midpoint of its side BC. Let γ be the incircle of triangle ABC. The median AM of triangle ABC intersects the incircle γ at two points K and L. Let the lines passing through K and L, parallel to BC, intersect the incircle γ again in two points X and Y. Let the lines AX and AY intersect BC again at the points P and Q. Prove that $BP = CQ$.

55. Let ABC be a triangle, and M an interior point such that $\angle MAB = 10°$, $\angle MBA = 20°$, $\angle MAC = 40°$ and $\angle MCA = 30°$. Prove that the triangle is isosceles.

56. Let ABC be a right-angle triangle ($AB \perp AC$). Define the middlepoint M of the side $[BC]$ and the point $D \in (BC)$, $\widehat{BAD} \equiv \widehat{CAD}$. Prove that exists a point $P \in (AD)$ so that $PB \perp PM$ and $PB = PM$ if and only if $AC = 2 \cdot AB$ and in this case $\dfrac{PA}{PD} = \dfrac{3}{5}$.

57. Consider a convex pentagon $ABCDE$ such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \angle ABC = \angle ACD = \angle ADE$$

Let P be the point of intersection of the lines BD and CE. Prove that the line AP passes through the midpoint of the side CD.

58. The perimeter of triangle ABC is equal to $3 + 2\sqrt{3}$. In the coordinate plane, any triangle congruent to triangle ABC has at least one lattice point in its interior or on its sides. Prove that triangle ABC is equilateral.

59. Let ABC be a triangle inscribed in a circle of radius R, and let P be a point in the interior of triangle ABC. Prove that

$$\frac{PA}{BC^2} + \frac{PB}{CA^2} + \frac{PC}{AB^2} \geq \frac{1}{R}.$$

60. Show that the plane cannot be represented as the union of the inner regions of a finite number of parabolas.

61. Let $ABCD$ be a circumscriptible quadrilateral, let $\{O\} = AC \cap BD$, and let K, L, M, and N be the feet of the perpendiculars from the point O to the sides AB, BC, CD, and DA. Prove that: $\dfrac{1}{|OK|} + \dfrac{1}{|OM|} = \dfrac{1}{|OL|} + \dfrac{1}{|ON|}$.

62. Let a triangle ABC . At the extension of the sides BC (to C) ,CA (to A) , AB (to B) we take points D, E, F such that $CD = AE = BF$. Prove that if the triangle DEF is equilateral then ABC is also equilateral.

63. Given triangle ABC, incenter I, incircle of triangle IBC touch IB, IC at I_a, I'_a resp similar we have I_b, I'_b, I_c, I'_c the lines $I_bI'_b \cap I_cI'_c = \{A'\}$ similarly we have B', C' prove that two triangle $ABC, A'B'C'$ are perspective.

64. Let AA_1, BB_1, CC_1 be the altitudes in acute triangle ABC, and let X be an arbitrary point. Let M, N, P, Q, R, S be the feet of the perpendiculars from X to the lines $AA_1, BC, BB_1, CA, CC_1, AB$. Prove that MN, PQ, RS are concurrent.

65. Let ABC be a triangle and let X, Y and Z be points on the sides $[BC], [CA]$ and $[AB]$, respectively, such that $AX = BY = CZ$ and $BX = CY = AZ$. Prove that triangle ABC is equilateral.

66. Let P and P' be two isogonal conjugate points with respect to triangle ABC. Let the lines AP, BP, CP meet the lines BC, CA, AB at the points A', B', C', respectively. Prove that the reflections of the lines AP', BP', CP' in the lines $B'C', C'A', A'B'$ concur.

67. In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies

$anglePBC = \angle DBA \quad and \quad \angle PDC = \angle BDA.$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

68. Let the tangents to the circumcircle of a triangle ABC at the vertices B and C intersect each other at a point X. Then, the line AX is the A-symmedian of triangle ABC.

69. Let the tangents to the circumcircle of a triangle ABC at the vertices B and C intersect each other at a point X, and let M be the midpoint of the side BC of triangle ABC. Then, $AM = AX \cdot |\cos A|$ (we don't use directed angles here).

70. Let ABC be an equilateral triangle (i. e., a triangle which satisfies $BC = CA = AB$). Let M be a point on the side BC, let N be a point on the side CA, and let P be a point on the side AB, such that $S(ANP) = S(BPM) = S(CMN)$, where $S(XYZ)$ denotes the area of a triangle XYZ. Prove that $\triangle ANP \cong \triangle BPM \cong \triangle CMN$.

71. Let $ABCD$ be a parallelogram. A variable line g through the vertex A intersects the rays BC and DC at the points X and Y, respectively. Let K and L be the A- excenters of the triangles ABX and ADY. Show that the angle $\measuredangle KCL$ is independent of the line g.

72. Triangle QAP has the right angle at A. Points B and R are chosen on the segments PA and PQ respectively so that BR is parallel to AQ. Points S and T are on AQ and BR respectively and AR is perpendicular to BS, and AT is perpendicular to BQ. The intersection of AR and BS is U, The intersection of AT and BQ is V. Prove that

(i) the points P, S and T are collinear;

(ii) the points P, U and V are collinear.

73. Let ABC be a triangle and m a line which intersects the sides AB and AC at interior points D and F, respectively, and intersects the line BC at a point E such that C lies between B and E. The parallel lines from the points A, B, C to the line m intersect the circumcircle of triangle ABC at the points A_1, B_1 and C_1, respectively (apart from A, B, C). Prove that the lines A_1E , B_1F and C_1D pass through the same point.

74. Let H is the orthocentre of triangle ABC. X is an arbitrary point in the plane. The circle with diameter XH again meets lines AH, BH, CH at a points A_1, B_1, C_1, and lines AX, BX, CX at a points A_2, B_2, C_2, respectively. Prove that the lines A_1A_2, B_1B_2, C_1C_2 meet at same point.

75. Determine the nature of a triangle ABC such that the incenter lies on HG where H is the orthocenter and G is the centroid of the triangle ABC.

76. ABC is a triangle. D is a point on line AB. (C) is the in circle of triangle BDC. Draw a line which is parallel to the bisector of angle ADC, And goes through I, the incenter of ABC and this line is tangent to circle (C). Prove that $AD = BD$.

77. Let M, N be the midpoints of the sides BC and AC of $\triangle ABC$, and BH be its altitude. The line through M, perpendicular to the bisector of $\angle HMN$, intersects the line AC at point P such that $HP = \frac{1}{2}(AB + BC)$ and $\angle HMN = 45$. Prove that ABC is isosceles.

78. Points D, E, F are on the sides BC, CA and AB, respectively which satisfy $EF||BC$, D_1 is a point on BC, Make $D_1E_1||D_E, D_1F_1||DF$ which intersect AC and AB at E_1 and F_1, respectively. Make $\triangle PBC \sim \triangle DEF$ such that P and A are on the same side of BC. Prove that E, E_1F_1, PD_1 are concurrent.

79. Let $ABCD$ be a rectangle. We choose four points P, M, N and Q on AB, BC, CD and DA respectively. Prove that the perimeter of $PMNQ$ is at least two times the diameter of $ABCD$.

80. In the following, the abbreviation $g \cap h$ will mean the point of intersection of two lines g and h.

Let $ABCDE$ be a convex pentagon. Let $A = BD \cap CE$, $B = CE \cap DA$, $C = DA \cap EB$, $D = EB \cap AC$ and $E = AC \cap BD$. Furthermore, let $A = AA \cap EB$, $B = BB \cap AC$, $C = CC \cap BD$, $D = DD \cap CE$ and $E = EE \cap DA$.

Prove that:

$$\frac{EA}{AB} \cdot \frac{AB}{BC} \cdot \frac{BC}{CD} \cdot \frac{CD}{DE} \cdot \frac{DE}{EA} = 1.$$

81. Let ABC be a triangle. The its incircle $i = C(I, r)$ touches the its sides in the points $D \in (BC), E \in (CA), F \in (AB)$ respectively. I note the second intersections M, N, P of the lines AI, BI, CI respectively with the its circumcircle $e = C(O, R)$. Prove that the lines MD, NE, PF are concurrently.

Remark. If the points A', B', C' are the second intersections of the lines AO, BO, CO respectively with the circumcircle e then the points $U \in MD \cap A'I, V \in NE \cap B'I, V \in PF \cap C'I$ belong to the circumcircle w.

82. let ABC be an acute triangle with $\angle BAC > \angle BCA$, and let D be a point on side AC such that $|AB| = |BD|$. Furthermore, let F be a point on the circumcircle of triangle ABC such that line FD is perpendicular to side BC and points F, B lie on different sides of line AC. Prove that line FB is perpendicular to side AC.

83. Let ABC be a triangle with orthocenter H, incenter I and centroid S, and let d be the diameter of the circumcircle of triangle ABC.

Prove the inequality

$9 \cdot HS^2 + 4\left(AH \cdot AI + BH \cdot BI + CH \cdot CI\right) \geq 3d^2$,

and determine when equality holds.

84. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

85. ABC inscribed triangle in circle (O, R). At AB we take point C' such that $AC = AC'$ and at AC we take point B' such that $AB' = AB$. The segment $B'C'$ intersects the circle at E, D respectively and and it intersects BC at M. Prove that when the point A moves on the arc BAC the AM pass from a standard point.

86. In an acute-angled triangle ABC, we consider the feet H_a and H_b of the altitudes from A and B, and the intersections W_a and W_b of the angle bisectors from A and B with the opposite sides BC and CA respectively. Show that the centre of the incircle I of triangle ABC lies on the segment H_aH_b if and only if the centre of the circumcircle O of triangle ABC lies on the segment W_aW_b.

87. Let ABC be a triangle and O a point in its plane. Let the lines BO and CO intersect the lines CA and AB at the points M and N, respectively. Let the parallels to the lines CN and BM through the points M and N intersect each other at E, and let the parallels to the lines CN and BM through the points B and C intersect each other at F.

88. In space, given a right-angled triangle ABC with the right angle at A, and given a point D such that the line CD is perpendicular to the plane

ABC. Denote $d = AB, h = CD$, $\alpha = \measuredangle DAC$ and $\beta = \measuredangle DBC$. Prove that $h = \dfrac{d \tan\alpha \tan\beta}{\sqrt{\tan^2\alpha - \tan^2\beta}}$.

89. A triangle ABC is given in a plane. The internal angle bisectors of the angles A, B, C of this triangle ABC intersect the sides BC, CA, AB at A', B', C'. Let P be the point of intersection of the angle bisector of the angle A with the line $B'C'$. The parallel to the side BC through the point P intersects the sides $AB$and AC in the points M and N. Prove that $2 \cdot MN = BM + CN$.

90. A triangle ABC has the sidelengths a, b, c and the angles A, B, C, where a lies opposite to A, where b lies opposite to B, and c lies opposite to C. If $a\,(1 - 2\cos A) + b\,(1 - 2\cos B) + c\,(1 - 2\cos C) = 0$, then prove that the triangle ABC is equilateral.

91. Circles $C(O_1)$ and $C(O_2)$ intersect at points A, B. CD passing through point O_1 intersects $C(O_1)$ at point D and tangents $C(O_2)$ at point C. AC tangents $C(O_1)$ at A. Draw $AE \perp CD$, and AE intersects $C(O_1)$ at E. Draw $AF \perp DE$, and AF intersects DE at F. Prove that BD bisects AF.

92. In a triangle ABC, let A_1, B_1, C_1 be the points where the excircles touch the sides BC, CA and AB respectively. Prove that AA_1, BB_1 and CC_1 are the sidelenghts of a triangle.

93. Let ABC be an acute-angled triangle, and let P and Q be two points on its side BC. Construct a point C_1 in such a way that the convex quadrilateral $APBC_1$ is cyclic, $QC_1 \parallel CA$, and the points C_1 and Q lie on opposite sides of the line AB. Construct a point B_1 in such a way that the convex quadrilateral $APCB_1$ is cyclic, $QB_1 \parallel BA$, and the points B_1 and Q lie on opposite sides of the line AC.

Prove that the points B_1, C_1, P, and Q lie on a circle.

94. Let $ABCD$ be an arbitrary quadrilateral. The bisectors of external angles A and C of the quadrilateral intersect at P; the bisectors of external angles B and D intersect at Q. The lines AB and CD intersect at E, and the lines BC and DA intersect at F. Now we have two new angles: E (this is the angle $\angle AED$) and F (this is the angle $\angle BFA$). We also consider a point R of intersection of the external bisectors of these angles. Prove that the points P, Q and R are collinear.

95. Let I be the incenter in triangle ABC and let triangle $A_1B_1C_1$ be its medial triangle (i.e. A_1 is the midpoint of BC, etc.). Prove that the centers of Euler's nine- point circles of triangle BIC, CIA, AIB lie on the angle bisectors of the medial triangle $A_1B_1C_1$.

96. Consider three circles equal radii R that have a common point H. They intersect also two by two in three other points different than H, denoted A, B, C. Prove that the circumradius of triangle ABC is also R.

97. Three congruent circles G_1, G_2, G_3 have a common point P. Further, define $G_2 \cap G_3 = \{A,\ P\}$, $G_3 \cap G_1 = \{B,\ P\}$, $G_1 \cap G_2 = \{C,\ P\}$. **1)** Prove that the point P is the orthocenter of triangle ABC. **2)** Prove that the circumcircle of triangle ABC is congruent to the given circles G_1, G_2, G_3.

98. Let $ABXY$ be a convex trapezoid such that $BX \parallel AY$. We call C the midpoint of its side XY, and we denote by P and Q the midpoints of the segments BC and CA, respectively. Let the lines XP and YQ intersect at a point N. Prove that the point N lies in the interior or on the boundary of triangle ABC if and only if $\dfrac{1}{3} \leq \dfrac{BX}{AY} \leq 3$.

99. Let P be a fixed point on a conic, and let M, N be variable points on that same conic s.t. $PM \perp PN$. Show that MN passes through a fixed point.

100. A triangle ABC is given. Let L be its Lemoine point and F its Fermat (Torricelli) point. Also, let H be its orthocenter and O its circumcenter. Let l be its Euler line and l' be a reflection of l with respect to the line AB. Call D the intersection of l' with the circumcircle different from H' (where H' is the reflection of H with respect to the line AB), and E the intersection of the line FL with OD. Now, let G be a point different from H such that the pedal triangle of G is similar to the cevian triangle of G (with respect to triangle ABC). Prove that angles ACB and GCE have either common or perpendicular bisectors.

101. Let ABC be a triangle with area S, and let P be a point in the plane. Prove that $AP + BP + CP \geq 2\sqrt[4]{3}\sqrt{S}$.

102. Suppose M is a point on the side AB of triangle ABC such that the incircles of triangle AMC and triangle BMC have the same radius. The two circles, centered at O_1 and O_2, meet AB at P and Q respectively. It is known that the area of triangle ABC is six times the area of the quadrilateral PQO_2O_1, determine the possible value(s) of $\dfrac{AC+BC}{AB}$. Justify your claim.

103. Let AB_1C_1, AB_2C_2, AB_3C_3 be directly congruent equilateral triangles. Prove that the pairwise intersections of the circumcircles of triangles AB_1C_2, AB_2C_3, AB_3C_1 form an equilateral triangle congruent to the first three.

104. Tried posting this in Pre-Olympiad but thought I'd get more feed back here: For acute triangle ABC, cevians AD, BE, and CF are concurrent at P. Prove $2\left(\dfrac{1}{AP}+\dfrac{1}{BP}+\dfrac{1}{CP}\right) \leq \dfrac{1}{PD}+\dfrac{1}{PE}+\dfrac{1}{PF}$ and determine when equality holds

105. Given a triangle ABC. Let O be the circumcenter of this triangle ABC. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of these altitudes AH, BK, CL, respectively. The incircle of triangle ABC has center I and touches the sides BC, CA, AB at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F and OI are concurrent. (When the point O concides with I, we consider the line OI as an arbitrary line passing through O.)

106. Given an equilateral triangle ABC and a point M in the plane (ABC). Let A', B', C' be respectively the symmetric through M of A, B, C.

. I. Prove that there exists a unique point P equidistant from A and B', from B and C' and from C and A'.

. II. Let D be the midpoint of the side AB. When M varies (M does not coincide with D), prove that the circumcircle of triangle MNP (N is the intersection of the line DM and AP) pass through a fixed point.

107. Let $ABCD$ be a square, and C the circle whose diameter is AB. Let Q be an arbitrary point on the segment CD. We know that QA meets C on E and QB meets it on F. Also CF and DE intersect in M. show that M belongs to C.

108. In a triangle, let a, b, c denote the side lengths and h_a, h_b, h_c the altitudes to the corresponding side. Prove that $(\frac{a}{h_a})^2 + (\frac{b}{h_b})^2 + (\frac{c}{h_c})^2 \geq 4$

109. Given a triangle ABC. A point X is chosen on a side AC. Some circle passes through X, touches the side AC and intersects the circumcircle of triangle ABC in points M and N such that the segment MN bisects BX and intersects sides AB and BC in points P and Q. Prove that the circumcircle of triangle PBQ passes through a fixed point different from B.

110. Let ABC be an isosceles triangle with $\angle ACB = \frac{\pi}{2}$, and let P be a point inside it.

. A) Show that $\angle PAB + \angle PBC \geq \min(\angle PCA, \angle PCB)$;

. B) When does equality take place in the inequality above?

111. Given a regular tetrahedron $ABCD$ with edge length 1 and a point P inside it. What is the maximum value of $|PA| + |PB| + |PC| + |PD|$.

112. Given the tetrahedron $ABCD$ whose faces are all congruent. The vertices A, B, C lie in the positive part of x-axis, y-axis, and z-axis, respectively, and $AB = 2l - 1$, $BC = 2l$, $CA = 2l + 1$, where $l > 2$. Let the volume of tetrahedron ABCD be $V(l)$.

Evaluate

$$\lim_{l \to 2} \frac{V(l)}{\sqrt{l-2}}$$

.

113. Let a triangle ABC . M , N , P are the midpoints of BC, CA, AB . a) d_1, d_2, d_3 are lines throughing M, N, P and dividing the perimeter of triangle ABC into halves . Prove that : d_1, d_2, d_3 are concurrent at K . b) Prove that : among the ratios : $\frac{KA}{BC}, \frac{KB}{AC}, \frac{KC}{AB}$, there exists at least one ratio $\geq \frac{1}{\sqrt{3}}$.

114. Given rectangle $ABCD$ $(AB = a, BC = b)$ find locus of points M , so that reflections of M in the sides are concyclic.

115. An incircle of a triangle ABC touches it's sides AB, BC and CA at C', A' and B' respectively. Let M, N, K, L be midpoints of $C'A$, $B'A$, $A'C$, $B'C$ respectively. The line $A'C'$ intersects lines MN and KL at E and F respectively; lines $A'B'$ and MN intersect at P; lines $B'C'$ and KL intersect at Q. Let Ω_A and Ω_C be outcircles of triangles EAP and FCQ respectively. a) Let l_1 and l_2 be common tangents of circles Ω_A and Ω_C. Prove that the lines l_1, l_2, EF and PQ have a common point. b) Let circles Ω_A and Ω_C intersect at X and Y. Prove that the points X, Y and B lie on the line.

116. Let two circles (O_1) and (O_2) cut each other at two points A and B. Let a point M move on the circle (O_1). Denote by K the point of intersection of the two tangents to the circle (O_1) at the points A and B. Let the line MK cut the circle (O_1) again at C. Let the line AC cut the circle (O_2) again at Q. Let the line MA cut the circle (O_2) again at P. **(a)** Prove that the line KM bisects the segment PQ. **(b)** When the point M moves on the circle (O_1), prove that the line PQ passes through a fixed point.

117. Given n balls $B_1, B_2, \cdots, B_n$ of radii $R_1, R_2, \cdots, R_n$ in space. Assume that there doesn't exist any plane separating these n balls. Then prove that there exists a ball of radius $R_1 + R_2 + \cdots + R_n$ which covers all of our n balls $B_1, B_2, \cdots, B_n$.

118. Let ABC be a triangle, and erect three rectangles ABB_1A_2, BCC_1B_2, CAA_1C_2 externally on its sides AB, BC, CA, respectively. Prove that the perpendicular bisectors of the segments A_1A_2, B_1B_2, C_1C_2 are concurrent.

119. On a line points A, B, C, D are given in this order s.t. $AB = CD$. Can we find the midpoint of BC using only a straightedge?

120. Let ABC be a triangle, and D, E, F the points where its incircle touches the sides BC, CA, AB, respectively. The parallel to AB through E meets DF at Q, and the parallel to AB through D meets EF at T. Prove that the lines CF, DE, QT are concurrent.

121. Given the triangle ABC. I and N are the incenter and the Nagel point of ABC, and r is the in radius of ABC. Prove that

$$IN = r \iff a + b = 3c \text{ or } b + c = 3a \text{ or } c + a = 3b$$

122. The centers of three circles isotomic with the Apollonian circles of triangle ABC located on a line perpendicular to the Euler line of ABC.

123. Let ABC be a triangle, and M and M' two points in its plane. Let X and X' be two points on the line BC, let Y and Y' be two points on the line CA, and let Z and Z' be two points on the line AB. Assume that

$$M'X \parallel AM; M'Y \parallel BM; M'Z \parallel CM; MX' \parallel AM'; MY' \parallel BM'; MZ' \parallel CM'.$$

Prove that the lines AX, BY, CZ concur if and only if the lines AX', BY', CZ' concur.

124. Let's call a sextuple of points (A, B, C, D, E, F) in the plane a Pascalian sextuple if and only if the points of intersection $AB \cap DE$, $BC \cap EF$ and $CD \cap FA$ are collinear. Prove that if a sextuple of points is Pascalian, then each permutation of this sextuple is Pascalian.

125. If P be any point on the circumcircle of a triangle ABC whose Lemoine point is K, show that the line PK will cut the sides BC, CA, AB of the triangle in points X, Y, Z so that

$$\frac{3}{PK} = \frac{1}{PX} + \frac{1}{PY} + \frac{1}{PZ}$$

where the segments are directed.

126. Given four distinct points A_1, A_2, B_1, B_2 in the plane, show that if every circle through A_1, A_2 meets every circle through B_1, B_2, then A_1, A_2, B_1, B_2 are either collinear or concyclic.

127. $ABCD$ is a convex quadrilateral s.t. AB and CD are not parallel. The circle through A, B touches CD at X, and a circle through C, D touches

AB at Y. These two circles intersect in U, V. Show that $AD \| BC \iff UV$ bisects XY.

128. Given R, r, construct circles with radi R, r s.t. the distance between their centers is equal to their common chord.

129. Construct triangle ABC, given the midpoint M of BC, the midpoint N of AH (H is the orthocenter), and the point A' where the incircle touches BC.

130. Let A', B', C' be the reflections of the vertices A, B, C in the sides BC, CA, AB respectively. Let O be the circumcenter of ABC. Show that the circles $(AOA'), (BOB'), (COC')$ concur again in a point P, which is the inverse in the circumcircle of the isogonal conjugate of the nine-point center.

131. Let ABC be an isosceles triangle with $\angle ACB = \dfrac{\pi}{2}$, and let P be a point inside it.

a) Show that $\angle PAB + \angle PBC \geq \min(\angle PCA, \angle PCB)$;

b) When does equality take place in the inequality above?

132. Let $\mathbb{S}$ be the set of all polygonal surfaces in the plane (a polygonal surface is the interior together with the boundary of a non-self-intersecting polygon; the polygons do not have to be convex). Show that we can find a function $f : \mathbb{S} \to (0, 1)$ such that, if $S_1, S_2, S_1 \cup S_2 \in \mathbb{S}$ and the interiors of S_1, S_2 are disjoint, then $f(S_1 \cup S_2) = f(S_1) + f(S_2)$.

133. Let $A'B'C'$ be the orthic triangle of ABC, and let A'', B'', C'' be the orthocenters of $AB'C', A'BC', A'B'C$ respectively. Show that $A'B'C', A''B''C''$ are homothetic.

134. Let O be the midpoint of a chord AB of an ellipse. Through O, we draw another chord PQ of the ellipse. The tangents in P, Q to the ellipse cut AB in S, T respectively. Show that $AS = BT$.

135. Given a parallelogram $ABCD$ with $AB < BC$, show that the circumcircles of the triangles APQ share a second common point (apart from A) as P, Q move on the sides BC, CD respectively s.t. $CP = CQ$.

136. We have an acute-angled triangle ABC, and AA', BB' are its altitudes. A point D is chosen on the arc ACB of the circumcircle of ABC. If $P = AA' \cap BD, Q = BB' \cap AD$, show that the midpoint of PQ lies on $A'B'$.

137. Let $(I), (O)$ be the incircle, and, respectiely, circumcircle of ABC. (I) touches BC, CA, AB in D, E, F respectively. We are also given three circles $\omega_a, \omega_b, \omega_c$, tangent to $(I), (O)$ in D, K (for ω_a), E, M (for ω_b), and F, N (for ω_c).

a) Show that DK, EM, FN are concurrent in a point P;

b) Show that the orthocenter of DEF lies on OP.

138. Given four points A, B, C, D in the plane and another point P, the polars of P wrt the conics passing through A, B, C, D pass through a fixed point (well, unless P is one of $AB \cap CD, AD \cap BC, AC \cap BD$, in which case the polar is fixed).

139. Prove that if the hexagon $A_1A_2A_3A_4A_5A_6$ has all sides of length ≤ 1, then at least one of the diagonals A_1A_4, A_2A_5, A_3A_6 has length ≤ 2.

140. Find the largest $k > 0$ with the property that for any convex polygon of area S and any line ℓ in the plane, we can inscribe a triangle with area $\geq kS$ and a side parallel to ℓ in the polygon.

141. Given a finite number of parallel segments in the plane s.t. for each three there is a line intersecting them, prove that there is a line intersecting all the segments.

142. Let $A_0A_1 \dots A_n$ be an n-dimensional simplex, and let r, R be its inradius and circumradius, respectively. Prove that $R \geq nr$.

143. Find those $n \geq 2$ for which the following holds:

For any $n+2$ points $P_1, \dots, P_{n+2} \in \mathbb{R}^n$, no three on a line, we can find $i \neq j \in \overline{1, n+2}$ such that P_iP_j is not an edge of the convex hull of the points P_i.

144. Given $n+1$ convex polytopes in $\mathbb{R}^n$, prove that the following two assertions are equivalent:

(a) There is no hyperplane which meets all $n+1$ polytopes;

(b) Every polytope can be separated from the other n by a hyperplane.

145. Find those convex polygons which can be covered by 3 strictly smaller homothetic images of themselves (i.e. images through homothecies with ratio in the interval $(0, 1)$).

146. Let ABC be a triangle inscribed in a circle of radius R, and let P be a point in the interior of triangle ABC. Prove that

$$\frac{PA}{BC^2} + \frac{PB}{CA^2} + \frac{PC}{AB^2} \geq \frac{1}{R}.$$

147. There is an odd number of soldiers, the distances between all of them being all distinct, which are training as follows: each one of them is looking at the one closest to them. Show that there is a soldier which nobody is looking at.

148. Let H be the orthocenter of the acute triangle ABC. Let BB' and CC' be altitudes of the triangle ($B' \in AC$, $C' \in AB$). A variable line ℓ passing through H intersects the segments $[BC']$ and $[CB']$ in M and N. The perpendicular lines of ℓ from M and N intersect BB' and CC' in P and Q. Determine the locus of the midpoint of the segment $[PQ]$.

149. Show that there are no regular polygons with more than 4 sides inscribed in an ellipse.

150. Given a cyclic $2n$-gon with a fixed circumcircle s.t. $2n-1$ of its sides pass through $2n-1$ fixed point lying on a line ℓ, show that the $2n$th side also passes through a fixed point on ℓ.

END.