LTE Challenge Problems

Amir Hossein Parvardi

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Abstract

These problems were solved by Fedja Nezarov, who helped me a lot during writing the Lifting The Exponent Lemma article.

Problem 1. Let k be a positive integer. Find all positive integers n such that $3^n | 2^n - 1$.

Solution. $2|n$ otherwise $2^n - 1 \equiv 1 \mod 3$. If $n = 2m$, we should have

$$v_3(4^m - 1) = v_3(m) + 1 \geq k,$$

i.e., $m = 3^{k-1}s$ for some $s \in \mathbb{N}$.

Problem 2. Let a, n be two positive integers and let p be an odd prime number such that

$$a^p \equiv 1 \pmod{p^n}.$$

Prove that

$$a \equiv 1 \pmod{p^{n-1}}.$$

Solution. By Fermat, $a \equiv a^p \equiv 1 \mod p$, so

$$v_p(a-1) = v_p(a^p - 1) - 1 \geq n - 1.$$

Problem 3. Show that the only positive integer value of a for which $4(a^n + 1)$ is a perfect cube for all positive integers n, is 1.

Solution. If $a > 1$, $a^2 + 1$ is not a power of 2 (because it is > 2 and either 1 or 2 modulo 4). Choose some odd prime $p|a^2 + 1$. Now, take some $n = 2m$ with odd m and notice that $v_p(4(a^n + 1)) = v_p(a^2 + 1) + v_p(m)$ but $v_p(m)$ can be anything we want modulo 3.

Problem 4. Let $k > 1$ be an integer. Show that there exists infinitely many positive integers n such that

$$n | 1^n + 2^n + 3^n + \cdots + k^n.$$

Solution. If $1 + k$ is not a power of 2, choose an odd prime $p|1 + k$ and take $n = p^m$. Then, for each j not divisible by p, we have

$$v_p(j^n + (k + 1 - j)^n) = v_p(k + 1) + v_p(n) \geq m + 1.$$

Also, if $p|j$ (and, thereby, $p|k+1-j$), then $n|p^m|p^n|j^n$ so the sum in question is divisible by $p^m = n$. If $1+k$ is a power of 2, then take an odd prime divisor p of k and repeat the above argument with $k - 1$ instead of k (the last term k^n is, obviously, not a problem)

Problem 5. Let p be a prime number, and a and n positive integers. Prove that if

$$2^p + 3^p = a^n.$$

then $n = 1$.

Solution. $2^2 + 3^2 = 13$, so assume that p is odd. Then $2^p + 3^p \equiv 2 \mod 3$, so it cannot be a square. But $v_5(2^p + 3^p) = 1 + v_5(p) \leq 2$.

Problem 6. Find all positive integers n for which there exist positive integers x, y and k such that $\gcd(x, y) = 1, k > 1$ and $3^n = x^k + y^k$.

Solution. k must be odd since the sum of 2 squares is divisible by 3 only if both squares are. If $p|x + y$, then p is odd and $v_p(3^n) = v_p(x^k + y^k) = v_p(k) + v_p(x + y)$, which means that $p = 3$, so $x + y = 3^m$ and $n = v_3(k) + m$. Now it is just cases. a) $m > 1$. Then $v_3(k) \leq k - 2$ for all $k > 1$, and $M = \max(x, y) \geq 5$ so

$$x^k + y^k \geq M^k > \frac{1}{2}3^m 5^{k-1} > 3^m 5^{k-2} \geq 3^{m+k-2} \geq 3^{m+v_3(k)} = 3^n,$$

which gives an immediate contradiction. b) $m = 1$. Then $x = 1, y = 2$ (or vice versa) and we get $3^{1+v_3(k)} = 1 + 2^k$, meaning $k \leq 2(1 + v_3(k))$ whence $v_3(k) = 1$, so $n = 2$ giving the only solution $3^2 = 1^3 + 2^3$.

Problem 7. Let x, y, p, n, k be positive integers such that n is odd and p is a prime. Prove that if $x^n + y^n = p^k$, then n is a power of p.

Solution. $x + y|x^n + y^n = p^k$, so $x + y = p^m$. Now divide x and y by the highest power of p they contain (it has to be the same). This may change k and m but not n in our condition. Then use the LTE to get $m + v_p(n) = k$, so $x^n + y^n = (x + y)p^{v_p(n)}$. If $n \neq p^{v_p(n)}$, we get $n \geq 2p^{v_p(n)} \geq 2$, so

$$M^n < x^n + y^n \leq \frac{n}{2}(x + y) \leq nM$$

and $M < n^{\frac{1}{n-1}}$, which is less than 2 for odd $n \geq 3$ but the case $M = 1$ is impossible.

Problem 8. Let p be a prime number. Solve the equation $a^p - 1 = p^k$ in the set of positive integers.

Solution. By Fermat, $a - 1 \equiv a^p - 1 \equiv 0 mod p$. a) p is odd. Then $v_p(a - 1) = k - 1$, so $a > p^{k-1}$ and, if $k > 1$, then $a^p > p^{p(k-1)} > p^{3(k-1)} > p^k$. Thus, in this case $k = 1$ and $a = (p + 1)^{1/p}$, which is never an integer (because it is strictly between 1 and 2). b) $p = 2$. Then $a^2 - 1 = 2^k$ but, unless $a = 3$, either $a - 1$ or $a + 1$ is not a power of 2. So, the only solution is $3^2 - 1 = 2^3$.

Problem 9. Find all solutions of the equation

$$(n - 1)! + 1 = n^m$$

in positive integers.

Solution. n must be 1 or prime (otherwise any nontrivial divisor d of n will divide both $(n - 1)!$ and n^m. Now, $n = 2$, $m = 1$ is a solution, $n = 1$ is not, so it suffices to consider the case when n is an odd prime. If m is odd, then $v_2(n^m - 1) = v_2(n - 1) < v_2((n - 1)!)$ if $n > 3$. $n = 3$ is not a solution, so we can consider only even m. Then

$$v_2((n - 1)!) = v_2(m) - 1 + v_2(n + 1) + v_2(n - 1)$$

or

$$v_2((n - 2)!) = v_2(n + 1) + v_2(m) - 1.$$

But the left hand side is at least $\frac{n-3}{2}$ (just count evens up to $n - 2$), so $m \geq 2^{(n-1)/2}/(n + 1)$, which is at least n for $n \geq 18$. It remains to note that $n^n > (n - 1)! + 1$ for $n > 1$. Now comes the remaining finite trial and error part: $4! + 1 = 5^2$ is good $6! + 1 = 721$ is bad $n = 11$ gives $v_2(m) = 9$ which is far too large. $n = 13$ gives $v_2(m) = 10$, which is too large too $n = 17$ gives some big $v_2(m)$ as well.

Problem 10. For some positive integer n, the number $3^n - 2^n$ is a perfect power of a prime. Prove that n is a prime.

Solution. Assume $n = ab$, $a, b > 1$. Then $3^a - 2^a = p^m$ with $m \geq 1$. Thus, by LTE, $v_p(3^n - 2^n) = m + v_p(b)$ and

$$p^{m+v_p(b)} = 3^{ab} - 2^{ab} > (3^a - 2^a)^b = p^{mb}$$

but for $b > 1$, one has $mb \geq m + b - 1 \geq m + v_p(b)$, which gives a contradiction.

Problem 11. Let m, n, b be three positive integers with $m \neq n$ and $b > 1$. Show that if prime divisors of the numbers $b^n - 1$ and $b^m - 1$ be the same, then $b + 1$ is a perfect power of 2.

Solution. I failed to find a way to use the LTE here. The way I solved it is as follows. Let $u = \gcd(m, n)$. Then $b^u - 1$ and $b^{2u} - 1$ have the same prime divisors (this uses the proof of the principle I mentioned rather than the statement itself: when you repeat going from m, n to $m - n, n$, you stop when you get two equal numbers). But then each prime dividing $b^u + 1$ has to divide $b^u - 1$ whence $b^u + 1$ is a power of 2. If u were even, the remainder of the LHS modulo 4 would be 2, so u is odd. Then $b + 1 | b^u + 1$ and must be a power of 2 too.

Problem 12. Find the highest degree k of 1991 for which 1991^k divides the number

$$1990^{1991^{1992}} + 1992^{1991^{1990}}.$$

Solution. Using $p = 11$ and $p = 181$, we get

$$v_p(1990^{1991^{1992}} + 1) = 1 + 1992 = 1993$$

and

$$v_p(1992^{1991^{1990}} - 1) = 1 + 1990 = 1991.$$

Since $v_p(a + b) = \min(v_p(a), v_p(b))$ when $v_p(a) \neq v_p(b)$, the answer is 1991.

Problem 13. Let p be a prime number and $m > 1$ be a positive integer. Show that if for some positive integers $x > 1, y > 1$ we have

$$\frac{x^p + y^p}{2} = \left(\frac{x + y}{2}\right)^m,$$

then $m = p$.

Solution. Since $\frac{x^p+y^p}{2} \geq \left(\frac{x+y}{2}\right)^p$, we must have $m \geq p$. Now, factoring out $d = \gcd(x, y)$ and writing $x = dx_1$, $y = dy_1$, we get

$$2^{m-1}(x_1^p + y_1^p) = d^{m-p}(x_1 + y_1)^m.$$

Assume that p is odd. Take any prime divisor $q | x_1 + y_1$ and let $v = v_q(x_1 + y_1)$. If q is odd, we get $v + 1 \geq v + v_q(p) \geq mv$ whence $m \leq 2$ and $p \leq 2$, giving an immediate contradiction. If $q = 2$, we get $m - 1 + v \geq mv$, so $v \leq 1$ and $x_1 + y_1 = 2$, i.e., $x = y$, which immediately implies $m = p$. If $p = 2$, we just notice that

$$\frac{x^2 + y^2}{2} < 2\left(\frac{x + y}{2}\right)^2 \leq \left(\frac{x + y}{2}\right)^3$$

if $x + y \geq 4$, so $m = 2$ is the only possibility unless $\{x, y\} = \{1, 2\}$, which is easy to outrule.

Problem 14. Find all positive integers x, y such that $p^x - y^p = 1$, where p is a prime.

Solution. The case $p = 2$ is easily done $\mod 4$ (y^2+1 has remainder 2, so $2^1 - 1^2 = 1$ is the only possibility), so assume that p is odd. Then $y + 1 = p^m$ and, by LTE, $y^p + 1 = p^{m+1}$. But $(p^m - 1)^p + 1 > p^{m+1}$ unless $p = 3$, $m = 1$, which gives the second solution.

Problem 15. Let x and y be two positive real numbers such that for each positive integer n, the number $x^n - y^n$ is a positive integer. Show that x and y are both positive integers.

Problem 16. Let x and y be two positive rational numbers such that for infinitely many positive integers n, the number $x^n - y^n$ is a positive integer. Show that x and y are both positive integers.

Solution. They are very much alike, so I'll combine the solutions. In 17, start with the observation that $x - y$ and $x + y = \frac{x^2-y^2}{x-y}$ are rational, so $x, y \in \mathbb{Q}$. Now we are in the conditions of 18. Write $x = \frac{a}{c}, y = \frac{b}{c}$ where c is the least common denominator of x, y. If $c > 1$, take any prime divisor p of c. Then $p^n | a^n - b_n$ for infinitely many n and p cannot divide a or b (otherwise it would divide them both and we could reduce the fractions). Led u be the least power such that $p | a^u - b^u$. Then all those n's are mu for some integer m and we get $n \leq v_p(a^n - b^n) \leq v_p(m) + v_p(a^u - b^u) + v_p(a^u + b^u)$ (just not to consider 2 separately). But $v_p(m)$ grows only logarithmically in n, so we get a contradiction.

Problem 17. Does there exist a positive integer n such that n has exactly 2000 prime divisors and n divides 2^n+1?

Problem 18. Note that $3^m|2^{3^m}+1$ by LTE. Thus, if $2^{3^m}+1$ has 1999 distinct prime divisors $q_1,\dots,q_{1999}>3$, $n=3^m q_1\dots q_{1999}$ will work. Note that each divisor of $2^{3^m}+1$ is also a divisor of $2^{3^{m+1}}+1$, so the set of prime divisors either grows without bound or saturates to some finite set P. In the latter case, we have $v_p(2^{3^m}+1)\le(m-m_p)+v_p(2^{3^{m_p}}+1)\le m+C_p$ where m_p is the least integer such that $p|2^{3^{m_p}}+1$. Thus, $2^{3^m}+1\le CA^m$ where A is the product of all primes in P, which is absurd.

Problem 19. Suppose that m and k are non-negative integers, and $p=2^{2^m}+1$ is a prime number. Prove that $2^{2^{m+1}p^k}\equiv 1 \pmod{p^{k+1}}$; $2^{m+1}p^k$ is the smallest positive integer n satisfying the congruence equation $2^n\equiv 1 \pmod{p^{k+1}}$.

Solution. $v_p(2^{2^m p^k}+1)=k+v_p(2^{2^m}+1)=k+1$ by LTE and $2^{2^m p^k}+1|2^{2^{m+1}p^k}-1$. Furthermore, $p|2^{2^{m+1}}-1$ but not $2^{2^m}-1$, so if $p|2^n-1$, we have $2^{m+1}|n$ (powers of 2 have only divisors that are powers of 2 themselves). If $n=Q2^{m+1}$, then

$$v_p(2^n-1)=v_p(Q)+v_p(2^{2^{m+1}}-1)=v_p(Q)+1,$$

so $v_p(Q)\ge k$.

Problem 20. Let $p\ge 5$ be a prime. Find the maximum value of positive integer k such that

$$p^k|(p-2)^{2(p-1)}-(p-4)^{p-1}.$$

Solution. Let $p-1=2^s m$. Then, since $v_p((p-2)^2+(p-4))=v_p(p^2-3p)=1$, $v_p((p-2)^2-(p-4))=v_p(p^2-5p+8)=0$, and $v_p(2^{s-1}m)=0$, we get

$$\begin{aligned} v_p((p-2)^{2(p-1)}-(p-4)^{p-1}) &= v_p((p-2)^2-(p-4)^2) \\ &= v_p((p-2)^2+(p-4))+v_p((p-2)^2-(p-4))=1. \end{aligned}$$

Problem 21. Find all positive integers a,b which are greater than 1 and

$$b^a|a^b-1.$$

Solution. Let p be the least prime divisor of b. Let m be the least positive integer for which $p|a^m-1$. Then $m|b$ and $m|p-1$, so any prime divisor of m divides b and is less than p. Thus, not to run into a contradiction, we must have $m=1$. Now, if p is odd, we have $av_p(b)\le v_p(a-1)+v_p(b)$, so

$$a-1\le(a-1)v_p(b)\le v_p(a-1),$$

which is impossible. Thus $p=2$, b is even, a is odd and $av_2(b)\le v_2(a-1)+v_2(a+1)+v_2(b)-1$ whence $a\le(a-1)v_2(b)+1\le v_2(a-1)+v_2(a+1)$, which is possible only if $a=3$, $v_2(b)=1$. Put $b=2B$ with odd B and rewrite the condition as $2^3B^3|3^{2B}-1$. Let q be the least prime divisor of B (now, surely, odd). Let n be the least positive integer such that $q|3^n-1$. Then $n|2B$ and $n|q-1$ whence n must be 2 (or B has a smaller prime divisor), so $q|3^2-1=8$, which is impossible. Thus $B=1$.

Problem 22. Let a,b be distinct real numbers such that the numbers

$$a-b,\ a^2-b^2,\ a^3-b^3,\dots$$

Are all integers. Prove that a,b are both integers.

Solution. If $a^2=b^2$, then $a=-b$ is either an integer or a half-integer, the latter case being impossible because then $a^3-b^3=2a^3$ is not an integer. Otherwise, a^2 and b^2 are integers by problem 17, so $a+b=a^2-b^2a-b$ is rational, so a,b are rational but all rational square roots of integers are integer.

Problem 23. Find all quadruples of positive integers (x,r,p,n) such that p is a prime number, $n,r>1$ and $x^r-1=p^n$.

Solution. Assume p is odd. Let $x-1=p^m$. Then $n=m+v_p(r)$, so $p^{m+v_p(r)}=(p^m+1)^r-1\geq p^{mr}$, implying $m+v_p(r)\geq mr\geq m+r-1$ and $v_p(r)\geq r-1$, which is impossible for $r>1$. Thus $p=2$. If r is odd, we get $n=v_2(x^r-1)=v_2(x-1)$ so $r=1$, which is outruled. Thus, r is even and $n=v_2(x-1)+v_2(x+1)+v_2(r)-1$. If $m=1$, we get the usual $3^2-1=2^3$. Otherwise $v_2(x+1)=1$, so $n=m+v_2(r)$ and we can finish just as we started.

Problem 24. Let $a>b>1$ be positive integers and b be an odd number, let n be a positive integer. If $b^n\mid a^n-1$, then show that $a^b>\frac{3^n}{n}$.

Solution. Let P be the set of all prime divisors of b. For $p\in P$, let s_p be the least integer such that $p|a^{s_p}-1$. We have $s_p|n$, $s_p|p-1$ and $n\leq v_p(a^{s_p}-1)+v_p(n/s_p)$. Now note that $b\geq\prod_{p\in P}p>\prod_{p\in P}s_p=S$ and that $a^{s_p}-1|a^S-1$ for all $p\in P$. Thus,

$$a^b>a^S-1\geq\prod_{p\in P}p^n\prod_{p\in P}p^{-v_p(n)}\geq 3^n/n.$$

Problem 25. Let p be a prime number, $p\neq 3$, and integers a,b such that $p\mid a+b$ and $p^2\mid a^3+b^3$. Prove that $p^2\mid a+b$ or $p^3\mid a^3+b^3$.

Solution. If $p|a,b$, then $p^3|a^3+b^3$. Otherwise LTE applies and $v_p(a+b)=v_p(a^3+b^3)\geq 2$.

Problem 26. Let m and n be positive integers. Prove that for each odd positive integer b there are infinitely many primes p such that $p^n\equiv 1\pmod{b^m}$ implies $b^{m-1}\mid n$.

Solution. Let q be any prime divisor of b. If $v_q(p^{q-1}-1)=1$, we get $mv_q(b)\leq v_q(n)+1$ (using, as before, that the minimal s satisfying $q|p^s-1$ divides both n and $q-1$) and $v_q(n)\geq mv_q(b)-1\geq(m-1)v_q(b)$. Thus we just need to show that there are infinitely many primes p satisfying the condition $v_q(p^{q-1}-1)=1$ for all prime divisors of b. If one knows Dirichlet, it is simple: just consider $p\equiv 1+q \mod q^2$ for all primes q dividing p. If not, I'm stuck for now.

Problem 27. Determine all integers $n>1$ such that

$$\frac{2^n+1}{n^2}$$

is an integer.

Solution. Let p be a prime divisor on n, $v=v_p(n)$, $Q=p^{-v}n$. We have $p^{2v}|2^{p^v}Q+1$. We also know that $2^{p^v}\equiv 2 \mod p$ by Fermat. Thus $p|2^Q+1$ and, by LTE $p^v|2^Q+1$. Let m be the least positive integer such that $p^v|2^m+1$. Then $m|Q,p^{v-1}(p-1)$. If p is the least prime divisor of n, we conclude that $m=1$ (because Q consists of primes larger than p) and $p=3$, $v=1$. Now, if p is the second smallest prime divisor in n, then m can be only 3 (the only factor in Q that can occur in $p-1$). But $2^3+1=9$ has no prime divisors greater than 3 and we are stuck in our attempt to acsend. Thus, the answer is $n=1$ or $n=3$.

Problem 28. Find all positive integers n such that

$$\frac{2^{n-1}+1}{n}$$

is an integer.

Solution. Let $n=p_1^{v_1}\dots {p_m}^{v^m}$. As before, put $Q_j=p_j^{-v_j}n$. Since $p_j|2^{p_j^{v_j}Q_j-1}+1$ and $2^{p_j^{v_j}}\equiv 2\pmod{p_j}$ by Fermat, we get $p_j|2^{Q_j-1}+1$. Let m_j be the least positive integer such that $p_j|2^{m_j}+1$. Then m_j divides both Q_j-1 and $\frac{p_j-1}{2}$ and Q_j-1 is an odd multiple of m_j. But the power of 2 in $Q_j-1=\prod_{k\neq j}(1+(p_k-1))-1$ is at least $\min_{k\neq j}v_2(p_k-1)$ while $v_2(m_j)\leq v_2(p_j-1)-1$, so choosing p_j so that $v_2(p_j-1)$ is minimal, we'll get a contradiction.

Problem 29. Find all primes p,q such that $\dfrac{(5^p-2^p)(5^q-2^q)}{pq}$ is an integer.

Solution. Assume $p \le q$. If $5^p - 2^p \equiv 5 - 2 = 3 \mod p$ by Fermat, so if $p | 5^p - 2^p$, then $p = 3$. In particular, if $p = q$, then $p = q = 3$. If $p < q$ and $p | 5^q - 2^q$, then $p | 5^{p-1} - 2^{p-1}$ by Fermat, so the least $m > 0$ such that $p | 5^m - 2^m$ must divide both $p - 1$ and q, i.e., $m = 1$, so $p = 3$ again. Now, either $q = 3$, or $q | 5^3 - 2^3 = 117 = 9 \cdot 13$, so $(3, 13)$ is the only other solution.

Problem 30. For some natural number n let a be the greatest natural nubmer for which $5^n - 3^n$ is divisible by 2^a. Also let b be the greatest natural number such that $2^b \le n$. Prove that $a \le b + 3$.

Solution. If n is odd, $a = 1$ and there is nothing to prove. If n is even, $a = v_2(5^n - 3^n) = v_2(5 - 3) + v_2(5 + 3) + v_2(n) - 1 = 3 + v_2(n)$. But, clearly, $b \ge v_2(n)$.

Problem 31. Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ and every prime p, the number $f(m + n)$ is divisible by p if and only if $f(m) + f(n)$ is divisible by p.

Solution. We start with the (well-known?) observation that every subset S of positive integers that is closed under addition is an eventual arithmetic progression. More precisely, there exists $d \ge 1$ (which actually is just the greatest common divisor of the elements of S) and N such that for $n \ge N$ we have $n \in S \iff d | n$. Now, for prime p, let $S_p = \{n : p | f(n)$. Let d_p be the corresponding difference. Thus $p | f(n) \iff d_p | n$ if n is large. Now take any n and take a huge m divisible by d_p (so $p | f(m)$). Then $d_p | m \iff d_p | n + m \iff p | f(n + m) \iff p | f(n)$, so the equivalence holds without the requirement that n is large too. The next step is to show that the remainder of n modulo d_p determines the remainder of $f(n)$ modulo p and vice versa. Let's take $A = \{1, 2, \dots d_p - 1$. For every n there exists the unique $a \in A$ such that $d_p | n + a$ determined by $n \mod d_p$. But then $p | f(n) + f(a)$ determining $f(n) \mod p$ uniquely. Conversely, let $B = \{b_1, \dots, b_p\}$ so that $f(b_j) = j$ (here is where we use surjectivity). Then once we know $f(n) \mod p$, we know j such that $p | f(n) + j \equiv f(n + b_j) \mod p$ whence we know that $d_p | n + b_j$. This one-to one correspondence between remainders implies that $p = d_p$ and that

$$p | n - m \iff p | f(n) - f(m).$$

In particular, $f(1) = 1$ and if $f(n + 1) - f(n) = \pm 1$ for all n. Now take a huge odd prime P and note that we can have $P | f(P)$ only if all ± 1 up to P are actually 1. Since P is arbitrarily large, $f(n) = n$ for all n.

Problem 32. Determine all sets of non-negative integers x, y and z which satisfy the equation

$$2^x + 3^y = z^2.$$

Solution. This is just a casework: If $x = 0$, we get $3^y = (z - 1)(z + 1)$, but 1 and 3 are the only two powers of 3 differing by 2, so $y = 1$, $z = 2$. If $y = 0$, then $2^x = (z - 1)(z + 1)$ giving $z = 3$, $x = 3$ in the same way. If $x, y > 0$, then x is even (z^2 cannot be $2 \mod 3$) whence y is even (z^2 cannot be $3 \mod 4$), so, letting $x = 2X, y = 2Y$, we get $3^{2Y} = (z - 2^X)(z + 2^X)$. Thus, we must have $z = 2^X + 1$ and $3^{2Y} - 1 = 2^{X+1}$. But then $X + 1 = v_2(Y) + 3$ by the LTE, so $2Y \ge 2^{X-1} > X + 1$ if $X \ge 4$. $X = 1$ gives nothing, $X = 2$ gives $Y = 1$, and $X = 3$ gives nothing.

Problem 33. Find all positive integer solutions of equation $x^{2009} + y^{2009} = 7^z$.

Solution. $7 | 2009$ so $7 | x + y$ by Fermat. Removing the highest possible power of 7 from x, y, we get

$$v_7(x^{2009} + y^{2009}) = v_7(x + y) + v_7(2009) = v_7(x + y) + 2,$$

so $x^{2009} + y^{2009} = 49(x + y)$ but the left hand side is much larger than the right hand one if $\max(x, y) > 1$.

Problem 34. Let n be an odd positive integer. Prove that $((n - 1)^n + 1)^2$ divides $n(n - 1)^{(n-1)^n+1} + n$.

Solution. $n | (n - 1)^n + 1$, so for every $p | (n - 1)^n + 1$, we have

$$\begin{aligned} v_p((n-1)^{(n-1)^n+1} + 1) \\ &= v_p((n-1)^n + 1) + v_p\left(\frac{(n-1)^n + 1 + 1}{n}\right) \\ &= 2v_p((n-1)^n + 1) - v_p(n), \end{aligned}$$

which is just what we need in terms of prime divisors.

Problem 35. Find all positive integers n such that $3^n - 1$ is divisible by 2^n.

Solution. $n \le v_2(3^n - 1) \le 3 + v_2(n)$, so $n \le 4$. $1, 2, 4$ work, 3 doesn't.

Problem 36. Let p be a prime and a, b be positive integers such that $a \equiv b \pmod{p}$. Prove that if $p^x \| a - b$ and $p^y \| n$, then $p^{x+y} \| a^n - b^n$.

Solution. LTE, odd prime case.

Problem 37. Let $a, n \ge 2$ be two integers, which have the following property: there exists an integer $k \ge 2$, such that n divides $(a-1)^k$. Prove that n also divides $a^{n-1} + a^{n-2} + \cdots + a + 1$.

Solution. If some prime $p|n$, then $p|a-1$ and $v_p(a^n - 1) \ge v_p(a-1) + v_p(n)$, which is a restatement of what we need in terms of prime divisors.

Problem 38. Find all positive integers a such that $\frac{5^a+1}{3^a}$ is a positive integer.

Solution. a must be odd (otherwise the numerator is $2 \mod 3$). Then $a \le v_3(5^a + 1) = 1 + v_3(a)$ giving $a = 1$ as the only solution.

Problem 39. Let a, b, n be positive integers such that $2^\alpha \| \frac{a^2-b^2}{2}$ and $2^\beta \| n$ (with $\beta \ge 1$). Prove that $2^{\alpha+\beta} \| a^n - b^n$.

Solution. LTE, even prime case.