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George A. Anastassiou

Advances on Fractional Inequalities

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Advances on Fractional Inequalities

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George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
USA
ganastass@memphis.edu

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To my wife Koula and my daughters
Angela and Peggy

The measure of success for a person is the magnitude of his/her ability to convert negative conditions to positive ones and achieve goals

—The author

Preface

This short monograph is a spin-off of the author's "Fractional Differentiation Inequalities," a research monograph published by Springer, New York, 2009. It continues and complements the earlier book to various interesting and important directions.

In this short monograph we use primarily the Caputo fractional derivative, as the most important in applications, and we present first fractional differentiation inequalities of Opial type where we involve the so-called balanced fractional derivatives. We continue with right and mixed fractional differentiation Ostrowski inequalities in the univariate and multivariate cases.

Then we present right and left, as well as mixed, Landau fractional differentiation inequalities in the univariate and multivariate cases.

The inequalities are given for various norms.

Fractional differentiation inequalities are by themselves an important and great mathematical topic for research. Furthermore they have many applications, the most important ones are in establishing uniqueness of solution in fractional differential equations and systems and in fractional partial differential equations. Also they provide upper bounds to the solutions of the above equations.

In this brief monograph we give several applications.

Each chapter is self-contained and can be read independently of the others and several graduate courses and seminars can be taught out of this monograph.

The final preparation of this book took place in Memphis, USA, during 2010–2011.

Fractional calculus has become very useful over the last 40 years due to its many applications in almost all applied sciences. We see now applications in acoustic wave propagation in inhomogeneous porous material, diffusive transport, fluid flow, dynamical processes in self-similar structures, dynamics of earthquakes, optics, geology, viscoelastic materials, biosciences, bioengineering, medicine, economics, probability and statistics, astrophysics, chemical engineering, physics, splines, tomography, fluid mechanics, electromagnetic waves, nonlinear control, signal processing, control of power electronics, converters, chaotic dynamics, polymer science, proteins, polymer physics, electrochemistry, statistical physics, rheology, thermodynamics, neural networks, etc. By now almost all fields of research in science and engineering use fractional calculus as better describing them.

So as expected this book being a part of fractional calculus is useful for researchers and graduate students for research, seminars and advanced graduate courses, in pure and applied mathematics, engineering, and all other applied sciences.

I would like to thank my family members for their support and for their tolerance to accept my continuous mathematics habit. Also I am greatly indebted and thankful to my Ph.D. student Razvan Mezei for the heroic and fantastic technical preparation of the manuscript in a very short time.

Memphis, TN, USA George A. Anastassiou

Contents

Chapter 1
Opial-Type Inequalities for Balanced Fractional Derivatives

Here we study L_p, $p > 0$, fractional Opial-type inequalities subject to high-order boundary conditions. They involve the right and left Caputo, Riemann–Liouville fractional derivatives. These derivatives are blended together into the balanced Caputo, Riemann–Liouville, respectively, fractional derivative. We give an application to a balanced fractional boundary value problem by proving uniqueness of the solution. This chapter relies on [7].

1.1 Introduction

The presented material here is related to [5].

This chapter is motivated by the famous theorem of Z. Opial [14], 1960, which follows.

Theorem 1.1. *Let* $x(t) \in C^1([0,h])$ *be such that* $x(0) = x(h) = 0$, *and* $x(t) > 0$ *in* $(0,h)$. *Then*

$$\int_0^h \left| x(t)x'(t) \right| \mathrm{d}t \le \frac{h}{4} \int_0^h (x'(t))^2 \mathrm{d}t. \qquad (*)$$

In $(*)$, *the constant* $\frac{h}{4}$ *is the best possible. Inequality* $(*)$ *holds as equality for the optimal function*

$$x(t) = \begin{cases} ct, & 0 \le t \le h/2 \\ c(h-t), & \frac{h}{2} \le t \le h, \end{cases}$$

where $c > 0$ *is an arbitrary constant.*

To prove easily Theorem 1.1, Beesack [8] gave the following well-known Opial-type inequality which is used very commonly.

This is another inspiration to this chapter.

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Theorem 1.2. *Let $x(t)$ be absolutely continuous in $[0,a]$, and $x(0)=0$.*
Then

$$\int_0^a \left|x(t)x'(t)\right| \mathrm{d}t \leq \frac{a}{2}\int_0^a (x'(t))^2 \mathrm{d}t. \qquad (**)$$

Inequality $(**)$ is sharp, it is attained by $x(t)=ct$, $c>0$ is an arbitrary constant.

See also [13].

Opial-type inequalities are used a lot in proving uniqueness of solutions to differential equations and also to give upper bounds to their solutions.

They have become a great subject of intensive research and there exists a great literature about them.

Typical and important sources on them are the monographs [1,3].

1.2 Background

We need

Definition 1.3 ([4,10–12,15]). Let $f \in AC^m([a,b])$ (space of functions from $[a,b]$ into $\mathbb{R}$ with $m-1$ derivative absolutely continuous function on $[a,b]$, $m \in \mathbb{N}$, where $m=\lceil\alpha\rceil$ ($\lceil\cdot\rceil$ is the ceilling of the number), $\alpha>0$.

We define the right Caputo fractional derivatives of order $\alpha>0$ by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)}\int_x^b (\xi-x)^{m-\alpha-1} f^{(m)}(\xi)\mathrm{d}\xi. \qquad (1.1)$$

We set $D_{b-}^{0} f(x)=f(x)$, $\forall x \in [a,b]$.

We mention the right Caputo fractional Taylor formula with integral remainder.

Theorem 1.4 ([4]). *Let $f \in AC^m([a,b]), x \in [a,b], \alpha>0, m=\lceil\alpha\rceil$. Then*

$$f(x)=\sum_{k=0}^{m-1}\frac{f^{(k)}(b)}{k!}(x-b)^k + \frac{1}{\Gamma(\alpha)}\int_x^b (\xi-x)^{\alpha-1} D_{b-}^{\alpha} f(\xi)\mathrm{d}\xi. \qquad (1.2)$$

We make

Definition 1.5 ([9, p. 38]). Let $f \in AC^m([a,b]), m \in \mathbb{N}$,where $m=\lceil\alpha\rceil, \alpha>0$. We define the left Caputo fractional derivative of order $\alpha>0$ by

$$D_{*a}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\mathrm{d}t. \qquad (1.3)$$

We set $D_{*a}^{0} f(x)=f(x), \forall x \in [a,b]$.

We mention the left Caputo fractional Taylor formula with integral remainder.

Theorem 1.6 ([9, p. 40]). *Let* $f \in AC^m([a,b]), m \in \mathbb{N}$,*where* $m = \lceil \alpha \rceil, \alpha > 0$, $x \in [a,b]$.

Then

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} D_{*a}^{\alpha} f(\tau) \mathrm{d}\tau. \quad (1.4)$$

Definition 1.7 ([5]). Let $f \in AC^m([a,b]), m \in \mathbb{N}$,where $m = \lceil \alpha \rceil, \alpha > 0, x \in [a,b]$.

We define the balanced Caputo fractional derivative,

$$D^{\alpha} f(x) := \begin{cases} D_{b-}^{\alpha} f(x), \text{ for } \frac{a+b}{2} \leq x \leq b, \\ D_{*a}^{\alpha} f(x), \text{ for } \alpha \leq x < \frac{a+b}{2}. \end{cases} \quad (1.5)$$

1.3 Main Results

We give

Theorem 1.8. *Let* $\alpha > 0, m = \lceil \alpha \rceil$, $x \in [a,b], f \in AC^m([a,b]), f^{(k)}(a) = f^{(k)}(b) = 0, k = 0, 1, ..., m-1; 0 < p < 1$ *and* $q < 0 : \frac{1}{p} + \frac{1}{q} = 1$.

Assume $D_{*a}^{\alpha} f \neq 0$ *on* $\left[a, \frac{a+b}{2}\right)$ *and of fixed sign, a.e.;* $D_{b-}^{\alpha} f \neq 0$ *on* $\left[\frac{a+b}{2}, b\right]$ *and of fixed sign, a.e.*

Then

$$\int_a^b |f(w)| \, |D^{\alpha} f(w)| \, \mathrm{d}w$$

$$\geq \left(\frac{(b-a)^{\left((\alpha-1)+\frac{2}{p}\right)}}{\left(2^{\left(\alpha+\frac{1}{p}\right)} \Gamma(\alpha)((p(\alpha-1)+1)(p(\alpha-1)+2)\right)^{1/p}} \right)$$

$$\times \left[\left(\int_a^{\frac{a+b}{2}} |D_{*a}^{\alpha} f(t)|^q \, \mathrm{d}t \right)^{2/q} + \left(\int_{\frac{a+b}{2}}^b |D_{b-}^{\alpha} f(t)|^q \, \mathrm{d}t \right)^{2/q} \right]. \quad (1.6)$$

Proof. Here $\alpha > 0, \mathbb{N} \ni m = \lceil \alpha \rceil, x \in [a,b], f \in AC^m([a,b]), f^{(k)}(a) = f^{(k)}(b) = 0$, for $k = 0, 1, ..., m-1$; $0 < p < 1$ and $q : \frac{1}{p} + \frac{1}{q} = 1$. Then by (1.4),

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} D_{*a}^{\alpha} f(t) \mathrm{d}t, \quad \forall x \in [a,b].$$

Also by (1.2),

$$f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} D_{b-}^{\alpha} f(t)\mathrm{d}t, \quad \forall x\in[a,b].$$

Assume $D_{*a}^{\alpha} f\neq 0$ over $\left[\alpha,\frac{a+b}{2}\right)$ and of fixed sign, a.e., and assume $D_{b-}^{\alpha} f\neq 0$ over $\left[\frac{a+b}{2},b\right]$ and of fixed sign, a.e. Then we have

$$|f(w)|=\frac{1}{\Gamma(\alpha)}\int_a^w (w-t)^{\alpha-1}\left|D_{*a}^{\alpha} f(t)\right|\mathrm{d}t, \quad \forall\, w\in\left[a,\frac{a+b}{2}\right),$$

and

$$|f(w)|=\frac{1}{\Gamma(\alpha)}\int_w^b (t-w)^{\alpha-1}\left|D_{b-}^{\alpha} f(t)\right|\mathrm{d}t, \quad \forall w\in\left[\frac{a+b}{2},b\right].$$

Therefore by Hölder's inequality $(q=p/(p-1)<0)$ we obtain

$$|f(w)|\geq\frac{1}{\Gamma(\alpha)}\left(\int_a^w (w-t)^{p(\alpha-1)}\mathrm{d}t\right)^{1/p}\left(\left(\int_a^w \left|D_{*a}^{\alpha} f(t)\right|^q\mathrm{d}t\right)\right)^{1/q}.$$

Hence

$$|f(w)|\geq\frac{1}{\Gamma(\alpha)}\frac{(w-a)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}}\left(\int_a^w \left|D_{*a}^{\alpha} f(t)\right|^q\mathrm{d}t\right)^{1/q}, \quad \forall w\in\left[a,\frac{a+b}{2}\right).$$

Similarly we derive

$$|f(w)|\geq\frac{1}{\Gamma(\alpha)}\frac{(b-w)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}}\left(\int_w^b \left|D_{b-}^{\alpha} f(t)\right|^q\mathrm{d}t\right)^{1/q}, \quad \forall w\in\left[\frac{a+b}{2},b\right].$$

Set

$$z_1(w):=\int_a^w \left|D_{*a}^{\alpha} f(t)\right|^q\mathrm{d}t,\ z_1(a)=0,$$

so that $z_1'(w)=\left|D_{*a}^{\alpha} f(w)\right|^q$ and $\left|D_{*a}^{\alpha} f(w)\right|=(z_1'(w))^{1/q}$, all $a\leq w<\frac{a+b}{2}$. Set

$$z_2(w):=\int_w^b \left|D_{b-}^{\alpha} f(t)\right|^q\mathrm{d}t, \quad z_2(b)=0,$$

so that $z_2'(w)=-\left|D_{b-}^{\alpha} f(w)\right|^q$ and $\left|D_{b-}^{\alpha} f(w)\right|=(-z_2'(w))^{1/q}$, $\frac{a+b}{2}\leq w\leq b$. Therefore

$$|f(w)|\left|D_{*a}^{\alpha} f(w)\right|\geq\frac{1}{\Gamma(\alpha)}\frac{(w-a)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}}\left(z_1(w)z_1'(w)\right)^{1/q},$$

all $a \le w < \frac{a+b}{2}$. Hence

$$\int_a^{\frac{a+b}{2}} |f(w)|\,|D_{*a}^{a} f(w)|\,\mathrm{d}w \ge \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}}$$

$$\int_a^{\frac{a+b}{2}} (w-a)^{\alpha-1+\frac{1}{p}} (z_1(w) z_1'(w)^{1/q} \mathrm{d}w$$

$$\ge \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}} \left(\int_a^{\frac{a+b}{2}} (w-a)^{p(\alpha-1)+1}\right)^{1/p} \left(\int_a^{\frac{a+b}{2}} z_1(w) z_1'(w) \mathrm{d}w\right)^{1/q}$$

$$= \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}(p(\alpha-1)+2)^{1/p}} \left(\frac{b-a}{2}\right)^{(\alpha-1)+(2/p)} \left(\frac{z_1^2(w)}{2}\bigg|_a^{\frac{a+b}{2}}\right)^{1/q}$$

$$= \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}(p(\alpha-1)+2)^{1/p}} \left(\frac{b-a}{2}\right)^{\left(\alpha-1+\frac{2}{p}\right)} \left(\frac{z_1^2\left(\frac{a+b)}{2}\right)}{2}\right)^{1/q}.$$

Therefore

$$\int_a^{\frac{a+b}{2}} |f(w)|\,|D_{*a}^{\alpha} f(w)|\,\mathrm{d}w \ge c^* \left(\int_a^{\frac{a+b}{2}} |D_{*a}^{\alpha} f(t)|^q\,\mathrm{d}t\right)^{2/q}, \tag{1.7}$$

where

$$c^* := \frac{1}{2^{1/q}\Gamma(\alpha)\left((p(\alpha-1)+1)\,(p(\alpha-1)+2)\right)^{1/p}} \left(\frac{b-a}{2}\right)^{\left(\alpha-1+\frac{2}{p}\right)}. \tag{1.8}$$

Similarly

$$|f(w)|\,|D_{b-}^{\alpha} f(w)| \ge \frac{1}{\Gamma(\alpha)} \frac{(b-w)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}} (z_2(w)(-z_2'(w)))^{1/q}$$

and

$$\int_{\frac{a+b}{2}}^{b} |f(w)|\,|D_{b-}^{\alpha} f(w) \mathrm{d}w| \ge \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}}$$

$$\times \int_{\frac{a+b}{2}}^{b} (b-w)^{(p(\alpha-1)+1)/p} \left(z_2\,(w)\left(-z_2'\,(w)\right)\right)^{1/q} \mathrm{d}w$$

$$\ge \frac{1}{2^{1/q}\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}(p(\alpha-1)+2)^{1/p}}$$

$$\times \left(\frac{b-a}{2}\right)^{(\alpha-1)+\frac{2}{p}} \left(\int_{\frac{a+b}{2}}^{b} |D_{b-}^{\alpha} f(t)|^q\,\mathrm{d}t\right)^{2/q},$$

i.e.,

$$\int_{\frac{a+b}{2}}^{b} |f(w)|\,|D_{b-}^{\alpha} f(w)|\,\mathrm{d}w \geq c^* \left(\int_{\frac{a+b}{2}}^{b} |D_{b-}^{\alpha} f(t)|^q\,\mathrm{d}t \right)^{2/q}. \tag{1.9}$$

adding (1.7) and (1.9) we get (1.6). □

We present and need

Theorem 1.9. *Let $x \leq s \leq 0$ and $f \in L_\infty([x, 0]), r > 0$. Define*

$$G(s) = \int_s^0 (t-s)^r f(t)\mathrm{d}t. \tag{1.10}$$

Then there exists

$$G'(s) = -r \int_s^0 (t-s)^{r-1} f(t)\mathrm{d}t. \tag{1.11}$$

Proof. Fix $s_0 \in [x, 0]$ and observe that

$$G(s_0) = \int_{s_0}^0 (t-s_0)^r f(t)\mathrm{d}t = \int_x^0 \chi_{[s_0,0]}(t)(t-s_0)^r f(t)\mathrm{d}t.$$

We call

$$g(s,t) := \chi_{[s,0]}(t)(t-s)^r f(t),$$

which is a Lebesgue integrable function for every $s \in [x, 0]$. That is, $g(s_0, t) = \chi_{[s_0,0]}(t)(t-s_0)^r f(t)$, all $t \in [x, 0]$ and $G(s_0) = \int_x^0 g(s_0, t)\mathrm{d}t$. We would like to study if there exists

$$\frac{\partial g(s_0,t)}{\partial s} = f(t) \left(\lim_{h \to 0} \frac{\chi(t)_{[s_0+h,o]}(t-s_0-h)^r - \chi(t)_{[s_0,0]}(t-s_0)^r}{h} \right). \tag{1.12}$$

We distinguish the following cases:

(1) Let $x \leq t < s_0$, then there exists small enough $h > 0$ such that $t < s_0 \pm h$. That is,

$$\chi_{[s_0 \pm h,0]}(t) = \chi_{[s_0,0]}(t) = 0.$$

Hence, there exists

$$\frac{\partial g(s_0,t)}{\partial s} = 0, \text{ all } t : x \leq t < s_0.$$

(2) Let $s_0 < t \leq 0$, then there exists small enough $h > 0$ such that $t > s_0 \pm h$. That is,

$$\chi_{[s_0 \pm h,0]}(t) = \chi_{[s_0,0]}(t) = 1.$$

In that case

$$\frac{\partial g(s_0,t)}{\partial s} = f(t)\left(\lim_{h\to 0}\frac{(t-(s_0+h))^r-(t-s_0)^r}{h}\right)$$
$$= -r(t-s_0)^{r-1}f(t)$$

exists for almost all $t : s_0 < t \le 0$.

(3) Let $t = s_0$. Then we observe that

$$\frac{\partial g_+(s_0,s_0)}{\partial s} = f(s_0)\left(\lim_{h\to 0+}\frac{0\cdot(-h)^r - 1\cdot 0^r}{h}\right) = 0.$$

Also we get

$$\frac{\partial g_-(s_0,s_0)}{\partial s} = f(s_0)\left(\lim_{h\to 0-}\frac{1\cdot(-h)^r - 1\cdot 0^r}{h}\right)$$
$$= f(s_0)\left(\lim_{h\to 0-}\frac{(-h)^r}{h}\right) = -f(s_0)\left(\lim_{h\to 0-}\frac{(-h)^r}{(-h)}\right)$$
$$= -f(s_0)\left(\lim_{h\to 0-}(-h)^{r-1}\right) = -f(s_0)\left(\lim_{h\to 0+}h^{r-1}\right).$$

The last limit does not exist if $0 < r < 1$, equals $-f(s_0)$ if $r = 1$ and may not exist, and equals 0 if $r > 1$.

In general as a conclusion we get that $\frac{\vartheta g(s_0,t)}{\vartheta s}$ exists for almost all $t \in [x,0]$.

Next we define the difference quotient at s_0:

$$D_{s_0}(h,t) := f(t)\left(\frac{\chi(t)_{[s_0+h,0]}(t-s_0-h)^r - \chi_{[s_0,0]}(t)(t-s_0)^r}{h}\right),$$

for $h \neq 0$, and $D_{s_0}(0,t) := 0$.

Again we distinguish the following cases:

(1) Let $x \le t < s_0$, then there exists small enough $h > 0$ such that $t < s_0 \pm h$. Clearly then $D_{s_0}(h,t) = 0$.

(2) Let $s_0 < t \le 0$, then there exists small enough h > 0 such that $t > s_0 \pm h$. In that case

$$D_{s_0}(\pm h,t) = f(t)\left(\frac{(t-(s_0 \pm h)^r - (t-s_0)^r}{\pm h}\right).$$

Call $\rho := t - s_0 > 0$; clearly $0 < \rho \le |x|$.

Define

$$\varphi(h) := \frac{(\rho - h)^r - \rho^r}{h} = \frac{(t-s_0-h)^r-(t-s_0)^r}{h}, \text{ for } h \text{ close to zero}, r > 0.$$

That is,

$$D_{s_0}(h,t) = f(t)\varphi(h).$$

If $r = 1$, then $\varphi(h) = -1$ and $D_{s_0}(h,t) = -f(t)$.

We now treat the following subcases:

$(2_{(i)})$ If $r > 1$ and $|h|$ small, then by mean value theorem we get

$$|D_{s_0}(h,t)| = |f(t)|\,|\varphi(h)| \le |f(t)|\, r2^{r-1}\,|x|^{r-1}.$$

That is, for $r \ge 1$ and small $|h|$ we derive

$$|D_{s_0}(h,t)| \le r2^{r-1}\,|x|^{r-1}\,|f(t)|.$$

$(2_{(ii)})$ If $0 < r < 1$ and $|h|$ small we get the following:
The function $\gamma(\rho) = \rho^r,\ 0 \le \rho \le |x|$ is concave and increasing. Let $h > 0$, then

$$|\varphi(h)| = \frac{\rho^r - (\rho-h)^r}{\rho-(\rho-h)} < \rho^{r-1} = (r-s_0)^{r-1},$$

and for $h < 0$, again

$$\varphi(h) = \frac{(\rho-h)^r - \rho^r}{(\rho-h)-\rho} < \rho^{r-1} = (t-s_0)^{r-1}.$$

Therefore we obtain

$$|D_{s_0}(h,t)| \le |f(t)|\,(t-s_0)^{r-1},$$

for $0 < r < 1$ and $|h|$ small.

(3) Case of $t = s_0$, then

$$D_{s_0}(h,s_0) = -f(s_0)\,|h|^{r-1},\ \text{for } h < 0,$$

and

$$D_{s_0}(h,s_0) = 0,\ \text{for } h > 0.$$

So, if $r \ge 1$ we obtain

$$|D_{s_0}(h,s_0)| \le |f(s_0)|\,|x|^{r-1},$$

for small $|h|$.
If $0 < r < 1$, then for small $|h|$ with $h < 0$, the function $D_{s_0}(h,s_0)$ may be unbounded.

In conclusion we get:

(I) For $r \geq 1$, that

$$|D_{s_0}(h,t)| \leq r2^{r-1}|x|^{r-1}\|f\|_\infty < +\infty, \text{ for almost all } t \in [x,0].$$

(II) For $0 < r < 1$, that
$|D_{s_0}(h,t)| \leq \lambda(t)$, for almost all $t \in [x,0]$, where

$$\lambda(t) := \begin{cases} 0, & x \leq t \leq s_0, \\ |f(t)|(t-s_0)^{r-1}, & s_0 < t \leq 0. \end{cases}$$

Clearly λ is integrable on $[x,0]$.
Then by Theorem 24.5, pp. 193–194 of [2], we get that $\frac{\partial g(s_0,\cdot)}{\partial s}$ defines an integrable function and there exists

$$\begin{aligned} G'(s_0) &= \int_x^0 \frac{\partial g(s_0,t)}{\partial s}\mathrm{d}t \\ &= -r\int_{s_0}^0 (t-s_0)^{r-1} f(t)\mathrm{d}t + \int_x^{s_0} 0\mathrm{d}t \\ &= -r\int_{s_0}^0 (t-s_0)^{r-1} f(t)\mathrm{d}t. \end{aligned}$$

That proves the claim. □

We use

Theorem 1.10 ([3] **Theorem 7.7, p. 117).** *Let* $0 \leq s \leq x$ *and* $f \in L_\infty([0,x])$, $r > 0$. *Define*

$$F(s) := \int_0^s (s-t)^r f(t)\mathrm{d}t. \tag{1.13}$$

Then there exists

$$F'(s) = r\int_0^s (s-t)^{r-1} f(t)\mathrm{d}t, \tag{1.14}$$

$\forall s \in [0,x]$.

We make

Remark 1.11. Let $f \in AC^m([a,b])$, $m \in, \mathbb{N}, m = \lceil\alpha\rceil$, $\alpha > 1$. Suppose $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0,1,...,m-1$.

Then we get by (1.4)

$$f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} D^\alpha_{*a} f(t)\mathrm{d}t, \quad \forall x \in [a,b], \tag{1.15}$$

also (by (1.2))

$$f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} D^\alpha_{b-} f(t)\mathrm{d}t, \quad \forall x \in [a,b]. \tag{1.16}$$

Suppose that

$$D_{*a}^{\alpha} f,\ D_{b-}^{\alpha} f \in L_{\infty}([a, b]). \tag{1.17}$$

Here $\alpha \notin \mathbb{N}$ and $m-1<\alpha<m,\ m \in \mathbb{N}$. So from $\alpha>m-1$ we get $(\alpha-1)-(m-2)>0$. Thus for $\ell=1, \ldots, m-1$, we obtain $0 \leq \ell-1 \leq m-2$, and $(\alpha-1)-(\ell-1) \geq(\alpha-1)-(m-2)>0$.

Hence $(\alpha-1)-(\ell-1)>0$ and $\alpha-\ell>0,\ \ell=1, \ldots, m-1$.

Therefore by Theorem 1.8 and (1.15) we get

$$\begin{aligned} f^{\prime}(x) & =\frac{(\alpha-1)}{\Gamma(\alpha)} \int_{a}^{x}(x-t)^{\alpha-2} D_{*a}^{\alpha} f(t) \mathrm{d} t \\ & =\frac{1}{\Gamma(\alpha-1)} \int_{a}^{x}(x-t)^{\alpha-2} D_{*a}^{\alpha} f(t) \mathrm{d} t . \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} & f^{(\ell)}(x)=\frac{1}{\Gamma(\alpha-\ell)} \int_{a}^{x}(x-t)^{\alpha-\ell-1} D_{*a}^{\alpha} f(t) \mathrm{d} t, \\ & \text{for } \ \ell=1, \ldots, m-1, \forall x \in[a, b]. \end{aligned} \tag{1.18}$$

Using Theorem 1.9 and (1.16), we derive

$$f^{(\ell)}(x)=\frac{(-1)^{\ell}}{\Gamma(\alpha-\ell)} \int_{x}^{b}(t-x)^{\alpha-\ell-1} D_{b-}^{\alpha} f(t) \mathrm{d} t, \quad \ell=1, \ldots, m-1, x \in[a, b]. \tag{1.19}$$

We prove ($p=1, q=\infty$ case).

Theorem 1.12. *Let* $\alpha>0,\ m=\lceil\alpha\rceil,\ x \in[a, b],\ f \in A C^{m}([a, b]),\ f^{(k)}(a)= f^{(k)}(b)=0$, *for* $k=0,1, \ldots,\ m-1$. *Assume*

$$D_{*a}^{\alpha} f \in L_{\infty}\left(\left[a, \frac{a+b}{2}\right]\right),\ D_{b-}^{\alpha} f \in L_{\infty}\left(\left[\frac{a+b}{2}, b\right]\right).$$

Then

$$\begin{aligned} \int_{a}^{b}|f(x)|\left|D^{\alpha} f(x)\right| \mathrm{d} x \leq & \left(\frac{b-a}{2}\right)^{\alpha+1} \cdot \frac{1}{\Gamma(\alpha+2)} \\ & \times\left(\left\|D_{*a}^{\alpha} f\right\|_{\infty,\left[a, \frac{a+b}{2}\right)}^{2}+\left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2}, b\right]}^{2}\right) \end{aligned} \tag{1.20}$$

Proof. Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $x \in [a, b]$, $f \in AC^m([a, b])$, $f^{(k)}(a) = f^{(k)}(b) = 0$, for $k = 0, 1, ..., m-1$. Suppose

$$D_{*a}^{\alpha} f \in L_{\infty}\left(\left[a, \frac{a+b}{2}\right)\right),$$

$$D_{b-}^{\alpha} f \in L_{\infty}\left(\left[\frac{a+b}{2}, b\right]\right).$$

Then

$$|f(x)| \leq \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \|D_{*a}^{\alpha} f\|_{\infty,\left[a, \frac{a+b}{2}\right)}, \quad a \leq x < \frac{a+b}{2},$$

and

$$|f(x)| \leq \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} \|D_{b-}^{\alpha} f\|_{\infty,\left[\frac{a+b}{2}, b\right]}, \quad \frac{a+b}{2} \leq x \leq b.$$

Thus

$$|f(x)| \, |D_{*a}^{\alpha} f(x)| \leq \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \|D_{*a}^{\alpha} f\|_{\infty,\left[a, \frac{a+b}{2}\right)}^{2}, \ \forall x \in \left[a, \frac{a+b}{2}\right).$$

Hence

$$\int_{a}^{\frac{a+b}{2}} |f(x)| \, |D_{*a}^{\alpha} f(x)| \, \mathrm{d}x \leq \frac{\|D_{*a}^{\alpha} f\|_{\infty,\left[a, \frac{a+b}{2}\right)}^{2}}{\Gamma(\alpha+2)} \cdot \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}}.$$

Similarly

$$|f(x)| \leq \|D_{b-}^{\alpha} f\|_{\infty,\left[\frac{a+b}{2}, b\right]} \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)}, \quad \forall x \in \left[\frac{a+b}{2}, b\right]$$

and

$$|f(x)| \, |D_{b-}^{\alpha} f(x)| \leq \frac{\left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2}, b\right]}^{2}}{\Gamma(\alpha+1)} (b-x)^{\alpha}, \ \forall x \in \left[\frac{a+b}{2}, b\right].$$

Hence

$$\int_{\frac{a+b}{2}}^{b} |f(x)| \, |D_{b-}^{\alpha} f(x)| \, \mathrm{d}x \leq \frac{\left\|D_{b-}^{\alpha}\right\|_{\infty,\left[\frac{a+b}{2}, b\right]}^{2}}{\Gamma(\alpha+2)} \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}}.$$

Consequently it holds

$$\int_a^b |f(x)|\,|D^\alpha f(x)|\,\mathrm{d}x \le \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+2)}\left(\|D^\alpha_{*a}f\|^2_{\infty,\left[a,\frac{a+b}{2}\right)} + \|D^\alpha_{b-}f\|_{\infty,\left[\frac{a+b}{2},b\right]}\right),$$

establishing the claim. □

We continue with

Theorem 1.13. *Let* $f \in AC^m([a,b])$, $N \ni m = \lceil \alpha \rceil$, $\alpha \ge 1$. *Assume* $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$. *Assume* $D^\alpha_{*a}f$, $D^\alpha_{b-}f \in L_\infty([a,b])$, $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > m - \frac{1}{p}$.

Then

(i) Case of $1 < q \le 2$. *Then*

$$\int_a^b \left|f^{(\ell)}(w)\right| |D^\alpha f(w)|\,\mathrm{d}w \le \frac{2^{-\left(\alpha-\ell+\frac{1}{p}\right)}(b-a)^{\left(\alpha-\ell-1+\frac{2}{p}\right)}}{\Gamma(\alpha-\ell)[(p(\alpha-\ell-1)+(p(\alpha-\ell-1)+2)]^{\frac{1}{p}}} \times \left(\int_a^b |D^\alpha f(w)|^q\,\mathrm{d}w\right)^{2/q}, \quad \ell = 1, ..., m-1. \tag{1.21}$$

(ii) Case of $q > 2$. *Then*

$$\int_a^b \left|f^{(\ell)}(w)\right| |D^\alpha f(w)|\,\mathrm{d}w \le \frac{2^{-\left(\alpha-\ell+\frac{1}{q}\right)}(b-a)^{\left(\alpha-\ell-1+\frac{2}{p}\right)}}{\Gamma(\alpha-\ell)[(p(\alpha-\ell-1)+1)(p(\alpha-\ell-1)+2)]^{\frac{1}{p}}} \times \left(\int_a^b |D^\alpha f(w)|^q\,\mathrm{d}w\right)^{2/q}, \quad \ell = 1, ..., m-1. \tag{1.22}$$

(iii) When $p = q = 2$, $\alpha > m - \frac{1}{2}$, $(\ell = 1, ..., m-1)$. *Then*

$$\int_a^b \left|f^{(\ell)}(w)\right| |D^\alpha f(w)|\,\mathrm{d}w \le \frac{2^{-\left(\alpha-\ell+\frac{1}{2}\right)}(b-a)^{\alpha-\ell}}{\Gamma(\alpha-\ell)[\sqrt{2(\alpha-\ell)(2\alpha-2\ell-1)}} \times \left(\int_a^b (D^\alpha f(w))^2\mathrm{d}w\right). \tag{1.23}$$

Proof. By the assumption of the theorem we have

$$f^{(\ell)}(x) = \frac{1}{\Gamma(\alpha-\ell)} \int_a^x (x-t)^{\alpha-\ell-1} D_{*a}^{\alpha} f(t) \mathrm{d}t,$$
$$\text{for } \ell = 1, ..., \ m-1, \ \forall x \in [a,b].$$

Similarly it holds

$$f^{(\ell)}(x) = \frac{(-1)^{\ell}}{\Gamma(\alpha-\ell)} \int_x^b (t-x)^{\alpha-\ell-1} D_{b-}^{\alpha} f(t) \mathrm{d}t,$$
$$\ell = l, ..., \ m-1, \ \forall x \in [a,b].$$

Hence

$$\left|f^{(\ell)}(x)\right| \le \frac{1}{\Gamma(\alpha-\ell)} \int_a^x (x-t)^{\alpha-\ell-1} \left|D_{*a}^{\alpha} f(t)\right| \mathrm{d}t,$$
$$\ell = 1, ..., \ m-1, \ \forall x \in [a,b]$$

and

$$\left|f^{(\ell)}(x)\right| \le \frac{1}{\Gamma(\alpha-\ell)} \int_x^b (t-x)^{\alpha-\ell-1} \left|D_{b-}^{\alpha} f(t)\right| \mathrm{d}t,$$
$$\ell = 1, ..., \ m-1, \ \forall x \in [a,b].$$

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Acting now as in the proof of Theorem 9 of [5], and replacing f by $f^{(\ell)}$ and α by $\alpha - \ell$ we prove our theorem. □

We further give

Theorem 1.14. *Let* $\alpha \ge 1, \ \ell = 1, ..., \ m-1; \ f \in AC^m([a,b]), m = \lceil \alpha \rceil, f^{(k)}(a) = f^{(k)}(b) = 0, \ k = 0, 1, ..., \ m-1.$

Assume $D_{*a}^{\alpha} f, D_{b-}^{\alpha} f \in L_{\infty}([a,b])$. *Assume* $D_{*a}^{\alpha} f \neq 0$ *over* $\left[a, \frac{a+b}{2}\right)$, *and of fixed sign, a.e.;* $D_{b-}^{\alpha} f \neq 0$ *over* $\left[\frac{a+b}{2}, b\right]$ *and of fixed sign, a.e.,* $0 < p < 1, \ q < 0;$ $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_a^b \left|f^{(\ell)}(w)\right| \left|D^{\alpha} f(w)\right| \mathrm{d}w$$
$$\ge \left(\frac{(b-a)^{\left((\alpha-\ell-1)+\frac{2}{p}\right)}}{2^{\left(\alpha-\ell+\frac{1}{p}\right)} \Gamma(\alpha-\ell) \left(\frac{(p(\alpha-\ell-1)+1)}{(p(\alpha-\ell-1)+2)}\right)^{1/p}} \right)$$
$$\times \left[\left(\int_a^{\frac{a+b}{2}} \left|D_{*a}^{\alpha} f(t)\right|^q \mathrm{d}t \right)^{\frac{2}{q}} + \left(\int_{\frac{a+b}{2}}^b \left|D_{b-}^{\alpha} f(t)\right|^q \mathrm{d}t \right)^{\frac{2}{q}} \right]. \tag{1.24}$$

Proof. As in the proof of Theorem 1.6, by replacing f by $f^{(\ell)}$ and α by $\alpha-\ell$, for $\ell=1,...,m-1$; $\alpha\geq 1$. □

Theorem 1.15. *The $p=1$, $q=\infty$ case again. Let $\alpha\geq 1, \ell=1,...,m-1$; $f\in AC^m([a,b]), m=\lceil\alpha\rceil$, $f^{(k)}(a)=f^{(k)}(b)=0$; $k=0,1,...,m-1$. Assume $D^\alpha_{*a}f$, $D^\alpha_{b-}f\in L_\infty([a,b])$.*

Then

$$\int_a^b \left|f^{(\ell)}(x)\right| |D^\alpha f(x)|\,\mathrm{d}x \leq \left(\frac{b-a}{2}\right)^{\alpha-\ell+1}\cdot\frac{1}{\Gamma(\alpha-\ell+2)} \times\left(\|D^\alpha_{*a}f\|^2_{\infty,\left[a,\frac{a+b}{2}\right)}+\|D^\alpha_{b-}f\|^2_{\infty,\left[\frac{a+b}{2},b\right]}\right). \tag{1.25}$$

Proof. As in the proof of Theorem 1.12, by replacing f by $f^{(\ell)}$ and α by $\alpha-\ell$, for $\ell=1,...,m-1$. □

We need ([4, 9–11], p. 22)

Definition 1.16. Let $\alpha>0$, $m=\lceil\alpha\rceil$, $f\in AC^m([a,b])$. We define the right Riemann–Liouville fractional derivative by

$$\mathcal{D}^\alpha_{b-}f(x):=\frac{(-1)^m}{\Gamma(m-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m\int_x^b(t-x)^{m-\alpha-1}f(t)\mathrm{d}t, \tag{1.26}$$

$D^0_{b-}f(x):=I\,(x)$ (the identity operator).

We also define the left Riemann–Liouville fractional derivative by

$$\mathcal{D}^\alpha_{a+}f(x):=\frac{1}{\Gamma(m-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m\int_a^x(x-t)^{m-a-1}f(t)\mathrm{d}t, \tag{1.27}$$

$$\mathcal{D}^0_{a+}f(x):=I(x).$$

We further define the balanced Riemann–Liouville fractional derivative ([5])

$$\mathcal{D}^\alpha f(x):=\begin{cases}\mathcal{D}^\alpha_{b-}f(x), & \text{for } \frac{a+b}{2}\leq x\leq b,\\ \mathcal{D}^\alpha_{a+}f(x), & \text{for } \alpha\leq x<\frac{a+b}{2}.\end{cases} \tag{1.28}$$

Remark 1.17. Let $f\in C^m([a,b])$, $m=\lceil\alpha\rceil$, $\alpha>0$. In [6] we have proved that $D^\alpha_{b-}f(x)$, $D^\alpha_{*a}f(x)$ are continuous functions in $x\in[a,b]$. Of course $C^m([a,b])\subset AC^m([a,b])$, so that $f\in AC^m([a,b])$.

Thus by Theorem 9 of [4], we get that also $\mathcal{D}_{b-}^{\alpha} f(x)$ exists and continuous for every $x \in [a,b]$. Furthermore if $f^{(k)}(b) = 0,\ k = 0, 1, ..., m-1$ we obtain

$$\mathcal{D}_{b-}^{\alpha} f(x) = D_{b-}^{\alpha} f(x), \tag{1.29}$$

$\forall x \in [a,b]$.

Similarly, by [9], p. 39, we get that $\mathcal{D}_{a+}^{\alpha} f(x)$ exists and continuous in $x \in [a,b]$. Furthermore if $f^{(k)}(\alpha) = 0,\ k = 0, 1, ..., m-1$ we get

$$\mathcal{D}_{a+}^{\alpha} f(x) = D_{*a}^{\alpha} f(x), \tag{1.30}$$

$\forall x \in [a,b]$.

So if $f^{(k)}(\alpha) = f^{(k)}(b) = 0,\ k = 0, 1, ..., m-1$ we find

$$\mathcal{D}^{\alpha} f(x) = D^{\alpha} f(x), \tag{1.31}$$

$\forall x \in [a,b]$.

So by Theorem 1.13 we obtain the corresponding results for the balanced Riemann–Liouville fractional derivative.

Theorem 1.18. *Let* $f \in C^m([a,b])$, $N \ni m = \lceil \alpha \rceil$, $\alpha \geq 1$. *Assume* $f^{(k)}(a) = f^{(k)}(b) = 0,\ k = 0, 1, ..., m-1$. *Suppose* $p,\ q > 1 : \frac{1}{p} + \frac{1}{q} = 1,\ \alpha > m - \frac{1}{p}$.

(i) Case of $1 < q \leq 2$. *Then*

$$\int_a^b \left| f^{(\ell)}(w) \right| |\mathcal{D}^{\alpha} f(w)| \, dw \leq \frac{2^{-\left(\alpha - \ell + \frac{1}{p}\right)} (b-a)^{\left(\alpha - \ell - 1 + \frac{2}{p}\right)}}{\Gamma(\alpha - \ell) [(p(\alpha - \ell - 1) + 1)(p(\alpha - \ell - 1) + 2)]^{\frac{1}{p}}} \times \left(\int_a^b |\mathcal{D}^{\alpha} f(w)|^q \, dw \right)^{\frac{2}{q}}, \quad \ell = 1, ..., m-1. \tag{1.32}$$

(ii) Case of $q > 2$. *Then*

$$\int_a^b \left| f^{(\ell)}(w) \right| |\mathcal{D}^{\alpha} f(w)| \, dw \leq \frac{2^{\left(\alpha - \ell + \frac{1}{q}\right)} (b-a)^{\left(\alpha - \ell - 1 + \frac{2}{p}\right)}}{\Gamma(\alpha - \ell) [(p(\alpha - \ell - 1) + 1)(p(\alpha - \ell - 1) + 2)]^{1/p}} \times \left(\int_a^b |\mathcal{D}^{\alpha} f(w)|^q \, dw \right)^{2/q}, \quad \ell = 1, ..., m-1. \tag{1.33}$$

(iii) When $p = q = 2,\ \alpha > m - \frac{1}{2},\ (\ell = 1, ..., m-1)$*. Then*

$$\int_a^b |f|^{(\ell)}(w)\, |\mathcal{D}^\alpha f(w)|\, \mathrm{d}w \leq \frac{2^{-\left(\alpha-\ell+\frac{1}{2}\right)}(b-a)^{\alpha-\ell}}{\Gamma(\alpha-\ell)[\sqrt{2(\alpha-\ell)(2\alpha-2\ell-1)}} \times \left(\int_a^b (\mathcal{D}^\alpha f(w))^2 \mathrm{d}w\right). \tag{1.34}$$

Remark 1.19. For $f \in C^m([a,b])$ $m = \lceil \alpha \rceil$, $\alpha > 0$, one can rewrite also Theorems 1.8, 1.12, 1.14, and 1.15 in the language Riemann–Liouville fractional derivatives, as we did in Theorem 1.18.

Application 1.20. Uniqueness of solution to fractional boundary value problem

$$(D^\nu f)(t) = F\left(t, f, f', ..., f^{(m-1)}\right):$$
$$f^{(i)}(a) = a_i,\ f^{(i)}(b) = b_i, 0 \leq i \leq m-1, m := \lceil \nu \rceil,\ \nu \geq 1, \nu > m - \frac{1}{2}. \tag{1.35}$$

Here $f \in AC^m([a,b])$, $D^\nu_{*a} f$, $D^\nu_{b-} f \in L_\infty([a,b])$; F is a continuous function on $[a,b] \times \mathbb{R}^m$, $t \in [a,b]$. Also F fulfills the Lipschitz condition

$$\left|F(t, z_0, z_1, ..., z_{m-1}) - F(t, z_0', z_1', ..., z_{m-1}')\right| \leq \sum_{i=0}^{m-1} q_i(t)\left|z_i - z_i'\right|,$$

where $q_i(t) \geq 0,\ 0 \leq i \leq m-1$ are continuous functions on $[a,b]$.

Call

$$\phi(a,b) := \sum_{i=0}^{m-1} \|q_i\|_\infty \frac{2^{-\left(\nu-i+\frac{1}{2}\right)}(b-a)^{\nu-i}}{\Gamma(\nu-i)\sqrt{2(\nu-i)(2\nu-2i-1)}.}$$

Suppose that

$$\phi(a,b) < 1. \tag{1.36}$$

Next we prove uniqueness to the solution of (1.35). Let f_1, f_2 as above fulfilling (1.35), that is,

$$(D^\nu f_j)(t) = F\left(t, f_j, f_j', ..., f_j^{(m-1)}\right) : f_j^{(i)}(a) = a_i, f_j^{(i)}(b) = b_i, 0 \leq i \leq m-1,$$

$j = 1, 2$.

Call $g := f_1 - f_2$. Then

$$(D^\nu g)(t) = F\left(t, f_1, f_1', ..., f_1^{(m-1)}\right) - F\left(t, f_2, f_2', ..., f_2^{(m-1)}\right),$$

$$\text{such that } g^{(i)}(a) = 0,\ g^{(i)}(b) = 0,\ 0 \leq i \leq m-1. \tag{1.37}$$

Here

$$\left|F\left(t, f_1, f_1', ..., f_1^{(m-1)}\right) - F\left(t, f_2, f_2', ..., f_2^{(m-1)}\right)\right|$$
$$\leq \sum_{i=0}^{m-1} q_i(t)\left|f_1^{(i)}(t) - f_2^{(i)}(t)\right| = \sum_{i=0}^{m-1} q_i(t)\left|g^{(i)}(t)\right|.$$

From

$$(D^\nu g(t))^2 = (D^\nu g(t))\left\{F\left(t, f_1, f_1', ..., f_1^{(m-1)}\right) - F\left(t, f_2, f_2', ..., f_2^{(m-1)}\right)\right\},$$

we obtain

$$(D^\nu g(t))^2 \leq |D^\nu g(t)|\left(\sum_{i=0}^{m-1} q_i(t)\left|g^{(i)}(t)\right|\right) = \sum_{i=0}^{m-1} q_i(t)(|D^\nu g(t)|\left|g^{(i)}(t)\right|).$$

Integrating the last inequality we get

$$\int_a^b (D^\nu g(t))^2 \mathrm{d}t \leq \sum_{i=0}^{m-1} \int_a^b q_i(t)\left|g^{(i)}(t)\right| |D^\nu g(t)|\, \mathrm{d}t$$
$$\leq \sum_{i=0}^{m-1} \|q_i\|_\infty \int_a^b \left|g^{(i)}(t)\right| |D^\nu g(t)|\, \mathrm{d}t. \tag{1.38}$$

From inequalities (23) and (13) of [5] we find that

$$\int_a^b \left|g^{(i)}(t)\right| |D^\nu g(t)|\, \mathrm{d}t$$
$$\leq \frac{2^{-\left(\nu-i+\frac{1}{2}\right)}(b-a)^{\nu-i}}{\Gamma(\nu-i)\sqrt{2(2\nu-2i-1)(\nu-i)}} \int_a^b ((D^\nu g)(t))^2 \mathrm{d}t, \text{ for } i = 0, 1, ..., m-1. \tag{1.39}$$

Combining (1.38) and (1.39) we derive

$$\int_a^b ((D^\nu g)(t))^2 \mathrm{d}t \leq \phi(a, b) \int_a^b ((D^\nu g(t))^2 \mathrm{d}t. \tag{1.40}$$

If $\int_a^b (D^\nu g(t))^2 \mathrm{d}t \neq 0$, then from (1.40) we get $\phi(a, b) \geq 1$, a contradiction by (1.36).

Thus

$$\int_a^b (D^\nu g(t))^2 \mathrm{d}t = 0,$$

which implies

$$(D^\nu g(t)^2 = 0,\ a.e.\ in\ t \in [a,b],$$

that is,

$$D^\nu g(t) = 0,\ a.e,\ in\ t \in [a,b].$$

Therefore

$$D^\nu_{b-} g(t) = 0,\ \text{for}\ \frac{a+b}{2} \le t \le b,\ a.e.$$

and

$$D^\nu_{*a} g(t) = 0, \text{for}\ a \le t < \frac{a+b}{2},\ a.e.$$

But

$$g^{(i)}(a) = g^{(i)}(b) = 0,\ \text{for all}\ 0 \le i \le m-1.$$

So from (1.2) and (1.4) we get $g(x) = 0$ on $\left[\frac{a+b}{2}, b\right]$, and $g(x) = 0$ on $\left[a, \frac{a+b}{2}\right]$, respectively.

Thus $g(x) = 0,\ \forall x \in [a,b]$, giving us $f_1(x) = f_2(x), \forall x \in [a,b]$, that is, proving the uniqueness of solution to fractional boundary value problem (1.35).

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Chapter 2
Univariate Right Caputo Fractional Ostrowski Inequalities

Here we present general univariate right Caputo fractional Ostrowski inequalities. One of them is proved sharp and attained. Estimates are with respect to $\|\cdot\|_p$, $1 \leq p \leq \infty$. This chapter is based on [4].

2.1 Introduction

In 1938, A. Ostrowski [8] proved the following important inequality:

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right]\cdot (b-a)\,\|f'\|_\infty, \tag{2.1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to numerical analysis and probability.

This chapter is greatly motivated and inspired also by the following result.

Theorem 2.2 (see [1]). *Let $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$ and $x \in [a,b]$ be fixed, such that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then it holds*

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+2)!}\cdot\left(\frac{(x-a)^{n+2} + (b-x)^{n+2}}{b-a}\right). \tag{2.2}$$

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Inequality (2.2) is sharp. In particular, when n is odd is attained by $f^*(y) := (y-x)^{n+1} \cdot (b-a)$, *while when n is even the optimal function is*

$$\overline{f}(y) := |y-x|^{n+\alpha} \cdot (b-a), \quad \alpha > 1.$$

Clearly inequality (2.2) generalizes inequality (2.1) for higher order derivatives of f.

Also in [2], see Chaps. 24–26, we gave a complete theory of left fractional Ostrowski inequalities.

2.2 Main Results

We need

Definition 2.3 ([3,5–7,9]). Let $f \in L_1([a,b])$, $\alpha > 0$. The right Riemann–Liouville fractional operator of order α by

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} f(J) \, \mathrm{d}J, \tag{2.3}$$

$\forall\ x \in [a,b]$, where Γ is the gamma function. We set $I_{b-}^0 := I$ (the identity operator).

Definition 2.4 ([3,5–7,9]). Let $f \in AC^m([a,b])$ ($f^{(m-1)}$ is in $AC([a,b])$), $m \in \mathbb{N}$, $m = \lceil \alpha \rceil$, $\alpha > 0$ ($\lceil \cdot \rceil$ the ceiling of the number). We define the right Caputo fractional derivative of order $\alpha > 0$ by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J) \, \mathrm{d}J, \quad \forall\ x \le b. \tag{2.4}$$

If $\alpha = m \in \mathbb{N}$, then

$$D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x), \quad \forall\ x \in [a,b].$$

If $x > b$ we define $D_{b-}^{\alpha} f(x) = 0$.

We also need

Theorem 2.5 ([3]). *Let* $f \in AC^m([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m = \lceil \alpha \rceil$. *Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} D_{b-}^{\alpha} f(J) \, \mathrm{d}J, \tag{2.5}$$

the right Caputo fractional Taylor formula with integral remainder.

We give

Theorem 2.6. *Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. Assume $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, and $D_{b-}^{\alpha} f \in L_{\infty}([a,b])$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+2)} (b-a)^{\alpha}. \tag{2.6}$$

Proof. Let $x \in [a,b]$. We have

$$f(x) - f(b) = \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} D_{b-}^{\alpha} f(J)\, dJ.$$

Then

$$\begin{aligned} |f(x) - f(b)| &\leq \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} \left| D_{b-}^{\alpha} f(J) \right| dJ \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_x^b (J-x)^{\alpha-1} dJ \right) \| D_{b-}^{\alpha} f \|_{\infty,[a,b]} \\ &= \frac{1}{\Gamma(\alpha)} \left(\left. \frac{(J-x)^{\alpha}}{\alpha} \right|_x^b \right) \| D_{b-}^{\alpha} f \|_{\infty,[a,b]} \\ &= \frac{1}{\Gamma(\alpha+1)} (b-x)^{\alpha} \| D_{b-}^{\alpha} f \|_{\infty,[a,b]}. \end{aligned}$$

Therefore

$$|f(x) - f(b)| \leq \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} \| D_{b-}^{\alpha} f \|_{\infty,[a,b]}, \quad \forall\, x \in [a,b].$$

Hence it holds

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right| = \left| \frac{1}{b-a} \int_a^b (f(x) - f(b))\, dx \right| \\ &\leq \frac{1}{b-a} \int_a^b |f(x) - f(b)|\, dx \leq \frac{1}{b-a} \int_a^b \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} \| D_{b-}^{\alpha} f \|_{\infty,[a,b]}\, dx \\ &= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(b-a)\, \Gamma(\alpha+1)} \int_a^b (b-x)^{\alpha}\, dx \\ &= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(b-a)\, \Gamma(\alpha+1)} \left(- \left(\left. \frac{(b-x)^{\alpha+1}}{\alpha+1} \right|_a^b \right) \right) \end{aligned}$$

$$= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(b-a)\,\Gamma(\alpha+1)} (-1) \left(0 - \frac{(b-a)^{\alpha+1}}{\alpha+1} \right)$$

$$= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(b-a)\,\Gamma(\alpha+2)} \cdot (b-a)^{\alpha+1} = \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]} \cdot (b-a)^{\alpha}}{\Gamma(\alpha+2)},$$

proving the claim. □

We present

Theorem 2.7. *Let* $\alpha \geq 1$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. *Assume that* $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, *and* $D_{b-}^{\alpha} f \in L_1([a,b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha+1)} (b-a)^{\alpha-1}. \tag{2.7}$$

Proof. We have again

$$|f(x) - f(b)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} \left| D_{b-}^{\alpha} f(J) \right| \mathrm{d}J$$

$$\leq \frac{1}{\Gamma(\alpha)} (b-x)^{\alpha-1} \int_x^b \left| D_{b-}^{\alpha} f(J) \right| \mathrm{d}J$$

$$\leq \frac{1}{\Gamma(\alpha)} (b-x)^{\alpha-1} \left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}.$$

Hence

$$|f(x) - f(b)| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha)} (b-x)^{\alpha-1}, \quad \forall\, x \in [a,b].$$

Therefore

$$\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - f(b) \right| \leq \frac{1}{b-a} \int_a^b |f(x) - f(b)|\,\mathrm{d}x$$

$$\leq \frac{1}{b-a} \int_a^b \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha)} (b-x)^{\alpha-1}\,\mathrm{d}x$$

$$= \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{(b-a)\,\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1}\,\mathrm{d}x$$

$$= \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{(b-a)\,\Gamma(\alpha)} \frac{(b-x)^{\alpha}}{\alpha} = \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha+1)} (b-x)^{\alpha-1},$$

proving the claim. □

We continue with

Theorem 2.8. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 1 - \frac{1}{p}$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. *Assume that* $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, *and* $D_{b-}^{\alpha} f \in L_q([a,b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - f(b) \right| \le \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} (b-a)^{\alpha - 1 + \frac{1}{p}}. \tag{2.8}$$

Proof. We have again

$$|f(x) - f(b)| \le \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} \left| D_{b-}^{\alpha} f(J) \right| \mathrm{d}J$$
$$\le \frac{1}{\Gamma(\alpha)} \left(\int_x^b (J-x)^{p(\alpha-1)} \mathrm{d}J \right)^{\frac{1}{p}} \left(\int_x^b \left| D_{b-}^{\alpha} f(J) \right|^q \mathrm{d}J \right)^{\frac{1}{q}}$$
$$\le \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_x^b \left| D_{b-}^{\alpha} f(J) \right|^q \mathrm{d}J \right)^{\frac{1}{q}}$$
$$\le \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}.$$

Therefore

$$|f(x) - f(b)| \le \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}}} (b-x)^{\alpha-1+\frac{1}{p}}, \quad \forall\, x \in [a,b].$$

Hence

$$\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - f(b) \right| \le \frac{1}{b-a} \int_a^b |f(x) - f(b)|\,\mathrm{d}x$$
$$\le \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{(b-a)\,\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}}} \int_a^b (b-x)^{\alpha-1+\frac{1}{p}}\,\mathrm{d}x$$
$$= \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\alpha-1+\frac{1}{p}}}{\left(\alpha + \frac{1}{p}\right)}.$$

□

Corollary 2.9. *Let* $\alpha > \frac{1}{2}$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. *Suppose* $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, $D_{b-}^{\alpha} f \in L_2([a,b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - f(b) \right| \le \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_2([a,b])}}{\Gamma(\alpha)\left(\sqrt{2\alpha-1}\right)\left(\alpha + \frac{1}{2}\right)} (b-a)^{\alpha-\frac{1}{2}}. \tag{2.9}$$

We finish the chapter with

Proposition 2.10. *Inequality (2.6) is sharp, namely it is attained by*

$$f(x) = (b-x)^{\alpha}, \ \alpha > 0, \alpha \notin \mathbb{N}, x \in [a,b].$$

Proof. Notice that $(b-x)^{\alpha} \in AC^m([a,b])$. We observe that

$$f'(x) = -\alpha (b-x)^{\alpha-1},$$

$$f''(x) = (-1)^2 \alpha (\alpha-1)(b-x)^{\alpha-2},$$

$$\vdots$$

$$f^{(m-1)}(x) = (-1)^{m-1} \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-m+2)(b-x)^{\alpha-m+1},$$

and

$$f^{(m)}(x) = (-1)^m \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-m+2)(\alpha-m+1)(b-x)^{\alpha-m}.$$

Thus

$$\begin{aligned} D_{b-}^{\alpha} f(x) &= \frac{(-1)^{2m}}{\Gamma(m-\alpha)} \alpha(\alpha-1)\cdots(\alpha-m+1) \int_x^b (J-x)^{m-\alpha-1} (b-J)^{\alpha-m} \, dJ \\ &= \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{\Gamma(m-\alpha)} \int_x^b (b-J)^{(\alpha-m+1)-1} (J-x)^{(m-\alpha)-1} \, dJ \\ &= \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{\Gamma(m-\alpha)} \frac{\Gamma(\alpha-m+1)\Gamma(m-\alpha)}{\Gamma(1)} \\ &= \alpha(\alpha-1)\cdots(\alpha-m+1)\Gamma(\alpha-m+1) = \Gamma(\alpha+1). \end{aligned}$$

That is,

$$D_{b-}^{\alpha} f(x) = \Gamma(\alpha+1), \quad \forall\, x \in [a,b].$$

Also we see that $f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$, and $D_{b-}^{\alpha} f \in L_{\infty}([a,b])$. So f fulfills all assumptions. Next we see

$$\text{R.H.S. (2.6)} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)}(b-a)^{\alpha} = \frac{(b-a)^{\alpha}}{(\alpha+1)}.$$

$$\begin{aligned} \text{L.H.S. (2.6)} &= \frac{1}{b-a}\int_a^b (b-x)^{\alpha} \, dx \\ &= \frac{1}{b-a} \frac{(b-a)^{\alpha+1}}{(\alpha+1)} = \frac{(b-a)^{\alpha}}{\alpha+1}, \end{aligned}$$

proving attainability and sharpness of (2.6). ☐

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Chapter 3
Multivariate Right Caputo Fractional Ostrowski Inequalities

Here we present general multivariate right Caputo fractional Ostrowski inequalities. Some of them are proved to be sharp and attained. Estimates are with respect to $\|\cdot\|_\infty$. This chapter relies on [5].

3.1 Introduction

In 1938, A. Ostrowski [8] proved the following important inequality:

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - f(x) \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \cdot (b-a) \|f'\|_\infty, \tag{3.1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to numerical analysis and probability.

This chapter is greatly motivated and inspired also by the following result.

Theorem 3.2 (see [1]). *Let $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$ and $x \in [a,b]$ be fixed, such that $f^{(k)}(x) = 0$, $k = 1, \ldots, n$. Then it holds*

$$\left| \frac{1}{b-a} \int_a^b f(y)\,\mathrm{d}y - f(x) \right| \le \frac{\|f^{(n+1)}\|_\infty}{(n+2)!} \cdot \left(\frac{(x-a)^{n+2} + (b-x)^{n+2}}{b-a} \right). \tag{3.2}$$

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Inequality (3.2) is sharp. In particular, when n is odd is attained by $f^*(y) := (y-x)^{n+1} \cdot (b-a)$, *while when n is even the optimal function is*

$$\overline{f}(y) := |y-x|^{n+\alpha} \cdot (b-a), \quad \alpha > 1.$$

Clearly inequality (3.2) generalizes inequality (3.1) for higher order derivatives of f.

Also in [2], see Chaps. 24–26, we gave a complete theory of left fractional Ostrowski inequalities.

3.2 Main Results

We need

Remark 3.3. We define the ball $B(0,R) = \{x \in \mathbb{R}^N : |x| < R\} \subseteq \mathbb{R}^N$, $N \geq 2$, $R > 0$, and the sphere

$$S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\},$$

where $|\cdot|$ is the Euclidean norm. Let $d\omega$ be the element of surface measure on S^{N-1} and let

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}.$$

For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$. Note that $\int_{B(0,R)} dy = \frac{\omega_N R^N}{N}$ is the Lebesgue measure of the ball.

Following [6, pp. 149–150, exercise 6] and [7, pp. 87–88, Theorem 5.2.2], we can write $F : \overline{B(0,R)} \to \mathbb{R}$ a Lebesgue integrable function that

$$\int_{B(0,R)} F(x)\, dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega)\, r^{N-1} dr \right) d\omega; \tag{3.3}$$

we use this formula a lot.

Initially the function $f : \overline{B(0,R)} \to \mathbb{R}$ is radial, that is, there exists a function g such that $f(x) = g(r)$, where $r = |x|$, $r \in [0,R]$, $\forall\, x \in \overline{B(0,R)}$. Here we assume that $g \in AC^m([0,R])$ (means $g^{(m-1)}$ is in $AC([0,R])$), $m = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), $\alpha > 0$, and $g^{(k)}(R) = 0$, $k = 1, ..., m-1$.

By [3] we obtain

$$g(s) - g(R) = \frac{1}{\Gamma(\alpha)} \int_s^R (J-s)^{\alpha-1} D_{R-}^{\alpha} g(J)\, dJ, \tag{3.4}$$

$\forall\ s \in [0, R]$, where $D^{\alpha}_{R-}g$ is the right Caputo derivative. Further suppose that $D^{\alpha}_{R-}g \in L_{\infty}([0, R])$.

We get

$$\begin{aligned}|g(s) - g(R)| &\leq \frac{1}{\Gamma(\alpha)} \int_s^R (J - s)^{\alpha - 1} \left| D^{\alpha}_{R-}g(J) \right| \mathrm{d}J \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_s^R (J - s)^{\alpha - 1} \mathrm{d}J \right) \left\| D^{\alpha}_{R-}g \right\|_{\infty,[0,R]} \\ &= \frac{(R - s)^{\alpha}}{\Gamma(\alpha + 1)} \left\| D^{\alpha}_{R-}g \right\|_{\infty,[0,R]} . \end{aligned} \tag{3.5}$$

That is,

$$|g(s) - g(R)| \leq \frac{\left\| D^{\alpha}_{R-}g \right\|_{\infty,[0,R]}}{\Gamma(\alpha + 1)} (R - s)^{\alpha}, \tag{3.6}$$

$\forall\ s \in [0, R]$.

Next observe that

$$\left| f(R\omega) - \frac{\int_{B(0,R)} f(y)\,\mathrm{d}y}{Vol(B(0,R))} \right| = \left| g(R) - \frac{\int_{S^{N-1}} \left(\int_0^R g(s)\, s^{N-1} \mathrm{d}s \right) \mathrm{d}\omega}{\int_{S^{N-1}} \left(\int_0^R s^{N-1} \mathrm{d}s \right) \mathrm{d}\omega} \right|$$

$$= \left| g(R) - \frac{N}{R^N} \int_0^R g(s)\, s^{N-1} \mathrm{d}s \right| = \frac{N}{R^N} \left| \int_0^R s^{N-1} (g(R) - g(s))\, \mathrm{d}s \right| \tag{3.7}$$

$$\leq \frac{N}{R^N} \int_0^R s^{N-1} |g(R) - g(s)|\, \mathrm{d}s \leq \frac{N}{R^N} \frac{\left\| D^{\alpha}_{R-}g \right\|_{\infty,[0,R]}}{\Gamma(\alpha + 1)} \int_0^R s^{N-1} (R - s)^{\alpha} \mathrm{d}s$$

$$= \frac{N}{R^N} \frac{\left\| D^{\alpha}_{R-}g \right\|_{\infty,[0,R]}}{\Gamma(\alpha + 1)} \int_0^R (R - s)^{(\alpha+1)-1} s^{N-1} \mathrm{d}s$$

$$= \frac{N}{R^N} \frac{\left\| D^{\alpha}_{R-}g \right\|_{\infty,[0,R]}}{\Gamma(\alpha + 1)} \frac{\Gamma(\alpha + 1)(N - 1)!}{\Gamma(\alpha + N + 1)} R^{\alpha + N}$$

$$= \left\| D^{\alpha}_{R-}g \right\|_{\infty,[0,R]} \frac{N!R^{\alpha}}{\Gamma(\alpha + N + 1)}. \tag{3.8}$$

So we have proved that

$$\left| f(R\omega) - \frac{\int_{B(0,R)} f(y)\,\mathrm{d}y}{Vol(B(0,R))} \right| = \left| g(R) - \frac{N}{R^N} \int_0^R g(s)\, s^{N-1} \mathrm{d}s \right| \tag{3.9}$$

$$\leq \left\| D^{\alpha}_{R-}g \right\|_{\infty,[0,R]} \frac{N!R^{\alpha}}{\Gamma(\alpha + N + 1)}. \tag{3.10}$$

The last inequality (3.10) is sharp, it is attained by $\overline{g}(r) = (R - r)^{\alpha}$, $\alpha > 0$, $r \in [0, R]$. As in [4] we get

$$D_{R-}^{\alpha}\overline{g}(r) = \Gamma(\alpha + 1), \quad \forall\, r \in [0, R].$$

Hence $\left\| D_{R-}^{\alpha}\overline{g} \right\|_{\infty,[0,R]} = \Gamma(\alpha + 1)$. And $\overline{g}(R) = 0$. Therefore

$$\begin{aligned} \text{L.H.S. (3.10)} &= \frac{N}{R^N} \int_0^R (R - s)^{\alpha} s^{N-1} \mathrm{d}s \\ &= \frac{N}{R^N} \int_0^R (R - s)^{(\alpha+1)-1} (s - 0)^{N-1} \mathrm{d}s \\ &= \frac{N}{R^N} \frac{\Gamma(\alpha + 1)\Gamma(N)}{\Gamma(\alpha + N + 1)} R^{\alpha+N} = \frac{\Gamma(\alpha + 1) N!}{\Gamma(\alpha + N + 1)} R^{\alpha}. \end{aligned}$$

And

$$\text{R.H.S. (3.10)} = \frac{\Gamma(\alpha + 1) N! R^{\alpha}}{\Gamma(\alpha + N + 1)},$$

proving attainability of (3.10).

We have established the following multivariate Ostrowski inequality.

Theorem 3.4. *Let $f : \overline{B(0, R)} \to \mathbb{R}$ which is radial, that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $\forall\, x \in \overline{B(0, R)}$. Suppose that $g \in AC^m([0, R])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, and $g^{(k)}(R) = 0$, $k = 1, ..., m - 1$, and $D_{R-}^{\alpha} g \in L_{\infty}([0, R])$. Then*

$$\left| f(R\omega) - \frac{\int_{B(0,R)} f(y)\,\mathrm{d}y}{Vol(B(0, R))} \right| = \left| g(R) - \frac{N}{R^N} \int_0^R g(s) s^{N-1} \mathrm{d}s \right|$$
$$\leq \left\| D_{R-}^{\alpha} g \right\|_{\infty,[0,R]} \frac{N! R^{\alpha}}{\Gamma(\alpha + N + 1)}. \tag{3.11}$$

The last inequality is sharp, that is attained by $\overline{g}(r) = (R - r)^{\alpha}$, $\alpha > 0$, $\forall\, r \in [0, R]$.

We also make

Remark 3.5. Let the spherical shell $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, $x \in \overline{A}$. Consider again that $f : \overline{A} \to \mathbb{R}$ is radial, that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\, x \in \overline{A}$. Here again x can be written uniquely as $x = r\omega$, where $r = |x| > 0$, and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$. We can write for $F : \overline{A} \to \mathbb{R}$ a Lebesgue integrable function that

$$\int_A F(x)\,\mathrm{d}x = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega) r^{N-1} \mathrm{d}r \right) \mathrm{d}\omega. \tag{3.12}$$

Here $Vol(A) = \frac{\omega_N \left(R_2^N - R_1^N\right)}{N}$, and we assume that $g \in AC^m([R_1, R_2])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, and $g^{(k)}(R_2) = 0$, $k = 1, ..., m-1$. We get (see [3])

$$g(s) - g(R_2) = \frac{1}{\Gamma(\alpha)} \int_s^{R_2} (J - s)^{\alpha - 1} D_{R_2-}^{\alpha} g(J) \, \mathrm{d}J, \tag{3.13}$$

$\forall\ s \in [R_1, R_2]$, where $D_{R_2-}^{\alpha} g$ is the right Caputo fractional derivative. Further suppose that $D_{R_2-}^{\alpha} g \in L_\infty([R_1, R_2])$. Hence

$$\begin{aligned} |g(s) - g(R_2)| &\leq \frac{1}{\Gamma(\alpha)} \int_s^{R_2} (J - s)^{\alpha - 1} \left| D_{R_2-}^{\alpha} g(J) \right| \mathrm{d}J \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_s^{R_2} (J - s)^{\alpha - 1} \mathrm{d}J \right) \left\| D_{R_2-}^{\alpha} g \right\|_{\infty,[R_1,R_2]} \\ &= \frac{1}{\Gamma(\alpha)} \frac{(R_2 - s)^{\alpha}}{\alpha} \left\| D_{R_2-}^{\alpha} g \right\|_{\infty,[R_1,R_2]}. \end{aligned} \tag{3.14}$$

Therefore

$$|g(s) - g(R_2)| \leq \frac{\left\| D_{R_2-}^{\alpha} g \right\|_{\infty,[R_1,R_2]}}{\Gamma(\alpha + 1)} (R_2 - s)^{\alpha}, \tag{3.15}$$

$\forall\ s \in [R_1, R_2]$.

Next we observe that

$$\begin{aligned} \left| f(R_2\omega) - \frac{\int_A f(y)\,\mathrm{d}y}{Vol(A)} \right| &= \left| g(R_2) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s)\, s^{N-1} \mathrm{d}s \right| \\ &= \left(\frac{N}{R_2^N - R_1^N} \right) \left| \int_{R_1}^{R_2} (g(R_2) - g(s))\, s^{N-1} \mathrm{d}s \right| \\ &\leq \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} |g(R_2) - g(s)|\, s^{N-1} \mathrm{d}s \\ &\leq \left(\frac{N}{R_2^N - R_1^N} \right) \frac{\left\| D_{R_2-}^{\alpha} g \right\|_{\infty,[R_1,R_2]}}{\Gamma(\alpha + 1)} \\ &\quad \times \int_{R_1}^{R_2} (R_2 - s)^{\alpha} s^{N-1} \mathrm{d}s =: (*). \end{aligned} \tag{3.16}$$

We evaluate

$$\begin{aligned} &\int_{R_1}^{R_2} (R_2 - s)^{\alpha} s^{N-1} \mathrm{d}s \\ &= \int_{R_1}^{R_2} (R_2 - s)^{\alpha} ((s - R_1) + R_1)^{N-1} \mathrm{d}s \end{aligned}$$

$$= \int_{R_1}^{R_2} (R_2 - s)^{\alpha} \left(\sum_{k=0}^{N-1} \binom{N-1}{k} (s - R_1)^k \, R_1^{N-1-k} \right) \mathrm{d}s$$

$$= \sum_{k=0}^{N-1} \binom{N-1}{k} R_1^{N-1-k} \int_{R_1}^{R_2} (R_2 - s)^{(\alpha+1)-1} (s - R_1)^{(k+1)-1} \, \mathrm{d}s \quad (3.17)$$

$$= \sum_{k=0}^{N-1} \binom{N-1}{k} R_1^{N-1-k} \frac{\Gamma(\alpha+1)\,\Gamma(k+1)}{\Gamma(\alpha+k+2)} (R_2 - R_1)^{\alpha+k+1}$$

$$= \sum_{k=0}^{N-1} \frac{(N-1)!}{k!\,(N-k-1)!} R_1^{N-1-k} \frac{\Gamma(\alpha+1)\,k!}{\Gamma(\alpha+k+2)} (R_2 - R_1)^{\alpha+k+1}.$$

Therefore we get

$$\int_{R_1}^{R_2} (R_2 - s)^{\alpha} \, s^{N-1} \mathrm{d}s = (N-1)!\,\Gamma(\alpha+1) \sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)!\,\Gamma(\alpha+2+k)}. \quad (3.18)$$

Consequently we find

$$(*) = \left(\frac{N!}{R_2^N - R_1^N} \right) \left\| D_{R_2-}^{\alpha} g \right\|_{\infty,[R_1,R_2]} \left(\sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)!\,\Gamma(\alpha+2+k)} \right). \quad (3.19)$$

So we have established

$$\left| f(R_2\omega) - \frac{\int_A f(y)\,\mathrm{d}y}{Vol(A)} \right| = \left| g(R_2) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s)\, s^{N-1} \mathrm{d}s \right|$$

$$\leq \left(\frac{N!}{R_2^N - R_1^N} \right) \left(\sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)!\,\Gamma(\alpha+2+k)} \right) \left\| D_{R_2-}^{\alpha} g \right\|_{\infty,[R_1,R_2]}. \quad (3.20)$$

The last inequality (3.20) is sharp, that is attained by $AC^m([R_1, R_2]) \ni \overline{g}(r) = (R_2 - r)^{\alpha}, \alpha > 0, m = \lceil \alpha \rceil, r \in [R_1, R_2]$. Indeed

$$D_{R_2-}^{\alpha} \overline{g}(r) = \Gamma(\alpha+1), \quad \forall\, r \in [R_1, R_2],$$

and

$$\left\| D_{R_2-}^{\alpha} \overline{g} \right\|_{\infty,[R_1,R_2]} = \Gamma(\alpha+1). \quad (3.21)$$

Also we have $\overline{g}^{(k)}(R_2) = 0, k = 0, 1, ..., m-1$, and $D_{R_2-}^{\alpha} \overline{g} \in L_{\infty}([R_1, R_2])$. So $\overline{g}$ fulfills all the assumptions here.

We see that

$$\text{L.H.S. } (3.20) = \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} (R_2 - s)^{\alpha} s^{N-1} \mathrm{d}s \frac{N!\Gamma(\alpha+1)}{R_2^N - R_1^N}$$
$$\times \sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)!\Gamma(\alpha+2+k)}$$
$$= \left(\frac{N!}{R_2^N - R_1^N}\right)\left(\sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)!\Gamma(\alpha+2+k)}\right)$$
$$\times \left\| D_{R_2-}^{\alpha} \overline{g} \right\|_{\infty,[R_1,R_2]}$$
$$= \text{R.H.S. } (3.20), \tag{3.22}$$

proving the optimality of (3.20).

We have established the Ostrowski inequality.

Theorem 3.6. *Let $f : \overline{A} \to \mathbb{R}$ be radial, that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $\forall\, x \in \overline{A}$; $\omega \in S^{N-1}$. Assume $g \in AC^m([R_1, R_2])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, and $g^{(k)}(R_2) = 0$, $k = 1, ..., m-1$, and $D_{R_2-}^{\alpha} g \in L_{\infty}([R_1, R_2])$. Then*

$$\left| f(R_2\omega) - \frac{\int_A f(y)\,\mathrm{d}y}{Vol(A)} \right| = \left| g(R_2) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} g(s) s^{N-1} \mathrm{d}s \right|$$

$$\leq \left(\frac{N!}{R_2^N - R_1^N}\right)\left(\sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)!\Gamma(\alpha+2+k)}\right) \left\| D_{R_2-}^{\alpha} g \right\|_{\infty,[R_1,R_2]}. \tag{3.23}$$

The last inequality (3.6) is sharp, that is attained by

$$g(s) = (R_2 - s)^{\alpha}, \quad \alpha > 0, s \in [R_1, R_2]. \tag{3.24}$$

We need

Definition 3.7. Let $F : \overline{A} \to \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$ such that $F(\cdot\omega) \in AC^m([R_1, R_2])$, for all $\omega \in S^{N-1}$. We call the Caputo right radial fractional derivative the following function

$$\frac{\partial_{R_2-}^{\alpha} F(x)}{\partial r^{\alpha}} = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_r^{R_2} (t-r)^{m-\alpha-1} \frac{\partial^m F(t\omega)}{\partial r^m} \mathrm{d}t, \tag{3.25}$$

where $x \in \overline{A}$, that is, $x = r\omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

Clearly

$$\frac{\partial_{R_2-}^{0} F(x)}{\partial r^{0}} = F(x), \tag{3.26}$$

$$\frac{\partial^{\alpha}_{R_2-}F(x)}{\partial r^{\alpha}} = \frac{\partial^{\alpha}F(x)}{\partial r^{\alpha}}, \quad \text{if } \alpha \in \mathbb{N}. \tag{3.27}$$

The above defined function exists almost everywhere for $x \in \overline{A}$.
We justify this next.

Remark 3.8. Call

$$\Lambda_1 := \left\{ r \in [R_1, R_2] : \frac{\partial^{\alpha}_{R_2-}F(x)}{\partial r^{\alpha}} \text{ does not exist} \right\}.$$

We have that Lebesgue measure $\lambda_{\mathbb{R}}(\Lambda_1) = 0$. Call $\Lambda_N := \Lambda_1 \times S^{N-1}$. So there exists a Borel set $\Lambda_1^* \subset [R_1, R_2]$, such that $\Lambda_1 \subset \Lambda_1^*$, $\lambda_{\mathbb{R}}(\Lambda_1^*) = \lambda_{\mathbb{R}}(\Lambda_1) = 0$; thus, $R_N(\Lambda_1^*) = 0$, see [2], pp. 419–422.

Consider now $\Lambda_N^* := \Lambda_1^* \times S^{N-1} \subset \overline{A}$, which is a Borel set of $\mathbb{R}^N - \{0\}$. Clearly then by Theorem 16.59, p. 420, [2], $\lambda_{\mathbb{R}^N}(\Lambda_N^*) = 0$, but $\Lambda_N \subset \Lambda_N^*$, implying $\lambda_{\mathbb{R}^N}(\Lambda_N) = 0$.

Consequently the above radial derivative exists a.e. in x w.r.t. $\lambda_{\mathbb{R}^N}$ on $\overline{A}$.

We make

Remark 3.9. We treat here the general, not necessarily radial, case of f. We apply last Theorem 3.6 to $f(r\omega)$, ω is fixed, $r \in [R_1, R_2]$, under the following assumptions: $f(\cdot\omega) \in AC^m([R_1, R_2])$, for all $\omega \in S^{N-1}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, where $f : \overline{A} \to \mathbb{R}$ is Lebesgue integrable; $\frac{\partial^k f}{\partial r^k}$, $k = 1, ..., m-1$ vanish on $\partial B(0, R_2)$, and $\frac{\partial^{\alpha}_{R_2-}f}{\partial r^{\alpha}} \in B(\overline{A})$, along with $D^{\alpha}_{R_2-}f(\cdot\omega) \in L_{\infty}([R_1, R_2])$, $\forall\, \omega \in S^{N-1}$.

So we have

$$\left| f(R_2\omega) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} f(s\omega)\, s^{N-1} \mathrm{d}s \right|$$
$$\leq \left(\frac{N!}{R_2^N - R_1^N}\right) \left(\sum_{k=0}^{N-1} \frac{R_1^{N-1-k}(R_2 - R_1)^{\alpha+1+k}}{(N-1-k)!\,\Gamma(\alpha+2+k)} \right) \left\| \frac{\partial^{\alpha}_{R_2-}f}{\partial r^{\alpha}} \right\|_{\infty,\overline{A}} =: \lambda_1. \tag{3.28}$$

Consequently it holds

$$\left| \frac{\int_{S^{N-1}} f(R_2\omega)\,\mathrm{d}\omega}{\omega_N} - \frac{N}{(R_2^N - R_1^N)\,\omega_N} \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega)\, s^{N-1} \mathrm{d}s \right) \mathrm{d}\omega \right| \leq \lambda_1. \tag{3.29}$$

That is,

$$\left| \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R_2\omega)\,\mathrm{d}\omega - \frac{\int_A f(x)\,\mathrm{d}x}{Vol(A)} \right| \leq \lambda_1. \tag{3.30}$$

Therefore, it holds for $x \in \overline{A}$, that

$$\left| f(x) - \frac{\int_A f(x)\,dx}{Vol(A)} \right| = \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R_2\omega)\,d\omega + \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \right.$$

$$\left. \times \int_{S^{N-1}} f(R_2\omega)\,d\omega - \frac{\int_A f(x)\,dx}{Vol(A)} \right|$$

$$\leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R_2\omega)\,d\omega \right| + \lambda_1. \quad (3.31)$$

We have proved

Theorem 3.10. *Let $f : \overline{A} \to \mathbb{R}$ be Lebesgue integrable with $f(\cdot\omega) \in AC^m([R_1, R_2])$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $\forall\, \omega \in S^{N-1}$; $\frac{\partial^k f}{\partial r^k}$, $k = 1, ..., m-1$ vanish on $\partial B(0, R_2)$; $\partial^{\alpha}_{R_2-} f(\cdot\omega) \in L_\infty([R_1, R_2])$, $\forall\, \omega \in S^{N-1}$; and $\frac{\partial^{\alpha}_{R_2-} f}{\partial r^\alpha} \in B\left(\overline{A}\right)$ (bounded functions on $\overline{A}$). Then, for $x \in \overline{A}$, we have*

$$\left| f(x) - \frac{\int_A f(x)\,dx}{Vol(A)} \right|$$

$$\leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R_2\omega)\,d\omega \right|$$

$$+ \left(\frac{N!}{R_2^N - R_1^N} \right) \left(\sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)!\,\Gamma(\alpha+2+k)} \right) \left\| \frac{\partial^{\alpha}_{R_2-} f}{\partial r^\alpha} \right\|_{\infty, \overline{A}}. \quad (3.32)$$

We also make

Remark 3.11. Let $f : \overline{B(0, R)} \to \mathbb{R}$ be a Lebesgue integrable function, that is, not necessarily a radial function. Assume $f(\cdot\omega) \in AC^1([0, R])$, $\forall\, \omega \in S^{N-1}$; $0 < \alpha < 1$, and $D^{\alpha}_{R-} f(\cdot\omega) \in L_\infty([0, R])$, $\forall\, \omega \in S^{N-1}$. Clearly here we obtain

$$f(s\omega) - f(R\omega) = \frac{1}{\Gamma(\alpha)} \int_s^R (J - s)^{\alpha-1} D^{\alpha}_{R-} f(J\omega)\,dJ, \quad (3.33)$$

$\forall\, \omega \in S^{N-1}$, $\forall\, s \in [0, R]$.

We further suppose that

$$\| D^{\alpha}_{R-} f(J\omega) \|_{\infty, (J \in [0, R])} \leq K, \ \ \forall\, \omega \in S^{N-1},$$

where $K > 0$.

Applying the earlier Theorem 3.4 we get

$$\left| f(R\omega) - \frac{N}{R^N} \int_0^R f(s\omega) s^{N-1} \mathrm{d}s \right| \leq \left(\| D_{R-}^{\alpha} f(t\omega) \|_{\infty,(t\in[0,R])} \right) \frac{N! R^{\alpha}}{\Gamma(\alpha+N+1)} \leq \frac{K N! R^{\alpha}}{\Gamma(\alpha+N+1)}. \tag{3.34}$$

Consequently we obtain

$$\left| \frac{\int_{S^{N-1}} f(R\omega) \mathrm{d}\omega}{\omega_N} - \frac{N}{R^N \omega_N} \int_{S^{N-1}} \left(\int_0^R f(s\omega) s^{N-1} \mathrm{d}s \right) \mathrm{d}\omega \right| \leq \frac{K N! R^{\alpha}}{\Gamma(\alpha+N+1)}. \tag{3.35}$$

Hence

$$\left| \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R\omega) \mathrm{d}\omega - \frac{\int_{B(0,R)} f(x) \mathrm{d}x}{Vol(B(0,R))} \right| \leq \frac{K N! R^{\alpha}}{\Gamma(\alpha+N+1)}. \tag{3.36}$$

Consequently it holds

$$\begin{aligned} &\left| f(R\omega) - \frac{\int_{B(0,R)} f(x) \mathrm{d}x}{Vol(B(0,R))} \right| \\ &= \left| f(R\omega) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R\omega) \mathrm{d}\omega \right. \\ &\quad \left. + \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R\omega) \mathrm{d}\omega - \frac{\int_{B(0,R)} f(x) \mathrm{d}x}{Vol(B(0,R))} \right| \\ &\leq \left| f(R\omega) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R\omega) \mathrm{d}\omega \right| + \frac{K N! R^{\alpha}}{\Gamma(\alpha+N+1)}. \end{aligned} \tag{3.37}$$

So we have proved the Ostrowski inequality.

Theorem 3.12. *Let* $f : \overline{B(0,R)} \to \mathbb{R}$ *be a Lebesgue integrable function, not necessarily radial. Assume* $f(\cdot\omega) \in AC^1([0,R])$, $R > 0$, $\forall\ \omega \in S^{N-1}$; $0 < \alpha < 1$, *and* $D_{R-}^{\alpha} f(\cdot\omega) \in L_{\infty}([0,R])$, $\forall\ \omega \in S^{N-1}$. *Suppose also that* $\left\| D_{R-}^{\alpha} f(t\omega) \right\|_{\infty,(t\in[0,R])} \leq K$, $\forall\ \omega \in S^{N-1}$, *where* $K > 0$. *Then*

$$\left| f(R\omega) - \frac{\int_{B(0,R)} f(x) \mathrm{d}x}{Vol(B(0,R))} \right| \leq \left| f(R\omega) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R\omega) \mathrm{d}\omega \right| + \frac{K N! R^{\alpha}}{\Gamma(\alpha+N+1)}. \tag{3.38}$$

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Chapter 4
Univariate Mixed Fractional Ostrowski Inequalities

Here we give general univariate mixed Caputo fractional Ostrowski inequalities, one is proved sharp and attained. Estimates are with respect to $\|\cdot\|_p$, $1 \leq p \leq \infty$. This chapter is based on [4].

4.1 Introduction

In 1938, A. Ostrowski [9] proved the following important inequality.

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{a-b} \int_a^b f(t)\mathrm{d}t - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \cdot (b-a) \left\| f' \right\|_\infty, \tag{4.1}$$

for any $x \in [a,b]$.The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to numerical analysis and probability.

This chapter is greatly motivated and inspired also by the following result.

Theorem 4.2 (see [1]). *Let $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$ and $x \in [a,b]$ be fixed, such that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then it holds*

$$\left| \frac{1}{b-a} \int_a^b f(y)\mathrm{d}y - f(x) \right| \leq \frac{\left\| f^{(n+1)} \right\|_\infty}{(n+2)!} \cdot \left(\frac{(x-a)^{n+2} + (b-x)^{n+2}}{b-a} \right). \tag{4.2}$$

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Inequality (4.2) is sharp. In particular, when n is odd is attained by $f^*(y) := (y-x)^{n+1} \cdot (b-a)$, *while when n is even the optimal function is*

$$\overline{f}(y) := |y-x|^{n+\alpha}(b-a), \quad \alpha > 1.$$

Clearly inequality (4.2) generalizes inequality (4.1) for higher order derivatives of f.

Also in [2], see Chaps. 24–26, we gave a complete theory of left fractional Ostrowski inequalities.

Here we combine both right and left Caputo fractional derivatives and produce Ostrowski inequalities.

For the concepts of right and left Caputo fractional Calculus we use here, we refer to [3–8, 10].

4.2 Main Results

We make

Remark 4.3. Let $[a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$. Let $f \in AC^m([a,b])$, $x_0 \in [a,b]$ (i.e., $f^{(m-1)} \in AC([a,b])$). Thus $f \in AC^m([a,x_0])$ and $f \in AC^m([x_0,b])$. Consequently, by [5, p. 40],

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-J)^{\alpha-1} D_{*x_0}^{\alpha} f(J)\mathrm{d}J, \quad \forall x \in [x_0,b].$$

And by [3],

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{\Gamma(\alpha)}\int_{x}^{x_0}(J-x)^{\alpha-1} D_{x_0-}^{\alpha} f(J)\mathrm{d}J, \quad \forall x \in [a,x_0].$$

Here $D_{*x_0}^{\alpha} f$, $D_{x_0-}^{\alpha} f$ are the left and right Caputo fractional derivatives of order α.

Suppose $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. Then

$$f(x) - f(x_0) = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-J)^{\alpha-1} D_{*x_0}^{\alpha} f(J)\mathrm{d}J, \quad \forall x \in [x_0,b] \tag{4.3}$$

and

$$f(x) - f(x_0) = \frac{1}{\Gamma(\alpha)}\int_{x}^{x_0}(J-x)^{\alpha-1} D_{x_0-}^{\alpha} f(J)\mathrm{d}J, \quad \forall x \in [a,x_0]. \tag{4.4}$$

Hence

$$\begin{aligned}|f(x)-f(x_0)| &\le \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-J)^{\alpha-1}\left|D_{*x_0}^{\alpha}f(J)\right|\mathrm{d}J\\ &\le \frac{1}{\Gamma(\alpha)}\left(\int_{x_0}^{x}(x-J)^{\alpha-1}\mathrm{d}J\right)\left\|D_{*x_0}^{\alpha}f\right\|_{\infty,[x_0,b]}\\ &= \frac{1}{\Gamma(\alpha)}\frac{(x-x_0)^{\alpha}}{\alpha}\left\|D_{*x_0}^{\alpha}f\right\|_{\infty,[x_0,b]}. \end{aligned} \tag{4.5}$$

Hence

$$|f(x)-f(x_0)| \le \frac{\left\|D_{*x_0}^{\alpha}f\right\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}, \quad \forall x\in[x_0,b]. \tag{4.6}$$

Similarly it holds

$$\begin{aligned}|f(x)-f(x_0)| &\le \frac{1}{\Gamma(\alpha)}\left(\int_{x}^{x_0}(J-x)^{\alpha-1}\mathrm{d}J\right)\left\|D_{x_0-}^{\alpha}f\right\|_{\infty,[a,x_0]}\\ &= \frac{1}{\Gamma(\alpha)}\frac{(x_0-x)^{\alpha}}{\alpha}\left\|D_{x_0-}^{\alpha}f\right\|_{\infty,[a,x_0]}\\ &= \frac{\left\|D_{x_0-}^{\alpha}f\right\|_{\infty,[\alpha,x_0]}}{\Gamma(\alpha+1)}(x_0-x)^{\alpha}, \end{aligned}$$

that is,

$$|f(x)-f(x_0)| \le \frac{\left\|D_{x_0-}^{\alpha}f\right\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)}(x_0-x)^{\alpha}, \ \forall x\in[a,x_0]. \tag{4.7}$$

Next we observe that

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-f(x_0)\right| &= \frac{1}{b-a}\left|\int_a^b (f(x)-f(x_0))\mathrm{d}x\right|\\ &\le \frac{1}{b-a}\int_a^b |f(x)-f(x_0)|\,\mathrm{d}x \qquad (4.8)\\ &= \frac{1}{b-a}\left\{\int_a^{x_0}|f(x)-f(x_0)|\,\mathrm{d}x\right.\\ &\qquad \left.+\int_{x_0}^{b}|f(x)-f(x_0)|\,\mathrm{d}x\right\}\end{aligned}$$

$$\leq \frac{1}{(b-a)\Gamma(\alpha+1)} \left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]} \right.$$

$$\left. \int_a^{x_0} (x_0-x)^{\alpha} \mathrm{d}x + \left\| D^{\alpha}_{*x_0} \right\|_{\infty,[x_0,b]} \int_{x_0}^{b} (x-x_0)^{\alpha} \mathrm{d}x \right\}$$

$$= \frac{1}{(b-a)\Gamma(\alpha+1)} \left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]} \left(\frac{(x_0-a)^{\alpha+1}}{\alpha+1} \right) \right.$$

$$\left. + \left\| D^{\alpha}_{*x_0} f \right\|_{\infty,[x_0,b]} \left(\frac{(b-x_0)^{\alpha+1}}{\alpha+1} \right) \right\}$$

$$= \frac{1}{(b-a)\Gamma(\alpha+2)} \left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]} (x_0-a)^{\alpha+1} \right.$$

$$\left. + \left\| D^{\alpha}_{*x_0} f \right\|_{\infty,[x_0,b]} (b-x_0)^{\alpha+1} \right\}. \tag{4.9}$$

We have established that

Theorem 4.4. *Let* $[a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$, *and* $\left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]}, \left\| D^{\alpha}_{*x_0} \right\|_{\infty,[x_0,b]} < \infty$, $x_0 \in [a,b]$. *Assume* $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x - f(x_0) \right| \leq \frac{1}{(b-a)\Gamma(\alpha+2)} \left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]} (x_0-a)^{\alpha+1} \right.$$

$$\left. + \left\| D^{\alpha}_{*x_0} f \right\|_{\infty,[x_0,b]} (b-x_0)^{\alpha+1} \right\}$$

$$\leq \frac{1}{\Gamma(\alpha+2)} \max \left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]}, \right.$$

$$\left. \times \left\| D^{\alpha}_{*x_0} f \right\|_{\infty,[x_0,b]} \right\} (b-a)^{\alpha}. \tag{4.10}$$

We also make

Remark 4.5. As before we have ($\alpha \geq 1$)

$$|f(x) - f(x_0)| \leq \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-J)^{\alpha-1} \left| D^{\alpha}_{*x_0} f(J) \right| \mathrm{d}J$$

$$\leq \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} \int_{x_0}^{x} \left| D^{\alpha}_{*x_0} f(J) \right| \mathrm{d}J$$

$$\leq \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} \left\| D^{\alpha}_{*x_0} \right\|_{L_1([x_0,b])}.$$

That is, we get

$$|f(x)-f(x_0)| \leq \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{*x_0}^{\alpha} f \right\|_{L_1([x_0,b])}, \quad \forall x \in [x_0,b]. \tag{4.11}$$

Similarly we derive ($\alpha \geq 1$)

$$\begin{aligned}
|f(x)-f(x_0)| &\leq \frac{1}{\Gamma(\alpha)} \int_x^{x_0} (J-x)^{\alpha-1} \left| D_{x_0-}^{\alpha} f(J) \right| \mathrm{d}J \\
&\leq \frac{1}{\Gamma(\alpha)} (x_0-x)^{\alpha-1} \int_x^{x_0} \left| D_{x_0-}^{\alpha} f(J) \right| \mathrm{d}J \\
&\leq \frac{1}{\Gamma(\alpha)} (x_0-x)^{\alpha-1} \left\| D_{x_0-}^{\alpha} f \right\|_{L_1([a,x_0])}.
\end{aligned}$$

That is,

$$|f(x)-f(x_0)| \leq \frac{(x_0-x)^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{x_0-}^{\alpha} f \right\|_{L_1([a,x_0])}, \quad \forall x \in [a,x_0]. \tag{4.12}$$

As before we have

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x - f(x_0) \right| \leq \frac{1}{b-a} \int_a^b |f(x)-f(x_0)| \,\mathrm{d}x \\
&= \frac{1}{b-a} \left\{ \int_a^{x_0} |f(x)-f(x_0)| \,\mathrm{d}x + \int_{x_0}^b |f(x)-f(x_0)| \,\mathrm{d}x \right\} \qquad (4.13) \\
&\leq \frac{1}{(b-a)\Gamma(\alpha)} \left\{ \left(\int_a^{x_0} (x_0-x)^{\alpha-1} \mathrm{d}x \right) \left\| D_{x_0-}^{\alpha} f \right\|_{L_1([a,x_0])} \right. \\
&\qquad \left. + \left(\int_{x_0}^b (x-x_0)^{\alpha-1} \mathrm{d}x \right) \left\| D_{*x_0}^{\alpha} f \right\|_{L_1([x_0,b])} \right\} \\
&= \frac{1}{(b-a)\Gamma(\alpha+1)} \left\{ (x_0-a)^{\alpha} \left\| D_{x_0-}^{\alpha} f \right\|_{L_1([a,x_0])} \right. \\
&\qquad \left. + (b-x_0)^{\alpha} \left\| D_{*x_0}^{\alpha} f \right\|_{L_1([x_0,b])} \right\}. \qquad (4.14)
\end{aligned}$$

We have established

Theorem 4.6. *Let* $\alpha \geq 1$, $m = \lceil \alpha \rceil$, *and* $f \in AC^m([a,b])$. *Suppose that* $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$, $x_0 \in [a,b]$ *and* $D_{x_0-}^{\alpha} f \in L_1([a,x_0])$, $D_{*x_0}^{\alpha} f \in L_1([x_0,b])$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f(x_0)\right|$$
$$\leq \frac{1}{(b-a)\Gamma(\alpha+1)}\left\{(x_0-a)^{\alpha}\left\|D_{x_0-}^{\alpha}f\right\|_{L_1([a,x_0])} + (b-x_0)^{\alpha}\left\|D_{*x_0}^{\alpha}f\right\|_{L_1([x_0,b])}\right\}$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\max\left\{\left\|D_{x_0-}^{\alpha}f\right\|_{L_1([a,x_0])}, \left\|D_{*x_0}^{\alpha}f\right\|_{L_1([x_0,b])}\right\} \cdot (b-a)^{\alpha-1}. \tag{4.15}$$

We further make

Remark 4.7. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > 1 - \frac{1}{p}$. Working as before on (4.3) and (4.4) and using Hölder's inequality we obtain

$$|f(x) - f(x_0)| \leq \frac{1}{\Gamma(\alpha)}\frac{(x-x_0)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}}\left\|D_{*x_0}^{\alpha}f\right\|_{L_q([x_0,b])}, \ \forall x \in [x_0, b]. \tag{4.16}$$

And also it holds

$$|f(x) - f(x_0)| \leq \frac{1}{\Gamma(\alpha)}\frac{(x_0-x)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}}\left\|D_{x_0-}^{\alpha}f\right\|_{L_q([a,x_0])}, \ \forall x \in [a, x_0]. \tag{4.17}$$

We observe that

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f(x_0)\right|$$
$$\leq \frac{1}{b-a}\left\{\int_a^{x_0}|f(x)-f(x_0)|\,\mathrm{d}x + \int_{x_0}^b |f(x)-f(x_0)|\,\mathrm{d}x\right\}$$
$$\leq \frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}}\left\{\left(\int_a^{x_0}(x_0-x)^{\alpha-1+\frac{1}{p}}\mathrm{d}x\right) \times \left\|D_{x_0-}^{\alpha}f\right\|_{L_q([a,x_0])} + \left(\int_{x_0}^b (x-x_0)^{\alpha-1+\frac{1}{p}}\mathrm{d}x\right)\left\|D_{*x_0}f\right\|_{L_q([x_0,b])}\right\} \tag{4.18}$$

$$= \frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}\left(\alpha+\frac{1}{p}\right)} \times \left\{(x_0-a)^{\alpha+\frac{1}{p}}\left\|D_{x_0-}^{\alpha}f\right\|_{L_q([a,x_0])} + (b-x_0)^{\alpha+\frac{1}{p}}\left\|D_{*x_0}f\right\|_{L_q([x_0,b])}\right\}. \tag{4.19}$$

We have proved

Theorem 4.8. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 1 - \frac{1}{p}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, *and* $f \in AC^m([a,b])$. *Suppose that* $f^{(k)}(x_0) = 0$, $k = 1, \ldots, m-1$, $x_0 \in [a,b]$. *Assume* $D^{\alpha}_{x_0-} f \in L_q([a, x_0])$, *and* $D^{\alpha}_{*x_0} f \in L_q([x_0, b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x - f(x_0) \right| \leq \frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}\left(\alpha + \frac{1}{p}\right)} \times \left\{ (x_0 - a)^{\alpha + \frac{1}{p}} \left\| D^{\alpha}_{x_0-} f \right\|_{L_q([a,x_0])} + (b - x_0)^{\alpha + \frac{1}{p}} \left\| D^{\alpha}_{*x_0} f \right\|_{L_q([x_0,b])} \right\} \tag{4.20}$$

$$\leq \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}\left(\alpha + \frac{1}{p}\right)} \max \left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{L_q([a,x_0])}, \left\| D^{\alpha}_{*x_0} f \right\|_{L_q([x_0,b])} \right\} \cdot (b-a)^{\alpha - 1/q}. \tag{4.21}$$

Corollary 4.9. *Let* $p = q = 2$, $\alpha > \frac{1}{2}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, *and* $f \in AC^m([a,b])$. *Assume* $f^{(k)}(x_0) = 0$, $k = 1, \ldots, m-1$, $x_0 \in [a,b]$. *Assume* $D^{\alpha}_{x_0-} f \in L_2([a, x_0])$, *and* $D^{\alpha}_{*x_0} f \in L_2([x_0, b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x - f(x_0) \right| \leq \frac{1}{(b-a)\Gamma(\alpha)(2\alpha-1)^{1/2}\left(\alpha + \frac{1}{2}\right)} \cdot \left\{ (x_0 - a)^{\alpha + \frac{1}{2}} \left\| D^{\alpha}_{x_0-} f \right\|_{L_2([a,x_0])} + (b - x_0)^{\alpha + \frac{1}{2}} \left\| D^{\alpha}_{*x_0} f \right\|_{L_2([x_0,b])} \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)\sqrt{2\alpha - 1}\left(\alpha + \frac{1}{2}\right)} \cdot \max \left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{L_2([a,x_0])}, \left\| D^{\alpha}_{*x_0} f \right\|_{L_2([x_0,b])} \right\} \cdot (b-a)^{\alpha - \frac{1}{2}}. \tag{4.22}$$

We further make

Remark 4.10. Here $\alpha > 0,\ a \leq x_0 \leq b$. Let

$$\overline{f}(x) := \begin{cases} (x - x_0)^{\alpha},\ x \in [x_0, b], \\ (x_0 - x)^{\alpha},\ x \in [a, x_0]. \end{cases} \tag{4.23}$$

See $\overline{f}$ is in $AC^m([x_0, b])$, and in $AC^m([a, x_0])$.

See that $\overline{f}_-^{(k)}(x_0) = \overline{f}_+^{(k)}(x_0) = 0,\ k = 0, 1, ...,\ m - 1$.

Hence there exists $\overline{f}^{(m-1)}$ at x_0, also $\overline{f}^{(m-1)} \in AC[a, b]$.

That is, $\overline{f} \in AC^m[a, b]$.

We find that

$$\left\| D_{x_0-}^{\alpha} \overline{f} \right\|_{\infty,[a,x_0]} = \Gamma(\alpha + 1) \tag{4.24}$$

and

$$\left\| D_{*x_0}^{\alpha} \overline{f} \right\|_{\infty,[x_0,b]} = \Gamma(\alpha + 1). \tag{4.25}$$

Consequently

$$\begin{aligned} \text{R.H.S. } (4.10) &= \frac{\Gamma(\alpha + 1)(\alpha + 1)}{(b - a)\Gamma(\alpha + 2)(\alpha + 1)} \left\{(x_0 - a)^{\alpha+1} + (b - x_0)^{\alpha+1}\right\} \\ &= \frac{1}{(b - a)(\alpha + 1)} \left\{(x_0 - \alpha)^{\alpha+1} + (b - x_0)^{\alpha+1}\right\}. \end{aligned}$$

Also we observe that

$$\begin{aligned} \text{L.H.S } (4.10) &= \frac{1}{b - a} \left\{ \int_a^{x_0} (x_0 - x)^{\alpha} \mathrm{d}x + \int_{x_0}^{b} (x - x_0)^{\alpha} \mathrm{d}x \right\} \\ &= \frac{1}{b - a)} \left\{ \frac{(x_0 - a)^{\alpha+1}}{\alpha + 1} + \frac{(b - x_0)^{\alpha+1}}{\alpha + 1} \right\} \\ &= \frac{1}{(b - a)(\alpha + 1)} \left\{(x_0 - a)^{\alpha+1} + (b - x_0)^{\alpha+1}\right\}. \end{aligned} \tag{4.26}$$

Therefore inequality (4.10) is sharp and attained.

We have proved

Proposition 4.11. *Inequality (4.10) is sharp, in particular it is attained.*

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Chapter 5
Multivariate Radial Mixed Fractional Ostrowski Inequalities

Here we give general multivariate radial mixed Caputo fractional Ostrowski inequalities. One of them is proved sharp and attained. Estimates are with respect to $\|\cdot\|_p$, $1 \leq p \leq \infty$. This chapter relies on [4].

5.1 Introduction

In 1938, A. Ostrowski [11] proved the following important inequality.

Theorem 5.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b),i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f(x)\right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\cdot(b-a)\,\|f'\|_\infty, \tag{5.1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to numerical analysis and probability.

This chapter is greatly motivated and inspired also by the following result.

Theorem 5.2 (see [1]). *Let $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$ and $x \in [a,b]$ fixed, such that $f^{(k)}(x) = 0$, $k = 1,...,n$. Then it holds*

$$\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y - f(x)\right| \leq \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+2)!}\cdot\left(\frac{(x-a)^{n+2}+(b-x)^{n+2}}{b-a}\right). \tag{5.2}$$

G.A. Anastassiou, *Advances on Fractional Inequalities*, SpringerBriefs in Mathematics,
DOI 10.1007/978-1-4614-0703-4_5,

Inequality (5.2) is sharp. In particular, when n is odd is attained by $f^*(y) := (y-x)^{n+1} \cdot (b-a)$, *while when n is even the optimal function is*

$$\overline{f}(y) := |y-x|^{n+\alpha} \cdot (b-a),\ \alpha > 1.$$

Clearly inequality (5.2) generalizes inequality (5.1) for higher order derivatives of f.

Also in [2] see Chaps. 24–26, we gave a complete theory of left fractional Ostrowski inequalities.

Here we combine both right and left Caputo fractional derivatives and produce Ostrowski inequalities in the multivariate setting for radial functions.

For the concepts of right and left Caputo fractional calculus we use here, we refer to [3–8, 12].

5.2 Main Results

We make

Remark 5.3. We define the ball

$$B(0,R) := \{x \in \mathbb{R}^N : |x| < R\} \subseteq \mathbb{R}^N,\ N \geq 2,\ R > 0$$

and the sphere

$$S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\},$$

where $|\cdot|$ is the Euclidean norm.

Let $d\omega$ be the element of surface measure on S^{N-1} and let

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{N/2}}{\Gamma(N/2)}.$$

For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$ and $\omega = x/r \in S^{N-1}$, $|\omega| = 1$. Note that

$$\int_{B(0,R)} dy = \frac{\omega_N R^N}{N}$$

is the Lebesgue measure of the ball. Following [9, pp. 149–150, Exercise 6] and [10, pp. 87–88, Theorem 5.2.2] we can write for $F : \overline{B(0,R)} \to \mathbb{R}$ a Lebesgue integrable function that

$$\int_{B(0,R)} F(x)dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega) r^{N-1} dr \right) d\omega; \tag{5.3}$$

we use this formula often.

Here the function $f : \overline{B(0,R)} \to \mathbb{R}$ is radial, that is, there exists a function g such that $f(x) = g(r)$, where $r = |x|$, $r \in [0, R]$, $\forall x \in \overline{B(0,R)}$. We further assume that $g \in AC^m([0, R])$ (i.e., $g^{(m-1)} \in AC([0, R])$), $m = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ the ceiling of the number), $\alpha > 0$ and $g^{(k)}(r_0) = 0$, $k = 1, ..., m-1$, $r_0 \in [0, R]$ fixed. We find by left Caputo Taylor's formula, [5], p. 40, that

$$|g(r) - g(r_0)| \leq \frac{\left\| D^{\alpha}_{*r_0} g \right\|_{\infty,[r_0,R]}}{\Gamma(\alpha+1)} (r - r_0)^{\alpha}, \tag{5.4}$$

$\forall r \in [r_0, R]$, where $D^{\alpha}_{*r_0} g$ is the left Caputo fractional derivative, [5] p. 38.

Similarly, by right Caputo Taylor formula, [3], we find that

$$|g(r) - g(r_0)| \leq \frac{\left\| D^{\alpha}_{r_0-} g \right\|_{\infty,[0,r_0]}}{\Gamma(\alpha+1)} (r_0 - r)^{\alpha}, \tag{5.5}$$

$\forall r \in [0, r_0]$, where $D^{\alpha}_{r_0-} g$ is the right Caputo fractional derivative see [6–8].

Next we observe

$$\begin{aligned}
&\left| f(r_0\omega) - \frac{\int_{B(0,R)} f(y)\mathrm{d}y}{Vol(B(0,R))} \right| \\
&= \left| g(r_0) - \frac{\int_{S^{N-1}} \left(\int_0^R g(s) s^{N-1} \mathrm{d}s \right) \mathrm{d}\omega}{\int_{S^{N-1}} \left(\int_0^R s^{N-1} \mathrm{d}s \right) \mathrm{d}\omega} \right| \\
&= \left| g(r_0) - \frac{N}{R^N} \int_0^R g(s) s^{N-1} \mathrm{d}s \right| \\
&= \frac{N}{R^N} \left| \int_0^R s^{N-1} (g(r_0) - g(s)) \mathrm{d}s \right| \\
&\leq \frac{N}{R^N} \left[\int_0^{r_0} s^{N-1} |g(r_0) - g(s)| \mathrm{d}s + \int_{r_0}^R s^{N-1} |g(r_0) - g(s)| \mathrm{d}s \right] \\
&\leq \frac{N}{\Gamma(\alpha+1) R^N} \left[\left\| D^{\alpha}_{r_0-} g \right\|_{\infty,[0,r_0]} \int_0^{r_0} s^{N-1} (r_0 - s)^{\alpha} \mathrm{d}s \right. \\
&\qquad \left. + \left\| D^{\alpha}_{*r_0} g \right\|_{\infty,[r_0,R]} \int_{r_0}^R s^{N-1} (s - r_0)^{\alpha} \mathrm{d}s \right] \qquad (5.6) \\
&= \frac{N}{\Gamma(\alpha+1) R^N} \left[\left\| D^{\alpha}_{r_0-} g \right\|_{\infty,[0,r_0]} \int_0^{r_0} (r_0 - s)^{(\alpha+1)-1} (s - 0)^{N-1} \mathrm{d}s \right. \\
&\qquad + \left\| D^{\alpha}_{*r_0} g \right\|_{\infty,[r_0,R]} \cdot (-1)^{N-1} \\
&\qquad \left. \times \int_{r_0}^R ((R - s) - R)^{N-1} (s - r_0)^{\alpha} \mathrm{d}s \right]
\end{aligned}$$

$$= \frac{N}{\Gamma(\alpha+1)R^N}\left[\left\|D_{r_0-}^{\alpha}g\right\|_{\infty,[0,r_0]}\frac{\Gamma(\alpha+1)(N-1)!}{\Gamma(\alpha+N+1)}r_0^{\alpha+N}\right.$$

$$+\left\|D_{*r_0}^{\alpha}g\right\|_{\infty,[r_0,R]}\cdot(-1)^{N-1}\sum_{k=0}^{N-1}(-1)^k R^k\binom{N-1}{k}$$

$$\left.\times\int_{r_0}^{R}(R-s)^{N-k-1}\cdot(s-r_0)^{(\alpha+1)-1}\mathrm{d}s\right] \tag{5.7}$$

$$= \frac{N}{\Gamma(\alpha+1)R^N}\left[\left\|D_{r_0-}^{\alpha}g\right\|_{\infty,[0,r_0]}\cdot\frac{\Gamma(\alpha+1)(N-1)!}{\Gamma(\alpha+N+1)}r_0^{\alpha+N}\right.$$

$$+\left\|D_{*r_0}^{\alpha}g\right\|_{\infty,[r_0,R]}\cdot\sum_{k=0}^{N-1}(-1)^{N+k-1}R^k\frac{(N-1)!}{k!(N-k-1)!}$$

$$\left.\times\frac{(N-k-1)!\Gamma(\alpha+1)}{\Gamma(N-k+\alpha+1)}(R-r_0)^{(N-k+\alpha)}\right]$$

$$= \frac{N!}{R^N}\left[\left\|D_{r_0-}^{\alpha}g\right\|_{\infty,[0,r_0]}\cdot\frac{r_0^{\alpha+N}}{\Gamma(\alpha+N+1)}\right.$$

$$+\left\|D_{*r_0}^{\alpha}g\right\|_{\infty,[r_0,R]}\cdot\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}}{k!\Gamma(N+1-k+\alpha)}\right.$$

$$\left.\left.\times R^k(R-r_0)^{(N-k+\alpha)}\right)\right]. \tag{5.8}$$

Hence in the radial mixed case we proved

$$\left|f(r_0\omega)-\frac{\int_{B(0,R)}f(y)\mathrm{d}y}{Vol(B(0,R))}\right|$$

$$=\left|g(r_0)-\frac{N}{R^N}\int_0^R g(s)s^{N-1}\mathrm{d}s\right|$$

$$\le\frac{N!}{R^N}\left[\left\|D_{r_0-}^{\alpha}g\right\|_{\infty,[0,r_0]}\frac{r_0^{\alpha+N}}{\Gamma(\alpha+N+1)}\right.$$

$$\left.+\left\|D_{*r_0}^{\alpha}g\right\|_{\infty,[r_0,R]}\left(\sum_{k=0}^{N-1}\frac{(-1)^{(N+k-1)}R^k(R-r_0)^{(N-k+\alpha)}}{k!\Gamma(N+1-k+\alpha)}\right)\right]. \tag{5.9}$$

The last inequality (5.9) is attained by

$$\overline{g}(s)=\begin{cases}(s-r_0)^{\alpha}, & s\in[r_0,R],\\ (r_0-s)^{\alpha}, & s\in[0,r_0].\end{cases} \tag{5.10}$$

Hence $\overline{g}\in AC^m([r_0,R])$ and $\overline{g}\in AC^m([0,r_0])$.

See that $\overline{g}_{-}^{(k)}(r_0) = \overline{g}_{+}^{(k)}(r_0) = 0,\ k = 0, 1, ..., m-1.$
Hence there exists

$$\overline{g}^{(m-1)} \text{ at } r_0, \text{ and also } \overline{g}^{(m-1)} \in AC([0, R]), \text{ therefore } \overline{g} \in AC^m([0, R]).$$

But we have that

$$\left\| D_{r_0-}^{\alpha} \overline{g} \right\|_{\infty,[0,r_0]} = \Gamma(\alpha+1) = \left\| D_{*r_0}^{\alpha} \overline{g} \right\|_{\infty,[r_0,R]}. \tag{5.11}$$

We observe that

$$\text{R.H.S. (5.9)} = \frac{N!\Gamma(\alpha+1)}{R^N} \left[\frac{r_0^{\alpha+N}}{\Gamma(\alpha+N+1)} + \sum_{k=0}^{N-1} \frac{(-1)^{(N+k-1)} R^k (R-r_0)^{(N-k+\alpha)}}{k!\Gamma(N+1-k+\alpha)} \right]. \tag{5.12}$$

And also it holds

$$\text{L.H.S. (5.9)} = \frac{N}{R^N} \int_0^R \overline{g}(s) s^{N-1} \mathrm{d}s$$

$$= \frac{N}{R^N} \times \left[\int_0^{r_0} (r_0 - s)^{\alpha} s^{N-1} \mathrm{d}s + \int_{r_0}^{R} (s - r_0)^{\alpha} s^{N-1} \mathrm{d}s \right] \tag{5.13}$$

$$= \frac{N}{R^N} \left[\frac{\Gamma(\alpha+1)(N-1)!}{\Gamma(\alpha+1+N)} r_0^{\alpha+N} + (-1)^{N-1} \int_{r_0}^{R} ((R-s)-R)^{N-1} (s-r_0)^{\alpha} \mathrm{d}s \right]$$

$$= \frac{N}{R^N} \left[\frac{\Gamma(\alpha+1)(N-1)!}{\Gamma(\alpha+1+N)} r_0^{\alpha+N} + (-1)^{N-1} \times \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k R^k \int_{r_0}^{R} (R-s)^{N-k-1} (s-r_0)^{(\alpha+1)-1} \mathrm{d}s \right]$$

$$= \frac{N}{R^N} \left[\frac{\Gamma(\alpha+1)(N-1)!}{\Gamma(\alpha+1+N)} r_0^{\alpha+N} + (-1)^{N-1} \times \sum_{k=0}^{N-1} \frac{(N-1)!}{k!(N-k-1)!} (-1)^k R^k (R-r_0)^{N-k+\alpha} \frac{(N-k-1)!\Gamma(\alpha+1)}{\Gamma(N-k+\alpha+1)} \right]$$

$$= \frac{N!\Gamma(\alpha+1)}{R^N} \left[\frac{r_0^{\alpha+N}}{\Gamma(\alpha+N+1)} + \sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R^k (R-r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)} \right], \tag{5.14}$$

proving (5.9) attained and sharp.

We have proved

Theorem 5.4. *Let* $f : \overline{B(0,R)} \to \mathbb{R}$ *which is radial; that is, there exists* g *such that* $f(x) = g(r)$, $r = |x|$, $\forall x \in \overline{B(0,R)}$. *Assume that* $g \in AC^m([0,R])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, *and* $g^{(k)}(r_0) = 0$, $k = 1, ..., m-1$, $r_0 \in [0,R]$ *fixed, and*

$$\left\| D_{r_0-}^{\alpha} g \right\|_{\infty,[0,r_0]}, \left\| D_{*r_0}^{\alpha} g \right\|_{\infty,[r_0,R]} < \infty.$$

Then $(\forall \omega \in s^{N-1})$

$$\left| f(r_0\omega) - \frac{\int_{B(0,R)} f(y)\mathrm{d}y}{Vol(B(0,R))} \right| \tag{5.15}$$

$$= \left| g(r_0) - \frac{N}{R^N} \int_0^R g(s) s^{N-1} \mathrm{d}s \right|$$

$$\leq \frac{N!}{R^N} \left[\left\| D_{r_0-}^{\alpha} g \right\|_{\infty,[0,r_0]} \frac{r_0^{\alpha+N}}{\Gamma(\alpha+N+1)} + \left\| D_{*r_0}^{\alpha} g \right\|_{\infty,[r_0,R]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{(N+k-1)} R^k (R-r_0)^{(N+\alpha-k)}}{k!\Gamma(N+\alpha+1-k)} \right) \right]. \tag{5.16}$$

Inequality (5.16) is sharp, it is attained by

$$\overline{g}(s) = \begin{cases} (s-r_0)^{\alpha}, \ s \in [r_0, R] \\ (r_0-s)^{\alpha}, \ s \in [0, r_0], \ \alpha > 0. \end{cases} \tag{5.17}$$

We make

Remark 5.5. (continuing Remark 5.3)
We have again (by [5], p. 40)

$$g(r) - g(r_0) = \frac{1}{\Gamma(\alpha)} \int_{r_0}^{r} (r-J)^{\alpha-1} D_{*r_0}^{\alpha} g(J) \mathrm{d}J, \tag{5.18}$$

$\forall r \in [r_0, R]$.
And (by [3]), it holds

$$g(r) - g(r_0) = \frac{1}{\Gamma(\alpha)} \int_{r}^{r_0} (J-r)^{\alpha-1} D_{r_0-}^{\alpha} g(J) \mathrm{d}J, \tag{5.19}$$

$\forall r \in [0, r_0]$. Hence for $\alpha \geq 1$ we get

$$|g(r) - g(r_0)| \leq \frac{1}{\Gamma(\alpha)} (r-r_0)^{\alpha-1} \left\| D_{*r_0}^{\alpha} g \right\|_{L_1([r_0,R])}, \tag{5.20}$$

$\forall r \in [r_0, R]$. Also

$$|g(r) - g(r_0)| \leq \frac{1}{\Gamma(\alpha)} (r_0 - r)^{\alpha-1} \left\| D_{r_0-}^{\alpha} g \right\|_{L_1([0,r_0])}, \tag{5.21}$$

$\forall r \in [0, r_0]$.

Hence

$$\begin{aligned}
&\left| f(r_0\omega) - \frac{\int_{B(0,R)} f(y)\mathrm{d}y}{Vol(B(0,R))} \right| \\
&= \left| g(r_0) - \frac{N}{R^N} \int_0^R g(s) s^{N-1} \mathrm{d}s \right| \\
&\leq \frac{N}{R^N} \left[\int_0^{r_0} s^{N-1} |g(r_0) - g(s)| \,\mathrm{d}s + \int_{r_0}^R s^{N-1} |g(r_0) - g(s)| \,\mathrm{d}s \right] \\
&\leq \frac{N}{R^N \Gamma(\alpha)} \left[\left\| D_{r_0-}^{\alpha} g \right\|_{L_1([0,r_0])} \int_0^{r_0} s^{N-1} (r_0 - s)^{\alpha-1} \mathrm{d}s \right. \\
&\qquad \left. + \left\| D_{*r_0}^{\alpha} g \right\|_{L_1([r_0,R])} (-1)^{N-1} \int_{r_0}^R ((R-s) - R)^{N-1} (s - r_0)^{\alpha-1} \mathrm{d}s \right]
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
&= \frac{N!}{R^N \Gamma(\alpha)} \left[\left\| D_{r_0-}^{\alpha} g \right\|_{L_1([0,r_0])} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+N)} r_0^{\alpha+N-1} \right. \\
&\qquad + \left\| D_{*r_0}^{\alpha} g \right\|_{L_1([r_0,R])} \cdot (-1)^{N-1} \\
&\qquad \left. \cdot \sum_{k=0}^{N-1} \frac{(-1)^k R^k (N-k-1)! \Gamma(\alpha) (R - r_0)^{N+\alpha-k-1}}{k!(N-k-1)! \Gamma(N+\alpha-k)} \right] \\
&= \frac{N!}{R^N} \left[\left\| D_{r_0-}^{\alpha} g \right\|_{L_1([0,r_0])} \cdot \frac{r_0^{\alpha+N-1}}{\Gamma(\alpha+N)} \right. \\
&\qquad \left. + \left\| D_{*r_0}^{\alpha} g \right\|_{L_1([r_0,R])} \cdot \sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R^k (R - r_0)^{N+\alpha-k-1}}{k! \Gamma(N+\alpha-k)} \right].
\end{aligned} \tag{5.23}$$

We have proved:

Theorem 5.6. *Let* $f : \overline{B(0,R)} \to \mathbb{R}$ *which is radial, that is, there exists such that* $f(x) = g(r)$, $r = |x|$, $\forall x \in \overline{B(0,R)}$. *Assume that* $g \in AC^m([0,R])$, $m = \lceil \alpha \rceil$, $\alpha \geq 1$, *and* $g^{(k)}(r_0) = 0$, $k = 1, ..., m-1$, $r_0 \in [0,R]$ *fixed, and*

$$\left\| D_{r_0-}^{\alpha} g \right\|_{L_1([0,r_0])}, \left\| D_{*r_0}^{\alpha} g \right\|_{L_1([r_0,R])} < \infty.$$

Then $(\forall \omega \in S^{N-1})$

$$\left| f(r_0\omega) - \frac{\int_{B(0,R)} f(y)\mathrm{d}y}{Vol(B(0,R))} \right| = \left| g(r_0) - \frac{N}{R^N} \int_0^R g(s)s^{N-1}\mathrm{d}s \right|$$
$$\leq \frac{N!}{R^N} \left[\left\| D^{\alpha}_{r_0-} g \right\|_{L_1([0,r_0])} \cdot \frac{r_0^{\alpha+N-1}}{\Gamma(\alpha+N)} + \left\| D^{\alpha}_{*r_0} g \right\|_{L_1([r_0,R])} \right.$$
$$\left. \times \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R^k (R-r_0)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) \right]. \tag{5.24}$$

We also make

Remark 5.7. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\alpha > 1 - \frac{1}{p}$.
We have

$$|g(r) - g(r_0)| \leq \frac{1}{\Gamma(\alpha)} \int_{r_0}^r (r-J)^{\alpha-1} \left| D^{\alpha}_{*r_0} g(J) \right| \mathrm{d}J$$
$$\leq \frac{1}{\Gamma(\alpha)} \left(\int_{r_0}^r (r-J)^{p(\alpha-1)} \mathrm{d}J \right)^{1/p} \left\| D^{\alpha}_{*r_0} g \right\|_{L_q([r_0,R])}$$
$$= \frac{1}{\Gamma(\alpha)} \frac{(r-r_0)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}} \left\| D^{\alpha}_{*r_0} g \right\|_{L_q([r_0,R])}. \tag{5.25}$$

That is,

$$|g(r) - g(r_0)| \leq \frac{(r-r_0)^{(\alpha-1)+\frac{1}{p}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}} \left\| D^{\alpha}_{*r_0} g \right\|_{L_q([r_0,R])}, \tag{5.26}$$

$\forall r \in [r_0, R]$. Also we have

$$|g(r) - g(r_0)| \leq \frac{1}{\Gamma(\alpha)} \int_r^{r_0} (J-r)^{\alpha-1} \left| D^{\alpha}_{r_0-} g(J) \right| \mathrm{d}J$$
$$\leq \frac{1}{\Gamma(\alpha)} \frac{(r_0-r)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{1/p}} \left\| D^{\alpha}_{r_0-} g \right\|_{L_q([0,r_0])}, \tag{5.27}$$

$\forall r \in [0, r_0]$. Consequently we obtain $(\forall \omega \in S^{N-1})$

$$\left| f(r_0\omega) - \frac{\int_{B(0,R)} f(y)\mathrm{d}y}{Vol(B(0,R))} \right| \tag{5.28}$$
$$= \left| g(r_0) - \frac{N}{R^N} \int_0^R g(s)s^{N-1}\mathrm{d}s \right|$$

$$\leq \frac{N}{R^N}\left[\int_0^{r_0} s^{N-1}\left|g(r_0)-g(s)\right|\mathrm{d}s+\int_{r_0}^{R} s^{N-1}\left|g(r_0)-g(s)\right|\mathrm{d}s\right]$$

$$\leq \frac{N}{\Gamma(\alpha)R^N(p(\alpha-1)+1)^{1/p}}\left[\left\|D_{r_0-}^{\alpha}g\right\|_{L_q([0,r_0])}\cdot\int_0^{r_0} s^{N-1}(r_0-s)^{(\alpha-1)+\frac{1}{p}}\mathrm{d}s\right.$$

$$\left.+\left\|D_{*r_0}^{\alpha}g\right\|_{L_q([r_0,R])}\cdot(-1)^{N-1}\cdot\int_{r_0}^{R}((R-s)-R)^{N-1}(s-r_0)^{(\alpha-1)+\frac{1}{p}}\mathrm{d}s\right] \tag{5.29}$$

$$= \frac{N}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}R^N}\left[\left\|D_{r_0-}^{\alpha}g\right\|_{L_q([0,r_0])}\cdot\frac{(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma\left(N+\alpha+\frac{1}{p}\right)}\right.$$

$$\times r_0^{N+\alpha+\frac{1}{p}-1}+\left\|D_{*r_0}^{\alpha}g\right\|_{L_q([r_0,R])}\sum_{k=0}^{N-1}\frac{(N-1)!}{k!(N-k-1)!}(-1)^{N+k-1}R^k$$

$$\left.\cdot\frac{(N-k-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma\left(N-k+\alpha+\frac{1}{p}\right)}(R-r_0)^{N-k+\alpha+\frac{1}{p}-1}\right]$$

$$= \frac{N!\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}R^N}\left[\left\|D_{r_0-}^{\alpha}g\right\|_{L_q([0,r_0])}\cdot\frac{r_0^{N+\alpha-1/q}}{\Gamma\left(N+\alpha+\frac{1}{p}\right)}\right.$$

$$\left.+\left\|D_{*r_0}^{\alpha}g\right\|_{L_q([r_0,R])}\cdot\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R^k(R-r_0)^{(N-k+\alpha-1/q)}}{k!\Gamma\left(N-k+\alpha+\frac{1}{p}\right)}\right]. \tag{5.30}$$

We have established

Theorem 5.8. *Let* $p,q>1$*:* $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>1-\frac{1}{p}$, $\lceil\alpha\rceil=m$. *Let* $f:\overline{B(0,R)}\to\mathbb{R}$ *which is radial, that is, there exists* g *such that* $f(x)=g(r)$, $r=|x|$, $\forall x\in\overline{B(0,R)}$. *Suppose that* $g\in AC^m([0,R])$, *and* $g^{(k)}(r_0)=0$, $k=1,\dots,m-1$, $r_0\in[0,R]$ *fixed, and*

$$\left\|D_{r_0-}^{\alpha}g\right\|_{L_q([0,r_0])},\left\|D_{*r_0}^{\alpha}g\right\|_{L_q([r_0,R])}<\infty.$$

Then $(\forall\omega\in S^{N-1})$

$$\left|f(r_0\omega)-\frac{\int_{B(0,R)}f(y)\mathrm{d}y}{Vol(B(0,R))}\right|=\left|g(r_0)-\frac{N}{R^N}\int_0^R g(s)s^{N-1}\mathrm{d}s\right|$$

$$\leq\frac{N!\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma(\alpha)(p(\alpha-1)+1)^{1/p}R^N}\left[\left\|D_{r_0-}^{\alpha}g\right\|_{L_q([0,r_0])}\cdot\right.$$

$$\times \frac{r_0^{N+\alpha-\frac{1}{q}}}{\Gamma(N+\alpha+\frac{1}{p})} + \left\| D_{*r_0}^{\alpha} g \right\|_{L_q([r_0,R])}$$

$$\cdot \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R^k (R-r_0)^{\left(N-k+\alpha-\frac{1}{q}\right)}}{k!\Gamma\left(N-k+\alpha+\frac{1}{p}\right)} \right) \Bigg]. \tag{5.31}$$

We give

Corollary 5.9. *Let* $\alpha > \frac{1}{2}$, $\lceil \alpha \rceil = m$, $f : \overline{B(0,R)} \to \mathbb{R}$ *is radial, with* $g : f(x) = g(r)$, $r = |x|$, $\forall x \in \overline{B(0,R)}$. *Suppose* $g \in AC^m([0,R]) : g^{(k)}(r_0) = 0$, $k = 1, ..., m-1$, $r_0 \in [0,R]$ *fixed, and*

$$\left\| D_{r_0-}^{\alpha} g \right\|_{L_2([0,r_0])}, \left\| D_{*r_0}^{\alpha} g \right\|_{L_2([r_0,R])} < \infty.$$

Then $(\forall \omega \in S^{N-1})$

$$\left| f(r_0\omega) - \frac{\int_{B(0,R)} f(y)\mathrm{d}y}{Vol(B(0,R))} \right| = \left| g(r_0) - \frac{N}{R^N} \int_0^R g(s) s^{N-1} \mathrm{d}s \right|$$

$$\leq \frac{N!\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)\sqrt{2\alpha-1}R^N} \Bigg[\left\| D_{r_0-}^{\alpha} g \right\|_{L_2([0,r_0])}$$

$$\times \frac{r_0^{N+\alpha-\frac{1}{2}}}{\Gamma\left(N+\alpha+\frac{1}{2}\right)} + \left\| D_{*r_0}^{\alpha} g \right\|_{L_2([r_0,R])} \cdot$$

$$\times \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R^k (R-r_0)^{\left(N-k+\alpha-\frac{1}{2}\right)}}{k!\Gamma\left(N-k+\alpha+\frac{1}{2}\right)} \right) \Bigg]. \tag{5.32}$$

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Chapter 6
Shell Mixed Caputo Fractional Ostrowski Inequalities

Here we present general shell mixed Caputo fractional Ostrowski inequalities, radial and nonradial cases. One of them is proved to be sharp and attained. Estimates are with respect to $\|\cdot\|_p$, $1 \leq p \leq \infty$. This chapter is based on [4].

6.1 Introduction

In 1938, A. Ostrowski [11] proved the following important inequality.

Theorem 6.1. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *whose derivative* $f' : (a,b) \to \mathbb{R}$ *is bounded on* (a,b), *i.e.,* $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < +\infty$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, \mathrm{d}t - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \cdot (b-a) \|f'\|_\infty, \tag{6.1}$$

for any $x \in [a,b]$. *The constant* $\frac{1}{4}$ *is the best possible.*

Since then there has been a lot of activity around these inequalities with important applications to numerical analysis and probability.

This chapter is greatly motivated and inspired by the following result.

Theorem 6.2 (see [1]). *Let* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$ *and* $x \in [a,b]$ *be fixed, such that* $f^{(k)}(x) = 0$, $k = 1, \ldots, n$. *Then it holds*

$$\left| \frac{1}{b-a} \int_a^b f(y)\, \mathrm{d}y - f(x) \right| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+2)!} \cdot \left(\frac{(x-a)^{n+2} + (b-x)^{n+2}}{b-a} \right). \tag{6.2}$$

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Inequality (6.2) is sharp. In particular, when n is odd is attained by $f^*(y) := (y-x)^{n+1} \cdot (b-a)$*, while when n is even the optimal function is*

$$\overline{f}(y) := |y-x|^{n+\alpha} \cdot (b-a), \quad \alpha > 1.$$

Clearly inequality (6.2) generalizes inequality (6.1) for higher order derivatives of f*.*

Also in [2], see Chaps. 24–26, we gave a complete theory of left fractional Ostrowski inequalities.

Here we combine both right and left Caputo fractional derivatives and produce Ostrowski inequalities in the multivariate setting of a shell for radial and nonradial functions. A nonradial case ball result is given at the end of the chapter. For the nonradial case results we use the left and right fractional radial derivatives. For the basic concepts of fractional calculus used here, we refer to [3, 6–9, 12].

6.2 Main Results

We make

Remark 6.3. Let the spherical shell $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, $x \in \overline{A}$. Consider that $f : \overline{A} \to \mathbb{R}$ is radial, that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\, x \in \overline{A}$. Here x can be written uniquely as $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$, see ([10], pp. 149–150 and [2], p. 421), furthermore for $F : \overline{A} \to \mathbb{R}$ a Lebesgue integrable function we have that

$$\int_A F(x)\,\mathrm{d}x = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega)\, r^{N-1} \mathrm{d}r \right) \mathrm{d}\omega. \tag{6.3}$$

Let $\mathrm{d}\omega$ be the element of surface measure on S^{N-1} and let

$$\omega_N = \int_{S^{N-1}} \mathrm{d}\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}.$$

Here $Vol(A) = \frac{\omega_N \left(R_2^N - R_1^N\right)}{N}$, and we suppose that $g \in AC^m([R_1, R_2])$ (i.e., $g^{(m-1)} \in AC([R_1, R_2])$), $m = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ ceilling of number), $\alpha > 0$, and $g^{(k)}(r_0) = 0$, $k = 1, ..., m-1$, $r_0 \in [R_1, R_2]$.

We get by [6], p. 40, that

$$g(r) - g(r_0) = \frac{1}{\Gamma(\alpha)} \int_{r_0}^{r} (r-J)^{\alpha-1} D_{*r_0}^{\alpha} g(J)\,\mathrm{d}J, \tag{6.4}$$

$\forall\ r \in [r_0, R_2]$, where $D^{\alpha}_{*r_0} g$ is the left Caputo fractional derivative of order α, see [6], p. 38. And, by [3],

$$g(r) - g(r_0) = \frac{1}{\Gamma(\alpha)} \int_r^{r_0} (J - r)^{\alpha - 1} D^{\alpha}_{r_0 -} g(J)\, \mathrm{d}J, \tag{6.5}$$

(where $D^{\alpha}_{r_0 -} g$ is the right Caputo fractional derivative of order α, see [7–9]), $\forall r \in [R_1, r_0]$. Here assume $D^{\alpha}_{*r_0} g \in L_\infty([r_0, R_2])$ and $D^{\alpha}_{r_0 -} g \in L_\infty([R_1, r_0])$. We obtain

$$|g(r) - g(r_0)| \leq \frac{(r - r_0)^{\alpha}}{\Gamma(\alpha + 1)} \left\| D^{\alpha}_{*r_0} g \right\|_{\infty, [r_0, R_2]}, \tag{6.6}$$

$\forall\ r \in [r_0, R_2]$ and

$$|g(r) - g(r_0)| \leq \frac{(r_0 - r)^{\alpha}}{\Gamma(\alpha + 1)} \left\| D^{\alpha}_{r_0 -} g \right\|_{\infty, [R_1, r_0]}, \tag{6.7}$$

$\forall\ r \in [R_1, r_0]$. Next we observe that

$$\left| f(r_0 \omega) - \frac{\int_A f(y)\, \mathrm{d}y}{Vol(A)} \right| = \left| g(r_0) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s)\, s^{N-1} \mathrm{d}s \right|$$

$$= \left(\frac{N}{R_2^N - R_1^N} \right) \left| \int_{R_1}^{R_2} (g(r_0) - g(s))\, s^{N-1} \mathrm{d}s \right|$$

$$\leq \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} |g(r_0) - g(s)|\, s^{N-1} \mathrm{d}s \tag{6.8}$$

$$= \left(\frac{N}{R_2^N - R_1^N} \right) \left\{ \int_{R_1}^{r_0} |g(r_0) - g(s)|\, s^{N-1} \mathrm{d}s + \int_{r_0}^{R_2} |g(r_0) - g(s)|\, s^{N-1} \mathrm{d}s \right\}$$

$$\leq \left(\frac{N}{R_2^N - R_1^N} \right) \frac{1}{\Gamma(\alpha + 1)} \left\{ \left\| D^{\alpha}_{r_0 -} g \right\|_{\infty, [R_1, r_0]} \int_{R_1}^{r_0} (r_0 - s)^{\alpha} s^{N-1} \mathrm{d}s \right.$$

$$\left. + \left\| D^{\alpha}_{*r_0} g \right\|_{\infty, [r_0, R_2]} \int_{r_0}^{R_2} (s - r_0)^{\alpha} s^{N-1} \mathrm{d}s \right\} =: (*). \tag{6.9}$$

Here we calculate

$$I_1 := \int_{R_1}^{r_0} (r_0 - s)^{\alpha} s^{N-1} \mathrm{d}s = \int_{R_1}^{r_0} (r_0 - s)^{\alpha} ((s - R_1) + R_1)^{N-1} \mathrm{d}s$$

$$= \sum_{k=0}^{N-1} \binom{N-1}{k} R_1^k \int_{R_1}^{r_0} (r_0 - s)^{(\alpha+1)-1} (s - R_1)^{N-k-1} \mathrm{d}s$$

$$= \sum_{k=0}^{N-1} \frac{(N-1)!}{k!\,(N-k-1)!} R_1^k \frac{\Gamma(\alpha + 1)\,(N-k-1)!}{\Gamma(\alpha + 1 + N - k)} (r_0 - R_1)^{\alpha + N - k}.$$

That is,

$$I_1 = (N-1)!\Gamma(\alpha+1)\sum_{k=0}^{N-1}\frac{R_1^k (r_0-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}. \tag{6.10}$$

Also

$$I_2 := \int_{r_0}^{R_2} s^{N-1}(s-r_0)^{\alpha}\,ds = (-1)^{N-1}\int_{r_0}^{R_2}((R_2-s)-R_2)^{N-1}(s-r_0)^{\alpha}\,ds$$

$$= (-1)^{N-1}\sum_{k=0}^{N-1}\frac{(N-1)!}{k!(N-k-1)!}(-1)^k R_2^k\int_{r_0}^{R_2}(R_2-s)^{(N-k)-1}(s-r_0)^{(\alpha+1)-1}ds$$

$$= \sum_{k=0}^{N-1}\frac{(N-1)!}{k!(N-k-1)!}(-1)^{N+k-1}R_2^k\frac{(N-k-1)!\Gamma(\alpha+1)}{\Gamma(N-k+\alpha+1)}(R_2-r_0)^{N-k+\alpha}.$$

That is,

$$I_2 = (N-1)!\Gamma(\alpha+1)\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R_2^k(R_2-r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}. \tag{6.11}$$

Consequently we obtain

$$(*) = \left(\frac{N!}{R_2^N-R_1^N}\right)\left\{\left\|D_{r_0-}^{\alpha}g\right\|_{\infty,[R_1,r_0]}\sum_{k=0}^{N-1}\frac{R_1^k(r_0-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right.$$
$$\left.+\left\|D_{*r_0}^{\alpha}g\right\|_{\infty,[r_0,R_2]}\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R_2^k(R_2-r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right\}. \tag{6.12}$$

So far we have proved that ($\forall\ \omega \in S^{N-1}$)

$$\left|f(r_0\omega)-\frac{\int_A f(y)\,dy}{Vol(A)}\right| = \left|g(r_0)-\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2}g(s)s^{N-1}ds\right|$$
$$\leq \left(\frac{N!}{R_2^N-R_1^N}\right)\left\{\left\|D_{r_0-}^{\alpha}g\right\|_{\infty,[R_1,r_0]}\left(\sum_{k=0}^{N-1}\frac{R_1^k(r_0-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right)\right.$$
$$\left.+\left\|D_{*r_0}^{\alpha}g\right\|_{\infty,[r_0,R_2]}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R_2^k(R_2-r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)\right\}. \tag{6.13}$$

Inequality (6.13) ($\alpha > 0$) is attained by

$$\overline{g}(s) := \begin{cases}(s-r_0)^{\alpha}, & s\in[r_0,R_2],\\ (r_0-s)^{\alpha}, & s\in[R_1,r_0].\end{cases} \tag{6.14}$$

Here $\overline{g}$ fulfills all assumptions and

$$\left\| D_{r_0-}^{\alpha} g \right\|_{\infty,[R_1,r_0]} = \left\| D_{*r_0}^{\alpha} g \right\|_{\infty,[r_0,R_2]} = \Gamma(\alpha+1). \tag{6.15}$$

Hence

$$\begin{aligned}
\text{L.H.S. (6.13)} &= \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} \overline{g}(s)\, s^{N-1} ds \\
&= \frac{N}{R_2^N - R_1^N} \left[\int_{R_1}^{r_0} (r_0 - s)^{\alpha} s^{N-1} ds + \int_{r_0}^{R_2} (s - r_0)^{\alpha} s^{N-1} ds \right] \\
&= \frac{N}{R_2^N - R_1^N} \left[(N-1)!\Gamma(\alpha+1) \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)} \right) \right. \\
&\quad + (N-1)!\Gamma(\alpha+1) \\
&\quad \left. \times \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)} \right) \right] \\
&= \left(\frac{N!}{R_2^N - R_1^N} \right) \left[(\Gamma(\alpha+1)) \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)} \right) \right. \\
&\quad + (\Gamma(\alpha+1)) \\
&\quad \left. \times \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)} \right) \right] \\
&= \text{R.H.S. (6.13)},
\end{aligned} \tag{6.16}$$

proving it attained and sharp (6.13).

We have established

Theorem 6.4. *Let $f : \overline{A} \to \mathbb{R}$ be radial, that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $\forall\, x \in \overline{A}$; $\omega \in S^{N-1}$. Assume that $g \in AC^m([R_1, R_2])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, and $g^{(k)}(r_0) = 0$ ($r_0 \in [R_1, R_2]$ fixed), $k = 1, ..., m-1$, and $D_{r_0-}^{\alpha} g \in L_{\infty}([R_1, r_0])$, $D_{*r_0}^{\alpha} g \in L_{\infty}([r_0, R_2])$. Then ($\forall\, \omega \in S^{N-1}$)*

$$\begin{aligned}
\left| f(r_0\omega) - \frac{\int_A f(y)\, dy}{Vol(A)} \right| &= \left| g(r_0) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s)\, s^{N-1} ds \right| \\
&\le \left(\frac{N!}{R_2^N - R_1^N} \right) \left\{ \left\| D_{r_0-}^{\alpha} g \right\|_{\infty,[R_1,r_0]} \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)} \right) \right. \\
&\quad \left. + \left\| D_{*r_0}^{\alpha} g \right\|_{\infty,[r_0,R_2]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)} \right) \right\}.
\end{aligned} \tag{6.17}$$

The last inequality (6.17) is sharp, that is attained by

$$\overline{g}(s) := \begin{cases} (s - r_0)^{\alpha}, & s \in [r_0, R_2], \\ (r_0 - s)^{\alpha}, & s \in [R_1, r_0]. \end{cases} \tag{6.18}$$

We need to make

Remark 6.5. Let $\alpha \geq 1$. We get easily by (6.4) that

$$|g(r) - g(r_0)| \leq \frac{(r - r_0)^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{*r_0}^{\alpha} g \right\|_{L_1([r_0, R_2])}, \tag{6.19}$$

$\forall\ r \in [r_0, R_2]$. Also by (6.5) we find that

$$|g(r) - g(r_0)| \leq \frac{(r_0 - r)^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{r_0-}^{\alpha} g \right\|_{L_1([R_1, r_0])}, \tag{6.20}$$

$\forall\ r \in [R_1, r_0]$.

Here we suppose $\left\| D_{*r_0}^{\alpha} g \right\|_{L_1([r_0, R_2])}, \left\| D_{r_0-}^{\alpha} g \right\|_{L_1([R_1, r_0])} < \infty$.

Then it holds

$$\left| f(r_0\omega) - \frac{\int_A f(y)\,dy}{Vol(A)} \right| = \left| g(r_0) - \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} g(s)\, s^{N-1} ds \right|$$

$$\leq \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} |g(r_0) - g(s)|\, s^{N-1} ds \tag{6.21}$$

$$= \left(\frac{N}{R_2^N - R_1^N} \right) \left\{ \int_{R_1}^{r_0} |g(r_0) - g(s)|\, s^{N-1} ds + \int_{r_0}^{R_2} |g(r_0) - g(s)|\, s^{N-1} ds \right\}$$

$$\leq \left(\frac{N}{R_2^N - R_1^N} \right) \frac{1}{\Gamma(\alpha)} \left\{ \left\| D_{r_0-}^{\alpha} g \right\|_{L_1([R_1, r_0])} \left(\int_{R_1}^{r_0} (r_0 - s)^{\alpha-1} s^{N-1} ds \right) \right.$$

$$\left. + \left\| D_{*r_0}^{\alpha} g \right\|_{L_1([r_0, R_2])} \int_{r_0}^{R_2} (s - r_0)^{\alpha-1} s^{N-1} ds \right\}$$

$$= \left(\frac{N!}{R_2^N - R_1^N} \right) \left\{ \left\| D_{r_0-}^{\alpha} g \right\|_{L_1([R_1, r_0])} \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha+N-k-1}}{k!\,\Gamma(\alpha + N - k)} \right) \right.$$

$$\left. + \left\| D_{*r_0}^{\alpha} g \right\|_{L_1([r_0, R_2])} \times \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha-1}}{k!\,\Gamma(N - k + \alpha)} \right) \right\}. \tag{6.22}$$

We have established

Theorem 6.6. *Let $f : \overline{A} \to \mathbb{R}$ be radial, that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $\forall\, x \in \overline{A}$; $\omega \in S^{N-1}$. Assume that $g \in AC^m([R_1, R_2])$, $m = \lceil \alpha \rceil$, $\alpha \geq 1$, and $g^{(k)}(r_0) = 0$, $k = 1, ..., m-1$; $r_0 \in [R_1, R_2]$ be fixed. Assume $D^{\alpha}_{r_0-}g \in L_1([R_1, r_0])$ and $D^{\alpha}_{*r_0}g \in L_1([r_0, R_2])$. Then ($\forall\, \omega \in S^{N-1}$) we get*

$$\left| f(r_0\omega) - \frac{\int_A f(y)\,\mathrm{d}y}{Vol(A)} \right| = \left| g(r_0) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} g(s)\, s^{N-1} \mathrm{d}s \right|$$

$$\leq \left(\frac{N!}{R_2^N - R_1^N}\right) \left\{ \left\| D^{\alpha}_{r_0-}g \right\|_{L_1([R_1,r_0])} \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha+N-k-1}}{k!\,\Gamma(\alpha+N-k)} \right) \right.$$

$$\left. + \left\| D^{\alpha}_{*r_0}g \right\|_{L_1([r_0,R_2])} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha-1}}{k!\,\Gamma(N-k+\alpha)} \right) \right\}. \tag{6.23}$$

We continue with

Remark 6.7. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$. Then by (6.4) we derive

$$|g(r) - g(r_0)| \leq \frac{1}{\Gamma(\alpha)} \left(\int_{r_0}^{r} (r - J)^{p(\alpha-1)} \mathrm{d}J \right)^{\frac{1}{p}} \left\| D^{\alpha}_{*r_0}g \right\|_{L_q([r_0,R_2])}, \tag{6.24}$$

$\forall\, r \in [r_0 R_2]$. That is,

$$|g(r) - g(r_0)| \leq \frac{1}{\Gamma(\alpha)} \frac{(r - r_0)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D^{\alpha}_{*r_0}g \right\|_{L_q([r_0,R_2])}, \tag{6.25}$$

$\forall\, r \in [r_0, R_2]$. Similarly by (6.5) we obtain

$$|g(r) - g(r_0)| \leq \frac{1}{\Gamma(\alpha)} \frac{(r_0 - r)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D^{\alpha}_{r_0-}g \right\|_{L_q([R_1,r_0])}, \tag{6.26}$$

$\forall\, r \in [R_1, r_0]$. Here we assume that $D^{\alpha}_{*r_0}g \in L_q([r_0, R_2])$, and $D^{\alpha}_{r_0-}g \in L_q([R_1, r_0])$.

Hence we derive

$$\left| f(r_0\omega) - \frac{\int_A f(y)\,\mathrm{d}y}{Vol(A)} \right| = \left| g(r_0) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} g(s)\, s^{N-1} \mathrm{d}s \right| \tag{6.27}$$

$$= \left(\frac{N}{R_2^N - R_1^N}\right) \left| \int_{R_1}^{R_2} (g(r_0) - g(s))\, s^{N-1} \mathrm{d}s \right|$$

$$\leq \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} |g(r_0) - g(s)| s^{N-1} \mathrm{d}s$$

$$= \left(\frac{N}{R_2^N - R_1^N}\right) \left\{ \int_{R_1}^{r_0} |g(r_0) - g(s)| s^{N-1} \mathrm{d}s + \int_{r_0}^{R_2} |g(r_0) - g(s)| s^{N-1} \mathrm{d}s \right\}$$

$$\leq \left(\frac{N}{R_2^N - R_1^N}\right) \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

$$\times \left\{ \left\| D_{r_0-}^{\alpha} g \right\|_{L_q([R_1, r_0])} \left(\int_{R_1}^{r_0} (r_0 - s)^{\alpha - \frac{1}{q}} s^{N-1} \mathrm{d}s \right) \right.$$

$$\left. + \left\| D_{*r_0}^{\alpha} g \right\|_{L_q([r_0, R_2])} \int_{r_0}^{R_2} (s - r_0)^{\alpha - \frac{1}{q}} s^{N-1} \mathrm{d}s \right\} \tag{6.28}$$

$$= \frac{N}{\left(R_2^N - R_1^N\right) \Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

$$\times \left\{ \left\| D_{r_0-}^{\alpha} g \right\|_{L_q([R_1, r_0])} (N-1)! \Gamma\left(\alpha - \frac{1}{q} + 1\right) \right.$$

$$\times \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha - \frac{1}{q} + N - k}}{k! \Gamma\left(\alpha - \frac{1}{q} + 1 + N - k\right)} \right)$$

$$+ \left\| D_{*r_0}^{\alpha} g \right\|_{L_q([r_0, R_2])} (N-1)! \Gamma\left(\alpha - \frac{1}{q} + 1\right) \cdot$$

$$\left. \times \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha-\frac{1}{q}}}{k! \Gamma\left(N - k + \alpha - \frac{1}{q} + 1\right)} \right) \right\}.$$

We have established

Theorem 6.8. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let $f : \overline{A} \to \mathbb{R}$ be radial, that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $\forall\ x \in \overline{A}$; $\omega \in S^{N-1}$. Assume that $g \in AC^m([R_1, R_2])$, $m = \lceil \alpha \rceil$, $\alpha > \frac{1}{q}$ and $g^{(k)}(r_0) = 0$, $k = 1, ..., m-1$; $r_0 \in [R_1, R_2]$ be fixed. Suppose also $D_{r_0-}^{\alpha} g \in L_q([R_1, r_0])$ and $D_{*r_0}^{\alpha} g \in L_q([r_0, R_2])$. Then $(\forall \omega \in S^{N-1})$ we obtain*

$$\left| f(r_0 \omega) - \frac{\int_A f(y) \mathrm{d}y}{Vol(A)} \right| \tag{6.29}$$

$$= \left| g(r_0) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} g(s) s^{N-1} \mathrm{d}s \right|$$

$$\leq \frac{\Gamma\left(\alpha+\frac{1}{p}\right)N!}{\Gamma(\alpha)\left(R_2^N-R_1^N\right)(p(\alpha-1)+1)^{\frac{1}{p}}}\cdot$$

$$\times\left\{\left\|D_{r_0-}^{\alpha}g\right\|_{L_q([R_1,r_0])}\left(\sum_{k=0}^{N-1}\frac{R_1^k(r_0-R_1)^{\alpha-\frac{1}{q}+N-k}}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right)\right.$$

$$\left.+\left\|D_{*r_0}^{\alpha}g\right\|_{L_q([r_0,R_2])}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R_2^k(R_2-r_0)^{N-k+\alpha-\frac{1}{q}}}{k!\Gamma\left(N-k+\alpha+\frac{1}{p}\right)}\right)\right\}. \tag{6.30}$$

We mention

Corollary 6.9 ($p=q=2$ **case).** *Let $f:\overline{A}\to\mathbb{R}$ be radial, that is, there exists g such that $f(x)=g(r)$, $r=|x|$, $\forall\ x\in\overline{A}$; $\omega\in S^{N-1}$. Assume that $g\in AC^m([R_1,R_2])$, $m=\lceil\alpha\rceil$, $\alpha>\frac{1}{2}$ and $g^{(k)}(r_0)=0$, $k=1,...,m-1$; $r_0\in[R_1,R_2]$. Suppose $D_{r_0-}^{\alpha}g\in L_2([R_1,r_0])$, $D_{*r_0}^{\alpha}g\in L_2([r_0,R_2])$. Then $(\forall\omega\in S^{N-1})$*

$$\left|f(r_0\omega)-\frac{\int_A f(y)\,dy}{Vol(A)}\right|=\left|g(r_0)-\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2}g(s)s^{N-1}ds\right|$$

$$\leq\frac{\Gamma\left(\alpha+\frac{1}{2}\right)N!}{\Gamma(\alpha)\left(R_2^N-R_1^N\right)(2\alpha-1)^{\frac{1}{2}}}$$

$$\times\left\{\left\|D_{r_0-}^{\alpha}g\right\|_{L_2([R_1,r_0])}\left(\sum_{k=0}^{N-1}\frac{R_1^k(r_0-R_1)^{\alpha-\frac{1}{2}+N-k}}{k!\Gamma\left(\alpha+\frac{1}{2}+N-k\right)}\right)\right.$$

$$\left.+\left\|D_{*r_0}^{\alpha}g\right\|_{L_2([r_0,R_2])}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R_2^k(R_2-r_0)^{N-k+\alpha-\frac{1}{2}}}{k!\Gamma\left(N-k+\alpha+\frac{1}{2}\right)}\right)\right\}. \tag{6.31}$$

Finally we deal with nonradial L_∞ Ostrowski inequalities on the shell.

We need (see also [2], p. 421)

Definition 6.10. Let $F:\overline{A}\to\mathbb{R}$, $\alpha>0$, $m=\lceil\alpha\rceil$ such that $F(\cdot\omega)\in AC^m([R_1,R_2])$, $\forall\ \omega\in S^{N-1}$. We call the Caputo left radial fractional derivative the following function

$$\frac{\partial_{*r_0}^{\alpha}F(x)}{\partial r^{\alpha}}:=\frac{1}{\Gamma(m-\alpha)}\int_{r_0}^{r}(r-t)^{m-\alpha-1}\frac{\partial^m F(t\omega)}{\partial r^m}dt, \tag{6.32}$$

where $x\in\overline{A}$, that is, $x=r\omega$; $r,r_0\in[R_1,R_2]$, r_0 is fixed, $\omega\in S^{N-1}$, $\forall\ r\geq r_0$.

Clearly

$$\frac{\partial^0_{*r_0} F(x)}{\partial r^0} = F(x), \tag{6.33}$$

$$\frac{\partial^\alpha_{*r_0} F(x)}{\partial r^\alpha} = \frac{\partial^\alpha F(x)}{\partial r^\alpha}, \quad \text{if } \alpha \in \mathbb{N}. \tag{6.34}$$

The above function exists almost everywhere for $x \in \overline{A}$.

We also need

Definition 6.11. Let $F : \overline{A} \to \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$ such that $F(\cdot\omega) \in AC^m([R_1, R_2])$, $\forall\ \omega \in S^{N-1}$. We call the Caputo right radial fractional derivative the following function

$$\frac{\partial^\alpha_{r_0-} F(x)}{\partial r^\alpha} := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_r^{r_0} (J-r)^{m-\alpha-1} \frac{\partial^m F(J\omega)}{\partial J^m} \mathrm{d}J, \tag{6.35}$$

where $x \in \overline{A}$, that is, $x = r\omega$; $r, r_0 \in [R_1, R_2]$, r_0 is fixed, $\omega \in S^{N-1}$, $\forall\ r \le r_0$.

Clearly

$$\frac{\partial^0_{r_0-} F(x)}{\partial r^0} = F(x), \tag{6.36}$$

$$\frac{\partial^\alpha_{r_0-} F(x)}{\partial r^\alpha} = (-1)^\alpha \frac{\partial^\alpha F(x)}{\partial r^\alpha}, \quad \text{if } \alpha \in \mathbb{N}. \tag{6.37}$$

The above function exists almost everywhere for $x \in \overline{A}$.

We need to make

Remark 6.12. We treat here the general, not necessarily radial, case of f. We apply Theorem 6.4 to $f(r\omega)$, ω fixed, $r \in [R_1, R_2]$, under the following assumptions: $f(\cdot\omega) \in AC^m([R_1, R_2])$, $\forall\ \omega \in S^{N-1}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, where $f : \overline{A} \to \mathbb{R}$ is Lebesgue integrable; $\frac{\partial^k f}{\partial r^k}$, $k = 1, ..., m-1$, vanish on $\partial B(0, r_0)$, r_0 is fixed in $[R_1, R_2]$; and $\frac{\partial^\alpha_{*r_0} f}{\partial r^\alpha} \in B(\overline{A_1})$ (bounded functions), where $A_1 := B(0, R_2) - \overline{B(0, r_0)}$, and $\frac{\partial^\alpha_{r_0-} f}{\partial r^\alpha} \in B(\overline{A_2})$, where $A_2 := B(0, r_0) - \overline{B(0, R_1)}$, along with $D^\alpha_{r_0-} f(\cdot\omega) \in L_\infty([R_1, r_0])$, $D^\alpha_{*r_0} f(\cdot\omega) \in L_\infty([r_0, R_2])$, $\forall\ \omega \in S^{N-1}$.

Then ($\forall\ \omega \in S^{N-1}$)

$$\left| f(r_0\omega) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} f(s\omega) s^{N-1} \mathrm{d}s \right|$$

$$\le \left(\frac{N!}{R_2^N - R_1^N}\right) \left\{ \left\| \frac{\partial^\alpha_{r_0-} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{\alpha+N-k}}{k! \Gamma(\alpha+1+N-k)} \right) + \left\| \frac{\partial^\alpha_{*r_0} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_1}} \right.$$

$$\left. \times \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+\alpha}}{k! \Gamma(N-k+\alpha+1)} \right) \right\} =: \lambda_1. \tag{6.38}$$

Therefore

$$\left| \frac{\int_{S^{N-1}} f\left(r_0\omega\right) \mathrm{d}\omega}{\omega_N} - \frac{N}{\left(R_2^N - R_1^N\right)\omega_N} \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f\left(s\omega\right) s^{N-1} \mathrm{d}s \right) \mathrm{d}\omega \right| \leq \lambda_1. \tag{6.39}$$

That is,

$$\left| \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f\left(r_0\omega\right) \mathrm{d}\omega - \frac{\int_A f\left(x\right) \mathrm{d}x}{Vol\left(A\right)} \right| \leq \lambda_1. \tag{6.40}$$

Therefore it holds for $x \in \overline{A}$ that

$$\left| f\left(x\right) - \frac{\int_A f\left(x\right) \mathrm{d}x}{Vol\left(A\right)} \right| \leq \left| f\left(x\right) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f\left(r_0\omega\right) \mathrm{d}\omega \right| + \lambda_1. \tag{6.41}$$

We have proved the following.

Theorem 6.13. *Let* $f : \overline{A} \to \mathbb{R}$ *be Lebesgue integrable with* $f\left(\cdot\omega\right) \in AC^m\left(\left[R_1, R_2\right]\right)$, $\alpha > 0$, $m = \lceil\alpha\rceil$, $\forall\ \omega \in S^{N-1}$; $\frac{\partial^k f}{\partial r^k}$, $k = 1, ..., m-1$, *vanish on* $\partial B\left(0, r_0\right)$, r_0 *fixed in* $\left[R_1, R_2\right]$; *and* $\frac{\partial^\alpha_{*r_0} f}{\partial r^\alpha} \in B\left(\overline{A_1}\right)$, $\frac{\partial^\alpha_{r_0-} f}{\partial r^\alpha} \in B\left(\overline{A_2}\right)$ *along with* $D^\alpha_{r_0-} f\left(\cdot\omega\right) \in L_\infty\left(\left[R_1, r_0\right]\right)$, $D^\alpha_{*r_0} f\left(\cdot\omega\right) \in L_\infty\left(\left[r_0, R_2\right]\right)$, $\forall\ \omega \in S^{N-1}$. *Then for* $x \in \overline{A}$ *we have*

$$\left| f\left(x\right) - \frac{\int_A f\left(x\right) \mathrm{d}x}{Vol\left(A\right)} \right| \leq \left| f\left(x\right) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f\left(r_0\omega\right) \mathrm{d}\omega \right|$$

$$\leq \left(\frac{N!}{R_2^N - R_1^N} \right) \left\{ \left\| \frac{\partial^\alpha_{r_0-} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{R_1^k \left(r_0 - R_1\right)^{\alpha+N-k}}{k!\Gamma\left(\alpha + 1 + N - k\right)} \right) \right.$$

$$\left. + \left\| \frac{\partial^\alpha_{*r_0} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{\left(-1\right)^{N+k-1} R_2^k \left(R_2 - r_0\right)^{N-k+\alpha}}{k!\Gamma\left(N - k + \alpha + 1\right)} \right) \right\}. \tag{6.42}$$

We also make

Remark 6.14. Let $f : \overline{B\left(0, R\right)} \to \mathbb{R}$ be a Lebesgue integrable function, that is, not necessarily a radial function. Suppose $f\left(\cdot\omega\right) \in AC\left(\left[0, R\right]\right)$, $\forall\ \omega \in S^{N-1}$, $0 < \alpha \leq 1$; $r_0 \in \left[0, R\right]$ fixed, $D^\alpha_{r_0-} f\left(\cdot\omega\right) \in L_\infty\left(\left[0, r_0\right]\right)$, $D^\alpha_{*r_0} f\left(\cdot\omega\right) \in L_\infty\left(\left[r_0, R\right]\right)$, $\forall\ \omega \in S^{N-1}$. We further assume that

$$\left\| D^\alpha_{r_0-} f\left(t\omega\right) \right\|_{\infty, \left(t \in \left[0, r_0\right]\right)}, \left\| D^\alpha_{*r_0} f\left(t\omega\right) \right\|_{\infty, \left(t \in \left[r_0, R\right]\right)} \leq K, \quad \forall\ \omega \in S^{N-1}, \tag{6.43}$$

where $K > 0$.

By inequality (16) of [5], applied on $f(\cdot\omega)$, $\forall\, \omega \in S^{N-1}$, we get

$$\left| f(r_0\omega) - \frac{N}{R^N}\int_0^R f(s\omega)\, s^{N-1} ds \right| \leq \frac{N!K}{R^N}\left[\frac{r_0^{\alpha+N}}{\Gamma(\alpha+N+1)} + \left(\sum_{k=0}^{N-1} \frac{(-1)^{(N+k-1)} R^k (R-r_0)^{(N+\alpha-k)}}{k!\Gamma(N+\alpha+1-k)} \right) \right] =: \lambda_2. \tag{6.44}$$

Therefore it holds

$$\left| \frac{\int_{S^{N-1}} f(r_0\omega)\, d\omega}{\omega_N} - \frac{N}{R^N\omega_N}\int_{S^{N-1}}\left(\int_0^R f(s\omega)\, s^{N-1} ds \right) d\omega \right| \leq \lambda_2. \tag{6.45}$$

That is, we find

$$\left| \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\int_{S^{N-1}} f(r_0\omega)\, d\omega - \frac{\int_{B(0,R)} f(x)\, dx}{Vol(B(0,R))} \right| \leq \lambda_2. \tag{6.46}$$

Consequently we derive

$$\left| f(x) - \frac{\int_{B(0,R)} f(x)\, dx}{Vol(B(0,R))} \right| \leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\int_{S^{N-1}} f(r_0\omega)\, d\omega \right| + \lambda_2. \tag{6.47}$$

We have proved

Theorem 6.15. *Let $f : \overline{B(0,R)} \to \mathbb{R}$ be a Lebesgue integrable function, that is, not necessarily a radial function. Assume $f(\cdot\omega) \in AC([0,R])$, $R > 0$, $\forall\, \omega \in S^{N-1}$, $0 < \alpha \leq 1$, $r_0 \in [0,R]$ fixed, $D^{\alpha}_{r_0-} f(\cdot\omega) \in L_\infty([0,r_0])$, $D^{\alpha}_{*r_0} f(\cdot\omega) \in L_\infty([r_0,R])$, $\forall\, \omega \in S^{N-1}$.*

Suppose also $\left\| D^{\alpha}_{r_0-} f(t\omega) \right\|_{\infty,(t\in[0,r_0])}$, $\left\| D^{\alpha}_{*r_0} f(t\omega) \right\|_{\infty,(t\in[r_0,R])} \leq K$, $\forall\, \omega \in S^{N-1}$, where $K > 0$. Then

$$\left| f(x) - \frac{\int_{B(0,R)} f(x)\, dx}{Vol(B(0,R))} \right| \leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\int_{S^{N-1}} f(r_0\omega)\, d\omega \right| + \frac{N!K}{R^N}\left[\frac{r_0^{\alpha+N}}{\Gamma(\alpha+N+1)} + \left(\sum_{k=0}^{N-1} \frac{(-1)^{(N+k-1)} R^k (R-r_0)^{(N+\alpha-k)}}{k!\Gamma(N+\alpha+1-k)} \right) \right]. \tag{6.48}$$

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Chapter 7
Left Caputo Fractional Uniform Landau Inequalities

Here we present left Caputo fractional uniform Landau-type inequalities. We give applications and we recover the original Landau inequality on $\mathbb{R}_+$. This chapter relies on [3].

7.1 Introduction

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$.

Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

$$\|f'\|_{p,I} \leq C_p(I) \, \|f\|_{p,I}^{1/2} \, \|f''\|_{p,I}^{1/2}, \tag{7.1}$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I, see [1, 4].

The research on these inequalities started by E. Landau [10] in 1914. For the case of $p = \infty$ he proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2} \tag{7.2}$$

are the best constants in (7.1).

In 1932, G. H. Hardy and J. E. Littlewood [7] proved (7.1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \text{ and } C_2(\mathbb{R}) = 1. \tag{7.3}$$

In 1935, G. H. Hardy, E. Landau, and J. E. Littlewood [8] showed that the best constant $C_p(\mathbb{R}_+)$ in (7.1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2 \text{ for } p \in [1, \infty), \tag{7.4}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$.

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In fact in [6] and [9], it was shown that $C_p(\mathbb{R}) \leq \sqrt{2}$.

In this chapter we prove fractional Landau inequalities with respect to $\|\cdot\|_\infty$ involving the left Caputo fractional derivative.

We need

Definition 7.1 ([5], p. 38). Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ the ceiling of the number), $f \in AC^n([A, B])$ (i.e., $f^{(n-1)} \in AC([A, B])$, $A, B \in \mathbb{R}$).

We define the left Caputo fractional derivative

$$D_{*A}^{\nu} f(x) := \frac{1}{\Gamma(n-\nu)} \int_A^x (x-t)^{n-\nu-1} f^{(n)}(t) \mathrm{d}t, \ (\forall)\ x \in [A, B], \tag{7.5}$$

and

$$D_{*A}^{\nu} f(x) := 0, \text{ if } x < A.$$

Here Γ is the gamma function.

Definition 7.2. Let $0 < \nu \leq 1$, $f \in AC([A, B])$ we define

$$D_{*A}^{\nu} f(x) := \frac{1}{\Gamma(1-\nu)} \int_A^x (x-t)^{-\nu} f'(t) \mathrm{d}t, \ (\forall)\ x \in [A, B]. \tag{7.6}$$

Notice

$$D_{*A}^{n} f := f^{(n)}, \text{ for } n \in \mathbb{N}.$$

We make

Remark 7.3. Let $f' \in AC([A, B])$, $0 < \nu \leq 1$, then $\lceil \nu + 1 \rceil = 2$, and

$$D_{*A}^{\nu} f'(x) = \frac{1}{\Gamma(1-\nu)} \int_A^x (x-t)^{-\nu} f''(t) \mathrm{d}t$$
$$= \frac{1}{\Gamma(2-(\nu+1))} \int_A^x (x-t)^{2-(\nu+1)-1} f''(t) \mathrm{d}t = D_{*A}^{\nu+1} f(x).$$

Hence it holds

$$D_{*A}^{\nu} f'(x) = D_{*A}^{\nu+1} f(x). \tag{7.7}$$

7.2 Main Results

From [2], p. 618, Theorem 26.8 there we obtain

Theorem 7.4. *Let* $0 < \nu \leq 1$, $A, B \in \mathbb{R}$, $A < B$, $f \in AC([A, B])$. *Suppose* $D_{*A}^{\nu} f \in L_\infty([A, B])$.

Then

$$\left|\frac{1}{B-A}\int_A^B f(x)\mathrm{d}x - f(A)\right| \le \frac{\|D^{\nu}_{*A} f\|_{\infty,[A,B)}}{\Gamma(\nu+2)}(B-A)^{\nu}. \quad (7.8)$$

We make

Remark 7.5. Let $0<\nu\le 1$, $A,B\in \mathrm{R}$, $A<B$, $f'\in AC([A,B])$. Suppose $D^{\nu}_{*A}f'\in L_{\infty}([A,B])$.

Then by (7.8),

$$\left|\frac{1}{B-A}\int_A^B f'(x)\mathrm{d}x - f'(A)\right| \le \frac{\|D^{\nu}_{*A} f'\|_{\infty,[A,B]}}{\Gamma(\nu+2)}(B-A)^{\nu}. \quad (7.9)$$

The above are equivalent to next statements.

Let $0<\nu\le 1$, $A,B\in \mathrm{R}$, $A<B$, $f\in AC^2([A,B])$. Assume $D^{\nu+1}_{*A}f\in L_{\infty}([A,B])$.

Then

$$\left|\frac{1}{B-A}(f(B)-f(A)) - f'(A)\right| \le \frac{\|D^{\nu+1}_{*A} f\|_{\infty,[A,B]}}{\Gamma(\nu+2)}(B-A)^{\nu}. \quad (7.10)$$

Hence

$$|f'(A)| - \frac{1}{B-A}|f(B)-f(A)| \le \frac{\|D^{\nu+1}_{*A} f\|_{\infty,[A,B]}}{\Gamma(\nu+2)}(B-A)^{\nu}. \quad (7.11)$$

We make

Remark 7.6. Let $0<\nu\le 1$, $A,b\in \mathrm{R}$ with $f\in AC^2([A,b])$, $(\forall)\ b>A$.

We fix A. Assume $D^{\nu+1}_{*A}f\in L_{\infty}([A,\ +\infty))$, thus $D^{\nu+1}_{*A}f\in L_{\infty}([A,b])$.

Let $A<a<b$. Then $f\in AC^2([a,b])$ and $D^{\nu+1}_{*A}f\in L_{\infty}([a,\ +\infty))$ and in particular $D^{\nu+1}_{*A}f\in L_{\infty}([a,b])$.

Here

$$D^{\nu+1}_{*A}f(x) = \frac{1}{\Gamma(1-\nu)}\int_a^x (x-t)^{-\nu} f''(t)\mathrm{d}t. \quad (7.12)$$

If $f''(t)\ge 0$ a.e., then

$$\frac{1}{\Gamma(1-\nu)}\int_A^x (x-t)^{-\nu} f''(t)\mathrm{d}t \ge \frac{1}{\Gamma(\nu-1)}\int_a^x (x-t)^{-\nu} f''(t)\mathrm{d}t, \quad (7.13)$$

i.e.,

$$D^{\nu+1}_{*A}f(x) \ge D^{\nu+1}_{*a}f(x) \ge 0,$$

a.e., for $x\ge a>A$.

Therefore it holds

$$\infty > \|D_{*A}^{\nu+1} f\|_{\infty,[A,\ +\infty)} \geq \|D_{*a}^{\nu+1} f\|_{\infty,[a,\ +\infty)}. \tag{7.14}$$

So it is not strange to suppose that

$$\|D_{*a}^{\nu+1} f\|_{\infty,[a,\ +\infty)} \leq \|D_{*A}^{\nu+1} f\|_{\infty,[A,\ +\infty)}, \tag{7.15}$$

$(\forall)\ a \geq A$ (it is obvious when $\nu = 1$).

We also make

Remark 7.7. Let $a, b \in [A,\ +\infty),\ a < b$. Then as before we obtain

$$|f'(a)| \leq \frac{1}{b-a}|f(b) - f(a)| + \frac{\|D_{*a}^{\nu+1} f\|_{\infty,[a,b]}}{\Gamma(\nu+2)}(b-a)^{\nu}. \tag{7.16}$$

We also assume that

$$\|f\|_{\infty,\ [A,+\infty)} < \infty.$$

Therefore

$$|f'(a)| \leq \frac{2\|f\|_{\infty,\ [A,+\infty)}}{b-a} + \frac{\|D_{*A}^{\nu+1} f\|_{\infty,\ [A,+\infty)}}{\Gamma(\nu+2)}(b-a)^{\nu}, \tag{7.17}$$

$(\forall)\ a, b \in [A, +\infty),\ a < b$. R.H.S. of (7.17) depends only on $b - a$.

Consequently, it holds

$$\|f'\|_{\infty,\ [A,+\infty)} \leq \frac{2\|f\|_{\infty,\ [A,+\infty)}}{b-a} + \frac{\|D_{*A}^{\nu+1} f\|_{\infty,\ [A,+\infty)}}{\Gamma(\nu+2)}(b-a)^{\nu}. \tag{7.18}$$

We may call $t = b - a > 0$. Thus by (7.18),

$$\|f'\|_{\infty,\ [A,+\infty)} \leq \frac{2\|f\|_{\infty,\ [A,+\infty)}}{t} + \frac{\|D_{*A}^{\nu+1} f\|_{\infty,\ [A,+\infty)}}{\Gamma(\nu+2)}t^{\nu},\ (\forall)\ t > 0. \tag{7.19}$$

Call

$$\mu := 2\|f\|_{\infty,\ [A,+\infty)},$$

$$\theta := \frac{\|D_{*A}^{\nu+1} f\|_{\infty,\ [A,+\infty)}}{\Gamma(\nu+2)}, \tag{7.20}$$

both are greater than 0.

We consider the function

$$y(t) = \frac{\mu}{t} + \theta t^{\nu},\ 0 < \nu \leq 1,\ t > 0.$$

We have

$$y'(t) = -\frac{\mu}{t^2} + \nu\theta t^{\nu-1} = 0,$$

then

$$\nu\theta t^{\nu-1} = \frac{\mu}{t^2},$$

and

$$\nu\theta t^{\nu+1} = \mu,$$

that is,

$$t^{\nu+1} = \frac{\mu}{\nu\theta}, \tag{7.21}$$

with a unique solution

$$t_0 := t_{crit.no} = \left(\frac{\mu}{\nu\theta}\right)^{1/(\nu+1)}. \tag{7.22}$$

We have

$$\begin{aligned} y'(t) &= -\mu t^{-2} + \nu\theta t^{\nu-1} \text{ and} \\ y''(t) &= 2\mu t^{-3} + \nu(\nu-1)\theta t^{\nu-2}. \end{aligned} \tag{7.23}$$

We see that

$$\begin{aligned} y''(t_0) = 2\mu t_0^{-3} + \nu(\nu-1)\theta t_0^{\nu-2} &= 2\mu\left(\frac{\mu}{\nu\theta}\right)^{-3/(\nu+1)} + \nu(\nu-1)\theta\left(\frac{\mu}{\nu\theta}\right)^{\frac{(\nu+1)-3}{(\nu+1)}} \\ &= \left(\frac{\mu}{\nu\theta}\right)^{-3/(\nu+1)}\left(2\mu + \nu(\nu-1)\theta\left(\frac{\mu}{\nu\theta}\right)\right) \\ &= \left(\frac{\mu}{\nu\theta}\right)^{-3/(\nu+1)}\mu(2^{\nu+1} + \nu - 1) > 0. \end{aligned} \tag{7.24}$$

Therefore y has a global minimum at

$$t_0 = \left(\frac{\mu}{\nu\theta}\right)^{1/(\nu+1)},$$

which is

$$\begin{aligned} y(t_0) &= \frac{\mu}{\left(\frac{\mu}{\nu\theta}\right)^{1/(\nu+1)}} + \theta\cdot\left(\frac{\mu}{\nu\theta}\right)^{\frac{\nu}{\nu+1}} \\ &= (\nu\theta)^{1/(\nu+1)}\frac{\mu}{\mu^{1/(\nu+1)}} + \theta\cdot\frac{\mu^{\frac{\nu}{\nu+1}}}{\nu^{\frac{\nu}{\nu+1}}\cdot\theta^{\frac{\nu}{\nu+1}}} \\ &= (\nu\theta)^{1/(\nu+1)}\mu^{1-\frac{1}{\nu+1}} + \frac{\theta^{1-\frac{\nu}{\nu+1}}\mu^{\frac{\nu}{\nu+1}}}{\nu^{\frac{\nu}{\nu+1}}} \end{aligned} \tag{7.25}$$

$$= \nu^{\frac{1}{\nu+1}} \theta^{\frac{1}{\nu+1}} \mu^{\frac{\nu}{\nu+1}} + \theta^{\frac{1}{\nu+1}} \nu^{\frac{-\nu}{\nu+1}} \mu^{\frac{\nu}{\nu+1}} = \theta^{\frac{1}{\nu+1}} \mu^{\frac{\nu}{\nu+1}} \left(\nu^{\frac{1}{\nu+1}} + \nu^{\frac{-\nu}{\nu+1}} \right)$$

$$= (\theta \mu^{\nu})^{\frac{1}{\nu+1}} \left(\nu^{\frac{1}{\nu+1}} + \frac{1}{\nu^{\frac{\nu}{\nu+1}}} \right) = (\theta \mu^{\nu})^{\frac{1}{\nu+1}} \frac{\left(\nu^{\frac{1}{\nu+1}} \nu^{\frac{\nu}{\nu+1}} + 1 \right)}{\nu^{\frac{\nu}{\nu+1}}}$$

$$= (\theta \mu^{\nu})^{\frac{1}{\nu+1}} \frac{(\nu + 1)}{\nu^{\frac{\nu}{\nu+1}}} = (\theta \mu^{\nu})^{\frac{1}{\nu+1}} (\nu + 1) \nu^{\frac{-\nu}{\nu+1}}. \tag{7.26}$$

That is,

$$y(t_0) = (\theta \mu^{\nu})^{\frac{1}{\nu+1}} (\nu + 1) \nu^{-\left(\frac{\nu}{\nu+1}\right)}. \tag{7.27}$$

Consequently

$$y(t_0) = \left(\frac{\| D_{*a}^{\nu+1} f \|_{\infty, [A,+\infty)}}{\Gamma(\nu + 2)} \right)^{\frac{1}{(\nu+1)}} \cdot \left(2 \| f \|_{\infty, [A,+\infty)} \right)^{\frac{\nu}{(\nu+1)}} (\nu + 1) \nu^{-\left(\frac{\nu}{\nu+1}\right)}. \tag{7.28}$$

We have proved that

$$\| f' \|_{\infty, [A,+\infty)} \leq (\nu + 1) \left(\frac{2}{\nu} \right)^{\frac{\nu}{(\nu+1)}} (\Gamma(\nu + 2))^{-\frac{1}{(\nu+1)}}$$

$$\left(\| f \|_{\infty, [A,+\infty)} \right)^{\frac{\nu}{(\nu+1)}} \left(\| D_{*A}^{\nu+1} f \|_{\infty, [A,+\infty)} \right)^{\frac{1}{(\nu+1)}}. \tag{7.29}$$

We have established the following result, left Caputo fractional Landau inequality for $\| \cdot \|_{\infty}$:

Theorem 7.8. *Let* $0 < \nu \leq 1$, $A, b \in \mathbb{R}$ *with* $f \in AC^2([A, b])$, $(\forall)\ b > A$, *where* A *is fixed. Assume* $\| f \|_{\infty, [A,+\infty)} < \infty$, $D_{*A}^{\nu+1} f \in L_{\infty}([A, +\infty))$, *and*

$$\| D_{*a}^{\nu+1} f \|_{\infty, [a,+\infty)} \leq \| D_{*A}^{\nu+1} f \|_{\infty, [A,+\infty)}, \ (\forall)\ a \geq A.$$

Then

$$\| f' \|_{\infty, [A,+\infty)} \leq (\nu + 1) \left(\frac{2}{\nu} \right)^{\frac{\nu}{(\nu+1)}} (\Gamma(\nu + 2))^{-\frac{1}{(\nu+1)}}$$

$$\left(\| f \|_{\infty, [A,+\infty)} \right)^{\frac{\nu}{(\nu+1)}} \left(\| D_{*A}^{\nu+1} f \|_{\infty, [A,+\infty)} \right)^{\frac{1}{(\nu+1)}}. \tag{7.30}$$

In the assumptions of last Theorem 7.8, for $A = 0$ we get

Corollary 7.9. *Let* $0 < \nu \leq 1$, $f \in AC^2([0, b])$, $(\forall)\ b > 0$. *Suppose* $\| f \|_{\infty, \mathbb{R}_+} < \infty$, $D_{*0}^{\nu+1} f \in L_{\infty}(\mathbb{R}_+)$, *and*

$$\| D_{*a}^{\nu+1} f \|_{\infty, [a,+\infty)} \leq \| D_{*0}^{\nu+1} f \|_{\infty, \mathbb{R}_+}, \ (\forall)\ a \geq 0.$$

Then

$$\|f'\|_{\infty,\,\mathbb{R}_+} \leq (\nu+1)\left(\frac{2}{\nu}\right)^{\frac{\nu}{(\nu+1)}} (\Gamma(\nu+2))^{-\frac{1}{(\nu+1)}}$$

$$\left(\|f\|_{\infty,\,\mathbb{R}_+}\right)^{\frac{\nu}{(\nu+1)}} \left(\|D_{*0}^{\nu+1} f\|_{\infty,\,\mathbb{R}_+}\right)^{\frac{1}{(\nu+1)}}. \tag{7.31}$$

When $\nu = 1$ *we get*

Corollary 7.10. *Let* $A \in \mathbb{R}$, $f \in AC^2([A,b])$, $(\forall)\ b > A$. *Assume* $f, f'' \in L_\infty([A,+\infty))$.
Then

$$\|f'\|_{\infty,\,[A,+\infty)} \leq 2\|f\|_{\infty,\,[A,+\infty)}^{\frac{1}{2}} \|f''\|_{\infty,\,[A,+\infty)}^{\frac{1}{2}}. \tag{7.32}$$

Also when $\nu = 1$ *we get*

Corollary 7.11. *Let* $f \in AC^2([0,b])$, $(\forall)\ b > 0$. *Suppose* $f, f'' \in L_\infty(\mathbb{R}_+)$.
Then

$$\|f'\|_{\infty,\,\mathbb{R}_+} \leq 2\|f\|_{\infty,\,\mathbb{R}_+}^{\frac{1}{2}} \|f''\|_{\infty,\,\mathbb{R}_+}^{\frac{1}{2}}, \tag{7.33}$$

which is the Landau [10] *inequality, where* 2 *is the best constant.*

7.3 Addendum

Let $g \in C^2([0,R])$, $R > 0$. Using integration by parts we obtain

$$\int_0^R g''(s)s\,\mathrm{d}s = Rg'(R) - g(R) + g(0). \tag{7.34}$$

If $g \in C^3([0,R])$, $R > 0$, then similarly we get

$$\int_0^R g^{(3)}(s)s^2\mathrm{d}s = R^2 g''(R) - 2Rg'(R) + 2g(R) - 2g(0). \tag{7.35}$$

For $g \in C^4([0,R])$, $R > 0$, we find

$$\int_0^R g^{(4)}(s)s^3\mathrm{d}s = R^3 g^{(3)}(R) - 3R^2 g''(R) + 6Rg'(R) - 6g(R) + 6g(0), \tag{7.36}$$

etc.

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Chapter 8
Left Caputo Fractional L_p-Landau-Type Inequalities

Here we present left Caputo fractional L_p-Landau-type inequalities and we give applications on $\mathbb{R}_+$. This chapter relies on [3].

8.1 Introduction

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$, and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

$$\left\| f' \right\|_{p,I} \leq C_p(I) \left\| f \right\|_{p,I}^{\frac{1}{2}} \left\| f'' \right\|_{p,I}^{\frac{1}{2}}, \tag{8.1}$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I, see [1, 4].

The research on these inequalities started by E. Landau [10] in 1914. For the case of $p = \infty$ he proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2} \tag{8.2}$$

are the best constants in (8.1).

In 1932, G.H. Hardy and J.E. Littlewood [7] proved (8.1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2}, \text{ and } C_2(\mathbb{R}) = 1. \tag{8.3}$$

In 1935, G.H. Hardy, E. Landau, and J.E. Littlewood [8] showed that the best constant $C_p(\mathbb{R}_+)$ in (8.1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2, \text{ for } p \in [1, \infty), \tag{8.4}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$. Infact in [6] and [9], it was shown that

$$C_p(\mathbb{R}) \leq \sqrt{2}.$$

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In this chapter we give fractional Landau inequalities with respect to $\|\cdot\|_p$, $p > 1$, involving the left Caputo fractional derivative.

We need

Definition 8.1 ([5], p. 38). Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ the ceiling of the number), $f \in AC^n([A,B])$ (i.e., $f^{(n-1)} \in AC([A,B])$, $A, B \in \mathbb{R}$). We define the left Caputo fractional derivative

$$D_{*A}^{\nu} f(x) := \frac{1}{\Gamma(n-\nu)} \int_A^x (x-t)^{n-\nu-1} f^{(n)}(t)\,dt, \tag{8.5}$$

$\forall\, x \in [A,B]$, and $D_{*A}^{\nu} f(x) := 0$, if $x < A$. Here Γ is the gamma function.

Definition 8.2. Let $0 < \nu \leq 1$, $f \in AC([A,B])$, we define

$$D_{*A}^{\nu} f(x) = \frac{1}{\Gamma(1-\nu)} \int_A^x (x-t)^{-\nu} f'(t)\,dt, \tag{8.6}$$

$\forall\, x \in [A,B]$.

Notice $D_{*A}^{n} f(x) = f^{(n)}$, for $n \in \mathbb{N}$.

We make

Remark 8.3. Let $f' \in AC([A,B])$, $0 < \nu \leq 1$, then $\lceil \nu + 1 \rceil = 2$, and

$$\begin{aligned} D_{*A}^{\nu} f'(x) &= \frac{1}{\Gamma(1-\nu)} \int_A^x (x-t)^{-\nu} f''(t)\,dt \\ &= \frac{1}{\Gamma(2-(\nu+1))} \int_A^x (x-t)^{2-(\nu+1)-1} f''(t)\,dt = D_{*A}^{\nu+1} f(x). \end{aligned}$$

Hence it holds

$$D_{*A}^{\nu} f'(x) = D_{*A}^{\nu+1} f(x). \tag{8.7}$$

8.2 Main Results

We need

Theorem 8.4 ([2], p. 620). *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $1 - \frac{1}{p} < \nu \leq 1$, $f \in AC([A,B])$, $A, B \in \mathbb{R}$, $A < B$. Assume $D_{*A}^{\nu} f \in L_q([A,B])$. Then*

$$\left| \frac{1}{B-A} \int_A^B f(x)\,dx - f(A) \right| \leq \frac{\left\| D_{*A}^{\nu} f \right\|_{L_q([A,B])}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (B-A)^{\nu-1+\frac{1}{p}}. \tag{8.8}$$

We make

Remark 8.5. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $1 - \frac{1}{p} < \nu \leq 1$, $f' \in AC([A, B])$, $A, B \in \mathbb{R}$, $A < B$. Suppose $D_{*A}^{\nu} f' \in L_q([A, B])$. Then

$$\left| \frac{1}{B-A} \int_A^B f'(x)\,\mathrm{d}x - f'(A) \right| \leq \frac{\left\| D_{*A}^{\nu} f' \right\|_{L_q([A,B])}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (B-A)^{\nu-1+\frac{1}{p}}. \tag{8.9}$$

Equivalently we have: For $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $1 - \frac{1}{p} < \nu \leq 1$, $f \in AC^2([A, B])$, $A, B \in \mathbb{R}$, $A < B$. Assume $D_{*A}^{\nu+1} f \in L_q([A, B])$. Then

$$\left| \frac{1}{B-A} (f(B) - f(A)) - f'(A) \right| \leq \frac{\left\| D_{*A}^{\nu+1} f \right\|_{L_q([A,B])}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (B-A)^{\nu-1+\frac{1}{p}}. \tag{8.10}$$

Therefore it holds

$$\left| f'(A) \right| - \frac{1}{B-A} |f(B) - f(A)| \leq \frac{\left\| D_{*A}^{\nu+1} f \right\|_{L_q([A,B])}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (B-A)^{\nu-1+\frac{1}{p}}. \tag{8.11}$$

We also make

Remark 8.6. Let $1 - \frac{1}{p} < \nu \leq 1$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $f \in AC^2([A, b])$, $\forall b > A$. Assume $D_{*A}^{\nu+1} f \in L_q([A, +\infty))$ (thus $D_{*A}^{\nu+1} \in L_q([A, b])$). Let $A < a < b$. Then $f \in AC^2([a, b])$ and $D_{*A}^{\nu+1} f \in L_q([a, +\infty))$ and $D_{*A}^{\nu+1} \in L_q([a, b])$. Here

$$D_{*a}^{\nu+1} f(x) = \frac{1}{\Gamma(1-\nu)} \int_a^x (x-t)^{-\nu} f''(t)\,\mathrm{d}t. \tag{8.12}$$

If $f''(t) \geq 0$ a.e., then

$$D_{*A}^{\nu+1} f(x) \geq D_{*a}^{\nu+1} f(x) \geq 0, \text{ a.e., for } x \geq a.$$

Thus

$$\infty > \left\| D_{*A}^{\nu+1} f \right\|_{q,[A,+\infty)} \geq \left\| D_{*A}^{\nu+1} f \right\|_{q,[a,+\infty)} \geq \left\| D_{*a}^{\nu+1} f \right\|_{q,[a,+\infty)}.$$

So it is not strange to suppose that

$$\left\| D_{*a}^{\nu+1} f \right\|_{q,[a,+\infty)} \leq \left\| D_{*A}^{\nu+1} f \right\|_{q,[A,+\infty)}, \tag{8.13}$$

$\forall\, a \geq A$ (it is obvious when $\nu \in \mathbb{N}$).

We need

Remark 8.7. Let $a, b \in [A, +\infty)$, $a < b$. Then we obtain

$$\left|f'(a)\right| \leq \frac{1}{b-a}\left|f(b)-f(a)\right|+\frac{\left\|D_{*a}^{\nu+1}f\right\|_{L_q([a,b])}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(b-a)^{\nu-1+\frac{1}{p}}. \tag{8.14}$$

We also assume that $\|f\|_{\infty,[A,+\infty)} < \infty$.

Therefore

$$\left|f'(a)\right| \leq \frac{2\|f\|_{\infty,[A,+\infty)}}{b-a}+\frac{\left\|D_{*A}^{\nu+1}f\right\|_{q,[A,+\infty)}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(b-a)^{\nu-1+\frac{1}{p}}, \tag{8.15}$$

$\forall\, a, b \in [A, +\infty)$, $a < b$.

The R.H.S. (8.15) depends only on $b-a$.

Therefore

$$\left\|f'\right\|_{\infty,[A,+\infty)} \leq \frac{2\|f\|_{\infty,[A,+\infty)}}{b-a}+\frac{\left\|D_{*A}^{\nu+1}f\right\|_{q,[A,+\infty)}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(b-a)^{\nu-1+\frac{1}{p}}. \tag{8.16}$$

We may call $t = b-a > 0$. Thus

$$\left\|f'\right\|_{\infty,[A,+\infty)} \leq \frac{2\|f\|_{\infty,[A,+\infty)}}{t}+\frac{\left\|D_{*A}^{\nu+1}f\right\|_{q,[A,+\infty)}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}t^{\nu-1+\frac{1}{p}}, \tag{8.17}$$

$\forall\, t \in (0, \infty)$.

Notice that $0 < \nu - 1 + \frac{1}{p} < 1$. Call

$$\widetilde{\mu} := 2\|f\|_{\infty,[A,+\infty)},$$
$$\widetilde{\theta} := \frac{\left\|D_{*A}^{\nu+1}f\right\|_{q,[A,+\infty)}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}, \tag{8.18}$$

both are positive, and

$$\widetilde{\nu} := \nu - 1 + \frac{1}{p}, \quad (\widetilde{\nu} \in (0,1)). \tag{8.19}$$

We consider the function

$$\widetilde{y}(t) = \frac{\widetilde{\mu}}{t} + \widetilde{\theta}t^{\widetilde{\nu}}, \quad t \in (0, \infty). \tag{8.20}$$

The only critical number here is

$$\widetilde{t_0} = \left(\frac{\widetilde{\mu}}{\widetilde{\nu}\widetilde{\theta}}\right)^{\widetilde{\frac{1}{\nu+1}}}, \tag{8.21}$$

and $\widetilde{y}$ has a global minimum at $\widetilde{t_0}$, which is

$$\widetilde{y}\left(\widetilde{t_0}\right) = \left(\widetilde{\theta}\widetilde{\mu}^{\nu}\right)^{\widetilde{\frac{1}{\nu+1}}} \left(\widetilde{\nu}+1\right)\widetilde{\nu}^{-\left(\widetilde{\frac{\nu}{\nu+1}}\right)}. \tag{8.22}$$

Consequently, we derive

$$\widetilde{y}\left(\widetilde{t_0}\right) = \left(\frac{\left\| D_{*A}^{\nu+1} f \right\|_{q,[A,+\infty)}}{\Gamma\left(\nu\right)\left(p\left(\nu-1\right)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}\right)^{\frac{1}{\nu+\frac{1}{p}}} \cdot$$

$$\times \left(2\left\|f\right\|_{\infty,[A,+\infty)}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \left(\nu+\frac{1}{p}\right)\left(\nu-1+\frac{1}{p}\right)^{-\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)}. \tag{8.23}$$

We have proved that

$$\left\|f'\right\|_{\infty,[A,+\infty)} \leq \left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-1+\frac{1}{p}}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \cdot \frac{1}{\left(\Gamma\left(\nu\right)\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}} \cdot \tag{8.24}$$

$$\frac{1}{\left(p\left(\nu-1\right)+1\right)^{\frac{1}{(p\nu+1)}}} \cdot \left(\left\|f\right\|_{\infty,[A,+\infty)}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \cdot \left(\left\|D_{*A}^{\nu+1} f\right\|_{q,[A,+\infty)}\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}.$$

We have established the following L_q result, left Caputo fractional Landau L_q inequality:

Theorem 8.8. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $1 - \frac{1}{p} < \nu \leq 1$, $f \in AC^2([A,b])$, $\forall b > A$. *Suppose* $D_{*A}^{\nu+1} f \in L_q([A,+\infty))$, $\|f\|_{\infty,[A,+\infty)} < \infty$, *and*

$$\left\|D_{*a}^{\nu+1} f\right\|_{q,[a,+\infty)} \leq \left\|D_{*A}^{\nu+1} f\right\|_{q,[A,+\infty)}, \tag{8.25}$$

$\forall\, a \geq A$. *Then*

$$\left\|f'\right\|_{\infty,[A,+\infty)} \leq \left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-1+\frac{1}{p}}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \cdot \tag{8.26}$$

$$\frac{1}{(\Gamma(\nu))^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}(p(\nu-1)+1)^{\frac{1}{(p\nu+1)}}}\cdot$$

$$\left(\|f\|_{\infty,[A,+\infty)}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)}\cdot\left(\left\|D_{*A}^{\nu+1}f\right\|_{q,[A,+\infty)}\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}.$$

We give

Corollary 8.9. *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $1-\frac{1}{p}<\nu\leq 1$, $f\in AC^2([0,b])$, $\forall b>0$. *Suppose* $D_{*0}^{\nu+1}f\in L_q([0,+\infty))$, $\|f\|_{\infty,\mathbb{R}_+}<\infty$, *and*

$$\left\|D_{*a}^{\nu+1}f\right\|_{q,[a,+\infty)}\leq\left\|D_{*0}^{\nu+1}f\right\|_{q,\mathbb{R}_+}, \tag{8.27}$$

$\forall\ a\geq 0$. *Then*

$$\left\|f'\right\|_{\infty,\mathbb{R}_+}\leq\left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-1+\frac{1}{p}}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)}\cdot \tag{8.28}$$

$$\frac{1}{(\Gamma(\nu))^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}(p(\nu-1)+1)^{\frac{1}{(p\nu+1)}}}\cdot$$

$$\left(\|f\|_{\infty,\mathbb{R}_+}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)}\cdot\left(\left\|D_{*0}^{\nu+1}f\right\|_{q,\mathbb{R}_+}\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}.$$

We mention the case of $p=q=2$.

Corollary 8.10. *Let* $\frac{1}{2}<\nu\leq 1$, $f\in AC^2([0,b])$, $\forall b>0$. *Assume* $D_{*0}^{\nu+1}f\in L_2(\mathbb{R}_+)$, $f\in L_\infty(\mathbb{R}_+)$, *and*

$$\left\|D_{*a}^{\nu+1}f\right\|_{2,[a,+\infty)}\leq\left\|D_{*0}^{\nu+1}f\right\|_{2,\mathbb{R}_+}, \tag{8.29}$$

$\forall\ a\geq 0$. *Then*

$$\left\|f'\right\|_{\infty,\mathbb{R}_+}\leq\left(\frac{2\nu+1}{\nu-\frac{1}{2}}\right)^{\left(\frac{\nu-\frac{1}{2}}{\nu+\frac{1}{2}}\right)}\cdot \tag{8.30}$$

$$\frac{1}{(\Gamma(\nu))^{\frac{1}{\left(\nu+\frac{1}{2}\right)}}(2\nu-1)^{\frac{1}{(2\nu+1)}}}\cdot$$

$$\left(\|f\|_{\infty,\mathbb{R}_+}\right)^{\left(\frac{\nu-\frac{1}{2}}{\nu+\frac{1}{2}}\right)}\cdot\left(\left\|D_{*0}^{\nu+1}f\right\|_{2,\mathbb{R}_+}\right)^{\frac{1}{\left(\nu+\frac{1}{2}\right)}}.$$

When $\nu = 1$ we derive

Corollary 8.11. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $f \in AC^2([0,b])$, $\forall b > 0$. *Assume* $f'' \in L_q(\mathbb{R}_+)$, $f \in L_\infty(\mathbb{R}_+)$. *Then*

$$\left\| f' \right\|_{\infty,\mathbb{R}_+} \leq (2(p+1))^{\frac{1}{p+1}} \cdot \tag{8.31}$$

$$\left(\|f\|_{\infty,\mathbb{R}_+}\right)^{\frac{1}{p+1}} \cdot \left(\left\|f''\right\|_{q,\mathbb{R}_+}\right)^{\frac{p}{p+1}}.$$

We finish chapter with

Corollary 8.12. *Let* $f \in AC^2([0,b])$, $\forall b > 0$. *Suppose* $f'' \in L_2(\mathbb{R}_+)$, $f \in L_\infty(\mathbb{R}_+)$. *Then*

$$\left\| f' \right\|_{\infty,\mathbb{R}_+} \leq \sqrt[3]{6} \cdot \left(\|f\|_{\infty,\mathbb{R}_+}\right)^{\frac{1}{3}} \cdot \left(\left\|f''\right\|_{2,\mathbb{R}_+}\right)^{\frac{2}{3}}. \tag{8.32}$$

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Chapter 9
Right Caputo Fractional L_p-Landau-Type Inequalities

We present right Caputo fractional $\|\cdot\|_p$-Landau type inequalities, $p \in (1, \infty]$ with applications on $\mathbb{R}_-$. This chapter is based on [4].

9.1 Introduction

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$, and $f : I \to \mathbb{R}$ be twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

$$\|f'\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \|f''\|_{p,I}^{\frac{1}{2}}, \tag{9.1}$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I, see [1, 5].

Development of these inequalities was initiated by E. Landau [13] in 1914. For the case of $p = \infty$ he proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2} \tag{9.2}$$

are the best constants in (9.1).

In 1932, G.H. Hardy and J.E. Littlewood [10] proved (9.1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2}, \text{ and } C_2(\mathbb{R}) = 1. \tag{9.3}$$

In 1935, G.H. Hardy, E. Landau, and J.E. Littlewood [11] showed that the best constant $C_p(\mathbb{R}_+)$ in (9.1) satisfies

$$C_p(\mathbb{R}_+) \leq 2, \text{ for } p \in [1, \infty), \tag{9.4}$$

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which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$. Infact in [6] and [12], it was shown that

$$C_p(\mathbb{R}) \leq \sqrt{2}.$$

In this chapter we give fractional Landau inequalities with respect to $\|\cdot\|_p$, $p \in (1, \infty]$, involving the right Caputo fractional derivative.

9.2 Main Results

We need

Definition 9.1 ([7–9, 14]). Let $f \in L_1([a, b])$, $\alpha > 0$. The right Riemann–Liouville fractional operator of order α is given by

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (J - x)^{\alpha - 1} f(J)\, dJ, \tag{9.5}$$

$\forall\ x \in [a, b]$, where Γ is the gamma function. We set $I_{b-}^0 := I$ (the identity operator).

Definition 9.2 ([7–9, 14]). Let $f \in AC^m([a, b])$ ($f^{(m-1)}$ is in $AC([a, b])$), $m \in \mathbb{N}$, $m = \lceil \alpha \rceil$, $\alpha > 0$ ($\lceil \cdot \rceil$ the ceiling of the number). We define the right Caputo fractional derivative of order $\alpha > 0$ by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^b (J - x)^{m - \alpha - 1} f^{(m)}(J)\, dJ, \quad \forall\ x \leq b. \tag{9.6}$$

If $\alpha = m \in \mathbb{N}$, then

$$D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x), \quad \forall\ x \in [a, b].$$

If $x > b$ we define $D_{b-}^{\alpha} f(x) = 0$.

In particular we have

Definition 9.3. Let $0 < \alpha \leq 1$, $f \in AC([A, B])$ and define

$$D_{B-}^{\alpha} f(x) := \frac{-1}{\Gamma(1 - \alpha)} \int_x^B (J - x)^{-\alpha} f'(J)\, dJ. \tag{9.7}$$

Remark 9.4. Let $f' \in AC([A, B])$, $0 < \alpha \leq 1$, then $\lceil \alpha + 1 \rceil = 2$ and

$$\begin{aligned} D_{B-}^{\alpha} f'(x) &= \frac{-1}{\Gamma(1 - \alpha)} \int_x^B (J - x)^{-\alpha} f''(J)\, dJ \\ &= \frac{-(-1)^2}{\Gamma(2 - (\alpha + 1))} \int_x^B (J - x)^{2 - (\alpha + 1) - 1} f''(J)\, dJ = -D_{B-}^{\alpha + 1} f(x), \end{aligned}$$

i.e.,

$$D_{B-}^{\alpha} f'(x) = -D_{B-}^{\alpha+1} f(x) \tag{9.8}$$

and hence

$$\left\| D_{B-}^{\alpha} f' \right\|_{\infty,[A,B]} = \left\| D_{B-}^{\alpha+1} f \right\|_{\infty,[A,B]} . \tag{9.9}$$

The focus is now on $(-\infty, B]$, $B \in \mathbb{R}$.

Let $a < b < B$, $a, b \in \mathbb{R}$. If $f''(J) \geq 0$, a.e., then

$$\frac{1}{\Gamma(1-\alpha)} \int_x^B (J-x)^{-\alpha} f''(J) \, dJ$$
$$\geq \frac{1}{\Gamma(1-\alpha)} \int_x^b (J-x)^{-\alpha} f''(J) \, dJ \geq 0, \text{ a.e.,}$$

i.e.,

$$D_{B-}^{\alpha+1} f(x) \geq D_{b-}^{\alpha+1} f(x) \geq 0, \text{ a.e., for } x \leq b. \tag{9.10}$$

It holds therefore that

$$\infty > \left\| D_{B-}^{\alpha+1} f \right\|_{\infty,(-\infty,B]} \geq \left\| D_{b-}^{\alpha+1} f \right\|_{\infty,(-\infty,b]} . \tag{9.11}$$

It is reasonable then to suppose that

$$\left\| D_{b-}^{\alpha+1} f \right\|_{\infty,(-\infty,b]} \leq \left\| D_{B-}^{\alpha+1} f \right\|_{\infty,(-\infty,B]} . \tag{9.12}$$

(it is obvious when $\alpha = 1$), $\forall\, b \leq B$.

Remark 9.5. Let $0 < \alpha \leq 1$; $B, a \in \mathbb{R}$, with $f \in AC^2([a, B])$, $\forall a < B$. Assume $D_{B-}^{\alpha+1} f \in L_\infty((-\infty, B])$, thus $D_{B-}^{\alpha+1} f \in L_\infty([a, B])$. Let $a < b < B$, then $f \in AC^2([a, b])$, and $D_{B-}^{\alpha+1} f \in L_\infty((-\infty, b])$ and $D_{B-}^{\alpha+1} f \in L_\infty([a, b])$.

By Theorem 6 of [2], applied on f', we obtain that

$$\left| f'(b) \right| \leq \frac{1}{b-a} \left| f(b) - f(a) \right| + \frac{\left\| D_{b-}^{\alpha+1} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+2)} (b-a)^{\alpha} \tag{9.13}$$

$$\leq \frac{2 \left\| f \right\|_{\infty,(-\infty,B]}}{b-a} + \frac{\left\| D_{B-}^{\alpha+1} f \right\|_{\infty,(-\infty,B]} (b-a)^{\alpha}}{\Gamma(\alpha+2)}, \tag{9.14}$$

where we also assume that $\| f \|_{\infty,(-\infty,B]} < \infty$.

Setting $t = b - a$,

$$\left\| f' \right\|_{\infty,(-\infty,B]} \leq \frac{2 \left\| f \right\|_{\infty,(-\infty,B]}}{t} + \frac{\left\| D_{B-}^{\alpha+1} f \right\|_{\infty,(-\infty,B]} \cdot t^{\alpha}}{\Gamma(\alpha+2)}, \tag{9.15}$$

$\forall\, t \in (0, \infty)$.

One now can prove easily that the R.H.S. (9.15) has a global minimum and find it (see also [3]) with a usual calculus method.

We have established the following.

Theorem 9.6. *Let* $0 < \alpha \leq 1$; $B, a \in \mathbb{R}$*, with* $f \in AC^2([a, B])$*,* $\forall a < B$*, where* B *is fixed. Assume* $\|f\|_{\infty,(-\infty,B]} < \infty$*,* $D_{B-}^{\alpha+1} f \in L_\infty((-\infty, B])$*, and*

$$\left\| D_{b-}^{\alpha+1} f \right\|_{\infty,(-\infty,b]} \leq \left\| D_{B-}^{\alpha+1} f \right\|_{\infty,(-\infty,B]}, \tag{9.16}$$

$\forall\, b < B$*, then*

$$\left\| f' \right\|_{\infty,(-\infty,B]} \leq (\alpha + 1) \cdot \left(\frac{2}{\alpha} \right)^{\left(\frac{\alpha}{\alpha+1}\right)} \cdot (\Gamma(\alpha + 2))^{-\frac{1}{(\alpha+1)}} \cdot \tag{9.17}$$

$$\left(\|f\|_{\infty,(-\infty,B]} \right)^{\left(\frac{\alpha}{\alpha+1}\right)} \cdot \left(\left\| D_{B-}^{\alpha+1} f \right\|_{\infty,(-\infty,B]} \right)^{\frac{1}{(\alpha+1)}}.$$

The case of $B = 0$ *is given by the following Corollary.*

Corollary 9.7. *Let* $0 < \alpha \leq 1$; $f \in AC^2([a, 0])$*,* $\forall\, a < 0$*. Suppose that* $\|f\|_{\infty,\mathbb{R}_-} < \infty$*,* $D_{0-}^{\alpha+1} f \in L_\infty(\mathbb{R}_-)$*, and*

$$\left\| D_{b-}^{\alpha+1} f \right\|_{\infty,(-\infty,b]} \leq \left\| D_{0-}^{\alpha+1} f \right\|_{\infty,\mathbb{R}_-}, \tag{9.18}$$

$\forall\, b < 0$*. Then*

$$\left\| f' \right\|_{\infty,\mathbb{R}_-} \leq (\alpha + 1) \cdot \left(\frac{2}{\alpha} \right)^{\left(\frac{\alpha}{\alpha+1}\right)} \cdot (\Gamma(\alpha + 2))^{-\frac{1}{(\alpha+1)}} \cdot \tag{9.19}$$

$$\left(\|f\|_{\infty,\mathbb{R}_-} \right)^{\left(\frac{\alpha}{\alpha+1}\right)} \cdot \left(\left\| D_{0-}^{\alpha+1} f \right\|_{\infty,\mathbb{R}_-} \right)^{\frac{1}{(\alpha+1)}}.$$

The case of $\alpha = 1$ *is given by the following corollary.*

Corollary 9.8. *Let* $B, a \in \mathbb{R}$*, with* $f \in AC^2([a, B])$*,* $\forall a < B$*,* B *is fixed. Assume* $\|f\|_{\infty,(-\infty,B]} < \infty$*,* $f'' \in L_\infty((-\infty, B])$*, then*

$$\left\| f' \right\|_{\infty,(-\infty,B]} \leq 2 \cdot \left(\|f\|_{\infty,(-\infty,B]} \right)^{\frac{1}{2}} \cdot \left(\left\| f'' \right\|_{\infty,(-\infty,B]} \right)^{\frac{1}{2}}. \tag{9.20}$$

Corollary 9.9. *Let* $f \in AC^2([a, 0])$*,* $\forall a < 0$*. Suppose* $f, f'' \in L_\infty(\mathbb{R}_-)$*. Then*

$$\left\| f' \right\|_{\infty,\mathbb{R}_-} \leq 2 \cdot \left(\|f\|_{\infty,\mathbb{R}_-} \right)^{\frac{1}{2}} \cdot \left(\left\| f'' \right\|_{\infty,\mathbb{R}_-} \right)^{\frac{1}{2}}. \tag{9.21}$$

Remark 9.10. Let any $a, b \in (-\infty, B]$, $a < b$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $1 - \frac{1}{p} < \alpha \le 1$, and $f \in AC^2([a, b])$. Assume also $D_{b-}^{\alpha+1} f \in L_q([a, b])$, then by [2], Theorem 8, for f' we get that

$$\left|f'(b)\right| \le \frac{1}{b-a} |f(b) - f(a)| + \frac{\left\| D_{b-}^{\alpha+1} f \right\|_{L_q([a,b])}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} (b-a)^{\alpha-1+\frac{1}{p}}. \tag{9.22}$$

Suppose here

$$\left\| D_{b-}^{\alpha+1} f \right\|_{q,(-\infty,b]} \le \left\| D_{B-}^{\alpha+1} f \right\|_{q,(-\infty,B]} < \infty, \tag{9.23}$$

$\forall\, b \le B$.

Therefore,

$$\left|f'(b)\right| \le \frac{2\|f\|_{\infty,(-\infty,B]}}{b-a} + \frac{\left\| D_{B-}^{\alpha+1} f \right\|_{q,(-\infty,B]}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} (b-a)^{\alpha-1+\frac{1}{p}}, \tag{9.24}$$

$\forall\, a, b : a < b \le B$, where B is fixed.

Let $t := b - a > 0$, then

$$\left\|f'\right\|_{\infty,(-\infty,B]} \le \frac{2\|f\|_{\infty,(-\infty,B]}}{t} + \frac{\left\| D_{B-}^{\alpha+1} f \right\|_{q,(-\infty,B]}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} t^{\alpha-1+\frac{1}{p}}, \tag{9.25}$$

$\forall\, t \in (0, \infty)$, where it is assumed that $\|f\|_{\infty,(-\infty,B]} < \infty$.

As before we have proved

Theorem 9.11. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $1 - \frac{1}{p} < \alpha \le 1$, $f \in AC^2([a, B])$, $\forall a < B$. Assume $D_{B-}^{\alpha+1} f \in L_q((-\infty, B])$, $\|f\|_{\infty,(-\infty,B]} < \infty$, and*

$$\left\| D_{b-}^{\alpha+1} f \right\|_{q,(-\infty,b]} \le \left\| D_{B-}^{\alpha+1} f \right\|_{q,(-\infty,B]}, \tag{9.26}$$

$\forall\, b \le B$, then

$$\left\|f'\right\|_{\infty,(-\infty,B]} \le \left(\frac{2\left(\alpha+\frac{1}{p}\right)}{\alpha-1+\frac{1}{p}}\right)^{\left(\frac{\alpha-1+\frac{1}{p}}{\alpha+\frac{1}{p}}\right)} \cdot \tag{9.27}$$

$$\frac{1}{(\Gamma(\alpha))^{\frac{1}{\left(\alpha+\frac{1}{p}\right)}}(p(\alpha-1)+1)^{\frac{1}{(p\alpha+1)}}} \cdot$$

$$\left(\|f\|_{\infty,(-\infty,B]}\right)^{\left(\frac{a-1+\frac{1}{p}}{\alpha+\frac{1}{p}}\right)} \cdot \left(\left\| D_{B-}^{\alpha+1} f \right\|_{q,(-\infty,B]}\right)^{\frac{1}{\left(\alpha+\frac{1}{p}\right)}}.$$

The case of $B = 0$ *is given by Corollary 9.12.*

Corollary 9.12. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $1 - \frac{1}{p} < \alpha \leq 1$, $f \in AC^2([a, 0])$, $\forall a < 0$. *Suppose* $D_{0-}^{\alpha+1} f \in L_q(\mathbb{R}_-)$, $\|f\|_{\infty,\mathbb{R}_-} < \infty$, *and*

$$\left\| D_{b-}^{\alpha+1} f \right\|_{q,(-\infty,b]} \leq \left\| D_{0-}^{\alpha+1} f \right\|_{q,\mathbb{R}_-}, \tag{9.28}$$

$\forall\, b \leq 0$. *Then*

$$\left\| f' \right\|_{\infty,\mathbb{R}_-} \leq \left(\frac{2\left(\alpha + \frac{1}{p}\right)}{\alpha - 1 + \frac{1}{p}} \right)^{\left(\frac{\alpha - 1 + \frac{1}{p}}{\alpha + \frac{1}{p}}\right)} \cdot \tag{9.29}$$

$$\frac{1}{\left(\Gamma(\alpha)\right)^{\frac{1}{\left(\alpha + \frac{1}{p}\right)}} \left(p(\alpha - 1) + 1\right)^{\frac{1}{(p\alpha + 1)}}} \cdot$$

$$\left(\|f\|_{\infty,\mathbb{R}_-}\right)^{\left(\frac{\alpha - 1 + \frac{1}{p}}{\alpha + \frac{1}{p}}\right)} \cdot \left(\left\| D_{0-}^{\alpha+1} f \right\|_{q,\mathbb{R}_-} \right)^{\frac{1}{\left(\alpha + \frac{1}{p}\right)}}.$$

The case for $p = q = 2$ follows.

Corollary 9.13. *Let* $\frac{1}{2} < \alpha \leq 1$, $f \in AC^2([a, 0])$, $\forall a < 0$. *Suppose* $D_{0-}^{\alpha+1} f \in L_2(\mathbb{R}_-)$, $\|f\|_{\infty,\mathbb{R}_-} < \infty$, *and*

$$\left\| D_{b-}^{\alpha+1} f \right\|_{2,(-\infty,b]} \leq \left\| D_{0-}^{\alpha+1} f \right\|_{2,\mathbb{R}_-}, \tag{9.30}$$

$\forall\, b \leq 0$, *then*

$$\left\| f' \right\|_{\infty,\mathbb{R}_-} \leq \left(\frac{2\alpha + 1}{\alpha - \frac{1}{2}} \right)^{\left(\frac{\alpha - \frac{1}{2}}{\alpha + \frac{1}{2}}\right)} \cdot \tag{9.31}$$

$$\frac{1}{\left(\Gamma(\alpha)\right)^{\frac{1}{\left(\alpha + \frac{1}{2}\right)}} (2\alpha - 1)^{\frac{1}{(2\alpha + 1)}}} \cdot$$

$$\left(\|f\|_{\infty,\mathbb{R}_-}\right)^{\left(\frac{\alpha - \frac{1}{2}}{\alpha + \frac{1}{2}}\right)} \cdot \left(\left\| D_{0-}^{\alpha+1} f \right\|_{2,\mathbb{R}_-} \right)^{\frac{1}{\left(\alpha + \frac{1}{2}\right)}}.$$

The case $\alpha = 1$ gives:

Corollary 9.14. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $f \in AC^2([a, 0])$, $\forall a < 0$. *Assume* $f'' \in L_q(\mathbb{R}_-)$, $f \in L_\infty(\mathbb{R}_-)$, *then*

$$\left\| f' \right\|_{\infty,\mathbb{R}_-} \leq \left(2(p + 1)\right)^{\frac{1}{p+1}} \cdot \left(\|f\|_{\infty,\mathbb{R}_-}\right)^{\frac{1}{p+1}} \cdot \left(\left\| f'' \right\|_{q,\mathbb{R}_-} \right)^{\frac{p}{p+1}}. \tag{9.32}$$

Finally, for $p = q = 2$ we have

Corollary 9.15. *Let* $f \in AC^2([a,0])$, $\forall a < 0$. *Assume* $f'' \in L_2(\mathbb{R}_-)$, $f \in L_\infty(\mathbb{R}_-)$, *then*

$$\|f'\|_{\infty,\mathbb{R}_-} \leq \sqrt[3]{6} \cdot \left(\|f\|_{\infty,\mathbb{R}_-}\right)^{\frac{1}{3}} \cdot \left(\|f''\|_{2,\mathbb{R}_-}\right)^{\frac{2}{3}}. \tag{9.33}$$

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Chapter 10
Mixed Caputo Fractional L_p-Landau-Type Inequalities

Here we give mixed Caputo fractional $||\cdot||_p$-Landau type inequalities, $p \in (1, \infty]$ with applications on $\mathbb{R}$. This chapter relies on [5].

10.1 Introduction

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

$$||f'||_{p,I} \leq C_p(I)||f||_{p,I}^{\frac{1}{2}}||f''||_{p,I}^{\frac{1}{2}}, \tag{10.1}$$

where $||.||_{p,I}$ is the p-norm on the interval I, see [1, 6].

The research on these inequalities started by E. Landau [15] in 1914. For the case of $p = \infty$ he proved that

$$C_\infty(\mathbb{R}_+) = 2 \quad \text{and} \quad C_\infty(\mathbb{R}) = \sqrt{2} \tag{10.2}$$

are the best constants in (10.1).

In 1932, G. H. Hardy and J. E. Littlewood [12] proved (10.1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \quad \text{and} \quad C_2(\mathbb{R}) = 1. \tag{10.3}$$

In 1935, G. H. Hardy, E. Landau, and J. E. Littlewood [13] showed that the best constant $C_p(\mathbb{R}_+)$ in (10.1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2, \quad \text{for} \quad p \in [1, \infty), \tag{10.4}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$.

In fact, in [8] and [14] it was shown that $C_p(\mathbb{R}) \leq \sqrt{2}$.

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In this chapter we prove mixed fractional Landau inequalities with respect to $\|\cdot\|_p,\ p \in (1, \infty]$, involving the right and left Caputo fractional derivatives.

10.2 Main Results

We need (see [9–11, 16])

Definition 10.1. Let $f \in AC^m([a,b])$ (space of functions from $[a,b]$ into $\mathbb{R}$ with $m-1$ derivative absolutely continuous function on $[a,b]$), $m \in \mathbb{N}$, where $m = \lceil\alpha\rceil$, $\alpha > 0$ ($\lceil . \rceil$ the ceiling of the number).

We define the right Caputo fractional derivative of order $\alpha > 0$ by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta) d\zeta. \tag{10.5}$$

We set $D_{b-}^{0} f(x) = f(x),\ \forall x \in [a,b]$.

Remark 10.2. Let $f \in AC^m([a,b]),\ m = \lceil\alpha\rceil$, with $\alpha > 0$ then $f^{(m-1)} \in AC([a,b])$, which implies that f^m exists a.e. on $[a,b]$ and that $f^m \in L_1([a,b])$.

Consequently, if $f \in AC^m([a,b])$, then $D_{b-}^{\alpha} f(x)$ exists a.e. on $[a,b]$ and $D_{b-}^{\alpha} f \in L_1([a,b])$, see [7], p. 13.

Observe that when $\alpha = m \in \mathbb{N}$, then

$$D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x),\ \forall x \in [a,b]. \tag{10.6}$$

If $x > b$ we define $D_{b-}^{\alpha} f(x) = 0$.

We also need

Definition 10.3 ([7], p. 38). Let $f \in AC^m([a,b])$, $m \in \mathbb{N}$, where $m = \lceil\alpha\rceil$, $\alpha > 0$. We define the left Caputo fractional derivative of order $\alpha > 0$, by

$$D_{*a}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \tag{10.7}$$

$\forall x \in [a,b]$. We set $D_{*a}^{0} f(x) = f(x)$, $\forall x \in [a,b]$.

Again here $D_{*a}^{\alpha} f$ exists a.e. on $[a,b]$ and $D_{*a}^{\alpha} f \in L_1([a,b])$, see [7], pp. 13 and 37–38. When $\alpha = m \in \mathbb{N}$ then

$$D_{*a}^{m} f(x) = f^{(m)}(x), \quad \forall x \in [a,b]. \tag{10.8}$$

If $x < \alpha$ we define $D_{*a}^{\alpha} f(x) = 0$.

We make

Remark 10.4. Let $\alpha > 0$, $m = \lceil \alpha \rceil, \alpha \notin \mathbb{N}$ and $f \in AC^m([a,b])$ with $x_0 \in [a,b]$. Then

$$D^{\alpha}_{x_0-} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^{x_0} (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta) \mathrm{d}\zeta, \quad \forall x \le x_0, \tag{10.9}$$

$$D^{\alpha}_{x_0-} f(x) = 0, \quad \forall x > x_0;$$

and

$$D^{\alpha}_{x_0*} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t) \mathrm{d}t, \quad \forall x \ge x_0, \tag{10.10}$$

$$D^{\alpha}_{x_0*} f(x) = 0, \quad \forall x < x_0.$$

Suppose first $f \in C^{\infty}(\mathbb{R})$ and of compact support $[A, B]$; $A < B$. Then $f^{(m)}$ is of support in $[A, B]$. Hence possibly $D^{\alpha}_{x_0*} f(x) \neq 0$ only for $x_0, x \in [A, B]$, i.e., only $[A, B]$ counts.

We proved earlier that $D^{\alpha}_{x_0*} f(x)$ is jointly continuous on $[A, B]^2$, see [3], so that $\sup_{x_o \in \mathbb{R}} ||D^{\alpha}_{x_0*} f||_{\infty,\mathbb{R}} < \infty$. In particular $\sup_{x_o \in \mathbb{R}} ||D^{\alpha}_{x_0*} f||_{\infty,[x_0,+\infty)} < \infty$.

Totally similar we have for this case that $\sup_{x_o \in \mathbb{R}} ||D^{\alpha}_{x_0-} f||_{\infty,(-\infty,x_0]} < +\infty$.

We also make

Remark 10.5. Let $x \ge x_0$, $x, x_0 \in \mathbb{R}$. We want to prove that

$$\int_{x_0}^{x} \frac{\mathrm{e}^{\mathrm{i}t}}{\sqrt{x-t}} \mathrm{d}t \tag{10.11}$$

is uniformly bounded for all $x, x_0 \in \mathbb{R}$, $x \ge x_0$.

We call $u := x - t$, then

$$\begin{aligned}
\int_{x_0}^{x} \frac{\mathrm{e}^{\mathrm{i}t}}{\sqrt{x-t}} \mathrm{d}t &= \int_0^{x-x_0} \frac{\mathrm{e}^{\mathrm{i}(x-u)}}{\sqrt{u}} \mathrm{d}u = \mathrm{e}^{\mathrm{i}x} \int_0^{x-x_0} \frac{\mathrm{e}^{-\mathrm{i}u}}{\sqrt{u}} \mathrm{d}u \\
&= \left(\text{calling} \quad w := u^{\frac{1}{2}}, 2\mathrm{d}w = \frac{\mathrm{d}u}{\sqrt{u}} \right) \\
&= 2\mathrm{e}^{\mathrm{i}x} \int_0^{\sqrt{x-x_0}} \mathrm{e}^{-\mathrm{i}w^2} \mathrm{d}w \quad \text{(the Fresnel integral)} \\
&\quad \text{(assume first } x - x_0 > 1) \\
&= 2\mathrm{e}^{\mathrm{i}x} \left[\int_0^1 \mathrm{e}^{-\mathrm{i}w^2} \mathrm{d}w + \int_1^{\sqrt{x-x_0}} \mathrm{e}^{-\mathrm{i}w^2} \mathrm{d}w \right].
\end{aligned} \tag{10.12}$$

In the last e^{ix} and $\int_0^1 e^{-iw^2}dw$ are bounded.

$$\text{Call} \quad \psi := \sqrt{x - x_0} > 1.$$

So, it is enough to bound

$$\int_1^{\psi} e^{-iw^2}dw = \int_1^{\psi} \frac{we^{-iw^2}}{w}dw = (*). \tag{10.13}$$

We observe that

$$\frac{de^{-iw^2}}{dw} = e^{-iw^2}(-2iw)$$

and

$$-\frac{de^{-iw^2}}{2i} = e^{-iw^2}wdw. \tag{10.14}$$

So that

$$(*) = -\frac{1}{2i}\int_1^{\psi} \frac{de^{-iw^2}}{w} = -\frac{1}{2i}\left[\frac{1}{w}e^{-iw^2}\Big|_1^w - \int_1^{\psi} e^{-iw^2}d\left(\frac{1}{w}\right)\right]$$
$$= -\frac{1}{2i}\left[\left[\frac{1}{\psi}e^{-iw^2} - e^{-i}\right] + \int_1^{\psi} \frac{e^{-iw^2}}{w^2}dw\right]. \tag{10.15}$$

In the last expression $\left(\frac{1}{\psi}e^{-iw^2} - e^{-i}\right)$ is bounded.

We further see that

$$\left|\int_1^{\psi} \frac{e^{-iw^2}}{w^2}dw\right| \le \int_1^{\psi} \frac{dw}{w^2} = \int_1^{\psi} w^{-2}dw = -\frac{1}{w}\Big|_1^{\psi} = \frac{1}{w}\Big|_{\psi}^1 = 1 - \frac{1}{w} < 1, \tag{10.16}$$

with $\left(1 - \frac{1}{w}\right) > 0$.

Hence the last integral is bounded.

If $\sqrt{x - x_0} \le 1$, we observe that

$$\left|\int_0^{\sqrt{x-x_0}} e^{-iw^2}dw\right| \le \int_0^{\sqrt{x-x_0}} \left|e^{-iw^2}\right|dw = \sqrt{x - x_0} \le 1, \tag{10.17}$$

a bounded integral.

Hence integral (10.11) is uniformly bounded for all $x, x_0 \in \mathbb{R}$, $x \ge x_0$.

Conclusion 10.6 *The integrals*

$$\int_{x_0}^{x} \frac{\sin t}{\sqrt{x-t}} \mathrm{d}t, \quad \int_{x_0}^{x} \frac{\cos t}{\sqrt{x-t}} \mathrm{d}t \tag{10.18}$$

are uniformly bounded for all $x, x_0 \in \mathbb{R}, \ x \geq x_0$.

Explanation: We observe that

$$\int_{x_0}^{x} \frac{\mathrm{e}^{\mathrm{i}t}}{\sqrt{x-t}} \mathrm{d}t = \int_{x_0}^{x} \frac{\cos t + \mathrm{i} \sin t}{\sqrt{x-t}} \mathrm{d}t = \left(\int_{x_0}^{x} \frac{\cos t}{\sqrt{x-t}} \mathrm{d}t \right) + \mathrm{i} \left(\int_{x_0}^{x} \frac{\sin t}{\sqrt{x-t}} \mathrm{d}t \right). \tag{10.19}$$

That is,

$$+\infty > \left| \int_{x_0}^{x} \frac{\mathrm{e}^{\mathrm{i}t}}{\sqrt{x-t}} \mathrm{d}t \right| = \sqrt{\left(\int_{x_0}^{x} \frac{\cos t}{\sqrt{x-t}} \mathrm{d}t \right)^2 + \left(\int_{x_0}^{x} \frac{\sin t}{\sqrt{x-t}} \mathrm{d}t \right)^2} \tag{10.20}$$

$$\geq \left| \int_{x_0}^{x} \frac{\cos t}{\sqrt{x-t}} \mathrm{d}t \right|, \quad \left| \int_{x_0}^{x} \frac{\sin t}{\sqrt{x-t}} \mathrm{d}t \right|. \tag{10.21}$$

Similarly, for $\sin At$, $\cos Bt$, and their linear combinations, etc.

Conclusion 10.7 *We demonstrated one more reason that it makes sense to suppose for* $\alpha > 0$ *that*

$$\sup_{x_o \in \mathbb{R}} ||D_{x_0-}^{\alpha} f||_{\infty,(-\infty,x_0]} < +\infty, \quad \sup_{x_o \in \mathbb{R}} ||D_{*x_0}^{\alpha} f||_{\infty,[x_0,+\infty)} < +\infty. \tag{10.22}$$

We now make

Remark 10.8. So we can suppose that there are big classes of functions f such that

$$\sup_{x_o \in \mathbb{R}} ||D_{*x_0}^{\alpha} f||_{\infty,[x_0,+\infty)} < +\infty, \quad \sup_{x_o \in \mathbb{R}} ||D_{x_0-}^{\alpha} f||_{\infty,(-\infty,x_0]} < +\infty. \tag{10.23}$$

So, for $0 < \alpha \leq 1$, $m = 1$ we treat $f \in AC([a,b])$, $\forall [a,b] \subset \mathbb{R}$. Then by [4], Theorem 6 and by [2], Theorem 26.8, p. 618 we derive

$$\left| \frac{1}{b-a} \int_a^b f(x) \mathrm{d}x - f(x_0) \right|$$

$$\leq \frac{1}{\Gamma(\alpha+2)} \max \left\{ ||D_{x_0-}^{\alpha} f||_{\infty,[\alpha,x_0]}, \ ||D_{*x_0}^{\alpha} f||_{\infty,[x_0,b]} \right\} \cdot (b-a)^{\alpha} \tag{10.24}$$

$$\leq \frac{1}{\Gamma(\alpha+2)} \max \left\{ \sup_{x_o \in \mathbb{R}} ||D_{x_0-}^{\alpha} f||_{\infty,(-\infty,x_0]}, \sup_{x_o \in \mathbb{R}} ||D_{x_0*}^{\alpha} f||_{\infty,[x_0,+\infty)} \right\} \cdot (b-a)^{\alpha}. \tag{10.25}$$

Next assume $f \in AC^2([a,b])$, $\forall [a,b] \subseteq \mathbb{R}$ and assume $||f||_{\infty,\mathbb{R}} < \infty$, along with

$$\sup_{x_o \in \mathbb{R}} ||D_{*x_0}^{\alpha+1} f||_{\infty,[x_0,+\infty)} < \infty, \quad \sup_{x_o \in \mathbb{R}} ||D_{x_0-}^{\alpha+1} f||_{\infty,(-\infty,x_0]} < \infty.$$

Hence it holds

$$\left| \frac{1}{b-a} \int_a^b f'(x)\mathrm{d}x - f'(x_0) \right|$$
$$\leq \frac{1}{\Gamma(\alpha+2)} \max \left\{ \sup_{x_o \in \mathbb{R}} ||D_{x_0-}^{\alpha} f'||_{\infty,(-\infty,x_0]} \sup_{x_o \in \mathbb{R}} ||D_{x_0*}^{\alpha} f'||_{\infty,[x_0,+\infty)} \right\} \cdot (b-a)^{\alpha}. \tag{10.26}$$

Equivalently, it holds

$$|f'(x_0)| - \frac{1}{b-a}|f(b) - f(a)| \leq \left| \frac{1}{b-a} (f(b) - f(a)) - f'(x_0) \right|$$
$$\leq \frac{1}{\Gamma(\alpha+2)} \max \left\{ \sup_{x_o \in \mathbb{R}} ||D_{x_0-}^{\alpha+1} f||_{\infty,(-\infty,x_0]} \sup_{x_o \in \mathbb{R}} ||D_{x_0*}^{\alpha+1} f||_{\infty,[x_0,+\infty)} \right\} \cdot (b-a)^{\alpha}. \tag{10.27}$$

Consequently,

$$|f'(x_0)| \leq \frac{2}{b-a}||f||_{\infty,\mathbb{R}} \frac{1}{\Gamma(\alpha+2)}$$
$$\max \left\{ \sup_{x_o \in \mathbb{R}} ||D_{x_0-}^{\alpha+1} f||_{\infty,(-\infty,x_0]}, \sup_{x_o \in \mathbb{R}} ||D_{x_0*}^{\alpha+1} f||_{\infty,[x_0,+\infty)} \right\} \cdot (b-a)^{\alpha}. \tag{10.28}$$

So that

$$||f'||_{\infty,\mathbb{R}} \leq \frac{2||f||_{\infty,\mathbb{R}}}{b-a}$$
$$+ \frac{\max \left\{ \sup_{x_o \in \mathbb{R}} ||D_{x_0-}^{\alpha+1} f||_{\infty,(-\infty,x_0]}, \sup_{x_o \in \mathbb{R}} ||D_{x_0*}^{\alpha+1} f||_{\infty,[x_0,+\infty)} \right\}}{\Gamma(\alpha+2)} \cdot (b-a)^{\alpha}. \tag{10.29}$$

The R.H.S. of (10.29) depends only on $t := b - a > 0$, that t can be anything positive.

Thus

$$||f'||_{\infty,\mathbb{R}} \leq \frac{2||f||_{\infty,\mathbb{R}}}{t}$$
$$+ \frac{\max \left\{ \sup_{x_o \in \mathbb{R}} ||D_{x_0-}^{\alpha+1} f||_{\infty,(-\infty,x_0]}, \sup_{x_o \in \mathbb{R}} ||D_{x_0*}^{\alpha+1} f||_{\infty,[x_0,+\infty)} \right\}}{\Gamma(\alpha+2)} \cdot t^{\alpha}, \quad \forall\, t > 0. \tag{10.30}$$

We call

$$M := 2||f||_{\infty,\mathbb{R}} > 0,$$

$$\theta := \frac{\max\left\{ \sup_{x_o \in \mathbb{R}} ||D_{x_0-}^{\alpha+1} f||_{\infty,(-\infty,x_0]}, \ \sup_{x_o \in \mathbb{R}} ||D_{x_0*}^{\alpha+1} f||_{\infty,[x_0,+\infty)} \right\}}{\Gamma(\alpha+2)} > 0. \quad (10.31)$$

Hence

$$||f'||_{\infty,\mathbb{R}} \leq \frac{M}{t} + \theta t^{\alpha}, \quad \forall\, t > 0. \quad (10.32)$$

Set

$$y(t) := \frac{M}{t} + \theta t^{\alpha}, \quad \forall\, t > 0. \quad (10.33)$$

Using basic calculus the function $y(t)$ has a global minimum which is

$$y(t_0) := (\theta M^{\alpha})^{\frac{1}{\alpha+1}} \cdot (\alpha+1) \cdot \alpha^{-\frac{\alpha}{\alpha+1}}. \quad (10.34)$$

Therefore, we have proved that

$$||f'||_{\infty,\mathbb{R}}$$

$$\leq \left[\frac{\max\left\{ \sup_{x_o \in \mathbb{R}} ||D_{x_0-}^{\alpha+1} f||_{\infty,(-\infty,x_0]}, \sup_{x_o \in \mathbb{R}} ||D_{x_0*}^{\alpha+1} f||_{\infty,[x_0,+\infty)} \right\} 2^{\alpha} ||f||_{\infty,\mathbb{R}}^{\alpha}}{\Gamma(\alpha+2)} \right]^{\frac{1}{\alpha+1}}$$

$$\cdot (\alpha+1)\alpha^{-\frac{\alpha}{\alpha+1}}. \quad (10.35)$$

We have established the following mixed fractional Landau inequality with respect to $||.||_{\infty}$ norm.

Theorem 10.9. *Let* $0 < \alpha \leq 1$, $f \in AC^2([a,b])$, $a < b$, $\forall [a,b] \subseteq \mathbb{R}$. *Assume* $||f||_{\infty,\mathbb{R}} < \infty$ *and* $\sup_{x_o \in \mathbb{R}} ||D_{*x_0}^{\alpha+1} f||_{\infty,[x_0,+\infty)} < \infty$, $\sup_{x_o \in \mathbb{R}} ||D_{x_0-}^{\alpha+1} f||_{\infty,(-\infty,x_0]} < \infty$. *Then*

$$||f'||_{\infty,\mathbb{R}} \leq (\alpha+1) \left(\frac{2}{\alpha}\right)^{\frac{\alpha}{\alpha+1}} \cdot \left(\Gamma(\alpha+2)\right)^{-\frac{1}{\alpha+1}} \cdot \left(||f||_{\infty,\mathbb{R}}\right)^{\frac{\alpha}{\alpha+1}}$$

$$\cdot \left(\max\left\{ \sup_{x_o \in \mathbb{R}} ||D_{x_0-}^{\alpha+1} f||_{\infty,(-\infty,x_0]}, \sup_{x_o \in \mathbb{R}} ||D_{x_0*}^{\alpha+1} f||_{\infty,[x_0,+\infty)} \right\} \right)^{\frac{1}{\alpha+1}}. \quad (10.36)$$

When $\alpha = 1$, we obtain

Corollary 10.10. *Let* $f \in AC^2([a,b])$, $\forall [a,b] \subseteq \mathbb{R}$. *Suppose* $||f||_{\infty,\mathbb{R}} < \infty$, $||f''||_{\infty,\mathbb{R}} < \infty$. *Then*

$$||f'||_{\infty,\mathbb{R}} \leq 2 \cdot \sqrt{||f||_{\infty,\mathbb{R}} \cdot ||f''||_{\infty,\mathbb{R}}}. \tag{10.37}$$

We continue with

Remark 10.11. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. If $f \in C^\infty(\mathbb{R})$ and of compact support $[A, B]$, then as we saw earlier, it only counts $x_0, x \in [A, B]$ for having

$$D^\alpha_{x_0 *} f(x) \neq 0 \quad \text{and} \quad \theta^* := ||D^\alpha_{* x_0} f(x)||_{\infty,[A,B]^2} < \infty.$$

Thus

$$||D^\alpha_{* x_0}||_{L_q(\mathbb{R})} = \left(\int_A^B \left| D^\alpha_{* x_0} f(x) \right|^q \mathrm{d}x \right)^{\frac{1}{q}} \leq (B - A)^{\frac{1}{q}} \cdot \theta^* < \infty, \ \forall x_0 \in [A, B]. \tag{10.38}$$

Hence

$$\sup_{x_o \in \mathbb{R}} ||D^\alpha_{* x_0} f||_{L_q([x_0,+\infty))} < \infty. \tag{10.39}$$

Similarly, we obtain

$$||D^\alpha_{x_0 -} f||_{L_q((-\infty,x_0])} < \infty, \tag{10.40}$$

for the above case; similarly for trigonometric polynomials, etc. So, we can assume f that

$$\sup_{x_o \in \mathbb{R}} ||D^\alpha_{* x_0} f||_{L_q([x_0,+\infty))} < \infty$$

and

$$\sup_{x_o \in \mathbb{R}} ||D^\alpha_{x_0 -} f||_{L_q((-\infty,x_0])} < \infty \tag{10.41}$$

is valid for large classes of functions f.

Remark 10.12. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $1 - \frac{1}{p} < \alpha \leq 1$, $f \in AC^2([a,b])$, $\forall [a,b] \subseteq \mathbb{R}$. Suppose $||f||_{\infty,\mathbb{R}} < \infty$, $x_0 \in [a,b]$ and

$$\sup_{x_o \in \mathbb{R}} ||D^{\alpha+1}_{x_0 -} f||_{L_q((-\infty,x_0])} < \infty;$$

$$\sup_{x_o \in \mathbb{R}} ||D^{\alpha+1}_{* x_0} f||_{L_q([x_0,+\infty))} < \infty.$$

Then by [4], Theorem 8 and by [2] Theorem 26.10, p. 620 we obtain that

$$|f'(x_0)| \leq \frac{2\|f\|_{\infty,\mathbb{R}}}{b-a} + \frac{1}{\Gamma(\alpha)\left[p(\alpha-1)+1\right]^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} \cdot$$

$$\max\left\{\sup_{x_o\in\mathbb{R}} \|D_{x_0-}^{\alpha+1} f\|_{L_q((-\infty,x_0])}, \sup_{x_o\in\mathbb{R}} \|D_{*x_0}^{\alpha+1} f\|_{L_q([x_0,+\infty))}\right\} \cdot (b-a)^{\alpha-\frac{1}{q}}. \quad (10.42)$$

Hence

$$\|f'\|_{\infty,\mathbb{R}} \leq \frac{2\|f\|_{\infty,\mathbb{R}}}{t} + \frac{\max\left\{\sup_{x_o\in\mathbb{R}} \|D_{x_0-}^{\alpha+1} f\|_{L_q((-\infty,x_0])}, \sup_{x_o\in\mathbb{R}} \|D_{*x_0}^{\alpha+1} f\|_{L_q([x_0,+\infty))}\right\}}{\Gamma(\alpha)\left[p(\alpha-1)+1\right]^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} \cdot t^{\alpha-\frac{1}{q}}, \quad \forall\, t \in (0,\infty). \quad (10.43)$$

Similarly as before we get

$$\|f'\|_{\infty,\mathbb{R}} \leq \left(\frac{2\left(\alpha+\frac{1}{p}\right)}{\alpha-\frac{1}{q}}\right)^{\frac{\alpha-\frac{1}{q}}{\alpha+\frac{1}{p}}} \cdot \frac{1}{(\Gamma(\alpha))^{\frac{1}{\alpha+\frac{1}{p}}}\left[p(\alpha-1)+1\right]^{\frac{1}{\alpha p+1}}} \cdot \left(\|f\|_{\infty,\mathbb{R}}\right)^{\frac{\alpha-\frac{1}{q}}{\alpha+\frac{1}{p}}} \cdot$$

$$\left(\max\left\{\sup_{x_o\in\mathbb{R}} \|D_{x_0-}^{\alpha+1} f\|_{L_q((-\infty,x_0])}, \sup_{x_o\in\mathbb{R}} \|D_{*x_0}^{\alpha+1} f\|_{L_q([x_0,+\infty))}\right\}\right)^{\frac{1}{\alpha+\frac{1}{p}}}. \quad (10.44)$$

We have established the mixed fractional L_p-Landau inequality.

Theorem 10.13. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{q} < \alpha \leq 1$, $f \in AC^2([a,b])$, $\forall\, [a,b] \subseteq \mathbb{R}$. *Assume* $\|f\|_{\infty,\mathbb{R}} < \infty$ *and*

$$\sup_{x_o\in\mathbb{R}} \|D_{x_0-}^{\alpha+1} f\|_{L_q((-\infty,x_0])} < \infty;$$

$$\sup_{x_o\in\mathbb{R}} \|D_{*x_0}^{\alpha+1} f\|_{L_q([x_0,+\infty))} < \infty.$$

Then

$$\|f'\|_{\infty,\mathbb{R}} \leq \left(\frac{2\left(\alpha+\frac{1}{p}\right)}{\alpha-\frac{1}{q}}\right)^{\frac{\alpha-\frac{1}{q}}{\alpha+\frac{1}{p}}} \cdot \frac{1}{(\Gamma(\alpha))^{\frac{1}{\alpha+\frac{1}{p}}}\left[p(\alpha-1)+1\right]^{\frac{1}{\alpha p+1}}} \cdot \left(\|f\|_{\infty,\mathbb{R}}\right)^{\frac{\alpha-\frac{1}{q}}{\alpha+\frac{1}{p}}} \cdot$$

$$\cdot \left(\max\left\{\sup_{x_o\in\mathbb{R}} \|D_{x_0-}^{\alpha+1} f\|_{L_q((-\infty,x_0])}, \sup_{x_o\in\mathbb{R}} \|D_{*x_0}^{\alpha+1} f\|_{L_q([x_0,+\infty))}\right\}\right)^{\frac{1}{\alpha+\frac{1}{p}}}. \quad (10.45)$$

Corollary 10.14. *Let* $\frac{1}{2} < \alpha \leq 1$, $f \in AC^2([a,b]), \forall [a,b] \subseteq \mathbb{R}$. *Assume* $\|f\|_{\infty,\mathbb{R}} < \infty$ *and*

$$\sup_{x_o \in \mathbb{R}} \|D_{x_0-}^{\alpha+1} f\|_{L_2((-\infty,x_0])} < \infty;$$

$$\sup_{x_o \in \mathbb{R}} \|D_{*x_0}^{\alpha+1} f\|_{L_2([x_0,+\infty))} < \infty.$$

Then

$$\|f'\|_{\infty,\mathbb{R}} \leq \left(\frac{2\alpha+1}{\alpha-\frac{1}{2}}\right)^{\frac{\alpha-\frac{1}{2}}{\alpha+\frac{1}{2}}} \cdot \frac{1}{(\Gamma(\alpha))^{\frac{1}{\alpha+\frac{1}{2}}}(2\alpha-1)^{\frac{1}{2\alpha+1}}} \cdot \left(\|f\|_{\infty,\mathbb{R}}\right)^{\frac{\alpha-\frac{1}{2}}{\alpha+\frac{1}{2}}} \cdot$$
$$\times \left(\max\left\{\sup_{x_o \in \mathbb{R}} \|D_{x_0-}^{\alpha+1} f\|_{L_2((-\infty,x_0])}, \sup_{x_o \in \mathbb{R}} \|D_{*x_0}^{\alpha+1} f\|_{L_2([x_0,+\infty))}\right\}\right)^{\frac{1}{\alpha+\frac{1}{2}}}. \tag{10.46}$$

Corollary 10.15. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $f \in AC^2([a,b]), \forall [a,b] \subseteq \mathbb{R}$. *Suppose that* $\|f\|_{\infty,\mathbb{R}} < \infty$, $\|f''\|_{L_q(\mathbb{R})} < \infty$. *Then*

$$\|f'\|_{\infty,\mathbb{R}} \leq \left(2(p+1)\right)^{\frac{1}{p+1}} \cdot \left(\|f\|_{\infty,\mathbb{R}}\right)^{\frac{1}{p+1}} \cdot \left(\|f''\|_{L_q(\mathbb{R})}\right)^{\frac{p}{p+1}}. \tag{10.47}$$

We finish chapter with

Corollary 10.16. *Let* $f \in AC^2([a,b]), \forall [a,b] \subseteq \mathbb{R}$. *Assume* $\|f\|_{\infty,\mathbb{R}} < \infty$, $\|f''\|_{L_2(\mathbb{R})} < \infty$. *Then*

$$\|f'\|_{\infty,\mathbb{R}} \leq \sqrt[3]{6} \cdot \left(\|f\|_{\infty,\mathbb{R}}\right)^{\frac{1}{3}} \cdot \left(\|f''\|_{L_2(\mathbb{R})}\right)^{\frac{2}{3}}. \tag{10.48}$$

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Chapter 11
Multivariate Caputo Fractional Landau Inequalities

Here we give multivariate left Caputo fractional L_p-Landau-type inequalities, $p \in (1, \infty]$ with applications on $\mathbb{R}^N$, $N \geq 1$. This chapter is based on [5].

11.1 Introduction

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

$$||f'||_{p,I} \leq C_p(I) ||f||_{p,I}^{\frac{1}{2}} ||f''||_{p,I}^{\frac{1}{2}} \tag{11.1}$$

where $||.||_{p,I}$ is the p-norm on the interval I, see [1, 6].

The research on these inequalities started by E. Landau [12] in 1914. For the case of $p = \infty$ he proved that

$$C_\infty(\mathbb{R}_+) = 2 \quad \text{and} \quad C_\infty(\mathbb{R}) = \sqrt{2} \tag{11.2}$$

are the best constants in (11.1).

In 1932, G. H. Hardy and J. E. Littlewood [9] proved (11.1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \quad \text{and} \quad C_2(\mathbb{R}) = 1. \tag{11.3}$$

In 1935, G. H. Hardy, E. Landau, and J. E. Littlewood [10] showed that the best constant $C_p(\mathbb{R}_+)$ in (11.1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2, \quad \text{for} \quad p \in [1, \infty), \tag{11.4}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$.

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In fact, in [8] and [11], it was shown that $C_p(\mathbb{R}) \leq \sqrt{2}$.

In this chapter we prove multivariate fractional Landau inequalities with respect to $||.||_p$, $p \in (1, \infty]$, involving the left Caputo fractional radial derivative.

11.2 Main Results

We need

Definition 11.1. ([7], p. 38) Let $f \in AC^m([a,b])$, $m \in \mathbb{N}$ (i.e., $f^{(m-1)} \in AC([a,b])$),where $m = \lceil \alpha \rceil$, $\alpha > 0$, ($\lceil . \rceil$ the ceiling of the number). We define the left Caputo fractional derivative of order $\alpha > 0$ by

$$D_{*a}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\mathrm{d}t, \tag{11.5}$$

$\forall\, x \in [a,b]$. *We set* $D_{*a}^0 f(x) = f(x)$, $\forall x \in [a,b]$.

Here $D_{*a}^{\alpha} f$ exists a.e. on $[a,b]$ and $D_{*a}^{\alpha} f \in L_1([a,b])$, see [7], pp. 13 and 37–38. When $\alpha = m \in \mathbb{N}$, then

$$D_{*a}^{m} f(x) = f^{(m)}(x), \ \forall x \in [a,b].$$

If $x < a$ we define $D_{*a}^{\alpha} f(x) = 0$.

We make

Remark 11.2. Here we follow [13], pp. 149–150.

For $x \in \mathbb{R}^N - \{0\}$, $N > 1$, we can write uniquely $x = rw$, $r = |x| > 0$, $w = x/r \in S^{N-1}$, $|w| = 1$. Clearly

$$\mathbb{R}^N - \{0\} = (0,\infty) \times S^{N-1}, \quad \text{where} \quad S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}.$$

Let $A > 0$, we consider

$$B(0,A) := \{x \in \mathbb{R}^N : |x| < A\} \subseteq \mathbb{R}^N, \quad |x| - \text{ Euclidean norm of } x \in \mathbb{R}^N.$$

We consider

$$\mathbb{R}^N - B(0,A) = [A, +\infty) \times S^{N-1} \tag{11.6}$$

on which we establish Landau fractional inequalities. We need to define the left Caputo radial fractional derivative for our case, see also [2], p. 421.

Definition 11.3. Let $f : \mathbb{R}^N - B(0,A) \to \mathbb{R}$, $\nu \geq 0$, $m := \lceil \nu \rceil$, such that $f(\cdot w) \in AC^m([A,b])$, $\forall b > A$, for all $w \in S^{N-1}$. We call the left Caputo radial fractional derivative the following function

$$\frac{\partial_{*A}^{\nu} f(x)}{\partial r^{\nu}} := \frac{1}{\Gamma(m-\nu)} \int_A^r (r-t)^{m-\nu-1} \frac{\partial^m f(tw)}{\partial r^m} \mathrm{d}t, \tag{11.7}$$

where $x \in \mathbb{R}^N - B(0,A)$, that is, $x = rw, r \in [A,\infty), w \in S^{N-1}$.

Clearly

$$\frac{\partial_{*A}^{0} f(x)}{\partial r^{0}} = f(x), \tag{11.8}$$

$$\frac{\partial_{*A}^{\nu} f(x)}{\partial r^{\nu}} = \frac{\partial^{\nu} f(x)}{\partial r^{\nu}}, \quad \text{if} \quad \nu \in \mathbb{N}. \tag{11.9}$$

The above function (11.7) exists almost everywhere for $x \in \mathbb{R}^N - B(0, A)$, see [2], p. 422.

We make

Remark 11.4. Let $0 < \nu \leq 1$; $A > 0$ be fixed, with $f(\cdot w) \in AC^2([A, b])$, $\forall b > A$, $\forall\, w \in S^{N-1}$. Suppose $||f||_{\infty,\mathbb{R}^N - B(0,A)} < \infty$,

$$\left\| \frac{\partial_{*A}^{\nu+1} f}{\partial r^{\nu+1}} \right\|_{\infty,\mathbb{R}^N - B(0,A)} < \infty$$

and

$$||D_{*a}^{\nu+1} f(\cdot w)||_{\infty,[a,+\infty)} \leq ||D_{*A}^{\nu+1} f(\cdot w)||_{\infty,[A,+\infty)}, \ \forall\, a \geq A, \forall\, w \in S^{N-1}. \tag{11.10}$$

Then, by Theorem 8 of [3], we obtain that

$$\begin{aligned} ||f'(\cdot w)||_{\infty,[A,+\infty)} &\leq (\nu+1) \left(\frac{2}{\nu}\right)^{\frac{\nu}{\nu+1}} \cdot \Big(\Gamma(\nu+2)\Big)^{-\frac{1}{\nu+1}} \\ &\times \Big(||f(\cdot w)||_{\infty,[A,+\infty)}\Big)^{\frac{\nu}{\nu+1}} \cdot \Big(||D_{*A}^{\nu+1} f(\cdot w)||_{\infty,[A,+\infty)}\Big)^{\frac{1}{\nu+1}} \\ &\leq (\nu+1) \left(\frac{2}{\nu}\right)^{\frac{\nu}{\nu+1}} \cdot \Big(\Gamma(\nu+2)\Big)^{-\frac{1}{\nu+1}} \cdot \Big(||f||_{\infty,\mathbb{R}^N - B(0,A)}\Big)^{\frac{\nu}{\nu+1}} \\ &\cdot \left(\left\| \frac{\partial_{*A}^{\nu+1} f}{\partial r^{\nu+1}} \right\|_{\infty,\mathbb{R}^N - B(0,A)} \right)^{\frac{1}{\nu+1}} =: \theta. \end{aligned} \tag{11.11}$$

Hence

$$\left\| \frac{\partial f}{\partial r} \right\|_{\infty,\mathbb{R}^N - B(0,A)} \leq \theta. \tag{11.12}$$

We have proved the multivariate fractional Landau inequality.

Theorem 11.5. *Let* $0 < \nu \leq 1$*;* $A > 0$ *be fixed, with* $f(\cdot w) \in AC^2([A, b])$, $\forall b > A$*,* $\forall\, w \in S^{N-1}$*. Suppose* $||f||_{\infty,\mathbb{R}^N - B(0,A)} < \infty$*,*

$$\left\| \frac{\partial_{*A}^{\nu+1} f}{\partial r^{\nu+1}} \right\|_{\infty,\mathbb{R}^N - B(0,A)} < \infty$$

and

$$||D_{*a}^{\nu+1} f(\cdot w)||_{\infty,[a,+\infty)} \leq ||D_{*A}^{\nu+1} f(\cdot w)||_{\infty,[A,+\infty)}, \;\; \forall\, a \geq A, \forall\, w \in S^{N-1}. \tag{11.13}$$

Then

$$\left\| \frac{\partial f}{\partial r} \right\|_{\infty,\mathbb{R}^N - B(0,A)} \leq (\nu+1) \left(\frac{2}{\nu}\right)^{\frac{\nu}{\nu+1}} \cdot \left(\Gamma(\nu+2)\right)^{-\frac{1}{\nu+1}} \times \left(||f||_{\infty,\mathbb{R}^N - B(0,A)}\right)^{\frac{\nu}{\nu+1}} \cdot \left(\left\| \frac{\partial_{*A}^{\nu+1} f}{\partial r^{\nu+1}} \right\|_{\infty,\mathbb{R}^N - B(0,A)} \right)^{\frac{1}{\nu+1}}. \tag{11.14}$$

We give when $\nu = 1$,

Corollary 11.6. *Let $A > 0$ fixed, with $f(\cdot w) \in AC^2([A,b])$, $\forall b > A$, $\forall\, w \in S^{N-1}$. Suppose $||f||_{\infty,\mathbb{R}^N - B(0,A)} < \infty$,*

$$\left\| \frac{\partial^2 f}{\partial r^2} \right\|_{\infty,\mathbb{R}^N - B(0,A)} < \infty.$$

Then

$$\left\| \frac{\partial f}{\partial r} \right\|_{\infty,\mathbb{R}^N - B(0,A)} \leq 2 \cdot \sqrt{||f||_{\infty,\mathbb{R}^N - B(0,A)} \cdot \left\| \frac{\partial^2 f}{\partial r^2} \right\|_{\infty,\mathbb{R}^N - B(0,A)}} \;\; . \tag{11.15}$$

We make

Remark 11.7. All entities here are assumed to make sense and to be well-defined. We see that ($q > 1$):

$$\begin{aligned} \left\| D_{*A}^{\nu+1} f(\cdot w) \right\|_{q,[A,+\infty)}^q &= \int_A^{+\infty} \left| D_{*A}^{\nu+1} f(rw) \right|^q \mathrm{d}r \\ &= \int_A^{+\infty} \left| D_{*A}^{\nu+1} f(rw) \right|^q \cdot r^{N-1} \cdot r^{1-N} \mathrm{d}r \end{aligned} \tag{11.16}$$

(see $r \geq A$ implies $r^{1-A} \leq A^{1-N}, \;\; N > 1$).

Hence it holds

$$\left\| D_{*A}^{\nu+1} f(\cdot w) \right\|_{q,[A,+\infty)}^q \leq A^{1-N} \cdot \int_A^{+\infty} \left| D_{*A}^{\nu+1} f(rw) \right|^q \cdot r^{N-1} \mathrm{d}r. \tag{11.17}$$

Observe that

$$\frac{\int_{S^{N-1}} dw}{w_N} = 1, \quad \text{where} \quad w_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \tag{11.18}$$

Therefore we find

$$\left\| D_{*A}^{\nu+1} f(\cdot w) \right\|_{q,[A,+\infty)}^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}$$

$$\leq A^{\left(\frac{1-N}{q}\right)\cdot\frac{1}{\left(\nu+\frac{1}{p}\right)}} \cdot \left(\int_A^{+\infty} \left| D_{*A}^{\nu+1} f(rw) \right|^q \cdot r^{N-1} dr \right)^{\frac{1}{q}\cdot\left(\frac{1}{\nu+\frac{1}{p}}\right)}. \tag{11.19}$$

Set

$$c_\nu := \left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-1+\frac{1}{p}} \right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \cdot \frac{1}{\left(\Gamma(\nu)\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}} \cdot (p(\nu-1)+1)^{\frac{1}{p\nu+1}}}. \tag{11.20}$$

So here we have $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $1 - \frac{1}{p} < \nu \leq 1$, $f(\cdot w) \in AC^2([A, b])$, $\forall b > A$, $\forall\, w \in S^{N-1}$, $A > 0$ is fixed.

Assume $D_{*A}^{\nu+1} f(\cdot w) \in L_q([A, +\infty))$, $\forall\, w \in S^{N-1}$, $||f||_{\infty,\mathbb{R}^N - B(0,A)} < \infty$ and

$$\left\| D_{*a}^{\nu+1} f(\cdot w) \right\|_{q,[a,+\infty)} \leq \left\| D_{*A}^{\nu+1} f(\cdot w) \right\|_{q,[A,+\infty)}, \forall\ a \geq A,\ \ \forall\, w \in S^{N-1}. \tag{11.21}$$

Hence by Theorem 8 of [4] and (11.19), we derive

$$||f'(\cdot w)||_{\infty,[A,+\infty)} \leq c_\nu \cdot A^{\left(\frac{1-N}{q}\right)\cdot\frac{1}{\left(\nu+\frac{1}{p}\right)}}.$$

$$\cdot \left(||f(\cdot w)||_{\infty,[A,+\infty)}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \cdot \left(\int_A^{+\infty} \left| D_{*A}^{\nu+1} f(rw) \right|^q \cdot r^{N-1} dr \right)^{\frac{1}{q\left(\nu+\frac{1}{p}\right)}}. \tag{11.22}$$

We call

$$\beta_\nu := c_\nu \cdot A^{\left(\frac{1-N}{q}\right)\cdot\frac{1}{\left(\nu+\frac{1}{p}\right)}}, \tag{11.23}$$

and

$$\gamma_\nu := \beta_\nu \cdot \left(||f||_{\infty,\mathbb{R}^N - B(0,A)}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)}. \tag{11.24}$$

Therefore

$$||f'(\cdot w)||_{\infty,[A,+\infty)} \leq \tag{11.25}$$

$$\beta_\nu \cdot \left(||f(\cdot w)||_{\infty,[A,+\infty)}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \cdot \left(\int_A^{+\infty} \left|D_{*A}^{\nu+1} f(rw)\right|^q \cdot r^{N-1} \mathrm{d}r\right)^{\frac{1}{q\left(\nu+\frac{1}{p}\right)}}$$

$$\leq \gamma_\nu \cdot \left(\int_A^{+\infty} \left|D_{*A}^{\nu+1} f(rw)\right|^q \cdot r^{N-1} \mathrm{d}r\right)^{\frac{1}{q\left(\nu+\frac{1}{p}\right)}}. \tag{11.26}$$

That is, we proved

$$|f'(rw)| \leq \gamma_\nu \cdot \left(\int_A^{+\infty} \left|D_{*A}^{\nu+1} f(rw)\right|^q \cdot r^{N-1} \mathrm{d}r\right)^{\frac{1}{q\left(\nu+\frac{1}{p}\right)}}, \tag{11.27}$$

$\forall r \in [A, +\infty), \quad \forall w \in S^{N-1}$.

Call

$$\delta_\nu := \gamma_\nu^{q\left(\nu+\frac{1}{p}\right)}. \tag{11.28}$$

Then by (11.27) we obtain

$$|f'(rw)|^{q\left(\nu+\frac{1}{p}\right)} \leq \delta_\nu \cdot \left(\int_A^{+\infty} \left|D_{*A}^{\nu+1} f(rw)\right|^q \cdot r^{N-1} \mathrm{d}r\right), \tag{11.29}$$

$\forall r \in [A, +\infty), \quad \forall w \in S^{N-1}$.

Therefore

$$\int_{S^{N-1}} |f'(rw)|^{q\left(\nu+\frac{1}{p}\right)} \mathrm{d}w \leq \delta_\nu \cdot \left(\int_{S^{N-1}} \left(\int_A^{+\infty} \left|D_{*A}^{\nu+1} f(rw)\right|^q \cdot r^{N-1} \mathrm{d}r\right) \mathrm{d}w\right) \tag{11.30}$$

$$= \delta_\nu \cdot \int_{\mathbb{R}^N - B(0,A)} \left|\frac{\partial_{*A}^{\nu+1} f(x)}{\partial r^{\nu+1}}\right|^q \mathrm{d}x. \tag{11.31}$$

Consequently we derive

$$\left(\int_{S^{N-1}} \left(|f'(rw)|^{\left(\nu+\frac{1}{p}\right)}\right)^q \mathrm{d}w\right)^{\frac{1}{q}} \leq (\delta_\nu)^{\frac{1}{q}} \cdot \left(\int_{\mathbb{R}^N - B(0,A)} \left|\frac{\partial_{*A}^{\nu+1} f(x)}{\partial r^{\nu+1}}\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \tag{11.32}$$

$$= (\delta_\nu)^{\frac{1}{q}} \cdot \left\|\frac{\partial_{*A}^{\nu+1} f}{\partial r^{\nu+1}}\right\|_{q,\mathbb{R}^N - B(0,A)}. \tag{11.33}$$

Therefore it holds

$$\left\| \left(f'(r\cdot)\right)^{\left(\nu+\frac{1}{p}\right)} \right\|_{q,S^{N-1}} \leq (\delta_\nu)^{\frac{1}{q}} \cdot \left\| \frac{\partial_{*A}^{\nu+1} f}{\partial r^{\nu+1}} \right\|_{q,\mathbb{R}^N - B(0,A)}, \tag{11.34}$$

$\forall r \in [A, +\infty)$.

Hence we get

$$\left\| \left\| \left(\frac{\partial f}{\partial r} (rw) \right)^{\left(\nu+\frac{1}{p}\right)} \right\|_{(q,S^{N-1},w)} \right\|_{(\infty,[A,+\infty),r)} \leq (\delta_\nu)^{\frac{1}{q}} \cdot \left\| \frac{\partial_{*A}^{\nu+1} f}{\partial r^{\nu+1}} \right\|_{q,\mathbb{R}^N - B(0,A)}. \tag{11.35}$$

Notice that

$$\begin{aligned}
(\delta_\nu)^{\frac{1}{q}} &= (\gamma_\nu)^{\left(\nu+\frac{1}{p}\right)} \\
&= (\beta_\nu)^{\left(\nu+\frac{1}{p}\right)} \cdot \left(||f||_{\infty,\mathbb{R}^N - B(0,A)}\right)^{\left(\nu-1+\frac{1}{p}\right)} \\
&= A^{\frac{1-N}{q}} \cdot (c_\nu)^{\left(\nu+\frac{1}{p}\right)} \cdot \left(||f||_{\infty,\mathbb{R}^N - B(0,A)}\right)^{\left(\nu-1+\frac{1}{p}\right)} \\
&= A^{\frac{1-N}{q}} \cdot \left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-1+\frac{1}{p}} \right)^{\left(\nu-1+\frac{1}{p}\right)} \\
&\quad \times \frac{1}{\Gamma(\nu) \cdot (p(\nu-1)+1)^{\left(\nu+\frac{1}{p}\right)/(p\nu+1)}} \cdot \left(||f||_{\infty,\mathbb{R}^N - B(0,A)}\right)^{\left(\nu-1+\frac{1}{p}\right)}.
\end{aligned} \tag{11.36}$$

We have proved the following multivariate L_p fractional Landau inequality.

Theorem 11.8. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *with* $\frac{1}{q} < \nu \leq 1$, $f(\cdot w) \in AC^2([A, b])$, $\forall b > A$, $A > 0$ *fixed,* $\forall\, w \in S^{N-1}$.

Assume that $D_{*A}^{\nu+1} f(\cdot w) \in L_q([A, +\infty))$, $\forall\, w \in S^{N-1}$, $||f||_{\infty,\mathbb{R}^N - B(0,A)} < \infty$ *and*

$$\left\| D_{*a}^{\nu+1} f(\cdot w) \right\|_{q,[a,+\infty)} \leq \left\| D_{*A}^{\nu+1} f(\cdot w) \right\|_{q,[A,+\infty)}, \ \forall\ a \geq A, \ \forall\, w \in S^{N-1}. \tag{11.37}$$

Then

$$\left\| \left\| \left(\frac{\partial f}{\partial r}(rw) \right)^{\left(\nu+\frac{1}{p}\right)} \right\|_{(q,S^{N-1},w)} \right\|_{(\infty,[A,+\infty),r)}$$

$$\leq \frac{A^{\frac{1-N}{q}}}{\Gamma(\nu)} \cdot \left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-1+\frac{1}{p}} \right)^{\left(\nu-1+\frac{1}{p}\right)} \cdot \frac{1}{(p(\nu-1)+1)^{\frac{\left(\nu+\frac{1}{p}\right)}{(p\nu+1)}}} \cdot$$

$$\times \left(\|f\|_{\infty,\mathbb{R}^N-B(0,A)} \right)^{\left(\nu-1+\frac{1}{p}\right)} \cdot \left\| \frac{\partial_{*A}^{\nu+1} f}{\partial r^{\nu+1}} \right\|_{q,\mathbb{R}^N-B(0,A)}. \quad (11.38)$$

It follows the $p = q = 2$ case.

Corollary 11.9. *Let* $\frac{1}{2} < \nu \leq 1$, $f(\cdot w) \in AC^2([A,b])$, $\forall b > A$, $A > 0$ *fixed,* $\forall\, w \in S^{N-1}$.

Suppose that $D_{*A}^{\nu+1} f(\cdot w) \in L_2([A,+\infty))$, $\forall\, w \in S^{N-1}$, $\|f\|_{\infty,\mathbb{R}^N-B(0,A)} < \infty$ *and*

$$\left\| D_{*a}^{\nu+1} f(\cdot w) \right\|_{2,[a,+\infty)} \leq \left\| D_{*A}^{\nu+1} f(\cdot w) \right\|_{2,[A,+\infty)}, \forall\; a \geq A, \;\; \forall\, w \in S^{N-1}. \quad (11.39)$$

Then

$$\left\| \left\| \left(\frac{\partial f}{\partial r}(rw) \right)^{\left(\nu+\frac{1}{2}\right)} \right\|_{(2,S^{N-1},w)} \right\|_{(\infty,[A,+\infty),r)}$$

$$\leq \frac{A^{\frac{1-N}{2}}}{\Gamma(\nu)} \cdot \left(\frac{2\left(\nu+\frac{1}{2}\right)}{\nu-\frac{1}{2}} \right)^{\left(\nu-\frac{1}{2}\right)} \cdot \frac{1}{\left(2\nu-1\right)^{\frac{\left(\nu+\frac{1}{2}\right)}{(2\nu+1)}}}$$

$$\times \left(\|f\|_{\infty,\mathbb{R}^N-B(0,A)} \right)^{\left(\nu-\frac{1}{2}\right)} \cdot \left\| \frac{\partial_{*A}^{\nu+1} f}{\partial r^{\nu+1}} \right\|_{2,\mathbb{R}^N-B(0,A)}. \quad (11.40)$$

We finish chapter with the $p = q = 2, \;\nu = 1$ case.

Corollary 11.10. *Let* $f(\cdot w) \in AC^2([A,b])$, $\forall b > A$, $A > 0$ *fixed* $,\forall\, w \in S^{N-1}$. *Assume that* $f''(\cdot w) \in L_2([A,+\infty))$, $\forall\, w \in S^{N-1}$ *and* $\|f\|_{\infty,\mathbb{R}^N-B(0,A)} < \infty$.

Then

$$\left\| \left\| \left(\frac{\partial f}{\partial r}(rw) \right)^{1.5} \right\|_{(2,S^{N-1},w)} \right\|_{(\infty,[A,+\infty),r)}$$

$$\leq \left(A^{\frac{1-N}{2}} \right) \cdot \left(\sqrt{6} \right) \cdot \sqrt{\|f\|_{\infty,\mathbb{R}^N-B(0,A)}} \cdot \left\| \frac{\partial^2 f}{\partial r^2} \right\|_{2,\mathbb{R}^N-B(0,A)}. \quad (11.41)$$

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