

Zdravko Cvetkovski

# Inequalities

Theorems, Techniques  
and Selected Problems

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and Selected Problems

Dipl. Math. Zdravko Cvetkovski  
Informatics Department  
European University-Republic of Macedonia  
Skopje, Macedonia  
[zdrcvet@gmail.com](mailto:zdrcvet@gmail.com)

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*Dedicated with great respect  
to the memory of Prof. Ilija Janev*



# Preface

This book has resulted from my extensive work with talented students in Macedonia, as well as my engagement in the preparation of Macedonian national teams for international competitions. The book is designed and intended for all students who wish to expand their knowledge related to the theory of inequalities and those fascinated by this field. The book could be of great benefit to all regular high school teachers and trainers involved in preparing students for national and international mathematical competitions as well. But first and foremost it is written for students—participants of all kinds of mathematical contests.

The material is written in such a way that it starts from elementary and basic inequalities through their application, up to mathematical inequalities requiring much more sophisticated knowledge. The book deals with almost all the important inequalities used as apparatus for proving more complicated inequalities, as well as several methods and techniques that are part of the apparatus for proving inequalities most commonly encountered in international mathematics competitions of higher rank. Most of the theorems and corollaries are proved, but some of them are not proved since they are easy and they are left to the reader, or they are too complicated for high school students.

As an integral part of the book, following the development of the theory in each section, solved examples have been included—a total of 175 in number—all intended for the student to acquire skills for practical application of previously adopted theory. Also should emphasize that as a final part of the book an extensive collection of 310 “high quality” solved problems has been included, in which various types of inequalities are developed. Some of them are mine, while the others represent inequalities assigned as tasks in national competitions and national olympiads as well as problems given in team selection tests for international competitions from different countries.

I have made every effort to acknowledge the authors of certain problems; therefore at the end of the book an index of the authors of some problems has been included, and I sincerely apologize to anyone who is missing from the list, since any omission is unintentional.

My great honour and duty is to express my deep gratitude to my colleagues Mirko Petrushevski and Đorđe Baralić for proofreading and checking the manuscript, so



that with their remarks and suggestions, the book is in its present form. Also I want to thank my wife Maja and my lovely son Gjorgji for all their love, encouragement and support during the writing of this book.

There are many great books about inequalities. But I truly hope and believe that this book will contribute to the development of our talented students—future national team members of our countries at international competitions in mathematics, as well as to upgrade their knowledge.

Despite my efforts there may remain some errors and mistakes for which I take full responsibility. There is always the possibility for improvement in the presentation of the material and removing flaws that surely exist. Therefore I should be grateful for any well-intentioned remarks and criticisms in order to improve this book.

Skopje

Zdravko Cvetkovski

# Contents

<b>1</b>	<b>Basic (Elementary) Inequalities and Their Application . . . . .</b>	<b>1</b>
<b>2</b>	<b>Inequalities Between Means (with Two and Three Variables) . . . . .</b>	<b>9</b>
<b>3</b>	<b>Geometric (Triangle) Inequalities . . . . .</b>	<b>19</b>
<b>4</b>	<b>Bernoulli's Inequality, the Cauchy–Schwarz Inequality, Chebishev's Inequality, Surányi's Inequality . . . . .</b>	<b>27</b>
<b>5</b>	<b>Inequalities Between Means (General Case) . . . . .</b>	<b>49</b>
	5.1 Points of Incidence in Applications of the <i>AM–GM</i> Inequality . . .	53
<b>6</b>	<b>The Rearrangement Inequality . . . . .</b>	<b>61</b>
<b>7</b>	<b>Convexity, Jensen's Inequality . . . . .</b>	<b>69</b>
<b>8</b>	<b>Trigonometric Substitutions and Their Application for Proving Algebraic Inequalities . . . . .</b>	<b>79</b>
	8.1 The Most Usual Forms of Trigonometric Substitutions . . . . .	86
	8.2 Characteristic Examples Using Trigonometric Substitutions . . . .	89
<b>9</b>	<b>Hölder's Inequality, Minkowski's Inequality and Their Variants . . .</b>	<b>95</b>
<b>10</b>	<b>Generalizations of the Cauchy–Schwarz Inequality, Chebishev's Inequality and the Mean Inequalities . . . . .</b>	<b>107</b>
<b>11</b>	<b>Newton's Inequality, Maclaurin's Inequality . . . . .</b>	<b>117</b>
<b>12</b>	<b>Schur's Inequality, Muirhead's Inequality and Karamata's Inequality . . . . .</b>	<b>121</b>
<b>13</b>	<b>Two Theorems from Differential Calculus, and Their Applications for Proving Inequalities . . . . .</b>	<b>133</b>
<b>14</b>	<b>One Method of Proving Symmetric Inequalities with Three Variables . . . . .</b>	<b>137</b>

<b>15</b>	<b>Method for Proving Symmetric Inequalities with Three Variables Defined on the Set of Real Numbers . . . . .</b>	<b>147</b>
<b>16</b>	<b>Abstract Concreteness Method (ABC Method) . . . . .</b>	<b>155</b>
	16.1 ABC Theorem . . . . .	155
<b>17</b>	<b>Sum of Squares (SOS Method) . . . . .</b>	<b>161</b>
<b>18</b>	<b>Strong Mixing Variables Method (SMV Theorem) . . . . .</b>	<b>169</b>
<b>19</b>	<b>Method of Lagrange Multipliers . . . . .</b>	<b>177</b>
<b>20</b>	<b>Problems . . . . .</b>	<b>183</b>
<b>21</b>	<b>Solutions . . . . .</b>	<b>217</b>
	<b>Index of Problems . . . . .</b>	<b>435</b>
	<b>Abbreviations . . . . .</b>	<b>441</b>
	<b>References . . . . .</b>	<b>443</b>

# Chapter 1

## Basic (Elementary) Inequalities and Their Application

There are many trivial facts which are the basis for proving inequalities. Some of them are as follows:

1. If  $x \geq y$  and  $y \geq z$  then  $x \geq z$ , for any  $x, y, z \in \mathbb{R}$ .
2. If  $x \geq y$  and  $a \geq b$  then  $x + a \geq y + b$ , for any  $x, y, a, b \in \mathbb{R}$ .
3. If  $x \geq y$  then  $x + z \geq y + z$ , for any  $x, y, z \in \mathbb{R}$ .
4. If  $x \geq y$  and  $a \geq b$  then  $xa \geq yb$ , for any  $x, y \in \mathbb{R}^+$  or  $a, b \in \mathbb{R}^+$ .
5. If  $x \in \mathbb{R}$  then  $x^2 \geq 0$ , with equality if and only if  $x = 0$ . More generally, for  $A_i \in \mathbb{R}^+$  and  $x_i \in \mathbb{R}, i = 1, 2, \dots, n$  holds  $A_1x_1^2 + A_2x_2^2 + \dots + A_nx_n^2 \geq 0$ , with equality if and only if  $x_1 = x_2 = \dots = x_n = 0$ .

These properties are obvious and simple, but are a powerful tool in proving inequalities, particularly *Property 5*, which can be used in many cases.

We'll give a few examples that will illustrate the strength of *Property 5*.

Firstly we'll prove few "elementary" inequalities that are necessary for a complete and thorough upgrade of each student who is interested in this area.

To prove these inequalities it is sufficient to know elementary inequalities that can be used in a certain part of the proof of a given inequality, but in the early stages, just basic operations are used.

The following examples, although very simple, are the basis for what follows later. Therefore I recommend the reader pay particular attention to these examples, which are necessary for further upgrading.

**Exercise 1.1** Prove that for any real number  $x > 0$ , the following inequality holds

$$x + \frac{1}{x} \geq 2.$$

*Solution* From the obvious inequality  $(x - 1)^2 \geq 0$  we have

$$x^2 - 2x + 1 \geq 0 \quad \Leftrightarrow \quad x^2 + 1 \geq 2x,$$

and since  $x > 0$  if we divide by  $x$  we get the desired inequality. Equality occurs if and only if  $x - 1 = 0$ , i.e.  $x = 1$ .

**Exercise 1.2** Let  $a, b \in \mathbb{R}^+$ . Prove the inequality

$$\frac{a}{b} + \frac{b}{a} \geq 2.$$

*Solution* From the obvious inequality  $(a - b)^2 \geq 0$  we have

$$a^2 - 2ab + b^2 \geq 0 \Leftrightarrow a^2 + b^2 \geq 2ab \Leftrightarrow \frac{a^2 + b^2}{ab} \geq 2 \Leftrightarrow \frac{a}{b} + \frac{b}{a} \geq 2.$$

Equality occurs if and only if  $a - b = 0$ , i.e.  $a = b$ .

**Exercise 1.3** (Nesbitt's inequality) Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

*Solution* According to Exercise 1.2 it is clear that

$$\frac{a+b}{b+c} + \frac{b+c}{a+b} + \frac{a+c}{c+b} + \frac{c+b}{a+c} + \frac{b+a}{a+c} + \frac{a+c}{b+a} \geq 2 + 2 + 2 = 6. \quad (1.1)$$

Let us rewrite inequality (1.1) as follows

$$\left( \frac{a+b}{b+c} + \frac{a+c}{c+b} \right) + \left( \frac{c+b}{a+c} + \frac{b+a}{a+c} \right) + \left( \frac{b+c}{a+b} + \frac{a+c}{b+a} \right) \geq 6,$$

i.e.

$$\frac{2a}{b+c} + 1 + \frac{2b}{c+a} + 1 + \frac{2c}{a+b} + 1 \geq 6$$

or

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2},$$

as required.

Equality occurs if and only if  $\frac{a+b}{b+c} = \frac{b+c}{a+b}$ ,  $\frac{a+c}{c+b} = \frac{c+b}{a+c}$ ,  $\frac{b+a}{a+c} = \frac{a+c}{b+a}$ , from where easily we deduce  $a = b = c$ .

The following inequality is very simple but it has a very important role, as we will see later.

**Exercise 1.4** Let  $a, b, c \in \mathbb{R}$ . Prove the inequality

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

*Solution* Since  $(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$  we deduce

$$2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca) \Leftrightarrow a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Equality occurs if and only if  $a = b = c$ .

As a consequence of the previous inequality we get following problem.

**Exercise 1.5** Let  $a, b, c \in \mathbb{R}$ . Prove the inequalities

$$3(ab + bc + ca) \leq (a + b + c)^2 \leq 3(a^2 + b^2 + c^2).$$

*Solution* We have

$$\begin{aligned} 3(ab + bc + ca) &= ab + bc + ca + 2(ab + bc + ca) \\ &\leq a^2 + b^2 + c^2 + 2(ab + bc + ca) = (a + b + c)^2 \\ &= a^2 + b^2 + c^2 + 2(ab + bc + ca) \\ &\leq a^2 + b^2 + c^2 + 2(a^2 + b^2 + c^2) = 3(a^2 + b^2 + c^2). \end{aligned}$$

Equality occurs if and only if  $a = b = c$ .

**Exercise 1.6** Let  $x, y, z > 0$  be real numbers such that  $x + y + z = 1$ . Prove that

$$\sqrt{6x+1} + \sqrt{6y+1} + \sqrt{6z+1} \leq 3\sqrt{3}.$$

*Solution* Let  $\sqrt{6x+1} = a, \sqrt{6y+1} = b, \sqrt{6z+1} = c$ .

Then

$$a^2 + b^2 + c^2 = 6(x + y + z) + 3 = 9.$$

Therefore

$$(a + b + c)^2 \leq 3(a^2 + b^2 + c^2) = 27, \quad \text{i.e.} \quad a + b + c \leq 3\sqrt{3}.$$

**Exercise 1.7** Let  $a, b, c \in \mathbb{R}$ . Prove the inequality

$$a^4 + b^4 + c^4 \geq abc(a + b + c).$$

*Solution* By Exercise 1.4 we have that: If  $x, y, z \in \mathbb{R}$  then

$$x^2 + y^2 + z^2 \geq xy + yz + zx.$$

Therefore

$$\begin{aligned} a^4 + b^4 + c^4 &\geq a^2b^2 + b^2c^2 + c^2a^2 = (ab)^2 + (bc)^2 + (ca)^2 \\ &\geq (ab)(bc) + (bc)(ca) + (ca)(ab) = abc(a + b + c). \end{aligned}$$

**Exercise 1.8** Let  $a, b, c \in \mathbb{R}$  such that  $a + b + c \geq abc$ . Prove the inequality

$$a^2 + b^2 + c^2 \geq \sqrt{3}abc.$$

*Solution* We have

$$\begin{aligned}(a^2 + b^2 + c^2)^2 &= a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 \\ &= a^4 + b^4 + c^4 + a^2(b^2 + c^2) + b^2(c^2 + a^2) + c^2(a^2 + b^2).\end{aligned}\quad (1.2)$$

By Exercise 1.7, it follows that

$$a^4 + b^4 + c^4 \geq abc(a + b + c). \quad (1.3)$$

Also

$$b^2 + c^2 \geq 2bc, \quad c^2 + a^2 \geq 2ca, \quad a^2 + b^2 \geq 2ab. \quad (1.4)$$

Now by (1.2), (1.3) and (1.4) we deduce

$$\begin{aligned}(a^2 + b^2 + c^2)^2 &\geq abc(a + b + c) + 2a^2bc + 2b^2ac + 2c^2ab \\ &= abc(a + b + c) + 2abc(a + b + c) = 3abc(a + b + c).\end{aligned}\quad (1.5)$$

Since  $a + b + c \geq abc$  in (1.5) we have

$$(a^2 + b^2 + c^2)^2 \geq 3abc(a + b + c) \geq 3(abc)^2,$$

i.e.

$$a^2 + b^2 + c^2 \geq \sqrt{3}abc.$$

Equality occurs if and only if  $a = b = c = \sqrt{3}$ .

**Exercise 1.9** Let  $a, b, c > 1$  be real numbers. Prove the inequality

$$abc + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} > a + b + c + \frac{1}{abc}.$$

*Solution* Since  $a, b, c > 1$  we have  $a > \frac{1}{b}, b > \frac{1}{c}, c > \frac{1}{a}$ , i.e.

$$\left(a - \frac{1}{b}\right)\left(b - \frac{1}{c}\right)\left(c - \frac{1}{a}\right) > 0.$$

After multiplying we get the required inequality.

**Exercise 1.10** Let  $a, b, c, d$  be real numbers such that  $a^4 + b^4 + c^4 + d^4 = 16$ . Prove the inequality

$$a^5 + b^5 + c^5 + d^5 \leq 32.$$

*Solution* We have  $a^4 \leq a^4 + b^4 + c^4 + d^4 = 16$ , i.e.  $a \leq 2$  from which it follows that  $a^4(a - 2) \leq 0$ , i.e.  $a^5 \leq 2a^4$ .

Similarly we obtain  $b^5 \leq 2b^4, c^5 \leq 2c^4$  and  $d^5 \leq 2d^4$ .

Hence

$$a^5 + b^5 + c^5 + d^5 \leq 2(a^4 + b^4 + c^4 + d^4) = 32.$$

Equality occurs iff  $a = 2, b = c = d = 0$  (up to permutation).

**Exercise 1.11** Prove that for any real number  $x$  the following inequality holds

$$x^{12} - x^9 + x^4 - x + 1 > 0.$$

*Solution* We consider two cases:  $x < 1$  and  $x \geq 1$ .

(1) Let  $x < 1$ . We have

$$x^{12} - x^9 + x^4 - x + 1 = x^{12} + (x^4 - x^9) + (1 - x).$$

Since  $x < 1$  we have  $1 - x > 0$  and  $x^4 > x^9$ , i.e.  $x^4 - x^9 > 0$ , so in this case

$$x^{12} - x^9 + x^4 - x + 1 > 0,$$

i.e. the desired inequality holds.

(2) For  $x \geq 1$  we have

$$\begin{aligned} x^{12} - x^9 + x^4 - x + 1 &= x^8(x^4 - x) + (x^4 - x) + 1 \\ &= (x^4 - x)(x^8 + 1) + 1 = x(x^3 - 1)(x^8 + 1) + 1. \end{aligned}$$

Since  $x \geq 1$  we have  $x^3 \geq 1$ , i.e.  $x^3 - 1 \geq 0$ .

Therefore

$$x^{12} - x^9 + x^4 - x + 1 > 0,$$

and the problem is solved.

**Exercise 1.12** Prove that for any real number  $x$  the following inequality holds

$$2x^4 + 1 \geq 2x^3 + x^2.$$

*Solution* We have

$$\begin{aligned} 2x^4 + 1 - 2x^3 - x^2 &= 1 - x^2 - 2x^3(1 - x) = (1 - x)(1 + x) - 2x^3(1 - x) \\ &= (1 - x)(x + 1 - 2x^3) = (1 - x)(x(1 - x^2) + 1 - x^3) \\ &= (1 - x) \left( x(1 - x)(1 + x) + (1 - x)(1 + x + x^2) \right) \\ &= (1 - x) \left( (1 - x)(x(1 + x) + 1 + x + x^2) \right) \\ &= (1 - x)^2((x + 1)^2 + x^2) \geq 0. \end{aligned}$$

Equality occurs if and only if  $x = 1$ .



**Exercise 1.13** Let  $x, y \in \mathbb{R}$ . Prove the inequality

$$x^4 + y^4 + 4xy + 2 \geq 0.$$

*Solution* We have

$$\begin{aligned} x^4 + y^4 + 4xy + 2 &= (x^4 - 2x^2y^2 + y^4) + (2x^2y^2 + 4xy + 2) \\ &= (x^2 - y^2)^2 + 2(xy + 1)^2 \geq 0, \end{aligned}$$

as desired.

Equality occurs if and only if  $x = 1, y = -1$  or  $x = -1, y = 1$ .

**Exercise 1.14** Prove that for any real numbers  $x, y, z$  the following inequality holds

$$x^4 + y^4 + z^2 + 1 \geq 2x(xy^2 - x + z + 1).$$

*Solution* We have

$$\begin{aligned} x^4 + y^4 + z^2 + 1 - 2x(xy^2 - x + z + 1) &= (x^4 - 2x^2y^2 + x^4) + (z^2 - 2xz + x^2) + (x^2 - 2x + 1) \\ &= (x^2 - y^2)^2 + (x - z)^2 + (x - 1)^2 \geq 0, \end{aligned}$$

from which we get the desired inequality.

Equality occurs if and only if  $x = y = z = 1$  or  $x = z = 1, y = -1$ .

**Exercise 1.15** Let  $x, y, z$  be positive real numbers such that  $x + y + z = 1$ . Prove the inequality

$$xy + yz + 2zx \leq \frac{1}{2}.$$

*Solution* We will prove that

$$2xy + 2yz + 4zx \leq (x + y + z)^2,$$

from which, since  $x + y + z = 1$  we'll obtain the required inequality.

The last inequality is equivalent to

$$x^2 + y^2 + z^2 - 2zx \geq 0, \quad \text{i.e.} \quad (x - z)^2 + y^2 \geq 0,$$

which is true.

Equality occurs if and only if  $x = z$  and  $y = 0$ , i.e.  $x = z = \frac{1}{2}, y = 0$ .

**Exercise 1.16** Let  $a, b \in \mathbb{R}^+$ . Prove the inequality

$$a^2 + b^2 + 1 > a\sqrt{b^2 + 1} + b\sqrt{a^2 + 1}.$$

*Solution* From the obvious inequality

$$(a - \sqrt{b^2 + 1})^2 + (b - \sqrt{a^2 + 1})^2 \geq 0, \quad (1.6)$$

we get the desired result.

Equality occurs if and only if

$$a = \sqrt{b^2 + 1} \quad \text{and} \quad b = \sqrt{a^2 + 1}, \quad \text{i.e.} \quad a^2 = b^2 + 1 \quad \text{and} \quad b^2 = a^2 + 1,$$

which is impossible, so in (1.6) we have strictly inequality.

**Exercise 1.17** Let  $x, y, z \in \mathbb{R}^+$  such that  $x + y + z = 3$ . Prove the inequality

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx.$$

*Solution* We have

$$3(x + y + z) = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx).$$

Hence it follows that

$$xy + yz + zx = \frac{1}{2}(3x - x^2 + 3y - y^2 + 3z - z^2).$$

Then

$$\begin{aligned} & \sqrt{x} + \sqrt{y} + \sqrt{z} - (xy + yz + zx) \\ &= \sqrt{x} + \sqrt{y} + \sqrt{z} + \frac{1}{2}(x^2 - 3x + y^2 - 3y + z^2 - 3z) \\ &= \frac{1}{2}((x^2 - 3x + 2\sqrt{x}) + (y^2 - 3y + 2\sqrt{y}) + (z^2 - 3z + 2\sqrt{z})) \\ &= \frac{1}{2}(\sqrt{x}(\sqrt{x} - 1)^2(\sqrt{x} + 2) + \sqrt{y}(\sqrt{y} - 1)^2(\sqrt{y} + 2) \\ & \quad + \sqrt{z}(\sqrt{z} - 1)^2(\sqrt{z} + 2)) \geq 0, \end{aligned}$$

i.e.

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx.$$



## Chapter 2

# Inequalities Between Means (with Two and Three Variables)

In this section, we'll first mention and give a proof of *inequalities between means*, which are of particular importance for a full upgrade of the student in solving tasks in this area. It ought to be mentioned that in this section we will discuss the case that treats two or three variables, while the general case will be considered later in Chap. 5.

**Theorem 2.1** Let  $a, b \in \mathbb{R}^+$ , and let us denote

$$QM = \sqrt{\frac{a^2 + b^2}{2}}, \quad AM = \frac{a + b}{2}, \quad GM = \sqrt{ab} \quad \text{and} \quad HM = \frac{2}{\frac{1}{a} + \frac{1}{b}}.$$

Then

$$QM \geq AM \geq GM \geq HM. \quad (2.1)$$

Equalities occur if and only if  $a = b$ .

*Proof* Firstly we'll show that  $QM \geq AM$ .

For  $a, b \in \mathbb{R}^+$  we have

$$\begin{aligned} (a - b)^2 &\geq 0 \\ \Leftrightarrow a^2 + b^2 &\geq 2ab \quad \Leftrightarrow 2(a^2 + b^2) \geq a^2 + b^2 + 2ab \\ \Leftrightarrow 2(a^2 + b^2) &\geq (a + b)^2 \quad \Leftrightarrow \frac{a^2 + b^2}{2} \geq \left(\frac{a + b}{2}\right)^2 \\ \Leftrightarrow \sqrt{\frac{a^2 + b^2}{2}} &\geq \frac{a + b}{2}. \end{aligned}$$

Equality holds if and only if  $a - b = 0$ , i.e.  $a = b$ .

Furthermore, for  $a, b \in \mathbb{R}^+$  we have

$$(\sqrt{a} - \sqrt{b})^2 \geq 0 \Leftrightarrow a + b - 2\sqrt{ab} \geq 0 \Leftrightarrow \frac{a+b}{2} \geq \sqrt{ab}.$$

So  $AM \geq GM$ , with equality if and only if

$$\sqrt{a} - \sqrt{b} = 0, \quad \text{i.e. } a = b.$$

Finally we'll show that

$$GM \geq HM, \quad \text{i.e. } \sqrt{ab} \geq \frac{2}{\frac{1}{a} + \frac{1}{b}}.$$

We have

$$\begin{aligned} (\sqrt{a} - \sqrt{b})^2 \geq 0 &\Leftrightarrow a + b \geq 2\sqrt{ab} \Leftrightarrow 1 \geq \frac{2\sqrt{ab}}{a+b} \Leftrightarrow \sqrt{ab} \geq \frac{2ab}{a+b} \\ &\Leftrightarrow \sqrt{ab} \geq \frac{2}{\frac{1}{a} + \frac{1}{b}}. \end{aligned}$$

Equality holds if and only if  $\sqrt{a} - \sqrt{b} = 0$ , i.e.  $a = b$ . □

*Remark* The numbers  $QM$ ,  $AM$ ,  $GM$  and  $HM$  are called the quadratic, arithmetic, geometric and harmonic mean for the numbers  $a$  and  $b$ , respectively; the inequalities (2.1) are called *mean inequalities*.

These inequalities usually will be used in the case when  $a, b \in \mathbb{R}^+$ .

Also similarly we can define the quadratic, arithmetic, geometric and harmonic mean for three variables as follows:

$$\begin{aligned} QM &= \sqrt{\frac{a^2 + b^2 + c^2}{3}}, \quad AM = \frac{a + b + c}{3}, \quad GM = \sqrt[3]{abc} \quad \text{and} \\ HM &= \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}. \end{aligned}$$

Analogous to Theorem 2.1, with three variables we have the following theorem.

**Theorem 2.2** Let  $a, b, c \in \mathbb{R}^+$ , and let us denote

$$\begin{aligned} QM &= \sqrt{\frac{a^2 + b^2 + c^2}{3}}, \quad AM = \frac{a + b + c}{3}, \quad GM = \sqrt[3]{abc} \quad \text{and} \\ HM &= \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}. \end{aligned}$$

Then

$$QM \geq AM \geq GM \geq HM.$$

Equalities occur if and only if  $a = b = c$ .

Over the next few exercises we will see how these inequalities can be put in use.

**Exercise 2.1** Let  $x, y, z \in \mathbb{R}^+$  such that  $x + y + z = 1$ . Prove the inequality

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \geq 1.$$

When does equality occur?

*Solution* We have

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} = \frac{1}{2} \left( \frac{xy}{z} + \frac{yz}{x} \right) + \frac{1}{2} \left( \frac{yz}{x} + \frac{zx}{y} \right) + \frac{1}{2} \left( \frac{zx}{y} + \frac{xy}{z} \right). \quad (2.2)$$

Since  $AM \geq GM$  we have

$$\frac{1}{2} \left( \frac{xy}{z} + \frac{yz}{x} \right) \geq \sqrt{\frac{xy}{z} \frac{yz}{x}} = y.$$

Analogously we get

$$\frac{1}{2} \left( \frac{yz}{x} + \frac{zx}{y} \right) \geq z \quad \text{and} \quad \frac{1}{2} \left( \frac{zx}{y} + \frac{xy}{z} \right) \geq x.$$

Adding these three inequalities we obtain

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \geq x + y + z = 1.$$

Equality holds if and only if  $\frac{xy}{z} = \frac{yz}{x} = \frac{zx}{y}$ , i.e.  $x = y = z$ . Since  $x + y + z = 1$  we get that equality holds iff  $x = y = z = 1/3$ .

**Exercise 2.2** Let  $x, y, z > 0$  be real numbers. Prove the inequality

$$\frac{x^2 - z^2}{y + z} + \frac{y^2 - x^2}{z + x} + \frac{z^2 - y^2}{x + y} \geq 0.$$

When does equality occur?

*Solution* Let  $a = x + y, b = y + z, c = z + x$ .

Then clearly  $a, b, c > 0$ , and it follows that

$$\begin{aligned} \frac{x^2 - z^2}{y + z} + \frac{y^2 - x^2}{z + x} + \frac{z^2 - y^2}{x + y} &= \frac{(a - b)c}{b} + \frac{(b - c)a}{c} + \frac{(c - a)b}{a} \\ &= \frac{ac}{b} + \frac{ba}{c} + \frac{cb}{a} - (a + b + c). \end{aligned} \quad (2.3)$$

Similarly as in Exercise 2.1, we can prove that for any  $a, b, c > 0$

$$\frac{ac}{b} + \frac{ba}{c} + \frac{cb}{a} \geq a + b + c. \quad (2.4)$$

By (2.3) and (2.4) we get

$$\begin{aligned} \frac{x^2 - z^2}{y + z} + \frac{y^2 - x^2}{z + x} + \frac{z^2 - y^2}{x + y} \\ = \frac{ac}{b} + \frac{ba}{c} + \frac{cb}{a} - (a + b + c) \geq (a + b + c) - (a + b + c) = 0. \end{aligned}$$

Equality occurs iff we have equality in (2.4), i.e.  $a = b = c$ , from which we deduce that  $x = y = z$ .

**Exercise 2.3** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right)\left(c + \frac{1}{a}\right) \geq 8.$$

When does equality occur?

*Solution* Applying  $AM \geq GM$  we get

$$a + \frac{1}{b} \geq 2\sqrt{\frac{a}{b}}, \quad b + \frac{1}{c} \geq 2\sqrt{\frac{b}{c}}, \quad c + \frac{1}{a} \geq 2\sqrt{\frac{c}{a}}.$$

Therefore

$$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right)\left(c + \frac{1}{a}\right) \geq 8\sqrt{\frac{a}{b}} \cdot \sqrt{\frac{b}{c}} \cdot \sqrt{\frac{c}{a}} = 8.$$

Equality occurs if and only if  $a = \frac{1}{b}$ ,  $b = \frac{1}{c}$ ,  $c = \frac{1}{a}$  i.e.  $a = \frac{1}{b} = c = \frac{1}{a}$ , from which we deduce that  $a = b = c = 1$ .

**Exercise 2.4** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{ab}{a + b + 2c} + \frac{bc}{b + c + 2a} + \frac{ca}{c + a + 2b} \leq \frac{a + b + c}{4}.$$

*Solution* Since  $AM \geq HM$  we have

$$\frac{ab}{a+b+2c} = \frac{ab}{(a+c) + (b+c)} \leq \frac{ab}{4} \left( \frac{1}{a+c} + \frac{1}{b+c} \right).$$

Similarly we get

$$\frac{bc}{b+c+2a} \leq \frac{bc}{4} \left( \frac{1}{a+b} + \frac{1}{a+c} \right) \quad \text{and} \quad \frac{ca}{c+a+2b} \leq \frac{ca}{4} \left( \frac{1}{a+b} + \frac{1}{b+c} \right).$$

By adding these three inequalities we obtain the required inequality.

**Exercise 2.5** Let  $x, y, z$  be positive real numbers such that  $x + y + z = 1$ . Prove the inequality

$$xy + yz + zx \geq 9xyz.$$

*Solution* Applying  $AM \geq GM$  we get

$$xy + yz + zx = (xy + yz + zx)(x + y + z) \geq 3\sqrt[3]{(xy)(yz)(zx)} \cdot 3\sqrt[3]{xyz} = 9xyz.$$

Equality occur if and only if  $x = y = z = \frac{1}{3}$ .

**Exercise 2.6** Let  $a, b, c \in \mathbb{R}^+$  such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \geq \frac{3}{2}.$$

*Solution* Applying  $AM \geq HM$  and the inequality  $a^2 + b^2 + c^2 \geq ab + bc + ca$ , we get

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \geq \frac{9}{3+ab+bc+ca} \geq \frac{9}{3+a^2+b^2+c^2} = \frac{3}{2}.$$

**Exercise 2.7** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\sqrt{\frac{a+b}{c}} + \sqrt{\frac{b+c}{a}} + \sqrt{\frac{c+a}{b}} \geq 3\sqrt{2}.$$

*Solution* We have

$$\begin{aligned} \sqrt{\frac{a+b}{c}} + \sqrt{\frac{b+c}{a}} + \sqrt{\frac{c+a}{b}} &\stackrel{A \geq G}{\geq} 3\sqrt[3]{\sqrt{\left(\frac{a+b}{c}\right)\left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right)}} \\ &= 3\sqrt[6]{\frac{(a+b)(b+c)(c+a)}{abc}} \\ &\stackrel{A \geq G}{\geq} 3\sqrt[6]{\frac{2^3 \sqrt{ab} \cdot \sqrt{bc} \cdot \sqrt{ca}}{abc}} = 3\sqrt{2}. \end{aligned}$$



Equality occurs if and only if  $a = b = c$ .

**Exercise 2.8** Let  $x, y, z$  be positive real numbers such that  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ . Prove the inequality

$$(x-1)(y-1)(z-1) \geq 8.$$

*Solution* The given inequality is equivalent to

$$\left(\frac{x-1}{x}\right)\left(\frac{y-1}{y}\right)\left(\frac{z-1}{z}\right) \geq \frac{8}{xyz}$$

or

$$\left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right)\left(1 - \frac{1}{z}\right) \geq \frac{8}{xyz}. \quad (2.5)$$

From the initial condition and  $AM \geq GM$  we have

$$1 - \frac{1}{x} = \frac{1}{y} + \frac{1}{z} \geq 2\sqrt{\frac{1}{y} \frac{1}{z}} = \frac{2}{\sqrt{yz}}.$$

Analogously we obtain  $1 - \frac{1}{y} \geq \frac{2}{\sqrt{zx}}$  and  $1 - \frac{1}{z} \geq \frac{2}{\sqrt{xy}}$ .

If we multiply the last three inequalities we get inequality (2.5), as required. Equality holds if and only if  $x = y = z = 3$ .

**Exercise 2.9** Let  $x, y, z \in \mathbb{R}^+$  such that  $x + y + z = 1$ . Prove the inequality

$$\frac{x^2 + y^2}{z} + \frac{y^2 + z^2}{x} + \frac{z^2 + x^2}{y} \geq 2.$$

*Solution* We have

$$\begin{aligned} & \frac{x^2 + y^2}{z} + \frac{y^2 + z^2}{x} + \frac{z^2 + x^2}{y} \\ & \geq 2\frac{xy}{z} + 2\frac{yz}{x} + 2\frac{zx}{y} = 2\left(\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}\right) \\ & = 2\left(\frac{1}{2}\left(\frac{xy}{z} + \frac{yz}{x}\right) + \frac{1}{2}\left(\frac{xy}{z} + \frac{zx}{y}\right) + \frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y}\right)\right) \\ & \geq 2\left(\sqrt{y^2} + \sqrt{x^2} + \sqrt{z^2}\right) = 2(x + y + z) = 2. \end{aligned}$$

**Exercise 2.10** Let  $x, y, z \in \mathbb{R}^+$  such that  $xyz = 1$ . Prove the inequality

$$\frac{x^2 + y^2 + z^2 + xy + yz + zx}{\sqrt{x} + \sqrt{y} + \sqrt{z}} \geq 2.$$

*Solution* We have

$$\begin{aligned} \frac{x^2 + y^2 + z^2 + xy + yz + zx}{\sqrt{x} + \sqrt{y} + \sqrt{z}} &= \frac{x^2 + yz + y^2 + zx + z^2 + xy}{\sqrt{x} + \sqrt{y} + \sqrt{z}} \\ &\geq \frac{2\sqrt{x^2yz} + 2\sqrt{xy^2z} + 2\sqrt{xyz^2}}{\sqrt{x} + \sqrt{y} + \sqrt{z}} \\ &= \frac{2(\sqrt{x} + \sqrt{y} + \sqrt{z})}{\sqrt{x} + \sqrt{y} + \sqrt{z}} = 2. \end{aligned}$$

Equality occurs if and only if  $x = y = z = 1$ .

**Exercise 2.11** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequalities

$$\frac{9abc}{2(a+b+c)} \leq \frac{ab^2}{a+b} + \frac{bc^2}{b+c} + \frac{ca^2}{c+a} \leq \frac{a^2 + b^2 + c^2}{2}.$$

*Solution* Since  $AM \geq HM$  and from the well-known inequality

$$ab + bc + ca \leq a^2 + b^2 + c^2,$$

we get

$$\begin{aligned} \frac{ab^2}{a+b} + \frac{bc^2}{b+c} + \frac{ca^2}{c+a} &= \frac{1}{1/b^2 + 1/ab} + \frac{1}{1/c^2 + 1/bc} + \frac{1}{1/a^2 + 1/ca} \\ &\leq \frac{b^2 + ab}{4} + \frac{c^2 + bc}{4} + \frac{a^2 + ca}{4} \\ &= \frac{a^2 + b^2 + c^2 + ab + bc + ca}{4} \\ &\leq \frac{2(a^2 + b^2 + c^2)}{4} = \frac{a^2 + b^2 + c^2}{2}. \end{aligned}$$

It remains to show the left inequality.

Since  $AM \geq GM$  we have

$$\frac{ab^2}{a+b} + \frac{bc^2}{b+c} + \frac{ca^2}{c+a} \geq \frac{3abc}{\sqrt[3]{(a+b)(b+c)(c+a)}}.$$

Therefore it suffices to show that

$$\frac{3abc}{\sqrt[3]{(a+b)(b+c)(c+a)}} \geq \frac{9abc}{2(a+b+c)},$$

i.e.

$$2(a+b+c) \geq 3\sqrt[3]{(a+b)(b+c)(c+a)},$$

which is true, since

$$2(a + b + c) = (a + b) + (b + c) + (c + a) \geq 3\sqrt[3]{(a + b)(b + c)(c + a)}.$$

The following exercises shows how we can use *mean inequalities* in a different, non-trivial way.

**Exercise 2.12** Prove that for every positive real number  $a, b, c$  we have

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c.$$

*Solution 1* From  $AM \geq GM$  we have

$$\frac{a^2}{b} + b \geq 2\sqrt{\frac{a^2}{b} \cdot b} = 2a.$$

Analogously we get

$$\frac{b^2}{c} + c \geq 2b \quad \text{and} \quad \frac{c^2}{a} + a \geq 2c.$$

After adding these three inequalities we obtain

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + (a + b + c) \geq 2(a + b + c),$$

i.e.

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c.$$

Equality occurs if and only if  $a = b = c$ .

*Solution 2* Observe that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} = \frac{a^2 - ab + b^2}{b} + \frac{b^2 - bc + c^2}{c} + \frac{c^2 - ca + a^2}{a}. \quad (2.6)$$

Since for any  $x, y \in \mathbb{R}$ , we have  $x^2 - xy + y^2 \geq xy$ , by (2.6) we get

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{ab}{b} + \frac{bc}{c} + \frac{ca}{a} = a + b + c.$$

**Exercise 2.13** Let  $x, y, z$  be positive real numbers. Prove the inequality

$$\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} \geq x + y + z.$$

*Solution* Since  $AM \geq GM$  we have

$$\frac{x^3}{yz} + y + z \geq 3\sqrt[3]{\frac{x^3}{yz} \cdot y \cdot z} = 3x.$$

Similarly we have

$$\frac{y^3}{zx} + z + x \geq 3y \quad \text{and} \quad \frac{z^3}{xy} + x + y \geq 3z.$$

After adding these inequalities we get the required result.

Equality holds if and only if  $x = y = z$ .

**Exercise 2.14** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{abc}{(1+a)(a+b)(b+c)(c+16)} \leq \frac{1}{81}.$$

*Solution* We have

$$\begin{aligned} & (1+a)(a+b)(b+c)(c+16) \\ &= \left(1 + \frac{a}{2} + \frac{a}{2}\right) \left(a + \frac{b}{2} + \frac{b}{2}\right) \left(b + \frac{c}{2} + \frac{c}{2}\right) (c+8+8) \\ &\geq 3\sqrt[3]{\frac{a^2}{4}} \cdot 3\sqrt[3]{\frac{ab^2}{4}} \cdot 3\sqrt[3]{\frac{bc^2}{4}} \cdot 3\sqrt[3]{\frac{64c}{4}} \geq 81abc. \end{aligned}$$

Thus

$$\frac{abc}{(1+a)(a+b)(b+c)(c+16)} \leq \frac{1}{81}.$$

**Exercise 2.15** Let  $x, y \in \mathbb{R}^+$  such that  $x + y = 2$ . Prove the inequality

$$x^3y^3(x^3 + y^3) \leq 2.$$

*Solution* Since  $AM \geq GM$  we have  $\sqrt{xy} \leq \frac{x+y}{2} = 1$ , i.e.  $xy \leq 1$ .

Hence  $0 \leq xy \leq 1$ .

Furthermore

$$\begin{aligned} x^3y^3(x^3 + y^3) &= (xy)^3(x+y)(x^2 - xy + y^2) = 2(xy)^3((x+y)^2 - 3xy) \\ &= 2(xy)^3(4 - 3xy). \end{aligned}$$

It's enough to show that

$$(xy)^3(4 - 3xy) \leq 1.$$

Let  $xy = z$  then  $0 \leq z \leq 1$  and clearly  $4 - 3z > 0$ .

Then using  $AM \geq GM$  we obtain

$$z^3(4 - 3z) = z \cdot z \cdot z(4 - 3z) \leq \left( \frac{z + z + z + 4 - 3z}{4} \right)^4 = 1,$$

as required.

Equality occurs if and only if  $z = 4 - 3z$ , i.e.  $z = 1$ , i.e.  $x = y = 1$ . (Why?)

**Exercise 2.16** Let  $a, b, c, d$  be positive real numbers such that  $a + b + c + d = 4$ . Prove the inequality

$$\frac{1}{a^2 + 1} + \frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} + \frac{1}{d^2 + 1} \geq 2.$$

*Solution* We have

$$\frac{1}{a^2 + 1} = 1 - \frac{a^2}{a^2 + 1} \geq 1 - \frac{a^2}{2a} = 1 - \frac{a}{2}.$$

Similarly we get

$$\frac{1}{b^2 + 1} \geq 1 - \frac{b}{2}, \frac{1}{c^2 + 1} \geq 1 - \frac{c}{2} \quad \text{and} \quad \frac{1}{d^2 + 1} \geq 1 - \frac{d}{2}.$$

After adding these inequalities we obtain

$$\frac{1}{a^2 + 1} + \frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} + \frac{1}{d^2 + 1} \geq 4 - \frac{a + b + c + d}{2} = 4 - 2 = 2.$$

Equality occurs if and only if  $a = b = c = 1$ .

## Chapter 3

# Geometric (Triangle) Inequalities

These inequalities in most cases have as variables the lengths of the sides of a given triangle; there are also inequalities in which appear other elements of the triangle, such as lengths of heights, lengths of medians, lengths of the bisectors, angles, etc.

First we will introduce some standard notation which will be used in this section:

- $h_a, h_b, h_c$ —lengths of the altitudes drawn to the sides  $a, b, c$ , respectively.
- $t_a, t_b, t_c$ —lengths of the medians drawn to the sides  $a, b, c$ , respectively.
- $l_\alpha, l_\beta, l_\gamma$ —lengths of the bisectors of the angles  $\alpha, \beta, \gamma$ , respectively.
- $P$ —area,  $s$ —semi-perimeter,  $R$ —circumradius,  $r$ —inradius.

Furthermore we will give relations between the lengths of medians and lengths of the bisectors of the angles with the sides of a given triangle.

Namely we have

$$t_a^2 = \frac{b^2 + c^2}{2} - \frac{a^2}{4}, \quad t_b^2 = \frac{a^2 + c^2}{2} - \frac{b^2}{4}, \quad t_c^2 = \frac{a^2 + b^2}{2} - \frac{c^2}{4}$$

and

$$l_\alpha^2 = bc \frac{((b+c)^2 - a^2)}{(b+c)^2}, \quad l_\beta^2 = ac \frac{((a+c)^2 - b^2)}{(a+c)^2},$$

$$l_\gamma^2 = ab \frac{((a+b)^2 - c^2)}{(a+b)^2}.$$

We can rewrite the last three identities in the following form

$$l_\alpha^2 = 4bc \frac{s(s-a)}{(b+c)^2}, \quad l_\beta^2 = 4ac \frac{s(s-b)}{(a+c)^2}, \quad l_\gamma^2 = 4ab \frac{s(s-c)}{(a+b)^2}.$$

Also we note that the following properties are true, and we'll present them without proof. (The first inequality follows by using geometric formulas and *mean inequalities*, and the second inequality immediately follows, for instance, according to *Leibniz's theorem*.)

**Proposition 3.1** *For an arbitrary triangle the following inequalities hold*

$$R \geq 2r \quad \text{and} \quad a^2 + b^2 + c^2 \leq 9R^2.$$

Basic inequalities which concern the lengths of the sides of a given triangle are well-known inequalities:  $a + b > c$ ,  $a + c > b$ ,  $b + c > a$ .

But also useful and frequent substitutions are:

$$a = x + y, \quad b = y + z, \quad c = z + x, \quad \text{where } x, y, z > 0. \quad (3.1)$$

The question is whether there are always positive real numbers  $x, y, z$ , such that the above identities (3.1) hold and  $a, b, c$  are the sides of the triangle.

The answer is positive.

Namely  $x, y, z$  are tangent segments dropped from the vertices to the inscribed circle of the given triangle.

From (3.1) we easily get that

$$x = \frac{a + c - b}{2}, \quad y = \frac{a + b - c}{2}, \quad z = \frac{c + b - a}{2},$$

and then clearly  $x, y, z > 0$ .

*Remark* The substitutions (3.1) are called *Ravi's substitutions*.

**Exercise 3.1** Let  $a, b, c$  be the lengths of the sides of given triangle. Prove the inequalities

$$\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

*Solution* Let's prove the right-hand inequality.

Since  $a + b > c$  we have  $2(a + b) > a + b + c$ , i.e.  $a + b > s$ .

Similarly we get  $b + c > s$  and  $a + c > s$ .

Therefore

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{b+a} < \frac{a}{s} + \frac{b}{s} + \frac{c}{s} = 2.$$

Let's consider the left-hand inequality.

If we denote  $b + c = x$ ,  $a + c = y$ ,  $a + b = z$  then we have

$$a = \frac{z + y - x}{2}, \quad b = \frac{z + x - y}{2}, \quad c = \frac{x + y - z}{2}.$$

Hence

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{b+a} = \frac{z + y - x}{2x} + \frac{z + x - y}{2y} + \frac{x + y - z}{2z},$$

i.e.

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{b+a} = \frac{1}{2} \left( \frac{z}{x} + \frac{y}{x} + \frac{z}{y} + \frac{x}{y} + \frac{x}{z} + \frac{y}{z} - 3 \right) \geq \frac{1}{2} (2+2+2-3) = \frac{3}{2},$$

as required.

*Remark* The left-hand inequality is known as *Nesbitt's inequality*, and is true for any positive real numbers  $a, b$  and  $c$  (Exercise 1.3).

**Exercise 3.2** Let  $a, b, c$  be the side lengths of a given triangle. Prove the inequality

$$\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \geq \frac{9}{s}.$$

*Solution* Since  $AM \geq HM$  we have

$$\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \geq \frac{9}{(s-a) + (s-b) + (s-c)} = \frac{9}{s}.$$

Equality occurs if and only if  $a = b = c$ .

**Exercise 3.3** Let  $s$  and  $r$  be the semi-perimeter and inradius, respectively, in an arbitrary triangle. Prove the inequality

$$s \geq 3r\sqrt{3}.$$

*Solution 1* We have

$$2s = a + b + c \geq 3\sqrt[3]{abc} = 3\sqrt[3]{4PR} = 3\sqrt[3]{4srR} \geq 3\sqrt[3]{8sr^2},$$

i.e.

$$s \geq 3\sqrt[3]{sr^2}$$

or

$$s \geq 3r\sqrt{3}.$$

Equality occurs if and only if  $a = b = c$ .

*Solution 2* We have

$$\frac{s}{3} = \frac{(s-a) + (s-b) + (s-c)}{3} \stackrel{AM \geq GM}{\geq} \sqrt[3]{(s-a)(s-b)(s-c)}. \quad (3.2)$$

Also

$$(s-a)(s-b)(s-c) = \frac{P^2}{s} = \frac{s^2 r^2}{s} = sr^2. \quad (3.3)$$



By (3.2) and (3.3) we obtain

$$s \geq 3\sqrt[3]{sr^2}, \quad \text{i.e.} \quad s \geq 3\sqrt{3}r.$$

Equality occurs if and only if  $a = b = c$ .

**Exercise 3.4** Let  $a, b, c$  be the side lengths of a given triangle. Prove the inequality

$$(a + b - c)(b + c - a)(c + a - b) \leq abc.$$

*Solution 1* We have

$$a^2 \geq a^2 - (b - c)^2 = (a + b - c)(a + c - b).$$

Analogously

$$b^2 \geq (b + a - c)(b + c - a) \quad \text{and} \quad c^2 \geq (c + a - b)(c + b - a).$$

If we multiply these inequalities we obtain

$$\begin{aligned} a^2 b^2 c^2 &\geq (a + b - c)^2 (b + c - a)^2 (c + a - b)^2 \\ \Leftrightarrow abc &\geq (a + b - c)(b + c - a)(c + a - b). \end{aligned}$$

Equality holds if and only if  $a = b = c$ , i.e. the triangle is equilateral.

*Solution 2* After setting  $a = x + y, b = y + z, c = z + x$ , where  $x, y, z > 0$ , the given inequality becomes

$$(x + y)(y + z)(z + x) \geq 8xyz.$$

Since  $AM \geq GM$  we have

$$(x + y)(y + z)(z + x) \geq 2\sqrt{xy} \cdot 2\sqrt{yz} \cdot 2\sqrt{zx} = 8xyz,$$

as required. Equality occurs if and only if  $x = y = z$  i.e.  $a = b = c$ .

*Remark* This inequality holds for any  $a, b, c \in \mathbb{R}^+$  (Problem 47).

**Exercise 3.5** Let  $a, b, c$  be the side lengths of a given triangle. Prove the inequality

$$a^2 + b^2 + c^2 < 2(ab + bc + ca).$$

*Solution* Let  $a = x + y, b = y + z, c = z + x, x, y, z > 0$ .

Then we have

$$\begin{aligned} (x + y)^2 + (y + z)^2 + (z + x)^2 \\ < 2((x + y)(y + z) + (y + z)(z + x) + (z + x)(x + y)) \end{aligned}$$

or

$$xy + yz + zx > 0,$$

which is clearly true.

**Exercise 3.6** Let  $a, b, c$  be the side lengths of a given triangle. Prove the inequality

$$8(a+b-c)(b+c-a)(c+a-b) \leq (a+b)(b+c)(c+a).$$

*Solution* Since  $AM \geq GM$  we have

$$(a+b)(b+c)(c+a) \geq 2\sqrt{ab}2\sqrt{bc}2\sqrt{ca} = 8abc.$$

So, it suffices to show that

$$8abc \geq 8(a+b-c)(b+c-a)(c+a-b),$$

i.e.

$$abc \geq (a+b-c)(b+c-a)(c+a-b),$$

which is true by Exercise 3.4.

Equality occurs if and only if  $a = b = c$ .

**Exercise 3.7** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{a+b-c} + \frac{1}{b+c-a} + \frac{1}{c+a-b}.$$

*Solution* Since  $AM \geq HM$  we have

$$\frac{1}{2} \left( \frac{1}{a+b-c} + \frac{1}{b+c-a} \right) \geq \frac{2}{a+b-c+b+c-a} = \frac{1}{b}.$$

Similarly we deduce

$$\frac{1}{2} \left( \frac{1}{a+b-c} + \frac{1}{c+a-b} \right) \geq \frac{1}{a} \quad \text{and} \quad \frac{1}{2} \left( \frac{1}{b+c-a} + \frac{1}{c+a-b} \right) \geq \frac{1}{c}.$$

Adding these inequalities we get the required inequality.

Equality occurs if and only if  $a = b = c$ .

**Exercise 3.8** Let  $ABC$  be a triangle with side lengths  $a, b, c$  and  $\triangle A_1B_1C_1$  with side lengths  $a + \frac{b}{2}, b + \frac{c}{2}, c + \frac{a}{2}$ . Prove that  $P_1 \geq \frac{9}{4}P$ , where  $P$  is the area of  $\triangle ABC$ , and  $P_1$  is the area of  $\triangle A_1B_1C_1$ .

*Solution* By Heron's formula for  $\triangle ABC$  and  $\triangle A_1B_1C_1$  we have

$$16P^2 = (a+b+c)(a+b-c)(b+c-a)(a+c-b)$$

and

$$16P_1^2 = \frac{3}{16}(a+b+c)(-a+b+3c)(-b+c+3a)(-c+a+3b).$$

Since  $a, b$  and  $c$  are the side lengths of triangle there exist positive real numbers  $p, q, r$  such that  $a = q + r, b = r + p, c = p + q$ .

Now we easily get that

$$\frac{P^2}{P_1^2} = \frac{16pqr}{3(2p+q)(2q+r)(2r+p)}. \quad (3.4)$$

So it suffices to show that

$$(2p+q)(2q+r)(2r+p) \geq 27pqr.$$

Applying  $AM \geq QM$  we obtain

$$\begin{aligned} (2p+q)(2q+r)(2r+p) &= (p+p+q)(q+q+r)(r+r+p) \\ &\geq 3\sqrt[3]{p^2q} \cdot 3\sqrt[3]{q^2r} \cdot 3\sqrt[3]{r^2p} = 27pqr. \end{aligned} \quad (3.5)$$

By (3.4) and (3.5) we get the desired result.

**Exercise 3.9** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove that: if  $2(ab^2 + bc^2 + ca^2) = a^2b + b^2c + c^2a + 3abc$  then the triangle is equilateral.

*Solution* We'll show that

$$a^2b + b^2c + c^2a + 3abc \geq 2(ab^2 + bc^2 + ca^2),$$

with equality if and only if  $a = b = c$ , i.e. the triangle is equilateral.

Let us use *Ravi's substitutions*, i.e.  $a = x + y, b = y + z, c = z + x$ . Then the given inequality becomes

$$x^3 + y^3 + z^3 + x^2y + y^2z + z^2x \geq 2(x^2z + y^2x + z^2y).$$

Since  $AM \geq GM$  we have

$$x^3 + z^2x \geq 2x^2z, y^3 + x^2y \geq 2y^2x, z^3 + y^2z \geq 2z^2y.$$

After adding these inequalities we obtain

$$x^3 + y^3 + z^3 + x^2y + y^2z + z^2x \geq 2(x^2z + y^2x + z^2y).$$

Equality holds if and only if  $x = y = z$ , i.e.  $a = b = c$ , as required.

**Exercise 3.10** Let  $a, b, c$  be the side lengths, and  $\alpha, \beta, \gamma$  be the respective angles (in radians) of a given triangle. Prove the inequalities

$$\frac{\pi}{3} \leq \frac{a\alpha + b\beta + c\gamma}{a + b + c} < \frac{\pi}{2}.$$

*Solution* First let's prove the left inequality.

We can assume that  $a \geq b \geq c$  and then clearly  $\alpha \geq \beta \geq \gamma$ .

So we have

$$\begin{aligned} (a-b)(\alpha-\beta) + (b-c)(\beta-\gamma) + (c-a)(\gamma-\alpha) &\geq 0 \\ \Leftrightarrow 2(a\alpha + b\beta + c\gamma) &\geq (b+c)\alpha + (c+a)\beta + (a+b)\gamma, \end{aligned}$$

i.e.

$$3(a\alpha + b\beta + c\gamma) \geq (a+b+c)(\alpha + \beta + \gamma).$$

Hence

$$\frac{a\alpha + b\beta + c\gamma}{a + b + c} \geq \frac{\alpha + \beta + \gamma}{3} = \frac{\pi}{3}.$$

Equality occurs if and only if  $a = b = c$ .

Let's consider the right inequality.

Since  $a, b$  and  $c$  are side lengths of a triangle we have  $a + b + c > 2a$ ,  $a + b + c > 2b$  and  $a + b + c > 2c$ .

If we multiply these inequalities by  $\alpha, \beta$  and  $\gamma$ , respectively, we obtain

$$(a + b + c)(\alpha + \beta + \gamma) > 2(a\alpha + b\beta + c\gamma),$$

i.e.

$$\frac{a\alpha + b\beta + c\gamma}{a + b + c} < \frac{\alpha + \beta + \gamma}{2} = \frac{\pi}{2}.$$



# Chapter 4

## Bernoulli's Inequality, the Cauchy–Schwarz Inequality, Chebishev's Inequality, Surányi's Inequality

These inequalities fill that part of the knowledge of students necessary for proving more complicated, characteristic inequalities such as mathematical inequalities containing more variables, and inequalities which are difficult to prove with already adopted elementary inequalities. These inequalities are often used for proving different inequalities for mathematical competitions.

**Theorem 4.1** (Bernoulli's inequality) *Let  $x_i, i = 1, 2, \dots, n$ , be real numbers with the same sign, greater than  $-1$ . Then we have*

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \geq 1 + x_1 + x_2 + \cdots + x_n. \quad (4.1)$$

*Proof* We'll prove the given inequality by induction.

For  $n = 1$  we have  $1 + x_1 \geq 1 + x_1$ .

Suppose that for  $n = k$ , and arbitrary real numbers  $x_i > -1, i = 1, 2, \dots, k$ , with the same signs, inequality (4.1) holds i.e.

$$(1 + x_1)(1 + x_2) \cdots (1 + x_k) \geq 1 + x_1 + x_2 + \cdots + x_k. \quad (4.2)$$

Let  $n = k + 1$ , and  $x_i > -1, i = 1, 2, \dots, k + 1$ , be arbitrary real numbers with the same signs.

Then, since  $x_1, x_2, \dots, x_{k+1}$  have the same signs, we have

$$(x_1 + x_2 + \cdots + x_k)x_{k+1} \geq 0. \quad (4.3)$$

Hence

$$\begin{aligned} & (1 + x_1)(1 + x_2) \cdots (1 + x_{k+1}) \\ & \stackrel{(4.2)}{\geq} (1 + x_1 + x_2 + \cdots + x_k)(1 + x_{k+1}) = 1 + x_1 + x_2 + \cdots + x_k + x_{k+1} \\ & \quad + (x_1 + x_2 + \cdots + x_k)x_{k+1} \stackrel{(4.3)}{\geq} 1 + x_1 + x_2 + \cdots + x_{k+1}, \end{aligned}$$

i.e. inequality (4.1) holds for  $n = k + 1$ , and we are done.  $\square$

**Corollary 4.1** (Bernoulli's inequality) *Let  $n \in \mathbb{N}$  and  $x > -1$ . Then  $(1+x)^n \geq 1+nx$ .*

*Proof* According to Theorem 4.1, for  $x_1 = x_2 = \dots = x_n = x$ , we obtain the required result.  $\square$

**Definition 4.1** We'll say that the function  $f(x_1, x_2, \dots, x_n)$  is *homogenous* with *coefficient of homogeneity*  $k$ , if for arbitrary  $t \in \mathbb{R}, t \neq 1$ , we have

$$f(tx_1, tx_2, \dots, tx_n) = t^k f(x_1, x_2, \dots, x_n).$$

*Example 4.1* The function  $f(x, y) = \frac{x^2+y^2}{2x+y}$  is homogenous with coefficient 1, since

$$f(tx, ty) = \frac{t^2x^2 + t^2y^2}{2tx + ty} = t \frac{x^2 + y^2}{2x + y} = t \cdot f(x, y).$$

The function  $f(x, y, z) = x^2 + xy + 3z$  is not homogenous.

If we consider the inequality  $f(x_1, x_2, \dots, x_n) \geq g(x_1, x_2, \dots, x_n)$  then for this inequality we'll say that it is homogenous if the function

$$h(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) - g(x_1, x_2, \dots, x_n) \text{ is homogenous.}$$

In other words, a given inequality is homogenous if all its summands have equal degree.

*Example 4.2* The inequality  $x^2 + y^2 + 2xy \geq z^2 + yz$  is homogenous, since all monomials have degree 2.

The inequality  $a^2b + b^2a \leq a^3 + b^3$  is also homogenous, but the inequality  $a^5 + b^5 + 1 \geq 5ab(1 - ab)$  is not homogenous.

In the case of a homogenous inequality, without loss of generality we may assume additional conditions, which can reduce the given inequality to a much simpler form. In this way we can always reduce the number of variables of the given inequality. This procedure of assigning additional conditions is called *normalization*. An inequality with variables  $a, b, c$  can be normalized in many different ways; for example we can assume  $a + b + c = 1$ , or  $abc = 1$  or  $ab + bc + ca = 1$ , etc. The choice of normalization depends on the problem and the available substitutions.

*Example 4.3* Let us consider the homogenous inequality  $a^2 + b^2 + c^2 \geq ab + bc + ca$ . We may use the additional condition  $abc = 1$ . The reason is explained below.

Suppose that  $abc = k^3$ .

Let  $a = kx$ ,  $b = ky$  and  $c = kz$ ; then clearly  $xyz = 1$  and the given inequality becomes  $x^2 + y^2 + z^2 \geq xy + yz + zx$ , which is the same as before. Therefore the restriction  $xyz = 1$  doesn't change anything in the inequality.

Alternatively, we can assume  $a + b + c = 1$  or we can assume  $ab + bc + ca = 1$ , etc.

In general if we have a homogenous inequality then without loss of generality we may assign an additional condition such as:  $abc$ ,  $a + b + c$ ,  $ab + bc + ca$ , etc. to be whatever non-zero constant (not necessarily 1) that we choose.

In the case of a conditional inequality, there is a procedure somewhat opposite to normalization. With this procedure (known as *homogenization*) the given condition can be used to homogenize the whole inequality. After that, the newly acquired homogenous inequality can be normalized with some additional condition. For successful homogenization many obvious substitutions can be helpful.

For example, if we have  $abc = 1$  then we can take  $a = \frac{x}{y}$ ,  $b = \frac{y}{z}$ ,  $c = \frac{z}{x}$ , if we have  $a + b + c = 1$  then we can take  $a = \frac{x}{x+y+z}$ ,  $b = \frac{y}{x+y+z}$ ,  $c = \frac{z}{x+y+z}$  and if  $a^2 + b^2 + c^2 = 1$  we can take  $a = \frac{x}{\sqrt{x^2+y^2+z^2}}$ ,  $b = \frac{y}{\sqrt{x^2+y^2+z^2}}$ ,  $c = \frac{z}{\sqrt{x^2+y^2+z^2}}$ , etc.

**Example 4.4** Consider the following conditional inequality

$$xy + yz + zx \geq 9xyz, \quad \text{when } x + y + z = 1.$$

Obviously, the given inequality is not homogenous.

We can homogenize it as follows: since  $x + y + z = 1$  by taking

$$x = \frac{a}{a+b+c}, \quad y = \frac{b}{a+b+c}, \quad z = \frac{c}{a+b+c},$$

the inequality becomes

$$\frac{ab}{(a+b+c)^2} + \frac{bc}{(a+b+c)^2} + \frac{ca}{(a+b+c)^2} \geq \frac{9abc}{(a+b+c)^3},$$

i.e.

$$(a+b+c)(ab+bc+ca) \geq 9abc.$$

Now it is homogenous and can be further normalized with  $abc = 1$ , which reduces it to the inequality

$$(ab+bc+ca)(a+b+c) \geq 9.$$

The last inequality is true since

$$\begin{aligned} (ab+bc+ca)(a+b+c) &= a^2b + a^2c + b^2a + b^2c + c^2b + c^2a + 3abc \\ &= \frac{a}{c} + \frac{a}{b} + \frac{b}{c} + \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + 3 \\ &= \frac{a}{c} + \frac{c}{a} + \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + 3 \\ &\geq 2 + 2 + 2 + 3 = 9. \end{aligned}$$

**Theorem 4.2** (Cauchy–Schwarz inequality) Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be real numbers. Then we have

$$\left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) \geq \left( \sum_{i=1}^n a_i b_i \right)^2,$$



i.e.

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2.$$

Equality occurs if and only if the sequences  $(a_1, a_2, \dots, a_n)$  and  $(b_1, b_2, \dots, b_n)$  are proportional, i.e.  $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$ .

*Proof 1* The given inequality is equivalent to

$$\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \cdot \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2} \geq |a_1b_1 + a_2b_2 + \cdots + a_nb_n|. \quad (4.4)$$

Let  $A = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$ ,  $B = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$ .

If  $A = 0$  then clearly  $a_1 = a_2 = \cdots = a_n = 0$ , and inequality (4.4) is true.

So let us assume that  $A, B > 0$ .

Inequality (4.4) is homogenous, so we may normalize with

$$a_1^2 + a_2^2 + \cdots + a_n^2 = 1 = b_1^2 + b_2^2 + \cdots + b_n^2, \quad (4.5)$$

i.e. we need to prove that

$$|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \leq 1, \text{ with conditions (4.5).}$$

Since  $QM \geq GM$  we have

$$\begin{aligned} |a_1b_1 + a_2b_2 + \cdots + a_nb_n| &\leq |a_1b_1| + |a_2b_2| + \cdots + |a_nb_n| \\ &\leq \frac{a_1^2 + b_1^2}{2} + \frac{a_2^2 + b_2^2}{2} + \cdots + \frac{a_n^2 + b_n^2}{2} \\ &= \frac{(a_1^2 + a_2^2 + \cdots + a_n^2) + (b_1^2 + b_2^2 + \cdots + b_n^2)}{2} = 1, \end{aligned}$$

as required.

Equality occurs if and only if  $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$ . (Why?) □

*Proof 2.* Consider the quadratic trinomial

$$\sum_{i=1}^n (a_i x - b_i)^2 = \sum_{i=1}^n (a_i^2 x^2 - 2a_i b_i x + b_i^2) = x^2 \sum_{i=1}^n a_i^2 - 2x \sum_{i=1}^n a_i b_i + \sum_{i=1}^n b_i^2.$$

This trinomial is non-negative for all  $x \in \mathbb{R}$ , so its discriminant is not positive, i.e.

$$\begin{aligned} 4 \left( \sum_{i=1}^n a_i b_i \right)^2 - 4 \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) &\leq 0 \\ \Leftrightarrow \left( \sum_{i=1}^n a_i b_i \right)^2 &\leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right), \end{aligned}$$

as required.

Equality holds if and only if  $a_i x - b_i = 0, i = 1, 2, \dots, n$ , i.e.  $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n}$ .  $\square$

Now we'll give several consequences of the *Cauchy–Schwarz inequality* which have broad use in proving other inequalities.

**Corollary 4.2** *Let  $a, b, x, y$  be real numbers and  $x, y > 0$ . Then we have*

$$(1) \frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}, \quad (2) \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}.$$

*Proof* (1) The given inequality is equivalent to

$$y(x+y)a^2 + x(x+y)b^2 \geq xy(a+b)^2, \quad \text{i.e.} \quad (ay - bx)^2 \geq 0,$$

which is clearly true.

Equality occurs iff  $ay = bx$  i.e.  $\frac{a}{x} = \frac{b}{y}$ .

(2) If we apply inequality from the first part twice, we get

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b)^2}{x+y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}.$$

Equality occurs iff  $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$ .  $\square$

Also as you can imagine there must be some generalization of the previous corollaries. Namely the following result is true.

**Corollary 4.3** *Let  $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n$  be real numbers such that  $b_1, b_2, \dots, b_n > 0$ . Then*

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{b_1 + b_2 + \dots + b_n},$$

*with equality if and only if  $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n}$ .*

*Proof* The proof is a direct consequence of the *Cauchy–Schwarz inequality*.  $\square$

**Corollary 4.4** *Let  $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n$  be real numbers. Then*

$$\begin{aligned} & \sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \dots + \sqrt{a_n^2 + b_n^2} \\ & \geq \sqrt{(a_1 + a_2 + \dots + a_n)^2 + (b_1 + b_2 + \dots + b_n)^2}. \end{aligned}$$

*Proof* By induction by  $n$ .

For  $n = 1$  we have equality.

For  $n = 2$  we have

$$\begin{aligned}\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} &\geq \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2} \\ \Leftrightarrow \sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2} &\geq (a_1 a_2 + b_1 b_2) \\ \Leftrightarrow (a_1^2 + b_1^2) \cdot (a_2^2 + b_2^2) &\geq (a_1 a_2 + b_1 b_2)^2,\end{aligned}$$

which is the *Cauchy–Schwarz inequality*.

For  $n = k$ , let the given inequality hold, i.e.

$$\begin{aligned}\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \cdots + \sqrt{a_k^2 + b_k^2} \\ \geq \sqrt{(a_1 + a_2 + \cdots + a_k)^2 + (b_1 + b_2 + \cdots + b_k)^2}.\end{aligned}$$

For  $n = k + 1$  we have

$$\begin{aligned}\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \cdots + \sqrt{a_{k+1}^2 + b_{k+1}^2} \\ = \sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \cdots + \sqrt{a_k^2 + b_k^2} + \sqrt{a_{k+1}^2 + b_{k+1}^2} \\ \geq \sqrt{(a_1 + a_2 + \cdots + a_k)^2 + (b_1 + b_2 + \cdots + b_k)^2} + \sqrt{a_{k+1}^2 + b_{k+1}^2} \\ \geq \sqrt{(a_1 + a_2 + \cdots + a_{k+1})^2 + (b_1 + b_2 + \cdots + b_{k+1})^2}.\end{aligned}$$

So the given inequality holds for every positive integer  $n$ . □

The next result is due to *Walter Janous*, and is considered by the author to be a very important result, which has broad use in proving inequalities.

**Corollary 4.5** *Let  $a, b, c$  and  $x, y, z$  be positive real numbers. Then*

$$\frac{x}{y+z}(b+c) + \frac{y}{z+x}(c+a) + \frac{z}{x+y}(a+b) \geq \sqrt{3(ab+bc+ca)}.$$

*Proof* The given inequality is homogenous, in the variables  $a, b$  and  $c$ , so we can normalize with  $a + b + c = 1$ .

And we can rewrite the inequality as

$$\frac{x}{y+z}(1-a) + \frac{y}{z+x}(1-b) + \frac{z}{x+y}(1-c) \geq \sqrt{3(ab+bc+ca)}.$$

Hence

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \sqrt{3(ab+bc+ca)} + \frac{ax}{y+z} + \frac{by}{z+x} + \frac{cz}{x+y}. \quad (4.6)$$

By the *Cauchy–Schwarz inequality* we have

$$\begin{aligned} & \frac{ax}{y+z} + \frac{by}{z+x} + \frac{cz}{x+y} + \sqrt{3(ab+bc+ca)} \\ & \leq \sqrt{\left(\frac{x}{y+z}\right)^2 + \left(\frac{y}{z+x}\right)^2 + \left(\frac{z}{x+y}\right)^2} \cdot \sqrt{a^2+b^2+c^2} \\ & \quad + \sqrt{\frac{3}{4}}\sqrt{ab+bc+ca} + \sqrt{\frac{3}{4}}\sqrt{ab+bc+ca}, \end{aligned}$$

and after one more usage of the *Cauchy–Schwarz inequality* we get

$$\begin{aligned} & \sqrt{\left(\frac{x}{y+z}\right)^2 + \left(\frac{y}{z+x}\right)^2 + \left(\frac{z}{x+y}\right)^2} \cdot \sqrt{a^2+b^2+c^2} \\ & \quad + \sqrt{\frac{3}{4}}\sqrt{ab+bc+ca} + \sqrt{\frac{3}{4}}\sqrt{ab+bc+ca} \\ & \leq \sqrt{\left(\frac{x}{y+z}\right)^2 + \left(\frac{y}{z+x}\right)^2 + \left(\frac{z}{x+y}\right)^2 + \frac{3}{2}} \\ & \quad \times \sqrt{a^2+b^2+c^2+2(ab+bc+ac)} \\ & = \sqrt{\left(\frac{x}{y+z}\right)^2 + \left(\frac{y}{z+x}\right)^2 + \left(\frac{z}{x+y}\right)^2 + \frac{3}{2}}. \end{aligned}$$

So we have

$$\begin{aligned} & \frac{ax}{y+z} + \frac{by}{z+x} + \frac{cz}{x+y} + \sqrt{3(ab+bc+ca)} \\ & \leq \sqrt{\left(\frac{x}{y+z}\right)^2 + \left(\frac{y}{z+x}\right)^2 + \left(\frac{z}{x+y}\right)^2 + \frac{3}{2}}. \end{aligned}$$

It suffices to show that

$$\left(\frac{x}{y+z}\right)^2 + \left(\frac{y}{z+x}\right)^2 + \left(\frac{z}{x+y}\right)^2 + \frac{3}{2} \leq \left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right)^2,$$

which is equivalent to

$$\frac{yz}{(x+y)(x+z)} + \frac{xz}{(y+x)(y+z)} + \frac{xy}{(z+x)(z+y)} \geq \frac{3}{4}. \quad (4.7)$$

After clearing the denominators inequality (4.7) becomes

$$x^2y + y^2x + y^2z + z^2y + z^2x + x^2z \geq 6xyz,$$

which is a direct consequence of  $AM \geq GM$ . □

**Theorem 4.3** (Chebishev's inequality) *Let  $a_1 \leq a_2 \leq \cdots \leq a_n$  and  $b_1 \leq b_2 \leq \cdots \leq b_n$  be real numbers. Then we have*

$$\left( \sum_{i=1}^n a_i \right) \left( \sum_{i=1}^n b_i \right) \leq n \sum_{i=1}^n a_i b_i,$$

i.e.

$$(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) \leq n(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n).$$

*Equality occurs if and only if  $a_1 = a_2 = \cdots = a_n$  or  $b_1 = b_2 = \cdots = b_n$ .*

*Proof* For all  $i, j \in \{1, 2, \dots, n\}$  we have

$$(a_i - a_j)(b_i - b_j) \geq 0, \quad (4.8)$$

i.e.

$$a_i b_i + a_j b_j \geq a_i b_j + a_j b_i. \quad (4.9)$$

By (4.9) we get

$$\begin{aligned} \left( \sum_{i=1}^n a_i \right) \left( \sum_{i=1}^n b_i \right) &= a_1 b_1 + a_1 b_2 + a_1 b_3 + \cdots + a_1 b_n \\ &\quad + a_2 b_1 + a_2 b_2 + a_2 b_3 + \cdots + a_2 b_n \\ &\quad + a_3 b_1 + a_3 b_2 + a_3 b_3 + \cdots + a_3 b_n \\ &\quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ &\quad + a_n b_1 + a_n b_2 + a_n b_3 + \cdots + a_n b_n \\ &\leq a_1 b_1 \\ &\quad + a_1 b_1 + a_2 b_2 + a_2 b_2 \\ &\quad + a_1 b_1 + a_3 b_3 + a_2 b_2 + a_3 b_3 + a_3 b_3 \\ &\quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ &\quad + a_1 b_1 + a_n b_n + a_2 b_2 + a_n b_n + \cdots + a_n b_n = n \sum_{i=1}^n a_i b_i. \end{aligned}$$

Equality holds iff we have equality in (4.8), i.e.  $a_1 = a_2 = \cdots = a_n$  or  $b_1 = b_2 = \cdots = b_n$ .  $\square$

**Note** *Chebishev's inequality* is also true in the case when  $a_1 \geq a_2 \geq \cdots \geq a_n$  and  $b_1 \geq b_2 \geq \cdots \geq b_n$ . But if  $a_1 \leq a_2 \leq \cdots \leq a_n$ ,  $b_1 \geq b_2 \geq \cdots \geq b_n$  (or the reverse) then we have

$$\left( \sum_{i=1}^n a_i \right) \left( \sum_{i=1}^n b_i \right) \geq n \sum_{i=1}^n a_i b_i.$$

Let us note that the inequality from Corollary 4.1 is true not just in case when  $n \in \mathbb{N}$ , but it is also true in the cases  $n > 1$ ,  $n \in \mathbb{Q}$  and  $n \in [1, \infty)$ ,  $n \in \mathbb{R}$ .

We prove this statement below in the case when  $n \geq 1$ ,  $n \in \mathbb{Q}$ , and the second case will be left to the reader.

**Corollary 4.6** *Let  $x > -1$  and  $r \geq 1$ ,  $r \in \mathbb{Q}$ . Then*

$$(1+x)^r \geq 1+rx.$$

*Proof* Let  $r = \frac{p}{q}$ ,  $\text{Gcd}(p, q) = 1$ . Then clearly  $p > q$ .

Let  $a_1 = a_2 = \dots = a_q = 1+rx$  and  $a_{q+1} = a_{q+2} = \dots = a_p = 1$ .

If  $1+rx \leq 0$ , then we are done.

So let us suppose that  $1+rx > 0$ .

Since  $AM \geq GM$  we have

$$\begin{aligned} 1+x &= \frac{px+p}{p} = \frac{q+rqx+p-q}{p} = \frac{q(1+rx)+p-q}{p} \\ &= \frac{a_1+a_2+\dots+a_q+a_{q+1}+\dots+a_p}{p} \geq \sqrt[p]{a_1a_2\cdots a_p} \\ &= \sqrt[p]{(1+rx)^q} = (1+rx)^{\frac{q}{p}} = (1+rx)^{\frac{1}{r}}, \end{aligned}$$

and we easily obtain  $(1+x)^r \geq 1+rx$ . □

**Corollary 4.7** *Let  $x > -1$  and  $\alpha \in [1, \infty)$ ,  $\alpha \in \mathbb{R}$ . Then*

$$(1+x)^\alpha \geq 1+\alpha x.$$

**Theorem 4.4** (Surányi's inequality) *Let  $a_1, a_2, \dots, a_n$  be non-negative real numbers, and let  $n$  be a positive integer. Then*

$$\begin{aligned} (n-1)(a_1^n + a_2^n + \dots + a_n^n) + na_1a_2\cdots a_n \\ \geq (a_1 + a_2 + \dots + a_n)(a_1^{n-1} + a_2^{n-1} + \dots + a_n^{n-1}). \end{aligned}$$

*Proof* We will use induction.

Due to the symmetry and homogeneity of the inequality we may assume that

$$a_1 \geq a_2 \geq \dots \geq a_{n+1} \quad \text{and} \quad a_1 + a_2 + \dots + a_n = 1.$$

For  $n = 1$  equality occurs.

Let us assume that for  $n = k$  the inequality holds, i.e.

$$(k-1)(a_1^k + a_2^k + \dots + a_k^k) + ka_1a_2\cdots a_k \geq a_1^{k-1} + a_2^{k-1} + \dots + a_k^{k-1}.$$

We need to prove that:

$$k \sum_{i=1}^k a_i^{k+1} + k a_{k+1}^{k+1} + k a_{k+1} \prod_{i=1}^k a_i + a_{k+1} \prod_{i=1}^k a_i - (1 + a_{k+1}) \left( \sum_{i=1}^k a_i^k + a_{k+1}^k \right) \geq 0.$$

But from the inductive hypothesis we have

$$(k-1)(a_1^k + a_2^k + \cdots + a_k^k) + k a_1 a_2 \cdots a_k \geq a_1^{k-1} + a_2^{k-1} + \cdots + a_k^{k-1}.$$

Hence

$$k a_{k+1} \prod_{i=1}^k a_i \geq a_{k+1} \sum_{i=1}^k a_i^{k-1} - (k-1) a_{k+1} \sum_{i=1}^k a_i^k.$$

Using this last inequality, it remains to prove that:

$$\begin{aligned} & \left( k \sum_{i=1}^k a_i^{k+1} - \sum_{i=1}^k a_i^k \right) - a_{k+1} \left( k \sum_{i=1}^k a_i^k - \sum_{i=1}^k a_i^{k-1} \right) \\ & + a_{k+1} \left( \prod_{i=1}^k a_i + (k-1) a_{k+1}^k - a_{k+1}^{k-1} \right) \geq 0. \end{aligned}$$

We prove that

$$a_{k+1} \left( \prod_{i=1}^k a_i + (k-1) a_{k+1}^k - a_{k+1}^{k-1} \right) \geq 0,$$

and

$$\left( k \sum_{i=1}^k a_i^{k+1} - \sum_{i=1}^k a_i^k \right) - a_{k+1} \left( k \sum_{i=1}^k a_i^k - \sum_{i=1}^k a_i^{k-1} \right) \geq 0.$$

We have

$$\begin{aligned} \prod_{i=1}^k a_i + (k-1) a_{k+1}^k - a_{k+1}^{k-1} &= \prod_{i=1}^k (a_i - a_{k+1} + a_{k+1}) + (k-1) a_{k+1}^k - a_{k+1}^{k-1} \\ &\geq a_{k+1}^k + a_{k+1}^{k-1} \cdot \sum_{i=1}^k (a_i - a_{k+1}) + (k-1) a_{k+1}^k - a_{k+1}^{k-1} \\ &= 0. \end{aligned}$$

The second inequality is equivalent to

$$k \sum_{i=1}^k a_i^{k+1} - \sum_{i=1}^k a_i^k \geq a_{k+1} \left( k \sum_{i=1}^k a_i^k - \sum_{i=1}^k a_i^{k-1} \right).$$

By *Chebyshev's inequality* we have

$$k \sum_{i=1}^k a_i^k \geq \sum_{i=1}^k a_i \sum_{i=1}^k a_i^{k-1} = \sum_{i=1}^k a_i^{k-1}, \quad \text{i.e.} \quad k \sum_{i=1}^k a_i^k - \sum_{i=1}^k a_i^{k-1} \geq 0,$$

and since  $a_1 + a_2 + \cdots + a_{k+1} = 1$ , by the assumption that  $a_1 \geq a_2 \geq \cdots \geq a_{k+1}$ , we deduce that

$$a_{k+1} \leq \frac{1}{k}.$$

So it is enough to prove that

$$k \sum_{i=1}^k a_i^{k+1} - \sum_{i=1}^k a_i^k \geq \frac{1}{k} \left( k \sum_{i=1}^k a_i^k - \sum_{i=1}^k a_i^{k-1} \right),$$

which is equivalent to

$$k \sum_{i=1}^k a_i^{k+1} + \frac{1}{k} \sum_{i=1}^k a_i^{k-1} \geq 2 \sum_{i=1}^k a_i^k.$$

Since  $AM \geq GM$  inequality we have that

$$k a_i^{k+1} + \frac{1}{k} a_i^{k-1} \geq 2 a_i^k \quad \text{for all } i.$$

Adding this inequalities for  $i = 1, 2, \dots, k$  we obtain the required inequality.  $\square$

**Exercise 4.1** Let  $x, y$  be positive real numbers. Prove the inequality

$$x^y + y^x \geq 1.$$

*Solution* We'll show that for every real number  $a, b \in (0, 1)$  we have

$$a^b \geq \frac{a}{a + b - ab}.$$

By *Bernoulli's inequality* we have

$$a^{1-b} = (1 + a - 1)^{1-b} \leq 1 + (a - 1)(1 - b) = a + b - ab,$$

i.e.

$$a^b \geq \frac{a}{a + b - ab}.$$

If  $x \geq 1$  or  $y \geq 1$  then the given inequality clearly holds.

So let  $0 < x, y < 1$ .

By the previous inequality we have

$$x^y + y^x \geq \frac{x}{x + y - xy} + \frac{y}{x + y - xy} = \frac{x + y}{x + y - xy} > \frac{x + y}{x + y} = 1.$$

**Exercise 4.2** Let  $a, b, c > 0$ . Prove *Nesbitt's inequality*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

*Solution 1* Applying the *Cauchy-Schwarz inequality* for



$$a_1 = \sqrt{b+c}, \quad a_2 = \sqrt{c+a}, \quad a_3 = \sqrt{a+b};$$

$$b_1 = \frac{1}{\sqrt{b+c}}, \quad b_2 = \frac{1}{\sqrt{c+a}}, \quad b_3 = \frac{1}{\sqrt{a+b}}$$

gives us

$$((b+c) + (c+a) + (a+b)) \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) \geq (1+1+1)^2 = 9,$$

i.e.

$$2(a+b+c) \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) \geq 9$$

$$\Leftrightarrow \frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b} \geq \frac{9}{2}$$

$$\Leftrightarrow \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{9}{2} - 3 = \frac{3}{2}.$$

Equality occurs iff  $(b+c)^2 = (c+a)^2 = (a+b)^2$ , i.e. iff  $a = b = c$ .

*Solution 2* We'll use *Chebyshev's inequality*.

Assume that  $a \geq b \geq c$ ; then  $\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$ .

Now by *Chebyshev's inequality* we get

$$(a+b+c) \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) \leq 3 \left( \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right). \quad (4.10)$$

Note that

$$(a+b+c) \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) = \frac{1}{2} ((b+c) + (c+a) + (a+b))$$

$$\times \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right).$$

Since  $AM \geq HM$  (the same thing in this case with *Cauchy–Schwarz*) we have

$$((b+c) + (c+a) + (a+b)) \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) \geq 9.$$

Therefore

$$(a+b+c) \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) \geq \frac{9}{2}. \quad (4.11)$$

By (4.10) and (4.11) we obtain

$$3 \left( \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right) \geq \frac{9}{2}, \quad \text{i.e.} \quad \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Equality occurs iff  $a = b = c$ .

**Exercise 4.3** Let  $a, b, c, d$  be positive real numbers. Prove the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}.$$

*Solution* By Corollary 4.3 we obtain

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{(1+1+2+4)^2}{a+b+c+d} = \frac{64}{a+b+c+d},$$

as required.

**Exercise 4.4** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{a^2}{3^3} + \frac{b^2}{4^3} + \frac{c^2}{5^3} \geq \frac{(a+b+c)^2}{6^3}.$$

*Solution* Note that  $3^3 + 4^3 + 5^3 = 6^3$ .

Taking

$$\begin{aligned} a_1 &= \frac{a}{\sqrt{3^3}}, & a_2 &= \frac{b}{\sqrt{4^3}}, & a_3 &= \frac{c}{\sqrt{5^3}}; \\ b_1 &= \sqrt{3^3}, & b_2 &= \sqrt{4^3}, & b_3 &= \sqrt{5^3}, \end{aligned}$$

by the *Cauchy–Schwarz inequality* we obtain

$$\left( \frac{a^2}{3^3} + \frac{b^2}{4^3} + \frac{c^2}{5^3} \right) (3^3 + 4^3 + 5^3) \geq (a+b+c)^2,$$

as required.

**Exercise 4.5** Let  $a, b, c$  be positive real numbers. Determine the minimal value of

$$\frac{3a}{b+c} + \frac{4b}{c+a} + \frac{5c}{a+b}.$$

*Solution* By the *Cauchy–Schwarz inequality* we have

$$\begin{aligned} & \frac{3a}{b+c} + \frac{4b}{c+a} + \frac{5c}{a+b} + (3+4+5) \\ &= (a+b+c) \left( \frac{3}{b+c} + \frac{4}{c+a} + \frac{5}{a+b} \right) \\ &= \frac{1}{2} ((b+c) + (c+a) + (a+b)) \left( \frac{3}{b+c} + \frac{4}{c+a} + \frac{5}{a+b} \right) \\ &\geq \frac{1}{2} (\sqrt{3} + \sqrt{4} + \sqrt{5})^2. \end{aligned}$$

Hence

$$\frac{3a}{b+c} + \frac{4b}{c+a} + \frac{5c}{a+b} \geq \frac{1}{2} (\sqrt{3} + \sqrt{4} + \sqrt{5})^2 - 12.$$

So the minimal value of the expression is  $\frac{1}{2}(\sqrt{3} + \sqrt{4} + \sqrt{5})^2 - 12$ , and it is reached if and only if  $\frac{b+c}{\sqrt{3}} = \frac{c+a}{2} = \frac{a+b}{\sqrt{5}}$ .

**Exercise 4.6** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a^2 + b^2}{a + b} + \frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} \geq a + b + c.$$

*Solution* By the *Cauchy–Schwarz inequality* (Corollary 4.3) we have

$$\begin{aligned} & \frac{a^2 + b^2}{a + b} + \frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} \\ &= \frac{a^2}{a + b} + \frac{b^2}{b + c} + \frac{c^2}{c + a} + \frac{b^2}{a + b} + \frac{c^2}{b + c} + \frac{a^2}{c + a} \\ &\geq \frac{(2(a + b + c))^2}{4(a + b + c)} = a + b + c. \end{aligned}$$

**Exercise 4.7** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{a}{b + 2c} + \frac{b}{c + 2a} + \frac{c}{a + 2b} \geq 1.$$

*Solution* Applying the *Cauchy–Schwarz inequality* we get

$$\begin{aligned} & \left( \frac{a}{b + 2c} + \frac{b}{c + 2a} + \frac{c}{a + 2b} \right) (a(b + 2c) + b(c + 2a) + c(a + 2b)) \\ &\geq (a + b + c)^2, \end{aligned}$$

hence

$$\frac{a}{b + 2c} + \frac{b}{c + 2a} + \frac{c}{a + 2b} \geq \frac{(a + b + c)^2}{3(ab + bc + ca)}.$$

So it suffices to show that

$$\frac{(a + b + c)^2}{3(ab + bc + ca)} \geq 1, \quad \text{i.e.} \quad (a + b + c)^2 \geq 3(ab + bc + ca),$$

which is equivalent to  $a^2 + b^2 + c^2 \geq ab + bc + ca$ , and clearly holds.

Equality occurs iff  $a = b = c$ .

**Exercise 4.8** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a}{b + 1} + \frac{b}{c + 1} + \frac{c}{a + 1} \geq \frac{3(a + b + c)}{3 + a + b + c}.$$

*Solution* By the *Cauchy–Schwarz inequality* (Corollary 4.3) we have

$$\begin{aligned}\frac{a}{b+1} + \frac{b}{c+1} + \frac{c}{a+1} &= \frac{a^2}{a(b+1)} + \frac{b^2}{b(c+1)} + \frac{c^2}{c(a+1)} \\ &\geq \frac{(a+b+c)^2}{a(b+1) + b(c+1) + c(a+1)} \\ &= \frac{(a+b+c)^2}{ab+bc+ca+a+b+c}.\end{aligned}\quad (4.12)$$

Also we have

$$ab+bc+ca \leq \frac{(a+b+c)^2}{3}.\quad (4.13)$$

By (4.12) and (4.13) we get

$$\frac{a}{b+1} + \frac{b}{c+1} + \frac{c}{a+1} \geq \frac{(a+b+c)^2}{\frac{(a+b+c)^2}{3} + a+b+c} = \frac{3(a+b+c)}{3+a+b+c}.$$

Equality occurs iff  $a = b = c$ .

**Exercise 4.9** Let  $a, b, c > 0$  be real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{\sqrt{3}}{2}.$$

*Solution* By the *Cauchy–Schwarz inequality* we have

$$\left( \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \right) ((b+c) + (c+a) + (a+b)) \geq (a+b+c)^2,$$

i.e.

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2}.\quad (4.14)$$

Furthermore

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca) \geq 3(ab+bc+ca) = 3,$$

i.e.

$$a+b+c \geq \sqrt{3}.\quad (4.15)$$

Using (4.14) and (4.15) we obtain the required inequality.

Equality occurs iff  $a = b = c = 1/\sqrt{3}$ .

**Exercise 4.10** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{a}{a+b^4+c^4} + \frac{b}{b+c^4+a^4} + \frac{c}{c+a^4+b^4} \leq 1.$$

*Solution* By the *Cauchy–Schwarz inequality* we have

$$\frac{a}{a+b^4+c^4} = \frac{a(a^3+2)}{(a+b^4+c^4)(a^3+1+1)} \leq \frac{a(a^3+2)}{(a^2+b^2+c^2)^2}.$$

Similarly we get

$$\frac{b}{b+c^4+a^4} \leq \frac{b(b^3+2)}{(a^2+b^2+c^2)^2} \quad \text{and} \quad \frac{c}{c+a^4+b^4} \leq \frac{c(c^3+2)}{(a^2+b^2+c^2)^2}.$$

Hence

$$\frac{a}{a+b^4+c^4} + \frac{b}{b+c^4+a^4} + \frac{c}{c+a^4+b^4} \leq \frac{a^4+b^4+c^4+2(a+b+c)}{(a^2+b^2+c^2)^2},$$

and we need to prove that

$$(a^2+b^2+c^2)^2 \geq a^4+b^4+c^4+2(a+b+c),$$

which is equivalent to

$$a^2b^2+b^2c^2+c^2a^2 \geq a+b+c.$$

By the well-known inequality  $a^2b^2+b^2c^2+c^2a^2 \geq abc(a+b+c)$  and  $abc = 1$ , we have

$$a^2b^2+b^2c^2+c^2a^2 \geq abc(a+b+c) = a+b+c,$$

as required.

**Exercise 4.11** Let  $a, b, c$  be positive real numbers such that  $a+b+c=1$ . Prove the inequality

$$(a+b)^2(1+2c)(2a+3c)(2b+3c) \geq 54abc.$$

*Solution* The given inequality can be rewritten as follows

$$(a+b)^2(1+2c)\left(2+3\frac{c}{a}\right)\left(2+3\frac{c}{b}\right) \geq 54c.$$

By the *Cauchy–Schwarz inequality* and  $AM \geq GM$  we have

$$\begin{aligned} \left(2+3\frac{c}{a}\right)\left(2+3\frac{c}{b}\right) &\geq \left(2+\frac{3c}{\sqrt{ab}}\right)^2 \geq \left(2+\frac{6c}{a+b}\right)^2 = \frac{(2(a+b)+6c)^2}{(a+b)^2} \\ &= \frac{(2(1-c)+6c)^2}{(a+b)^2} = \frac{4(1+2c)^2}{(a+b)^2}. \end{aligned}$$

Then we have

$$\begin{aligned} (a+b)^2(1+2c)\left(2+3\frac{c}{a}\right)\left(2+3\frac{c}{b}\right) &\geq (a+b)^2(1+2c)\frac{4(1+2c)^2}{(a+b)^2} \\ &= 4(1+2c)^3, \end{aligned}$$

and it remains to prove that

$$4(1+2c)^3 \geq 54c, \quad \text{i.e.} \quad (1+2c)^3 \geq \frac{27c}{2}.$$

By the  $AM \geq GM$  inequality we have

$$(1+2c)^3 = \left(\frac{1}{2} + \frac{1}{2} + 2c\right)^3 \geq 27 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 2c = \frac{27c}{2},$$

as required.

Equality occurs iff  $a = b = \frac{3}{8}, c = \frac{1}{4}$ .

**Exercise 4.12** Let  $a, b, c, d, e, f$  be positive real numbers. Prove the inequality

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+e} + \frac{d}{e+f} + \frac{e}{f+a} + \frac{f}{a+b} \geq 3.$$

*Solution* By the *Cauchy-Schwarz inequality* we have

$$\begin{aligned} & \frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+e} + \frac{d}{e+f} + \frac{e}{f+a} + \frac{f}{a+b} \\ &= \frac{a^2}{ab+ad} + \frac{b^2}{bc+bd} + \frac{c^2}{cd+ce} + \frac{d^2}{de+df} + \frac{e^2}{ef+ea} + \frac{f^2}{fa+fb} \\ &\geq \frac{(a+b+c+d+e+f)^2}{ab+ac+bc+bd+cd+ce+de+df+ef+ea+fa+fb}. \end{aligned} \quad (4.16)$$

Let

$$S = ab + ac + bc + bd + cd + ce + de + df + ef + ea + fa + fb.$$

Then

$$\begin{aligned} 2S &= (a+b+c+d+e+f)^2 \\ &\quad - (a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + 2ad + 2bd + 2cf). \end{aligned} \quad (4.17)$$

Also we have

$$\begin{aligned} & a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + 2ad + 2be + 2cf \\ &= (a+d)^2 + (b+e)^2 + (c+f)^2 \\ &\stackrel{QM \geq AM}{\geq} \frac{1}{3}(a+b+c+d+e+f)^2. \end{aligned} \quad (4.18)$$

Using (4.17) and (4.18) we get

$$\begin{aligned} 2S &= (a+b+c+d+e+f)^2 \\ &\quad - (a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + 2ad + 2bd + 2cf) \\ &\leq (a+b+c+d+e+f)^2 - \frac{1}{3}(a+b+c+d+e+f)^2 \\ &= \frac{2}{3}(a+b+c+d+e+f)^2, \end{aligned}$$

i.e.

$$\frac{(a+b+c+d+e+f)^2}{S} \geq 3. \quad (4.19)$$

Finally from (4.16) and (4.19) we obtain the required inequality.

Equality occurs iff  $a = b = c = d = e = f$ .

**Exercise 4.13** Let  $a, b, c \in \mathbb{R}^+$  such that  $\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \geq 1$ . Prove the inequality

$$a + b + c \geq ab + bc + ca.$$

*Solution* We'll use the *Cauchy–Schwarz inequality*.

We have

$$(a+b+1)(a+b+c^2) \geq (a+b+c)^2, \quad \text{i.e.} \quad \frac{1}{a+b+1} \leq \frac{a+b+c^2}{(a+b+c)^2}.$$

Analogously

$$\frac{1}{b+c+1} \leq \frac{b+c+a^2}{(a+b+c)^2} \quad \text{and} \quad \frac{1}{c+a+1} \leq \frac{c+a+b^2}{(a+b+c)^2}.$$

By the given condition we have

$$1 \leq \frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq \frac{a+b+c^2+b+c+a^2+c+a+b^2}{(a+b+c)^2},$$

i.e.

$$\begin{aligned} 2(a+b+c) &\geq (a+b+c)^2 - (a^2+b^2+c^2) \\ \Leftrightarrow \quad a+b+c &\geq ab+bc+ca. \end{aligned}$$

**Exercise 4.14** Let  $a, b, c$  be positive real numbers such that  $ab+bc+ca = 3$ . Prove the inequality

$$\frac{a(b^2+c^2)}{a^2+bc} + \frac{b(c^2+a^2)}{b^2+ca} + \frac{c(a^2+b^2)}{c^2+ab} \geq 3.$$

*Solution* Let  $x = a(b^2+c^2)$ ,  $y = b(c^2+a^2)$  and  $z = c(a^2+b^2)$ .

Then we have

$$\frac{x}{y+z}(b+c) = \frac{a(b^2+c^2)(b+c)}{b(c^2+a^2)+c(a^2+b^2)} = \frac{a(b^2+c^2)}{a^2+bc}.$$

Analogously we get

$$\frac{y}{z+x}(c+a) = \frac{b(c^2+a^2)}{b^2+ca} \quad \text{and} \quad \frac{z}{x+y}(a+b) = \frac{c(a^2+b^2)}{c^2+ab}.$$

By Corollary 4.5 and the previous identities we have

$$\begin{aligned} & \frac{a(b^2 + c^2)}{a^2 + bc} + \frac{b(c^2 + a^2)}{b^2 + ca} + \frac{c(a^2 + b^2)}{c^2 + ab} \\ &= \frac{x}{y+z}(b+c) + \frac{y}{z+x}(c+a) + \frac{z}{x+y}(a+b) \geq \sqrt{3(ab+bc+ca)} = 3. \end{aligned}$$

**Exercise 4.15** Let  $x, y, z \geq 0$  be real numbers. Prove the inequality

$$\sqrt{x^2 + 1} + \sqrt{y^2 + 1} + \sqrt{z^2 + 1} \geq \sqrt{6(x + y + z)}.$$

*Solution* According to Corollary 4.4 we have

$$\sqrt{x^2 + 1} + \sqrt{y^2 + 1} + \sqrt{z^2 + 1} \geq \sqrt{(x + y + z)^2 + 9}. \quad (4.20)$$

Applying  $AM \geq GM$  we deduce

$$(x + y + z)^2 + 9 \geq 2\sqrt{9(x + y + z)^2} = 6(x + y + z). \quad (4.21)$$

From (4.20) and (4.21) we get the required inequality.

Equality occurs if and only if  $x = y = z = 1$ .

**Exercise 4.16** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequalities

- (1)  $2(a^8 + b^8) \geq (a^3 + b^3)(a^5 + b^5)$ ;
- (2)  $3(a^8 + b^8 + c^8) \geq (a^3 + b^3 + c^3)(a^5 + b^5 + c^5)$ .

*Solution* (1) Let  $a \geq b$ . Then  $a^3 \geq b^3$  and  $a^5 \geq b^5$ .

Due to Chebishev's inequality we have

$$(a^3 + b^3)(a^5 + b^5) \leq 2(a^8 + b^8).$$

(2) Similarly to (1).

**Exercise 4.17** Let  $a, b$  and  $c$  be the lengths of the sides of a triangle, and  $\alpha, \beta, \gamma$  be its angles (in radians), respectively. Let  $s$  be the semi-perimeter of the triangle. Prove the inequality

$$\frac{b+c}{\alpha} + \frac{c+a}{\beta} + \frac{a+b}{\gamma} \geq \frac{12s}{\pi}.$$

*Solution* Without loss of generality we may assume that  $a \leq b \leq c$ . Then clearly  $\alpha \leq \beta \leq \gamma$ ,  $a+b \leq a+c \leq b+c$  and  $\frac{1}{\gamma} \leq \frac{1}{\beta} \leq \frac{1}{\alpha}$ .

Now by Chebishev's inequality we have

$$\begin{aligned} & ((a+b) + (b+c) + (c+a)) \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right) \\ & \leq 3 \left( (b+c) \frac{1}{\alpha} + (c+a) \frac{1}{\beta} + (a+b) \frac{1}{\gamma} \right), \end{aligned}$$



i.e.

$$\frac{b+c}{\alpha} + \frac{c+a}{\beta} + \frac{a+b}{\gamma} \geq \frac{4s}{3} \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right). \quad (4.22)$$

Using (4.22) and  $AM \geq HM$  we obtain

$$\frac{b+c}{\alpha} + \frac{c+a}{\beta} + \frac{a+b}{\gamma} \geq \frac{4s}{3} \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right) \geq \frac{4s}{3} \cdot \frac{9}{\alpha + \beta + \gamma} = \frac{12s}{\pi}.$$

Equality occurs iff  $a = b = c$ .

**Exercise 4.18** Let  $a, b, c, d \in \mathbb{R}^+$ . Prove the inequality

$$\begin{aligned} & \frac{a^3 + b^3 + c^3}{a + b + c} + \frac{a^3 + b^3 + d^3}{a + b + d} + \frac{a^3 + c^3 + d^3}{a + c + d} + \frac{b^3 + c^3 + d^3}{b + c + d} \\ & \geq a^2 + b^2 + c^2 + d^2. \end{aligned}$$

*Solution* Without loss of generality we may assume that  $a \geq b \geq c \geq d$ . Then clearly  $a^2 \geq b^2 \geq c^2 \geq d^2$ .

We'll use *Chebyshev's inequality*, i.e. we have

$$\begin{aligned} (a + b + c)(a^2 + b^2 + c^2) & \leq 3(a^3 + b^3 + c^3) \\ \Leftrightarrow \frac{a^3 + b^3 + c^3}{a + b + c} & \geq \frac{a^2 + b^2 + c^2}{3}. \end{aligned}$$

Similarly we get

$$\begin{aligned} \frac{a^3 + b^3 + d^3}{a + b + d} & \geq \frac{a^2 + b^2 + d^2}{3}, & \frac{a^3 + c^3 + d^3}{a + c + d} & \geq \frac{a^2 + c^2 + d^2}{3}, \\ \frac{b^3 + c^3 + d^3}{b + c + d} & \geq \frac{b^2 + c^2 + d^2}{3}. \end{aligned}$$

After adding these inequalities we get the required inequality.

**Exercise 4.19** Let  $a_1, a_2, \dots, a_n \in \mathbb{R}^+$  such that  $a_1 + a_2 + \dots + a_n = 1$ . Prove the inequality

$$\frac{a_1}{2 - a_1} + \frac{a_2}{2 - a_2} + \dots + \frac{a_n}{2 - a_n} \geq \frac{n}{2n - 1}.$$

*Solution* Without loss of generality we may assume that  $a_1 \geq a_2 \geq \dots \geq a_n$ .

Then

$$\frac{1}{2 - a_1} \geq \frac{1}{2 - a_2} \geq \dots \geq \frac{1}{2 - a_n}.$$

Now by *Chebyshev's inequality* we have

$$\begin{aligned} & (a_1 + a_2 + \dots + a_n) \left( \frac{1}{2 - a_1} + \frac{1}{2 - a_2} + \dots + \frac{1}{2 - a_n} \right) \\ & \leq n \left( \frac{a_1}{2 - a_1} + \frac{a_2}{2 - a_2} + \dots + \frac{a_n}{2 - a_n} \right), \end{aligned}$$

hence

$$\begin{aligned} \frac{a_1}{2-a_1} + \frac{a_2}{2-a_2} + \cdots + \frac{a_n}{2-a_n} &\geq \frac{1}{n} \left( \frac{1}{2-a_1} + \frac{1}{2-a_2} + \cdots + \frac{1}{2-a_n} \right) \\ &\geq \frac{1}{n} \cdot \frac{n^2}{2n - (a_1 + a_2 + \cdots + a_n)} = \frac{n}{2n-1}. \end{aligned}$$

Equality occurs if and only if  $a_1 = a_2 = \cdots = a_n = 1/n$ .

**Exercise 4.20** Let  $a, b, c, d$  be positive real numbers such that  $a + b + c + d = 4$ . Prove the inequality

$$\frac{1}{11+a^2} + \frac{1}{11+b^2} + \frac{1}{11+c^2} + \frac{1}{11+d^2} \leq \frac{1}{3}.$$

*Solution* Rewrite the given inequality as follows

$$\frac{1}{11+a^2} - \frac{1}{12} + \frac{1}{11+b^2} - \frac{1}{12} + \frac{1}{11+c^2} - \frac{1}{12} + \frac{1}{11+d^2} - \frac{1}{12} \leq 0,$$

i.e.

$$\frac{a^2-1}{11+a^2} + \frac{b^2-1}{11+b^2} + \frac{c^2-1}{11+c^2} + \frac{d^2-1}{11+d^2} \geq 0,$$

i.e.

$$(a-1)\frac{a+1}{11+a^2} + (b-1)\frac{b+1}{11+b^2} + (c-1)\frac{c+1}{11+c^2} + (d-1)\frac{d+1}{11+d^2} \geq 0. \quad (4.23)$$

Without loss of generality we may assume that  $a \geq b \geq c \geq d$ .

Then we have

$$a-1 \geq b-1 \geq c-1 \geq d-1 \quad \text{and} \quad \frac{a+1}{11+a^2} \geq \frac{b+1}{11+b^2} \geq \frac{c+1}{11+c^2} \geq \frac{d+1}{11+d^2}.$$

Now inequality (4.23) is a direct consequences of *Chebyshev's inequality*.

Equality occurs if and only if  $a = b = c = d = 1$ .



## Chapter 5

# Inequalities Between Means (General Case)

In Chap. 2 we discussed *mean inequalities* of two and three variables. In this section we will develop their generalization, i.e. we'll present an analogous theorem for an arbitrary number of variables.

These inequalities are of particular importance because they are part of the basic apparatus for proving more complicated inequalities.

**Theorem 5.1** (Mean inequalities) *Let  $a_1, a_2, \dots, a_n$  be positive real numbers. The numbers*

$$QM = \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}}, \quad AM = \frac{a_1 + a_2 + \dots + a_n}{n},$$

$$GM = \sqrt[n]{a_1 a_2 \dots a_n} \quad \text{and} \quad HM = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}},$$

*are called the quadratic, arithmetic, geometric and harmonic mean for the numbers  $a_1, a_2, \dots, a_n$ , respectively, and we have*

$$QM \geq AM \geq GM \geq HM.$$

*Equalities occur if and only if  $a_1 = a_2 = \dots = a_n$ .*

*Proof* Firstly, we'll show that  $AM \geq GM$ , i.e.

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n}. \quad (5.1)$$

Let

$$x_i = \frac{a_i}{\sqrt[n]{a_1 a_2 \dots a_n}}, \quad \text{for } i = 1, 2, \dots, n. \quad (5.2)$$

Then  $x_i > 0$  for each  $i = 1, 2, \dots, n$  and we have

$$x_1 x_2 \cdots x_n = 1.$$

Inequality (5.1) is equivalent to

$$\frac{a_1}{\sqrt[n]{a_1 a_2 \cdots a_n}} + \frac{a_2}{\sqrt[n]{a_1 a_2 \cdots a_n}} + \cdots + \frac{a_n}{\sqrt[n]{a_1 a_2 \cdots a_n}} \geq n,$$

i.e. to

$$x_1 + x_2 + \cdots + x_n \geq n, \quad \text{when } x_1 x_2 \cdots x_n = 1, \quad (5.3)$$

with equality if and only if  $x_1 = x_2 = \cdots = x_n = 1$ .

We'll prove inequality (5.3) by induction.

For  $n = 1$ , inequality (5.3) is true; it becomes equality.

If  $n = 2$  then  $x_1 x_2 = 1$  and since  $x_1 + x_2 \geq 2\sqrt{x_1 x_2}$  we get  $x_1 + x_2 \geq 2$ .

Hence (5.3) is true, and equality occurs iff  $x_1 = x_2 = 1$ .

Let assume that for  $n = k$ , and arbitrary positive real numbers  $x_1, x_2, \dots, x_k$  such that  $x_1 x_2 \cdots x_k = 1$ , we have  $x_1 + x_2 + \cdots + x_k \geq k$ , with equality if and only if  $x_1 = x_2 = \cdots = x_k = 1$ .

Let  $n = k + 1$  and  $x_1, x_2, \dots, x_{k+1}$  be arbitrary positive real numbers such that

$$x_1 x_2 \cdots x_{k+1} = 1.$$

If  $x_1 = x_2 = \cdots = x_{k+1} = 1$  then inequality (5.3) clearly holds.

Therefore, let us assume that there are numbers smaller than 1. Then clearly, there are also numbers which are greater than 1.

Without loss of generality we may assume that  $x_1 < 1$  and  $x_2 > 1$ .

Then, for the sequences  $x_1 x_2, x_3, \dots, x_{k+1}$  which contain  $k$  terms we have  $(x_1 x_2) x_3 \cdots x_{k+1} = 1$ , and according to the induction hypothesis we have that  $x_1 x_2 + x_3 + \cdots + x_{k+1} \geq k$ , and equality occurs iff  $x_1 x_2 = x_3 = \cdots = x_{k+1} = 1$ .

Now we have

$$\begin{aligned} x_1 + x_2 + \cdots + x_{k+1} &\geq x_1 x_2 + x_3 + \cdots + x_{k+1} + 1 + (x_2 - 1)(1 - x_1) \\ &\geq k + 1 + (x_2 - 1)(1 - x_1) \geq k + 1, \end{aligned}$$

with equality if and only if  $x_1 x_2 = x_3 = \cdots = x_{k+1} = 1$  and  $(x_2 - 1)(1 - x_1) = 0$ , i.e. iff  $x_1 = x_2 = \cdots = x_{k+1} = 1$ .

So, due to the *principle of mathematical induction*, we conclude that (5.3) is proved.

Thus by (5.2) we have  $\frac{a_1}{\sqrt[n]{a_1 a_2 \cdots a_n}} = \frac{a_2}{\sqrt[n]{a_1 a_2 \cdots a_n}} = \cdots = \frac{a_n}{\sqrt[n]{a_1 a_2 \cdots a_n}}$ , i.e.

$$a_1 = a_2 = \cdots = a_n.$$

Hence we have proved (5.1), and we are done.

We'll show that  $GM \geq HM$ , i.e.

$$\sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

By  $AM \geq GM$  it follows that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq n \sqrt[n]{\frac{1}{a_1} \frac{1}{a_2} \cdots \frac{1}{a_n}} = \frac{n}{\sqrt[n]{a_1 a_2 \cdots a_n}},$$

i.e. we have

$$\sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}},$$

and clearly equality holds if and only if  $\frac{1}{a_1} = \frac{1}{a_2} = \cdots = \frac{1}{a_n}$ , i.e.  $a_1 = a_2 = \cdots = a_n$ .

It is left to be shown that  $QM \geq AM$ , i.e.

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

We'll use the *Cauchy–Schwarz inequality* for the sequences  $(a_1, a_2, \dots, a_n)$  and  $(1, 1, \dots, 1)$ .

So we have

$$\begin{aligned} (a_1^2 + a_2^2 + \cdots + a_n^2)(1^2 + 1^2 + \cdots + 1^2) &\geq (a_1 + a_2 + \cdots + a_n)^2 \\ \Leftrightarrow n(a_1^2 + a_2^2 + \cdots + a_n^2) &\geq (a_1 + a_2 + \cdots + a_n)^2 \\ \Leftrightarrow \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} &\geq \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)^2 \\ \Leftrightarrow \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} &\geq \frac{a_1 + a_2 + \cdots + a_n}{n}. \end{aligned}$$

Equality holds if and only if  $\frac{a_1}{1} = \frac{a_2}{1} = \cdots = \frac{a_n}{1}$ , i.e.  $a_1 = a_2 = \cdots = a_n$ . □

**Exercise 5.1** Let  $a, b, c, d \in \mathbb{R}^+$  such that  $abcd = 1$ . Prove the inequality

$$a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd \geq 10.$$

*Solution* Since  $AM \geq GM$  we have

$$a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd \geq 10 \sqrt[10]{a^5 b^5 c^5 d^5} = 10.$$

Equality holds if and only if  $a = b = c = d = 1$ .

**Exercise 5.2** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$(a + b + c)^3 \geq a^3 + b^3 + c^3 + 24abc.$$

*Solution* We have

$$\begin{aligned} (a + b + c)^3 &= a^3 + b^3 + c^3 + 6abc + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) \\ &\geq a^3 + b^3 + c^3 + 6abc + 3 \cdot 6\sqrt[6]{a^6b^6c^6} = a^3 + b^3 + c^3 + 24abc. \end{aligned}$$

Equality holds if and only if  $a = b = c$ .

**Exercise 5.3** Let  $k \in \mathbb{N}$ , and  $a_1, a_2, \dots, a_n$  be positive real numbers such that  $a_1 + a_2 + \dots + a_n = 1$ . Prove the inequality

$$a_1^{-k} + a_2^{-k} + \dots + a_n^{-k} \geq n^{k+1}.$$

*Solution* Since  $AM \geq GM$  we have

$$\sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{a_1 + a_2 + \dots + a_n}{n} = \frac{1}{n}$$

or

$$n \leq \sqrt[n]{\frac{1}{a_1} \frac{1}{a_2} \dots \frac{1}{a_n}}.$$

Hence

$$n^k \leq \sqrt[n]{a_1^{-k} a_2^{-k} \dots a_n^{-k}} \leq \frac{a_1^{-k} + a_2^{-k} + \dots + a_n^{-k}}{n},$$

i.e.

$$a_1^{-k} + a_2^{-k} + \dots + a_n^{-k} \geq n^{k+1},$$

as required.

**Exercise 5.4** Let  $a, b, c, d \in \mathbb{R}^+$ . Prove the inequality

$$a^6 + b^6 + c^6 + d^6 \geq abcd(ab + bc + cd + da).$$

*Solution* We have

$$\begin{aligned} a^6 + b^6 + c^6 + d^6 &= \frac{1}{6}((2a^6 + 2b^6 + c^6 + d^6) + (2b^6 + 2c^6 + d^6 + a^6) \\ &\quad + (2c^6 + 2d^6 + a^6 + b^6) + (2d^6 + 2a^6 + b^6 + c^6)). \end{aligned}$$

Since  $AM \geq GM$  we have

$$\frac{2a^6 + 2b^6 + c^6 + d^6}{6} = \frac{a^6 + a^6 + b^6 + b^6 + c^6 + d^6}{6} \geq \sqrt[6]{a^{12}b^{12}c^6d^6} = a^2b^2cd.$$

Similarly we get

$$\frac{2b^6 + 2c^6 + d^6 + a^6}{6} \geq b^2 c^2 ad,$$

$$\frac{2c^6 + 2d^6 + a^6 + b^6}{6} \geq c^2 d^2 ab$$

and

$$\frac{2d^6 + 2a^6 + b^6 + c^6}{6} \geq d^2 a^2 bc.$$

Adding the last four inequalities we obtain the required inequality.

Equality holds if and only if  $a = b = c = d$ .

**Exercise 5.5** Let  $x, y, z \geq 2$  be real numbers. Prove the inequality

$$(y^3 + x)(z^3 + y)(x^3 + z) \geq 125xyz.$$

*Solution* We have

$$y^3 + x \geq 4y + x = y + y + y + y + x \geq 5\sqrt[5]{y^4x}.$$

Analogously

$$z^3 + y \geq 5\sqrt[5]{z^4y} \quad \text{and} \quad x^3 + z \geq 5\sqrt[5]{x^4z}.$$

Multiplying the last three inequalities gives us the required inequality.

## 5.1 Points of Incidence in Applications of the AM–GM Inequality

In this subsection we will consider characteristic examples in which we can use incorrectly the inequality  $AM \geq GM$ . Namely, a possible major route for the proper use of this inequality (the means inequalities) will be the fact that equality in these inequalities is achieved when all variables are equal. These points at which equality (all their coordinates are equal) of a given inequality is satisfied are called points of incidence. It is also important to note that symmetrical expressions achieve a minimum or maximum at a point of incidence.

**Exercise 5.6** Let  $x > 0$  be a real number. Find the minimum value of the expression

$$x + \frac{1}{x}.$$

*Solution* Since  $AM \geq GM$  we have

$$x + \frac{1}{x} \geq 2\sqrt{x \cdot \frac{1}{x}} = 2,$$

with equality iff  $x = \frac{1}{x}$ , i.e.  $x = 1$ .

$$\text{Thus } \min\{x + \frac{1}{x}\} = 2.$$



**Exercise 5.7** Let  $x \geq 3$  be a real number. Find the minimum value of the expression

$$x + \frac{1}{x}.$$

*Solution* In this case we cannot directly use the inequality  $AM \geq GM$  since the point  $x = 1$  doesn't belong to the domain  $[3, +\infty)$ .

We can easily show that the function  $f(x) = x + \frac{1}{x}$  is an increasing function on  $[3, +\infty)$ , so it follows that  $\min\{x + \frac{1}{x}\} = 3 + \frac{1}{3} = \frac{10}{3}$ .

Now we will show how we can use  $AM \geq GM$ .

Since we have equality in  $AM \geq GM$  if and only if all variables are equal, we deduce that we cannot use this inequality for the numbers  $x$  and  $\frac{1}{x}$  at the point of incidence  $x = 3$  since  $3 \neq \frac{1}{3}$ .

Assume that  $AM \geq GM$  is used for the couple  $(\frac{x}{\alpha}, \frac{1}{x})$  such that at the point of incidence  $x = 3$ , equality occurs, i.e.  $\frac{x}{\alpha} = \frac{1}{x}$ .

So it follows that  $\alpha = x^2 = 3^2 = 9$ .

According to this we transform  $x + \frac{1}{x}$  as follows

$$A = x + \frac{1}{x} = \frac{x}{9} + \frac{1}{x} + \frac{8}{9}x \geq 2\sqrt{\frac{x}{9} \cdot \frac{1}{x}} + \frac{8}{9}x = \frac{2}{3} + \frac{8}{9} \cdot 3 = \frac{10}{3}.$$

**Exercise 5.8** Let  $a, b > 0$  be real numbers such that  $a + b \leq 1$ . Find the minimum value of the expression

$$A = ab + \frac{1}{ab}.$$

*Solution* If we use  $AM \geq GM$  we get

$$A = ab + \frac{1}{ab} \geq 2\sqrt{ab \cdot \frac{1}{ab}} = 2,$$

and equality occurs if and only if  $ab = \frac{1}{ab}$ , i.e.  $ab = 1$ .

But then we have  $a + b \geq 2\sqrt{ab} = 2$ , contradicting  $a + b \leq 1$ .

If we take  $x = \frac{1}{ab}$ , then we have  $x = \frac{1}{ab} \geq \frac{4}{(a+b)^2} \geq \frac{4}{1^2} = 4$ .

Thus we may consider an equivalent problem of the given problem:

*Find the minimum of the function  $A = x + \frac{1}{x}$ , with  $x \geq 4$ .*

Point of incidence is  $x = 4$ .

So we have  $\frac{x}{\alpha} = \frac{1}{x}$ , from which it follows that  $\alpha = x^2 = 16$ .

Then we transform as follows

$$A = x + \frac{1}{x} = \frac{x}{16} + \frac{1}{x} + \frac{15}{16}x \geq 2\sqrt{\frac{x}{16} \cdot \frac{1}{x}} + \frac{15}{16}x \geq 2 \cdot \frac{1}{4} + \frac{15}{16} \cdot 4 = \frac{17}{4}.$$

Equality holds if and only if  $x = 4$ , i.e.  $a = b = 1/2$ .

**Exercise 5.9** Let  $a, b, c > 0$  be real numbers such that  $a + b + c \leq \frac{3}{2}$ . Find the minimum value of the expression

$$A = a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

*Solution* If we use  $AM \geq GM$  we get

$$A = a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 6\sqrt[6]{abc \cdot \frac{1}{abc}} = 6,$$

with equality if and only if  $a = b = c = 1$ .

But then  $a + b + c = 3 > \frac{3}{2}$ , a contradiction.

Since  $A$  is a symmetrical expression on  $a, b$  and  $c$  we estimate that  $\min A$  occurs at  $a = b = c$ , i.e. at  $a = b = c = 1/2$ .

Therefore for a point of incidence we have  $\frac{1}{aa} = \frac{1}{ab} = \frac{1}{ac} = a = b = c = 1/2$ , and it follows that  $\alpha = \frac{1}{a^2} = 4$ .

Now we have

$$\begin{aligned} A &= a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \left( a + b + c + \frac{1}{4a} + \frac{1}{4b} + \frac{1}{4c} \right) + \frac{3}{4} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \\ &\geq 6\sqrt[6]{abc \cdot \frac{1}{(4a)(4b)(4c)}} + \frac{3}{4} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = 3 + \frac{3}{4} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \\ &\geq 3 + \frac{3}{4} \cdot \frac{9}{a + b + c} \geq 3 + \frac{27}{4} \cdot \frac{1}{3/2} = \frac{15}{2}. \end{aligned}$$

So  $\min A = \frac{15}{2}$ , for  $a = b = c = 1/2$ .

**Exercise 5.10** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Find the minimum value of the expression

$$abc + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

*Solution* By the inequality  $AM \geq GM$  we get

$$abc + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 4\sqrt[4]{abc \cdot \frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{c}} = 4,$$

with equality if and only if  $abc = \frac{1}{a} = \frac{1}{b} = \frac{1}{c}$ , from which we easily deduce that  $a = b = c = 1$  and then  $a + b + c = 3$ , a contradiction since  $a + b + c = 1$ .

Since  $abc + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$  is symmetrical with respect to  $a, b$  and  $c$  we estimate that the minimal value occurs when  $a = b = c$ , i.e.  $a = b = c = 1/3$ , since  $a + b + c = 1$ .

Let  $abc = \frac{1}{aa} = \frac{1}{ab} = \frac{1}{ac}$ , from which we obtain  $\alpha = \frac{1}{a^2bc} = 81$ .

Therefore let us rewrite the given expression as follows

$$abc + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = abc + \frac{1}{81a} + \frac{1}{81b} + \frac{1}{81c} + \frac{80}{81} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right). \quad (5.4)$$

By  $AM \geq GM$  and  $AM \geq HM$  we have

$$abc + \frac{1}{81a} + \frac{1}{81b} + \frac{1}{81c} \geq 4 \sqrt[4]{abc \cdot \frac{1}{81a} \cdot \frac{1}{81b} \cdot \frac{1}{81c}} = \frac{4}{27} \quad (5.5)$$

and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c} = 9. \quad (5.6)$$

By (5.4), (5.5) and (5.6) we have

$$abc + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{4}{27} + \frac{80}{9} = \frac{244}{27},$$

with equality if and only if  $a = b = c = \frac{1}{3}$ .

**Exercise 5.11** Let  $a, b, c, d > 0$  be real numbers. Find the minimum value of the expression

$$\begin{aligned} & \frac{a}{b+c+d} + \frac{b}{c+d+a} + \frac{c}{d+a+b} + \frac{d}{a+b+c} + \frac{b+c+d}{a} + \frac{c+d+a}{b} \\ & + \frac{a+b+d}{c} + \frac{a+b+c}{d}. \end{aligned}$$

*Solution* Let us denote

$$\begin{aligned} A = & \frac{a}{b+c+d} + \frac{b}{c+d+a} + \frac{c}{d+a+b} + \frac{d}{a+b+c} + \frac{b+c+d}{a} + \frac{c+d+a}{b} \\ & + \frac{a+b+d}{c} + \frac{a+b+c}{d}. \end{aligned}$$

If we use  $AM \geq GM$  we get  $A \geq 8$ , with equality iff

$$\begin{aligned} \frac{a}{b+c+d} &= \frac{b}{c+d+a} = \frac{c}{d+a+b} = \frac{d}{a+b+c} = \frac{b+c+d}{a} = \frac{c+d+a}{b} \\ &= \frac{a+b+d}{c} = \frac{a+b+c}{d}, \end{aligned}$$

i.e.

$$a = b + c + d, \quad b = c + d + a, \quad c = d + a + b \quad \text{and} \quad d = a + b + c.$$

After adding the last identities we deduce  $a + b + c + d = 3(a + b + c + d)$ , i.e.  $3 = 1$ , a contradiction.

Since  $A$  is a symmetrical expression with variables  $a, b, c, d$ , it follows that the minimum (maximum) will occur at the point of incidence  $a = b = c = d > 0$ .

Suppose  $a = b = c = d > 0$ .

We have

$$\frac{a}{b+c+d} = \frac{b}{c+d+a} = \frac{c}{d+a+b} = \frac{d}{a+b+c} = \frac{1}{3}$$

and

$$\frac{b+c+d}{\alpha a} = \frac{c+d+a}{\alpha b} = \frac{a+b+d}{\alpha c} = \frac{a+b+c}{\alpha d} = \frac{3}{\alpha},$$

i.e.  $\frac{1}{3} = \frac{3}{\alpha}$ , and it follows that  $\alpha = 9$ .

Therefore

$$\begin{aligned} A &= \frac{a}{b+c+d} + \frac{b}{c+d+a} + \frac{c}{d+a+b} + \frac{d}{a+b+c} + \frac{b+c+d}{9a} \\ &\quad + \frac{c+d+a}{9b} + \frac{a+b+d}{9c} + \frac{a+b+c}{9d} \\ &\quad + \frac{8}{9} \left( \frac{b+c+d}{a} + \frac{c+d+a}{b} + \frac{a+b+d}{c} + \frac{a+b+c}{d} \right) \\ &\geq \frac{8}{3} + \frac{8}{9}(2+2+2+2+2+2) = \frac{40}{3}. \end{aligned}$$

**Exercise 5.12** Let  $a, b, c \geq 0$  be real numbers such that  $a + b + c = 1$ . Find the maximum value of the expression  $A = \sqrt[3]{a+b} + \sqrt[3]{b+c} + \sqrt[3]{c+a}$ .

*Solution* Since  $AM \geq GM$  we have

$$\sqrt[3]{a+b} = \sqrt[3]{(a+b) \cdot 1 \cdot 1} \leq \frac{a+b+1+1}{3} = \frac{a+b+2}{3}.$$

Similarly

$$\sqrt[3]{b+c} \leq \frac{b+c+2}{3} \quad \text{and} \quad \sqrt[3]{c+a} \leq \frac{c+a+2}{3}.$$

Thus it follows that

$$A \leq \frac{a+b+2}{3} + \frac{b+c+2}{3} + \frac{c+a+2}{3} = \frac{2(a+b+c)}{3} + 2 = \frac{8}{3},$$

with equality iff  $a+b = b+c = c+a = 1$ , i.e.  $a = b = c = 1/2$ .

But then  $a+b+c = 3/2 \neq 1$ , a contradiction.

Since  $A$  is symmetrical expression in  $a, b, c$ , we estimate that the minimum (maximum) will occur at the point of incidence  $a = b = c$ , i.e.  $a = b = c = 1/3$ .

Clearly  $a+b = b+c = c+a = 2/3$ .

Since  $AM \geq GM$  we have

$$\sqrt[3]{a+b} = \sqrt[3]{(a+b) \cdot \frac{2}{3} \cdot \frac{2}{3}} \cdot \sqrt[3]{\frac{9}{4}} \leq \sqrt[3]{\frac{9}{4}} \cdot \frac{a+b+\frac{2}{3}+\frac{2}{3}}{3} = \sqrt[3]{\frac{9}{4}} \cdot \frac{3(a+b)+4}{9}.$$

Similarly we get

$$\sqrt[3]{b+c} \leq \sqrt[3]{\frac{9}{4}} \cdot \frac{3(b+c)+4}{9} \quad \text{and} \quad \sqrt[3]{c+a} \leq \sqrt[3]{\frac{9}{4}} \cdot \frac{3(c+a)+4}{9}.$$

Adding the last three inequalities gives us

$$\begin{aligned} A &\leq \sqrt[3]{\frac{9}{4}} \cdot \left( \frac{3(a+b)+4}{9} + \frac{3(b+c)+4}{9} + \frac{3(c+a)+4}{9} \right) \\ &= \sqrt[3]{\frac{9}{4}} \cdot \frac{6(a+b+c)+12}{9} = \sqrt[3]{18}. \end{aligned}$$

So  $\max A = \sqrt[3]{18}$ , and it occurs iff  $a+b = b+c = c+a = 2/3$ , i.e.  $a = b = c = 1/3$ .

**Exercise 5.13** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 6$ . Prove the inequality

$$\sqrt[3]{ab+bc} + \sqrt[3]{bc+ca} + \sqrt[3]{ca+ab} \leq 6.$$

*Solution 1* Since we have a symmetrical expression we estimate that the maximum value of  $\sqrt[3]{ab+bc} + \sqrt[3]{bc+ca} + \sqrt[3]{ca+ab}$  will occur at the point of incidence  $a = b = c = 2$  and then clearly we have  $ab + bc = 8$ .

By the inequality  $AM \geq GM$  we get

$$\sqrt[3]{ab+bc} = \frac{\sqrt[3]{(ab+bc) \cdot 8 \cdot 8}}{4} \leq \frac{1}{4} \left( \frac{(ab+bc) + 8 + 8}{3} \right).$$

Similarly we obtain

$$\sqrt[3]{bc+ca} \leq \frac{1}{4} \left( \frac{(bc+ca) + 8 + 8}{3} \right) \quad \text{and} \quad \sqrt[3]{ca+ab} \leq \frac{1}{4} \left( \frac{(ca+ab) + 8 + 8}{3} \right).$$

Adding the last three inequalities gives us

$$\sqrt[3]{ab+bc} + \sqrt[3]{bc+ca} + \sqrt[3]{ca+ab} \leq \frac{1}{4} \left( \frac{2(ab+bc+ca) + 48}{3} \right). \quad (5.7)$$

Since  $ab + bc + ca \leq \frac{(a+b+c)^2}{3} = 12$  by (5.7) we get

$$\sqrt[3]{ab+bc} + \sqrt[3]{bc+ca} + \sqrt[3]{ca+ab} \leq \frac{1}{4} \left( \frac{24 + 48}{3} \right) = 6.$$

Equality occurs if and only if  $ab + bc = bc + ca = ca + ab = 8$ , i.e.  $a = b = c = 2$ .

**Solution 2** The given inequality is equivalent to

$$\sqrt[3]{b(a+c)} + \sqrt[3]{c(b+a)} + \sqrt[3]{a(c+b)} \leq 6$$

i.e.

$$\sqrt[3]{b(6-b)} + \sqrt[3]{c(6-c)} + \sqrt[3]{a(6-a)} \leq 6. \quad (5.8)$$

Since at the point of incidence  $a = b = c = 2$  we have  $2a = 6 - a = 4$  by  $AM \geq GM$  we deduce

$$\sqrt[3]{a(6-a)} = \frac{\sqrt[3]{2a \cdot (6-a) \cdot 4}}{2} \leq \frac{2a + 6 - a + 4}{6} = \frac{a+10}{6}.$$

Analogously we obtain

$$\sqrt[3]{b(6-b)} \leq \frac{b+10}{6} \quad \text{and} \quad \sqrt[3]{c(6-c)} \leq \frac{c+10}{6}.$$

After adding the last three inequalities we get

$$\sqrt[3]{b(6-b)} + \sqrt[3]{c(6-c)} + \sqrt[3]{a(6-a)} \leq \frac{a+b+c+30}{6} = \frac{36}{6} = 6.$$

Equality occurs if and only if  $a = b = c = 2$ .

**Exercise 5.14** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\sqrt[3]{a^2 + bc} + \sqrt[3]{b^2 + ca} + \sqrt[3]{c^2 + ab} \leq 3\sqrt[3]{2}.$$

**Solution** Since we have a symmetrical expression we estimate that the maximum value will occur at the point of incidence  $a = b = c = 1$ . Then we have  $a^2 + bc = 2$ .

By the inequality  $AM \geq GM$  we get

$$\sqrt[3]{a^2 + bc} = \frac{1}{\sqrt[3]{4}} \cdot \sqrt[3]{(a^2 + bc) \cdot 2 \cdot 2} \leq \frac{1}{\sqrt[3]{4}} \left( \frac{a^2 + bc + 4}{3} \right).$$

Similarly we obtain

$$\sqrt[3]{b^2 + ca} \leq \frac{1}{\sqrt[3]{4}} \left( \frac{b^2 + ca + 4}{3} \right) \quad \text{and} \quad \sqrt[3]{c^2 + ab} \leq \frac{1}{\sqrt[3]{4}} \left( \frac{c^2 + ab + 4}{3} \right).$$

Adding the last three inequalities gives us

$$\begin{aligned} \sqrt[3]{a^2 + bc} + \sqrt[3]{b^2 + ca} + \sqrt[3]{c^2 + ab} &\leq \frac{1}{3\sqrt[3]{4}} (a^2 + b^2 + c^2 + ab + bc + ca + 12) \\ &\leq \frac{1}{3\sqrt[3]{4}} (2(a^2 + b^2 + c^2) + 12) = \frac{18}{3\sqrt[3]{4}} = 3\sqrt[3]{2}. \end{aligned}$$

Equality occurs if and only if  $a = b = c = 1$ .

**Exercise 5.15** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\sqrt[4]{5a^2 + 4(b+c) + 3} + \sqrt[4]{5b^2 + 4(c+a) + 3} + \sqrt[4]{5c^2 + 4(a+b) + 3} \leq 6.$$

*Solution* At the point of incidence  $a = b = c = 1$  we have  $5a^2 + 4(b+c) + 3 = 16$ .

By the inequality  $AM \geq GM$  we get

$$\begin{aligned} \sqrt[4]{5a^2 + 4(b+c) + 3} &= \frac{\sqrt[4]{(5a^2 + 4(b+c) + 3) \cdot 16^3}}{8} \\ &\leq \frac{1}{32} (5a^2 + 4(b+c) + 3 + 3 \cdot 16) \\ &= \frac{5a^2 + 4(b+c) + 51}{32}. \end{aligned}$$

Similarly we obtain

$$\sqrt[4]{5b^2 + 4(c+a) + 3} \leq \frac{5b^2 + 4(c+a) + 51}{32}$$

and

$$\sqrt[4]{5c^2 + 4(a+b) + 3} \leq \frac{5c^2 + 4(a+b) + 51}{32}.$$

Adding the last three inequalities gives us

$$\begin{aligned} &\sqrt[4]{5a^2 + 4(b+c) + 3} + \sqrt[4]{5b^2 + 4(c+a) + 3} + \sqrt[4]{5c^2 + 4(a+b) + 3} \\ &\leq \frac{5(a^2 + b^2 + c^2) + 8(a+b+c) + 153}{32}. \end{aligned}$$

Since  $a^2 + b^2 + c^2 = 3$  we have  $a + b + c \leq \sqrt{3(a^2 + b^2 + c^2)} = 3$ , and by the last inequality we obtain

$$\begin{aligned} &\sqrt[4]{5a^2 + 4(b+c) + 3} + \sqrt[4]{5b^2 + 4(c+a) + 3} + \sqrt[4]{5c^2 + 4(a+b) + 3} \\ &\leq \frac{5 \cdot 3 + 8 \cdot 3 + 153}{32} = \frac{192}{32} = 6. \end{aligned}$$

Equality occurs if and only if  $a = b = c = 1$ .

## Chapter 6

# The Rearrangement Inequality

In this section we will introduce one really useful inequality called the *rearrangement inequality*. This inequality has a very broad and easy use in proving other inequalities.

**Theorem 6.1** (Rearrangement inequality) *Let  $a_1 \leq a_2 \leq \dots \leq a_n$  and  $b_1 \leq b_2 \leq \dots \leq b_n$  be real numbers. For any permutation  $(x_1, x_2, \dots, x_n)$  of  $(a_1, a_2, \dots, a_n)$  we have the following inequalities:*

$$\begin{aligned} a_1 b_1 + a_2 b_2 + \dots + a_n b_n &\geq x_1 b_1 + x_2 b_2 + \dots + x_n b_n \\ &\geq a_n b_1 + a_{n-1} b_2 + \dots + a_1 b_n. \end{aligned}$$

In case when  $a_1 < a_2 < \dots < a_n$  and  $b_1 < b_2 < \dots < b_n$  there is a simple necessary and sufficient condition for equality in either of the inequalities. The left inequality becomes equality only if  $(x_1, x_2, \dots, x_n)$  matches  $(a_1, a_2, \dots, a_n)$ , and the right inequality becomes equality only if  $(x_1, x_2, \dots, x_n)$  matches  $(a_n, a_{n-1}, \dots, a_1)$ .

**Corollary 6.1** *Let  $a_1, a_2, \dots, a_n$  be real numbers and let  $(x_1, x_2, \dots, x_n)$  be a permutation of  $(a_1, a_2, \dots, a_n)$ . Then*

$$a_1^2 + a_2^2 + \dots + a_n^2 \geq a_1 x_1 + a_2 x_2 + \dots + a_n x_n.$$

**Exercise 6.1** Let  $a, b$  and  $c$  be positive real numbers. Prove *Nesbitt's inequality*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$



**Solution** Without loss of generality we may assume that  $a \geq b \geq c$ . Then clearly

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}.$$

By the *rearrangement inequality* we deduce

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{b}{b+c} + \frac{c}{c+a} + \frac{a}{a+b}$$

and

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{c}{b+c} + \frac{a}{c+a} + \frac{b}{a+b}.$$

Adding the last two inequalities gives us

$$2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \geq 3 \quad \text{or} \quad \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

**Exercise 6.2** Let  $a_1 \leq a_2 \leq \dots \leq a_n$  and  $b_1 \leq b_2 \leq \dots \leq b_n$  be two sequences of real numbers and let  $(c_1, c_2, \dots, c_n)$  be a permutation of  $(b_1, b_2, \dots, b_n)$ . Prove that

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 + \dots + (a_n - b_n)^2 \leq (a_1 - c_1)^2 + (a_2 - c_2)^2 + \dots + (a_n - c_n)^2.$$

**Solution** Note that  $b_1^2 + b_2^2 + \dots + b_n^2 = c_1^2 + c_2^2 + \dots + c_n^2$ .

So it suffices to prove that

$$a_1 c_1 + a_2 c_2 + \dots + a_n c_n \leq a_1 b_1 + a_2 b_2 + \dots + a_n b_n,$$

which is true due to the *rearrangement inequality*.

**Exercise 6.3** Let  $a_1, a_2, \dots, a_n$  be different positive integers. Prove the inequality

$$\frac{a_1}{1^2} + \frac{a_2}{2^2} + \dots + \frac{a_n}{n^2} \geq 1 + \frac{1}{2} + \dots + \frac{1}{n}.$$

**Solution** Let  $(x_1, x_2, \dots, x_n)$  be a permutation of  $(a_1, a_2, \dots, a_n)$  such that  $x_1 \leq x_2 \leq \dots \leq x_n$ .

Then clearly  $x_i \geq i$  for each  $i = 1, 2, \dots, n$  and  $\frac{1}{1^2} \geq \frac{1}{2^2} \geq \dots \geq \frac{1}{n^2}$ .

By the *rearrangement inequality* and the previous conclusion we obtain

$$\frac{a_1}{1^2} + \frac{a_2}{2^2} + \dots + \frac{a_n}{n^2} \geq \frac{x_1}{1^2} + \frac{x_2}{2^2} + \dots + \frac{x_n}{n^2} \geq 1 + \frac{2}{2^2} + \dots + \frac{n}{n^2} = 1 + \frac{1}{2} + \dots + \frac{1}{n}.$$

**Exercise 6.4** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove the inequality

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3abc.$$

**Solution** Without loss of generality we may assume that  $a \geq b \geq c$ . Then very easily we can verify that

$$c(a + b - c) \geq b(c + a - b) \geq a(b + c - a).$$

Applying the *rearrangement inequality* we obtain the following inequalities

$$\begin{aligned} & a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \\ & \leq ba(b + c - a) + cb(c + a - b) + ac(a + b - c) \end{aligned}$$

and

$$\begin{aligned} & a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \\ & \leq ca(b + c - a) + ab(c + a - b) + bc(a + b - c). \end{aligned}$$

Adding the last two inequalities gives us the required result.

**Exercise 6.5** Let  $a, b, c$  be real numbers. Prove the inequality

$$a^5 + b^5 + c^5 \geq a^4b + b^4c + c^4a.$$

**Solution 1** Without loss of generality we may assume that  $a \geq b \geq c$ , and then clearly  $a^4 \geq b^4 \geq c^4$  (since the given inequality is cyclic we also need to consider the case when  $c \geq b \geq a$ , which is analogous).

Now by the *rearrangement inequality* we get the required inequality. Equality occurs iff  $a = b = c$ .

**Solution 2** Since  $AM \geq GM$  we obtain the following inequalities:

$$\begin{aligned} a^5 + a^5 + a^5 + a^5 + b^5 & \geq 5a^4b, \\ b^5 + b^5 + b^5 + b^5 + c^5 & \geq 5b^4c, \\ c^5 + c^5 + c^5 + c^5 + a^5 & \geq 5c^4a, \end{aligned}$$

and adding the previous three inequalities yields required inequality. Equality occurs iff  $a = b = c$ .

**Exercise 6.6** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

**Solution** Without loss of generality we may assume that  $a \geq b \geq c$ .

Let  $x = \frac{1}{a}$ ,  $y = \frac{1}{b}$ ,  $z = \frac{1}{c}$ . Then clearly  $xyz = 1$ .

We have

$$\begin{aligned}\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} &= \frac{x^3}{1/y + 1/z} + \frac{y^3}{1/z + 1/x} + \frac{z^3}{1/x + 1/y} \\ &= \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}.\end{aligned}$$

Since  $c \leq b \leq a$  we have  $x \leq y \leq z$ .

So clearly  $x + y \leq z + x \leq y + z$  and  $\frac{x}{y+z} \leq \frac{y}{z+x} \leq \frac{z}{x+y}$ .

Now by the *rearrangement inequality* we get the following inequalities

$$\begin{aligned}\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} &\geq \frac{xy}{y+z} + \frac{yz}{z+x} + \frac{zx}{x+y}, \\ \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} &\geq \frac{xz}{y+z} + \frac{yx}{z+x} + \frac{zy}{x+y}.\end{aligned}$$

So we obtain

$$\begin{aligned}&2 \left( \frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \right) \\ &= 2 \left( \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \right) \\ &\geq \frac{xy}{y+z} + \frac{yz}{z+x} + \frac{zx}{x+y} + \frac{xz}{y+z} + \frac{yx}{z+x} + \frac{zy}{x+y} \\ &= x + y + z \geq 3\sqrt[3]{xyz} = 3,\end{aligned}$$

as required.

**Exercise 6.7** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a^2 + c^2}{b} + \frac{b^2 + a^2}{c} + \frac{c^2 + b^2}{a} \geq 2(a + b + c).$$

*Solution* Since the given inequality is symmetric, without loss of generality we may assume that  $a \geq b \geq c$ . Then clearly

$$a^2 \geq b^2 \geq c^2 \quad \text{and} \quad \frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a}.$$

By the *rearrangement inequality* we have

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} = a^2 \cdot \frac{1}{b} + b^2 \cdot \frac{1}{c} + c^2 \cdot \frac{1}{a} \geq a^2 \cdot \frac{1}{a} + b^2 \cdot \frac{1}{b} + c^2 \cdot \frac{1}{c} = a + b + c \quad (6.1)$$

and

$$\frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b} = a^2 \cdot \frac{1}{c} + b^2 \cdot \frac{1}{a} + c^2 \cdot \frac{1}{b} \geq a^2 \cdot \frac{1}{a} + b^2 \cdot \frac{1}{b} + c^2 \cdot \frac{1}{c} = a + b + c. \quad (6.2)$$

Adding (6.1) and (6.2) yields the required inequality.

Equality occurs if and only if  $a = b = c$ .

**Exercise 6.8** Let  $x, y, z > 0$  be real numbers. Prove the inequality

$$\frac{x^2 - z^2}{y + z} + \frac{y^2 - x^2}{z + x} + \frac{z^2 - y^2}{x + y} \geq 0.$$

*Solution* We need to prove that  $\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{z^2}{y+z} + \frac{x^2}{z+x} + \frac{y^2}{x+y}$ .

Without loss of generality we may assume that  $x \geq y \geq z$  (since the given inequality is cyclic we also will consider the case  $z \geq y \geq x$ ).

Then clearly  $x^2 \geq y^2 \geq z^2$  and  $\frac{1}{y+z} \geq \frac{1}{z+x} \geq \frac{1}{x+y}$ .

By the *rearrangement inequality* we have

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{z^2}{y+z} + \frac{x^2}{z+x} + \frac{y^2}{x+y},$$

as required.

If we assume that  $z \geq y \geq x$ , then  $z^2 \geq y^2 \geq x^2$  and  $\frac{1}{x+y} \geq \frac{1}{x+z} \geq \frac{1}{z+y}$ .

By the *rearrangement inequality* we obtain

$$\begin{aligned} \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} &= z^2 \cdot \frac{1}{x+y} + x^2 \cdot \frac{1}{y+z} + y^2 \cdot \frac{1}{z+x} \\ &\geq z^2 \cdot \frac{1}{y+z} + x^2 \cdot \frac{1}{z+x} + y^2 \cdot \frac{1}{x+y} \\ &= \frac{z^2}{y+z} + \frac{x^2}{z+x} + \frac{y^2}{x+y}. \end{aligned}$$

Equality occurs if and only if  $x = y = z$ .

**Exercise 6.9** Let  $x, y, z$  be positive real numbers. Prove the inequality

$$\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} \geq x + y + z.$$

*Solution* Since the given inequality is symmetric we may assume that  $x \geq y \geq z$ .

Then

$$x^3 \geq y^3 \geq z^3 \quad \text{and} \quad \frac{1}{yz} \geq \frac{1}{zx} \geq \frac{1}{xy}.$$

By the *rearrangement inequality* we have

$$\begin{aligned}\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} &= x^3 \cdot \frac{1}{yz} + y^3 \cdot \frac{1}{zx} + z^3 \cdot \frac{1}{xy} \\ &\geq x^3 \cdot \frac{1}{xy} + y^3 \cdot \frac{1}{yz} + z^3 \cdot \frac{1}{zx} = \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}.\end{aligned}\quad (6.3)$$

We will prove that

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x + y + z. \quad (6.4)$$

Let  $x \geq y \geq z$ .

Then  $x^2 \geq y^2 \geq z^2$  and  $\frac{1}{z} \geq \frac{1}{y} \geq \frac{1}{x}$  (since inequality (6.4) is cyclic we also need to consider the case  $z \geq y \geq x$ ).

By the *rearrangement inequality* we obtain

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq \frac{x^2}{x} + \frac{y^2}{y} + \frac{z^2}{z} = x + y + z.$$

The case when  $z \geq y \geq x$  is analogous to the previous case.

Now by (6.3) and (6.4) we obtain

$$\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} \geq x + y + z.$$

Equality occurs if and only if  $x = y = z$ .

**Exercise 6.10** Let  $a, b, c, d$  be positive real numbers such that  $a + b + c + d = 4$ . Prove the inequality

$$a^2bc + b^2cd + c^2da + d^2ab \leq 4.$$

*Solution* Let  $(x, y, z, t)$  be a permutation of  $(a, b, c, d)$  such that  $x \geq y \geq z \geq t$ . Then clearly  $xyz \geq xyt \geq xzt \geq yzt$ .

By the *rearrangement inequality* we obtain

$$x \cdot xyz + y \cdot xyt + z \cdot xzt + t \cdot yzt \geq a^2bc + b^2cd + c^2da + d^2ab. \quad (6.5)$$

Since  $AM \geq GM$  we deduce

$$x \cdot xyz + y \cdot xyt + z \cdot xzt + t \cdot yzt = (xy + zt)(xz + yt) \leq \frac{(xy + xz + yt + zt)^2}{4}. \quad (6.6)$$

Since

$$xy + xz + yt + zt = (x + z)(y + t) \leq \frac{(x + y + z + t)^2}{4} = 4$$

by (6.6) we deduce that

$$x \cdot xyz + y \cdot xyt + z \cdot xzt + t \cdot yzt \leq 4.$$

Finally by (6.5) we obtain

$$a^2bc + b^2cd + c^2da + d^2ab \leq 4,$$

and we are done.

Equality holds iff  $a = b = c = d = 1$  or  $a = 2, b = c = 1, d = 0$  (up to permutation).



## Chapter 7

# Convexity, Jensen's Inequality

The main purpose of this section is to acquaint the reader with one of the most important theorems, that is widely used in proving inequalities, *Jensen's inequality*. This is an inequality regarding so-called convex functions, so firstly we will give some definitions and theorems whose proofs are subject to mathematical analysis, and therefore we'll present them here without proof.

Also we will consider that the reader has an elementary knowledge of differential calculus.

**Definition 7.1** For the function  $f : [a, b] \rightarrow \mathbb{R}$  we'll say that it is convex on the interval  $[a, b]$  if for any  $x, y \in [a, b]$  and any  $\alpha \in (0, 1)$  we have

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y). \quad (7.1)$$

If in (7.1) we have strict inequality then we'll say that  $f$  is strictly convex.

For the function  $f$  we'll say that it is concave if  $-f$  is a convex function.

If the function  $f$  is defined on  $\mathbb{R}$ , it can happen that on some interval this function is a convex function, but on another interval it is a concave function. For this reason, we will consider functions defined on intervals.

*Example 7.1* The function  $f(x) = x^2$  is convex on  $\mathbb{R}$ , moreover  $f(x) = x^n$  is convex on  $\mathbb{R}$  for even  $n$ . Also  $f(x) = x^n$  is convex on  $\mathbb{R}^+$  for  $n$  odd, and it is concave on  $\mathbb{R}^-$ .

The function  $f(x) = \sin x$  on  $(\pi, 2\pi)$  is convex, but on  $(0, \pi)$  it is concave.

Now we will state a theorem that will give a criterion for determining whether and when a function is convex, respectively concave.



**Theorem 7.1** Let  $f : (a, b) \rightarrow \mathbb{R}$  and for any  $x \in (a, b)$  suppose there exists a second derivative  $f''(x)$ . The function  $f(x)$  is convex on  $(a, b)$  if and only if for each  $x \in (a, b)$  we have  $f''(x) \geq 0$ . If  $f''(x) > 0$  for each  $x \in (a, b)$ , then  $f$  is strictly convex on  $(a, b)$ .

Clearly, according to Definition 7.1 and Theorem 7.1 we have that the function  $f(x)$  is concave on  $(a, b)$  if and only if  $f''(x) \leq 0$ , for all  $x \in (a, b)$ .

**Example 7.2** Consider the power function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  defined as  $f(x) = x^\alpha$ . For the second derivative we have  $f''(x) = \alpha(\alpha - 1)x^{\alpha-2}$ , and clearly  $f''(x) > 0$  for  $\alpha > 1$  or  $\alpha < 0$  and  $f''(x) < 0$  for  $0 < \alpha < 1$ . So  $f$  is (strictly) convex for  $\alpha > 1$  or  $\alpha < 0$  and  $f$  is (strictly) concave for  $0 < \alpha < 1$ .

**Example 7.3** For the function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \ln(1 + e^x)$  we have  $f'(x) = \frac{e^x}{1+e^x}$ , and  $f''(x) = \frac{e^x}{(1+e^x)^2} > 0$  for  $x \in \mathbb{R}$ , and therefore  $f$  is convex on  $\mathbb{R}$ .

**Example 7.4** For the function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,  $f(x) = (1 + x^\alpha)^{\frac{1}{\alpha}}$  for  $\alpha \neq 0$  we have  $f''(x) = (\alpha - 1)x^{\alpha-2}(1 + x^\alpha)^{\frac{1}{\alpha} - 2}$ , from where it follows that for  $\alpha < 1$  the function  $f$  is strictly concave and for  $\alpha > 1$  the function  $f$  is strictly convex.

**Theorem 7.2** Let  $f_1, f_2, \dots, f_n$  be convex functions on  $(a, b)$ . Then the function  $c_1 f_1 + c_2 f_2 + \dots + c_n f_n$  is also convex on  $(a, b)$ , for any  $c_1, c_2, \dots, c_n \in (0, \infty)$ .

**Theorem 7.3 (Jensen's inequality)** Let  $f : (a, b) \rightarrow \mathbb{R}$  be a convex function on the interval  $(a, b)$ . Let  $n \in \mathbb{N}$  and  $\alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1)$  be real numbers such that  $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ . Then for any  $x_1, x_2, \dots, x_n \in (a, b)$  we have

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) \leq \sum_{i=1}^n \alpha_i f(x_i),$$

i.e.

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \dots + \alpha_n f(x_n). \quad (7.2)$$

*Proof* We'll prove inequality (7.2) by mathematical induction.

For  $n = 1$  we have  $\alpha_1 = 1$  and since  $f(x_1) = f(x_1)$  we get  $f(\alpha_1 x_1) = \alpha_1 f(x_1)$ , so (7.2) is true.

Let  $n = 2$ . Then (7.2) holds due to Definition 7.1.

Suppose that for  $n = k$ , and any real numbers  $\alpha_1, \alpha_2, \dots, \alpha_k \in [0, 1]$  such that  $\alpha_1 + \alpha_2 + \dots + \alpha_k = 1$  and any  $x_1, x_2, \dots, x_k \in (a, b)$ , we have

$$f(\alpha_1 x_1 + \dots + \alpha_k x_k) \leq \alpha_1 f(x_1) + \dots + \alpha_k f(x_k). \quad (7.3)$$

Let  $n = k + 1$ , and let  $\alpha_1, \alpha_2, \dots, \alpha_{k+1} \in [0, 1]$  such that  $\alpha_1 + \alpha_2 + \dots + \alpha_{k+1} = 1$ .

Let  $x_1, x_2, \dots, x_{k+1} \in (a, b)$ .

Then we have

$$\begin{aligned} & \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_{k+1} x_{k+1} \\ &= (\alpha_1 x_1 + \dots + \alpha_k x_k) + \alpha_{k+1} x_{k+1} \\ &= (1 - \alpha_{k+1}) \left( \frac{\alpha_1}{1 - \alpha_{k+1}} x_1 + \frac{\alpha_2}{1 - \alpha_{k+1}} x_2 + \dots + \frac{\alpha_k}{1 - \alpha_{k+1}} x_k \right) + \alpha_{k+1} x_{k+1}. \end{aligned} \quad (7.4)$$

Let

$$\frac{\alpha_1}{1 - \alpha_{k+1}} x_1 + \frac{\alpha_2}{1 - \alpha_{k+1}} x_2 + \dots + \frac{\alpha_k}{1 - \alpha_{k+1}} x_k = y_{k+1}.$$

Then since  $x_1, x_2, \dots, x_k \in (a, b)$  we deduce

$$\begin{aligned} y_{k+1} &= \frac{\alpha_1}{1 - \alpha_{k+1}} x_1 + \frac{\alpha_2}{1 - \alpha_{k+1}} x_2 + \dots + \frac{\alpha_k}{1 - \alpha_{k+1}} x_k \\ &< \frac{\alpha_1}{1 - \alpha_{k+1}} b + \frac{\alpha_2}{1 - \alpha_{k+1}} b + \dots + \frac{\alpha_k}{1 - \alpha_{k+1}} b \\ &< \frac{b}{1 - \alpha_{k+1}} (\alpha_1 + \alpha_2 + \dots + \alpha_k) = \frac{b}{1 - \alpha_{k+1}} (1 - \alpha_{k+1}) = b. \end{aligned}$$

Similarly we deduce that  $y_{k+1} > a$ .

Thus  $y_{k+1} \in (a, b)$ .

According to Definition 7.1 and by (7.4) we obtain

$$\begin{aligned} f(\alpha_1 x_1 + \dots + \alpha_k x_k + \alpha_{k+1} x_{k+1}) &= f((1 - \alpha_{k+1}) y_{k+1} + \alpha_{k+1} x_{k+1}) \\ &\leq (1 - \alpha_{k+1}) f(y_{k+1}) + \alpha_{k+1} f(x_{k+1}). \end{aligned} \quad (7.5)$$

By inequality (7.3) and since

$$\frac{\alpha_1}{1 - \alpha_{k+1}} + \frac{\alpha_2}{1 - \alpha_{k+1}} + \dots + \frac{\alpha_k}{1 - \alpha_{k+1}} = 1$$

we obtain

$$\begin{aligned} f(y_{k+1}) &= f\left(\frac{\alpha_1}{1 - \alpha_{k+1}} x_1 + \frac{\alpha_2}{1 - \alpha_{k+1}} x_2 + \dots + \frac{\alpha_k}{1 - \alpha_{k+1}} x_k\right) \\ &\leq \frac{\alpha_1}{1 - \alpha_{k+1}} f(x_1) + \frac{\alpha_2}{1 - \alpha_{k+1}} f(x_2) + \dots + \frac{\alpha_k}{1 - \alpha_{k+1}} f(x_k). \end{aligned} \quad (7.6)$$

Finally according to (7.5) and (7.6) we deduce

$$f(\alpha_1 x_1 + \cdots + \alpha_{k+1} x_{k+1}) \leq \alpha_1 f(x_1) + \cdots + \alpha_{k+1} f(x_{k+1}).$$

So by the principle of mathematical induction inequality, (7.2) holds for any positive integer  $n$ , any  $\alpha_1, \alpha_2, \dots, \alpha_n \in [0, 1]$  such that  $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ , and arbitrary  $x_1, x_2, \dots, x_n \in (a, b)$ .  $\square$

*Remark* If  $f$  is strictly convex then equality in *Jensen's inequality* occurs only for  $x_1 = x_2 = \cdots = x_n$ .

If the function  $f(x)$  is concave then in *Jensen's inequality* we have the reverse inequality, i.e.

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \geq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n).$$

It is important to note that *Jensen's inequality* can also be written in the equivalent form:

If  $f : I \rightarrow \mathbb{R}$  is convex on  $I$ ,  $x_1, x_2, \dots, x_n \in I$  and  $m_1, m_2, \dots, m_n \geq 0$  are real numbers such that  $m_1 + m_2 + \cdots + m_n > 0$ . Then

$$f\left(\frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}\right) \leq \frac{m_1 f(x_1) + m_2 f(x_2) + \cdots + m_n f(x_n)}{m_1 + m_2 + \cdots + m_n}.$$

*Example 7.5* Consider the function  $f(x) = -\ln x$ , on the interval  $(0, +\infty)$ . For the second derivative we have  $f''(x) = \frac{1}{x^2} > 0$ , which means that  $f(x)$  is a strictly convex on  $x \in (0, +\infty)$ .

By *Jensen's inequality* for  $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{1}{n}$ , and  $x_i \in (0, +\infty)$ ,  $i = 1, 2, \dots, n$ , we obtain

$$\begin{aligned} -\ln\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) &\leq -\left(\frac{\ln x_1 + \ln x_2 + \cdots + \ln x_n}{n}\right) \\ \Leftrightarrow \frac{\ln x_1 + \ln x_2 + \cdots + \ln x_n}{n} &\leq \ln\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \\ \Leftrightarrow \ln(x_1 x_2 \cdots x_n)^{1/n} &\leq \ln\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right), \end{aligned}$$

i.e.

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n},$$

which is the well-known inequality  $AM \geq GM$ .

*Example 7.6* Let us consider the function  $f(x) = x^2$ . Since  $f''(x) = 2 > 0$  it follows that  $f$  is convex on  $\mathbb{R}$ . Then by *Jensen's inequality*

$$f\left(\frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}\right) \leq \frac{m_1 f(x_1) + m_2 f(x_2) + \cdots + m_n f(x_n)}{m_1 + m_2 + \cdots + m_n},$$

we obtain

$$\left( \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} \right)^2 \leq \frac{m_1 x_1^2 + m_2 x_2^2 + \cdots + m_n x_n^2}{m_1 + m_2 + \cdots + m_n},$$

i.e.

$$(m_1 x_1 + m_2 x_2 + \cdots + m_n x_n)^2 \leq (m_1 x_1^2 + m_2 x_2^2 + \cdots + m_n x_n^2)(m_1 + m_2 + \cdots + m_n).$$

By taking  $m_i = b_i^2$ ,  $x_i = \frac{a_i}{b_i}$  for  $i = 1, 2, \dots, n$  in the last inequality, we obtain

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2),$$

which is the well-known *Cauchy–Schwarz inequality*.

On this occasion we will present *Popoviciu's inequality*, which will be used in the same manner as *Jensen's inequality*. But we must note that this inequality is stronger than *Jensen's inequality*, i.e. in some cases this inequality can be a powerful tool for proving other inequalities, where *Jensen's inequality* does not work.

**Theorem 7.4** (Popoviciu's inequality) *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a convex function on the interval  $[a, b]$ . Then for any  $x, y, z \in [a, b]$  we have*

$$\begin{aligned} & f\left(\frac{x+y+z}{3}\right) + \frac{f(x) + f(y) + f(z)}{3} \\ & \geq \frac{2}{3} \left( f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \right). \end{aligned}$$

*Proof* Without loss of generality we assume that  $x \leq y \leq z$ .

If  $y \leq \frac{x+y+z}{3}$  then  $\frac{x+y+z}{3} \leq \frac{x+z}{2} \leq z$  and  $\frac{x+y+z}{3} \leq \frac{y+z}{2} \leq z$ .

Therefore there exist  $s, t \in [0, 1]$  such that

$$\frac{x+z}{2} = \left( \frac{x+y+z}{3} \right) s + z(1-s) \quad \text{and} \quad (7.7)$$

$$\frac{y+z}{2} = \left( \frac{x+y+z}{3} \right) t + z(1-t). \quad (7.8)$$

Summing (7.7) and (7.8) gives

$$\frac{x+y-2z}{2} = \frac{x+y-2z}{3}(s+t),$$

from which we obtain  $s+t = \frac{3}{2}$ .

Because the function  $f$  is convex, we have

$$\begin{aligned} f\left(\frac{x+z}{2}\right) &\leq s \cdot f\left(\frac{x+y+z}{3}\right) + (1-s)f(z), \\ f\left(\frac{y+z}{2}\right) &\leq t \cdot f\left(\frac{x+y+z}{3}\right) + (1-t)f(z) \end{aligned}$$

and

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y).$$

After adding together the last three inequalities we obtain the required inequality.

The case when  $\frac{x+y+z}{3} < y$  is considered similarly, bearing in mind that  $x \leq \frac{x+z}{2} \leq \frac{x+y+z}{3}$  and  $x \leq \frac{y+z}{2} \leq \frac{x+y+z}{3}$ .  $\square$

**Note** If  $f$  is a concave function on  $[a, b]$  then in *Popoviciu's inequality* for all  $x, y, z \in [a, b]$  we have the reverse inequality, i.e. we have

$$\begin{aligned} f\left(\frac{x+y+z}{3}\right) + \frac{f(x) + f(y) + f(z)}{3} \\ \leq \frac{2}{3} \left( f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \right). \end{aligned}$$

**Theorem 7.5** (Generalized Popoviciu's inequality) *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a convex function on the interval  $[a, b]$  and  $a_1, a_2, \dots, a_n \in [a, b]$ . Then*

$$\begin{aligned} f(a_1) + f(a_2) + \dots + f(a_n) + n(n-2)f(a) \\ \geq (n-1)(f(b_1) + f(b_2) + \dots + f(b_n)), \end{aligned}$$

where  $a = \frac{a_1 + a_2 + \dots + a_n}{n}$ , and  $b_i = \frac{1}{n-1} \sum_{j \neq i} a_j$  for all  $i$ .

**Theorem 7.6** (Weighted AM–GM inequality) *Let  $a_i \in (0, \infty)$ ,  $i = 1, 2, \dots, n$ , and  $\alpha_i \in [0, 1]$ ,  $i = 1, 2, \dots, n$ , be such that  $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ . Then*

$$a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n} \leq a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n. \quad (7.9)$$

*Proof* For the function  $f(x) = -\ln x$  we have  $f'(x) = -\frac{1}{x}$  and  $f''(x) = \frac{1}{x^2}$ , i.e.  $f''(x) > 0$ , for  $x \in (0, \infty)$ .

So due to Theorem 7.1 we conclude that the function  $f$  is convex on  $(0, \infty)$ .

Let  $a_i \in (0, \infty)$ ,  $i = 1, 2, \dots, n$ , and  $\alpha_i \in [0, 1]$ ,  $i = 1, 2, \dots, n$ , be arbitrary real numbers such that  $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ .

By *Jensen's inequality* we deduce

$$\begin{aligned}
 -\ln\left(\sum_{i=1}^n a_i \alpha_i\right) &= f\left(\sum_{i=1}^n a_i \alpha_i\right) \leq \sum_{i=1}^n \alpha_i f(a_i) = -\sum_{i=1}^n \alpha_i \ln a_i \\
 &\Leftrightarrow -\ln(a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n) \leq -\alpha_1 \ln a_1 - \alpha_2 \ln a_2 - \dots - \alpha_n \ln a_n \\
 &\Leftrightarrow \alpha_1 \ln a_1 + \alpha_2 \ln a_2 + \dots + \alpha_n \ln a_n \leq \ln(a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n) \\
 &\Leftrightarrow \ln a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n} \leq \ln(a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n) \\
 &\Leftrightarrow a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n} \leq a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n,
 \end{aligned}$$

as required.  $\square$

**Note** By inequality (7.9) for  $\alpha_1 = \alpha_2 = \dots = \alpha_n = \frac{1}{n}$ , we obtain the inequality  $AM \geq GM$ .

**Exercise 7.1** Let  $\alpha, \beta, \gamma$  be the angles of a triangle. Prove the inequality

$$\sin \alpha \sin \beta \sin \gamma \leq \frac{3\sqrt{3}}{8}.$$

*Solution* Since  $\alpha, \beta, \gamma \in (0, \pi)$  it follows that  $\sin \alpha, \sin \beta, \sin \gamma > 0$ .

Therefore since  $AM \geq GM$  we obtain

$$\sqrt[3]{\sin \alpha \sin \beta \sin \gamma} \leq \frac{\sin \alpha + \sin \beta + \sin \gamma}{3}. \quad (7.10)$$

Since  $f(x) = \sin x$  is concave on  $(0, \pi)$ , by *Jensen's inequality* we deduce

$$\frac{\sin \alpha + \sin \beta + \sin \gamma}{3} \leq \sin \frac{\alpha + \beta + \gamma}{3} = \frac{\sqrt{3}}{2}. \quad (7.11)$$

Due to (7.10) and (7.11) we get

$$\sqrt[3]{\sin \alpha \sin \beta \sin \gamma} \leq \frac{\sqrt{3}}{2} \quad \Leftrightarrow \quad \sin \alpha \sin \beta \sin \gamma \leq \frac{3\sqrt{3}}{8}.$$

Equality occurs iff  $\alpha = \beta = \gamma$ , i.e. the triangle is equilateral.

**Exercise 7.2** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequalities:

- (1)  $4(a^3 + b^3) \geq (a + b)^3$ ;
- (2)  $9(a^3 + b^3 + c^3) \geq (a + b + c)^3$ .

**Solution** (1) The function  $f(x) = x^3$  is convex on  $(0, +\infty)$ , thus from *Jensen's inequality* it follows that

$$\left(\frac{a+b}{2}\right)^3 \leq \frac{a^3+b^3}{2} \Leftrightarrow 4(a^3+b^3) \geq (a+b)^3.$$

(2) Similarly as in (1) we deduce that

$$\left(\frac{a+b+c}{3}\right)^3 \leq \frac{a^3+b^3+c^3}{3} \Leftrightarrow 9(a^3+b^3+c^3) \geq (a+b+c)^3.$$

**Exercise 7.3** Let  $\alpha_i > 0, i = 1, 2, \dots, n$ , be real numbers such that  $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ . Prove the inequality

$$\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \dots \alpha_n^{\alpha_n} \geq \frac{1}{n}.$$

**Solution** If we take  $a_i = \frac{1}{\alpha_i}, i = 1, 2, \dots, n$ , by the *Weighted AM–GM inequality* we get

$$\frac{1}{\alpha_1^{\alpha_1}} \frac{1}{\alpha_2^{\alpha_2}} \dots \frac{1}{\alpha_n^{\alpha_n}} \leq \frac{1}{\alpha_1} \alpha_1 + \frac{1}{\alpha_2} \alpha_2 + \dots + \frac{1}{\alpha_n} \alpha_n = n,$$

i.e.

$$\frac{1}{n} \leq \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \dots \alpha_n^{\alpha_n}.$$

**Exercise 7.4** Find the minimum value of  $k$  such that for arbitrary  $a, b > 0$  we have

$$\sqrt[3]{a} + \sqrt[3]{b} \leq k \sqrt[3]{a+b}.$$

**Solution** Consider the function  $f(x) = \sqrt[3]{x}$ .

We have  $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$  and  $f''(x) = -\frac{2}{9}x^{-\frac{5}{3}} < 0$ , for any  $x \in (0, \infty)$ . Thus  $f(x)$  is concave on the interval  $(0, \infty)$ .

By *Jensen's inequality* we deduce

$$\begin{aligned} \frac{1}{2}f(a) + \frac{1}{2}f(b) &\leq f\left(\frac{a+b}{2}\right) \\ \Leftrightarrow \frac{\sqrt[3]{a} + \sqrt[3]{b}}{2} &\leq \sqrt[3]{\frac{a+b}{2}} \\ \Leftrightarrow \sqrt[3]{a} + \sqrt[3]{b} &\leq \frac{2}{\sqrt[3]{2}} \sqrt[3]{a+b} = \sqrt[3]{4} \cdot \sqrt[3]{a+b}. \end{aligned}$$

Therefore  $k_{\min} = \sqrt[3]{4}$ , and for instance we reach this value for  $a = b$ .

**Exercise 7.5** Let  $x, y, z \geq 0$  be real numbers. Prove the inequality

$$\sqrt{x^2 + 1} + \sqrt{y^2 + 1} + \sqrt{z^2 + 1} \geq \sqrt{6(x + y + z)}.$$

*Solution* Consider the function  $f(t) = \sqrt{t^2 + 1}$ ,  $t \geq 0$ .

Since  $f''(t) = \frac{1}{(\sqrt{t^2 + 1})^3} > 0$ ,  $f$  is convex on  $[0, \infty)$ .

Therefore by *Jensen's inequality* we have

$$\frac{\sqrt{x^2 + 1} + \sqrt{y^2 + 1} + \sqrt{z^2 + 1}}{3} \geq \sqrt{\left(\frac{x + y + z}{3}\right)^2 + 1},$$

i.e.

$$\sqrt{x^2 + 1} + \sqrt{y^2 + 1} + \sqrt{z^2 + 1} \geq \sqrt{(x + y + z)^2 + 9}. \quad (7.12)$$

From the obvious inequality  $((x + y + z) - 3)^2 \geq 0$  it follows that

$$(x + y + z)^2 + 9 \geq 6(x + y + z). \quad (7.13)$$

By (7.12) and (7.13) we obtain

$$\sqrt{x^2 + 1} + \sqrt{y^2 + 1} + \sqrt{z^2 + 1} \geq \sqrt{(x + y + z)^2 + 9} \geq \sqrt{6(x + y + z)}.$$

Equality occurs if and only if  $x = y = z = 1$ .

**Exercise 7.6** Let  $x, y, z$  be positive real numbers. Prove the inequality

$$\frac{x + y}{z} + \frac{y + z}{x} + \frac{z + x}{y} \geq 4 \left( \frac{z}{x + y} + \frac{x}{y + z} + \frac{y}{z + x} \right).$$

*Solution* Consider the function  $f(x) = x + \frac{1}{x}$ .

Since  $f'(x) = 1 - \frac{1}{x^2}$  and  $f''(x) = \frac{2}{x^3} > 0$  for any  $x > 0$  it follows that  $f$  is convex on  $\mathbb{R}^+$ .

Now by *Popoviciu's inequality* we can easily obtain the required inequality.





# Chapter 8

## Trigonometric Substitutions and Their Application for Proving Algebraic Inequalities

Very often, for proving a given algebraic inequality we can use trigonometric substitutions that work amazingly well, and can almost always lead the solver to a solution.

Using such substitutions, a given inequality may simplify to the point, where the final part of the proof will be only routine, and will need previous results (usually *Jensen's inequality* and elements of trigonometry). Therefore it is necessary to possess a knowledge of trigonometry.

We will give some basic facts that must be known and which are of benefit when *Jensen's inequality* is being used. Namely, the function  $\sin x$  is concave on  $(0, \pi)$ , the function  $\cos x$  is concave on  $(-\pi/2, \pi/2)$ , hence also on  $(0, \pi/2)$ ,  $\tan x$  is convex on  $(0, \pi/2)$ , while the function  $\cot x$  is convex on  $(0, \pi/2)$ .

Furthermore, without proof (the proofs are “pure” trigonometry, and some of them can be found in standard collections of problems in mathematics at secondary level) we will give several trigonometric identities relating the angles of a triangle, which the reader should certainly know.

**Proposition 8.1** *Let  $\alpha, \beta, \gamma$  be the angles of a given triangle. Then we have the following identities:*

$$\begin{aligned} I_1: & \cos \alpha + \cos \beta + \cos \gamma = 1 + 4 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2} \\ I_2: & \sin \alpha + \sin \beta + \sin \gamma = 4 \cos \frac{\alpha}{2} \cdot \cos \frac{\beta}{2} \cdot \cos \frac{\gamma}{2} \\ I_3: & \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4 \sin \alpha \cdot \sin \beta \cdot \sin \gamma \\ I_4: & \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2 + 2 \cos \alpha \cdot \cos \beta \cdot \cos \gamma \\ I'_4: & \sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} + 2 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2} = 1 \\ I_5: & \tan \frac{\alpha}{2} \cdot \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \cdot \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \cdot \tan \frac{\alpha}{2} = 1 \\ I_6: & \tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \cdot \tan \beta \cdot \tan \gamma \\ I_7: & \cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} = \cot \frac{\alpha}{2} \cdot \cot \frac{\beta}{2} \cdot \cot \frac{\gamma}{2}. \end{aligned}$$

**Proposition 8.2** Let  $\alpha, \beta, \gamma$  be arbitrary real numbers. Then we have:

$$I_8: \sin \alpha + \sin \beta + \sin \gamma - \sin(\alpha + \beta + \gamma) = 4 \sin \frac{\alpha+\beta}{2} \cdot \sin \frac{\beta+\gamma}{2} \cdot \sin \frac{\gamma+\alpha}{2}$$

$$I_9: \cos \alpha + \cos \beta + \cos \gamma + \cos(\alpha + \beta + \gamma) = 4 \cos \frac{\alpha+\beta}{2} \cdot \cos \frac{\beta+\gamma}{2} \cdot \cos \frac{\gamma+\alpha}{2}.$$

Now we will give several inequalities concerning the angles of a given triangle, which will be used in proving inequalities by using trigonometric substitutions, and which are of great importance. The method of introducing certain substitutions and knowledge of these inequalities are the essence of this way of proving algebraic inequalities.

**Proposition 8.3** Let  $\alpha, \beta, \gamma$  be the angles of a given triangle. Then we have the following inequalities:

$$N_1: \sin \alpha + \sin \beta + \sin \gamma \leq \frac{3\sqrt{3}}{2}$$

$$N_2: \sin \alpha \cdot \sin \beta \cdot \sin \gamma \leq \frac{3\sqrt{3}}{8}$$

$$N_3: \sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \leq \frac{3}{2}$$

$$N_4: \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2} \leq \frac{1}{8}$$

$$N_5: \cos \alpha + \cos \beta + \cos \gamma \leq \frac{3}{2}$$

$$N_6: \cos \alpha \cdot \cos \beta \cdot \cos \gamma \leq \frac{1}{8}$$

$$N_7: \cos \frac{\alpha}{2} + \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \leq \frac{3\sqrt{3}}{2}$$

$$N_8: \cos \frac{\alpha}{2} \cdot \cos \frac{\beta}{2} \cdot \cos \frac{\gamma}{2} \leq \frac{3\sqrt{3}}{8}$$

$$N_9: \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma \leq \frac{9}{4}$$

$$N_{10}: \sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} \geq \frac{3}{4}$$

$$N_{11}: \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma \geq \frac{3}{4}$$

$$N_{12}: \cos^2 \frac{\alpha}{2} + \cos^2 \frac{\beta}{2} + \cos^2 \frac{\gamma}{2} \leq \frac{9}{4}$$

$$N_{13}: \tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} \geq \sqrt{3}$$

$$N_{14}: \cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} \geq 3\sqrt{3}$$

$$N_{15}: \cot \alpha + \cot \beta + \cot \gamma \geq \sqrt{3}.$$

*Proof*  $N_1$ : The function  $\sin x$  is concave on the interval  $(0, \pi)$ , thus from *Jensen's inequality* we obtain

$$\frac{\sin \alpha + \sin \beta + \sin \gamma}{3} \leq \sin \left( \frac{\alpha + \beta + \gamma}{3} \right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\Leftrightarrow \sin \alpha + \sin \beta + \sin \gamma \leq \frac{3\sqrt{3}}{2}.$$

$N_2$ : Since  $\sin x > 0$  for any  $x \in (0, \pi)$  we can apply the inequality  $AM \geq GM$ , and we obtain

$$\sin \alpha \cdot \sin \beta \cdot \sin \gamma \leq \left( \frac{\sin \alpha + \sin \beta + \sin \gamma}{3} \right)^3 \stackrel{N_1}{\leq} \left( \frac{\sqrt{3}}{2} \right)^3 = \frac{3\sqrt{3}}{8}.$$

$N_3$ : Similarly as in the proof of  $N_1$  we have

$$\frac{\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2}}{3} \leq \sin \left( \frac{\alpha + \beta + \gamma}{6} \right) = \sin \frac{\pi}{6} = \frac{1}{2}$$

or

$$\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \leq \frac{3}{2},$$

since the function  $\sin x$  is concave on  $(0, \pi/2)$ .

$N_4$ : Similarly as in proof of  $N_2$  and since  $AM \geq GM$  we have

$$\sqrt[3]{\sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2}} \leq \frac{\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2}}{3} \stackrel{N_4}{\leq} \frac{1}{2},$$

i.e.

$$\sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2} \leq \frac{1}{8}.$$

$N_5$ : Since  $\alpha + \beta = \pi - \gamma$  it follows that

$$\cos \gamma = -\cos(\alpha + \beta) = -\cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

Thus

$$\begin{aligned} 3 - 2(\cos \alpha + \cos \beta + \cos \gamma) &= 3 - 2(\cos \alpha + \cos \beta - \cos \alpha \cos \beta + \sin \alpha \sin \beta) \\ &= \sin^2 \alpha + \sin^2 \beta - 2 \sin \alpha \sin \beta + 1 + \cos^2 \alpha \\ &\quad + \cos^2 \beta - 2 \cos \alpha - 2 \cos \beta + 2 \cos \alpha \cos \beta \\ &= (\sin \alpha - \sin \beta)^2 + (1 - \cos \alpha - \cos \beta)^2 \geq 0, \end{aligned}$$

which is equivalent to

$$\cos \alpha + \cos \beta + \cos \gamma \leq \frac{3}{2}.$$

$N_6$ : Since  $\cos(\alpha + \beta) = -\cos \gamma$ , we have

$$\begin{aligned} \cos \alpha \cos \beta \cos \gamma &= \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta)) \cos \gamma \\ &= \frac{1}{2}(\cos(\alpha - \beta) - \cos \gamma) \cos \gamma = \frac{1}{2} \cos(\alpha - \beta) \cos \gamma - \frac{\cos^2 \gamma}{2} \\ &= -\frac{1}{2} \left( \cos \gamma - \frac{\cos(\alpha - \beta)}{2} \right)^2 + \frac{\cos^2(\alpha - \beta)}{8} \\ &\leq \frac{\cos^2(\alpha - \beta)}{8} \leq \frac{1}{8}. \end{aligned}$$

$N_7$ : Since  $\alpha, \beta, \gamma \in (0, \pi)$  it follows that  $\alpha/2, \beta/2, \gamma/2 \in (0, \pi/2)$ .

The function  $\cos x$  is concave on the interval  $(0, \pi/2)$ .

Thus by *Jensen's inequality* we get

$$\frac{\cos \frac{\alpha}{2} + \cos \frac{\beta}{2} + \cos \frac{\gamma}{2}}{3} \leq \cos \frac{\alpha + \beta + \gamma}{6} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2},$$

i.e.

$$\cos \frac{\alpha}{2} + \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \leq \frac{3\sqrt{3}}{2}.$$

$N_8$ : Since  $\alpha, \beta, \gamma \in (0, \pi)$  it follows that  $\alpha/2, \beta/2, \gamma/2 \in (0, \pi/2)$ , i.e.

$$\cos \alpha, \cos \beta, \cos \gamma > 0,$$

so we can apply  $AM \geq GM$  to conclude that

$$\sqrt[3]{\cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}} \leq \frac{\cos \frac{\alpha}{2} + \cos \frac{\beta}{2} + \cos \frac{\gamma}{2}}{3} \stackrel{N_7}{\leq} \frac{\sqrt{3}}{2},$$

from which it follows that

$$\cos \frac{\alpha}{2} \cdot \cos \frac{\beta}{2} \cdot \cos \frac{\gamma}{2} \leq \frac{3\sqrt{3}}{8}.$$

$N_9$ : By identity  $I_4$  and inequality  $N_6$  we obtain

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2 + 2 \cos \alpha \cdot \cos \beta \cdot \cos \gamma \leq 2 + 2 \cdot \frac{1}{8} = \frac{9}{4}.$$

$N_{10}$ : By  $I'_4$  we have that

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} + 2 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2} = 1,$$

i.e.

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} = 1 - 2 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2}.$$

According to  $N_4$ :  $\sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2} \leq \frac{1}{8}$  and the previous identity we obtain

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} \geq 1 - \frac{2}{8} = \frac{3}{4}.$$

$N_{11}$ : We have

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 3 - (\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma) \stackrel{N_9}{\geq} 3 - \frac{9}{4} = \frac{3}{4}.$$

$N_{12}$ : We have

$$\cos^2 \frac{\alpha}{2} + \cos^2 \frac{\beta}{2} + \cos^2 \frac{\gamma}{2} = 3 - \left( \sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} \right) \stackrel{N_{10}}{\leq} 3 - \frac{3}{4} = \frac{9}{4}.$$

$N_{13}$ : Since  $\tan x$  is convex on the interval  $(0, \pi/2)$ , by *Jensen's inequality* we deduce

$$\frac{\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2}}{3} \geq \tan \frac{\alpha + \beta + \gamma}{6} = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}},$$

i.e.

$$\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} \geq \sqrt{3}.$$

$N_{14}$ : Due to the convexity of  $\cot x$  on  $(0, \pi/2)$  by *Jensen's inequality* we obtain

$$\cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} \geq 3 \cot \frac{\alpha + \beta + \gamma}{6} = 3\sqrt{3}.$$

$N_{15}$ : Firstly we have

$$\cot \alpha + \cot \beta = \frac{\cos \alpha}{\sin \alpha} + \frac{\cos \beta}{\sin \beta} = \frac{\cos \alpha \sin \beta + \sin \alpha \cos \beta}{\sin \alpha \sin \beta} = \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta}. \quad (8.1)$$

Also

$$1 \geq \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta, \quad (8.2)$$

$$\cos \gamma = -\cos(\alpha + \beta) = -\cos \alpha \cos \beta + \sin \alpha \sin \beta. \quad (8.3)$$

Adding (8.2) and (8.3) gives us

$$\begin{aligned} 2 \sin \alpha \sin \beta &\leq 1 + \cos \gamma \\ \Leftrightarrow 2 \sin \alpha \sin \beta \sin(\alpha + \beta) &\leq (1 + \cos \gamma) \sin(\alpha + \beta) \\ \Leftrightarrow 2 \sin \alpha \sin \beta \sin \gamma &\leq (1 + \cos \gamma) \sin(\alpha + \beta) \\ \Leftrightarrow \frac{2 \sin \alpha \sin \beta \sin \gamma}{\sin \alpha \sin \beta (1 + \cos \gamma)} &\leq \frac{(1 + \cos \gamma) \sin(\alpha + \beta)}{\sin \alpha \sin \beta (1 + \cos \gamma)} \\ \Leftrightarrow \frac{2 \sin \gamma}{1 + \cos \gamma} &\leq \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta}. \end{aligned} \quad (8.4)$$

Therefore

$$\begin{aligned} \cot \alpha + \cot \beta + \cot \gamma &\stackrel{(8.1)}{=} \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta} + \frac{\cos \gamma}{\sin \gamma} \stackrel{(8.4)}{\geq} \frac{2 \sin \gamma}{1 + \cos \gamma} + \frac{\cos \gamma}{\sin \gamma} \\ &= \frac{1}{2} \left( \frac{4 \sin^2 \gamma + 2 \cos^2 \gamma + 2 \cos \gamma}{(1 + \cos \gamma) \sin \gamma} \right) \\ &= \frac{1}{2} \left( \frac{3 \sin^2 \gamma + (1 + \cos \gamma)^2}{(1 + \cos \gamma) \sin \gamma} \right) \\ &\geq \frac{1}{2} \left( \frac{2\sqrt{3 \sin^2 \gamma (1 + \cos \gamma)^2}}{(1 + \cos \gamma) \sin \gamma} \right) = \frac{2\sqrt{3}}{2} = \sqrt{3}, \end{aligned}$$

as required. □

**Proposition 8.4** *Let  $\alpha, \beta, \gamma$  be the angles of an acute triangle. Then*

$$N_{16}: \tan \alpha + \tan \beta + \tan \gamma \geq 3\sqrt{3}.$$

*Proof* Since the triangle is acute it follows that  $\alpha, \beta, \gamma \in (0, \pi/2)$ . The function  $f(x) = \tan x$  is convex on  $(0, \pi/2)$ , so by *Jensen's inequality* we obtain

$$\tan \alpha + \tan \beta + \tan \gamma \geq 3 \tan \frac{\alpha + \beta + \gamma}{3} = 3 \tan \frac{\pi}{3} = 3\sqrt{3}.$$

Furthermore, we'll give two theorems that will be the basis for the introduction of trigonometric substitutions.  $\square$

**Theorem 8.1** *Let  $\alpha, \beta, \gamma \in (0, \pi)$ . Then  $\alpha, \beta$  and  $\gamma$  are the angles of a triangle if and only if*

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\alpha}{2} \tan \frac{\gamma}{2} = 1.$$

*Proof* Let  $\alpha, \beta, \gamma$  be the angles of an arbitrary triangle. Then  $\alpha + \beta + \gamma = \pi$ , i.e.  $\frac{\gamma}{2} = \frac{\pi}{2} - \frac{\alpha + \beta}{2}$ .

Therefore

$$\begin{aligned} \tan \frac{\gamma}{2} &= \tan \left( \frac{\pi}{2} - \frac{\alpha + \beta}{2} \right) = \cot \left( \frac{\alpha}{2} + \frac{\beta}{2} \right) = \frac{\cot \frac{\alpha}{2} \cot \frac{\beta}{2} - 1}{\cot \frac{\alpha}{2} + \cot \frac{\beta}{2}} = \frac{1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}{\tan \frac{\alpha}{2} + \tan \frac{\beta}{2}} \\ \Leftrightarrow \quad \tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\alpha}{2} \tan \frac{\gamma}{2} &= 1. \end{aligned}$$

Conversely, let us suppose that for some  $\alpha, \beta, \gamma \in (0, \pi)$  we have

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\alpha}{2} \tan \frac{\gamma}{2} = 1. \quad (8.5)$$

If  $\alpha = \beta = \gamma$  then  $3 \tan^2 \frac{\alpha}{2} = 1$ , and since  $\tan \frac{\alpha}{2} > 0$  we get  $\tan \frac{\alpha}{2} = \frac{1}{\sqrt{3}}$ , i.e.  $\alpha = \beta = \gamma = 60^\circ$ , from which it follows that  $\alpha + \beta + \gamma = \pi$ , i.e.  $\alpha, \beta$  and  $\gamma$  are the angles of a triangle.

Without loss of generality let us assume that  $\alpha \neq \beta$ .

Since  $0 < \alpha + \beta < 2\pi$  it follows that there exists  $\gamma_1 \in (-\pi, \pi)$  such that  $\alpha + \beta + \gamma_1 = \pi$ .

Then by the previous part of this proof we must have

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma_1}{2} + \tan \frac{\alpha}{2} \tan \frac{\gamma_1}{2} = 1. \quad (8.6)$$

We'll show that  $\gamma = \gamma_1$ , from which it will follow that  $\alpha + \beta + \gamma = \pi$ , i.e.  $\alpha$ ,  $\beta$  and  $\gamma$  are the angles of a triangle.

If we subtract (8.5) and (8.6) we get

$$\tan \frac{\gamma}{2} = \tan \frac{\gamma_1}{2}, \quad \text{i.e.} \quad \left| \frac{\gamma - \gamma_1}{2} \right| = k\pi, \quad \text{for some } k \geq 0, k \in \mathbb{Z}.$$

But  $|\frac{\gamma - \gamma_1}{2}| \leq \frac{\gamma}{2} + \frac{\gamma_1}{2} < \frac{\pi}{2} + \frac{\pi}{2} = \pi$ , so it follows that  $k = 0$ , i.e.  $\gamma = \gamma_1$ , and the proof is finished.  $\square$

**Theorem 8.2** *Let  $\alpha, \beta, \gamma \in (0, \pi)$ . Then  $\alpha, \beta$  and  $\gamma$  are the angles of a triangle if and only if*

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} + 2 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2} = 1.$$

*Proof* Let  $\alpha, \beta, \gamma$  be the angles of a triangle. Then we have

$$\begin{aligned} & \sin^2 \frac{\gamma}{2} + 2 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2} \\ &= \cos^2 \frac{\alpha + \beta}{2} + 2 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \cos \frac{\alpha + \beta}{2} \\ &= \cos \frac{\alpha + \beta}{2} \left( \cos \frac{\alpha + \beta}{2} + 2 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \right) \\ &= \cos \frac{\alpha + \beta}{2} \left( \cos \frac{\alpha + \beta}{2} + \cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta}{2} \right) \\ &= \cos \frac{\alpha + \beta}{2} \cdot \cos \frac{\alpha - \beta}{2} = \frac{1}{2} (\cos \alpha + \cos \beta) \\ &= \frac{1 - 2 \sin^2 \frac{\alpha}{2} + 1 - \sin^2 \frac{\beta}{2}}{2} = 1 - \sin^2 \frac{\alpha}{2} - \sin^2 \frac{\beta}{2}, \end{aligned}$$

i.e.

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} + 2 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2} = 1.$$

Conversely, let  $\alpha, \beta, \gamma \in (0, \pi)$  be such that

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} + 2 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2} = 1. \quad (8.7)$$

We'll show that  $\alpha + \beta + \gamma = \pi$ .

Since  $0 < \alpha + \beta < 2\pi$  it follows that there exists  $\gamma_1 \in (-\pi, \pi)$  such that  $\alpha + \beta + \gamma_1 = \pi$ .



Then clearly

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma_1}{2} + 2 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma_1}{2} = 1. \quad (8.8)$$

By subtracting (8.7) and (8.8) we obtain

$$\left( \sin \frac{\gamma}{2} - \sin \frac{\gamma_1}{2} \right) \left( \sin \frac{\gamma}{2} + \sin \frac{\gamma_1}{2} + 2 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \right) = 0. \quad (8.9)$$

But

$$\begin{aligned} \sin \frac{\gamma}{2} + \sin \frac{\gamma_1}{2} + 2 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} &= \sin \frac{\gamma}{2} + \sin \frac{\gamma_1}{2} + \cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta}{2} \\ &= \sin \frac{\gamma}{2} + \sin \frac{\gamma_1}{2} + \cos \frac{\alpha - \beta}{2} - \sin \frac{\gamma_1}{2} \\ &= \sin \frac{\gamma}{2} + \cos \frac{\alpha - \beta}{2}. \end{aligned}$$

Since  $\frac{\gamma}{2} \in (0, \pi/2)$  and  $\frac{\alpha - \beta}{2} \in (-\pi/2, \pi/2)$  it follows that

$$\sin \frac{\gamma}{2} + \cos \frac{\alpha - \beta}{2} > 0, \quad \text{i.e.} \quad \sin \frac{\gamma}{2} + \sin \frac{\gamma_1}{2} + 2 \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} > 0.$$

From the last inequality and (8.9) we have

$$\sin \frac{\gamma}{2} = \sin \frac{\gamma_1}{2}, \quad \text{i.e.} \quad \gamma = \gamma_1.$$

Thus  $\alpha + \beta + \gamma = \pi$ , as required.  $\square$

Now, based on these two theorems we will give basic cases, how a given algebraic inequality can be transformed by trigonometric substitutions. These substitutions, with the inequalities of Propositions 8.3 and 8.4 will be a powerful apparatus for proving algebraic inequalities.

## 8.1 The Most Usual Forms of Trigonometric Substitutions

**Case 1.** Let  $\alpha, \beta$  and  $\gamma$  be the angles of an arbitrary triangle.

Let  $A = \frac{\pi - \alpha}{2}$ ,  $B = \frac{\pi - \beta}{2}$  and  $C = \frac{\pi - \gamma}{2}$ .

Then  $A + B + C = \pi$ ; moreover  $0 < A, B, C < \pi/2$ , i.e. this substitution allows us to transfer angles of an arbitrary triangle to angles of an acute triangle. (This is especially important when we use *Jensen's inequality*, since “Jensen” could not be used for the function  $\cos x$  on the interval  $(0, \pi)$ , but only on the interval  $(0, \pi/2)$ .) Observe that we have:

$$\sin \frac{\alpha}{2} = \cos A, \quad \sin \frac{\beta}{2} = \cos B, \quad \sin \frac{\gamma}{2} = \cos C.$$

*Note:* There are similar identities for the functions  $\cos x$ ,  $\tan x$  and  $\cot x$ .

**Case 2.** Let  $x, y$  and  $z$  be positive real numbers. Then there exist triangle with length-sides  $a = x + y, b = y + z, c = z + x$ .

Clearly  $(x, y, z) = (s - b, s - c, s - a)$ , where  $s = \frac{a+b+c}{2} = x + y + z$ .

**Case 3.** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 1$ .

Since  $\tan x \in (0, \infty)$  for  $x \in (0, \pi/2)$ , and due to Theorem 8.1 we can use the substitutions

$$a = \tan \frac{\alpha}{2}, \quad b = \tan \frac{\beta}{2}, \quad c = \tan \frac{\gamma}{2},$$

where  $\alpha, \beta$  and  $\gamma$  are the angles of a triangle, i.e.  $\alpha, \beta, \gamma \in (0, \pi)$  and  $\alpha + \beta + \gamma = \pi$ .

**Case 4.** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 1$ .

Then according to *Case 3* and *Case 1* we can use the following substitutions

$$a = \cot \alpha, \quad b = \cot \beta, \quad c = \cot \gamma,$$

where  $\alpha, \beta$  and  $\gamma$  are the angles of an acute triangle, i.e.  $\alpha, \beta, \gamma \in (0, \pi/2)$  and  $\alpha + \beta + \gamma = \pi$ .

**Case 5.** Let  $a, b$  and  $c$  be positive real numbers such that

$$a + b + c = abc, \quad \text{i.e.} \quad \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = 1.$$

Then according to *Case 3* we can take

$$\begin{aligned} \frac{1}{a} &= \tan \frac{\alpha}{2}, & \frac{1}{b} &= \tan \frac{\beta}{2}, & \frac{1}{c} &= \tan \frac{\gamma}{2}, \\ \text{i.e.} \quad a &= \cot \frac{\alpha}{2}, & b &= \cot \frac{\beta}{2}, & c &= \cot \frac{\gamma}{2}, \end{aligned}$$

where  $\alpha, \beta$  and  $\gamma$  are the angles of an arbitrary triangle.

**Case 6.** Let  $a, b, c$  be positive real numbers such that  $a + b + c = abc$ .

Then according to *Case 5* and *Case 1* we can use the following substitutions

$$a = \tan \alpha, \quad b = \tan \beta, \quad c = \tan \gamma,$$

where  $\alpha, \beta$  and  $\gamma$  are the angles of an acute triangle, i.e.  $\alpha, \beta, \gamma \in (0, \pi/2)$  and  $\alpha + \beta + \gamma = \pi$ .

**Case 7.** Let  $a, b, c$  be positive real numbers such that

$$a^2 + b^2 + c^2 + 2abc = 1.$$

Note that since the numbers  $a, b, c$  are positive we must have  $a, b, c < 1$ . Therefore due to Theorem 8.2 we can use the substitutions

$$a = \sin \frac{\alpha}{2}, \quad b = \sin \frac{\beta}{2}, \quad c = \sin \frac{\gamma}{2},$$

where  $\alpha, \beta$  and  $\gamma$  are the angles of an arbitrary triangle, i.e.  $\alpha, \beta, \gamma \in (0, \pi/2)$  and  $\alpha + \beta + \gamma = \pi$ .

**Case 8.** Let  $a, b, c$  be positive real numbers such that

$$a^2 + b^2 + c^2 + 2abc = 1.$$

Then according to *Case 7* and *Case 1* we can make the following substitutions

$$a = \cos \alpha, \quad b = \cos \beta, \quad c = \cos \gamma,$$

where  $\alpha, \beta$  and  $\gamma$  are the angles of an acute triangle.

**Case 9.** Let  $x, y, z$  be positive real numbers.

Then the expressions:

$$\sqrt{\frac{yz}{(x+y)(x+z)}}, \sqrt{\frac{zx}{(y+z)(y+x)}}, \sqrt{\frac{xy}{(z+x)(z+y)}}$$

with the substitutions from *Case 2* become

$$\sqrt{\frac{(s-b)(s-c)}{bc}}, \sqrt{\frac{(s-c)(s-a)}{ca}}, \sqrt{\frac{(s-a)(s-b)}{ab}},$$

where  $a, b, c$  are the length-sides of a triangle.

But let us notice that

$$\begin{aligned} \sin \frac{\alpha}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}}, & \sin \frac{\beta}{2} &= \sqrt{\frac{(s-c)(s-a)}{ca}}, \\ \sin \frac{\gamma}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}}. \end{aligned}$$

Therefore for the given expressions we can simply make the substitutions:  $\sin \frac{\alpha}{2}, \sin \frac{\beta}{2}, \sin \frac{\gamma}{2}$  (respectively), where  $\alpha, \beta$  and  $\gamma$  are the angles of a triangle.

**Case 10.** Similarly as in *Case 9*, for the expressions:

$$\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}}, \sqrt{\frac{y(x+y+z)}{(y+z)(y+x)}}, \sqrt{\frac{z(x+y+z)}{(z+x)(z+y)}},$$

we can make the substitutions  $\cos \frac{\alpha}{2}, \cos \frac{\beta}{2}, \cos \frac{\gamma}{2}$  (respectively), where  $\alpha, \beta$  and  $\gamma$  are the angles of a triangle.

Now we will give practical applications of this material, through exercises that will demonstrate how it works, and how useful is this apparatus, which is based on the aforementioned substitutions in certain cases.

## 8.2 Characteristic Examples Using Trigonometric Substitutions

**Exercise 8.1** Let  $x, y, z > 0$  be real numbers. Prove the inequality

$$\sqrt{x(y+z)} + \sqrt{y(z+x)} + \sqrt{z(x+y)} \geq 2\sqrt{\frac{(x+y)(y+z)(z+x)}{x+y+z}}.$$

*Solution* The given inequality is equivalent to

$$\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}} + \sqrt{\frac{y(x+y+z)}{(y+z)(y+x)}} + \sqrt{\frac{z(x+y+z)}{(z+x)(z+y)}} \geq 2.$$

According to *Case 10*, it suffices to show that

$$\cos \frac{\alpha}{2} + \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \geq 2,$$

where  $\alpha, \beta$  and  $\gamma$  are the angles of a triangle, i.e.  $\alpha, \beta, \gamma \in (0, \pi)$  and  $\alpha + \beta + \gamma = \pi$ . Due to *Case 1*, it remains to prove that

$$\sin \alpha + \sin \beta + \sin \gamma \geq 2,$$

where  $\alpha, \beta$  and  $\gamma$  are the angles of an acute triangle, i.e.  $\alpha, \beta, \gamma \in (0, \pi/2)$  and  $\alpha + \beta + \gamma = \pi$ .

Since  $\alpha \in (0, \pi/2]$  it follows that  $0 < \sin \alpha \leq 1$ , i.e.  $\sin \alpha \geq \sin^2 \alpha$ , and equality occurs if and only if  $\alpha = \pi/2$ .

Similarly for  $\beta, \gamma \in (0, \pi/2]$  we conclude that

$$\sin \beta \geq \sin^2 \beta \quad \text{and} \quad \sin \gamma \geq \sin^2 \gamma.$$

Thus we have

$$\begin{aligned} & \sin \alpha + \sin \beta + \sin \gamma \\ & \geq \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma \\ & = \frac{1 - \cos 2\alpha}{2} + \frac{1 - \cos 2\beta}{2} + \sin^2 \gamma = 1 - \frac{1}{2}(\cos 2\alpha + \cos 2\beta) + \sin^2 \gamma \\ & = 1 - \frac{1}{2}2\cos(\alpha + \beta)\cos(\alpha - \beta) + 1 - \cos^2 \gamma \\ & = 2 - \cos(\pi - \gamma)\cos(\alpha - \beta) - \cos^2 \gamma = 2 + \cos \gamma \cos(\alpha - \beta) - \cos^2 \gamma \\ & = 2 + \cos \gamma (\cos(\alpha - \beta) - \cos \gamma) = 2 + \cos \gamma [\cos(\alpha - \beta) - \cos(\pi - (\alpha + \beta))] \\ & = 2 + \cos \gamma (\cos(\alpha - \beta) + \cos(\alpha + \beta)) = 2 + 2\cos \gamma \cos \alpha \cos \beta > 2. \end{aligned}$$

**Exercise 8.2** Let  $a, b$  and  $c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$a^2 + b^2 + c^2 + 2\sqrt{3abc} \leq 1.$$

*Solution* After taking  $a = xy, b = yz, c = zx$ , inequality becomes

$$x^2y^2 + y^2z^2 + z^2x^2 + 2\sqrt{3}xyz \leq 1, \quad (8.10)$$

where  $x, y, z$  are positive real numbers such that

$$xy + yz + zx = 1. \quad (8.11)$$

Inequality (8.10) is equivalent to

$$(xy + yz + zx)^2 + 2\sqrt{3}xyz \leq 1 + 2xyz(x + y + z)$$

or

$$\sqrt{3} \leq x + y + z. \quad (8.12)$$

By (8.11) and according to *Case 3*, we can take

$$x = \tan \frac{\alpha}{2}, \quad y = \tan \frac{\beta}{2}, \quad z = \tan \frac{\gamma}{2},$$

where  $\alpha, \beta, \gamma \in (0, \pi)$  and  $\alpha + \beta + \gamma = \pi$ .

Then inequality (8.12) is equivalent to  $\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} \geq \sqrt{3}$ , which is  $N_{13}$ .

**Exercise 8.3** Let  $a, b, c \in (0, 1)$  be positive real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} \geq \frac{3}{4} \left( \frac{1-a^2}{a} + \frac{1-b^2}{b} + \frac{1-c^2}{c} \right).$$

*Solution* Since  $ab + bc + ca = 1$  and by *Case 3*, we use the following substitutions

$$a = \tan \frac{\alpha}{2}, \quad b = \tan \frac{\beta}{2}, \quad c = \tan \frac{\gamma}{2},$$

where  $\alpha, \beta$  and  $\gamma$  are the angles of a triangle.

Since  $a, b, c \in (0, 1)$ , it follows that  $\tan \frac{\alpha}{2}, \tan \frac{\beta}{2}, \tan \frac{\gamma}{2} \in (0, 1)$ , i.e. it follows that  $\alpha, \beta$  and  $\gamma$  are the angles of an acute triangle.

Also we have

$$\frac{a}{1-a^2} = \frac{\tan \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}} = \frac{\sin \frac{\alpha}{2} \cdot \cos \frac{\alpha}{2}}{\cos \alpha} = \frac{\sin \alpha}{2 \cos \alpha} = \frac{\tan \alpha}{2}.$$

Similarly

$$\frac{b}{1-b^2} = \frac{\tan \beta}{2} \quad \text{and} \quad \frac{c}{1-c^2} = \frac{\tan \gamma}{2}.$$

Therefore the given inequality becomes

$$\frac{\tan \alpha + \tan \beta + \tan \gamma}{2} \geq \frac{3}{4} \left( \frac{2}{\tan \alpha} + \frac{2}{\tan \beta} + \frac{2}{\tan \gamma} \right)$$

or

$$\tan \alpha + \tan \beta + \tan \gamma \geq 3 \left( \frac{1}{\tan \alpha} + \frac{1}{\tan \beta} + \frac{1}{\tan \gamma} \right). \quad (8.13)$$

By  $I_6$  we have that  $\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \cdot \tan \beta \cdot \tan \gamma$ . Thus it suffices to show that

$$\begin{aligned} (\tan \alpha + \tan \beta + \tan \gamma)^2 &\geq 3(\tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \gamma \tan \alpha) \\ \Leftrightarrow \frac{1}{2}((\tan \alpha - \tan \beta)^2 + (\tan \beta - \tan \gamma)^2 + (\tan \gamma - \tan \alpha)^2) &\geq 0. \end{aligned}$$

We are done.

**Exercise 8.4** Let  $x, y, z$  be positive real numbers. Prove the inequality

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(y+z)(y+x)}} + \frac{z}{z + \sqrt{(z+x)(z+y)}} \leq 1.$$

*Solution* Rewrite the given inequality as follows

$$\frac{1}{1 + \sqrt{\frac{(x+y)(x+z)}{x^2}}} + \frac{1}{1 + \sqrt{\frac{(y+z)(y+x)}{y^2}}} + \frac{1}{1 + \sqrt{\frac{(z+x)(z+y)}{z^2}}} \leq 1. \quad (8.14)$$

Since this is homogenous we may take  $xy + yz + zx = 1$ .

Therefore by *Case 3*, we can take

$$a = \tan \frac{\alpha}{2}, \quad b = \tan \frac{\beta}{2}, \quad c = \tan \frac{\gamma}{2},$$

where  $\alpha, \beta$  and  $\gamma$  are the angles of a triangle.

We have

$$\frac{(x+y)(x+z)}{x^2} = \frac{(\tan \frac{\alpha}{2} + \tan \frac{\beta}{2})(\tan \frac{\alpha}{2} + \tan \frac{\gamma}{2})}{\tan^2 \frac{\alpha}{2}} = \frac{1}{\sin^2 \frac{\alpha}{2}}.$$

Similarly

$$\frac{(y+z)(y+x)}{y^2} = \frac{1}{\sin^2 \frac{\beta}{2}} \quad \text{and} \quad \frac{(z+x)(z+y)}{z^2} = \frac{1}{\sin^2 \frac{\gamma}{2}}.$$

Thus inequality (8.14) becomes

$$\frac{\sin \frac{\alpha}{2}}{1 + \sin \frac{\alpha}{2}} + \frac{\sin \frac{\beta}{2}}{1 + \sin \frac{\beta}{2}} + \frac{\sin \frac{\gamma}{2}}{1 + \sin \frac{\gamma}{2}} \leq 1,$$

i.e.

$$\frac{1}{1 + \sin \frac{\alpha}{2}} + \frac{1}{1 + \sin \frac{\beta}{2}} + \frac{1}{1 + \sin \frac{\gamma}{2}} \geq 2. \quad (8.15)$$

Since  $AM \geq HM$  we obtain

$$\frac{1}{1 + \sin \frac{\alpha}{2}} + \frac{1}{1 + \sin \frac{\beta}{2}} + \frac{1}{1 + \sin \frac{\gamma}{2}} \geq \frac{9}{3 + \sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2}}. \quad (8.16)$$

According to  $N_3$ , we have that  $\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \leq \frac{3}{2}$ .

Finally, by the previous inequality and (8.16) we obtain

$$\frac{1}{1 + \sin \frac{\alpha}{2}} + \frac{1}{1 + \sin \frac{\beta}{2}} + \frac{1}{1 + \sin \frac{\gamma}{2}} \geq \frac{9}{3 + \frac{3}{2}} = 2$$

as required.

**Exercise 8.5** Let  $a, b, c$  ( $a, b, c \neq 1$ ) be non-negative real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$\frac{a}{1 - a^2} + \frac{b}{1 - b^2} + \frac{c}{1 - c^2} \geq \frac{3\sqrt{3}}{2}.$$

*Solution* Since  $ab + bc + ca = 1$  (Case 3) we take:

$$a = \tan \frac{\alpha}{2}, \quad b = \tan \frac{\beta}{2} \quad \text{and} \quad c = \tan \frac{\gamma}{2},$$

where  $\alpha, \beta, \gamma \in (0, \pi)$  and  $\alpha + \beta + \gamma = \pi$ .

Using the well-known identity  $\frac{\tan \frac{\alpha}{2}}{\tan^2 \frac{\alpha}{2} - 1} = \tan \alpha$ , we get that the given inequality is equivalent to  $\tan \alpha + \tan \beta + \tan \gamma \geq 3\sqrt{3}$ , which is  $N_{16}$ .

Equality occurs if and only if  $a = b = c = 1/\sqrt{3}$ .

**Exercise 8.6** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + bc + ac).$$

*Solution* Let  $a = \sqrt{2} \tan \alpha, b = \sqrt{2} \tan \beta, c = \sqrt{2} \tan \gamma$  where  $\alpha, \beta, \gamma \in (0, \pi/2)$ . Then using the well-known identity  $1 + \tan^2 x = \frac{1}{\cos^2 x}$  the given inequality becomes

$$\frac{8}{\cos^2 \alpha \cdot \cos^2 \beta \cdot \cos^2 \gamma} \geq 9 \left( \frac{2}{\tan \alpha \tan \beta} + \frac{2}{\tan \beta \tan \gamma} + \frac{2}{\tan \gamma \tan \alpha} \right),$$

i.e.

$$\cos \alpha \cos \beta \cos \gamma (\cos \alpha \sin \beta \sin \gamma + \sin \alpha \cos \beta \sin \gamma + \sin \alpha \sin \beta \cos \gamma) \leq \frac{4}{9}. \quad (8.17)$$

Also since

$$\begin{aligned} \cos(\alpha + \beta + \gamma) &= \cos \alpha \cos \beta \cos \gamma - \cos \alpha \sin \beta \sin \gamma - \sin \alpha \cos \beta \sin \gamma \\ &\quad - \sin \alpha \sin \beta \cos \gamma \end{aligned}$$

inequality (8.17) is equivalent to

$$\cos \alpha \cos \beta \cos \gamma (\cos \alpha \cos \beta \cos \gamma - \cos(\alpha + \beta + \gamma)) \leq \frac{4}{9}. \quad (8.18)$$

Let  $\theta = \frac{\alpha + \beta + \gamma}{3}$ .

Since  $\cos \alpha, \cos \beta, \cos \gamma > 0$ , and since the function  $\cos x$  is concave on  $(0, \pi/2)$  by the inequality  $AM \geq GM$  and *Jensen's inequality*, we obtain

$$\cos \alpha \cos \beta \cos \gamma \leq \left( \frac{\cos \alpha + \cos \beta + \cos \gamma}{3} \right)^3 \leq \cos^3 \theta.$$

Therefore according to (8.18) we need to prove that

$$\cos^3 \theta (\cos^3 \theta - \cos 3\theta) \leq \frac{4}{9}. \quad (8.19)$$

Using the trigonometric identity

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta, \quad \text{i.e.} \quad \cos^3 \theta - \cos 3\theta = 3 \cos \theta - 3 \cos^3 \theta$$

inequality (8.19) becomes

$$\cos^4 \theta (1 - \cos^2 \theta) \leq \frac{4}{27},$$

which follows by the inequality  $AM \geq GM$ :

$$\left( \frac{\cos^2 \theta}{2} \cdot \frac{\cos^2 \theta}{2} \cdot (1 - \cos^2 \theta) \right)^3 \leq \frac{1}{3} \left( \frac{\cos^2 \theta}{2} + \frac{\cos^2 \theta}{2} + (1 - \cos^2 \theta) \right) = \frac{1}{3}.$$

Equality occurs iff  $\tan \alpha = \tan \beta = \tan \gamma = \frac{1}{\sqrt{2}}$ , i.e. iff  $a = b = c = 1$ .

**Exercise 8.7** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\frac{a}{a+bc} + \frac{b}{b+ca} + \frac{\sqrt{abc}}{c+ab} \leq 1 + \frac{3\sqrt{3}}{4}.$$

*Solution* Since  $a + b + c = 1$  we use the following substitutions  $a = xy, b = yz, c = zx$ , where  $x, y, z > 0$  and the given inequality becomes

$$\frac{xy}{xy + (yz)(zx)} + \frac{yz}{yz + (zx)(xy)} + \frac{xyz}{zx + (xy)(yz)} \leq 1 + \frac{3\sqrt{3}}{4},$$

i.e.

$$\frac{1}{1+z^2} + \frac{1}{1+x^2} + \frac{y}{1+y^2} \leq 1 + \frac{3\sqrt{3}}{4} \quad (8.20)$$

where  $xy + yz + zx = 1$ .



Since  $xy + yz + zx = 1$  according to *Case 3* we may set  $x = \tan \frac{\alpha}{2}$ ,  $y = \tan \frac{\beta}{2}$ ,  $z = \tan \frac{\gamma}{2}$  where  $\alpha, \beta, \gamma \in (0, \pi)$ , and  $\alpha + \beta + \gamma = \pi$ .

Then inequality (8.20) becomes

$$\frac{1}{1 + \tan^2 \frac{\gamma}{2}} + \frac{1}{1 + \tan^2 \frac{\alpha}{2}} + \frac{\tan \frac{\beta}{2}}{1 + \tan^2 \frac{\beta}{2}} \leq 1 + \frac{3\sqrt{3}}{4},$$

i.e.

$$\cos^2 \frac{\gamma}{2} + \cos^2 \frac{\alpha}{2} + \frac{\sin \beta}{2} \leq 1 + \frac{3\sqrt{3}}{4}.$$

Using the trigonometric identity  $\cos x = 2 \cos^2 \frac{x}{2} - 1$  the last inequality becomes

$$\frac{\cos \gamma + 1}{2} + \frac{\cos \alpha + 1}{2} + \frac{\sin \beta}{2} \leq 1 + \frac{3\sqrt{3}}{4},$$

i.e.

$$\cos \gamma + \cos \alpha + \sin \beta \leq \frac{3\sqrt{3}}{2}. \quad (8.21)$$

We have

$$\begin{aligned} \cos \alpha + \cos \gamma + \sin \beta &= \cos \alpha + \cos \gamma + \sin(\pi - (\alpha + \gamma)) \\ &= \frac{2}{\sqrt{3}} \left( \frac{\sqrt{3}}{2} \cos \alpha + \frac{\sqrt{3}}{2} \cos \gamma \right) \\ &\quad + \frac{1}{\sqrt{3}} (\sqrt{3} \sin \alpha \cos \gamma + \sqrt{3} \cos \alpha \sin \gamma) \\ &\leq \frac{1}{\sqrt{3}} \left( \frac{3}{4} + \cos^2 \alpha + \frac{3}{4} + \cos^2 \gamma \right) \\ &\quad + \frac{1}{2\sqrt{3}} (3 \sin^2 \alpha + \cos^2 \gamma + \cos^2 \alpha + 3 \sin^2 \gamma) \\ &= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} (\cos^2 \alpha + \sin^2 \alpha) + \frac{\sqrt{3}}{2} (\cos^2 \gamma + \sin^2 \gamma) \\ &= \frac{3\sqrt{3}}{2}. \end{aligned}$$

## Chapter 9

# Hölder's Inequality, Minkowski's Inequality and Their Variants

In this chapter we'll introduce two very useful inequalities with broad practical usage: *Hölder's inequality* and *Minkowski's inequality*. We'll also present few variants of these inequalities. For that purpose we will firstly introduce the following theorem.

**Theorem 9.1** (Young's inequality) *Let  $a, b > 0$  and  $p, q > 1$  be real numbers such that  $1/p + 1/q = 1$ . Then  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ . Equality occurs if and only if  $a^p = b^q$ .*

*Proof* For  $f(x) = e^x$  we have  $f'(x) = f''(x) = e^x > 0$ , for any  $x \in \mathbb{R}$ .

Thus  $f(x)$  is convex on  $(0, \infty)$ .

If we put  $x = p \ln a$  and  $y = q \ln b$  then due to *Jensen's inequality* we obtain

$$\begin{aligned} f\left(\frac{x}{p} + \frac{y}{q}\right) &\leq \frac{1}{p}f(x) + \frac{1}{q}f(y) \\ \Leftrightarrow e^{\frac{x}{p} + \frac{y}{q}} &\leq \frac{e^x}{p} + \frac{e^y}{q} \\ \Leftrightarrow e^{\ln a + \ln b} &\leq \frac{e^{p \ln a}}{p} + \frac{e^{q \ln b}}{q} \\ \Leftrightarrow e^{\ln ab} &\leq \frac{e^{\ln a^p}}{p} + \frac{e^{\ln b^q}}{q} \\ \Leftrightarrow ab &\leq \frac{a^p}{p} + \frac{b^q}{q}. \end{aligned}$$

Equality occurs iff  $x = y$ , i.e. iff  $a^p = b^q$ . □

**Theorem 9.2** (Hölder's inequality) *Let  $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n$  be positive real numbers and  $p, q > 1$  be such that  $1/p + 1/q = 1$ .*

*Then*

$$\sum_{i=1}^n a_i b_i \leq \left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n b_i^q \right)^{\frac{1}{q}}.$$

*Equality occurs if and only if  $\frac{a_1^p}{b_1^q} = \frac{a_2^p}{b_2^q} = \dots = \frac{a_n^p}{b_n^q}$ .*

*Proof 1* By Young's inequality for

$$a = \frac{a_i}{\left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}}}, \quad b = \frac{b_i}{\left( \sum_{i=1}^n b_i^q \right)^{\frac{1}{q}}}, \quad i = 1, 2, \dots, n,$$

we obtain

$$\frac{a_i b_i}{\left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n b_i^q \right)^{\frac{1}{q}}} \leq \frac{1}{p} \frac{a_i^p}{\sum_{i=1}^n a_i^p} + \frac{1}{q} \frac{b_i^q}{\sum_{i=1}^n b_i^q}. \quad (9.1)$$

Adding the inequalities (9.1), for  $i = 1, 2, \dots, n$ , we obtain

$$\frac{\sum_{i=1}^n a_i b_i}{\left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n b_i^q \right)^{\frac{1}{q}}} \leq \frac{1}{p} \frac{\sum_{i=1}^n a_i^p}{\sum_{i=1}^n a_i^p} + \frac{1}{q} \frac{\sum_{i=1}^n b_i^q}{\sum_{i=1}^n b_i^q} = \frac{1}{p} + \frac{1}{q} = 1,$$

i.e.

$$\sum_{i=1}^n a_i b_i \leq \left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n b_i^q \right)^{\frac{1}{q}}.$$

Obviously equality occurs if and only if  $\frac{a_1^p}{b_1^q} = \frac{a_2^p}{b_2^q} = \dots = \frac{a_n^p}{b_n^q}$ . □

*Proof 2* The function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ ,  $f(x) = x^p$  for  $p > 1$  and  $p < 0$  is strictly convex, and for  $0 < p < 1$ ,  $f$  is strictly concave (Example 7.2).

Let  $p > 1$ , then by Jensen's inequality we obtain

$$\left( \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n} \right)^p \leq \frac{m_1 x_1^p + m_2 x_2^p + \dots + m_n x_n^p}{m_1 + m_2 + \dots + m_n},$$

i.e.

$$\left( \sum_{i=1}^n m_i x_i \right)^p \leq \left( \sum_{i=1}^n m_i \right)^{p-1} \cdot \left( \sum_{i=1}^n m_i x_i^p \right),$$

i.e.

$$\sum_{i=1}^n m_i x_i \leq \left( \sum_{i=1}^n m_i \right)^{\frac{p-1}{p}} \cdot \left( \sum_{i=1}^n m_i x_i^p \right)^{\frac{1}{p}}.$$

Since  $\frac{1}{p} + \frac{1}{q} = 1$  we obtain  $\frac{p-1}{p} = \frac{1}{q}$  and the last inequality becomes

$$\sum_{i=1}^n m_i x_i \leq \left( \sum_{i=1}^n m_i \right)^{\frac{1}{q}} \cdot \left( \sum_{i=1}^n m_i x_i^p \right)^{\frac{1}{p}}.$$

By taking  $m_i = b_i^q$  and  $x_i = a_i b_i^{1-q}$ , for  $i = 1, 2, \dots, n$  we obtain

$$\sum_{i=1}^n a_i b_i \leq \left( \sum_{i=1}^n b_i^q \right)^{\frac{1}{q}} \cdot \left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}}.$$

□

*Remark* For  $p = q = 2$  by Hölder's inequality we get the *Cauchy–Schwarz inequality*.

We'll introduce, without proof, two generalizations of Hölder's inequality.

**Theorem 9.3** (Weighted Hölder's inequality) *Let  $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n; m_1, m_2, \dots, m_n$  be three sequences of positive real numbers and  $p, q > 1$  be such that  $1/p + 1/q = 1$ .*

*Then*

$$\sum_{i=1}^n a_i b_i m_i \leq \left( \sum_{i=1}^n a_i^p m_i \right)^{\frac{1}{p}} \left( \sum_{i=1}^n b_i^q m_i \right)^{\frac{1}{q}}.$$

*Equality occurs iff  $\frac{a_1^p}{b_1^q} = \frac{a_2^p}{b_2^q} = \dots = \frac{a_n^p}{b_n^q}$ .*

**Theorem 9.4** (Generalized Hölder's inequality) *Let  $a_{ij}, i = 1, 2, \dots, m; j = 1, 2, \dots, n$ , be positive real numbers, and  $\alpha_1, \alpha_2, \dots, \alpha_n$  be positive real numbers such that  $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ .*

*Then*

$$\sum_{i=1}^m \left( \prod_{j=1}^n a_{ij}^{\alpha_j} \right) \leq \prod_{j=1}^n \left( \sum_{i=1}^m a_{ij} \right)^{\alpha_j}.$$

A very useful and frequently used form of Hölder's inequality is given in the next corollary.

**Corollary 9.1** Let  $a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3$  be positive real numbers. Then we have

$$(a_1^3 + a_2^3 + a_3^3)(b_1^3 + b_2^3 + b_3^3)(c_1^3 + c_2^3 + c_3^3) \geq (a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3)^3.$$

**Theorem 9.5** (First Minkowski's inequality) Let  $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n$  be positive real numbers and  $p > 1$ . Then

$$\left( \sum_{i=1}^n (a_i + b_i)^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n b_i^p \right)^{\frac{1}{p}}.$$

Equality occurs if and only if  $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n}$ .

*Proof* For  $p > 1$ , we choose  $q > 1$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , i.e.  $q = \frac{p}{p-1}$ .

By Hölder's inequality we have

$$\begin{aligned} \sum_{i=1}^n (a_i + b_i)^p &= \sum_{i=1}^n (a_i + b_i)(a_i + b_i)^{p-1} \\ &= \sum_{i=1}^n a_i (a_i + b_i)^{p-1} + \sum_{i=1}^n b_i (a_i + b_i)^{p-1} \\ &\leq \left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n ((a_i + b_i)^{p-1})^q \right)^{\frac{1}{q}} \\ &\quad + \left( \sum_{i=1}^n b_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n ((a_i + b_i)^{p-1})^q \right)^{\frac{1}{q}} \\ &= \left( \sum_{i=1}^n ((a_i + b_i)^{p-1})^q \right)^{\frac{1}{q}} \left( \left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n b_i^p \right)^{\frac{1}{p}} \right) \\ &= \left( \sum_{i=1}^n ((a_i + b_i)^{p-1})^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left( \left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n b_i^p \right)^{\frac{1}{p}} \right) \\ &= \left( \sum_{i=1}^n (a_i + b_i)^p \right)^{\frac{p-1}{p}} \left( \left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n b_i^p \right)^{\frac{1}{p}} \right), \end{aligned}$$

i.e. we obtain

$$\begin{aligned}
 \sum_{i=1}^n (a_i + b_i)^p &\leq \left( \sum_{i=1}^n (a_i + b_i)^p \right)^{\frac{p-1}{p}} \left( \left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n b_i^p \right)^{\frac{1}{p}} \right) \\
 &\Leftrightarrow \left( \sum_{i=1}^n (a_i + b_i)^p \right)^{1-\frac{p-1}{p}} \leq \left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n b_i^p \right)^{\frac{1}{p}} \\
 &\Leftrightarrow \left( \sum_{i=1}^n (a_i + b_i)^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n b_i^p \right)^{\frac{1}{p}}.
 \end{aligned}$$

Equality occurs if and only if  $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n}$ . (Why?) □

**Theorem 9.6** (Second Minkowski's inequality) *Let  $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n$  be positive real numbers and  $p > 1$ . Then*

$$\left( \left( \sum_{i=1}^n a_i \right)^p + \left( \sum_{i=1}^n b_i \right)^p \right)^{\frac{1}{p}} \leq \sum_{i=1}^n (a_i^p + b_i^p)^{\frac{1}{p}}.$$

*Equality occurs if and only if  $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n}$ .*

*Proof* The function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,  $f(x) = (1 + x^\alpha)^{\frac{1}{\alpha}}$ ,  $\alpha \neq 0$  for  $\alpha > 1$  is a strictly convex and for  $\alpha < 1$  is a strictly concave (Example 7.4).

By *Jensen's inequality* for  $p > 1$  we obtain

$$\begin{aligned}
 &\left( 1 + \left( \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n} \right)^p \right)^{1/p} \\
 &\leq \frac{m_1 (1 + x_1^p)^{1/p} + m_2 (1 + x_2^p)^{1/p} + \dots + m_n (1 + x_n^p)^{1/p}}{m_1 + m_2 + \dots + m_n},
 \end{aligned}$$

i.e.

$$\left( \left( \sum_{i=1}^n m_i \right)^p + \left( \sum_{i=1}^n m_i x_i \right)^p \right)^{1/p} \leq \sum_{i=1}^n (m_i^p + (m_i x_i)^p)^{1/p}.$$

If we take  $m_i = a_i$  and  $x_i = \frac{b_i}{a_i}$  for  $i = 1, 2, \dots, n$ , by the last inequality we obtain

$$\left( \left( \sum_{i=1}^n a_i \right)^p + \left( \sum_{i=1}^n b_i \right)^p \right)^{\frac{1}{p}} \leq \sum_{i=1}^n (a_i^p + b_i^p)^{\frac{1}{p}}.$$

□

**Theorem 9.7** (Third Minkowski's inequality) *Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be positive real numbers. Then*

$$\sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{b_1 b_2 \cdots b_n} \leq \sqrt[n]{(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)}.$$

*Equality occurs if and only if  $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$ .*

*Proof* The proof is a direct consequence of *Jensen's inequality* for the convex function  $f(x) = \ln(1 + e^x)$  (Example 7.3), with  $x_i = \ln \frac{b_i}{a_i}$ ,  $i = 1, 2, \dots, n$ .  $\square$

**Theorem 9.8** (Weighted Minkowski's inequality) *Let  $a_1, a_2, \dots, a_n$ ;  $b_1, b_2, \dots, b_n$ ;  $m_1, m_2, \dots, m_n$  be three sequences of positive real numbers and let  $p > 1$ . Then*

$$\left( \sum_{i=1}^n (a_i + b_i)^p m_i \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^n a_i^p m_i \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n b_i^p m_i \right)^{\frac{1}{p}}.$$

*Equality occurs if and only if  $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$ .*

*Remark* If  $0 < p < 1$  then in Theorem 9.5, Theorem 9.6 and Theorem 9.8 the inequality is reversed.

**Exercise 9.1** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$3(a^3 + b^3 + c^3)^2 \geq (a^2 + b^2 + c^2)^3.$$

*Solution* By Corollary 9.1 (or simply *Hölder's inequality*) we obtain

$$(a^3 + b^3 + c^3)(a^3 + b^3 + c^3)(1 + 1 + 1) \geq (a^2 + b^2 + c^2)^3,$$

i.e.

$$3(a^3 + b^3 + c^3)^2 \geq (a^2 + b^2 + c^2)^3.$$

**Exercise 9.2** Let  $a, b, c, x, y, z \in \mathbb{R}^+$ . Prove the inequality

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a + b + c)^3}{3(x + y + z)}.$$

*Solution* By the *generalized Hölder's inequality* (or simply *Hölder's inequality*) we have

$$\left( \frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \right)^{\frac{1}{3}} (1 + 1 + 1)^{\frac{1}{3}} (x + y + z)^{\frac{1}{3}} \geq a + b + c,$$

and the conclusion follows.

**Exercise 9.3** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$(a^a + b^a + c^a)(a^b + b^b + c^b)(a^c + b^c + c^c) \geq (\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c})^3.$$

*Solution* By Hölder's inequality we obtain

$$(a^a + b^a + c^a)^{\frac{1}{3}}(a^b + b^b + c^b)^{\frac{1}{3}}(a^c + b^c + c^c)^{\frac{1}{3}} \geq a^{\frac{a+b+c}{3}} + b^{\frac{a+b+c}{3}} + c^{\frac{a+b+c}{3}}.$$

Since  $a + b + c = 1$ , the conclusion follows.

**Exercise 9.4** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$3(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq (ab + bc + ca)^3.$$

*Solution* By Hölder's inequality for the triples:

$$\begin{aligned}(a_1, a_2, a_3) &= (1, 1, 1), & (b_1, b_2, b_3) &= (\sqrt[3]{a^2b}, \sqrt[3]{b^2c}, \sqrt[3]{c^2a}), \\ (c_1, c_2, c_3) &= (\sqrt[3]{b^2a}, \sqrt[3]{c^2b}, \sqrt[3]{a^2c}),\end{aligned}$$

we obtain the given inequality.

**Exercise 9.5** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{abc}{(1+a)(a+b)(b+c)(c+16)} \leq \frac{1}{81}.$$

*Solution* By Hölder's inequality we have

$$\begin{aligned}(1+a)(a+b)(b+c)(c+16) &\geq (\sqrt[4]{1 \cdot a \cdot b \cdot c} + \sqrt[4]{a \cdot b \cdot c \cdot 16})^4 \\ &= (3\sqrt[4]{abc})^4 = 81abc.\end{aligned}$$

Equality occurs if and only if  $\frac{1}{a} = \frac{a}{b} = \frac{b}{c} = \frac{c}{16}$ , i.e.  $a = 2, b = 4, c = 8$ .

**Exercise 9.6** Let  $x, y, z$  be positive real numbers such that  $xy + yz + zx + xyz = 4$ . Prove the inequality

$$\sqrt{x+2} + \sqrt{y+2} + \sqrt{z+2} \geq 3\sqrt{3}.$$

*Solution* Let us denote  $x+2 = a, y+2 = b$  and  $z+2 = c$ . Then the condition  $xy + yz + zx + xyz = 4$  becomes

$$abc = ab + bc + ca,$$

i.e.

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$



By Hölder's inequality we have

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 3^3,$$

and since  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$  we get

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 \geq 3^3,$$

i.e.

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq 3\sqrt{3},$$

as required.

**Exercise 9.7** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{a}{\sqrt{7+b+c}} + \frac{b}{\sqrt{7+c+a}} + \frac{c}{\sqrt{7+a+b}} \geq 1.$$

*Solution* Let us denote

$$A = \frac{a}{\sqrt{7+b+c}} + \frac{b}{\sqrt{7+c+a}} + \frac{c}{\sqrt{7+a+b}}$$

and

$$B = a(7+b+c) + b(7+c+a) + c(7+a+b).$$

By Hölder's inequality we obtain

$$A^2 B \geq (a+b+c)^3.$$

It remains to prove that

$$(a+b+c)^3 \geq B = 7(a+b+c) + 2(ab+bc+ca).$$

Since  $AM \geq GM$  we deduce that

$$a+b+c \geq 3\sqrt[3]{abc} = 3,$$

so it follows that

$$\begin{aligned} (a+b+c)^3 &\geq 3(a+b+c)^2 = \frac{7}{3}(a+b+c)^2 + \frac{2}{3}(a+b+c)^2 \\ &\geq \frac{7}{3} \cdot 3(a+b+c) + 2(ab+bc+ca) \\ &= 7(a+b+c) + 2(ab+bc+ca). \end{aligned}$$

**Exercise 9.8** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\frac{a}{\sqrt[3]{a+2b}} + \frac{b}{\sqrt[3]{b+2c}} + \frac{c}{\sqrt[3]{c+2a}} \geq 1.$$

*Solution* Let us denote

$$A = \frac{a}{\sqrt[3]{a+2b}} + \frac{b}{\sqrt[3]{b+2c}} + \frac{c}{\sqrt[3]{c+2a}}$$

and

$$B = a(a+2b) + b(b+2c) + c(c+2a) = (a+b+c)^2 = 1.$$

By Hölder's inequality we have

$$A^3 B \geq (a+b+c)^4, \quad \text{i.e.} \quad A^3 \geq (a+b+c)^2 = 1,$$

from which it follows that  $A \geq 1$ .

Equality occurs iff  $a = b = c = 1/3$ .

**Exercise 9.9** Let  $p \geq 1$  be an arbitrary real number. Prove that for any positive integer  $n$  we have

$$1^p + 2^p + \cdots + n^p \geq n \cdot \left(\frac{n+1}{2}\right)^p.$$

*Solution* If  $p = 1$  then the given inequality is true, i.e. it becomes equality.

So let  $p > 1$ .

We take  $x_1 = 1, x_2 = 2, \dots, x_n = n$  and  $y_1 = n, y_2 = n-1, \dots, y_n = 1$ .

By Minkowski's inequality we have

$$((1+n)^p + (1+n)^p + \cdots + (1+n)^p)^{\frac{1}{p}} \leq 2(1^p + 2^p + \cdots + n^p)^{\frac{1}{p}},$$

i.e.

$$n(1+n)^p \leq 2^p(1^p + 2^p + \cdots + n^p)$$

or

$$1^p + 2^p + \cdots + n^p \geq n \cdot \left(\frac{n+1}{2}\right)^p,$$

as required.

Equality occurs iff  $n = 1$ . (Why?)

**Exercise 9.10** Let  $x, y, z$  be positive real numbers. Prove the inequality

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(y+z)(y+x)}} + \frac{z}{z + \sqrt{(z+y)(z+x)}} \leq 1.$$

*Solution* By Hölder's inequality for  $n = 2$  and  $p = q = 2$ , we obtain

$$\begin{aligned}\sqrt{(x+y)(x+z)} &= \left((\sqrt{x})^2 + (\sqrt{y})^2\right)^{\frac{1}{2}} \left((\sqrt{z})^2 + (\sqrt{x})^2\right)^{\frac{1}{2}} \\ &\geq \sqrt{x} \cdot \sqrt{z} + \sqrt{y} \cdot \sqrt{x} = \sqrt{xz} + \sqrt{xy},\end{aligned}$$

i.e.

$$\frac{1}{\sqrt{(x+y)(x+z)}} \leq \frac{1}{\sqrt{xz} + \sqrt{xy}} = \frac{1}{\sqrt{x}(\sqrt{y} + \sqrt{z})}.$$

So it follows that

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} \leq \frac{x}{x + \sqrt{x}(\sqrt{y} + \sqrt{z})} = \frac{\sqrt{x}}{\sqrt{x} + \sqrt{y} + \sqrt{z}}.$$

Similarly

$$\begin{aligned}\frac{y}{y + \sqrt{(y+z)(y+x)}} &\leq \frac{\sqrt{y}}{\sqrt{x} + \sqrt{y} + \sqrt{z}} \quad \text{and} \\ \frac{z}{z + \sqrt{(z+y)(z+x)}} &\leq \frac{\sqrt{z}}{\sqrt{x} + \sqrt{y} + \sqrt{z}}.\end{aligned}$$

Adding the last three inequalities yields

$$\begin{aligned}\frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(y+z)(y+x)}} + \frac{z}{z + \sqrt{(z+y)(z+x)}} \\ \leq \frac{\sqrt{x} + \sqrt{y} + \sqrt{z}}{\sqrt{x} + \sqrt{y} + \sqrt{z}} = 1,\end{aligned}$$

as required. Equality occurs iff  $x = y = z$ .

**Exercise 9.11** Let  $x, y, z > 0$  be real numbers. Prove the inequality

$$\begin{aligned}\sqrt{x^2 + xy + y^2} + \sqrt{y^2 + yz + z^2} + \sqrt{z^2 + zx + x^2} \\ \geq 3\sqrt{xy + yz + zx}.\end{aligned}$$

*Solution* By Hölder's inequality we have

$$\begin{aligned}xy + yz + zx &= (x^2)^{1/3}(xy)^{1/3}(y^2)^{1/3} + (y^2)^{1/3}(yz)^{1/3}(z^2)^{1/3} \\ &\quad + (z^2)^{1/3}(zx)^{1/3}(x^2)^{1/3} \\ &\leq (x^2 + xy + y^2)^{1/3}(y^2 + yz + z^2)^{1/3}(z^2 + zx + x^2)^{1/3},\end{aligned}$$

i.e.

$$(xy + yz + zx)^3 \leq (x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2). \quad (9.2)$$

Since  $AM \geq GM$  and by (9.2) we obtain

$$\begin{aligned} & \left( \frac{1}{3} (\sqrt{x^2 + xy + y^2} + \sqrt{y^2 + yz + z^2} + \sqrt{z^2 + zx + x^2}) \right)^3 \\ & \geq \sqrt{x^2 + xy + y^2} \cdot \sqrt{y^2 + yz + z^2} \sqrt{z^2 + zx + x^2} \geq \sqrt{(xy + yz + zx)^3}, \end{aligned}$$

i.e. we have

$$\sqrt{x^2 + xy + y^2} + \sqrt{y^2 + yz + z^2} + \sqrt{z^2 + zx + x^2} \geq 3\sqrt{xy + yz + zx},$$

as required. Equality occurs iff  $x = y = z$ .



# Chapter 10

## Generalizations of the Cauchy–Schwarz Inequality, Chebishev’s Inequality and the Mean Inequalities

In Chap. 4 we presented the *Cauchy–Schwarz inequality*, *Chebichev’s inequality* and the *mean inequalities*. In this section we will give their generalizations. The proof of first theorem is left to the reader, since it is similar to the proof of *Cauchy–Schwarz inequality*.

**Theorem 10.1** (Weighted Cauchy–Schwarz inequality) *Let  $a_i, b_i \in \mathbb{R}, i = 1, 2, \dots, n$ , be real numbers and let  $m_i \in \mathbb{R}^+, i = 1, 2, \dots, n$ . Then we have the inequality*

$$\left( \sum_{i=1}^n a_i b_i m_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 m_i \right) \left( \sum_{i=1}^n b_i^2 m_i \right).$$

*Equality occurs iff  $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n}$ .*

**Theorem 10.2** *Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be two sequences of non-negative real numbers and  $c_i > 0, i = 1, 2, \dots, n$ , such that  $\frac{a_1}{c_1} \geq \frac{a_2}{c_2} \geq \dots \geq \frac{a_n}{c_n}$  and  $\frac{b_1}{c_1} \geq \frac{b_2}{c_2} \geq \dots \geq \frac{b_n}{c_n}$  (the sequences  $(\frac{a_i}{c_i})$  and  $(\frac{b_i}{c_i})$  have the same orientation). Then*

$$\sum_{i=1}^n \frac{a_i b_i}{c_i} \geq \frac{\sum_{i=1}^n a_i \cdot \sum_{i=1}^n b_i}{\sum_{i=1}^n c_i} \quad (10.1)$$

*i.e.*

$$\frac{a_1 b_1}{c_1} + \frac{a_2 b_2}{c_2} + \dots + \frac{a_n b_n}{c_n} \geq \frac{(a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n)}{c_1 + c_2 + \dots + c_n}.$$

*Proof* In the proof we shall use the following lemma which can be easily proved using the principle of mathematical induction.

**Lemma 10.1** *Let  $a_1, a_2, \dots, a_n$  be non-negative real numbers, and  $c_i > 0$ ,  $i = 1, 2, \dots, n$ , such that  $\frac{a_1}{c_1} \geq \frac{a_2}{c_2} \geq \dots \geq \frac{a_n}{c_n}$ . Then*

$$\frac{a_1 + a_2 + \dots + a_k}{c_1 + c_2 + \dots + c_k} \geq \frac{a_n}{c_n}, \quad \text{for any } k = 1, 2, \dots, n.$$

We shall prove inequality (10.1) by mathematical induction.

For  $n = 1$ , we have equality in (10.1).

For  $n = 2$  we need to prove that

$$\frac{a_1 b_1}{c_1} + \frac{a_2 b_2}{c_2} \geq \frac{(a_1 + a_2)(b_1 + b_2)}{c_1 + c_2},$$

which is equivalent to  $(a_1 c_2 - a_2 c_1)(b_1 c_2 - b_2 c_1) \geq 0$ .

The last inequality holds since we have  $\frac{a_1}{c_1} \geq \frac{a_2}{c_2}$  and  $\frac{b_1}{c_1} \geq \frac{b_2}{c_2}$ .

Let us assume that for non-negative real numbers  $a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_k$  and  $c_i > 0, i = 1, 2, \dots, k$ , such that  $\frac{a_1}{c_1} \geq \frac{a_2}{c_2} \geq \dots \geq \frac{a_k}{c_k}$  and  $\frac{b_1}{c_1} \geq \frac{b_2}{c_2} \geq \dots \geq \frac{b_k}{c_k}$  inequality (10.1) holds for  $n = k$ , i.e.

$$\frac{a_1 b_1}{c_1} + \frac{a_2 b_2}{c_2} + \dots + \frac{a_k b_k}{c_k} \geq \frac{(a_1 + a_2 + \dots + a_k)(b_1 + b_2 + \dots + b_k)}{c_1 + c_2 + \dots + c_k}. \quad (10.2)$$

For  $n = k + 1$ , for non-negative real numbers  $a_1, a_2, \dots, a_{k+1}; b_1, b_2, \dots, b_{k+1}$  and  $c_i > 0, i = 1, 2, \dots, k + 1$  such that

$$\frac{a_1}{c_1} \geq \frac{a_2}{c_2} \geq \dots \geq \frac{a_{k+1}}{c_{k+1}} \quad \text{and} \quad \frac{b_1}{c_1} \geq \frac{b_2}{c_2} \geq \dots \geq \frac{b_{k+1}}{c_{k+1}},$$

we have

$$\begin{aligned} & \frac{a_1 b_1}{c_1} + \frac{a_2 b_2}{c_2} + \dots + \frac{a_k b_k}{c_k} + \frac{a_{k+1} b_{k+1}}{c_{k+1}} \\ & \stackrel{(10.2)}{\geq} \frac{(a_1 + a_2 + \dots + a_k)(b_1 + b_2 + \dots + b_k)}{c_1 + c_2 + \dots + c_k} + \frac{a_{k+1} b_{k+1}}{c_{k+1}} \\ & \geq \frac{(a_1 + a_2 + \dots + a_k + a_{k+1})(b_1 + b_2 + \dots + b_{k+1})}{c_1 + c_2 + \dots + c_{k+1}}, \end{aligned}$$

where the last inequality is true according to the case  $n = 2$  and Lemma 10.1.  $\square$

**Remark 10.1** If the sequences  $(\frac{a_i}{c_i})$  and  $(\frac{b_i}{c_i})$  have opposite orientation then in Theorem 10.1 we have the reverse inequality, i.e., we have  $\sum_{i=1}^n \frac{a_i b_i}{c_i} \leq \frac{\sum_{i=1}^n a_i \cdot \sum_{i=1}^n b_i}{\sum_{i=1}^n c_i}$ .

**Remark 10.2** For  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ ,  $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$  and  $0 < c_1 \leq c_2 \leq \cdots \leq c_n$  the required condition from Theorem 10.2 is satisfied, so we also have that

$$\sum_{i=1}^n \frac{a_i b_i}{c_i} \geq \frac{\sum_{i=1}^n a_i \cdot \sum_{i=1}^n b_i}{\sum_{i=1}^n c_i}.$$

If in Theorem 10.2 we put  $a_i = c_i x_i$ ,  $b_i = c_i y_i$  and  $m_i = \frac{c_i}{\sum_{i=1}^n c_i}$ ,  $i = 1, 2, \dots, n$ , then clearly  $\sum_{i=1}^n m_i = 1$  and the following theorem is obtained:

**Theorem 10.3** (Weighted Chebishev’s inequality) *Let  $a_1 \leq a_2 \leq \cdots \leq a_n$ ;  $b_1 \leq b_2 \leq \cdots \leq b_n$  be real numbers and let  $m_1, m_2, \dots, m_n$  be non-negative real numbers such that  $m_1 + m_2 + \cdots + m_n = 1$ .*

*Then*

$$\left( \sum_{i=1}^n a_i m_i \right) \left( \sum_{i=1}^n b_i m_i \right) \leq \sum_{i=1}^n a_i b_i m_i.$$

*Equality occurs iff  $a_1 = a_2 = \cdots = a_n$  or  $b_1 = b_2 = \cdots = b_n$ .*

**Note** If in the above Theorem 10.1 (Theorem 10.3) we choose  $m_1 = m_2 = \cdots = m_n$  ( $m_1 = m_2 = \cdots = m_n = \frac{1}{n}$ ), we get the *Cauchy–Schwarz inequality*, and *Chebichev’s inequality*, respectively.

**Exercise 10.1** Let  $a_1, a_2, \dots, a_n$  be the lengths of the sides of a given  $n$ -gon ( $n \geq 3$ ) and let  $s = a_1 + a_2 + \cdots + a_n$ . Prove the inequality

$$\frac{a_1}{s - 2a_1} + \frac{a_2}{s - 2a_2} + \cdots + \frac{a_n}{s - 2a_n} \geq \frac{n}{n - 2}.$$

**Solution** Without loss of generality we may assume that  $a_1 \geq a_2 \geq \cdots \geq a_n$ . Then clearly  $0 < s - 2a_1 \leq s - 2a_2 \leq \cdots \leq s - 2a_n$ .

According to Theorem 10.2 we obtain

$$\begin{aligned} \frac{a_1}{s - 2a_1} + \frac{a_2}{s - 2a_2} + \cdots + \frac{a_n}{s - 2a_n} &= \frac{a_1 \cdot 1}{s - 2a_1} + \frac{a_2 \cdot 1}{s - 2a_2} + \cdots + \frac{a_n \cdot 1}{s - 2a_n} \\ &\geq \frac{(a_1 + a_2 + \cdots + a_n)n}{ns - 2(a_1 + a_2 + \cdots + a_n)} \\ &= \frac{ns}{s(n - 2)} = \frac{n}{n - 2}. \end{aligned}$$

**Exercise 10.2** Let  $M$  be the centroid of the triangle  $ABC$ , and let  $k$  be its circumscribed circle. Let  $MA \cap k = \{A_1\}$ ,  $MB \cap k = \{B_1\}$  and  $MC \cap k = \{C_1\}$ . Prove the inequality

$$\overline{MA} + \overline{MB} + \overline{MC} \leq \overline{MA_1} + \overline{MB_1} + \overline{MC_1}.$$



*Solution* Denote  $\overline{BC} = a$ ,  $\overline{AC} = b$  and  $\overline{AB} = c$ . Let  $A'$ ,  $B'$  and  $C'$  be the midpoints of the sides  $BC$ ,  $AC$  and  $AB$ , respectively.

Without loss of generality we may assume that  $a \leq b \leq c$ , and then we may easily conclude that  $\overline{MC} \leq \overline{MB} \leq \overline{MA}$ .

Also by the power of a point we have  $\frac{3}{2}\overline{MA} \cdot \overline{A'A_1} = \frac{1}{4}a^2$  from which it follows that

$$\overline{A'A_1} = \frac{a^2}{6\overline{MA}}, \quad \text{i.e.} \quad \overline{MA_1} = \frac{1}{2}\overline{MA} + \overline{A'A_1} = \frac{1}{2}\overline{MA} + \frac{a^2}{6\overline{MA}}.$$

Analogously we obtain

$$\overline{MB_1} = \frac{1}{2}\overline{MB} + \frac{b^2}{6\overline{MB}} \quad \text{and} \quad \overline{MC_1} = \frac{1}{2}\overline{MC} + \frac{c^2}{6\overline{MC}}.$$

So it suffices to prove the inequality

$$\frac{a^2}{3\overline{MA}} + \frac{b^2}{3\overline{MB}} + \frac{c^2}{3\overline{MC}} \geq \overline{MA} + \overline{MB} + \overline{MC}.$$

According to Theorem 10.2 we have

$$\begin{aligned} \frac{a^2}{3\overline{MA}} + \frac{b^2}{3\overline{MB}} + \frac{c^2}{3\overline{MC}} &= \frac{a^2 \cdot 1}{3\overline{MA}} + \frac{b^2 \cdot 1}{3\overline{MB}} + \frac{c^2 \cdot 1}{3\overline{MC}} \geq \frac{3(a^2 + b^2 + c^2)}{3(\overline{MA} + \overline{MB} + \overline{MC})} \\ &= \frac{a^2 + b^2 + c^2}{\overline{MA} + \overline{MB} + \overline{MC}} \geq \frac{3(\overline{MA}^2 + \overline{MB}^2 + \overline{MC}^2)}{\overline{MA} + \overline{MB} + \overline{MC}} \\ &\geq \overline{MA} + \overline{MB} + \overline{MC}, \end{aligned}$$

as required.

Before introducing the *power mean inequality* we'll give following definition.

**Definition 10.1** Let  $a = (a_1, a_2, \dots, a_n)$  be a sequence of positive real numbers and  $r \neq 0$  be real number. Then the *power mean*  $M_r(a)$ , of order  $r$ , is defined as follows:  $M_r(a) = \left( \frac{a_1^r + a_2^r + \dots + a_n^r}{n} \right)^{\frac{1}{r}}$ .

For  $r = 1, r = 2, r = -1$  we get  $M_1(a), M_2(a), M_{-1}(a)$ , which represent the arithmetic, quadratic and harmonic means of the numbers  $a_1, a_2, \dots, a_n$ , respectively.

If  $r$  tends to 0 then it may be shown that  $M_r(a)$  tends to the geometric mean of the numbers  $a_1, a_2, \dots, a_n$ , i.e.  $M_0(a) = \sqrt[n]{a_1 a_2 \cdots a_n}$ .

Also if  $r \rightarrow -\infty$  then  $M_r(a) \rightarrow \min\{a_1, a_2, \dots, a_n\}$ , and if  $r \rightarrow \infty$  then  $M_r(a) \rightarrow \max\{a_1, a_2, \dots, a_n\}$ .

**Theorem 10.4** (Power mean inequality) *Let  $a = (a_1, a_2, \dots, a_n)$  be a sequence of positive real numbers and  $r \neq 0$  be real number. Then  $M_r(a) \leq M_s(a)$ , for any real numbers  $r \leq s$ .*

**Exercise 10.3** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$(a^2 + b^2 + c^2)^3 \leq 3(a^3 + b^3 + c^3)^2.$$

*Solution* By the power mean inequality we have that

$$\begin{aligned} M_2(a, b, c) &\leq M_3(a, b, c) \\ \Leftrightarrow \sqrt{\frac{a^2 + b^2 + c^2}{3}} &\leq \sqrt[3]{\frac{a^3 + b^3 + c^3}{3}} \\ \Leftrightarrow (a^2 + b^2 + c^2)^3 &\leq 3(a^3 + b^3 + c^3)^2. \end{aligned}$$

**Definition 10.2** Let  $m = (m_1, m_2, \dots, m_n)$  be a sequence of non-negative real numbers such that  $m_1 + m_2 + \dots + m_n = 1$ . Then the *weighted power mean*  $M_r^m(a)$ , of order  $r$  ( $r \neq 0$ ), for the sequence  $a = (a_1, a_2, \dots, a_n)$  is defined as  $M_r^m(a) = (a_1^r m_1 + a_2^r m_2 + \dots + a_n^r m_n)^{\frac{1}{r}}$ .

*Example 10.1* If  $m_1 = m_2 = \dots = m_n = \frac{1}{n}$  then  $M_r^m(x) = M_r(x)$ .

*Example 10.2* If  $n = 3, r = 4; m_1 = \frac{1}{2}, m_2 = \frac{1}{3}, m_3 = \frac{1}{6}$ , then

$$M_4^m(x, y, z) = \left( \frac{1}{2} \cdot x^4 + \frac{1}{3} \cdot y^4 + \frac{1}{6} \cdot z^4 \right)^{\frac{1}{4}}.$$

**Theorem 10.5** (Weighted power mean inequality) *Let  $a = (a_1, a_2, \dots, a_n)$  be a sequence of positive real numbers, and let  $m = (m_1, m_2, \dots, m_n)$  also be a sequence of positive real numbers such that  $m_1 + m_2 + \dots + m_n = 1$ . Then for each  $r \leq s$  we have*

$$M_r^m(a) \leq M_s^m(a),$$

*i.e.*

$$(m_1 a_1^r + m_2 a_2^r + \dots + m_n a_n^r)^{\frac{1}{r}} \leq (m_1 a_1^s + m_2 a_2^s + \dots + m_n a_n^s)^{\frac{1}{s}}.$$

*Proof* We shall use the fact that the power function  $f(x) = x^\alpha$  is convex for  $\alpha > 1$  or  $\alpha < 0$ , and it is concave for  $0 < \alpha < 1$ .

First we prove the inequality in the case  $r < s$  where both  $s$  and  $r$  are different from 0.

Three sub-cases may to be considered: 1°  $0 < r < s$ , 2°  $r < 0 < s$  and 3°  $r < s < 0$ .

1°  $0 < r < s$ . Since  $\frac{s}{r} > 1$  we conclude that  $f(x) = x^{\frac{s}{r}}$  is convex, so according to *Jensen's inequality*:

$$f(m_1x_1 + m_2x_2 + \cdots + m_nx_n) \leq m_1f(x_1) + m_2f(x_2) + \cdots + m_nf(x_n),$$

where  $m_1 + m_2 + \cdots + m_n = 1$  we have

$$(m_1x_1 + m_2x_2 + \cdots + m_nx_n)^{s/r} \leq m_1x_1^{s/r} + m_2x_2^{s/r} + \cdots + m_nx_n^{s/r}.$$

For  $x_i = a_i^r$ ,  $i = 1, 2, \dots, n$ , from the last inequality we obtain

$$(m_1a_1^r + m_2a_2^r + \cdots + m_na_n^r)^{s/r} \leq m_1a_1^s + m_2a_2^s + \cdots + m_na_n^s,$$

i.e.

$$(m_1a_1^r + m_2a_2^r + \cdots + m_na_n^r)^{1/r} \leq (m_1a_1^s + m_2a_2^s + \cdots + m_na_n^s)^{1/s},$$

so inequality holds in this case.

2°  $r < 0 < s$ . Then since  $\frac{s}{r} < 0$  we have that  $f(x) = x^{\frac{s}{r}}$  is a convex function. The rest of the proof in this case is the same as in case 1°.

3°  $r < s < 0$ . Then since  $0 < \frac{s}{r} < 1$  we have that  $f(x) = x^{\frac{s}{r}}$  is a concave function and according to *Jensen's inequality* for concave functions we obtain

$$(m_1x_1 + m_2x_2 + \cdots + m_nx_n)^{s/r} \geq m_1x_1^{s/r} + m_2x_2^{s/r} + \cdots + m_nx_n^{s/r}.$$

For  $x_i = a_i^r$ ,  $i = 1, 2, \dots, n$ , from the last inequality we obtain

$$(m_1a_1^r + m_2a_2^r + \cdots + m_na_n^r)^{s/r} \geq m_1a_1^s + m_2a_2^s + \cdots + m_na_n^s,$$

and since  $r < s < 0$  we obtain

$$(m_1a_1^r + m_2a_2^r + \cdots + m_na_n^r)^{1/r} \leq (m_1a_1^s + m_2a_2^s + \cdots + m_na_n^s)^{1/s}.$$

The cases when some values of  $s$  and  $r$  equal 0 are covered by the fact that the function  $t \rightarrow M_t^m(a)$  is a continuous function.  $\square$

**Exercise 10.4** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{(a + 2b + 3c)^2}{a^2 + 2b^2 + 3c^2} \leq 6.$$

**Solution** For  $m_1 = \frac{1}{6}, m_2 = \frac{2}{6}, m_3 = \frac{3}{6}, n = 3$  by the inequality

$$M_1^m(a, b, c) \leq M_2^m(a, b, c),$$

which is true due to the *weighted power mean inequality*, we obtain

$$\frac{a + 2b + 3c}{6} \leq \sqrt{\frac{a^2 + 2b^2 + 3c^2}{6}}, \quad \text{i.e.} \quad \frac{(a + 2b + 3c)^2}{a^2 + 2b^2 + 3c^2} \leq 6.$$

**Exercise 10.5** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 6$ . Prove the inequality

$$\sqrt[3]{ab + bc} + \sqrt[3]{bc + ca} + \sqrt[3]{ca + ab} \leq 6.$$

**Solution** By the *power mean inequality* we have

$$\frac{\sqrt[3]{ab + bc} + \sqrt[3]{bc + ca} + \sqrt[3]{ca + ab}}{3} \leq \sqrt[3]{\frac{(ab + bc) + (bc + ca) + (ca + ab)}{3}},$$

i.e.

$$\sqrt[3]{ab + bc} + \sqrt[3]{bc + ca} + \sqrt[3]{ca + ab} \leq \sqrt[3]{18(ab + bc + ca)}. \quad (10.3)$$

Since  $ab + bc + ca \leq \frac{(a+b+c)^2}{3} = 12$  by (10.3) we obtain

$$\sqrt[3]{ab + bc} + \sqrt[3]{bc + ca} + \sqrt[3]{ca + ab} \leq \sqrt[3]{18 \cdot 12} = 6.$$

Equality occurs if and only if  $a = b = c = 2$ .

**Exercise 10.6** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\sqrt[3]{a^2 + bc} + \sqrt[3]{b^2 + ca} + \sqrt[3]{c^2 + ab} \leq 3\sqrt[3]{2}.$$

**Solution** By the *power mean inequality* and the well-known inequality  $ab + bc + ca \leq a^2 + b^2 + c^2$  we have

$$\begin{aligned} \sqrt[3]{a^2 + bc} + \sqrt[3]{b^2 + ca} + \sqrt[3]{c^2 + ab} &\leq \sqrt[3]{9(a^2 + b^2 + c^2 + ab + bc + ca)} \\ &\leq \sqrt[3]{18(a^2 + b^2 + c^2)} = \sqrt[3]{18 \cdot 3} = 3\sqrt[3]{2}. \end{aligned}$$

Equality occurs if and only if  $a = b = c = 1$ .

**Exercise 10.7** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\sqrt[4]{5a^2 + 4(b + c) + 3} + \sqrt[4]{5b^2 + 4(c + a) + 3} + \sqrt[4]{5c^2 + 4(a + b) + 3} \leq 6.$$

*Solution* By the *power mean inequality* we have

$$\begin{aligned} & \sqrt[4]{5a^2 + 4(b+c) + 3} + \sqrt[4]{5b^2 + 4(c+a) + 3} + \sqrt[4]{5c^2 + 4(a+b) + 3} \\ & \leq \sqrt[4]{27(5(a^2 + b^2 + c^2) + 8(a+b+c) + 9)}. \end{aligned}$$

Since  $a^2 + b^2 + c^2 = 3$  we have  $a + b + c \leq \sqrt{3(a^2 + b^2 + c^2)} = 3$  and therefore

$$\begin{aligned} & \sqrt[4]{5a^2 + 4(b+c) + 3} + \sqrt[4]{5b^2 + 4(c+a) + 3} + \sqrt[4]{5c^2 + 4(a+b) + 3} \\ & \leq \sqrt[4]{27(5 \cdot 3 + 8 \cdot 3 + 9)} = 6. \end{aligned}$$

Equality occurs if and only if  $a = b = c = 1$ .

**Exercise 10.8** Let  $x, y, z$  be non-negative real numbers. Prove the inequality

$$8(x^3 + y^3 + z^3)^2 \geq 9(x^2 + yz)(y^2 + xz)(z^2 + xy).$$

*Solution* If one of the numbers  $x, y, z$  is zero, let us say  $z = 0$ , then the above inequality is equivalent to

$$8(x^3 + y^3)^2 \geq 9x^3y^3 \quad \text{or} \quad 8(x^6 + y^6) + 7x^3y^3 \geq 0,$$

which clearly holds.

Equality occurs iff  $x = y = 0$ .

So let us assume that  $x, y, z > 0$ .

Then

$$x^2 + yz \leq x^2 + \frac{y^2 + z^2}{2} = \frac{2x^2 + y^2 + z^2}{2}.$$

Similarly

$$y^2 + xz \leq \frac{2y^2 + x^2 + z^2}{2} \quad \text{and} \quad z^2 + xy \leq \frac{2z^2 + x^2 + y^2}{2}.$$

Hence

$$\begin{aligned} & 9(x^2 + yz)(y^2 + xz)(z^2 + xy) \\ & \leq \frac{9}{8}(2x^2 + y^2 + z^2)(2y^2 + x^2 + z^2)(2z^2 + x^2 + y^2) \\ & \leq \frac{9}{8} \left( \frac{(2x^2 + y^2 + z^2) + (2y^2 + x^2 + z^2) + (2z^2 + x^2 + y^2)}{3} \right)^3 \\ & = \frac{9}{8} \left( \frac{4(x^2 + y^2 + z^2)}{3} \right)^3 = \frac{9 \cdot 4^3}{8} \left( \frac{x^2 + y^2 + z^2}{3} \right)^3. \end{aligned} \tag{10.4}$$

By the *power mean inequality* we have

$$\sqrt{\frac{x^2 + y^2 + z^2}{3}} \leq \sqrt[3]{\frac{x^3 + y^3 + z^3}{3}},$$

i.e.

$$\left(\frac{x^2 + y^2 + z^2}{3}\right)^3 \leq \left(\frac{x^3 + y^3 + z^3}{3}\right)^2. \quad (10.5)$$

Finally, by (10.4) and (10.5) it follows that

$$\begin{aligned} 9(x^2 + yz)(y^2 + xz)(z^2 + xy) &\leq \frac{9 \cdot 4^3}{8} \left(\frac{x^2 + y^2 + z^2}{3}\right)^3 \\ &\leq \frac{9 \cdot 4^3}{8} \left(\frac{x^3 + y^3 + z^3}{3}\right)^2 = 8(x^3 + y^3 + z^3)^2. \end{aligned}$$

**Exercise 10.9** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$a^b b^c c^a \leq \left(\frac{a + b + c}{3}\right)^{a+b+c}.$$

*Solution* By the *weighted power mean inequality* we have

$$\begin{aligned} (a^b b^c c^a)^{\frac{1}{a+b+c}} &= a^{\frac{b}{a+b+c}} \cdot b^{\frac{c}{a+b+c}} \cdot c^{\frac{a}{a+b+c}} \leq \frac{ba + cb + ac}{a + b + c} \leq \frac{(a + b + c)^2}{3(a + b + c)} \\ &= \frac{a + b + c}{3}. \end{aligned}$$

**Exercise 10.10** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove the inequality

$$(a + b - c)^a (b + c - a)^b (c + a - b)^c \leq a^a b^b c^c.$$

*Solution* By the *weighted power mean inequality* we have

$$\begin{aligned} &\sqrt[a+b+c]{\left(\frac{a+b-c}{a}\right)^a \left(\frac{b+c-a}{b}\right)^b \left(\frac{c+a-b}{c}\right)^c} \\ &\leq \frac{1}{a+b+c} \left(a \cdot \frac{a+b-c}{a} + b \cdot \frac{b+c-a}{b} + c \cdot \frac{c+a-b}{c}\right) = 1, \end{aligned}$$

i.e.

$$(a + b - c)^a (b + c - a)^b (c + a - b)^c \leq a^a b^b c^c.$$

Equality occurs iff  $a = b = c$ .

**Exercise 10.11** Let  $a, b \in \mathbb{R}^+$  and  $n \in \mathbb{N}$ . Prove the inequality

$$(a + b)^n (a^n + b^n) \leq 2^n (a^{2n} + b^{2n}).$$

*Solution* By the *power mean inequality*, for any  $x, y \in \mathbb{R}^+$ ,  $n \in \mathbb{N}$ , we have

$$\left( \frac{x + y}{2} \right)^n \leq \frac{x^n + y^n}{2}.$$

Therefore

$$\begin{aligned} (a + b)^n (a^n + b^n) &= 2^n \left( \frac{a + b}{2} \right)^n (a^n + b^n) \\ &\leq 2^n \left( \frac{a^n + b^n}{2} \right) (a^n + b^n) = 2^n \frac{(a^n + b^n)^2}{2} \\ &\leq 2^n \frac{2(a^{2n} + b^{2n})}{2} = 2^n (a^{2n} + b^{2n}). \end{aligned}$$

**Exercise 10.12** Let  $a, b, c \in \mathbb{R}^+$ ,  $n \in \mathbb{N}$ . Prove the inequality

$$a^n + b^n + c^n \geq \left( \frac{a + 2b}{3} \right)^n + \left( \frac{b + 2c}{3} \right)^n + \left( \frac{c + 2a}{3} \right)^n.$$

*Solution* By the *power mean inequality* for any  $a, b, c \in \mathbb{R}^+$  and  $n \in \mathbb{N}$ , we have

$$\frac{a^n + b^n + c^n}{3} \geq \left( \frac{a + b + c}{3} \right)^n.$$

So it follows that

$$\frac{a^n + b^n + b^n}{3} \geq \left( \frac{a + b + b}{3} \right)^n = \left( \frac{a + 2b}{3} \right)^n.$$

Similarly we obtain

$$\frac{b^n + c^n + c^n}{3} \geq \left( \frac{b + 2c}{3} \right)^n \quad \text{and} \quad \frac{c^n + a^n + a^n}{3} \geq \left( \frac{c + 2a}{3} \right)^n.$$

After adding these we get the required inequality.

# Chapter 11

## Newton's Inequality, Maclaurin's Inequality

Let  $a_1, a_2, \dots, a_n$  be arbitrary real numbers.

Consider the polynomial

$$P(x) = (x + a_1)(x + a_2) \cdots (x + a_n) = c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n.$$

Then the coefficients  $c_0, c_1, \dots, c_n$  can be expressed as functions of  $a_1, a_2, \dots, a_n$ , i.e. we have

$$c_0 = 1,$$

$$c_1 = a_1 + a_2 + \cdots + a_n,$$

$$c_2 = a_1 a_2 + a_1 a_3 + \cdots + a_{n-1} a_n,$$

$$c_3 = a_1 a_2 a_3 + a_1 a_2 a_4 + \cdots + a_{n-2} a_{n-1} a_n,$$

$$\dots$$

$$c_n = a_1 a_2 \cdots a_n.$$

For each  $k = 1, 2, \dots, n$  we define  $p_k = \frac{c_k}{\binom{n}{k}} = \frac{k!(n-k)!}{n!} c_k$ .

**Theorem 11.1** (Newton's inequality) *Let  $a_1, a_2, \dots, a_n > 0$  be arbitrary real numbers. Then for each  $k = 1, 2, \dots, n - 1$ , we have*

$$p_{k-1} p_{k+1} \leq p_k^2.$$

*Equality occurs if and only if  $a_1 = a_2 = \cdots = a_n$ .*

*Proof* By induction. □



*Example 11.1* For  $n = 3$  we have

$$\begin{aligned} p_1 p_3 \leq p_2^2 &\Leftrightarrow \frac{c_1}{\binom{3}{1}} \frac{c_3}{\binom{3}{3}} \leq \frac{c_2^2}{\binom{3}{2}^2} \Leftrightarrow \frac{c_1 c_3}{3} \leq \frac{c_2^2}{9} \\ &\Leftrightarrow 3c_1 c_3 \leq c_2^2, \end{aligned}$$

i.e.

$$3abc(a + b + c) \leq (ab + ac + bc)^2.$$

Equality occurs iff  $a = b = c$ .

**Theorem 11.2** (Maclaurin's inequality) *Let  $a_1, a_2, \dots, a_n > 0$ . Then*

$$p_1 \geq p_2^{\frac{1}{2}} \geq \dots \geq p_k^{\frac{1}{k}} \geq \dots \geq p_n^{\frac{1}{n}}.$$

*Equality occurs if and only if  $a_1 = a_2 = \dots = a_n$ .*

*Proof* By Newton's inequality. □

**Exercise 11.1** Let  $a, b, c, d > 0$  be real numbers. Let  $u = ab + ac + ad + bc + bd + cd$  and  $v = abc + abd + acd + bcd$ . Prove the inequality

$$2u^3 \geq 27v^2.$$

*Solution* We have  $p_2 = \frac{u}{\binom{4}{2}} = \frac{u}{6}$  and  $p_3 = \frac{v}{\binom{4}{3}} = \frac{v}{4}$ .

By Maclaurin's inequality we have

$$p_2^{\frac{1}{2}} \geq p_3^{\frac{1}{3}} \Leftrightarrow p_2^3 \geq p_3^2 \Leftrightarrow \left(\frac{u}{6}\right)^3 \geq \left(\frac{v}{4}\right)^2 \Leftrightarrow 2u^3 \geq 27v^2.$$

Equality occurs iff  $a = b = c = d$ .

**Exercise 11.2** Let  $a, b, c, d > 0$  be real numbers. Prove the inequality

$$\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{ad} + \frac{1}{bc} + \frac{1}{bd} + \frac{1}{cd}\right) \leq \frac{3}{8} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)^2.$$

*Solution* If we multiply both sides by  $(abcd)^2$  the above inequality becomes

$$abcd(cd + bd + bc + ad + ac + ab) \leq \frac{3}{8}(bcd + acd + abd + abc)^2$$

$$\Leftrightarrow abcd \left( \frac{cd + bd + bc + ad + ac + ab}{6} \right) \leq \left( \frac{bcd + acd + abd + abc}{4} \right)^2$$

$$\Leftrightarrow p_4 p_2 \leq p_3^2.$$

The last inequality is true, due to *Newton's inequality*.

Equality occurs iff  $a = b = c = d$ .



# Chapter 12

## Schur's Inequality, Muirhead's Inequality and Karamata's Inequality

In this chapter we will present three very important theorems, which have broad usage in solving symmetric inequalities. In that way we'll start with following definition.

**Definition 12.1** Let  $x_1, x_2, \dots, x_n$  be a sequence of positive real numbers and let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be arbitrary real numbers. Let us denote  $F(x_1, x_2, \dots, x_n) = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ , and by  $T[\alpha_1, \alpha_2, \dots, \alpha_n]$  we'll denote the sum of all possible products  $F(x_1, x_2, \dots, x_n)$ , over all permutations of  $\alpha_1, \alpha_2, \dots, \alpha_n$ .

*Example 12.1*

$$\begin{aligned} T[1, 0, \dots, 0] &= (n-1)! \cdot (x_1 + x_2 + \cdots + x_n), \\ T[a, a, \dots, a] &= n! x_1^a x_2^a \cdots x_n^a, \quad T[1, 2] = x^2 y + x y^2, \\ T[1, 2, 1] &= 2x^2 y z + 2y^2 x z + 2z^2 y x, \quad T[3, 0, 0] = 2(x^3 + y^3 + z^3), \\ T[2, 1, 0] &= x^2 y + x^2 z + y^2 x + y^2 z + z^2 x + z^2 y. \end{aligned}$$

**Theorem 12.1** (Schur's inequality) *Let  $\alpha \in \mathbb{R}$  and  $\beta > 0$ . Then we have*

$$T[\alpha + 2\beta, 0, 0] + T[\alpha, \beta, \beta] \geq 2T[\alpha + \beta, \beta, 0].$$

*Proof* Let  $(x, y, z)$  be the sequence of variables.

By Definition 12.1, and with elementary algebraic transformations we have

$$\begin{aligned} & \frac{1}{2} T[\alpha + 2\beta, 0, 0] + \frac{1}{2} T[\alpha, \beta, \beta] - T[\alpha + \beta, \beta, 0] \\ &= x^\alpha (x^\beta - y^\beta)(x^\beta - z^\beta) + y^\alpha (y^\beta - x^\beta)(y^\beta - z^\beta) + z^\alpha (z^\beta - x^\beta)(z^\beta - y^\beta). \end{aligned}$$

Thus the given inequality is equivalent to

$$x^\alpha(x^\beta - y^\beta)(x^\beta - z^\beta) + y^\alpha(y^\beta - x^\beta)(y^\beta - z^\beta) + z^\alpha(z^\beta - x^\beta)(z^\beta - y^\beta) \geq 0.$$

Without loss of generality we may assume that  $x \geq y \geq z$ .

Then clearly only the second term can be negative.

If  $\alpha \geq 0$  then we have

$$\begin{aligned} x^\alpha(x^\beta - y^\beta)(x^\beta - z^\beta) &\geq x^\alpha(x^\beta - y^\beta)(y^\beta - z^\beta) \\ &\geq y^\alpha(x^\beta - y^\beta)(y^\beta - z^\beta) \\ &= -y^\alpha(y^\beta - x^\beta)(y^\beta - z^\beta), \end{aligned}$$

i.e.

$$x^\alpha(x^\beta - y^\beta)(x^\beta - z^\beta) + y^\alpha(y^\beta - x^\beta)(y^\beta - z^\beta) \geq 0,$$

and since  $z^\alpha(z^\beta - x^\beta)(z^\beta - y^\beta) \geq 0$  we get the required result.

Similarly we consider the case when  $\alpha < 0$ . □

Let us notice that for  $\beta = 1$  we get a special form of *Schur's inequality*, which is very useful. Therefore we have the next theorem.

**Theorem 12.2** *Let  $x, y, z \geq 0$  be real numbers and let  $t \in \mathbb{R}$ . Then we have*

$$x^t(x - y)(x - z) + y^t(y - x)(y - z) + z^t(z - x)(z - y) \geq 0,$$

*with equality if and only if  $x = y = z$  or  $x = y, z = 0$  (up to permutation).*

*Proof* Without loss of generality let us assume that  $x \geq y \geq z$ .

Suppose that  $t > 0$ .

Then we have

$$(z - x)(z - y) \geq 0, \quad \text{i.e.} \quad z^t(z - x)(z - y) \geq 0 \quad (12.1)$$

and

$$x^t(x - z) - y^t(y - z) = (x^{t+1} - y^{t+1}) + z(x^t - y^t) \geq 0$$

i.e.

$$x^t(x - y)(x - z) + y^t(y - x)(y - z) \geq 0. \quad (12.2)$$

By (12.1) and (12.2) clearly we have

$$x^t(x - y)(x - z) + y^t(y - x)(y - z) + z^t(z - x)(z - y) \geq 0.$$

Let  $t \leq 0$ . Then we have

$$(x - y)(x - z) \geq 0 \quad \text{i.e.} \quad x^t(x - y)(x - z) \geq 0 \quad (12.3)$$

and

$$z^t(x-z) - y^t(x-y) \geq z^t(x-y) - y^t(x-y) = (z^t - y^t)(x-y) \geq 0,$$

i.e.

$$y^t(y-x)(y-z) + z^t(z-x)(z-y) \geq 0. \quad (12.4)$$

By adding (12.3) and (12.4) we get

$$x^t(x-y)(x-z) + y^t(y-x)(y-z) + z^t(z-x)(z-y) \geq 0.$$

Equality occurs if and only if  $x = y = z$  or  $x = y, z = 0$  (up to permutation). □

**Corollary 12.1** *Let  $x, y, z$  and  $a, b, c$  be positive real numbers such that  $a \geq b \geq c$  or  $a \leq b \leq c$ . Then we have*

$$a(x-y)(x-z) + b(y-x)(y-z) + c(z-x)(z-y) \geq 0.$$

*Proof* Similar to the proof of Theorem 12.1. □

**Example 12.2** If we take  $\alpha = \beta = 1$  in Schur's inequality we get

$$T[3, 0, 0] + T[1, 1, 1] \geq 2T[2, 1, 0],$$

i.e.

$$2(x^3 + y^3 + z^3) + 6xyz \geq 2(x^2y + x^2z + y^2x + y^2z + z^2x + z^2y),$$

i.e.

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + x^2z + y^2x + y^2z + z^2x + z^2y.$$

Note that this inequality is a direct consequence of Surányi's inequality for  $n = 3$ .

**Corollary 12.2** *Let  $x, y, z > 0$ . Then  $3xyz + x^3 + y^3 + z^3 \geq 2((xy)^{3/2} + (yz)^{3/2} + (zx)^{3/2})$ .*

*Proof* By Schur's inequality and  $AM \geq GM$  we obtain

$$\begin{aligned} x^3 + y^3 + z^3 + 3xyz &\geq (x^2y + y^2x) + (z^2y + y^2z) + (x^2z + z^2x) \\ &\geq 2((xy)^{3/2} + (yz)^{3/2} + (zx)^{3/2}). \end{aligned} \quad \square$$

**Corollary 12.3** Let  $k \in (0, 3]$ . Then for any  $a, b, c \in \mathbb{R}^+$  we have

$$(3 - k) + k(abc)^{2/k} + a^2 + b^2 + c^2 \geq 2(ab + bc + ca).$$

*Proof* After setting  $x = a^{2/3}$ ,  $y = b^{2/3}$ ,  $z = c^{2/3}$ , the given inequality becomes

$$(3 - k) + k(xyz)^{3/k} + x^3 + y^3 + z^3 \geq 2((xy)^{3/2} + (yz)^{3/2} + (zx)^{3/2}),$$

and due to Corollary 12.1, it suffices to show that

$$(3 - k) + k(xyz)^{3/k} \geq 3xyz.$$

By the *weighted power mean inequality* we have

$$\frac{3-k}{3} \cdot 1 + \frac{k}{3}(xyz)^{3/k} \geq 1^{(3-k)/3}((xyz)^{3/k})^{k/3} = xyz,$$

i.e.

$$(3 - k) + k(xyz)^{3/k} \geq 3xyz,$$

as required. □

**Definition 12.2** We'll say that the sequence  $(\beta_i)_{i=1}^n$  is majorized by  $(\alpha_i)_{i=1}^n$ , denoted  $(\beta_i) < (\alpha_i)$ , if we can rearrange the terms of the sequences  $(\alpha_i)$  and  $(\beta_i)$  in such a way as to satisfy the following conditions:

- (1)  $\beta_1 + \beta_2 + \cdots + \beta_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n$
- (2)  $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$  and  $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$
- (3)  $\beta_1 + \beta_2 + \cdots + \beta_s \leq \alpha_1 + \alpha_2 + \cdots + \alpha_s$  for any  $1 \leq s < n$ .

Without proofs we'll give the following two very important theorems.

**Theorem 12.3** (Muirhead's theorem) Let  $x_1, x_2, \dots, x_n$  be a sequence of non-negative real numbers and let  $(\alpha_i)$  and  $(\beta_i)$  be sequences of positive real numbers such that  $(\beta_i) < (\alpha_i)$ . Then

$$T[\beta_i] \leq T[\alpha_i].$$

Equality occurs iff  $(\alpha_i) = (\beta_i)$  or  $x_1 = x_2 = \cdots = x_n$ .

**Example 12.3** Let  $(x, y, z)$  be the sequence of variables.

Consider the sequences  $(2, 2, 1), (3, 1, 1)$ . Then clearly  $(2, 2, 1) \prec (3, 1, 1)$ .

So by *Muirhead's theorem* we obtain

$$T[2, 2, 1] \leq T[3, 1, 1],$$

i.e.

$$2(x^2y^2z + x^2z^2y + y^2z^2x) \leq 2(x^3yz + y^3zx + z^3yx),$$

i.e.

$$x^2y^2z + x^2z^2y + y^2z^2x \leq x^3yz + y^3zx + z^3yx,$$

i.e.

$$xy + yz + zx \leq x^2 + y^2 + z^2,$$

which clearly holds.

**Theorem 12.4** (Karamata's inequality) *Let  $f : I \rightarrow \mathbb{R}$  be a convex function on the interval  $I \subseteq \mathbb{R}$  and let  $(a_i)_{i=1}^n, (b_i)_{i=1}^n$ , where  $a_i, b_i \in I, i = 1, 2, \dots, n$ , are two sequences, such that  $(a_i) \succ (b_i)$ . Then*

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq f(b_1) + f(b_2) + \dots + f(b_n).$$

**Remark** If  $f : I \rightarrow \mathbb{R}$  is strictly convex on the interval  $I \subseteq \mathbb{R}$ , and  $(a_i) \neq (b_i)$  are such that  $(a_i) \succ (b_i)$  then in *Karamata's inequality* we have strict inequality, i.e.

$$f(a_1) + f(a_2) + \dots + f(a_n) > f(b_1) + f(b_2) + \dots + f(b_n).$$

Also if  $f : I \rightarrow \mathbb{R}$  is concave (strictly concave) in *Karamata's inequality* we have the reverse inequalities.

**Exercise 12.1** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove the inequality

$$a^3(s - a) + b^3(s - b) + c^3(s - c) \leq abc s.$$

**Solution** The given inequality is equivalent to

$$a^2(a - b)(a - c) + b^2(b - c)(b - a) + c^2(c - a)(c - b) \geq 0,$$

which clearly holds by *Schur's inequality*.

**Exercise 12.2** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$27 + \left(2 + \frac{a^2}{bc}\right)\left(2 + \frac{b^2}{ca}\right)\left(2 + \frac{c^2}{ab}\right) \geq 6(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$



**Solution** The given inequality is equivalent to

$$2abc(a^3 + b^3 + c^3 + 3abc - a^2b - a^2c - b^2a - b^2c - c^2a - c^2b) + (a^3b^3 + b^3c^3 + c^3a^3 + 3a^2b^2c^2 - ab^3c^2 - ab^2c^3 - a^2b^1c^2 - a^3b^1c^2) \geq 0,$$

which is true due to *Schur's inequality*, for variables  $a, b, c$  and  $ab, bc, ca$ .

**Exercise 12.3** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{a^2+bc}{(a+b)(a+c)} + \frac{b^2+ca}{(b+a)(b+c)} + \frac{c^2+ab}{(c+a)(c+b)}.$$

**Solution** The given inequality is equivalent to

$$\frac{a^3 + b^3 + c^3 + 3abc - ab(a+b) - bc(b+c) - ca(c+a)}{(a+b)(b+c)(c+a)} \geq 0,$$

i.e.

$$a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \geq 0,$$

which is *Schur's inequality*.

**Exercise 12.4** Let  $a, b, c$  be non-negative real numbers. Prove the inequality

$$\frac{a}{4b^2 + bc + 4c^2} + \frac{b}{4c^2 + ca + 4a^2} + \frac{c}{4a^2 + ab + 4b^2} \geq \frac{1}{a+b+c}.$$

**Solution** By the *Cauchy-Schwarz inequality* we have

$$\begin{aligned} & \frac{a}{4b^2 + bc + 4c^2} + \frac{b}{4c^2 + ca + 4a^2} + \frac{c}{4a^2 + ab + 4b^2} \\ & \geq \frac{(a+b+c)^2}{4a(b^2+c^2) + 4b(c^2+a^2) + 4c(a^2+b^2) + 3abc}. \end{aligned}$$

So we need to prove that

$$\frac{(a+b+c)^2}{4a(b^2+c^2) + 4b(c^2+a^2) + 4c(a^2+b^2) + 3abc} \geq \frac{1}{a+b+c},$$

which is equivalent to

$$(a+b+c)^3 \geq 4a(b^2+c^2) + 4b(c^2+a^2) + 4c(a^2+b^2) + 3abc,$$

i.e.

$$a^3 + b^3 + c^3 + 3abc \geq a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2),$$

which is *Schur's inequality*.

**Exercise 12.5** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ac).$$

*Solution* By Schur's inequality we deduce

$$2(ab + bc + ac) - (a^2 + b^2 + c^2) \leq \frac{9abc}{a + b + c}.$$

So it remains to prove that

$$\frac{9abc}{a + b + c} \leq 2abc + 1.$$

Since  $AM \geq GM$  we have

$$2abc + 1 = abc + abc + 1 \geq 3\sqrt[3]{(abc)^2}.$$

Therefore we only need to prove that  $3\sqrt[3]{(abc)^2} \geq \frac{9abc}{a+b+c}$ , which is equivalent to  $a + b + c \geq 3\sqrt[3]{abc}$ , and clearly holds.

**Exercise 12.6** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a^2 + bc}{(b + c)^2} + \frac{b^2 + ca}{(c + a)^2} + \frac{c^2 + ab}{(a + b)^2} \geq \frac{3}{2}.$$

*Solution* To begin we'll show that

$$\frac{a^2 + bc}{(b + c)^2} + \frac{b^2 + ca}{(c + a)^2} + \frac{c^2 + ab}{(a + b)^2} \geq \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b}. \quad (12.5)$$

We have

$$\frac{a^2 + bc}{(b + c)^2} - \frac{a}{b + c} = \frac{(a - b)(a - c)}{(b + c)^2};$$

similarly we get

$$\frac{b^2 + ca}{(c + a)^2} - \frac{b}{c + a} = \frac{(b - c)(b - a)}{(c + a)^2} \quad \text{and} \quad \frac{c^2 + ab}{(a + b)^2} - \frac{c}{a + b} = \frac{(c - a)(c - b)}{(a + b)^2}.$$

Let

$$x = \frac{1}{(b + c)^2}, \quad y = \frac{1}{(c + a)^2} \quad \text{and} \quad z = \frac{1}{(a + b)^2}.$$

Then we can rewrite inequality (12.5) as follows

$$x(a - b)(a - c) + y(b - c)(b - a) + z(c - a)(c - b) \geq 0. \quad (12.6)$$

Without loss of generality we may assume that  $a \geq b \geq c$  from which it follows that  $x \geq y \geq z$ , and now inequality (12.6) i.e. inequality (12.5), will follow due to Corollary 12.1 from *Schur's inequality*. Equality occurs iff  $a = b = c$ .

**Exercise 12.7** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + ac + bc).$$

*Solution* The given inequality is equivalent to

$$8 + (abc)^2 + 2(a^2b^2 + b^2c^2 + c^2a^2) + 4(a^2 + b^2 + c^2) \geq 9(ab + ac + bc). \quad (12.7)$$

From the obvious inequality

$$(ab - 1)^2 + (bc - 1)^2 + (ca - 1)^2 \geq 0$$

we deduce that

$$6 + 2(a^2b^2 + b^2c^2 + c^2a^2) \geq 4(ab + ac + bc) \quad (12.8)$$

and clearly

$$3(a^2 + b^2 + c^2) \geq 3(ab + ac + bc). \quad (12.9)$$

For  $k = 1$  by Corollary 12.2, we obtain

$$2 + (abc)^2 + a^2 + b^2 + c^2 \geq 2(ab + ac + bc). \quad (12.10)$$

By adding (12.8), (12.9) and (12.10) we obtain inequality (12.7), as required.

**Exercise 12.8** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$a^4 + b^4 + c^4 \geq abc(a + b + c).$$

*Solution* We have

$$\begin{aligned} a^4 + b^4 + c^4 &\geq abc(a + b + c) \\ \Leftrightarrow a^4 + b^4 + c^4 &\geq a^2bc + b^2ac + c^2ab \\ \Leftrightarrow \frac{T[4, 0, 0]}{2} &\geq \frac{T[2, 1, 1]}{2}, \end{aligned}$$

i.e.

$$T[4, 0, 0] \geq T[2, 1, 1],$$

which is true according to *Muirhead's theorem*.

**Exercise 12.9** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \leq \frac{1}{abc}.$$

*Solution* After multiplying both sides by

$$abc(a^3 + b^3 + abc)(b^3 + c^3 + abc)(c^3 + a^3 + abc),$$

above inequality becomes

$$\begin{aligned} & \frac{3}{2}T[4, 4, 1] + 2T[5, 2, 2] + \frac{1}{2}T[7, 1, 1] + \frac{1}{2}T[3, 3, 3] \\ & \leq \frac{1}{2}T[3, 3, 3] + T[6, 3, 0] + \frac{3}{2}T[4, 4, 1] + \frac{1}{2}T[7, 1, 1] + T[5, 2, 2], \end{aligned}$$

i.e.

$$T[5, 2, 2] \leq T[6, 3, 0],$$

which is true according to *Muirhead's theorem*.

Equality occurs iff  $a = b = c$ .

**Exercise 12.10** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$0 \leq ab + bc + ca - 2abc \leq \frac{7}{27}.$$

*Solution* The left-hand inequality follows from the identity

$$\begin{aligned} ab + bc + ca - 2abc &= (a + b + c)(ab + bc + ca) - 2abc \\ &= a^2b + a^2c + b^2a + b^2c + c^2a + c^2b + abc \\ &= T[2, 1, 0] + \frac{1}{6}T[1, 1, 1], \end{aligned}$$

since  $T[2, 1, 0] + \frac{1}{6}T[1, 1, 1] \geq 0$ .

We have

$$\frac{7}{27} = \frac{7}{27}(x + y + z)^3 = \frac{7}{27} \left( \frac{1}{2}T[3, 0, 0] + 3T[2, 1, 0] + T[1, 1, 1] \right).$$

Therefore the given inequality is equivalent to

$$T[2, 1, 0] + \frac{1}{6}T[1, 1, 1] \leq \frac{7}{27} \left( \frac{1}{2}T[3, 0, 0] + 3T[2, 1, 0] + T[1, 1, 1] \right),$$

i.e.

$$12T[2, 1, 0] \leq 7T[3, 0, 0] + 5T[1, 1, 1]. \quad (12.11)$$

*Muirhead's theorem* we have

$$2T[2, 1, 0] \leq 2T[3, 0, 0], \quad (12.12)$$

and by *Schur's inequality* for  $\alpha = \beta = 1$  (third degree) we get

$$10T[2, 1, 0] \leq 5T[3, 0, 0] + 5T[1, 1, 1]. \quad (12.13)$$

Adding (12.12) and (12.13) gives us inequality (12.11), as required.

**Exercise 12.11** Let  $a, b, c \in \mathbb{R}^+$  such that  $abc = 1$ . Prove the inequality

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

*Solution* If we divide both sides by  $(abc)^{4/3} = 1$ , and after clearing the denominators, the given inequality will be equivalent to

$$2T\left[\frac{16}{3}, \frac{13}{3}, \frac{7}{3}\right] + T\left[\frac{16}{3}, \frac{16}{3}, \frac{4}{3}\right] + T\left[\frac{13}{3}, \frac{13}{3}, \frac{10}{3}\right] \geq 3T[5, 4, 3] + T[4, 4, 4].$$

Now according to *Muirhead's inequality* we have

$$\begin{aligned} 2T\left[\frac{16}{3}, \frac{13}{3}, \frac{7}{3}\right] &\geq 2T[5, 4, 3], & T\left[\frac{16}{3}, \frac{16}{3}, \frac{4}{3}\right] &\geq T[5, 4, 3], \\ T\left[\frac{13}{3}, \frac{13}{3}, \frac{10}{3}\right] &\geq T[4, 4, 4]. \end{aligned}$$

If we add the last three inequalities we obtain the required result.

Equality occurs iff  $a = b = c = 1$ .

**Exercise 12.12** (Schur's inequality) Let  $a, b, c$  be positive real numbers. Prove the inequality

$$a^3 + b^3 + c^3 + 3abc \geq a^2b + a^2c + b^2a + b^2c + c^2a + c^2b.$$

*Solution* Since the given inequality is symmetric, without loss of generality we can assume that  $a \geq b \geq c$ .

After taking  $x = \ln a$ ,  $y = \ln b$  and  $z = \ln c$  the given inequality becomes

$$\begin{aligned} e^{3x} + e^{3y} + e^{3z} + e^{x+y+z} + e^{x+y+z} + e^{x+y+z} \\ \geq e^{2x+y} + e^{2x+z} + e^{2y+x} + e^{2y+z} + e^{2z+x} + e^{2z+y}. \end{aligned}$$

The function  $f(x) = e^x$  is convex on  $\mathbb{R}$ , so by *Karamata's inequality* it suffices to prove that the sequence  $a = (3x, 3y, 3z, x+y+z, x+y+z, x+y+z)$  majorizes the sequence  $b = (2x+y, 2x+z, 2y+x, 2y+z, 2z+x, 2z+y)$ .

Since  $a \geq b \geq c$  it follows that  $x \geq y \geq z$  and clearly  $3x \geq x + y + z \geq 3z$ .

If  $x + y + z \geq 3y$  (the case when  $3y \geq x + y + z$  is analogous) then we obtain the following inequalities

$$\begin{aligned} 3x &\geq x + y + z \geq 3y \geq 3z, \\ 2x + y &\geq 2x + z \geq 2y + x \geq 2z + x \geq 2y + z \geq 2z + y, \end{aligned}$$

which means that  $a \succ b$ , and we are done.

**Exercise 12.13** Let  $a_1, a_2, \dots, a_n$  be positive real numbers. Prove the inequality

$$\frac{a_1^3}{a_2} + \frac{a_2^3}{a_3} + \dots + \frac{a_n^3}{a_1} \geq a_1^2 + a_2^2 + \dots + a_n^2.$$

*Solution* Let  $x_i = \ln a_i$ . Then the given inequality becomes

$$e^{3x_1 - x_2} + e^{3x_2 - x_3} + \dots + e^{3x_n - x_1} \geq e^{2x_1} + e^{2x_2} + \dots + e^{2x_n}.$$

Let us consider the sequences  $a : 3x_1 - x_2, 3x_2 - x_3, \dots, 3x_n - x_1$  and  $b : 2x_1, 2x_2, \dots, 2x_n$ .

Since  $f(x) = e^x$  is a convex function on  $\mathbb{R}$  by *Karamata's inequality* it suffices to prove that  $a$  (ordered in some way) majorizes the sequences  $b$  (ordered in some way).

For that purpose, let us assume that

$$\begin{aligned} 3x_{m_1} - x_{m_1+1} &\geq 3x_{m_2} - x_{m_2+1} \geq \dots \geq 3x_{m_n} - x_{m_n+1} \quad \text{and} \\ 2x_{k_1} &\geq 2x_{k_2} \geq \dots \geq 2x_{k_n}, \end{aligned}$$

for some indexes  $m_i, k_i \in \{1, 2, \dots, n\}$ .

Clearly

$$3x_{m_1} - x_{m_1+1} \geq 3x_{k_1} - x_{k_1+1} \geq 2x_{k_1}$$

and

$$(3x_{m_1} - x_{m_1+1}) + (3x_{m_2} - x_{m_2+1}) \geq (3x_{k_1} - x_{k_1+1}) + (3x_{k_2} - x_{k_2+1}) \geq 2x_{k_1} + 2x_{k_2}.$$

Analogously the sum of the first  $s$  terms of  $(a)$  is not less than the sum of an arbitrary  $s$  terms of  $(a)$ , hence it is not less than  $(3x_{k_1} - x_{k_1+1}) + (3x_{k_2} - x_{k_2+1}) + \dots + (3x_{k_s} - x_{k_s+1})$ , which, on the other hand, is not less than  $2x_{k_1} + 2x_{k_2} + \dots + 2x_{k_s}$ .

So  $a \succ b$ , and we are done.

**Exercise 12.14** (Turkevicius inequality) Let  $a, b, c, d$  be positive real numbers. Prove the inequality

$$a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2.$$

*Solution* Because of symmetry without loss of generality we can assume  $a \geq b \geq c \geq d$ .

Let  $x = \ln a, y = \ln b, z = \ln c, t = \ln d$ ; then clearly  $x \geq y \geq z \geq t$  and given inequality becomes

$$\begin{aligned} & e^{4x} + e^{4y} + e^{4z} + e^{4t} + e^{x+y+z+t} + e^{x+y+z+t} \\ & \geq e^{2(x+y)} + e^{2(x+z)} + e^{2(x+t)} + e^{2(y+z)} + e^{2(y+t)} + e^{2(z+t)}. \end{aligned}$$

The function  $f(x) = e^x$  is a convex on  $\mathbb{R}$ , so according to *Karamata's inequality* it suffices to prove that  $(4x, 4y, 4z, 4t, x + y + z + t, x + y + z + t)$  (ordered in some way) majorizes the sequences  $(2(x + y), 2(x + z), 2(x + t), 2(y + z), 2(y + t), 2(z + t))$  (ordered in some way).

Clearly  $4x \geq 4y \geq 4z$  and  $4x \geq x + y + z + t \geq 4t$ .

We need to consider four cases:

If  $4z \geq x + y + z + t$  then we can easily show that

$$2(x + y) \geq 2(x + z) \geq 2(y + z) \geq 2(x + t) \geq 2(y + t) \geq 2(z + t)$$

and we can check that the sequence  $(4x, 4y, 4z, x + y + z + t, x + y + z + t, 4t)$  majorize the sequence  $(2(x + y), 2(x + z), 2(y + z), 2(x + t), 2(y + t), 2(z + t))$ .

The cases when  $x + y + z + t \geq 4z, 4y \geq x + y + z + t$  or  $x + y + z + t \geq 4y$  are analogous as the first case and therefore are left to the reader.

## Chapter 13

# Two Theorems from Differential Calculus, and Their Applications for Proving Inequalities

In this section we'll give two theorems (without proof), whose origins are part of differential calculus, and which are widely used in proving certain inequalities. We assume that the reader has basic knowledge of differential calculus.

**Definition 13.1** For the function  $f : (a, b) \rightarrow \mathbb{R}$  we'll say that it is a monotone increasing function on the interval  $(a, b)$  if for all  $x, y \in (a, b)$  such that  $x \geq y$  we have  $f(x) \geq f(y)$ .

If we have strict inequalities, i.e. if for all  $x, y \in (a, b)$  such that  $x > y$  we have  $f(x) > f(y)$  then we'll say that  $f$  is strictly increasing on  $(a, b)$ .

Similarly we define a monotone decreasing function and a strictly decreasing function. Therefore we have the following definition.

**Definition 13.2** For the function  $f : (a, b) \rightarrow \mathbb{R}$  we'll say that it is a monotone decreasing function on the interval  $(a, b)$  if for all  $x, y \in (a, b)$  such that  $x \geq y$  we have  $f(x) \leq f(y)$ .

If we have strict inequalities, i.e. if for all  $x, y \in (a, b)$  such that  $x > y$  we have  $f(x) < f(y)$  then we'll say that  $f$  is strictly decreasing on  $(a, b)$ .

**Theorem 13.1** (Characterization of monotonic functions) *Let  $f : (a, b) \rightarrow \mathbb{R}$  be a differentiable function on  $(a, b)$ .*

*If, for all  $x \in (a, b)$ ,  $f'(x) \geq 0$ , then  $f$  is a monotone increasing function on the interval  $(a, b)$ .*



If, for all  $x \in (a, b)$ , we have  $f'(x) \leq 0$ , then  $f$  is a monotone decreasing function on the interval  $(a, b)$ .

If we have strict inequalities then  $f$  is a strictly increasing, respectively, strictly decreasing function on  $(a, b)$ .

**Theorem 13.2** Let  $f : [a, b] \rightarrow \mathbb{R}$  and  $g : [a, b] \rightarrow \mathbb{R}$  be functions such that:

- (i)  $f$  and  $g$  are continuous on  $[a, b]$  and  $f(a) = g(a)$ ;
- (ii)  $f$  and  $g$  are differentiable on  $(a, b)$ ;
- (iii)  $f'(x) > g'(x)$ , for all  $x \in (a, b)$ .

Then, for all  $x \in (a, b)$ , we have  $f(x) > g(x)$ .

**Exercise 13.1** Let  $x, y \geq 0$  be real numbers such that  $x + y = 2$ . Prove the inequality

$$x^2 y^2 (x^2 + y^2) \leq 2.$$

*Solution* We homogenize as follows

$$x^2 y^2 (x^2 + y^2) \leq 2 \left( \frac{x+y}{2} \right)^6 \Leftrightarrow (x+y)^6 \geq 32 x^2 y^2 (x^2 + y^2). \quad (13.1)$$

If  $xy = 0$  then the given inequality clearly holds.

Therefore let us assume that  $xy \neq 0$ .

Since (13.1) is homogenous, we may normalize with  $xy = 1$ .

So  $y = \frac{1}{x}$ , and inequality (13.1) becomes

$$\left( x + \frac{1}{x} \right)^6 \geq 32 \left( x^2 + \frac{1}{x^2} \right). \quad (13.2)$$

Let  $t = \left( x + \frac{1}{x} \right)^2$ , then clearly  $x^2 + \frac{1}{x^2} = t - 2$ .

Therefore (13.2) is equivalent to

$$t^3 \geq 32(t - 2).$$

Clearly  $t = \left( x + \frac{1}{x} \right)^2 \geq 2^2 = 4$ .

Let us consider the function  $f(t) = t^3 - 32(t - 2)$  on the interval  $[4, \infty)$ .

Since  $f'(t) = 3t^2 - 32$  we have that  $f'(t) \geq 0$  for all  $t \geq \sqrt{\frac{32}{3}} > 4$ , i.e. it follows that  $f$  is increasing on  $[4, \infty)$ , which implies that

$$f(t) \geq f(4) = 0$$

$$\Leftrightarrow t^3 - 32(t - 2) \geq 0$$

$$\Leftrightarrow t^3 \geq 32(t - 2), \quad \text{for all } t \in [4, \infty),$$

as required.

**Exercise 13.2** Let  $x, y, z$  be non-negative real numbers such that  $x + y + z = 1$ . Prove the inequality

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

*Solution* Let  $f(x, y, z) = xy + yz + zx - 2xyz$ .

Without loss of generality we may assume that  $0 \leq x \leq y \leq z \leq 1$ .

Since  $x + y + z = 1$  we have

$$3x \leq x + y + z = 1, \quad \text{i.e.} \quad x \leq \frac{1}{3}. \quad (13.3)$$

Furthermore we have

$$f(x, y, z) = (1 - 3x)yz + xy + zx + xyz \stackrel{(1)}{\geq} 0,$$

and we are done with the left inequality.

It remains to prove the right inequality.

Since  $AM \geq GM$  we obtain

$$yz \leq \left( \frac{y+z}{2} \right)^2 = \left( \frac{1-x}{2} \right)^2.$$

Since  $1 - 2x > 0$  we get

$$\begin{aligned} f(x, y, z) &= x(y+z) + yz(1-2x) \leq x(1-x) + \left( \frac{1-x}{2} \right)^2 (1-2x) \\ &= \frac{-2x^3 + x^2 + 1}{4}. \end{aligned}$$

We'll show that

$$f(x) = \frac{-2x^3 + x^2 + 1}{4} \leq \frac{7}{27}, \quad \text{for all } x \in \left[ 0, \frac{1}{3} \right].$$

We have

$$f'(x) = \frac{-6x^2 + 2x}{4} = \frac{3x}{2} \left( \frac{1}{3} - x \right) \geq 0, \quad \text{for all } x \in \left[ 0, \frac{1}{3} \right].$$

Thus  $f$  is an increasing function on  $[0, \frac{1}{3}]$ , so it follows that

$$f(x) \leq f\left(\frac{1}{3}\right) = \frac{7}{27}, \quad x \in \left[0, \frac{1}{3}\right],$$

as required.

**Exercise 13.3** Let  $x > 0$  be a real number. Prove that  $x - \frac{x^2}{2} < \ln(x + 1)$ .

*Solution* Let us consider the functions

$$f(x) = \ln(x + 1) \quad \text{and} \quad g(x) = x - \frac{x^2}{2} \quad \text{on the interval } [0, \alpha), \text{ where } \alpha \in \mathbb{R}.$$

We have

$$f(0) = 0 = g(0) \quad \text{and} \quad f'(x) = \frac{1}{1+x}, \quad g'(x) = 1 - x.$$

For  $x \in (0, \alpha)$  it follows that  $\frac{1}{1+x} > 1 - x$ , i.e.

$$f'(x) > g'(x), \quad \text{for all } x \in (0, \alpha).$$

According to Theorem 13.2 we have  $f(x) > g(x)$ , for all  $x \in (0, \alpha)$  i.e.

$$\ln(x + 1) > x - \frac{x^2}{2}, \quad x \in (0, \alpha).$$

Since  $\alpha$  is arbitrary we conclude that  $\ln(x + 1) > x - \frac{x^2}{2}$ , for all  $x \in (0, \infty)$ .

**Exercise 13.4** Prove that, for all  $0 < x < \frac{\pi}{2}$ , we have  $\tan x > x$ .

*Solution* Let  $f(x) = \tan x$ ,  $g(x) = x$  where  $x \in (0, \frac{\pi}{2})$ .

We have

$$f(0) = 0 = g(0) \quad \text{and} \quad f'(x) = \frac{1}{\cos^2 x} > 1 = g'(x), \quad \text{for all } x \in \left(0, \frac{\pi}{2}\right).$$

According to Theorem 13.2, we have  $f(x) > g(x)$ , i.e.  $\tan x > x$  for all  $x \in (0, \frac{\pi}{2})$ .

**Exercise 13.5** Prove that, for all  $0 < x < \frac{\pi}{2}$  we have  $\tan x > x + \frac{x^3}{3}$ .

*Solution* Let  $f(x) = \tan x$ ,  $g(x) = x + \frac{x^3}{3}$ ,  $x \in (0, \frac{\pi}{2})$ .

Then  $f(0) = 0 = g(0)$  and we have

$$f'(x) = \frac{1}{\cos^2 x} = 1 + \tan^2 x > 1 + x^2 = g'(x), \quad \text{for all } x \in \left(0, \frac{\pi}{2}\right).$$

Thus, due to Theorem 13.2, we get  $f(x) > g(x)$ , i.e.  $\tan x > x + \frac{x^3}{3}$  for all  $x \in (0, \frac{\pi}{2})$ .

## Chapter 14

# One Method of Proving Symmetric Inequalities with Three Variables

In this section we'll give a wonderful method that will be used in proving symmetrical inequalities with three variables. I must emphasize that this method is a powerful instrument which can be used for proving inequalities of varying difficulty which can't be proved with previous methods and techniques. Also I must say that I respect this method so much, because it can be very valuable and workable for all symmetric inequalities.

Let  $x, y, z \in \mathbb{R}^+$ , and  $p = x + y + z, q = xy + yz + zx, r = xyz$ . Clearly  $p, q, r \in \mathbb{R}^+$ .

Using these notations we can easily prove the following identities:

$$\begin{aligned}
 I_1: & x^2 + y^2 + z^2 = p^2 - 2q \\
 I_2: & x^3 + y^3 + z^3 = p(p^2 - 3q) + 3r \\
 I_3: & x^2y^2 + y^2z^2 + z^2x^2 = q^2 - 2pr \\
 I_4: & x^4 + y^4 + z^4 = (p^2 - 2q)^2 - 2(q^2 - 2pr) \\
 I_5: & (x + y)(y + z)(z + x) = pq - r \\
 I_6: & (x + y)(y + z) + (y + z)(z + x) + (z + x)(x + y) = p^2 + q \\
 I_7: & (x + y)^2(y + z)^2 + (y + z)^2(z + x)^2 + (z + x)^2(x + y)^2 = (p^2 + q)^2 - 4p(pq - r) \\
 I_8: & xy(x + y) + yz(y + z) + zx(z + x) = pq - 3r \\
 I_9: & (1 + x)(1 + y)(1 + z) = 1 + p + q + r \\
 I_{10}: & (1 + x)(1 + y) + (1 + y)(1 + z) + (1 + z)(1 + x) = 3 + 2p + q \\
 I_{11}: & (1 + x)^2(1 + y)^2 + (1 + y)^2(1 + z)^2 + (1 + z)^2(1 + x)^2 = (3 + 2p + q)^2 - 2(3 + p)(1 + p + q + r) \\
 I_{12}: & x^2(y + z) + y^2(z + x) + z^2(x + y) = pq - 3r \\
 I_{13}: & x^3y^3 + y^3z^3 + z^3x^3 = q^3 - 3pqr - 3r^2 \\
 I_{14}: & xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) = p^2q - 2q^2 - pr \\
 I_{15}: & (1 + x^2)(1 + y^2)(1 + z^2) = p^2 + q^2 + r^2 - 2pr - 2q + 1 \\
 I_{16}: & (1 + x^3)(1 + y^3)(1 + z^3) = p^3 + q^3 + r^3 - 3pqr - 3pq - 3r^2 + 3r + 1.
 \end{aligned}$$

The proofs, as mentioned, are quite simple, and are therefore left to the reader. Also, we will give some inequalities which will be used later, and which should be well-known.

Some of them follow by the *mean inequalities* but some of them are direct consequences of *Schur's* and *Muirhead's inequalities*.

We will prove some of them, and some are left to the reader.

**Theorem 14.1** *Let  $x, y, z \geq 0$  and  $p = x + y + z, q = xy + yz + zx, r = xyz$ . Then we have:*

$$N_1: p^3 - 4pq + 9r \geq 0, \quad N_2: p^4 - 5p^2q + 4q^2 + 6pr \geq 0.$$

*Proof* According to *Schur's inequality* we have: For any real numbers  $x, y, z \geq 0, t \in \mathbb{R}$  we have  $x^t(x - y)(x - z) + y^t(y - z)(y - x) + z^t(z - x)(z - y) \geq 0$ .

For  $t = 1$  and  $t = 2$ , we obtain the required inequalities  $N_1$  and  $N_2$ , respectively.  $\square$

**Theorem 14.2** *Let  $x, y, z \geq 0$ , and  $p = x + y + z, q = xy + yz + zx, r = xyz$ . Then we have the following inequalities:*

$$\begin{aligned} N_3: pq - 9r &\geq 0, & N_9: p^4 + 3q^2 &\geq 4p^2q, \\ N_4: p^2 &\geq 3q, & N_{10}: 2p^3 + 9r^2 &\geq 7pqr, \\ N_5: p^3 &\geq 27r, & N_{11}: p^2q + 3pr &\geq 4q^2, \\ N_6: q^3 &\geq 27r^2, & N_{12}: q^3 + 9r^2 &\geq 4pqr, \\ N_7: q^2 &\geq 3pr, & N_{13}: pq^2 &\geq 2p^2r + 3qr. \\ N_8: 2p^3 + 9r &\geq 7pq, \end{aligned}$$

*Proof* We have

$$\begin{aligned} N_3: pq &= (x + y + z)(xy + yz + zx) \geq 3\sqrt[3]{xyz} \cdot 3\sqrt[3]{x^2y^2z^2} = 9r \\ &\Leftrightarrow pq - 9r \geq 0, \end{aligned}$$

$$\begin{aligned} N_4: p^2 &\geq 3q \quad \Leftrightarrow \quad (x + y + z)^2 \geq 3(xy + yz + zx) \\ &\Leftrightarrow x^2 + y^2 + z^2 \geq xy + yz + zx, \end{aligned}$$

which clearly holds.

$$\begin{aligned} N_5: p &= x + y + z \geq 3\sqrt[3]{xyz} = 3\sqrt[3]{r} \quad \Leftrightarrow \quad p^3 \geq 27r, \\ N_6: q &= xy + yz + zx \geq 3\sqrt[3]{x^2y^2z^2} = 3\sqrt[3]{r^2} \quad \Leftrightarrow \quad q^3 \geq 27r^2, \\ N_7: q^2 &= (xy + yz + zx)^2 = x^2y^2 + y^2z^2 + z^2x^2 + 2xyz(x + y + z) \\ &\geq (xy)(yz) + (yz)(zx) + (zx)(xy) + 2xyz(x + y + z) \\ &= 3xyz(x + y + z) = 3pr, \end{aligned}$$

$$N_8: 2p^3 + 9r \geq 7pq$$

$$\Leftrightarrow 2(x + y + z)^3 + 9xyz \geq 7(x + y + z)(xy + yz + zx)$$

$$\Leftrightarrow 2(x^3 + y^3 + z^3) \geq x^2y + x^2z + y^2z + y^2x + z^2x + z^2y$$

$$\Leftrightarrow T[3, 0, 0] \geq T[2, 1, 0],$$

which is true due to *Muirhead's theorem*.  $\square$

**Exercise 14.1** Let  $x, y, z > 0$  such that  $x + y + z = 1$ . Prove the inequality

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)\left(1 + \frac{1}{z}\right) \geq 64.$$

*Solution* Let  $p = x + y + z = 1, q = xy + yz + zx, r = xyz$ .

Then the given inequality becomes

$$(1 + x)(1 + y)(1 + z) \geq 64xyz. \quad (14.1)$$

Using  $I_9$ :  $(1 + x)(1 + y)(1 + z) = 1 + p + q + r$  we deduce

$$(1 + x)(1 + y)(1 + z) = 2 + q + r.$$

So (14.1) is equivalent to

$$2 + q + r \geq 64r \quad \text{i.e.} \quad 2 + q \geq 63r. \quad (14.2)$$

By  $N_5$ :  $p^3 \geq 27r$  we get

$$r \leq \frac{1}{27}. \quad (14.3)$$

By  $N_3$ :  $pq - 9r \geq 0$  we get

$$pq \geq 9r, \quad \text{i.e.} \quad q \geq 9r. \quad (14.4)$$

Now using (14.4) we deduce that  $2 + q \geq 2 + 9r$ .

So it suffices to show that  $2 + 9r \geq 63r$ , which is  $2 \geq 54r \Leftrightarrow r \leq \frac{1}{27}$ , which clearly holds, by (14.3).

We have proved (14.2), and we are done.

**Exercise 14.2** Let  $x, y, z > 0$  be real numbers. Prove the inequality

$$(xy + yz + zx) \left( \frac{1}{(x + y)^2} + \frac{1}{(y + z)^2} + \frac{1}{(z + x)^2} \right) \geq \frac{9}{4}.$$

**Solution** The given inequality is equivalent to

$$\begin{aligned} & 4(xy + yz + zx)((z + x)^2(y + z)^2 + (x + y)^2(z + x)^2 + (x + y)^2(y + z)^2) \\ & \geq 9(x + y)^2(y + z)^2(z + x)^2. \end{aligned} \quad (14.5)$$

Let us denote  $p = x + y + z$ ,  $q = xy + yz + zx$ ,  $r = xyz$ .

By  $I_5$  and  $I_7$  we have

$$(x + y)^2(y + z)^2(z + x)^2 = (pq - r)^2$$

and

$$(x + y)^2(y + z)^2 + (y + z)^2(z + x)^2 + (z + x)^2(x + y)^2 = (p^2 + q)^2 - 4p(pq - r).$$

So we can rewrite inequality (14.5) as follows

$$\begin{aligned} & 4q((p^2 + q)^2 - 4p(pq - r)) \geq 9(pq - r)^2 \\ \Leftrightarrow & 4p^4q - 17p^2q^2 + 4q^3 + 34pqr - 9r^2 \geq 0 \\ \Leftrightarrow & 3pq(p^3 - 4pq + 9r) + q(p^4 - 5p^2q + 4q^2 + 6pr) + r(pq - 9r) \geq 0. \end{aligned}$$

The last inequality follows from  $N_1$ ,  $N_2$  and  $N_3$ , and the fact that  $p, q, r > 0$ . Equality occurs if and only if  $x = y = z$ .

**Exercise 14.3** Let  $x, y, z \in \mathbb{R}^+$  such that  $x + y + z = 1$ . Prove the inequality

$$\frac{1}{1 - xy} + \frac{1}{1 - yz} + \frac{1}{1 - zx} \leq \frac{27}{8}.$$

**Solution** Let  $p = x + y + z = 1$ ,  $q = xy + yz + zx$ ,  $r = xyz$ .

It can easily be shown that

$$(1 - xy)(1 - yz)(1 - zx) = 1 - q + pr - r^2$$

and

$$(1 - xy)(1 - yz) + (1 - yz)(1 - zx) + (1 - zx)(1 - xy) = 3 - 2q + pr.$$

So the given inequality becomes

$$\begin{aligned} & 8(3 - 2q + pr) \leq 27(1 - q + pr - r^2) \\ \Leftrightarrow & 3 - 11q + 19pr - 27r^2 \geq 0. \end{aligned}$$

Since  $p = 1$ , we need to show that

$$3 - 11q + 19r - 27r^2 \geq 0.$$

By  $N_5$ :  $p^3 \geq 27r$  we have  $1 \geq 27r$ , i.e.  $r \geq 27r^2$ .

Therefore

$$3 - 11q + 19r - 27r^2 \geq 3 - 11q + 19r - r = 3 - 11q + 18r.$$

So it suffices to prove that

$$3 - 11q + 18r \geq 0.$$

We have

$$\begin{aligned} 3 - 11q + 18r &\geq 0 \\ \Leftrightarrow 3 - 11(xy + yz + zx) + 18xyz &\geq 0 \\ \Leftrightarrow 11(xy + yz + zx) - 18xyz &\leq 3. \end{aligned}$$

Applying  $AM \geq GM$  we deduce

$$\begin{aligned} 11(xy + yz + zx) - 18xyz &= xy(11 - 18z) + 11z(x + y) \\ &\leq \frac{(x + y)^2}{4}(11 - 18z) + 11z(x + y) \\ &= \frac{(1 - z)^2}{4}(11 - 18z) + 11z(1 - z) \\ &= \frac{(1 - z)((1 - z)(11 - 18z) + 44z)}{4} \\ &= \frac{4z + 3z^2 - 18z^3 + 11}{4}. \end{aligned}$$

So it remains to show that

$$\begin{aligned} \frac{4z + 3z^2 - 18z^3 + 11}{4} &\leq 3 \\ \Leftrightarrow 4z + 3z^2 - 18z^3 &\leq 1 \\ \Leftrightarrow 18z^3 - 3z^2 - 4z + 1 &\geq 0 \\ \Leftrightarrow (3z - 1)^2(2z + 1) &\geq 0, \end{aligned}$$

which is obvious.

**Exercise 14.4** Let  $a, b, c \in \mathbb{R}^+$  such that  $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$ . Prove the inequality

$$\frac{1}{8ab+1} + \frac{1}{8bc+1} + \frac{1}{8ca+1} \geq 1. \quad (14.6)$$

*Solution* Let  $p = a + b + c$ ,  $q = ab + bc + ca$ ,  $r = abc$ .



Since  $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$  we have

$$(a+1)(b+1) + (b+1)(c+1) + (c+1)(a+1) = 2(a+1)(b+1)(c+1). \quad (14.7)$$

Using the identities  $I_9$  and  $I_{10}$ , identity (14.7) becomes  $3 + 2p + q = 2(1 + p + q + r)$ , from which it follows that

$$q + 2r = 1. \quad (14.8)$$

It can easily be shown that

$$(8ab+1)(8bc+1) + (8bc+1)(8ca+1) + (8ca+1)(8ab+1) = 64pr + 16q + 3$$

and

$$(8ab+1)(8bc+1)(8ca+1) = 512r^2 + 64pr + 8q + 1.$$

We need to prove that

$$64pr + 16q + 3 \geq 512r^2 + 64pr + 8q + 1,$$

which is equivalent to

$$8q + 2 \geq 512r^2. \quad (14.9)$$

By  $q^3 \geq 27r^2$  and since  $q = 1 - 2r$  we obtain

$$\begin{aligned} (1-2r)^3 &\geq 27r^2 \\ \Leftrightarrow 8r^3 + 15r^2 + 6r - 1 &\leq 0 \\ \Leftrightarrow (8r-1)(r^2+2r+1) &\leq 0. \end{aligned}$$

Thus

$$8r - 1 \leq 0, \quad \text{i.e.} \quad r \leq \frac{1}{8}. \quad (14.10)$$

Now since  $q + 2r = 1$ , inequality (14.9) becomes

$$\begin{aligned} 8(1-2r) + 2 &\geq 512r^2 \\ \Leftrightarrow 512r^2 + 16r - 10 &\leq 0 \\ \Leftrightarrow (8r-1)(64r+10) &\leq 0, \end{aligned}$$

which follows due to (14.10).

**Exercise 14.5** Let  $x, y, z$  be positive real numbers such that  $x + y + z = 1$ . Prove the inequality

$$\frac{z-xy}{x^2+xy+y^2} + \frac{y-zx}{x^2+xz+z^2} + \frac{x-yz}{y^2+yz+z^2} \geq 2.$$

**Solution** Let  $p = x + y + z = 1$ ,  $q = xy + yz + zx$ ,  $r = xyz$ .

We have

$$\begin{aligned} x^2 + xy + y^2 &= (x + y)^2 - xy = (1 - z)^2 - xy = 1 - 2z + z^2 - xy \\ &= 1 - z - z(1 - z) - xy = 1 - z - z(x + y) - xy = 1 - z - q. \end{aligned}$$

Similarly we deduce that

$$x^2 + xz + z^2 = 1 - y - q \quad \text{and} \quad y^2 + yz + z^2 = 1 - x - q.$$

According to the previous identities,  $I_3$  and  $I_{12}$ , by using elementary algebraic transformations the given inequality becomes

$$q^3 + q^2 - 4q + 3qr + 4r + 1 \geq 0,$$

i.e.

$$27q^3 + 27q^2 - 108q + 27r(3q + 4) + 27 \geq 0. \quad (14.11)$$

By  $N_1$ :  $p^3 - 4pq + 9r \geq 0$ , since  $p = 1$  we get

$$9r \geq 4q - 1. \quad (14.12)$$

According to inequality (14.12) we obtain

$$\begin{aligned} 27q^3 + 27q^2 - 108q + 27r(3q + 4) + 27 &\geq 27q^3 + 27q^2 - 108q + 3(4q - 1)(3q + 4) + 27 \\ &= (3q - 1)(9q^2 + 24q - 15). \end{aligned} \quad (14.13)$$

Since  $p = 1$  due to  $N_1$ :  $p^2 \geq 3q$  it follows that

$$q \leq \frac{1}{3}. \quad (14.14)$$

Finally by (14.13) and (14.14) we obtain

$$\begin{aligned} 27q^3 + 27q^2 - 108q + 27r(3q + 4) + 27 &\geq (3q - 1)(9q^2 + 24q - 15) \geq 0, \\ \text{since } 3q - 1 &\leq 0 \text{ and } 9q^2 + 24q - 15 \leq 9 \cdot \frac{1}{9} + 24 \cdot \frac{1}{3} - 15 = -6, \text{ as required.} \end{aligned}$$

**Exercise 14.6** Let  $a, b, c$  be non-negative real numbers such that  $a + b + c = 1$ . Prove the inequality

$$7(ab + bc + ca) \leq 2 + 9abc.$$

**Solution** Let  $p = a + b + c = 1$ ,  $q = ab + bc + ca$ ,  $r = abc$ .

Then according to  $N_8$ :  $2p^3 + 9r \geq 7pq$  we have

$$2 + 9r \geq 7q \quad \text{i.e.} \quad 2 + 9abc \geq 7(ab + bc + ca),$$

as required.

**Exercise 14.7** Let  $x, y, z \geq 0$  be real numbers such that  $x + y + z = 1$ . Prove the inequality

$$12(x^2y^2 + y^2z^2 + z^2x^2)(x^3 + y^3 + z^3) \leq xy + yz + zx.$$

*Solution* Let  $p = x + y + z = 1$ ,  $q = xy + yz + zx$ ,  $r = xyz$ .

By  $I_2$  and  $I_3$  we have

$$x^3 + y^3 + z^3 = p(p^2 - 3q) + 3r = 1 - 3q + 3r$$

and

$$x^2y^2 + y^2z^2 + z^2x^2 = q^2 - 2pr = q^2 - 2r.$$

Clearly  $q \leq \frac{1}{3}$ .

So the given inequality becomes

$$12(1 - 3q + 3r)(q^2 - 2r) \leq q. \quad (14.15)$$

Suppose that  $q \geq \frac{1}{4}$ .

By  $N_3$ :  $pq - 9r \geq 0$  it follows that  $r \leq \frac{q}{9}$ , i.e.

$$0 \leq r \leq \frac{q}{9}. \quad (14.16)$$

Since  $q \leq \frac{1}{3}$  we have

$$(1 - 3q + 3r)r \geq 0. \quad (14.17)$$

We'll prove that

$$12\left(1 - 3q + 3\frac{q}{9}\right)q^2 \leq q, \quad (14.18)$$

from which, together with (14.16) and (14.17), we'll have

$$12(1 - 3q + 3r)(q^2 - 2r) \leq 12(1 - 3q + 3r)q^2 \leq 12\left(1 - 3q + 3\frac{q}{9}\right)q^2 \leq q.$$

Hence

$$\begin{aligned} q &\geq 12\left(1 - 3q + 3\frac{q}{9}\right)q^2 \\ \Leftrightarrow 1 &\geq 12\left(1 - 3q + \frac{q}{3}\right)q \\ \Leftrightarrow 1 &\geq 12q - 32q^2. \end{aligned} \quad (14.19)$$

Let  $f(q) = 12q - 32q^2$ . Then  $f'(q) = 12 - 64q$ .

Since  $q \geq \frac{1}{4}$  we deduce that  $f'(q) = 12 - 64q \leq 12 - \frac{64}{4} = -4 < 0$ , so it follows that  $f$  decreases on the interval  $[1/4, 1/3]$ , i.e. we have

$$f(q) \leq f\left(\frac{1}{4}\right) = 12 \cdot \frac{1}{4} - 32 \cdot \frac{1}{16} = 3 - 2 = 1,$$

and inequality (14.18) follows.

Now let us suppose that  $0 \leq q \leq \frac{1}{4}$ .

Let's rewrite inequality (14.15) as follows

$$q \geq 12q^2(1 - 3q) + 12r(3q^2 + 6q - 2) - 72r^2. \quad (14.20)$$

Since

$$12q(1 - 3q) = 4 \cdot 3q(1 - 3q) \leq 4 \left( \frac{3q + (1 - 3q)}{2} \right)^2 = 1,$$

it follows that

$$12q^2(1 - 3q) \leq q. \quad (14.21)$$

Since  $0 \leq q \leq \frac{1}{4}$  we get

$$3q^2 + 6q - 2 \leq 3 \cdot \frac{1}{16} + 6 \cdot \frac{1}{4} - 2 < 0. \quad (14.22)$$

By (14.21) and (14.22) we obtain

$$12q^2(1 - 3q) + 12r(3q^2 + 6q - 2) - 72r^2 \leq 12q^2(1 - 3q) \leq q,$$

as required.



# Chapter 15

## Method for Proving Symmetric Inequalities with Three Variables Defined on the Set of Real Numbers

This section will consider one method that is similar to the previous method of Chap. 14, for proving symmetrical inequalities with three variables that will be solvable only by elementary transformations and without major knowledge of inequalities (in the sense that for some of them the student has no need to know the powerful *Cauchy–Schwarz*, *Chebyshev*, *Minkowski* and *Hölder* inequalities).

We must note that this method is suitable for proving inequalities that are defined on the set of real numbers, not just on the set of positive real numbers. For this purpose we will first state (without proof) two theorems from differential calculus.

**Theorem** Let  $f : I \rightarrow \mathbb{R}$  be a differentiable function on  $I$ . Then  $f$  is an increasing function on  $I$  if and only if  $f'(x) \geq 0$  for all  $x \in I$ , and  $f$  is a decreasing function on  $I$  if and only if  $f'(x) \leq 0$  for all  $x \in I$ .

**Theorem** Let  $f(x)$  be a continuous function and twice differentiable on some interval that contains the point  $x_0$ .

Suppose that  $f'(x_0) = 0$ . Then:

- (1) If  $f''(x_0) < 0$ , then  $f$  has a local maximum at  $x_0$ .
- (2) If  $f''(x_0) > 0$ , then  $f$  has a local minimum at  $x_0$ .

Let  $a, b, c$  be real numbers such that  $a + b + c = 1$ .

According to the obvious inequality  $a^2 + b^2 + c^2 \geq ab + bc + ca$  (equality occurs iff  $a = b = c$ ) it follows that

$$1 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \geq 3(ab + bc + ca),$$

i.e.

$$ab + bc + ca \leq \frac{1}{3}.$$

Let  $ab + bc + ca = \frac{1-q^2}{3}$ , ( $q \geq 0$ ). We will find the maximum and minimum values of  $abc$  in terms of  $q$ .

If  $q = 0$  then  $ab + bc + ca = \frac{1}{3}$ , i.e.  $a = b = c = \frac{1}{3}$ .

Thus

$$abc = \frac{1}{27}.$$

If  $q \neq 0$  then

$$\begin{aligned} ab + bc + ca &= \frac{1-q^2}{3} < \frac{1}{3} = \frac{(a+b+c)^2}{3} \Leftrightarrow a^2 + b^2 + c^2 > ab + bc + ca \\ \Leftrightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 &> 0, \end{aligned}$$

i.e. at least two of the numbers  $a, b, c$  are different.

Consider the function

$$f(x) = (x-a)(x-b)(x-c) = x^3 - x^2 + \frac{1-q^2}{3}x - abc.$$

We have

$$f'(x) = 3x^2 - 2x + \frac{1-q^2}{3}, \quad \text{with zeros } x_1 = \frac{1+q}{3} \quad \text{and} \quad x_2 = \frac{1-q}{3}.$$

Hence  $f'(x) < 0$  for  $x_2 < x < x_1$ , and  $f'(x) > 0$  for  $x < x_2$  or  $x > x_1$ .

For  $f''(x)$  we have

$$f''(x) = 6x - 2, \quad \text{i.e.} \quad f''(x_1) = 6\left(\frac{1+q}{3}\right) - 2 = 6q > 0,$$

so it follows that  $f(x)$  at  $x_1$  has a local minimum.

Similarly  $f''(x_2) = 6\left(\frac{1-q}{3}\right) - 2 = -6q < 0$ , i.e.  $f(x)$  at  $x_2$  has a local maximum.

Furthermore  $f(x)$  has three zeros:  $a, b, c$ .

Then it follows that

$$\begin{aligned} f\left(\frac{1+q}{3}\right) &= \frac{(1+q)^2(1-2q)}{27} - abc \leq 0 \quad \text{and} \\ f\left(\frac{1-q}{3}\right) &= \frac{(1-q)^2(1+2q)}{27} - abc \geq 0. \end{aligned}$$

Hence

$$\frac{(1+q)^2(1-2q)}{27} \leq abc \leq \frac{(1-q)^2(1+2q)}{27}.$$

Therefore we have the following theorem.

**Theorem 15.1** Let  $a, b, c$  be real numbers such that  $a + b + c = 1$  and let

$$ab + bc + ca = \frac{1 - q^2}{3} \quad (q \geq 0).$$

Then we have the following inequalities

$$\frac{(1 + q)^2(1 - 2q)}{27} \leq abc \leq \frac{(1 - q)^2(1 + 2q)}{27}.$$

**Theorem 15.2** (Generalized) Let  $a, b, c$  be real numbers such that  $a + b + c = p$ .

Let  $ab + bc + ca = \frac{p^2 - q^2}{3}$ , ( $q \geq 0$ ) and  $abc = r$ .

Then

$$\frac{(p + q)^2(p - 2q)}{27} \leq r \leq \frac{(p - q)^2(p + 2q)}{27}.$$

Equality occurs if and only if  $(a - b)(b - c)(c - a) = 0$ .

If  $a + b + c = p$  and  $ab + bc + ca = \frac{p^2 - q^2}{3}$  then we can easily show the following identities.

$$1^\circ \quad a^2 + b^2 + c^2 = \frac{p^2 + 2q^2}{3}$$

$$2^\circ \quad a^3 + b^3 + c^3 = pq^2 + 3r$$

$$3^\circ \quad ab(a + b) + bc(b + c) + ca(c + a) = \frac{p(p^2 - q^2)}{3} - 3r$$

$$4^\circ \quad (a + b)(b + c)(c + a) = \frac{p(p^2 - q^2)}{3} - r$$

$$5^\circ \quad a^2b^2 + b^2c^2 + c^2a^2 = \frac{(p^2 - q^2)^2}{9} - 2pr$$

$$6^\circ \quad ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) = \frac{(p^2 + 2q^2)(p^2 - q^2)}{9} - pr$$

$$7^\circ \quad a^4 + b^4 + c^4 = \frac{-p^4 + 8p^2q^2 + 2q^4}{9} + 4pr.$$

**Exercise 15.1** Let  $a, b, c$  be real numbers. Prove the inequality

$$a^4 + b^4 + c^4 \geq abc(a + b + c).$$

*Solution* Since the given inequality is homogenous, we may assume that  $a + b + c = 1$ .

Then it becomes

$$\frac{-1 + 8q^2 + 2q^4}{9} + 4r \geq r \quad \Leftrightarrow \quad -1 + 8q^2 + 2q^4 + 27r \geq 0.$$



According to Theorem 15.1, it follows that it suffices to show that

$$-1 + 8q^2 + 2q^4 + 27 \frac{(1+q)^2(1-2q)}{27} \geq 0.$$

We have

$$\begin{aligned} & -1 + 8q^2 + 2q^4 + 27 \frac{(1+q)^2(1-2q)}{27} \\ &= -1 + 8q^2 + 2q^4 + (1+q)^2(1-2q) \\ &= -1 + 8q^2 + 2q^4 + (1+2q+q^2)(1-2q) \\ &= -1 + 8q^2 + 2q^4 + (1-3q^2-2q^3) \\ &= 2q^4 + 5q^2 - 2q^3 = q^2(2q^2 - 2q + 5) \\ &= q^2 \frac{(4q^2 - 4q + 10)}{2} = q^2 \frac{((2q-1)^2 + 9)}{2} \geq 0, \end{aligned}$$

as required. Equality occurs iff  $a = b = c$ .

**Exercise 15.2** Let  $a, b, c \in \mathbb{R}$ . Prove the inequality

$$(a+b)^4 + (b+c)^4 + (c+a)^4 \geq \frac{4}{7}(a^4 + b^4 + c^4).$$

*Solution* Since

$$\begin{aligned} & (a+b)^4 + (b+c)^4 + (c+a)^4 \\ &= 2(a^4 + b^4 + c^4) + 4(a^3b + b^3a + b^3c + c^3b + c^3a + a^3c) \\ & \quad + 6(a^2b^2 + b^2c^2 + c^2a^2), \end{aligned}$$

the given inequality becomes

$$\begin{aligned} & 5(a^4 + b^4 + c^4) + 14(a^3b + b^3a + b^3c + c^3b + c^3a + a^3c) \\ & \quad + 21(a^2b^2 + b^2c^2 + c^2a^2) \geq 0. \end{aligned}$$

After setting  $a + b + c = p$ ,  $ab + bc + ca = \frac{p^2 - q^2}{3}$ ,  $r = abc$ , due to  $5^\circ$ ,  $6^\circ$  and  $7^\circ$  we deduce that the previous inequality is equivalent to

$$\begin{aligned} & 5 \left( \frac{-p^4 + 8p^2q^2 + 2q^4}{9} + 4pr \right) + 14 \left( \frac{(p^2 + 2q^2)(p^2 - q^2)}{9} - pr \right) \\ & \quad + 21 \left( \frac{(p^2 - q^2)^2}{9} - 2pr \right) \geq 0, \end{aligned}$$

i.e.

$$\begin{aligned} & 5(-p^4 + 8p^2q^2 + 2q^4 + 36pr) + 14((p^2 + 2q^2)(p^2 - q^2) - 9pr) \\ & + 21((p^2 - q^2)^2 - 18pr) \geq 0. \end{aligned}$$

If  $p = 0$  then  $10q^4 - 28q^4 + 21q^4 \geq 0$ , i.e.  $3q^4 \geq 0$ , which is obvious.

Let  $p \neq 0$ .

Without loss of generality we may assume that  $p = 1$ .

So we need to prove that

$$5(-1 + 8q^2 + 2q^4 + 36r) + 14((1 + 2q^2)(1 - q^2) - 9r) + 21((1 - q^2)^2 - 18r) \geq 0,$$

i.e.

$$3q^4 + 4q^2 + 10 - 108r \geq 0.$$

Using Theorem 15.1, we obtain

$$\begin{aligned} 3q^4 + 4q^2 + 10 - 108r & \geq 3q^4 + 4q^2 + 10 - 108 \frac{(1 - q)^2(1 + 2q)}{27} \\ & = 3q^4 + 4q^2 + 10 - 4(1 - q)^2(1 + 2q) \\ & = q^2(q - 4)^2 + 2q^4 + 6 \geq 0, \end{aligned}$$

which clearly holds.

Equality occurs if and only if  $a = b = c = 0$ .

**Exercise 15.3** Let  $a, b, c$  be real numbers such that  $a^2 + b^2 + c^2 = 9$ . Prove the inequality

$$2(a + b + c) - abc \leq 10.$$

*Solution* Let  $a + b + c = p$ ,  $ab + bc + ca = \frac{1 - q^2}{3}$ ,  $abc = r$ .

Then using identity 1°, the condition can be rewritten as

$$9 = a^2 + b^2 + c^2 = \frac{p^2 + 2q^2}{3},$$

i.e.

$$p^2 + 2q^2 = 27. \tag{15.1}$$

By Theorem 15.2 we deduce

$$\begin{aligned} 2(a + b + c) - abc & = 2p - r \leq 2p - \frac{(p + q)^2(p - 2q)}{27} \\ & = \frac{54p - p^3 + 3pq^2 + 2q^3}{27} \end{aligned}$$

$$\begin{aligned}
&= \frac{54p - p(p^2 + 2q^2) + 5pq^2 + 2q^3}{27} \\
&\stackrel{(15.1)}{=} \frac{54p - 27p + 5pq^2 + 2q^3}{27} \\
&= \frac{27p + 5pq^2 + 2q^3}{27} = \frac{p(27 + 5q^2) + 2q^3}{27}.
\end{aligned}$$

So it remains to prove that

$$\frac{p(27 + 5q^2) + 2q^3}{27} \leq 10 \quad \text{or} \quad p(27 + 5q^2) \leq 270 - 2q^3.$$

We have

$$\begin{aligned}
(270 - 2q^3)^2 &\geq (p(27 + 5q^2))^2 \\
\Leftrightarrow 27(q - 3)^2(2q^4 + 12q^3 + 49q^2 + 146q + 219) &\geq 0
\end{aligned}$$

as required.

Equality occurs if and only if  $(a, b, c) = (2, 2, -1)$  (up to permutation).

**Exercise 15.4** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca).$$

*Solution* The given inequality is equivalent to

$$\frac{p^2 + 2q^2}{3} + 2r + 1 \geq 2\frac{1 - q^2}{3}$$

i.e.

$$6r + 3 + 4q^2 - p^2 \geq 0.$$

If  $2q \geq p$  then we are done.

Therefore suppose that  $p \geq 2q$ .

By Theorem 15.2, it suffices to prove that

$$6r + 3 + 4q^2 - p^2 \geq 6\frac{(p+q)^2(p-2q)}{27} + 3 + 4q^2 - p^2 \geq 0,$$

i.e.

$$\begin{aligned}
&\frac{2(p+q)^2(p-2q)}{9} + 3 + 4q^2 - p^2 \geq 0 \\
\Leftrightarrow (p-3)^2(2p+3) &\geq 2q^2(2q+3p-18). \tag{15.2}
\end{aligned}$$

If  $2p \leq 9$  it follows that  $2q + 3p \leq 4p \leq 18$ , and we are done.

If  $2p \geq 9$  we have

$$\begin{aligned} 2q^2(2q + 3p - 18) &\leq 2q^2(p + 3p - 18) = 4q^2(2p - 9) \\ &\leq p^2(2p - 9) = (p - 3)^2(2p + 3) - 27 < (p - 3)^2(2p + 3), \end{aligned}$$

so inequality (15.2) is true, as required.

Equality occurs if and only if  $a = b = c = 1$ .

**Exercise 15.5** (Schur's inequality) Prove that for any non-negative real numbers  $a, b, c$  we have

$$a^3 + b^3 + c^3 + 3abc \geq ab(a + b) + bc(b + c) + ca(c + a).$$

*Solution* Since the above inequality is homogenous, we may assume that  $a + b + c = 1$ .

Then clearly  $q \in [0, 1]$  and the given inequality becomes

$$27r + 4q^2 - 1 \geq 0.$$

If  $q \geq \frac{1}{2}$ , then we are done.

If  $q \leq \frac{1}{2}$ , by Theorem 15.1, we have

$$27r + 4q^2 - 1 \geq 27 \frac{(1 + q)^2(1 - 2q)}{27} + 4q^2 - 1 = q^2(1 - 2q) \geq 0,$$

as required.

Equality occurs iff  $(a, b, c) = (t, t, t)$  or  $(a, b, c) = (t, t, 0)$ , where  $t \geq 0$  is an arbitrary real number (up to permutation).



## Chapter 16

# Abstract Concreteness Method (ABC Method)

In this section we will present three theorems without proofs (the proofs can be found in [27]) which are the basis of a very useful method, the *Abstract Concreteness Method (ABC method)*.

For this purpose we'll consider the function  $f(abc, ab + bc + ca, a + b + c)$ , as a one-variable function with variable  $abc$  on  $\mathbb{R}$ , i.e. on  $\mathbb{R}^+$ .

### 16.1 ABC Theorem

**Theorem 16.1** *If the function  $f(abc, ab + bc + ca, a + b + c)$  is monotonic then  $f$  achieves its maximum and minimum values on  $\mathbb{R}$  when  $(a - b)(b - c)(c - a) = 0$ , and on  $\mathbb{R}^+$  when  $(a - b)(b - c)(c - a) = 0$  or  $abc = 0$ .*

**Theorem 16.2** *If the function  $f(abc, ab + bc + ca, a + b + c)$  is a convex function then it achieves its maximum and minimum values on  $\mathbb{R}$  when  $(a - b)(b - c)(c - a) = 0$ , and on  $\mathbb{R}^+$  when  $(a - b)(b - c)(c - a) = 0$  or  $abc = 0$ .*

**Theorem 16.3** *If the function  $f(abc, ab + bc + ca, a + b + c)$  is a concave function then it achieves its maximum and minimum values on  $\mathbb{R}$  when  $(a - b)(b - c)(c - a) = 0$ , and on  $\mathbb{R}^+$  when  $(a - b)(b - c)(c - a) = 0$  or  $abc = 0$ .*

**Consequence 16.1** *Let  $f(abc, ab + bc + ca, a + b + c)$  be a linear function with variable  $abc$ . Then  $f$  achieves it's maximum and minimum values on  $\mathbb{R}$  if and only if  $(a - b)(b - c)(c - a) = 0$ , and on  $\mathbb{R}^+$  if and only if  $(a - b)(b - c)(c - a) = 0$  or  $abc = 0$ .*

**Consequence 16.2** *Let  $f(abc, ab + bc + ca, a + b + c)$  be a quadratic trinomial with variable  $abc$ , then  $f$  achieves it's maximum on  $\mathbb{R}$  if and only if  $(a - b)(b - c)(c - a) = 0$ , and on  $\mathbb{R}^+$  if and only if  $(a - b)(b - c)(c - a) = 0$  or  $abc = 0$ .*

**Consequence 16.3** *All symmetric three-variable polynomials of degree less than or equal to 5 achieves their maximum and minimum values on  $\mathbb{R}$  if and only if  $(a - b)(b - c)(c - a) = 0$ , and on  $\mathbb{R}^+$  if and only if  $(a - b)(b - c)(c - a) = 0$  or  $abc = 0$ .*

**Consequence 16.4** *All symmetric three-variables polynomials of degree less than or equal to 8 with non-negative coefficient of  $(abc)^2$  in the representation form  $f(abc, ab + bc + ca, a + b + c)$ , achieves their maximum on  $\mathbb{R}$  if and only if  $(a - b)(b - c)(c - a) = 0$ , and on  $\mathbb{R}^+$  if and only if  $(a - b)(b - c)(c - a) = 0$  or  $abc = 0$ .*

Also we'll introduce some additional identities which will be very useful for the correct presentation of this method.

For that purpose, let  $a = x + y + z$ ,  $b = xy + yz + zx$ ,  $c = xyz$ . Then we have.

$$\begin{aligned}
 I_1: & x^2 + y^2 + z^2 = a^2 - 2b \\
 I_2: & x^3 + y^3 + z^3 = a^3 - 3ab + 3c \\
 I_3: & x^4 + y^4 + z^4 = a^4 - 4a^2b + 2b^2 + 4ac \\
 I_4: & x^5 + y^5 + z^5 = a^5 - 5a^3b + 5ab^2 + 5a^2c - 5bc \\
 I_5: & x^6 + y^6 + z^6 = a^6 - 6a^4b + 6a^3c + 9a^2b^2 - 12abc + 3c^2 - 2b^3 \\
 I_6: & (xy)^2 + (yz)^2 + (zx)^2 = b^2 - 2ac \\
 I_7: & (xy)^3 + (yz)^3 + (zx)^3 = b^3 - 3abc + 3c^2 \\
 I_8: & (xy)^4 + (yz)^4 + (zx)^4 = b^4 - 4ab^2c + 2a^2c^2 + 4bc^2 \\
 I_9: & (xy)^5 + (yz)^5 + (zx)^5 = b^5 - 5ab^3c + 5a^2bc^2 + 5b^2c^2 - 5ac^3 \\
 I_{10}: & xy(x + y) + yz(y + z) + zx(z + x) = ab - 3c \\
 I_{11}: & xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) = a^2b - 2b^2 - ac \\
 I_{12}: & xy(x^3 + y^3) + yz(y^3 + z^3) + zx(z^3 + x^3) = a^3b - 3ab^2 - a^2c + 5bc \\
 I_{13}: & x^2y^2(x + y) + y^2z^2(y + z) + z^2x^2(z + x) = ab^2 - 2a^2c - bc \\
 I_{14}: & x^3y^3(x + y) + y^3z^3(y + z) + z^3x^3(z + x) = ab^3 - 3a^2bc + 5ac^2 - b^2c
 \end{aligned}$$

$$I_{15}: (x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) = 9c^2 + (a^3 - 6ab)c + b^3$$

$$I_{16}: (x^3y + y^3z + z^3x)(xy^3 + yz^3 + zx^3) = 7a^2c^2 + (a^5 - 5a^3b + ab^2)c + b^4.$$

**Exercise 16.1** Let  $a, b, c > 0$  be real numbers. Prove the inequality

$$\frac{abc}{a^3 + b^3 + c^3} + \frac{2}{3} \geq \frac{ab + bc + ca}{a^2 + b^2 + c^2}.$$

*Solution* The given inequality is equivalent to the following one

$$\begin{aligned} F &= abc(a^2 + b^2 + c^2) + \frac{2}{3}(a^3 + b^3 + c^3)(a^2 + b^2 + c^2) \\ &\quad - (a^3 + b^3 + c^3)(ab + bc + ca) \geq 0. \end{aligned}$$

The polynomial  $F$  is of third degree so it will achieve its minimum when

$$(a - b)(b - c)(c - a) = 0 \quad \text{or} \quad abc = 0.$$

If  $(a - b)(b - c)(c - a) = 0$ , then without loss of generality we may assume that  $a = c$  and the given inequality becomes

$$\begin{aligned} \frac{a^2b}{2a^3 + b^3} + \frac{2}{3} &\geq \frac{a^2 + 2ab}{2a^2 + b^2} \quad \Leftrightarrow \quad (a - b)^2 \left( \frac{1}{2a^2 + b^2} - \frac{2a + b}{3(2a^3 + b^3)} \right) \geq 0 \\ &\Leftrightarrow \quad (a - b)^4(a + b) \geq 0, \end{aligned}$$

which is obvious.

If  $abc = 0$  then without loss of generality we may assume that  $c = 0$  and the given inequality becomes

$$\frac{2}{3} \geq \frac{ab}{a^2 + b^2} \quad \Leftrightarrow \quad a^2 + b^2 + 3(a - b)^2 \geq 0,$$

which is true. And we are done.

**Exercise 16.2** Let  $a, b, c > 0$  be real numbers. Prove the inequality

$$\frac{a^3 + b^3 + c^3}{4abc} + \frac{1}{4} \geq \left( \frac{a^2 + b^2 + c^2}{ab + bc + ca} \right)^2.$$

*Solution* Observe that by applying the previous identities the given inequality can be rewritten as a seventh-degree symmetric polynomial with variables  $a, b, c$ , but it's only a first-degree polynomial with variable  $abc$ .

Therefore by Consequence 16.1, we need to consider only the following two cases.



*First case:* If  $(a - b)(b - c)(c - a) = 0$ , then without loss of generality we may assume that  $a = c$  and the given inequality becomes

$$\begin{aligned} \frac{2a^3 + b^3}{4a^2b} + \frac{1}{4} &\geq \left( \frac{2a^2 + b^2}{a^2 + 2ab} \right)^2 \Leftrightarrow \frac{2a^3 + b^3}{4a^2b} - \frac{3}{4} \geq \left( \frac{2a^2 + b^2}{a^2 + 2ab} \right)^2 - 1 \\ \Leftrightarrow \frac{(a - b)^2(2a + b)}{4a^2b} &\geq \frac{(a - b)^2(3a^2 + b^2 + 2ab)}{(a^2 + 2ab)^2} \\ \Leftrightarrow (a - b)^2((2b - a)^2 + a^2) &\geq 0, \end{aligned}$$

which is obvious.

*Second case:* If  $abc = 0$  then the given inequality is trivially correct.

**Exercise 16.3** Let  $a, b, c > 0$  be real numbers. Prove the inequality

$$(ab + bc + ca) \left( \frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} \right) \geq \frac{9}{4}.$$

*Solution* We can rewrite the given inequality in the following form

$$\begin{aligned} f(a + b + c, ab + bc + ca, abc) \\ &= 9((a + b)(b + c)(c + a))^2 \\ &\quad - 4(ab + bc + ca)((a + b)^2(b + c)^2 + (b + c)^2(c + a)^2 + (c + a)^2(a + b)^2) \\ &= k(abc)^2 + mabc + n, \end{aligned}$$

where  $k \geq 0$  and  $k, m, n$  are quantities containing constants or  $a + b + c, ab + bc + ca, abc$ , which we also consider as constants, i.e. in the form as a sixth-degree symmetric polynomial with variables  $a, b, c$  and a second-degree polynomial with variable  $abc$  and positive coefficients.

Let us explain this:

The expression  $(a + b)(b + c)(c + a)$  has the form  $kabc + m$  so it follows that  $9((a + b)(b + c)(c + a))^2$  has the form  $k^2(abc)^2 + mabc + n$ .

Furthermore

$$4(ab + bc + ca)((a + b)^2(b + c)^2 + (b + c)^2(c + a)^2 + (c + a)^2(a + b)^2) = 4kA,$$

where  $k = ab + bc + ca$ , and  $A$  is a fourth-degree polynomial and also has the form  $kabc + m$ .

Therefore the expression of the left side of  $f(a + b + c, ab + bc + ca, abc)$  has the form  $k(abc)^2 + mabc + n$ .

Then the function achieves its minimum value when  $(a - b)(b - c)(c - a) = 0$  or when  $abc = 0$ .

If  $(a - b)(b - c)(c - a) = 0$ , then without loss of generality we may assume that  $a = c$ , and the given inequality is equivalent to

$$\begin{aligned} (a^2 + 2ab) \left( \frac{1}{4a^2} + \frac{2}{(a+b)^2} \right) &\geq \frac{9}{4} \\ \Leftrightarrow (a-b)^2 \left( \frac{2a+b}{2a(a+b)^2} - \frac{1}{(a+b)^2} \right) &\geq 0 \\ \Leftrightarrow b(a-b)^2 &\geq 0, \end{aligned}$$

as required.

If  $abc = 0$ , we may assume that  $c = 0$  and the given inequality becomes

$$\begin{aligned} ab \left( \frac{1}{(a+b)^2} + \frac{1}{a^2} + \frac{1}{b^2} \right) &\geq \frac{9}{4} \quad \Leftrightarrow \quad (a-b)^2 \left( \frac{1}{ab} - \frac{1}{4(a+b)^2} \right) \geq 0 \\ \Leftrightarrow (a-b)^2 (4a^2 + 4b^2 + 7ab) &\geq 0, \end{aligned}$$

and the problem is solved.

**Exercise 16.4** Let  $a, b, c > 0$  be real numbers such that  $a^2 + b^2 + c^2 = 1$ . Prove the inequality

$$\frac{a}{a^3 + bc} + \frac{b}{b^3 + ca} + \frac{c}{c^3 + ab} \geq 3.$$

*Solution* If we transform the given inequality as a symmetric polynomial we obtain a ninth-degree polynomial with variables  $a, b, c$ , and a third-degree polynomial with variable  $abc$ . But, as we know, this case is not in the previously mentioned consequences, so the problem cannot be solved with *ABC* (for now).

Therefore we'll make some algebraic transformations.

If we take

$$x = \frac{bc}{a}, \quad y = \frac{ac}{b}, \quad z = \frac{ab}{c},$$

then clearly  $xy + yz + zx = a^2 + b^2 + c^2 = 1$ , and the given inequality becomes

$$\frac{1}{xy + z} + \frac{1}{yz + x} + \frac{1}{zx + y} \geq 3. \quad (16.1)$$

If we transform the inequality (16.1) we'll get a second-degree polynomial with variable  $xyz$ , with a non-negative coefficient in front of  $(xyz)^2$ .

So we need to consider just the following cases:

If  $x = z$  then inequality (16.1) becomes

$$\frac{2}{xy + x} + \frac{1}{x^2 + y} \geq 3.$$

Since  $2xy + x^2 = 1$  it follows that  $y = \frac{1-x^2}{2x}$ , and after using these, the previous inequality easily follows.

If  $z = 0$  then inequality (16.1) becomes

$$\frac{1}{xy} + \frac{1}{x} + \frac{1}{y} \geq 3, \quad \text{with } xy = 1.$$

We have  $\frac{1}{xy} + \frac{1}{x} + \frac{1}{y} \geq 1 + \frac{2}{\sqrt{xy}} = 3$ , as required.

**Exercise 16.5** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca + abc = 4$ . Prove the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq a + b + c.$$

*Solution* Since  $ab + bc + ca + abc = 4$  there exist real numbers  $x, y, z$  such that

$$a = \frac{2x}{y+z}, \quad b = \frac{2y}{z+x}, \quad c = \frac{2z}{x+y},$$

and the given inequality becomes

$$\frac{x+y}{z} + \frac{z+x}{y} + \frac{y+z}{x} \geq 4 \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right). \quad (16.2)$$

Inequality (16.2) is homogenous, so we may assume that  $x + y + z = 1$ ,  $xy + yz + zx = u$ ,  $xyz = v$ .

After some algebraic transformations we find that inequality (16.2) can be rewritten as follows

$$9v^2 + 4(1-u)v - v^2 \geq 0.$$

So, according to the *ABC theorem*, we need to consider just two cases:

If  $z = 0$  then inequality (16.2) is trivially correct.

If  $y = z = 1$  (we can do this because of the homogenous property) inequality (16.2) becomes

$$2(x+1) + \frac{2}{x} \geq 4 \left( \frac{x}{2} + \frac{2}{x+1} \right) \quad \text{i.e.} \quad 2(x-1)^2 \geq 0,$$

which is obvious.

# Chapter 17

## Sum of Squares (SOS Method)

One of the basic procedures for proving inequalities is to rewrite them as a sum of squares (SOS) and then, according to the most elementary property that the square of a real number is non-negative, to prove a certain inequality. This property is the basis of the SOS method.

The advantage of the *method of squares* is that it requires knowledge only of basic inequalities, which we met earlier, and basic skills in elementary operations.

Let's start with one well-known inequality.

*Example 17.1* Let  $a, b, c \geq 0$ . Prove the inequality

$$a^3 + b^3 + c^3 \geq 3abc.$$

*Solution* We have

$$a^3 + b^3 + c^3 - 3abc = \frac{a+b+c}{2}((a-b)^2 + (b-c)^2 + (c-a)^2) \geq 0,$$

which is obviously true.

The whole idea is to rewrite the given inequality in the form

$$S_a(b-c)^2 + S_b(a-c)^2 + S_c(a-b)^2,$$

where  $S_a, S_b, S_c$  are functions of  $a, b, c$ .

We must mention that this method works well for proving symmetrical inequalities where we can assume that  $a \geq b \geq c$ , while if we work with cyclic inequalities we need to consider the additional case  $c \geq b \geq a$ .

We will discuss symmetrical inequalities with three variables, and for that purpose firstly we'll give three properties that we will use for the proof of the main theorem.

**Proposition 17.1** Let  $a, b, c \in \mathbb{R}$ . Then  $(a-c)^2 \leq 2(a-b)^2 + 2(b-c)^2$ .

*Proof* We have

$$\begin{aligned}
 (a - c)^2 &\leq 2(a - b)^2 + 2(b - c)^2 \\
 \Leftrightarrow a^2 - 2ac + c^2 &\leq 2(a^2 - 2ab + b^2) + 2(b^2 - 2bc + c^2) \\
 \Leftrightarrow a^2 + 4b^2 + c^2 - 4ab - 4bc + 2ac &\geq 0 \\
 \Leftrightarrow (a + c - 2b)^2 &\geq 0,
 \end{aligned}$$

which clearly holds.  $\square$

**Proposition 17.2** *Let  $a \geq b \geq c$ . Then  $(a - c)^2 \geq (a - b)^2 + (b - c)^2$ .*

*Proof* We have

$$\begin{aligned}
 (a - c)^2 &\geq (a - b)^2 + (b - c)^2 \\
 \Leftrightarrow a^2 - 2ac + c^2 &\geq (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) \\
 \Leftrightarrow b^2 + ac - ab - bc &\leq 0 \\
 \Leftrightarrow (b - a)(b - c) &\leq 0,
 \end{aligned}$$

which is true since  $a \geq b \geq c$ .  $\square$

**Proposition 17.3** *Let  $a \geq b \geq c$ . Then  $\frac{a-c}{b-c} \geq \frac{a}{b}$ .*

*Proof* We have

$$\frac{a - c}{b - c} \geq \frac{a}{b} \Leftrightarrow b(a - c) \geq a(b - c) \Leftrightarrow ac \geq bc \Leftrightarrow a \geq b. \quad \square$$

**Theorem 17.1** (SOS method) *Consider the expression  $S = S_a(b - c)^2 + S_b(a - c)^2 + S_c(a - b)^2$ , where  $S_a, S_b, S_c$  are functions of  $a, b, c$ .*

- 1° If  $S_a, S_b, S_c \geq 0$  then  $S \geq 0$ .
- 2° If  $a \geq b \geq c$  or  $a \leq b \leq c$  and  $S_b, S_b + S_a, S_b + S_c \geq 0$  then  $S \geq 0$ .
- 3° If  $a \geq b \geq c$  or  $a \leq b \leq c$  and  $S_a, S_c, S_a + 2S_b, S_c + 2S_b \geq 0$  then  $S \geq 0$ .
- 4° If  $a \geq b \geq c$  and  $S_b, S_c, a^2S_b + b^2S_a \geq 0$  then  $S \geq 0$ .
- 5° If  $S_a + S_b \geq 0$  or  $S_b + S_c \geq 0$  or  $S_c + S_a \geq 0$  ( $S_a + S_b + S_c \geq 0$ ) and  $S_aS_b + S_bS_c + S_cS_a \geq 0$  then  $S \geq 0$ .

*Proof* 1° If  $S_a, S_b, S_c \geq 0$  then clearly  $S \geq 0$ .

2° Let us assume that  $a \geq b \geq c$  and  $S_b, S_b + S_a, S_b + S_c \geq 0$ .

By Proposition 17.2, it follows that  $(a - c)^2 \geq (a - b)^2 + (b - c)^2$ , so we have

$$\begin{aligned} S &= S_a(b - c)^2 + S_b(a - c)^2 + S_c(a - b)^2 \\ &\geq S_a(b - c)^2 + S_b((a - b)^2 + (b - c)^2) + S_c(a - b)^2 \\ &= (b - c)^2(S_a + S_b) + (a - b)^2(S_b + S_c). \end{aligned}$$

Now since  $S_b + S_a, S_b + S_c \geq 0$  it follows that  $S \geq 0$ .

3° Let  $a \geq b \geq c$  and  $S_a, S_c, S_a + 2S_b, S_c + 2S_b \geq 0$ .

Then if  $S_b \geq 0$  clearly  $S \geq 0$ .

Suppose that  $S_b \leq 0$ .

By Proposition 17.1, we have that  $(a - c)^2 \leq 2(a - b)^2 + 2(b - c)^2$ .

Therefore

$$\begin{aligned} S &= S_a(b - c)^2 + S_b(a - c)^2 + S_c(a - b)^2 \\ &\geq S_a(b - c)^2 + S_b(2(a - b)^2 + 2(b - c)^2) + S_c(a - b)^2 \\ &= (b - c)^2(S_a + 2S_b) + (a - b)^2(S_c + 2S_b), \end{aligned}$$

and since  $S_a + 2S_b, S_c + 2S_b \geq 0$  it follows that  $S \geq 0$ .

4° Let  $a \geq b \geq c$  and suppose that  $S_b, S_c, a^2S_b + b^2S_a \geq 0$ .

By Proposition 17.3, it follows that  $\frac{a-c}{b-c} \geq \frac{a}{b}$ .

Therefore

$$\begin{aligned} S &= S_a(b - c)^2 + S_b(a - c)^2 + S_c(a - b)^2 \geq S_a(b - c)^2 + S_b(a - c)^2 \\ &= (b - c)^2 \left( S_a + S_b \left( \frac{a - c}{b - c} \right)^2 \right) \geq (b - c)^2 \left( S_a + S_b \left( \frac{a}{b} \right)^2 \right) \\ &= (b - c)^2 \left( \frac{b^2S_a + a^2S_b}{b^2} \right), \end{aligned}$$

since  $a^2S_b + b^2S_a \geq 0$  we obtain  $S \geq 0$ .

5° Assume that  $S_b + S_c \geq 0$ .

We have

$$\begin{aligned} S &= S_a(b - c)^2 + S_b(a - c)^2 + S_c(a - b)^2 \\ &= S_a(b - c)^2 + S_b((c - b) + (b - a))^2 + S_c(a - b)^2 \\ &= (S_b + S_c)(a - b)^2 + 2S_b(c - b)(b - a) + (S_a + S_b)(b - c)^2 \\ &= (S_b + S_c) \left( b - a + \frac{S_b}{S_b + S_c}(c - b) \right)^2 + \frac{S_aS_b + S_bS_c + S_cS_a}{S_b + S_c}(c - b)^2 \\ &\geq 0. \end{aligned}$$

□

The main difficulty with using the S.O.S. method is the transformation of the given inequality into mentioned (S.O.S.) form.

Every difference  $\sum_{cyc} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} - \sum_{cyc} x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$  where  $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \beta_1 + \beta_2 + \cdots + \beta_n$  can be written in S.O.S. form, so almost all symmetrical or permutation homogeneous inequalities can be written in S.O.S. form. In fact there is a huge class of algebraic expressions which can be written in S.O.S. form (the algorithm which helps to transform algebraic expressions into S.O.S. form is explicitly explained for example in [27]).

Here we will introduce the reader to the simplest and most often used forms which are as follows:

$$\begin{aligned}
 1^\circ \quad & a^2 + b^2 + c^2 - ab - bc - ca = \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2} \\
 2^\circ \quad & a^3 + b^3 + c^3 - 3abc = (a+b+c) \cdot \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2} \\
 3^\circ \quad & a^2b + b^2c + c^2a - ab^2 - bc^2 - ca^2 = \frac{(a-b)^3 + (b-c)^3 + (c-a)^3}{3} \\
 4^\circ \quad & a^3 + b^3 + c^3 - a^2b - b^2c - c^2a = \frac{(2a+b)(a-b)^2 + (2b+c)(b-c)^2 + (2c+a)(c-a)^2}{3} \\
 5^\circ \quad & a^3b + b^3c + c^3a - ab^3 - bc^3 - ca^3 = (a+b+c) \cdot \frac{(b-a)^3 + (c-b)^3 + (a-c)^3}{3} \\
 6^\circ \quad & a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 = \frac{(a+b)^2(a-b)^2 + (b+c)^2(b-c)^2 + (c+a)^2(c-a)^2}{2}
 \end{aligned}$$

**Exercise 17.1** Let  $x, y, z \in \mathbb{R}$  such that  $xyz \geq 1$ . Prove the inequality

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$

*Solution* We'll homogenize as follows

$$\begin{aligned}
 \frac{x^5 - x^2}{x^5 + y^2 + z^2} &\geq \frac{x^5 - x^2 \cdot xyz}{x^5 + xyz(y^2 + z^2)} = \frac{x^4 - x^2yz}{x^4 + yz(y^2 + z^2)} \\
 &\geq \frac{x^4 - x^2(\frac{y^2+z^2}{2})}{x^4 + (\frac{y^2+z^2}{2})(y^2 + z^2)} = \frac{2x^4 - x^2(y^2 + z^2)}{2x^4 + (y^2 + z^2)^2}.
 \end{aligned}$$

Similarly we get

$$\frac{y^5 - y^2}{y^5 + z^2 + x^2} \geq \frac{2y^4 - y^2(z^2 + x^2)}{2y^4 + (z^2 + x^2)^2} \quad \text{and} \quad \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq \frac{2z^4 - z^2(x^2 + y^2)}{2z^4 + (x^2 + y^2)^2}.$$

So it suffices to show that

$$\frac{2x^4 - x^2(y^2 + z^2)}{2x^4 + (y^2 + z^2)^2} + \frac{2y^4 - y^2(z^2 + x^2)}{2y^4 + (z^2 + x^2)^2} + \frac{2z^4 - z^2(x^2 + y^2)}{2z^4 + (x^2 + y^2)^2} \geq 0. \quad (17.1)$$

Let  $x^2 = a, y^2 = b, z^2 = c$ . Then inequality (17.1) becomes

$$\frac{2a^2 - a(b+c)}{2a^2 + (b+c)^2} + \frac{2b^2 - b(c+a)}{2b^2 + (c+a)^2} + \frac{2c^2 - c(a+b)}{2c^2 + (a+b)^2} \geq 0. \quad (17.2)$$

After some algebraic operations we can rewrite inequality (17.2) as follows

$$\begin{aligned}
 & (b-c)^2 \frac{a^2 + a(b+c) + b^2 - bc + c^2}{(2b^2 + (c+a)^2)(2c^2 + (a+b)^2)} \\
 & + (c-a)^2 \frac{b^2 + b(a+c) + c^2 - ca + a^2}{(2a^2 + (b+c)^2)(2c^2 + (a+b)^2)} \\
 & + (a-b)^2 \frac{c^2 + c(a+b) + a^2 - ab + b^2}{(2a^2 + (b+c)^2)(2b^2 + (c+a)^2)} \geq 0,
 \end{aligned}$$

which is true due to the obvious inequality: if  $x, y \in \mathbb{R}$  then  $x^2 - xy + y^2 \geq 0$ .

**Exercise 17.2** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2. \quad (17.3)$$

*Solution* Observe that

$$a^2 + b^2 + c^2 - (ab + bc + ca) = \frac{1}{2}((a-b)^2 + (b-c)^2 + (c-a)^2)$$

and

$$(a+b)(b+c)(c+a) - 8abc = a(b-c)^2 + b(c-a)^2 + c(a-b)^2.$$

Inequality (17.3) becomes

$$\begin{aligned}
 & \frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 \geq 1 - \frac{8abc}{(a+b)(b+c)(c+a)} \\
 \Leftrightarrow & \frac{a^2 + b^2 + c^2 - (ab + bc + ca)}{ab + bc + ca} \geq \frac{(a+b)(b+c)(c+a) - 8abc}{(a+b)(b+c)(c+a)} \\
 \Leftrightarrow & \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{ab + bc + ca} \geq \frac{2a(b-c)^2 + 2b(c-a)^2 + 2c(a-b)^2}{(a+b)(b+c)(c+a)} \\
 \Leftrightarrow & (b-c)^2 \left( \frac{(a+b)(b+c)(c+a)}{ab + bc + ca} - 2a \right) \\
 & + (c-a)^2 \left( \frac{(a+b)(b+c)(c+a)}{ab + bc + ca} - 2b \right) \\
 & + (a-b)^2 \left( \frac{(a+b)(b+c)(c+a)}{ab + bc + ca} - 2c \right) \geq 0.
 \end{aligned}$$

Let

$$S_a = \frac{(a+b)(b+c)(c+a)}{ab + bc + ca} - 2a = b + c - a - \frac{abc}{ab + bc + ca},$$



$$S_b = \frac{(a+b)(b+c)(c+a)}{ab+bc+ca} - 2b = a+c-b - \frac{abc}{ab+bc+ca},$$

$$S_c = \frac{(a+b)(b+c)(c+a)}{ab+bc+ca} - 2c = a+b-c - \frac{abc}{ab+bc+ca}.$$

Since inequality (17.3) is symmetric, we may assume that  $a \geq b \geq c$ .

Then clearly

$$S_b, S_c \geq 0, \quad \text{i.e.} \quad S_b + S_c \geq 0.$$

According to 2° from Theorem 17.1, it suffices to show that  $S_b + S_a \geq 0$ .

We have

$$S_b + S_a = 2c - 2\frac{abc}{ab+bc+ca} = \frac{2c^2(a+b)}{ab+bc+ca} \geq 0,$$

as required.

**Exercise 17.3** Let  $a, b, c$  be positive real numbers such that  $ab+bc+ac = 1$ . Prove the inequality

$$\frac{1+a^2b^2}{(a+b)^2} + \frac{1+b^2c^2}{(b+c)^2} + \frac{1+c^2a^2}{(c+a)^2} \geq \frac{5}{2}.$$

*Solution* The given inequality is equivalent to

$$\begin{aligned} & \sum_{cyc} \frac{(ab+bc+ac)^2 + a^2b^2}{(a+b)^2} \geq \frac{5}{2}(ab+bc+ac) \\ \Leftrightarrow & 2 \sum_{cyc} \frac{2ab(ab+bc+ac) + (bc+ca)^2}{(a+b)^2} \geq 5(ab+bc+ac) \\ \Leftrightarrow & 4(ab+bc+ca) \left( \frac{ab}{(a+b)^2} + \frac{bc}{(b+c)^2} + \frac{ca}{(c+a)^2} \right) \\ & + 2(a^2+b^2+c^2) \geq 5(ab+bc+ac) \\ \Leftrightarrow & (ab+bc+ca) \left( \frac{4ab}{(a+b)^2} + \frac{4bc}{(b+c)^2} + \frac{4ca}{(c+a)^2} - 3 \right) \\ & + 2(a^2+b^2+c^2 - ab - bc - ca) \geq 0 \\ \Leftrightarrow & -(ab+bc+ca) \left( \frac{(a-b)^2}{(a+b)^2} + \frac{(b-c)^2}{(b+c)^2} + \frac{(c-a)^2}{(c+a)^2} \right) \\ & + ((a-b)^2 + (b-c)^2 + (c-a)^2) \geq 0 \end{aligned}$$

$$\Leftrightarrow \left(1 - \frac{ab+bc+ca}{(a+b)^2}\right)(a-b)^2 + \left(1 - \frac{ab+bc+ca}{(b+c)^2}\right)(b-c)^2 \\ + \left(1 - \frac{ab+bc+ca}{(c+a)^2}\right)(c-a)^2 \geq 0.$$

Let

$$S_a = 1 - \frac{ab+bc+ca}{(b+c)^2}, \quad S_b = 1 - \frac{ab+bc+ca}{(c+a)^2} \quad \text{and} \\ S_c = 1 - \frac{ab+bc+ca}{(a+b)^2}.$$

Without loss of generality we may assume that  $a \geq b \geq c$ , and then clearly  $S_a \leq S_b \leq S_c$ .

We have

$$S_c = 1 - \frac{ab+bc+ca}{(a+b)^2} = \frac{a^2 + (a+b)(b-c)}{(a+b)^2} > 0,$$

and it follows that  $S_b \geq S_c > 0$ .

Also we have

$$\begin{aligned} a^2 S_b + b^2 S_a &= a^2 \left(1 - \frac{ab+bc+ca}{(c+a)^2}\right) + b^2 \left(1 - \frac{ab+bc+ca}{(b+c)^2}\right) \\ &= a^2 \frac{c^2 + (c+a)(a-b)}{(c+a)^2} + b^2 \frac{c^2 + (b+c)(b-a)}{(b+c)^2} \\ &= c^2 \left(\frac{a^2}{(c+a)^2} + \frac{b^2}{(b+c)^2}\right) + (a-b) \left(\frac{a^2}{c+a} - \frac{b^2}{b+c}\right) \\ &= c^2 \left(\frac{a^2}{(c+a)^2} + \frac{b^2}{(b+c)^2}\right) + (a-b)^2 \frac{ab+bc+ca}{(c+a)(b+c)} > 0, \end{aligned}$$

and according to 4° from Theorem 17.1 we are done.

Equality occurs iff  $a = b = c = \frac{1}{\sqrt{3}}$ .



# Chapter 18

## Strong Mixing Variables Method (SMV Theorem)

This method is very useful in proving symmetric inequalities with more than two variables. The *SMV method* (strong mixing variables method) is a simple and concise method that “works” in proving inequalities that have either a too complicated or a too long proof. In order to better describe the given method, first we will give a *lemma* (without proof) and then we will introduce the reader to the *SMV theorem* and its applications through exercises. We should point out that this theorem is part of a more comprehensive method, the *Mixing Variable method* (MV method), which can be found in [27].

**Lemma 18.1** *Let  $(x_1, x_2, \dots, x_n)$  be an arbitrary real sequence.*

- 1° *Choose  $i, j \in \{1, 2, \dots, n\}$ , such that  $x_i = \min\{x_1, x_2, \dots, x_n\}, x_j = \max\{x_1, x_2, \dots, x_n\}$ .*
- 2° *Replace  $x_i$  and  $x_j$  by it's average  $\frac{x_i + x_j}{2}$  (their orders don't change).*

*After infinitely many of the above transformations, each number  $x_i, i = 1, 2, \dots, n$ , tends to the same limit  $x = \frac{x_1 + x_2 + \dots + x_n}{n}$ .*

**Theorem 18.1** (SMV theorem) *Let  $F : I \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be a symmetric, continuous, function satisfying  $F(a_1, a_2, \dots, a_n) \geq F(b_1, b_2, \dots, b_n)$ , where the sequence  $(b_1, b_2, \dots, b_n)$  is a sequence obtained from the sequence  $(a_1, a_2, \dots, a_n)$  by some predefined transformation (a  $\Delta$ -transformation). Then we have  $F(x_1, x_2, \dots, x_n) \geq F(x, x, \dots, x)$ , with  $x = \frac{x_1 + x_2 + \dots + x_n}{n}$ .*

Lets us note that the transformation  $\Delta$  can be different, i.e.  $\Delta$  can be defined according to the current problem; for example it can be defined as  $\frac{a+b}{2}, \sqrt{ab}, \sqrt{\frac{a^2+b^2}{2}}$ , etc.

**Exercise 18.1** Let  $a, b, c > 0$  be real numbers. Prove the inequality

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

*Solution* Let  $f(a, b, c) = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$ .

We have

$$\begin{aligned} f(a, b, c) - f\left(\frac{a+b}{2}, \frac{a+b}{2}, c\right) &= \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \left(\frac{a+b}{a+b+2c} + \frac{a+b}{a+b+2c} + \frac{c}{a+b}\right) \\ &= \frac{a}{b+c} + \frac{b}{c+a} - \frac{2(a+b)}{a+b+2c} = \frac{a^3 + ca^2 + cb^2 + b^3 - 2abc - ab^2 - a^2b}{(b+c)(a+c)(a+b+2c)}. \end{aligned} \quad (18.1)$$

Since  $AM \geq GM$  we obtain

$$\begin{aligned} a^3 + ca^2 + cb^2 + b^3 &= \frac{a^3 + a^3 + b^3}{3} + \frac{a^3 + b^3 + b^3}{3} \\ &\quad + ca^2 + cb^2 \geq a^2b + ab^2 + 2abc. \end{aligned} \quad (18.2)$$

From (18.1) and (18.2) it follows that

$$f(a, b, c) - f\left(\frac{a+b}{2}, \frac{a+b}{2}, c\right) \geq 0,$$

i.e.

$$f(a, b, c) \geq f\left(\frac{a+b}{2}, \frac{a+b}{2}, c\right).$$

Therefore by the *SMV theorem* it suffices to prove that  $f(t, t, c) \geq \frac{3}{2}$ .

We have

$$f(t, t, c) \geq \frac{3}{2} \Leftrightarrow \frac{t}{t+c} + \frac{t}{t+c} + \frac{c}{2t} \geq \frac{3}{2} \Leftrightarrow 2(t-c)^2 \geq 0,$$

which is obviously true.

Equality occurs if and only if  $a = b = c$ .

**Exercise 18.2** (Turkevicius inequality) Let  $a, b, c, d$  be non-negative real numbers. Prove the inequality

$$a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2 + a^2c^2 + b^2d^2.$$

*Solution* Without loss of generality we may assume that  $a \geq b \geq c \geq d$ .

Let us denote

$$\begin{aligned} f(a, b, c, d) &= a^4 + b^4 + c^4 + d^4 + 2abcd - a^2b^2 - b^2c^2 - c^2d^2 \\ &\quad - d^2a^2 - a^2c^2 - b^2d^2 \\ &= a^4 + b^4 + c^4 + d^4 + 2abcd - a^2c^2 - b^2d^2 - (a^2 + c^2)(b^2 + d^2). \end{aligned}$$

We have

$$\begin{aligned} f(a, b, c, d) - f(\sqrt{ac}, b, \sqrt{ac}, d) &= a^4 + b^4 + c^4 + d^4 + 2abcd - a^2c^2 - b^2d^2 - (a^2 + c^2)(b^2 + d^2) \\ &\quad - (a^2c^2 + b^4 + a^2c^2 + d^4 + 2abcd - a^2c^2 - b^2d^2 - 2ac(b^2 + d^2)) \\ &= a^4 + c^4 - 2a^2c^2 - (b^2 + d^2)(a^2 + c^2 + 2ac) \\ &= (a^2 - c^2)^2 - (b^2 + d^2)(a - c)^2 = (a - c)^2((a + c)^2 - (b^2 + d^2)) \geq 0. \end{aligned}$$

Thus

$$f(a, b, c, d) \geq f(\sqrt{ac}, b, \sqrt{ac}, d).$$

By the *SMV theorem* we only need to prove that  $f(a, b, c, d) \geq 0$ , in the case when  $a = b = c = t \geq d$ .

We have

$$f(t, t, t, d) \geq 0 \Leftrightarrow 3t^4 + d^4 + 2t^3d \geq 3t^4 + 3t^2d^2 \Leftrightarrow d^4 + 2t^3d \geq 3t^2d^2,$$

which immediately follows from  $AM \geq GM$ .

Equality occurs iff  $a = b = c = d$  or  $a = b = c, d = 0$  (up to permutation).

**Exercise 18.3** Let  $a, b, c, d$  be non-negative real numbers such that  $a + b + c + d = 4$ . Prove the inequality

$$(1 + 3a)(1 + 3b)(1 + 3c)(1 + 3d) \leq 125 + 131abcd.$$

*Solution* Let us denote

$$f(a, b, c, d) = (1 + 3a)(1 + 3b)(1 + 3c)(1 + 3d) - 131abcd.$$

Without loss of generality we may assume that  $a \geq b \geq c \geq d$ .

We have

$$\begin{aligned} f(a, b, c, d) - f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right) &= 9(1 + 3b)(1 + 3d)\left(ac - \frac{(a+c)^2}{4}\right) - 131bd\left(ac - \frac{(a+c)^2}{4}\right) \\ &= \frac{(a-c)^2}{4}(131bd - 9(1 + 3b)(1 + 3d)). \end{aligned} \tag{18.3}$$

Note that

$$b + d \leq \frac{1}{2}(a + b + c + d) = 2,$$

and clearly

$$bd \leq \frac{(b + d)^2}{4} = 1, \quad (18.4)$$

therefore

$$\begin{aligned} & 131bd - 9(1 + 3b)(1 + 3d) \\ &= 131bd - 9 - 27(b + d) - 81bd \\ &= 50bd - 27(b + d) - 9 = 50bd - 27(b + d) - 9 \stackrel{A \geq G}{\leq} 50bd - 54\sqrt{bd} \\ &\stackrel{(18.4)}{\leq} 50bd - 54bd = -4bd \leq 0. \end{aligned}$$

By (18.3) and the last inequality we deduce that

$$f(a, b, c, d) - f\left(\frac{a + c}{2}, b, \frac{a + c}{2}, d\right) \leq 0,$$

i.e.

$$f(a, b, c, d) \leq f\left(\frac{a + c}{2}, b, \frac{a + c}{2}, d\right).$$

According to the *SMV theorem* it follows that it's enough to prove that

$$f(a, b, c, d) \leq 125,$$

when  $a = b = c = t \geq d$ , i.e.

$$f(t, t, t, d) \leq 125, \quad \text{when } 3t + d = 4.$$

Clearly  $3t \leq 4$ .

We have

$$\begin{aligned} & f(t, t, t, d) \leq 125 \\ \Leftrightarrow & (1 + 3t)^3(1 + 3(4 - 3t)) - 131t^3(4 - 3t) \leq 125 \\ \Leftrightarrow & 150t^4 - 416t^3 + 270t^2 + 108t - 112 \leq 0 \\ \Leftrightarrow & (t - 1)^2(3t - 4)(50t + 28) \leq 0, \text{ which is true.} \end{aligned}$$

Equality occurs iff  $a = b = c = d = 1$  or  $a = b = c = \frac{4}{3}, d = 0$  (up to permutation).

**Exercise 18.4** Let  $a, b, c, d$  be non-negative real numbers such that  $a + b + c + d = 4$ . Prove the inequality

$$16 + 2abcd \geq 3(ab + ac + ad + bc + bd + cd).$$

*Solution* Without loss of generality we may assume that  $a \geq b \geq c \geq d$ .

Let us denote

$$f(a, b, c, d) = 3(ab + ac + ad + bc + bd + cd) - 2abcd.$$

We have

$$\begin{aligned} & f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right) - f(a, b, c, d) \\ &= 3\left(\left(\frac{a+c}{2}\right)b + \left(\frac{a+c}{2}\right)^2 + \left(\frac{a+c}{2}\right)d + \left(\frac{a+c}{2}\right)b + bd + \left(\frac{a+c}{2}\right)d\right) \\ &\quad - 2bd\left(\frac{a+c}{2}\right)^2 - (3(ab + ac + ad + bc + bd + cd) - 2abcd) \\ &= 3\left(\left(\frac{a+c}{2}\right)^2 - ac\right) - 2bd\left(\left(\frac{a+c}{2}\right)^2 - ac\right) \\ &= \left(\frac{a-c}{2}\right)^2 (3 - 2bd). \end{aligned} \tag{18.5}$$

Also  $2\sqrt{bd} \leq b + d \leq \frac{1}{2}(a + b + c + d) = 2$ , from which it follows that  $bd \leq 1$ .

By (18.5) and the last conclusion we get

$$\begin{aligned} f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right) - f(a, b, c, d) &= \left(\frac{a-c}{2}\right)^2 (3 - 2bd) \\ &\geq \left(\frac{a-c}{2}\right)^2 (3 - 2) \geq 0, \end{aligned}$$

i.e. it follows that

$$f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right) \geq f(a, b, c, d).$$

By the *SMV theorem* it follows that we only need to prove the inequality  $f(a, b, c, d) \leq 16$ , in the case when  $a = b = c = t \geq d$ , i.e. we need to prove that  $f(t, t, t, d) \leq 16$ , when  $3t + d = 4$ .

Clearly  $3t \leq 4$ .

Thus we have

$$\begin{aligned} & f(t, t, t, d) \leq 16 \\ \Leftrightarrow & 9(t^2 + dt) - 2t^3d - 16 \leq 0 \end{aligned}$$



$$\begin{aligned} \Leftrightarrow \quad & 9t^2 + 9t(4 - 3t) - 2t^3(4 - 3t) - 16 \leq 0 \\ \Leftrightarrow \quad & 2(3t - 4)(t - 1)^2(t + 2) \leq 0, \quad \text{which is true.} \end{aligned}$$

Equality occurs if and only if  $a = b = c = d = 1$  or  $a = b = c = 4/3, d = 0$  (up to permutation).

**Exercise 18.5** Let  $a, b, c, d$  be non-negative real numbers such that  $a + b + c + d = 1$ . Prove the inequality

$$abc + bcd + cda + dab \leq \frac{1}{27} + \frac{176}{27}abcd.$$

*Solution* Without loss of generality we may assume that  $a \leq b \leq c \leq d$ .

Let  $f(a, b, c, d) = abc + bcd + cda + dab - \frac{176}{27}abcd$  i.e.

$$f(a, b, c, d) = ac(b + d) + bd\left(a + c - \frac{176}{27}ac\right).$$

Since  $a \leq b \leq c \leq d$  we have

$$a + c \leq \frac{1}{2}(a + b + c + d) = \frac{1}{2},$$

from which it follows that

$$\frac{1}{a} + \frac{1}{c} \geq \frac{4}{a + c} \geq 8 > \frac{176}{27}. \quad (18.6)$$

We have

$$\begin{aligned} & f(a, b, c, d) - f\left(a, \frac{b+d}{2}, c, \frac{b+d}{2}\right) \\ &= ac(b + d) + bd\left(a + c - \frac{176}{27}ac\right) \\ &\quad - ac(b + d) - \left(\frac{b+d}{2}\right)^2\left(a + c - \frac{176}{27}ac\right) \\ &= \left(a + c - \frac{176}{27}ac\right)\left(bd - \left(\frac{b+d}{2}\right)^2\right) \\ &= -\left(a + c - \frac{176}{27}ac\right)\frac{(b-d)^2}{4} \stackrel{(18.6)}{\leq} 0. \end{aligned}$$

Therefore

$$f(a, b, c, d) \leq f\left(a, \frac{b+d}{2}, c, \frac{b+d}{2}\right).$$

By the *SMV theorem* we have

$$f(a, b, c, d) \leq f(a, t, t, t), \quad \text{when } t = \frac{b + c + d}{3}.$$

Now we need to prove only the inequality

$$f(a, t, t, t) \leq \frac{1}{27}, \quad \text{with } a + 3t = 1.$$

Let us note that  $3t \leq a + 3t = 1$ .

The inequality  $f(a, t, t, t) \leq \frac{1}{27}$  is equivalent to

$$3at^2 + t^3 \leq \frac{1}{27} + \frac{176}{27}at^3. \quad (18.7)$$

After putting  $a = 1 - 3t$  by (18.7) we get  $(1 - 3t)(4t - 1)^2(11t + 1) \geq 0$ , which is obviously true (since  $3t \leq 1$ ), and the problem is solved.

Equality occurs if and only if  $a = b = c = d = 1/4$  or  $a = b = c = 1/3, d = 0$  (up to permutation).



## Chapter 19

# Method of Lagrange Multipliers

This method is intended for conditional inequalities. It requires elementary skills of differential calculus but it is very easy to apply. We'll give the main theorem, without proof, and we'll introduce some exercises to see how this method works.

**Theorem 19.1** (Lagrange multipliers theorem) *Let  $f(x_1, x_2, \dots, x_m)$  be a continuous and differentiable function on  $I \subseteq \mathbb{R}^m$ , and let  $g_i(x_1, x_2, \dots, x_m) = 0, i = 1, 2, \dots, k$ , where  $(k < m)$  are the conditions that must be satisfied. Then the maximum or minimum values of  $f$  with the conditions  $g_i(x_1, x_2, \dots, x_m) = 0, i = 1, 2, \dots, k$ , occur at the bounds of the interval  $I$  or occur at the points at which the partial derivatives (according to the variables  $x_1, x_2, \dots, x_m$ ) of the function  $L = f - \sum_{i=1}^k \lambda_i g_i$ , are all zero.*

**Exercise 19.1** Let  $x_1, x_2, \dots, x_n$  be positive real numbers such that  $x_1 + x_2 + \dots + x_n = a$ . Find the maximal value of the expression  $A = \sqrt[n]{x_1 x_2 \dots x_n}$ .

*Solution* Let  $g = x_1 + x_2 + \dots + x_n - a$ . Then *Lagrange's function* is

$$F = A - \lambda g = \sqrt[n]{x_1 x_2 \dots x_n} - \lambda(x_1 + x_2 + \dots + x_n - a).$$

For the first partial derivatives we have

$$\begin{cases} F'_{x_1} = \frac{\sqrt[n]{x_1 x_2 \dots x_n}}{x_1} - \lambda, \\ F'_{x_2} = \frac{\sqrt[n]{x_1 x_2 \dots x_n}}{x_2} - \lambda, \\ \vdots \\ F'_{x_n} = \frac{\sqrt[n]{x_1 x_2 \dots x_n}}{x_n} - \lambda \end{cases}$$

from which easily we deduce that we must have  $x_1 = x_2 = \dots = x_n = \frac{a}{n}$ .

Hence  $\max A = \frac{a}{n}$ , i.e.  $\sqrt[n]{x_1 x_2 \dots x_n} \leq \frac{x_1 + x_2 + \dots + x_n}{n}$  which is the well-known inequality  $AM \geq GM$ .

**Exercise 19.2** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 1$ . Prove the inequality

$$7(ab + bc + ca) \leq 9abc + 2.$$

*Solution* Let

$$f(a, b, c) = 7(ab + bc + ca) - 9abc - 2, \quad g(a, b, c) = a + b + c - 1$$

and

$$L = f - \lambda g = 7(ab + bc + ca) - 9abc - 2 - \lambda(a + b + c - 1).$$

We have

$$\frac{\partial L}{\partial a} = 7(b + c) - 9bc - \lambda = 0 \quad \Rightarrow \quad \lambda = 7(b + c) - 9bc,$$

$$\frac{\partial L}{\partial b} = 7(c + a) - 9ca - \lambda = 0 \quad \Rightarrow \quad \lambda = 7(c + a) - 9ca,$$

$$\frac{\partial L}{\partial c} = 7(a + b) - 9ab - \lambda = 0 \quad \Rightarrow \quad \lambda = 7(a + b) - 9ab.$$

So

$$7(b + c) - 9bc = \lambda = 7(c + a) - 9ca \quad \Leftrightarrow \quad (b - a)(7 - 9c) = 0. \quad (19.1)$$

In the same way we obtain

$$(c - b)(7 - 9a) = 0 \quad (19.2)$$

and

$$(a - c)(7 - 9b) = 0. \quad (19.3)$$

Let us consider the identity (19.1).

If  $a = b$  then if  $b = c$  we get  $a = b = c = 1/3$ , and then

$$f(a, b, c) = 7(ab + bc + ca) - 9abc - 2 = \frac{21}{9} - \frac{9}{27} - 2 = 0.$$

If  $a = b$  and  $b \neq c$  then by (19.2) we must have  $a = \frac{7}{9} = b$  and then  $a + b = \frac{14}{9} > 1$ , a contradiction, since  $a + b < a + b + c = 1$ .

If  $7 - 9c = 0$  then we can't have  $7 - 9a = 0$  or  $7 - 9b = 0$  for the same reasons as before, so according to (19.2) and (19.3) we must have  $b = c$  and  $a = c$ , i.e.  $a = b = c = 7/9$ , which is impossible.

Therefore  $\min L = 0$ , i.e.  $7(ab + bc + ca) \leq 9abc + 2$ .

**Exercise 19.3** Let  $a, b, c \in \mathbb{R}$  such that  $a^2 + b^2 + c^2 + abc = 4$ . Find the minimal value of the expression  $a + b + c$ .

*Solution* Let

$$f(a, b, c) = a + b + c, g(a, b, c) = a^2 + b^2 + c^2 + abc - 4$$

and

$$L = f - \lambda g = a + b + c - \lambda(a^2 + b^2 + c^2 + abc - 4).$$

We have

$$\begin{aligned} \frac{\partial L}{\partial a} = 1 - \lambda a - \lambda bc = 0 &\Rightarrow \lambda = \frac{1}{2a + bc}, \\ \frac{\partial L}{\partial b} = 1 - \lambda b - \lambda ac = 0 &\Rightarrow \lambda = \frac{1}{2b + ac}, \\ \frac{\partial L}{\partial c} = 1 - \lambda c - \lambda ab = 0 &\Rightarrow \lambda = \frac{1}{2c + ab}. \end{aligned}$$

So

$$\frac{1}{2a + bc} = \frac{1}{2b + ac} \Leftrightarrow (a - b)(2 - c) = 0. \quad (19.4)$$

In the same way we obtain

$$(b - c)(2 - a) = 0 \quad (19.5)$$

and

$$(c - a)(2 - b) = 0. \quad (19.6)$$

If  $a = b = 2$  then since  $a^2 + b^2 + c^2 + abc = 4$  we get  $c = -2$ , and therefore  $a + b + c = 2$ .

If  $a = b = c \neq 2$  then from the given condition we deduce that

$$3a^2 + a^3 = 4 \Leftrightarrow (a - 1)(a + 2)^2 = 0,$$

and therefore  $a = b = c = 1$  or  $a = b = c = -2$ , i.e.  $a + b + c = 3$  or  $a + b + c = -6$ .

Thus  $\min\{a + b + c\} = -6$ .

**Exercise 19.4** Let  $a, b, c, d \in \mathbb{R}^+$  such that  $a + b + c + d = 1$ . Prove the inequality

$$abc + bcd + cda + dab \leq \frac{1}{27} + \frac{176}{27}abcd.$$

*Solution* Let  $f = abc + bcd + cda + dab - \frac{176}{27}abcd$ .

We'll prove that

$$f \leq \frac{1}{27}.$$

Define  $g = a + b + c + d - 1$  and

$$L = f - \lambda g = abc + bcd + cda + dab - \frac{176}{27}abcd - \lambda(a + b + c + d - 1).$$

For the first partial derivatives we have

$$\frac{\partial L}{\partial a} = bc + cd + db - \frac{176}{27}bcd - \lambda = 0,$$

$$\frac{\partial L}{\partial b} = ac + cd + da - \frac{176}{27}acd - \lambda = 0,$$

$$\frac{\partial L}{\partial c} = ab + bd + da - \frac{176}{27}abd - \lambda = 0,$$

$$\frac{\partial L}{\partial d} = bc + ac + ab - \frac{176}{27}abc - \lambda = 0.$$

Therefore

$$\begin{aligned}\lambda &= bc + cd + db - \frac{176}{27}bcd = ac + cd + da - \frac{176}{27}acd \\ &= ab + bd + da - \frac{176}{27}abd = bc + ac + ab - \frac{176}{27}abc.\end{aligned}$$

Since

$$bc + cd + db - \frac{176}{27}bcd = ac + cd + da - \frac{176}{27}acd,$$

we deduce that

$$(b - a)\left(c + d - \frac{176}{27}cd\right) = 0.$$

Similarly we get

$$(b - c)\left(a + d - \frac{176}{27}ad\right) = 0,$$

$$(b - d)\left(a + c - \frac{176}{27}ac\right) = 0,$$

$$(a - c)\left(b + d - \frac{176}{27}bd\right) = 0,$$

$$(a - d)\left(c + b - \frac{176}{27}cb\right) = 0,$$

$$(c - d)\left(a + b - \frac{176}{27}ab\right) = 0.$$

By solving these equations we must have  $a = b = c = d$ , and since  $a + b + c + d = 1$  it follows that  $a = b = c = d = 1/4$ .

Then

$$f(1/4, 1/4, 1/4, 1/4) = 1/27,$$

and we are done.

**Exercise 19.5** Let  $a, b, c \in \mathbb{R}$  be real numbers such that  $a + b + c > 0$ . Prove the inequality

$$a^3 + b^3 + c^3 \leq (a^2 + b^2 + c^2)^{3/2} + 3abc.$$

*Solution* If we define

$$x = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad y = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad z = \frac{c}{\sqrt{a^2 + b^2 + c^2}},$$

then the given inequality becomes

$$x^3 + y^3 + z^3 \leq (x^2 + y^2 + z^2)^{3/2} + 3xyz, \quad \text{with } x^2 + y^2 + z^2 = 1.$$

So it suffices to prove that

$$a^3 + b^3 + c^3 \leq (a^2 + b^2 + c^2)^{3/2} + 3abc, \quad \text{with condition } a^2 + b^2 + c^2 = 1,$$

i.e.

$$a^3 + b^3 + c^3 \leq 1 + 3abc, \quad \text{with } a^2 + b^2 + c^2 = 1.$$

Let us define

$$f = a^3 + b^3 + c^3 - 3abc, \quad g = a^2 + b^2 + c^2 - 1$$

and

$$L = f - \lambda g = a^3 + b^3 + c^3 - 3abc - \lambda(a^2 + b^2 + c^2 - 1).$$

We obtain

$$\frac{\partial L}{\partial a} = 3a^2 - 3bc - 2\lambda a = 0,$$

$$\frac{\partial L}{\partial b} = 3b^2 - 3ac - 2\lambda b = 0,$$

$$\frac{\partial L}{\partial c} = 3c^2 - 3ab - 2\lambda c = 0$$

i.e.

$$\lambda = \frac{3(a^2 - bc)}{2a} = \frac{3(b^2 - ac)}{2b} = \frac{3(c^2 - ab)}{2c}.$$

Thus

$$\frac{3(a^2 - bc)}{2a} = \frac{3(b^2 - ac)}{2b} \Leftrightarrow (a - b)(ab + bc + ca) = 0.$$



Similarly we deduce

$$(b - c)(ab + bc + ca) = 0 \quad \text{and} \quad (c - a)(ab + bc + ca) = 0.$$

By solving these equations we deduce that we must have  $a = b = c$  or  $ab + bc + ca = 0$ .

If  $a = b = c$  then  $f(a, a, a) = 0 < 1$ .

If  $ab + bc + ca = 0$  then

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) = 1,$$

and since  $a + b + c > 0$  we obtain  $a + b + c = 1$ .

Therefore

$$f(a, b, c) = a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 1,$$

and the problem is solved.

## Chapter 20

### Problems

**1** Let  $n$  be a positive integer. Prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2.$$

**2** Let  $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ . Prove that for any  $n \in \mathbb{N}$  we have

$$\frac{1}{a_1^2} + \frac{1}{2a_2^2} + \frac{1}{3a_3^2} + \cdots + \frac{1}{na_n^2} < 2.$$

**3** Let  $x, y, z$  be real numbers. Prove the inequality

$$x^4 + y^4 + z^4 \geq 4xyz - 1.$$

**4** Prove that for any real number  $x$ , the following inequality holds

$$x^{2002} - x^{1999} + x^{1996} - x^{1995} + 1 > 0.$$

**5** Let  $x, y$  be real numbers. Prove the inequality

$$3(x + y + 1)^2 + 1 \geq 3xy.$$

**6** Let  $a, b, c$  be positive real numbers such that  $a + b + c \geq abc$ . Prove that at least two of the following inequalities

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6$$

are true.

**7** Let  $a, b, c, x, y, z > 0$ . Prove the inequality

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(a+b+c)(x+y+z)}{a+b+c+x+y+z}.$$

**8** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{2a}{a^2+bc} + \frac{2b}{b^2+ac} + \frac{2c}{c^2+ab} \leq \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}.$$

**9** Let  $a, b, c, x, y, z \in \mathbb{R}^+$  such that  $a + x = b + y = c + z = 1$ . Prove the inequality

$$(abc + xyz) \left( \frac{1}{ay} + \frac{1}{bz} + \frac{1}{cx} \right) \geq 3.$$

**10** Let  $a_1, a_2, \dots, a_n$  be positive real numbers and let  $b_1, b_2, \dots, b_n$  be their permutation. Prove the inequality

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geq a_1 + a_2 + \dots + a_n.$$

**11** Let  $x \in \mathbb{R}^+$ . Find the minimum value of the expression  $\frac{x^2+1}{x+1}$ .

**12** Let  $a, b, c \in \mathbb{R}^+$  such that  $abc = 1$ . Prove the inequality

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4}.$$

**13** Let  $x, y \geq 0$  be real numbers such that  $y(y+1) \leq (x+1)^2$ . Prove the inequality

$$y(y-1) \leq x^2.$$

**14** Let  $x, y \in \mathbb{R}^+$  such that  $x^3 + y^3 \leq x - y$ . Prove that

$$x^2 + y^2 \leq 1.$$

**15** Let  $a, b, x, y \in \mathbb{R}$  such that  $ay - bx = 1$ . Prove that

$$a^2 + b^2 + x^2 + y^2 + ax + by \geq \sqrt{3}.$$

**16** Let  $a, b, c, d$  be non-negative real numbers such that  $a^2 + b^2 + c^2 + d^2 = 1$ . Prove the inequality

$$(1-a)(1-b)(1-c)(1-d) \geq abcd.$$

**17** Let  $x, y$  be non-negative real numbers. Prove the inequality

$$4(x^9 + y^9) \geq (x^2 + y^2)(x^3 + y^3)(x^4 + y^4).$$

**18** Let  $x, y, z \in \mathbb{R}^+$  such that  $xyz = 1$  and  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq x + y + z$ . Prove that for any natural number  $n$  the inequality

$$\frac{1}{x^n} + \frac{1}{y^n} + \frac{1}{z^n} \geq x^n + y^n + z^n$$

is true.

**19** Let  $x, y, z$  be real numbers different from 1, such that  $xyz = 1$ . Prove the inequality

$$\left(\frac{3-x}{1-x}\right)^2 + \left(\frac{3-y}{1-y}\right)^2 + \left(\frac{3-z}{1-z}\right)^2 > 7.$$

**20** Let  $x, y, z \leq 1$  be real numbers such that  $x + y + z = 1$ . Prove the inequality

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \leq \frac{27}{10}.$$

**21** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}.$$

**22** Let  $x, y, z$  be positive real numbers. Prove the inequality

$$9(a+b)(b+c)(c+a) \geq 8(a+b+c)(ab+bc+ca).$$

**23** Let  $a, b, c$  be real numbers. Prove the inequality

$$(a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^3a).$$

**24** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$a^3(b+c) + b^3(c+a) + c^3(a+b) \leq 6.$$

**25** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} > 2.$$

**26** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} \leq 1.$$

**27** Let  $x, y, z$  be distinct nonnegative real numbers. Prove the inequality

$$\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} \geq \frac{4}{xy + yz + zx}.$$

**28** Let  $a, b, c$  be non-negative real numbers. Prove the inequality

$$3(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq 1 + abc + (abc)^2.$$

**29** Let  $a, b \in \mathbb{R}$ ,  $a \neq 0$ . Prove the inequality

$$a^2 + b^2 + \frac{1}{a^2} + \frac{b}{a} \geq \sqrt{3}.$$

**30** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq 3.$$

**31** Let  $x, y, z$  be positive real numbers such that  $xy + yz + zx = 5$ . Prove the inequality

$$3x^2 + 3y^2 + z^2 \geq 10.$$

**32** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca > a + b + c$ . Prove the inequality

$$a + b + c > 3.$$

**33** Let  $a, b$  be real numbers such that  $9a^2 + 8ab + 7b^2 \leq 6$ . Prove that

$$7a + 5b + 12ab \leq 9.$$

**34** Let  $x, y, z \in \mathbb{R}^+$ , such that  $xyz \geq xy + yz + zx$ . Prove the inequality

$$xyz \geq 3(x + y + z).$$

**35** Let  $a, b, c \in \mathbb{R}^+$  with  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq 3.$$

**36** Let  $a, b, c$  be positive real numbers such that  $a + b + c = \sqrt{abc}$ . Prove the inequality

$$ab + bc + ca \geq 9(a + b + c).$$

**37** Let  $a, b, c$  be positive real numbers such that  $abc \geq 1$ . Prove the inequality

$$\left(a + \frac{1}{a+1}\right)\left(b + \frac{1}{b+1}\right)\left(c + \frac{1}{c+1}\right) \geq \frac{27}{8}.$$

**38** Let  $a, b, c, d \in \mathbb{R}^+$  such that  $a^2 + b^2 + c^2 + d^2 = 4$ . Prove the inequality

$$a + b + c + d \geq ab + bc + cd + da.$$

**39** Let  $a, b, c \in (-3, 3)$  such that  $\frac{1}{3+a} + \frac{1}{3+b} + \frac{1}{3+c} = \frac{1}{3-a} + \frac{1}{3-b} + \frac{1}{3-c}$ .  
Prove the inequality

$$\frac{1}{3+a} + \frac{1}{3+b} + \frac{1}{3+c} \geq 1.$$

**40** Let  $a, b, c \in \mathbb{R}^+$  such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{1}{a+bc+abc} + \frac{1}{b+ca+bca} + \frac{1}{c+ab+cab} \geq 1.$$

**41** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 3$ . Prove the inequality

$$\frac{a^2b^2 + a^2 + b^2}{ab+1} + \frac{b^2c^2 + b^2 + c^2}{bc+1} + \frac{c^2a^2 + c^2 + a^2}{ca+1} \geq \frac{9}{2}.$$

**42** Let  $a, b, c, d$  be positive real numbers such that  $a^2 + b^2 + c^2 + d^2 = 4$ . Prove the inequality

$$\frac{a^2 + b^2 + 3}{a+b} + \frac{b^2 + c^2 + 3}{b+c} + \frac{c^2 + d^2 + 3}{c+d} + \frac{d^2 + a^2 + 3}{d+a} \geq 10.$$

**43** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{1}{ab(a+b)} + \frac{1}{bc(b+c)} + \frac{1}{ca(c+a)} \geq \frac{9}{2(a^3 + b^3 + c^3)}.$$

**44** Let  $a, b, c \in \mathbb{R}^+$  such that  $a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \geq 1$ . Prove the inequality

$$a + b + c \geq \sqrt{3}.$$

**45** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{c}} \geq \sqrt{a} + \sqrt{b} + \sqrt{c} + 3.$$

**46** Let  $x, y, z$  be positive real numbers such that  $x + y + z = 4$ . Prove the inequality

$$\frac{1}{2xy+xz+yz} + \frac{1}{xy+2xz+yz} + \frac{1}{xy+xz+2yz} \leq \frac{1}{xyz}.$$

**47** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$abc \geq (a+b-c)(b+c-a)(c+a-b).$$

**48** Let  $a, b, c$  be positive real numbers such that  $a+b+c=3$ . Prove the inequality

$$abc + \frac{12}{ab+bc+ac} \geq 5.$$

**49** Let  $a, b, c$  be positive real numbers such that  $abc=1$ . Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

**50** Let  $a, b, c$  be positive real numbers such that  $abc=1$ . Prove the inequality

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

**51** Let  $a, b, c > 0$ . Prove the inequality

$$(a+b)^2 + (a+b+4c)^2 \geq \frac{100abc}{a+b+c}.$$

**52** Let  $a, b, c > 0$  such that  $abc=1$ . Prove the inequality

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ac}{1+a} \geq 3.$$

**53** Let  $a, b, c$  be real numbers such that  $ab+bc+ca=1$ . Prove the inequality

$$\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \geq 16.$$

**54** Let  $a, b, c$  be positive real numbers such that  $abc \geq 1$ . Prove the inequality

$$a+b+c \geq \frac{1+a}{1+b} + \frac{1+b}{1+c} + \frac{1+c}{1+a}.$$

**55** Let  $a, b \in \mathbb{R}^+$ . Prove the inequality

$$\left(a^2 + b + \frac{3}{4}\right)\left(b^2 + a + \frac{3}{4}\right) \geq \left(2a + \frac{1}{2}\right)\left(2b + \frac{1}{2}\right).$$

**56** Let  $a, b, c \in \mathbb{R}^+$  such that  $abc=1$ . Prove the inequality

$$\frac{a}{a^2+2} + \frac{b}{b^2+2} + \frac{c}{c^2+2} \leq 1.$$

**57** Let  $x, y, z > 0$  be real numbers such that  $x + y + z = xyz$ . Prove the inequality

$$(x - 1)(y - 1)(z - 1) \leq 6\sqrt{3} - 10.$$

**58** Let  $a, b, c \in (1, 2)$  be real numbers. Prove the inequality

$$\frac{b\sqrt{a}}{4b\sqrt{c} - c\sqrt{a}} + \frac{c\sqrt{b}}{4c\sqrt{a} - a\sqrt{b}} + \frac{a\sqrt{c}}{4a\sqrt{b} - b\sqrt{c}} \geq 1.$$

**59** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 3$ . Prove the inequality

$$\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)} \geq 3\sqrt{2abc}.$$

**60** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\sqrt{a+bc} + \sqrt{b+ca} + \sqrt{c+ab} \leq 2.$$

**61** Let  $a, b, c$  be positive real numbers such that  $a + b + c + 1 = 4abc$ . Prove that

$$\frac{b^2 + c^2}{a} + \frac{c^2 + a^2}{b} + \frac{a^2 + b^2}{c} \geq 2(ab + bc + ca).$$

**62** Let  $a, b, c \in (-1, 1)$  be real numbers such that  $ab + bc + ac = 1$ . Prove the inequality

$$6\sqrt[3]{(1-a^2)(1-b^2)(1-c^2)} \leq 1 + (a+b+c)^2.$$

**63** Let  $a, b, c, d$  be positive real numbers such that  $a^2 + b^2 + c^2 + d^2 = 1$ . Prove the inequality

$$\sqrt{1-a} + \sqrt{1-b} + \sqrt{1-c} + \sqrt{1-d} \geq \sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}.$$

**64** Let  $x, y, z$  be positive real numbers such that  $xyz = 1$ . Prove the inequality

$$\frac{1}{(x+1)^2 + y^2 + 1} + \frac{1}{(y+1)^2 + z^2 + 1} + \frac{1}{(z+1)^2 + x^2 + 1} \leq \frac{1}{2}.$$

**65** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\sqrt{\frac{a^3}{a^3 + (b+c)^3}} + \sqrt{\frac{a^3}{a^3 + (b+c)^3}} + \sqrt{\frac{a^3}{a^3 + (b+c)^3}} \geq 1.$$

**66** Let  $x, y, z \in \mathbb{R}^+$ . Prove the inequality

$$(x+y+z)^2(xy+yz+zx)^2 \leq 3(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2).$$



**67** Let  $a, b, c$  be real numbers such that  $a + b + c = 3$ . Prove the inequality

$$2(a^2b^2 + b^2c^2 + c^2a^2) + 3 \leq 3(a^2 + b^2 + c^2).$$

**68** Let  $a, b, c, d$  be positive real numbers. Prove the inequality

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \geq 0.$$

**69** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 1$ . Prove the inequality

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{9}{4}.$$

**70** Let  $a, b, c \in \mathbb{R}^+$  such that  $abc = 1$ . Prove the inequality

$$\frac{a^3c}{(b+c)(c+a)} + \frac{b^3a}{(c+a)(a+b)} + \frac{c^3b}{(a+b)(b+c)} \geq \frac{3}{4}.$$

**71** Let  $a, b, c > 0$  be real numbers such that  $abc = 1$ . Prove that

$$(a+b)(b+c)(c+a) \geq 4(a+b+c-1).$$

**72** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$1 + \frac{3}{a+b+c} \geq \frac{6}{ab+bc+ca}.$$

**73** Let  $x, y, z$  be positive real numbers such that  $x^2 + y^2 + z^2 = xyz$ . Prove the following inequalities:

$$\begin{array}{ll} 1^\circ \quad xyz \geq 27 & 2^\circ \quad xy + yz + zx \geq 27 \\ 3^\circ \quad x + y + z \geq 9 & 4^\circ \quad xy + yz + zx \geq 2(x + y + z) + 9. \end{array}$$

**74** Let  $a, b, c$  be real numbers such that  $a^3 + b^3 + c^3 - 3abc = 1$ . Prove the inequality

$$a^2 + b^2 + c^2 \geq 1.$$

**75** Let  $a, b, c, d \in \mathbb{R}^+$  such that  $\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+d^4} = 1$ . Prove that

$$abcd \geq 3.$$

**76** Let  $a, b, c$  be non-negative real numbers. Prove the inequality

$$\sqrt{\frac{ab+bc+ca}{3}} \leq \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}.$$

**77** Let  $a, b, c, d$  be positive real numbers such that  $a + b + c + d = 1$ . Prove that

$$16(abc + bcd + cda + dab) \leq 1.$$

**78** Let  $a, b, c, d, e$  be positive real numbers such that  $a + b + c + d + e = 5$ . Prove the inequality

$$abc + bcd + cde + dea + eab \leq 5.$$

**79** Let  $a, b, c > 0$  be real numbers. Prove the inequality

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b}{b+c} + \frac{b+c}{c+a} + 1.$$

**80** Let  $a, b, c > 0$  be real numbers such that  $abc = 1$ . Prove the inequality

$$\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right) \geq 2(1 + a + b + c).$$

**81** Let  $a, b, c$  be positive real numbers such that  $a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ . Prove the inequality

$$a + b + c \geq \frac{3}{a + b + c} + \frac{2}{abc}.$$

**82** Let  $a, b, c, d$  be positive real numbers such that  $abcd = 1$ . Prove the inequality

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+cd}{1+c} + \frac{1+da}{1+d} \geq 4.$$

**83** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}.$$

**84** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$\frac{a^2}{b^2 - 2b + 3} + \frac{b^2}{c^2 - 2c + 3} + \frac{c^2}{a^2 - 2a + 3} \geq \frac{3}{2}.$$

**85** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 3$ . Prove the inequality

$$\frac{1}{1+a^2(b+c)} + \frac{1}{1+b^2(c+a)} + \frac{1}{1+c^2(a+b)} \leq \frac{1}{abc}.$$

**86** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$a\sqrt[3]{1+b-c} + b\sqrt[3]{1+c-a} + c\sqrt[3]{1+a-b} \leq 1.$$

**87** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 1$ . Prove the inequality

$$\frac{1-2ab}{c} + \frac{1-2bc}{a} + \frac{1-2ca}{b} \geq 7.$$

**88** Let  $a, b, c$  be non negative real numbers such that  $a^2 + b^2 + c^2 = 1$ . Prove the inequality

$$\frac{1-ab}{7-3ac} + \frac{1-ab}{7-3ac} + \frac{1-ab}{7-3ac} \geq \frac{1}{3}.$$

**89** Let  $x, y, z \in \mathbb{R}^+$  such that  $x + y + z = 1$ . Prove the inequality

$$\frac{xy}{\sqrt{\frac{1}{3} + z^2}} + \frac{zx}{\sqrt{\frac{1}{3} + y^2}} + \frac{yz}{\sqrt{\frac{1}{3} + x^2}} \leq \frac{1}{2}.$$

**90** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\frac{a-bc}{a+bc} + \frac{b-ca}{b+ca} + \frac{c-ab}{c+ab} \leq \frac{3}{2}.$$

**91** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\sqrt{\frac{a+b}{a+1}} + \sqrt{\frac{b+c}{c+1}} + \sqrt{\frac{c+a}{a+1}} \geq 3.$$

**92** Let  $x, y, z \geq 0$  be real numbers such that  $xy + yz + zx = 1$ . Prove the inequality

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} \leq \frac{3\sqrt{3}}{4}.$$

**93** Let  $a, b, c$  be non-negative real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \geq \frac{3\sqrt{3}}{\sqrt{3}+1}.$$

**94** Let  $a, b, c$  be non-negative real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$\frac{a^2}{1+a} + \frac{b^2}{1+b} + \frac{c^2}{1+c} \geq \frac{\sqrt{3}}{\sqrt{3}+1}.$$

**95** Let  $a, b, c \in \mathbb{R}^+$  such that  $(a+b)(b+c)(c+a) = 8$ . Prove the inequality

$$\frac{a+b+c}{3} \geq \sqrt[27]{\frac{a^3+b^3+c^3}{3}}.$$

**96** Find the maximum value of  $\frac{x^4 - x^2}{x^6 + 2x^3 - 1}$ , where  $x \in \mathbb{R}$ ,  $x > 1$ .

**97** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a + \sqrt{ab} + \sqrt[3]{abc}}{3} \leq \sqrt[3]{a \cdot \frac{a+b}{2} \cdot \frac{a+b+c}{3}}.$$

**98** Let  $a, b, c$  be positive real numbers such that  $abc(a + b + c) = 3$ . Prove the inequality

$$(a + b)(b + c)(c + a) \geq 8.$$

**99** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\sqrt{\frac{2a}{b+c}} + \sqrt{\frac{2b}{c+a}} + \sqrt{\frac{2c}{a+b}} \leq 3.$$

**100** Let  $a, b, c \in \mathbb{R}^+$  such that  $ab + bc + ca = 1$ . Prove the inequality

$$\frac{1}{a(a+b)} + \frac{1}{b(b+c)} + \frac{1}{c(c+a)} \geq \frac{9}{2}.$$

**101** Let  $0 \leq a \leq b \leq c \leq 1$  be real numbers. Prove that

$$a^2(b-c) + b^2(c-b) + c^2(1-c) \leq \frac{108}{529}.$$

**102** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 1$ . Prove the inequality

$$S = a^4b + b^4c + c^4a \leq \frac{256}{3125}.$$

**103** Let  $a, b, c > 0$  be real numbers. Prove the inequality

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

**104** Prove that for all positive real numbers  $a, b, c$  we have

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq a + b + c.$$

**105** Prove that for all positive real numbers  $a, b, c$  we have

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}.$$

**106** Prove that for all positive real numbers  $a, b, c$  we have

$$\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} \geq ab + bc + ca.$$

**107** Prove that for all positive real numbers  $a, b, c$  we have

$$\frac{a^5}{b^3} + \frac{b^5}{c^3} + \frac{c^5}{a^3} \geq a^2 + b^2 + c^2.$$

**108** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 3$ . Prove the inequality

$$\frac{a^3}{b(2c+a)} + \frac{b^3}{c(2a+b)} + \frac{c^3}{a(2b+c)} \geq 1.$$

**109** Let  $a, b, c \in \mathbb{R}^+$  and  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{a^3}{b+2c} + \frac{b^3}{c+2a} + \frac{c^3}{a+2b} \geq 1.$$

**110** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{1}{a^3+2} + \frac{1}{b^3+2} + \frac{1}{c^3+2} \geq 1.$$

**111** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 1$ . Prove the inequality

$$\frac{a^3}{a^2+b^2} + \frac{b^3}{b^2+c^2} + \frac{c^3}{c^2+a^2} \geq \frac{1}{2}.$$

**112** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$\frac{1}{1+2a^2b} + \frac{1}{1+2b^2c} + \frac{1}{1+2c^2a} \geq 1.$$

**113** Let  $a, b, c, d$  be positive real numbers such that  $a + b + c + d = 4$ . Prove the inequality

$$\frac{a}{1+b^2c} + \frac{b}{1+c^2d} + \frac{c}{1+d^2a} + \frac{d}{1+a^2b} \geq 2.$$

**114** Let  $a, b, c, d$  be positive real numbers. Prove the inequality

$$\frac{a^3}{a^2+b^2} + \frac{b^3}{b^2+c^2} + \frac{c^3}{c^2+d^2} + \frac{d^3}{d^2+a^2} \geq \frac{a+b+c+d}{2}.$$

**115** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$\frac{a^2}{a + 2b^2} + \frac{b^2}{b + 2c^2} + \frac{c^2}{c + 2a^2} \geq 1.$$

**116** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$\frac{a^2}{a + 2b^3} + \frac{b^2}{b + 2c^3} + \frac{c^2}{c + 2a^3} \geq 1.$$

**117** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Find the minimum value of the expression

$$a + b + c + \frac{16}{a + b + c}.$$

**118** Let  $a, b, c \geq 0$  be real numbers such that  $a^2 + b^2 + c^2 = 1$ . Find the minimal value of the expression

$$A = a + b + c + \frac{1}{abc}.$$

**119** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 6$ . Prove the inequality

$$\sqrt[3]{ab + bc} + \sqrt[3]{bc + ca} + \sqrt[3]{ca + ab} + \sqrt[3]{\frac{9}{4}(a^2 + b^2 + c^2)} \leq 9.$$

**120** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + 2b + 3c \geq 20$ . Prove the inequality

$$S = a + b + c + \frac{3}{a} + \frac{9}{2b} + \frac{4}{c} \geq 13.$$

**121** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$S = 30a + 3b^2 + \frac{2c^3}{9} + 36\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) \geq 84.$$

**122** Let  $a, b, c \in \mathbb{R}^+$  such that  $ac \geq 12$  and  $bc \geq 8$ . Prove the inequality

$$S = a + b + c + 2\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) + \frac{8}{abc} \geq \frac{121}{12}.$$

**123** Let  $a, b, c, d > 0$  be real numbers. Determine the minimal value of the expression

$$A = \left(1 + \frac{2a}{3b}\right)\left(1 + \frac{2b}{3c}\right)\left(1 + \frac{2c}{3d}\right)\left(1 + \frac{2d}{3a}\right).$$

**124** Let  $a, b, c > 0$  be real numbers such that  $a^2 + b^2 + c^2 = 12$ . Determine the maximal value of the expression

$$A = a\sqrt[3]{b^2 + c^2} + b\sqrt[3]{c^2 + a^2} + c\sqrt[3]{a^2 + b^2}.$$

**125** Let  $a, b, c \geq 0$  such that  $a + b + c = 3$ . Prove the inequality

$$(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \leq 12.$$

**126** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

**127** Let  $x, y, z \in \mathbb{R}^+$  such that  $x + y + z = 1$ . Prove the inequality

$$\frac{xy}{\sqrt{1+z^2}} + \frac{zx}{\sqrt{1+y^2}} + \frac{yz}{\sqrt{1+x^2}} \leq \frac{1}{\sqrt{10}}.$$

**128** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$(a + b + c)^6 \geq 27(a^2 + b^2 + c^2)(ab + bc + ca)^2.$$

**129** Let  $a, b, c \in [1, 2]$  be real numbers. Prove the inequality

$$a^3 + b^3 + c^3 \leq 5abc.$$

**130** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 3$ . Prove the inequality

$$(a^7 - a^4 + 3)(b^5 - b^2 + 3)(c^4 - c + 3) \geq 27.$$

**131** Let  $a, b, c \in [1, 2]$  be real numbers. Prove the inequality

$$(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \leq 10.$$

**132** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 1$ . Prove the inequality

$$10(a^3 + b^3 + c^3) - 9(a^5 + b^5 + c^5) \geq 1.$$

**133** Let  $n \in \mathbb{N}$  and  $x_1, x_2, \dots, x_n \in (0, \pi)$ . Find the maximum value of the expression

$$\sin x_1 \cos x_2 + \sin x_2 \cos x_3 + \dots + \sin x_n \cos x_1.$$

**134** Let  $\alpha_i \in [\frac{\pi}{4}, \frac{5\pi}{4}]$ , for  $i = 1, 2, \dots, n$ . Prove the inequality

$$\left( \sin \alpha_1 + \sin \alpha_2 + \dots + \sin \alpha_n + \frac{1}{4} \right)^2 \geq (\cos \alpha_1 + \cos \alpha_2 + \dots + \cos \alpha_n).$$

**135** Let  $a_1, a_2, \dots, a_n$ ;  $a_{n+1} = a_1, a_{n+2} = a_2$  be positive real numbers. Prove the inequality

$$\sum_{i=1}^n \frac{a_i - a_{i+2}}{a_{i+1} + a_{i+2}} \geq 0.$$

**136** Let  $n \geq 2, n \in \mathbb{N}$  and  $x_1, x_2, \dots, x_n$  be positive real numbers such that

$$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + \dots + \frac{1}{x_n + 1998} = \frac{1}{1998}.$$

Prove the inequality

$$\sqrt[n]{x_1 x_2 \cdots x_n} \geq 1998(n-1).$$

**137** Let  $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ . Prove the inequality

$$\sum_{k=1}^n k a_k \leq \binom{n}{2} + \sum_{k=1}^n a_k^k.$$

**138** Let  $a_1, a_2, \dots, a_n$  be positive real numbers such that  $a_1 + a_2 + \dots + a_n = n$ . Prove that for every natural number  $k$  the following inequality holds

$$a_1^k + a_2^k + \dots + a_n^k \geq a_1^{k-1} + a_2^{k-1} + \dots + a_n^{k-1}.$$

**139** Let  $a, b, c, d$  be positive real numbers. Prove the inequality

$$\left(\frac{a}{a+b}\right)^5 + \left(\frac{b}{b+c}\right)^5 + \left(\frac{c}{c+d}\right)^5 + \left(\frac{d}{d+a}\right)^5 \geq \frac{1}{8}.$$

**140** Let  $x_1, x_2, \dots, x_n$  be positive real numbers not greater than 1. Prove the inequality

$$(1+x_1)^{\frac{1}{x_2}} (1+x_2)^{\frac{1}{x_3}} \cdots (1+x_n)^{\frac{1}{x_1}} \geq 2^n.$$

**141** Let  $x_1, x_2, \dots, x_n$  be non-negative real numbers such that  $x_1 + x_2 + \dots + x_n \leq \frac{1}{2}$ . Prove the inequality

$$(1-x_1)(1-x_2) \cdots (1-x_n) \geq \frac{1}{2}.$$

**142** Let  $a, b, c \in \mathbb{R}^+$  such that  $abc = 1$ . Prove the inequality

$$\frac{1}{a^3 + b^3 + 1} + \frac{1}{b^3 + c^3 + 1} + \frac{1}{c^3 + a^3 + 1} \leq 1.$$



**143** Let  $0 \leq a, b, c \leq 1$ . Prove the inequality

$$\frac{c}{7+a^3+b^3} + \frac{b}{7+c^3+a^3} + \frac{a}{7+b^3+c^3} \leq \frac{1}{3}.$$

**144** Let  $a, b, c \in \mathbb{R}^+$  such that  $abc = 1$ . Prove the inequality

$$\frac{ab}{a^5+ab+b^5} + \frac{bc}{b^5+bc+c^5} + \frac{ca}{c^5+ca+a^5} \leq 1.$$

**145** Let  $a, b, c \in \mathbb{R}^+$  such that  $a+b+c=3$ . Prove the inequality

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq 1.$$

**146** Let  $a, b, c$  be positive real numbers such that  $a^2+b^2+c^2=3abc$ . Prove the inequality

$$\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} \geq \frac{9}{a+b+c}.$$

**147** Let  $a, b, c, x, y, z$  be positive real number, and let  $a+b=3$ . Prove the inequality

$$\frac{x}{ay+bz} + \frac{y}{az+bx} + \frac{z}{ax+by} \geq 1.$$

**148** Let  $x, y, z > 0$  be real numbers. Prove the inequality

$$\frac{x}{x+2y+3z} + \frac{y}{y+2z+3x} + \frac{z}{z+2x+3y} \geq \frac{1}{2}.$$

**149** Let  $a, b, c, d \in \mathbb{R}^+$ . Prove the inequality

$$\frac{c}{a+3b} + \frac{d}{b+3c} + \frac{a}{c+3d} + \frac{b}{d+3a} \geq 1.$$

**150** Let  $a, b, c, d, e$  be positive real numbers. Prove the inequality

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+e} + \frac{d}{e+a} + \frac{e}{a+b} \geq \frac{5}{2}.$$

**151** Prove that for all positive real numbers  $a, b, c$  the following inequality holds

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{a^2+b^2+c^2}{a+b+c}.$$

**152** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$\frac{1}{4a^2 - bc + 1} + \frac{1}{4b^2 - ca + 1} + \frac{1}{4c^2 - ab + 1} \geq \frac{3}{2}.$$

**153** Let  $a, b, c$  be positive real numbers such that

$$\frac{1}{a^2 + b^2 + 1} + \frac{1}{b^2 + c^2 + 1} + \frac{1}{c^2 + a^2 + 1} \geq 1.$$

Prove the inequality

$$ab + bc + ca \leq 3.$$

**154** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 1/3$ . Prove the inequality

$$\frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \geq \frac{1}{a + b + c}.$$

**155** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a^3}{a^3 + b^3 + abc} + \frac{b^3}{b^3 + c^3 + abc} + \frac{c^3}{c^3 + a^3 + abc} \geq 1.$$

**156** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{a}{a^2 + 2b + 3} + \frac{b}{b^2 + 2c + 3} + \frac{c}{c^2 + 2a + 3} \leq \frac{1}{2}.$$

**157** Let  $a, b, c, d > 1$  be real numbers. Prove the inequality

$$\sqrt{a-1} + \sqrt{b-1} + \sqrt{c-1} + \sqrt{d-1} \leq \sqrt{(ab+1)(cd+1)}.$$

**158** Let  $a_1, a_2, \dots, a_n \in \mathbb{R}^+$  such that  $a_1 a_2 \cdots a_n = 1$ . Prove the inequality

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} \leq a_1 + a_2 + \cdots + a_n.$$

**159** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$a\sqrt{b} + b\sqrt{c} + c\sqrt{a} \leq \frac{1}{\sqrt{3}}.$$

**160** Let  $a, b, c \in (0, 1)$  be real numbers. Prove the inequality

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1.$$

**161** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$\frac{a^3 + 2}{b + 2} + \frac{b^3 + 2}{c + 2} + \frac{c^3 + 2}{a + 2} \geq 3.$$

**162** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{1}{2-a} + \frac{1}{2-b} + \frac{1}{2-c} \geq 3.$$

**163** Let  $a, b, c$  be positive real numbers such that  $abc = 8$ . Prove the inequality

$$\frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \leq 0.$$

**164** Let  $a, b, c \in \mathbb{R}^+$  such that  $a^2 + b^2 + c^2 = 1$ . Prove the inequality

$$a + b + c - 2abc \leq \sqrt{2}.$$

**165** Let  $x, y, z \in \mathbb{R}^+$  such that  $x^2 + y^2 + z^2 = 2$ . Prove the inequality

$$x + y + z \leq 2 + xyz.$$

**166** Let  $x, y, z > -1$  be real numbers. Prove the inequality

$$\frac{1+x^2}{1+y+z^2} + \frac{1+y^2}{1+z+x^2} + \frac{1+z^2}{1+x+y^2} \geq 2.$$

**167** Let  $a, b, c, d$  be positive real numbers such that  $abcd = 1$ . Prove the inequality

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2) \geq (a+b+c+d)^2.$$

**168** Let  $a, b, c, d \in \mathbb{R}^+$  such that  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 4$ . Prove the inequality

$$\sqrt[3]{\frac{a^3+b^3}{2}} + \sqrt[3]{\frac{b^3+c^3}{2}} + \sqrt[3]{\frac{c^3+d^3}{2}} + \sqrt[3]{\frac{d^3+a^3}{2}} \leq 2(a+b+c+d) - 4.$$

**169** Let  $x, y, z \in [-1, 1]$  be real numbers such that  $x + y + z + xyz = 0$ . Prove the inequality

$$\sqrt{x+1} + \sqrt{y+1} + \sqrt{z+1} \leq 3.$$

**170** Let  $a, b, c > 0$  be positive real numbers such that  $a + b + c = abc$ . Prove the inequality

$$ab + bc + ca \geq 3 + \sqrt{a^2 + 1} + \sqrt{b^2 + 1} + \sqrt{c^2 + 1}.$$

**171** Let  $a, b, c, x, y, z$  be positive real numbers such that  $ax + by + cz = xyz$ . Prove the inequality

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} < x + y + z.$$

**172** Let  $a, b, c$  be non-negative real numbers such that  $a^2 + b^2 + c^2 = 1$ . Prove the inequality

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \geq \frac{3}{4}(a\sqrt{a} + b\sqrt{b} + c\sqrt{c})^2.$$

**173** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}.$$

**174** Let  $x \geq y \geq z > 0$  be real numbers. Prove the inequality

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2.$$

**175** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} \leq 1.$$

**176** Let  $a, b, c$  be positive real numbers such that  $abc \geq 1$ . Prove the inequality

$$\frac{1}{a^4+b^3+c^2} + \frac{1}{b^4+c^3+a^2} + \frac{1}{c^4+a^3+b^2} \leq 1.$$

**177** Let  $a, b, c, d$  be positive real numbers such that  $abcd = 1$ . Prove the inequality

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+d)} + \frac{1}{d(1+a)} \geq 2.$$

**178** Let  $a, b, c$  be non-negative real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\frac{ab}{c+1} + \frac{bc}{a+1} + \frac{ca}{b+1} \leq \frac{1}{4}.$$

**179** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{1}{(a+1)^2(b+c)} + \frac{1}{(b+1)^2(c+a)} + \frac{1}{(c+1)^2(a+b)} \leq \frac{3}{8}.$$

**180** Let  $x, y, z$  be positive real numbers. Prove the inequality

$$xy(x+y-z) + yz(y+z-x) + zx(z+x-y) \geq \sqrt{3(x^3y^3 + y^3z^3 + z^3x^3)}.$$

**181** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{ab(a^3 + b^3)}{a^2 + b^2} + \frac{bc(b^3 + c^3)}{b^2 + c^2} + \frac{ca(c^3 + a^3)}{c^2 + a^2} \geq \sqrt{3abc(a^3 + b^3 + c^3)}.$$

**182** Let  $a, b, c$  be positive real numbers. Prove the inequality.

$$ab \frac{a+c}{b+c} + bc \frac{b+a}{c+a} + ca \frac{c+b}{a+b} \geq \sqrt{3abc(a+b+c)}.$$

**183** Let  $a, b, c$  and  $x, y, z$  be positive real numbers. Prove the inequality

$$a(y+z) + b(z+x) + c(x+y) \geq 2\sqrt{(xy + yz + zx)(ab + bc + ca)}.$$

**184** Let  $a, b, c$  be positive real numbers such that  $abc \geq 1$ . Prove the inequality

$$a^3 + b^3 + c^3 \geq ab + bc + ca.$$

**185** Let  $a, b, c > 0$  be real numbers such that  $a^{2/3} + b^{2/3} + c^{2/3} = 3$ . Prove the inequality

$$a^2 + b^2 + c^2 \geq a^{4/3} + b^{4/3} + c^{4/3}.$$

**186** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$\frac{1}{c^2 + a + b} + \frac{1}{a^2 + b + c} + \frac{1}{b^2 + c + a} \leq 1.$$

**187** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{2a^2}{b+c} + \frac{2b^2}{c+a} + \frac{2c^2}{a+b} \geq a + b + c.$$

**188** Let  $a, b, c$  be positive real numbers such that  $abc = 2$ . Prove the inequality

$$a^3 + b^3 + c^3 \geq a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}.$$

**189** Let  $a_1, a_2, \dots, a_n$  be positive real numbers. Prove the inequality

$$\frac{1}{\frac{1}{1+a_1} + \frac{1}{1+a_2} + \dots + \frac{1}{1+a_n}} - \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} \geq \frac{1}{n}.$$

**190** Let  $a, b, c, d \in \mathbb{R}^+$  such that  $ab + bc + cd + da = 1$ . Prove the inequality

$$\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{b+d+a} + \frac{d^3}{b+c+a} \geq \frac{1}{3}.$$

**191** Let  $\alpha, x, y, z$  be positive real numbers such that  $xyz = 1$  and  $\alpha \geq 1$ . Prove the inequality

$$\frac{x^\alpha}{y+z} + \frac{y^\alpha}{z+x} + \frac{z^\alpha}{x+y} \geq \frac{3}{2}.$$

**192** Let  $x_1, x_2, \dots, x_n$  be positive real numbers such that

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \dots + \frac{1}{1+x_n} = 1.$$

Prove the inequality

$$\frac{\sqrt{x_1} + \sqrt{x_2} + \dots + \sqrt{x_n}}{n-1} \geq \frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \dots + \frac{1}{\sqrt{x_n}}.$$

**193** Let  $x_1, x_2, \dots, x_n > 0$  be real numbers. Prove the inequality

$$x_1^{x_1} x_2^{x_2} \dots x_n^{x_n} \geq (x_1 x_2 \dots x_n)^{\frac{x_1 + x_2 + \dots + x_n}{n}}.$$

**194** Let  $a, b, c > 0$  be real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\frac{a^2 + b}{b + c} + \frac{b^2 + c}{c + a} + \frac{c^2 + a}{a + b} \geq 2.$$

**195** Let  $a, b, c > 1$  be positive real numbers such that  $\frac{1}{a^2-1} + \frac{1}{b^2-1} + \frac{1}{c^2-1} = 1$ . Prove the inequality

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \leq 1.$$

**196** Let  $a, b, c, d$  be positive real numbers such that  $a^2 + b^2 + c^2 + d^2 = 4$ . Prove the inequality

$$\frac{1}{5-a} + \frac{1}{5-b} + \frac{1}{5-c} + \frac{1}{5-d} \leq 1.$$

**197** Let  $a, b, c, d, e \in \mathbb{R}$  such that  $\frac{1}{4+a} + \frac{1}{4+b} + \frac{1}{4+c} + \frac{1}{4+d} + \frac{1}{4+e} = 1$ . Prove the inequality

$$\frac{a}{4+a^2} + \frac{b}{4+b^2} + \frac{c}{4+c^2} + \frac{d}{4+d^2} + \frac{e}{4+e^2} \leq 1.$$

**198** Let  $a, b, c$  be real numbers different from 1, such that  $a + b + c = 1$ . Prove the inequality

$$\frac{1+a^2}{1-a^2} + \frac{1+b^2}{1-b^2} + \frac{1+c^2}{1-c^2} \geq \frac{15}{4}.$$

**199** Let  $x, y, z > 0$ , such that  $xyz = 1$ . Prove the inequality

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

**200** Let  $a, b, c, d > 0$  be real numbers. Prove the inequality

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geq \frac{2}{3}.$$

**201** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq a+b+c.$$

**202** Let  $a, b > 0, n \in \mathbb{N}$ . Prove the inequality

$$\left(1 + \frac{a}{b}\right)^n + \left(1 + \frac{b}{a}\right)^n \geq 2^{n+1}.$$

**203** Let  $a, b, c > 0$  be real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 + \left(c + \frac{1}{c}\right)^2 \geq \frac{100}{3}.$$

**204** Let  $x, y, z > 0$  be real numbers. Prove the inequality

$$\frac{x}{2x+y+z} + \frac{y}{x+2y+z} + \frac{z}{x+y+2z} \leq \frac{3}{4}.$$

**205** Let  $a, b, c, d > 0$  be real numbers such that  $a \leq 1, a + b \leq 5, a + b + c \leq 14, a + b + c + d \leq 30$ . Prove that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d} \leq 10.$$

**206** Let  $a, b, c, d$  be positive real numbers such that  $a + b + c + d = 4$ . Prove the inequality

$$\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{d^2+d} + \frac{d}{a^2+a} \geq \frac{8}{(a+c)(b+d)}.$$

**207** Let  $x_1, x_2, \dots, x_n > 0$  and  $n \in \mathbb{N}, n > 1$ , such that  $x_1 + x_2 + \dots + x_n = 1$ . Prove the inequality

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \dots + \frac{x_n}{\sqrt{1-x_n}} \geq \frac{\sqrt{x_1} + \sqrt{x_2} + \dots + \sqrt{x_n}}{\sqrt{n-1}}.$$

**208** Let  $n \in \mathbb{N}$ ,  $n \geq 2$ . Determine the minimal value of

$$\frac{x_1^5}{x_2 + x_3 + \cdots + x_n} + \frac{x_2^5}{x_1 + x_3 + \cdots + x_n} + \cdots + \frac{x_n^5}{x_1 + x_2 + \cdots + x_{n-1}},$$

where  $x_1, x_2, \dots, x_n \in \mathbb{R}^+$  such that  $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ .

**209** Let  $P, L, R$  denote the area, perimeter and circumradius of  $\triangle ABC$ , respectively. Determine the maximum value of the expression  $\frac{LP}{R^3}$ .

**210** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = abc$ . Prove the inequality

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2}.$$

**211** Let  $a, b, c \in \mathbb{R}$  such that  $abc + a + c = b$ . Prove the inequality

$$\frac{2}{a^2+1} - \frac{2}{b^2+1} + \frac{3}{c^2+1} \leq \frac{10}{3}.$$

**212** Let  $x, y, z > 1$  be real numbers such that  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$ . Prove the inequality

$$\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1} \leq \sqrt{x+y+z}.$$

**213** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\sqrt{\frac{1}{a}-1}\sqrt{\frac{1}{b}-1} + \sqrt{\frac{1}{b}-1}\sqrt{\frac{1}{c}-1} + \sqrt{\frac{1}{c}-1}\sqrt{\frac{1}{a}-1} \geq 6.$$

**214** Let  $a, b, c$  be positive real numbers such that  $a + b + c + 1 = 4abc$ . Prove the inequalities

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3 \geq \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}.$$

**215** Let  $a, b, c$  be non-negative real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$\frac{a}{1+a^2} + \frac{b}{1+b^2} + \frac{c}{1+c^2} \leq \frac{3\sqrt{3}}{4}.$$

**216** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\sqrt{\frac{ab}{c+ab}} + \sqrt{\frac{bc}{a+bc}} + \sqrt{\frac{ca}{b+ca}} \leq \frac{3}{2}.$$



**217** Let  $a, b, c > 0$  be real numbers such that  $(a+b)(b+c)(c+a) = 1$ . Prove the inequality

$$ab + bc + ca \leq \frac{3}{4}.$$

**218** Let  $a, b, c \geq 0$  be real numbers such that  $a^2 + b^2 + c^2 + abc = 4$ . Prove the inequality

$$0 \leq ab + bc + ca - abc \leq 2.$$

**219** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$a^2 + b^2 + c^2 + 2abc + 3 \geq (1+a)(1+b)(1+c).$$

**220** Let  $a, b, c$  be real numbers. Prove the inequality

$$\sqrt{a^2 + (1-b)^2} + \sqrt{b^2 + (1-c)^2} + \sqrt{c^2 + (1-a)^2} \geq \frac{3\sqrt{2}}{2}.$$

**221** Let  $a_1, a_2, \dots, a_n \in \mathbb{R}^+$  such that  $\sum_{i=1}^n a_i^3 = 3$  and  $\sum_{i=1}^n a_i^5 = 5$ . Prove the inequality

$$\sum_{i=1}^n a_i > \frac{3}{2}.$$

**222** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 3$ . Prove the inequality

$$(1+a^2)(1+b^2)(1+c^2) \geq 8.$$

**223** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 1.$$

**224** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{a}{\sqrt{7+b^2+c^2}} + \frac{b}{\sqrt{7+c^2+a^2}} + \frac{c}{\sqrt{7+a^2+b^2}} \geq 1.$$

**225** Let  $a_1, a_2, \dots, a_n$  be positive real numbers such that  $a_1 + a_2 + \dots + a_n = 1$ . Prove the inequality

$$\frac{a_1}{\sqrt{1-a_1}} + \frac{a_2}{\sqrt{1-a_2}} + \dots + \frac{a_n}{\sqrt{1-a_n}} \geq \sqrt{\frac{n}{n-1}}.$$

**226** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a}{\sqrt{2b^2 + 2c^2 - a^2}} + \frac{b}{\sqrt{2c^2 + 2a^2 - b^2}} + \frac{c}{\sqrt{2a^2 + 2b^2 - c^2}} \geq \sqrt{3}.$$

**227** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca \geq 3$ . Prove the inequality

$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \geq \frac{3}{\sqrt{2}}.$$

**228** Let  $a, b, c \geq 1$  be real numbers such that  $a + b + c = 2abc$ . Prove the inequality

$$\sqrt[3]{(a+b+c)^2} \geq \sqrt[3]{ab-1} + \sqrt[3]{bc-1} + \sqrt[3]{ca-1}.$$

**229** Let  $t_a, t_b, t_c$  be the lengths of the medians, and  $a, b, c$  be the lengths of the sides of a given triangle. Prove the inequality

$$t_a t_b + t_b t_c + t_c t_a < \frac{5}{4}(ab + bc + ca).$$

**230** Let  $a, b, c$  and  $t_a, t_b, t_c$  be the lengths of the sides and lengths of the medians of an arbitrary triangle, respectively. Prove the inequality

$$at_a + bt_b + ct_c \leq \frac{\sqrt{3}}{2}(a^2 + b^2 + c^2).$$

**231** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove the inequality

$$\sqrt{a+b-c} + \sqrt{c+a-b} + \sqrt{b+c-a} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

**232** Let  $P$  be the area of the triangle with side lengths  $a, b$  and  $c$ , and  $T$  be the area of the triangle with side lengths  $a+b, b+c$  and  $c+a$ . Prove that  $T \geq 4P$ .

**233** Let  $a, b, c$  be the lengths of the sides of a triangle, such that  $a + b + c = 3$ . Prove the inequality

$$a^2 + b^2 + c^2 + \frac{4abc}{3} \geq \frac{13}{3}.$$

**234** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove that

$$\sqrt[3]{\frac{a^3 + b^3 + c^3 + 3abc}{2}} \geq \max\{a, b, c\}.$$

**235** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove the inequality

$$abc < a^2(s-a) + b^2(s-a) + c^2(s-a) \leq \frac{3}{2}abc.$$

**236** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove that

$$\frac{1}{\sqrt{a} + \sqrt{b} - \sqrt{c}} + \frac{1}{\sqrt{b} + \sqrt{c} - \sqrt{a}} + \frac{1}{\sqrt{c} + \sqrt{a} - \sqrt{b}} \geq \frac{3(\sqrt{a} + \sqrt{b} + \sqrt{c})}{a + b + c}.$$

**237** Let  $a, b, c$  be the lengths of the sides of a triangle with area  $P$ . Prove that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}P.$$

**238** (*Hadwinger–Finsler*) Let  $a, b, c$  be the lengths of the sides of a triangle. Prove the inequality

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}P + (a - b)^2 + (b - c)^2 + (c - a)^2.$$

**239** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove that

$$\frac{1}{8abc + (a + b - c)^3} + \frac{1}{8abc + (b + c - a)^3} + \frac{1}{8abc + (c + a - b)^3} \leq \frac{1}{3abc}.$$

**240** In the triangle  $ABC$ ,  $\overline{AC}^2$  is the arithmetic mean of  $\overline{BC}^2$  and  $\overline{AB}^2$ . Prove that

$$\cot^2 \beta \geq \cot \alpha \cdot \cot \gamma.$$

**241** Let  $d_1, d_2$  and  $d_3$  be the distances from an arbitrary point to the sides  $BC, CA, AB$ , respectively, of the triangle  $ABC$ . Prove the inequality

$$\frac{9}{4}(d_1^2 + d_2^2 + d_3^2) \geq \left(\frac{P}{R}\right)^2.$$

**242** Let  $a, b, c$  be the side lengths, and  $h_a, h_b, h_c$  be the lengths of the altitudes (respectively) of a given triangle. Prove the inequality

$$\frac{h_a + h_b + h_c}{a + b + c} \leq \frac{\sqrt{3}}{2}.$$

**243** Let  $O$  be an arbitrary point in the interior of  $\triangle ABC$ . Let  $x, y$  and  $z$  be the distances from  $O$  to the sides  $BC, CA, AB$ , respectively, and let  $R$  be the circumradius of the triangle  $\triangle ABC$ . Prove the inequality

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \leq 3\sqrt{\frac{R}{2}}.$$

**244** Let  $D, E$  and  $F$  be the feet of the altitudes of the triangle  $ABC$  dropped from the vertices  $A, B$  and  $C$ , respectively. Prove the inequality

$$\left(\frac{EF}{a}\right)^2 + \left(\frac{FD}{b}\right)^2 + \left(\frac{DE}{c}\right)^2 \geq \frac{3}{4}.$$

**245** Let  $a, b, c$  be the side-lengths,  $h_a, h_b, h_c$  be the lengths of the respective altitudes, and  $s$  be the semi-perimeter of a given triangle. Prove the inequality

$$\frac{h_a}{a} + \frac{h_b}{b} + \frac{h_c}{c} \leq \frac{s}{2r}.$$

**246** Let  $a, b, c$  be the side lengths,  $h_a, h_b, h_c$  be the altitudes, respectively, of a triangle. Prove the inequality

$$\frac{a^2}{h_b^2 + h_c^2} + \frac{b^2}{h_a^2 + h_c^2} + \frac{c^2}{h_a^2 + h_b^2} \geq 2.$$

**247** Let  $a, b, c$  be the side lengths,  $h_a, h_b, h_c$  be the altitudes, respectively, and  $r$  be the inradius of a triangle. Prove the inequality

$$\frac{1}{h_a - 2r} + \frac{1}{h_b - 2r} + \frac{1}{h_c - 2r} \geq \frac{3}{r}.$$

**248** Let  $a, b, c; l_\alpha, l_\beta, l_\gamma$  be the lengths of the sides and the bisectors of respective angles. Let  $s$  be the semi-perimeter and  $r$  denote the inradius of a given triangle. Prove the inequality

$$\frac{l_\alpha}{a} + \frac{l_\beta}{b} + \frac{l_\gamma}{c} \leq \frac{s}{2r}.$$

**249** Let  $a, b, c; l_\alpha, l_\beta, l_\gamma$  be the lengths of the sides and of the bisectors of respective angles. Let  $R$  and  $r$  be the circumradius and inradius, respectively, of a given triangle. Prove the inequality

$$18r^2\sqrt{3} \leq al_\alpha + bl_\beta + cl_\gamma < 9R^2.$$

**250** Let  $a, b, c$  be the lengths of the sides of a triangle, with circumradius  $r = 1/2$ . Prove the inequality

$$\frac{a^4}{b+c-a} + \frac{b^4}{a+c-b} + \frac{c^4}{a+b-c} \geq 9\sqrt{3}.$$

**251** Let  $a, b, c$  be the side-lengths of a triangle. Prove the inequality

$$\frac{a}{3a-b+c} + \frac{b}{3b-c+a} + \frac{c}{3c-a+b} \geq 1.$$

**252** Let  $h_a, h_b$  and  $h_c$  be the lengths of the altitudes, and  $R$  and  $r$  be the circumradius and inradius, respectively, of a given triangle. Prove the inequality

$$h_a + h_b + h_c \leq 2R + 5r.$$

**253** Let  $a, b, c$  be the side-lengths, and  $\alpha, \beta$  and  $\gamma$  be the angles of a given triangle, respectively. Prove the inequality

$$a\left(\frac{1}{\beta} + \frac{1}{\gamma}\right) + b\left(\frac{1}{\gamma} + \frac{1}{\alpha}\right) + c\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \geq 2\left(\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}\right).$$

**254** Let  $a, b, c$  be the lengths of the sides of a given triangle, and  $\alpha, \beta, \gamma$  be the respective angles (in radians). Prove the inequalities

$$\begin{aligned} 1^\circ \quad & \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \geq \frac{9}{\pi} \\ 2^\circ \quad & \frac{b+c-a}{\alpha} + \frac{c+a-b}{\beta} + \frac{a+b-c}{\gamma} \geq \frac{6s}{\pi}, \text{ where } s = \frac{a+b+c}{2} \\ 3^\circ \quad & \frac{b+c-a}{a\alpha} + \frac{c+a-b}{b\beta} + \frac{a+b-c}{c\gamma} \geq \frac{9}{\pi}. \end{aligned}$$

**255** Let  $X$  be an arbitrary interior point of a given regular  $n$ -gon with side-length  $a$ . Let  $h_1, h_2, \dots, h_n$  be the distances from  $X$  to the sides of the  $n$ -gon. Prove that

$$\frac{1}{h_1} + \frac{1}{h_2} + \dots + \frac{1}{h_n} > \frac{2\pi}{a}.$$

**256** Prove that among the lengths of the sides of an arbitrary  $n$ -gon ( $n \geq 3$ ), there always exist two of them (let's denote them by  $b$  and  $c$ ) such that  $1 \leq \frac{b}{c} < 2$ .

**257** Let  $a_1, a_2, a_3, a_4$  be the lengths of the sides, and  $s$  be the semi-perimeter of arbitrary quadrilateral. Prove that

$$\sum_{i=1}^4 \frac{1}{s+a_i} \leq \frac{2}{9} \sum_{1 \leq i < j \leq 4} \frac{1}{\sqrt{(s-a_i)(s-a_j)}}.$$

**258** Let  $n \in \mathbb{N}$ , and  $\alpha, \beta, \gamma$  be the angles of a given triangle. Prove the inequality

$$\cot^n \frac{\alpha}{2} + \cot^n \frac{\beta}{2} + \cot^n \frac{\gamma}{2} \geq 3^{\frac{n+2}{2}}.$$

**259** Let  $\alpha, \beta, \gamma$  be the angles of an arbitrary acute triangle. Prove that

$$2(\sin \alpha + \sin \beta + \sin \gamma) > 3(\cos \alpha + \cos \beta + \cos \gamma).$$

**260** Let  $\alpha, \beta, \gamma$  be the angles of a triangle. Prove the inequality

$$\sin \alpha + \sin \beta + \sin \gamma \geq \sin 2\alpha + \sin 2\beta + \sin 2\gamma.$$

**261** Let  $\alpha, \beta, \gamma$  be the angles of a triangle. Prove the inequality

$$\cos \alpha + \sqrt{2}(\cos \beta + \cos \gamma) \leq 2.$$

**262** Let  $\alpha, \beta, \gamma$  be the angles of a triangle and let  $t$  be a real number. Prove the inequality

$$\cos \alpha + t(\cos \beta + \cos \gamma) \leq 1 + \frac{t^2}{2}.$$

**263** Let  $0 \leq \alpha, \beta, \gamma \leq 90^\circ$  such that  $\sin \alpha + \sin \beta + \sin \gamma = 1$ . Prove the inequality

$$\tan^2 \alpha + \tan^2 \beta + \tan^2 \gamma \geq \frac{3}{8}.$$

**264** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$(1 + a + a^2)(1 + b + b^2)(1 + c + c^2) \geq 9(ab + bc + ca).$$

**265** Let  $a, b, c > 0$  such that  $a + b + c = 1$ . Prove the inequality

$$6(a^3 + b^3 + c^3) + 1 \geq 5(a^2 + b^2 + c^2).$$

**266** Let  $x, y, z \in \mathbb{R}^+$  such that  $x + y + z = 1$ . Prove the inequality

$$(1 - x^2)^2 + (1 - y^2)^2 + (1 - z^2)^2 \leq (1 + x)(1 + y)(1 + z).$$

**267** Let  $x, y, z$  be non-negative real numbers such that  $x^2 + y^2 + z^2 = 1$ . Prove the inequality

$$(1 - xy)(1 - yz)(1 - zx) \geq \frac{8}{27}.$$

**268** Let  $a, b, c \in \mathbb{R}^+$  such that  $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$ . Prove the inequalities:

$$1^\circ \quad \frac{1}{8a^2+1} + \frac{1}{8b^2+1} + \frac{1}{8c^2+1} \geq 1$$

$$2^\circ \quad \frac{1}{4ab+1} + \frac{1}{4bc+1} + \frac{1}{4ca+1} \geq \frac{3}{2}.$$

**269** Let  $a, b, c > 0$  be real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} - \frac{1}{a+b+c} \geq 2.$$

**270** Let  $a, b, c \geq 0$  be real numbers. Prove the inequality

$$\frac{ab + 4bc + ca}{a^2 + bc} + \frac{bc + 4ca + ab}{b^2 + ca} + \frac{ca + 4ab + bc}{c^2 + ab} \geq 6.$$

**271** Let  $a, b, c$  be positive real numbers such that  $a + b + c + 1 = 4abc$ . Prove the inequality

$$\frac{1}{a^4 + b + c} + \frac{1}{b^4 + c + a} + \frac{1}{c^4 + a + b} \leq \frac{3}{a + b + c}.$$

**272** Let  $x, y, z > 0$  be real numbers such that  $x + y + z = 1$ . Prove the inequality

$$(x^2 + y^2)(y^2 + z^2)(z^2 + x^2) \leq \frac{1}{32}.$$

**273** Let  $x, y, z \in \mathbb{R}^+$  such that  $x + y + z = 1$ . Prove the inequalities:

$$1 \leq \frac{x}{1-yz} + \frac{y}{1-zx} + \frac{z}{1-xy} \leq \frac{9}{8}.$$

**274** Let  $x, y, z \in \mathbb{R}^+$ , such that  $xyz = 1$ . Prove the inequality

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} + \frac{1}{(1+z)^2} + \frac{2}{(1+x)(1+y)(1+z)} \geq 1.$$

**275** Let  $a, b, c \geq 0$  such that  $a + b + c = 1$ . Prove the inequalities:

$$1^\circ \quad ab + bc + ca \leq a^3 + b^3 + c^3 + 6abc$$

$$2^\circ \quad a^3 + b^3 + c^3 + 6abc \leq a^2 + b^2 + c^2$$

$$3^\circ \quad a^2 + b^2 + c^2 \leq 2(a^3 + b^3 + c^3) + 3abc.$$

**276** Let  $x, y, z \geq 0$  be real numbers such that  $xy + yz + zx + xyz = 4$ . Prove the inequality

$$3(x^2 + y^2 + z^2) + xyz \geq 10.$$

**277** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$x^4(y+z) + y^4(z+x) + z^4(x+y) \leq \frac{1}{12}(x+y+z)^5.$$

**278** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 1$ . Prove the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 48(ab + bc + ca) \geq 25.$$

**279** Let  $a, b, c$  be non-negative real numbers such that  $a + b + c = 2$ . Prove the inequality

$$a^4 + b^4 + c^4 + abc \geq a^3 + b^3 + c^3.$$

**280** Let  $a, b, c$  be non-negative real numbers. Prove the inequality

$$2(a^2 + b^2 + c^2) + abc + 8 \geq 5(a + b + c).$$

**281** Let  $a, b, c$  be non-negative real numbers. Prove the inequality

$$a^3 + b^3 + c^3 + 4(a + b + c) + 9abc \geq 8(ab + bc + ca).$$

**282** Let  $a, b, c$  be non-negative real numbers. Prove the inequality

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq a + b + c.$$

**283** Let  $a, b, c$  be non-negative real numbers such that  $a + b + c = 2$ . Prove the inequality

$$a^3 + b^3 + c^3 + \frac{15abc}{4} \geq 2.$$

**284** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{a^2 + bc}{a^2(b + c)} + \frac{b^2 + ca}{b^2(c + a)} + \frac{c^2 + ab}{c^2(a + b)} \geq ab + bc + ca.$$

**285** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{a^3 + abc}{(b + c)^2} + \frac{b^3 + abc}{(c + a)^2} + \frac{c^3 + abc}{(a + b)^2} \geq \frac{3}{2}.$$

**286** Let  $a, b, c$  be positive real numbers such that  $a^4 + b^4 + c^4 = 3$ . Prove the inequality

$$\frac{1}{4 - ab} + \frac{1}{4 - bc} + \frac{1}{4 - ca} \leq 1.$$

**287** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 3$ . Prove the inequality

$$(a^3 - a + 5)(b^5 - b^3 + 5)(c^7 - c^5 + 5) \geq 125.$$

**288** Let  $x, y, z$  be positive real numbers. Prove the inequality

$$\frac{1}{x^2 + xy + y^2} + \frac{1}{y^2 + yz + z^2} + \frac{1}{z^2 + zx + x^2} \geq \frac{9}{(x + y + z)^2}.$$

**289** Let  $x, y, z$  be positive real numbers such that  $xyz = x + y + z + 2$ . Prove the inequalities

$$1^\circ \quad xy + yz + zx \geq 2(x + y + z)$$

$$2^\circ \quad \sqrt{x} + \sqrt{y} + \sqrt{z} \leq \frac{3\sqrt{xyz}}{2}.$$



**290** Let  $x, y, z$  be positive real numbers. Prove the inequality

$$8(x^3 + y^3 + z^3) \geq (x + y)^3 + (y + z)^3 + (z + x)^3.$$

**291** Let  $a, b, c$  be non-negative real numbers. Prove the inequality

$$a^3 + b^3 + c^3 + abc \geq \frac{1}{7}(a + b + c)^3.$$

**292** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$a^2 + b^2 + c^2 + 3abc \geq \frac{4}{9}.$$

**293** Let  $a_1, a_2, \dots, a_n$  be positive real numbers. Prove the inequality

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \leq \left(1 + \frac{a_1^2}{a_2}\right) \left(1 + \frac{a_2^2}{a_3}\right) \cdots \left(1 + \frac{a_n^2}{a_1}\right).$$

**294** Let  $a, b, c, d$  be positive real numbers such that  $abcd = 1$ . Prove the inequality

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1.$$

**295** Let  $a, b, c, d \geq 0$  be real numbers such that  $a + b + c + d = 4$ . Prove the inequality

$$abc + bcd + cda + dab + (abc)^2 + (bcd)^2 + (cda)^2 + (dab)^2 \leq 8.$$

**296** Let  $a, b, c, d \geq 0$  such that  $a + b + c + d = 1$ . Prove the inequality

$$a^4 + b^4 + c^4 + d^4 + \frac{148}{27}abcd \geq \frac{1}{27}.$$

**297** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$a^2b^2 + b^2c^2 + c^2a^2 \leq a + b + c.$$

**298** Let  $a, b, c, d \geq 0$  be real numbers such that  $a + b + c + d = 4$ . Prove the inequality

$$(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) \geq (1 + a)(1 + b)(1 + c)(1 + d).$$

**299** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{6}{a + b + c} \geq 5.$$

**300** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$12\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 4(a^3 + b^3 + c^3) + 21.$$

**301** Let  $a, b, c, d$  be non-negative real numbers such that  $a + b + c + d + e = 5$ . Prove the inequality

$$4(a^2 + b^2 + c^2 + d^2 + e^2) + 5abcd \geq 25.$$

**302** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$\frac{1}{2 + a^2 + b^2} + \frac{1}{2 + b^2 + c^2} + \frac{1}{2 + c^2 + a^2} \leq \frac{3}{4}.$$

**303** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$ab + bc + ca \leq abc + 2.$$

**304** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{a+c}{a+b}.$$

**305** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

**306** Let  $a, b, c$  be positive real numbers such that  $a \geq b \geq c$ . Prove the inequality

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

**307** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove the inequality

$$\frac{(b+c)^2}{a^2+bc} + \frac{(c+a)^2}{b^2+ca} + \frac{(a+b)^2}{c^2+ab} \geq 6.$$

**308** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b} + 3\frac{ab+bc+ca}{(a+b+c)^2} \geq 4.$$

**309** Let  $a, b, c$  be real numbers. Prove the inequality

$$3(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \geq a^3b^3 + b^3c^3 + c^3a^3.$$

**310** Let  $a, b, c, d \in \mathbb{R}^+$  such that  $a + b + c + d + abcd = 5$ . Prove the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \geq 4.$$

## Chapter 21

### Solutions

**1** Let  $n$  be a positive integer. Prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2.$$

*Solution* For each  $k \geq 2$  we have

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

So

$$\begin{aligned} 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} &< 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 2 - \frac{1}{n} < 2. \end{aligned}$$

■

**2** Let  $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ . Prove that for any  $n \in \mathbb{N}$  we have

$$\frac{1}{a_1^2} + \frac{1}{2a_2^2} + \frac{1}{3a_3^2} + \cdots + \frac{1}{na_n^2} < 2.$$

*Solution* Note that for any  $k \geq 2$  we have

$$\frac{1}{a_{k-1}} - \frac{1}{a_k} = \frac{a_k - a_{k-1}}{a_{k-1}a_k} = \frac{1}{ka_k a_{k-1}} > \frac{1}{ka_k^2}.$$

Adding these inequalities for  $k = 2, 3, \dots, n$  we get

$$\frac{1}{2a_2^2} + \frac{1}{3a_3^2} + \cdots + \frac{1}{na_n^2} < \frac{1}{a_1} - \frac{1}{a_n} < \frac{1}{a_1},$$

and since  $a_1 = 1$ , we obtain

$$\frac{1}{a_1^2} + \frac{1}{2a_2^2} + \frac{1}{3a_3^2} + \cdots + \frac{1}{na_n^2} < \frac{2}{a_1} = 2. \quad \blacksquare$$

**3** Let  $x, y, z$  be real numbers. Prove the inequality

$$x^4 + y^4 + z^4 \geq 4xyz - 1.$$

*Solution* We have

$$\begin{aligned} x^4 + y^4 + z^4 - 4xyz + 1 &= (x^4 - 2x^2 + 1) + (y^4 - 2y^2z^2 + z^4) + (2y^2z^2 - 4xyz + 2x^2) \\ &= (x^2 - 1)^2 + (y^2 - z^2)^2 + 2(yz - x)^2 \geq 0, \end{aligned}$$

so it follows that

$$x^4 + y^4 + z^4 \geq 4xyz - 1.$$

When does equality occur?  $\blacksquare$

**4** Prove that for any real number  $x$ , the following inequality holds

$$x^{2002} - x^{1999} + x^{1996} - x^{1995} + 1 > 0.$$

*Solution* Denote

$$x^{2002} - x^{1999} + x^{1996} - x^{1995} + 1 > 0. \quad (1)$$

We will consider five cases:

- 1° If  $x < 0$ , then all summands on the left side of the inequality (1) are positive, so the inequality is true.
- 2° If  $x = 0$ , inequality (1) is equivalent to  $1 > 0$ , which is obviously true.
- 3° If  $0 < x < 1$ , then (1) is

$$x^{2002} + x^{1996}(1 - x^3) + (1 - x^{1995}) > 0.$$

Since

$$\begin{aligned} 1 - x^3 &= (1 - x)(1 + x + x^2) > 0 \quad \text{and} \\ 1 - x^5 &= (1 - x)(1 + x + x^2 + x^3 + x^4) > 0, \end{aligned}$$

we deduce that the required inequality is true.

- 4° If  $x = 1$ , then (1) is equivalent to  $1 > 0$ , which is clearly true.

5° If  $x > 1$ , rewrite (1) in following way

$$x^{1999}(x^3 - 1) + x^{1995}(x - 1) + 1 > 0.$$

Since  $x > 1$  we have  $x^3 > 1$ .

So  $x^{1999}(x^3 - 1) + x^{1995}(x - 1) + 1 > 0$ , and we are done. ■

5 Let  $x, y$  be real numbers. Prove the inequality

$$3(x + y + 1)^2 + 1 \geq 3xy.$$

*Solution* Observe that for any real numbers  $a$  and  $b$  we have

$$a^2 + ab + b^2 = \left(a + \frac{b}{2}\right)^2 + \frac{3b^2}{4} \geq 0,$$

with equality if and only if  $a = b = 0$ .

Let  $x, y$  be real numbers. Then according to the above inequality we have

$$\begin{aligned} \left(x + \frac{2}{3}\right)^2 + \left(x + \frac{2}{3}\right)\left(y + \frac{2}{3}\right) + \left(y + \frac{2}{3}\right)^2 &\geq 0, \quad \text{i.e.} \\ 3x^2 + 3y^2 + 3xy + 6x + 6y + 4 &\geq 0, \end{aligned}$$

which is equivalent to

$$3(x + y + 1)^2 + 1 \geq 3xy.$$

Equality occurs iff  $x + \frac{2}{3} = y + \frac{2}{3} = 0$ , i.e.  $x = y = -\frac{2}{3}$ . ■

6 Let  $a, b, c$  be positive real numbers such that  $a + b + c \geq abc$ . Prove that at least two of the following inequalities

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6$$

are true.

*Solution* Set  $\frac{1}{a} = x, \frac{1}{b} = y, \frac{1}{c} = z$ .

Then  $x, y, z > 0$  and the initial condition becomes  $xy + yz + zx \geq 1$ .

We need to prove that at least two of the following inequalities  $2x + 3y + 6z \geq 6, 2y + 3z + 6x \geq 6, 2z + 3x + 6y \geq 6$ , hold.

Assume the contrary, i.e. we may assume that  $2x + 3y + 6z < 6$  and  $2z + 3x + 6y < 6$ .

Adding these inequalities we get  $5x + 9y + 8z < 12$ .

But we have  $x \geq \frac{1-yz}{y+z}$ .

Thus,  $12 > \frac{5-5yz}{y+z} + 9y + 8z$ , i.e.

$$12(y+z) > 5 + 9y^2 + 8z^2 + 12yz \Leftrightarrow (2z-1)^2 + (3y+2z-2)^2 < 0,$$

which is impossible, and the conclusion follows. ■

**7** Let  $a, b, c, x, y, z > 0$ . Prove the inequality

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(a+b+c)(x+y+z)}{a+b+c+x+y+z}.$$

*Solution* We'll use the following lemma.

**Lemma 21.1** For every  $p, q, \alpha, \beta > 0$  we have

$$\frac{pq}{p+q} \leq \frac{\alpha^2 p + \beta^2 q}{(\alpha + \beta)^2}.$$

*Proof* The given inequality is equivalent to  $(\alpha p - \beta q)^2 \geq 0$ . □

Now let  $\alpha = x + y + z$ ,  $\beta = a + b + c$ , and applying Lemma 21.1, we obtain

$$\begin{aligned} \frac{ax}{a+x} &\leq \frac{(x+y+z)^2 a + (a+b+c)^2 x}{(x+y+z+a+b+c)^2}, \\ \frac{by}{b+y} &\leq \frac{(x+y+z)^2 b + (a+b+c)^2 y}{(x+y+z+a+b+c)^2} \end{aligned}$$

and

$$\frac{cz}{c+z} \leq \frac{(x+y+z)^2 c + (a+b+c)^2 z}{(x+y+z+a+b+c)^2}.$$

Adding these inequalities we get the required result. ■

**8** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{2a}{a^2+bc} + \frac{2b}{b^2+ac} + \frac{2c}{c^2+ab} \leq \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}.$$

*Solution* Notice that  $\frac{2a}{a^2+bc} \leq \frac{1}{2}(\frac{1}{b} + \frac{1}{c})$ , which is equivalent to

$$b(a-c)^2 + c(a-b)^2 \geq 0.$$

Also  $\frac{1}{b} + \frac{1}{c} \leq \frac{1}{2}(\frac{2a}{bc} + \frac{b}{ac} + \frac{c}{ab})$ , which is equivalent to

$$(a-b)^2 + (a-c)^2 \geq 0.$$

Hence

$$\frac{2a}{a^2+bc} \leq \frac{1}{4} \left( \frac{2a}{bc} + \frac{b}{ac} + \frac{c}{ab} \right). \quad (1)$$

Analogously, we obtain

$$\frac{2b}{b^2 + ac} \leq \frac{1}{4} \left( \frac{2b}{ac} + \frac{c}{ab} + \frac{a}{bc} \right), \quad (2)$$

$$\frac{2c}{c^2 + ab} \leq \frac{1}{4} \left( \frac{2c}{ab} + \frac{a}{bc} + \frac{b}{ac} \right). \quad (3)$$

Adding (1), (2) and (3) we obtain the required inequality.

Equality occurs if and only if  $a = b = c$ . ■

**9** Let  $a, b, c, x, y, z \in \mathbb{R}^+$  such that  $a + x = b + y = c + z = 1$ . Prove the inequality

$$(abc + xyz) \left( \frac{1}{ay} + \frac{1}{bz} + \frac{1}{cx} \right) \geq 3.$$

*Solution* We have

$$abc + xyz = abc + (1-a)(1-b)(1-c) = (1-b)(1-c) + ac + ab - a.$$

So

$$\frac{abc + xyz}{a(1-b)} = \frac{1-c}{a} + \frac{c}{1-b} - 1,$$

and analogously we obtain  $\frac{abc+xyz}{b(1-c)}$  and  $\frac{abc+xyz}{c(1-a)}$ .

Hence

$$\begin{aligned} & (abc + xyz) \left( \frac{1}{ay} + \frac{1}{bz} + \frac{1}{cx} \right) \\ &= \frac{a}{1-c} + \frac{b}{1-a} + \frac{c}{1-b} + \frac{1-c}{a} + \frac{1-b}{c} + \frac{1-a}{b} - 3 \geq 6 - 3 = 3. \quad \blacksquare \end{aligned}$$

**10** Let  $a_1, a_2, \dots, a_n$  be positive real numbers and let  $b_1, b_2, \dots, b_n$  be their permutation. Prove the inequality

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geq a_1 + a_2 + \dots + a_n.$$

*Solution* For each  $x, y \in \mathbb{R}^+$  we have  $\frac{x^2}{y} \geq 2x - y$ .

Hence

$$\frac{a_i^2}{b_i} \geq 2a_i - b_i, \quad i = 1, 2, \dots, n.$$



After summing for  $i = 1, 2, \dots, n$  we obtain

$$\begin{aligned} \frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} &\geq 2(a_1 + a_2 + \dots + a_n) - (b_1 + b_2 + \dots + b_n) \\ &= a_1 + a_2 + \dots + a_n, \end{aligned}$$

and we are done.

Equality occurs if and only if  $a_i = b_i, i = 1, 2, \dots, n$ . ■

**11** Let  $x \in \mathbb{R}^+$ . Find the minimum value of the expression  $\frac{x^2+1}{x+1}$ .

*Solution* Denote  $A = \frac{x^2+1}{x+1}$ .

We have

$$A = \frac{x^2 - 1 + 2}{x + 1} = (x - 1) + \frac{2}{x + 1} = \left( (x + 1) + \frac{2}{x + 1} \right) - 2. \quad (1)$$

For any  $a, b \geq 0$  we have  $a + b \geq 2\sqrt{ab}$  (equality occurs iff  $a = b$ ).

Now from (1) we get  $A \geq 2\sqrt{2} - 2$ .

Equality occurs if and only if  $x = \sqrt{2} - 1$ . ■

**12** Let  $a, b, c \in \mathbb{R}^+$  such that  $abc = 1$ . Prove the inequality

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4}.$$

*Solution* After expanding we get

$$ab + ac + bc + a + b + c \geq 3(abc + 1)$$

i.e.

$$ab + ac + bc + a + b + c \geq 6.$$

Since

$$\begin{aligned} ab + ac + bc + a + b + c &= \frac{1}{c} + \frac{1}{b} + \frac{1}{a} + a + b + c \\ &= \left( \frac{1}{a} + a \right) + \left( \frac{1}{b} + b \right) + \left( \frac{1}{c} + c \right) \geq 2 + 2 + 2 = 6, \end{aligned}$$

we are done.

Equality occurs if and only if  $\frac{1}{a} + a = \frac{1}{b} + b = \frac{1}{c} + c = 1$ , i.e.  $a = b = c = 1$ . ■

**13** Let  $x, y \geq 0$  be real numbers such that  $y(y+1) \leq (x+1)^2$ . Prove the inequality

$$y(y-1) \leq x^2.$$

**Solution** If  $0 \leq y \leq 1$ , then  $y(y-1) \leq 0 \leq x^2$ .

Suppose that  $y > 1$ .

If  $x + \frac{1}{2} \leq y$ , then

$$y(y-1) = y(y+1) - 2y \leq (x+1)^2 - 2\left(x + \frac{1}{2}\right) = x^2.$$

If  $x + \frac{1}{2} > y$  then we have  $x > y - \frac{1}{2} > 0$ , i.e.

$$x^2 > \left(y - \frac{1}{2}\right)^2 = y(y-1) + \frac{1}{4} > y(y-1). \quad \blacksquare$$

**14** Let  $x, y \in \mathbb{R}^+$  such that  $x^3 + y^3 \leq x - y$ . Prove that

$$x^2 + y^2 \leq 1.$$

**Solution** From  $x^3 + y^3 \leq x - y$  we have

$$0 \leq y \leq x$$

and

$$0 \leq x^3 \leq x^3 + y^3 \leq x - y \leq x,$$

i.e.

$$x^3 \leq x,$$

from where we deduce that  $x \leq 1$ .

Thus  $0 \leq y \leq x \leq 1$ .

Now we have  $x(x+y) \leq 1 \cdot 2 = 2$  and  $xy(x+y) \leq 2y$ .

From  $x^3 + y^3 \leq x - y$  we obtain

$$\begin{aligned} (x+y)(x^2 - xy + y^2) &\leq x - y \Leftrightarrow x^2 - xy + y^2 \leq \frac{x-y}{x+y} \\ \Leftrightarrow x^2 + y^2 &\leq \frac{x-y}{x+y} + xy = \frac{x-y+xy(x+y)}{x+y} \leq \frac{x-y+2y}{x+y} = \frac{x+y}{x+y} = 1. \quad \blacksquare \end{aligned}$$

**15** Let  $a, b, x, y \in \mathbb{R}$  such that  $ay - bx = 1$ . Prove that

$$a^2 + b^2 + x^2 + y^2 + ax + by \geq \sqrt{3}.$$

**Solution** Let us denote  $u = a^2 + b^2$ ,  $v = x^2 + y^2$  and  $w = ax + by$ .

Then

$$\begin{aligned} uv &= (a^2 + b^2)(x^2 + y^2) = a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2 \\ &= a^2x^2 + b^2y^2 + 2axby + a^2y^2 + b^2x^2 - 2axby \\ &= (ax + by)^2 + (ay - bx)^2 = w^2 + 1. \end{aligned}$$

From the obvious inequality  $(t\sqrt{3} + 1)^2 \geq 0$  we deduce

$$3t^2 + 1 \geq -2t\sqrt{3},$$

i.e.

$$4t^2 + 4 \geq 3 - 2t\sqrt{3} + t^2,$$

i.e.

$$4t^2 + 4 \geq (\sqrt{3} - t)^2. \quad (1)$$

Now we have

$$(u + v)^2 \geq 4uv = 4(w^2 + 1) \stackrel{(1)}{\geq} (\sqrt{3} - w)^2,$$

from which we get  $u + v \geq \sqrt{3} - w$ , which is equivalent to  $u + v + w \geq \sqrt{3}$ . ■

**16** Let  $a, b, c, d$  be non-negative real numbers such that  $a^2 + b^2 + c^2 + d^2 = 1$ . Prove the inequality

$$(1 - a)(1 - b)(1 - c)(1 - d) \geq abcd.$$

*Solution* We have  $2cd \leq c^2 + d^2 = 1 - a^2 - b^2$ .

Hence

$$2(1 - a)(1 - b) - 2cd \geq 2(1 - a)(1 - b) - 1 + a^2 + b^2 = (1 - a - b)^2 \geq 0,$$

i.e.

$$(1 - a)(1 - b) \geq cd. \quad (1)$$

Similarly we get

$$(1 - c)(1 - d) \geq ab. \quad (2)$$

After multiplying (1) and (2) we obtain  $(1 - a)(1 - b)(1 - c)(1 - d) \geq abcd$ , as required. Equality occurs iff  $a = b = c = d = 1/2$  or  $a = 1, b = c = d = 0$  (up to permutation). ■

**17** Let  $x, y$  be non-negative real numbers. Prove the inequality

$$4(x^9 + y^9) \geq (x^2 + y^2)(x^3 + y^3)(x^4 + y^4).$$

*Solution* Since the given inequality is symmetric we may assume that  $x \geq y \geq 0$ .

Let  $a, b \in \mathbb{N}$ . Then we have  $x^a \geq y^a$  and  $x^b \geq y^b$ .

Hence

$$\begin{aligned} (x^a - y^a)(x^b - y^b) &\geq 0 \\ \Leftrightarrow x^{a+b} + y^{a+b} &\geq x^a y^b + x^b y^a \\ \Leftrightarrow 2(x^{a+b} + y^{a+b}) &\geq (x^a + y^a)(x^b + y^b). \end{aligned} \quad (1)$$

For  $a = 2, b = 3$  in (1) we get

$$2(x^5 + y^5) \geq (x^2 + y^2)(x^3 + y^3). \quad (2)$$

For  $a = 5, b = 4$  in (1) we get

$$2(x^9 + y^9) \geq (x^5 + y^5)(x^4 + y^4). \quad (3)$$

From (2) and (3) we get

$$4(x^9 + y^9) = 2 \cdot 2(x^9 + y^9) \geq 2(x^5 + y^5)(x^4 + y^4) \geq (x^2 + y^2)(x^3 + y^3)(x^4 + y^4),$$

and we are done. ■

**18** Let  $x, y, z \in \mathbb{R}^+$  such that  $xyz = 1$  and  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq x + y + z$ . Prove that for any natural number  $n$  the inequality

$$\frac{1}{x^n} + \frac{1}{y^n} + \frac{1}{z^n} \geq x^n + y^n + z^n$$

is true.

*Solution* After setting  $x = \frac{a}{b}, y = \frac{b}{c}$  and  $z = \frac{c}{a}$ , the initial condition

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq x + y + z$$

becomes

$$\begin{aligned} \frac{b}{a} + \frac{c}{b} + \frac{a}{c} &\geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \\ \Leftrightarrow a^2b + b^2c + c^2a &\geq ab^2 + bc^2 + ca^2 \\ \Leftrightarrow (a-b)(b-c)(c-a) &\leq 0. \end{aligned}$$

Let  $n \in \mathbb{N}$ , and take  $A = a^n, B = b^n, C = c^n$ .

Then  $a \geq b \Leftrightarrow A \geq B$  and  $a \leq b \Leftrightarrow A \leq B$ , etc.

So we have

$$\begin{aligned} (A-B)(B-C)(C-A) &\leq 0 \\ \Leftrightarrow \frac{B}{A} + \frac{C}{B} + \frac{A}{C} &\geq \frac{A}{B} + \frac{B}{C} + \frac{C}{A} \\ \Leftrightarrow \frac{1}{x^n} + \frac{1}{y^n} + \frac{1}{z^n} &\geq x^n + y^n + z^n. \end{aligned} \quad \blacksquare$$

**19** Let  $x, y, z$  be real numbers different from 1, such that  $xyz = 1$ . Prove the inequality

$$\left(\frac{3-x}{1-x}\right)^2 + \left(\frac{3-y}{1-y}\right)^2 + \left(\frac{3-z}{1-z}\right)^2 > 7.$$

**Solution** Denote  $A = \left(\frac{3-x}{1-x}\right)^2 + \left(\frac{3-y}{1-y}\right)^2 + \left(\frac{3-z}{1-z}\right)^2 - 7$ .

We have

$$A = \left(1 + \frac{2}{1-x}\right)^2 + \left(1 + \frac{2}{1-y}\right)^2 + \left(1 + \frac{2}{1-z}\right)^2 - 7.$$

Let  $\frac{1}{1-x} = a$ ,  $\frac{1}{1-y} = b$ ,  $\frac{1}{1-z} = c$ .

Then  $A = (1 + 2a)^2 + (1 + 2b)^2 + (1 + 2c)^2 - 7$ , i.e.

$$A = 4a^2 + 4b^2 + 4c^2 + 4a + 4b + 4c - 4. \quad (1)$$

Furthermore, the condition  $xyz = 1$  is equivalent to  $abc = (a-1)(b-1)(c-1)$ , i.e.

$$a + b + c - 1 = ab + bc + ca. \quad (2)$$

Using (1) and (2) we get

$$A = 4a^2 + 4b^2 + 4c^2 + 4(ab + bc + ca) = 2((a+b)^2 + (b+c)^2 + (c+a)^2),$$

i.e.  $A \geq 0$ .

Equality occurs if and only if  $a = b = c = 0$ , which is clearly impossible.

So we have strict inequality, i.e.  $A > 0$ , i.e.

$$\left(\frac{3-x}{1-x}\right)^2 + \left(\frac{3-y}{1-y}\right)^2 + \left(\frac{3-z}{1-z}\right)^2 - 7 > 0$$

and we are done. ■

**20** Let  $x, y, z \leq 1$  be real numbers such that  $x + y + z = 1$ . Prove the inequality

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \leq \frac{27}{10}.$$

**Solution** We'll prove that for every  $t \leq 1$  we have  $\frac{1}{1+t^2} \leq \frac{27}{50}(2-t)$ .

The last inequality is equivalent to  $(4-3t)(1-3t)^2 \geq 0$ , which is clearly true.

Hence

$$\begin{aligned} \frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} &\leq \frac{27}{50}((2-x) + (2-y) + (2-z)) \\ &= \frac{27}{50}(6 - (x+y+z)) = \frac{27}{10}. \end{aligned} \quad \blacksquare$$

**21** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}.$$

*Solution* We can easily check the following identities

$$\frac{1+abc}{a(1+b)} = \frac{1+a}{a(1+b)} + \frac{b(1+c)}{1+b} - 1, \quad \frac{1+abc}{b(1+c)} = \frac{1+b}{b(1+c)} + \frac{c(1+a)}{1+c} - 1$$

and

$$\frac{1+abc}{a(1+b)} = \frac{1+c}{c(1+a)} + \frac{a(1+b)}{1+a} - 1.$$

Adding these identities we obtain

$$\begin{aligned} & \frac{1+abc}{a(1+b)} + \frac{1+abc}{b(1+c)} + \frac{1+abc}{c(1+a)} \\ &= \left( \frac{1+a}{a(1+b)} + \frac{a(1+b)}{1+a} \right) + \left( \frac{1+b}{b(1+c)} + \frac{b(1+c)}{1+b} \right) \\ & \quad + \left( \frac{1+c}{c(1+a)} + \frac{c(1+a)}{1+c} \right) - 3 \geq 2 + 2 + 2 - 3 = 3, \end{aligned}$$

i.e.

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}.$$

Equality occurs if and only if  $a = b = c = 1$ . ■

**22** Let  $x, y, z$  be positive real numbers. Prove the inequality

$$9(a+b)(b+c)(c+a) \geq 8(a+b+c)(ab+bc+ca).$$

*Solution* The given inequality is equivalent to  $a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \geq 0$ , which is obviously true. Equality occurs iff  $a = b = c$ . ■

**23** Let  $a, b, c$  be real numbers. Prove the inequality

$$(a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^3a).$$

*Solution* By the well-known inequality  $(x+y+z)^2 \geq 3(xy+yz+zx)$  for

$$x = a^2 + bc - ab, \quad y = b^2 + ca - bc, \quad z = c^2 + ab - ca,$$

we obtain the required inequality. ■

**24** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$a^3(b+c) + b^3(c+a) + c^3(a+b) \leq 6.$$

*Solution* We'll show that

$$a^3(b+c) + b^3(c+a) + c^3(a+b) \leq \frac{2}{3}(a^2+b^2+c^2)^2. \quad (1)$$

Inequality (1) is equivalent to

$$\begin{aligned} 2(a^4+b^4+c^4) + 4(a^2b^2+b^2c^2+c^2a^2) \\ \geq 3ab(a^2+b^2) + 3bc(b^2+c^2) + 3ca(c^2+a^2). \end{aligned} \quad (2)$$

We have

$$a^4+b^4+4a^2b^2 \geq 3ab(a^2+b^2) \Leftrightarrow (a-b)^4 + ab(a-b)^2 \geq 0,$$

which is clearly true.

Analogously we get

$$b^4+c^4+4b^2c^2 \geq 3bc(b^2+c^2) \quad \text{and} \quad c^4+a^4+4c^2a^2 \geq 3ca(c^2+a^2).$$

Adding the last inequalities we get (2), i.e. (1).

Finally using  $a^2+b^2+c^2=3$  we obtain the required result.

Equality holds if and only if  $a=b=c$ . ■

**25** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} > 2.$$

*Solution* We'll show that  $\sqrt{\frac{x}{y+z}} \geq \frac{2x}{x+y+z}$ , for every  $x, y, z \in \mathbb{R}^+$ .

We have

$$\begin{aligned} \sqrt{\frac{x}{y+z}} \geq \frac{2x}{x+y+z} &\Leftrightarrow \frac{x}{y+z} \geq \left(\frac{2x}{x+y+z}\right)^2 \\ \Leftrightarrow (x+y+z)^2 \geq 4x(y+z) &\Leftrightarrow (y+z-x)^2 \geq 0, \end{aligned}$$

with equality iff  $x = y + z$ .

Now we easily obtain

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \geq \frac{2(a+b+c)}{a+b+c} = 2,$$

with equality if and only if  $a=b+c, b=a+c, c=a+b$ , i.e.  $a=b=c=0$ , which is impossible.

So we have strict inequality, i.e.  $\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} > 2$ , as required. ■

**26** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} \leq 1.$$

*Solution* The given inequality is equivalent to

$$ab^2 + bc^2 + ca^2 \leq 2 + abc.$$

We may assume that  $a \geq b \geq c$  (since the inequality is cyclic we must also consider the case  $c \geq b \geq a$ , which is analogous).

Then we have  $a(b-a)(b-c) \leq 0$  from which we have  $a^2b + abc \geq ab^2 + ca^2$ .  
Thus

$$ab^2 + bc^2 + ca^2 \leq a^2b + abc + bc^2.$$

We'll show that

$$a^2b + bc^2 \leq 2.$$

We have

$$a^2b + bc^2 \leq 2 \Leftrightarrow b(3 - b^2) \leq 2 \Leftrightarrow (b-1)^2(b+2) \geq 0,$$

which is clearly true, and we are done.

Equality occurs iff  $a = b = c = 1$  or  $a = 0, b = 1, c = \sqrt{2}$  (over all permutations). ■

**27** Let  $x, y, z$  be distinct non-negative real numbers. Prove the inequality

$$\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} \geq \frac{4}{xy + yz + zx}.$$

*Solution* If  $a, b > 0$ , then  $\frac{1}{(a-b)^2} + \frac{1}{a^2} + \frac{1}{b^2} \geq \frac{4}{ab}$ .

The last inequality is true since

$$\frac{1}{(a-b)^2} + \frac{1}{a^2} + \frac{1}{b^2} - \frac{4}{ab} = \frac{(a^2 + b^2 - 3ab)^2}{a^2b^2(a-b)^2}.$$

Without loss of generality we may assume that  $z = \min\{x, y, z\}$ .

By the previous inequality for  $a = x - z$  and  $b = y - z$  we get

$$\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} \geq \frac{4}{(x-z)(y-z)}.$$

So it suffices to show that

$$\frac{4}{(x-z)(y-z)} \geq \frac{4}{xy + yz + zx},$$



i.e.

$$xy + yz + zx \geq (x - z)(y - z),$$

i.e.

$$2z(y + x) \geq z^2,$$

which is true since  $z = \min\{x, y, z\}$ . ■

**28** Let  $a, b, c$  be non-negative real numbers. Prove the inequality

$$3(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq 1 + abc + (abc)^2.$$

*Solution* Since

$$2(a^2 - a + 1)(b^2 - b + 1) = 1 + a^2b^2 + (a - b)^2 + (1 - a)^2(1 - b)^2$$

we deduce that

$$2(a^2 - a + 1)(b^2 - b + 1) \geq 1 + a^2b^2.$$

It follows that

$$3(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq \frac{3}{2}(1 + a^2b^2)(c^2 - c + 1),$$

and it remains to prove that

$$3(1 + a^2b^2)(c^2 - c + 1) \geq 2(1 + abc + (abc)^2),$$

which is equivalent to the following quadratic in  $c$

$$(3 + a^2b^2)c^2 - (3 + 2ab + 3a^2b^2)c + 1 + 3a^2b^2 \geq 0,$$

and clearly the last inequality is true, since  $3 + a^2b^2 > 0$  and  $D = -3(1 - ab)^4 \leq 0$ .

Equality occurs iff  $a = b = c = 1$ . ■

**29** Let  $a, b \in \mathbb{R}, a \neq 0$ . Prove the inequality

$$a^2 + b^2 + \frac{1}{a^2} + \frac{b}{a} \geq \sqrt{3}.$$

*Solution* We have

$$a^2 + b^2 + \frac{1}{a^2} + \frac{b}{a} = \left(b + \frac{1}{2a}\right)^2 + a^2 + \frac{3}{4a^2}. \quad (1)$$

Since  $(b + \frac{1}{2a})^2 \geq 0$ , using (1) we get

$$a^2 + b^2 + \frac{1}{a^2} + \frac{b}{a} \geq a^2 + \frac{3}{4a^2}. \quad (2)$$

Using  $AM \geq GM$  we have

$$a^2 + \frac{3}{4a^2} \geq 2\sqrt{a^2 \frac{3}{4a^2}} = \sqrt{3}. \quad (3)$$

From (2) and (3) we get

$$a^2 + b^2 + \frac{1}{a^2} + \frac{b}{a} \geq \sqrt{3}.$$

Equality occurs iff  $b + \frac{1}{2a} = 0$  and  $a^2 = \frac{3}{4a^2}$ , i.e.  $a = \pm\sqrt[4]{\frac{3}{4}}$  and  $b = \mp\frac{1}{2}\sqrt[4]{\frac{4}{3}}$ . ■

**30** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq 3.$$

*Solution* For each  $x \in \mathbb{R}$  we have  $x^2 + 1 \geq 2x$ .

So we have

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq \frac{2a}{b + c} + \frac{2b}{c + a} + \frac{2c}{a + b}.$$

It's enough to prove that  $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$ , which is *Nesbitt's inequality*.

Equality occurs if and only if  $a = b = c = 1$ . ■

**31** Let  $x, y, z$  be positive real numbers such that  $xy + yz + zx = 5$ . Prove the inequality

$$3x^2 + 3y^2 + z^2 \geq 10.$$

*Solution* Using the inequality  $AM \geq GM$  we obtain

$$4x^2 + z^2 \geq 4xz, \quad 4y^2 + z^2 \geq 4yz \quad \text{and} \quad 2x^2 + 2y^2 \geq 4xy.$$

Adding these inequalities and using  $xy + yz + zx = 5$  we get the required inequality.

Equality occurs iff  $x = y = 1, z = 2$ . ■

**32** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca > a + b + c$ . Prove the inequality

$$a + b + c > 3.$$

*Solution* We have

$$\begin{aligned} (a + b + c)^2 &= a^2 + b^2 + c^2 + 2(ab + ac + bc) \\ &\geq ab + ac + bc + 2(ab + ac + bc) \\ &= 3(ab + ac + bc) > 3(a + b + c), \end{aligned}$$

from which we get  $a + b + c > 3$ . ■

**33** Let  $a, b$  be real numbers such that  $9a^2 + 8ab + 7b^2 \leq 6$ . Prove that

$$7a + 5b + 12ab \leq 9.$$

*Solution* By the inequality  $AM \geq GM$  we have

$$\begin{aligned} 7a + 5b + 12ab &\leq 7\left(a^2 + \frac{1}{4}\right) + 5\left(a^2 + \frac{1}{4}\right) + 12ab \\ &= 7a^2 + 5b^2 + 12ab + 3 \\ &= 9a^2 + 8ab + 7b^2 - 2a^2 + 4ab - 2b^2 + 3 \\ &= 9a^2 + 8ab + 7b^2 - 2(a-b)^2 + 3 \leq 6 + 3 = 9, \end{aligned}$$

as required. Equality holds iff  $a = b = 1/2$ . ■

**34** Let  $x, y, z \in \mathbb{R}^+$ , such that  $xyz \geq xy + yz + zx$ . Prove the inequality

$$xyz \geq 3(x + y + z).$$

*Solution* Letting  $\frac{1}{x} = a, \frac{1}{y} = b, \frac{1}{z} = c$ , the initial condition  $xyz \geq xy + yz + zx$  becomes

$$a + b + c \leq 1. \tag{1}$$

We need to show that

$$xyz \geq 3(x + y + z) \Leftrightarrow 3(ab + bc + ca) \leq 1. \tag{2}$$

Clearly

$$(a + b + c)^2 \geq 3(ab + bc + ca). \tag{3}$$

Now from (1) and (3) we obtain (2). ■

**35** Let  $a, b, c \in \mathbb{R}^+$  with  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq 3.$$

*Solution* The given inequality is equivalent to

$$\begin{aligned} \left(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\right)^2 &\geq 9 \\ \Leftrightarrow \frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} + 2(a^2 + b^2 + c^2) &\geq 3(a^2 + b^2 + c^2), \end{aligned}$$

i.e.

$$\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} \geq a^2 + b^2 + c^2.$$

Furthermore, applying  $AM \geq GM$  we get

$$\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} \geq 2b^2, \quad \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} \geq 2c^2, \quad \frac{a^2b^2}{c^2} + \frac{c^2a^2}{b^2} \geq 2a^2.$$

After adding these inequalities we obtain

$$\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} \geq a^2 + b^2 + c^2$$

and we are done. ■

**36** Let  $a, b, c$  be positive real numbers such that  $a + b + c = \sqrt{abc}$ . Prove the inequality

$$ab + bc + ca \geq 9(a + b + c).$$

*Solution* By the inequality  $AM \geq GM$  we have

$$\sqrt{abc} = a + b + c \geq 3\sqrt[3]{abc},$$

which implies

$$abc \geq 3^6 \quad \text{and} \quad a + b + c = \sqrt{abc} \geq \sqrt{3^6} = 3^3. \quad (1)$$

Once more, the inequality  $AM \geq GM$  gives us

$$ab + bc + ca \geq 3\sqrt[3]{(abc)^2},$$

i.e.

$$(ab + bc + ca)^3 \geq 3^3(abc)^2 = 3^3(a + b + c)^4 \stackrel{(1)}{\geq} 3^6(a + b + c)^3.$$

Hence

$$ab + bc + ca \geq 9(a + b + c),$$

as required.

Equality occurs if and only if  $a = b = c = 9$ . ■

**37** Let  $a, b, c$  be positive real numbers such that  $abc \geq 1$ . Prove the inequality

$$\left(a + \frac{1}{a+1}\right)\left(b + \frac{1}{b+1}\right)\left(c + \frac{1}{c+1}\right) \geq \frac{27}{8}.$$

*Solution* By the inequality  $AM \geq GM$  we have

$$\frac{a+1}{4} + \frac{1}{a+1} \geq 2\sqrt{\frac{a+1}{4} \cdot \frac{1}{a+1}} = 1 \quad \text{and} \quad \frac{3a}{4} + \frac{3}{4} \geq 2\sqrt{\frac{3a}{4} \cdot \frac{3}{4}} = \frac{3}{2}\sqrt{a}.$$

Adding these two inequalities we get

$$a + \frac{1}{a+1} \geq \frac{3}{2}\sqrt{a}.$$

Analogously we obtain

$$b + \frac{1}{b+1} \geq \frac{3}{2}\sqrt{b} \quad \text{and} \quad c + \frac{1}{c+1} \geq \frac{3}{2}\sqrt{c}.$$

Multiplying the last three inequalities gives us

$$\left(a + \frac{1}{a+1}\right)\left(b + \frac{1}{b+1}\right)\left(c + \frac{1}{c+1}\right) \geq \frac{27}{8}\sqrt{abc} \geq \frac{27}{8},$$

as required.

Equality occurs iff  $a = b = c = 1$ . ■

**38** Let  $a, b, c, d \in \mathbb{R}^+$  such that  $a^2 + b^2 + c^2 + d^2 = 4$ . Prove the inequality

$$a + b + c + d \geq ab + bc + cd + da.$$

*Solution* We have

$$a + b + c + d \geq ab + bc + cd + da \quad \Leftrightarrow \quad a + b + c + d \geq (a + c)(b + d),$$

i.e.

$$\frac{1}{a+c} + \frac{1}{b+d} \geq 1.$$

Since  $AM \geq HM$  we have

$$\frac{1}{a+c} + \frac{1}{b+d} \geq \frac{4}{a+b+c+d}. \tag{1}$$

Applying  $QM \geq AM$  we have

$$\frac{a+b+c+d}{4} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}} = 1,$$

i.e.

$$a + b + c + d \leq 4.$$

Now by (1) we get

$$\frac{1}{a+c} + \frac{1}{b+d} \geq \frac{4}{a+b+c+d} \geq \frac{4}{4} = 1.$$

Equality holds if and only if  $a = b = c = d = 1$ . ■

**39** Let  $a, b, c \in (-3, 3)$  such that  $\frac{1}{3+a} + \frac{1}{3+b} + \frac{1}{3+c} = \frac{1}{3-a} + \frac{1}{3-b} + \frac{1}{3-c}$ .  
Prove the inequality

$$\frac{1}{3+a} + \frac{1}{3+b} + \frac{1}{3+c} \geq 1.$$

*Solution* By the inequality  $AM \geq HM$  we have

$$((3+a) + (3+b) + (3+c)) \left( \frac{1}{3+a} + \frac{1}{3+b} + \frac{1}{3+c} \right) \geq 9 \quad (1)$$

and

$$\begin{aligned} & ((3-a) + (3-b) + (3-c)) \left( \frac{1}{3-a} + \frac{1}{3-b} + \frac{1}{3-c} \right) \geq 9 \\ \Leftrightarrow & ((3-a) + (3-b) + (3-c)) \left( \frac{1}{3+a} + \frac{1}{3+b} + \frac{1}{3+c} \right) \geq 9. \quad (2) \end{aligned}$$

After adding (1) and (2) we obtain

$$18 \left( \frac{1}{3+a} + \frac{1}{3+b} + \frac{1}{3+c} \right) \geq 18, \quad \text{i.e.} \quad \frac{1}{3+a} + \frac{1}{3+b} + \frac{1}{3+c} \geq 1. \quad \blacksquare$$

**40** Let  $a, b, c \in \mathbb{R}^+$  such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{1}{a+bc+abc} + \frac{1}{b+ca+bca} + \frac{1}{c+ab+cab} \geq 1.$$

*Solution* By  $AM \geq HM$  we have:

$$\begin{aligned} & \frac{1}{a+bc+abc} + \frac{1}{b+ca+bca} + \frac{1}{c+ab+cab} \\ & \geq \frac{9}{a+b+c+ab+bc+ca+3abc}. \end{aligned} \quad (1)$$

Using the well known inequalities:

$$a^2 + b^2 + c^2 \geq ab + bc + ac \quad \text{and} \quad (a+b+c)^2 \geq 3(a^2 + b^2 + c^2)$$

and according to  $a^2 + b^2 + c^2 = 3$ , we deduce

$$ab + bc + ca \leq 3 \quad \text{and} \quad a + b + c \leq 3. \quad (2)$$

By  $AM \geq GM$  we have  $a^2 + b^2 + c^2 \geq 3\sqrt[3]{(abc)^2}$  and since  $a^2 + b^2 + c^2 = 3$  we easily deduce that

$$abc \leq 1 \quad (3)$$

Now according to (1), (2) and (3) we obtain

$$\begin{aligned} & \frac{1}{a+bc+abc} + \frac{1}{b+ca+bca} + \frac{1}{c+ab+cab} \\ & \geq \frac{9}{a+b+c+ab+bc+ca+3abc} \geq \frac{9}{3+3+3} = 1 \end{aligned}$$

Equality occurs if and only if  $a = b = c = 1$ . ■

**41** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 3$ . Prove the inequality.

$$\frac{a^2b^2 + a^2 + b^2}{ab + 1} + \frac{b^2c^2 + b^2 + c^2}{bc + 1} + \frac{c^2a^2 + c^2 + a^2}{ca + 1} \geq \frac{9}{2}.$$

*Solution* Let  $a, b \in \mathbb{R}^+$  then we have

$$\begin{aligned} & (a-1)^2(b-1)^2 \geq 0 \\ \Leftrightarrow & a^2b^2 - 2a^2b + a^2 - 2ab^2 + 4ab - 2a + b^2 - 2b + 1 \geq 0 \\ \Leftrightarrow & a^2b^2 + a^2 + b^2 \geq 2a^2b + 2ab^2 + 2a + 2b - 4ab - 1 \\ \Leftrightarrow & a^2b^2 + a^2 + b^2 \geq 2a(ab+1) + 2b(ab+1) - 4(ab+1) + 3 \\ & = (ab+1)(2a+2b-4) + 3. \end{aligned}$$

Hence

$$\frac{a^2b^2 + a^2 + b^2}{ab + 1} \geq 2a + 2b - 4 + \frac{3}{ab + 1}. \quad (1)$$

Similarly we obtain

$$\frac{b^2c^2 + b^2 + c^2}{bc + 1} \geq 2b + 2c - 4 + \frac{3}{bc + 1} \quad (2)$$

and

$$\frac{c^2a^2 + c^2 + a^2}{ca + 1} \geq 2c + 2a - 4 + \frac{3}{ca + 1}. \quad (3)$$

Adding (1), (2) and (3) gives us

$$\begin{aligned} & \frac{a^2b^2 + a^2 + b^2}{ab + 1} + \frac{b^2c^2 + b^2 + c^2}{bc + 1} + \frac{c^2a^2 + c^2 + a^2}{ca + 1} \\ & \geq 4(a + b + c) - 12 + \frac{3}{ab + 1} + \frac{3}{bc + 1} + \frac{3}{ca + 1} \\ & = \frac{3}{ab + 1} + \frac{3}{bc + 1} + \frac{3}{ca + 1}. \end{aligned} \quad (4)$$

Applying  $AM \geq HM$  we obtain

$$\frac{1}{1 + ab} + \frac{1}{1 + bc} + \frac{1}{1 + ca} \geq \frac{9}{3 + ab + bc + ca}. \quad (5)$$

Using the well known inequality  $(a + b + c)^2 \geq 3(ab + bc + ca)$  and  $a + b + c = 3$  we deduce

$$ab + bc + ca \leq 3. \quad (6)$$

Finally by (4), (5) and (6) we obtain

$$\begin{aligned} & \frac{a^2b^2 + a^2 + b^2}{ab + 1} + \frac{b^2c^2 + b^2 + c^2}{bc + 1} + \frac{c^2a^2 + c^2 + a^2}{ca + 1} \\ & \geq \frac{3}{ab + 1} + \frac{3}{bc + 1} + \frac{3}{ca + 1} \geq \frac{27}{3 + ab + bc + ca} \geq \frac{27}{3 + 3} = \frac{9}{2}. \end{aligned}$$

Equality occurs iff  $a = b = c = 1$ . ■

**42** Let  $a, b, c, d$  be positive real numbers such that  $a^2 + b^2 + c^2 + d^2 = 4$ . Prove the inequality

$$\frac{a^2 + b^2 + 3}{a + b} + \frac{b^2 + c^2 + 3}{b + c} + \frac{c^2 + d^2 + 3}{c + d} + \frac{d^2 + a^2 + 3}{d + a} \geq 10.$$

*Solution* Observe that for any real numbers  $x, y$  we have

$$x^2 + xy + y^2 = \left(x + \frac{y}{2}\right)^2 + \frac{3y^2}{4} \geq 0,$$

equality achieves if and only if  $x = y = 0$ .

Hence  $(a - 1)^2 + (a - 1)(b - 1) + (b - 1)^2 \geq 0$ , which is equivalent to

$$a^2 + b^2 + ab - 3a - 3b + 3 \geq 0,$$

from which we obtain

$$a^2 + b^2 + 3 \geq 3a + 3b - ab,$$



i.e.

$$\frac{a^2 + b^2 + 3}{a + b} \geq 3 - \frac{ab}{a + b}.$$

By  $AM \geq GM$  we easily deduce that

$$\frac{a + b}{4} \geq \frac{ab}{a + b}.$$

Therefore by previous inequality we get

$$\frac{a^2 + b^2 + 3}{a + b} \geq 3 - \frac{a + b}{4}.$$

Similarly we obtain

$$\begin{aligned} \frac{b^2 + c^2 + 3}{b + c} &\geq 3 - \frac{b + c}{4}, & \frac{c^2 + d^2 + 3}{c + d} &\geq 3 - \frac{c + d}{4} \quad \text{and} \\ \frac{d^2 + a^2 + 3}{d + a} &\geq 3 - \frac{d + a}{4}. \end{aligned}$$

Adding the last four inequality yields

$$\frac{a^2 + b^2 + 3}{a + b} + \frac{b^2 + c^2 + 3}{b + c} + \frac{c^2 + d^2 + 3}{c + d} + \frac{d^2 + a^2 + 3}{d + a} \geq 12 - \frac{a + b + c + d}{2}. \quad (1)$$

According to inequality  $QM \geq AM$  we deduce that

$$\sqrt{\frac{a^2 + b^2 + c^2 + d^2}{4}} \geq \frac{a + b + c + d}{4}$$

and since  $a^2 + b^2 + c^2 + d^2 = 4$  we obtain

$$a + b + c + d \leq 4. \quad (2)$$

By (1) and (2) we get

$$\begin{aligned} \frac{a^2 + b^2 + 3}{a + b} + \frac{b^2 + c^2 + 3}{b + c} + \frac{c^2 + d^2 + 3}{c + d} + \frac{d^2 + a^2 + 3}{d + a} &\geq 12 - \frac{a + b + c + d}{2} \\ &\geq 12 - \frac{4}{2} = 10, \end{aligned}$$

as required.

Equality occurs if and only if  $a = b = c = d = 1$ . ■

**43** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{1}{ab(a + b)} + \frac{1}{bc(b + c)} + \frac{1}{ca(c + a)} \geq \frac{9}{2(a^3 + b^3 + c^3)}.$$

**Solution** According to the obvious inequality  $(a+b)(a-b)^2 \geq 0$  we get the inequality

$$a^3 + b^3 \geq ab(a+b).$$

Thus

$$\frac{1}{ab(a+b)} \geq \frac{1}{a^3 + b^3}.$$

Similarly we get

$$\frac{1}{bc(b+c)} \geq \frac{1}{b^3 + c^3} \quad \text{and} \quad \frac{1}{ca(c+a)} \geq \frac{1}{c^3 + a^3}.$$

After adding the last three inequalities we obtain

$$\frac{1}{ab(a+b)} + \frac{1}{bc(b+c)} + \frac{1}{ca(c+a)} \geq \frac{1}{a^3 + b^3} + \frac{1}{b^3 + c^3} + \frac{1}{c^3 + a^3}. \quad (1)$$

Now since  $AM \geq HM$  we have

$$\begin{aligned} \frac{1}{a^3 + b^3} + \frac{1}{b^3 + c^3} + \frac{1}{c^3 + a^3} &\geq \frac{9}{(a^3 + b^3) + (b^3 + c^3) + (c^3 + a^3)} \\ &= \frac{9}{2(a^3 + b^3 + c^3)}. \end{aligned} \quad (2)$$

From (1) and (2) we get the required inequality.

Equality holds if and only if  $a = b = c$ . ■

**44** Let  $a, b, c \in \mathbb{R}^+$  such that  $a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \geq 1$ . Prove the inequality

$$a + b + c \geq \sqrt{3}.$$

**Solution** We have

$$\begin{aligned} 1 &\leq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \leq a\frac{b+c}{2} + b\frac{c+a}{2} + c\frac{a+b}{2} \\ &= ab + ac + bc \leq \frac{(a+b+c)^2}{3}, \end{aligned}$$

i.e.

$$(a+b+c)^2 \geq 3 \quad \Leftrightarrow \quad a+b+c \geq \sqrt{3}. \quad \blacksquare$$

**45** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{c}} \geq \sqrt{a} + \sqrt{b} + \sqrt{c} + 3.$$

*Solution* By  $AM \geq GM$  we get

$$\begin{aligned}
 & \frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{c}} \\
 & \geq 2\sqrt{\frac{bc}{a}} + 2\sqrt{\frac{ca}{b}} + 2\sqrt{\frac{ab}{c}} \\
 & = \left(\sqrt{\frac{bc}{a}} + \sqrt{\frac{ca}{b}}\right) + \left(\sqrt{\frac{ca}{b}} + \sqrt{\frac{ab}{c}}\right) + \left(\sqrt{\frac{ab}{c}} + \sqrt{\frac{bc}{a}}\right) \\
 & \geq 2(\sqrt{a} + \sqrt{b} + \sqrt{c}) \geq \sqrt{a} + \sqrt{b} + \sqrt{c} + 3\sqrt[3]{abc} \\
 & = \sqrt{a} + \sqrt{b} + \sqrt{c} + 3.
 \end{aligned}$$

■

**46** Let  $x, y, z$  be positive real numbers such that  $x + y + z = 4$ . Prove the inequality

$$\frac{1}{2xy + xz + yz} + \frac{1}{xy + 2xz + yz} + \frac{1}{xy + xz + 2yz} \leq \frac{1}{xyz}.$$

*Solution* By  $AM \geq HM$  we have that  $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$ , for any  $a, b \in \mathbb{R}^+$ .

Therefore

$$\begin{aligned}
 \frac{1}{2xy + xz + yz} &= \frac{1}{(xy + xz) + (xy + yz)} \leq \frac{1}{4} \left( \frac{1}{xy + xz} + \frac{1}{xy + yz} \right) \\
 &\leq \frac{1}{4} \left( \frac{1}{4} \left( \frac{1}{xy} + \frac{1}{xz} \right) + \frac{1}{4} \left( \frac{1}{xy} + \frac{1}{yz} \right) \right) \\
 &= \frac{1}{16} \left( \frac{2}{xy} + \frac{1}{xz} + \frac{1}{yz} \right) = \frac{2z + y + x}{16xyz}.
 \end{aligned}$$

Similarly,

$$\frac{1}{xy + 2xz + yz} \leq \frac{z + 2y + x}{16xyz} \quad \text{and} \quad \frac{1}{xy + xz + 2yz} \leq \frac{z + y + 2x}{16xyz}.$$

Adding the three inequalities yields that

$$\frac{1}{2xy + xz + yz} + \frac{1}{xy + 2xz + yz} + \frac{1}{xy + xz + 2yz} \leq \frac{1}{16} \left( \frac{4(x + y + z)}{xyz} \right) = \frac{1}{xyz}.$$

Equality occurs iff  $x = y = z = 4/3$ .

■

**47** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$abc \geq (a + b - c)(b + c - a)(c + a - b).$$

*Solution* Setting  $a + b - c = x$ ,  $b + c - a = y$ ,  $c + a - b = z$  the inequality becomes

$$(x + y)(y + z)(z + x) \geq 8xyz.$$

Let us assume that  $x \leq 0$ . Then  $c \geq a + b$ , and clearly  $y$  and  $z$  are positive and the right-hand side of the given inequality is negative or zero, but the left-hand side is positive, i.e. the inequality holds.

So we may assume that  $x, y, z > 0$ . Then using  $AM \geq GM$  we get

$$(x + y)(y + z)(z + x) \geq 2\sqrt{xy} \cdot 2\sqrt{yz} \cdot 2\sqrt{xz} = 8xyz$$

and we are done. ■

**48** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$abc + \frac{12}{ab + bc + ac} \geq 5.$$

*Solution* Recalling the well-known inequality  $abc \geq (b + c - a)(c + a - b)(a + b - c)$  (Problem 47) we obtain

$$\begin{aligned} abc &\geq (3 - 2a)(3 - 2b)(3 - 2c) \\ \Leftrightarrow abc &\geq 27 - 18(a + b + c) + 12(ab + bc + ca) - 8abc \\ \Leftrightarrow 3abc &\geq 4(ab + bc + ca) - 9 \\ \Leftrightarrow abc &\geq \frac{4(ab + bc + ca)}{3} - 3. \end{aligned}$$

Therefore we have

$$abc + \frac{12}{ab + bc + ac} \geq \frac{4(ab + bc + ca)}{3} + \frac{12}{ab + bc + ac} - 3 \geq 8 - 3 = 5,$$

where the last inequality follows since  $AM \geq GM$ . ■

**49** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

*Solution* Since  $abc = 1$ , it is natural to take  $a = \frac{x}{y}$ ,  $b = \frac{y}{z}$ ,  $c = \frac{z}{x}$  where  $x, y, z > 0$ . Now the given inequality becomes

$$\begin{aligned} \left(\frac{x}{y} - 1 + \frac{z}{y}\right)\left(\frac{y}{z} - 1 + \frac{x}{z}\right)\left(\frac{z}{x} - 1 + \frac{y}{x}\right) &\leq 1 \quad \text{i.e.} \\ (x + y - z)(z + x - y)(y + z - x) &\leq xyz, \end{aligned}$$

which is true (Problem 47). Equality occurs iff  $a = b = c = 1$ . ■

**50** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

*Solution* Let  $x = a + b + c$  and  $y = ab + ac + bc$ .

Clearly  $x, y \geq 3$  (these are immediate consequences of  $AM \geq GM$ ).

Now the given inequality is equivalent to

$$\frac{3+4x+y+x^2}{2x+y+x^2+xy} \leq \frac{12+4x+y}{9+4x+2y},$$

i.e.

$$3x^2y + xy^2 + 6xy - 5x^2 - y^2 - 24x - 3y - 27 \geq 0,$$

i.e.

$$(3x^2y - 5x^2 - 12x) + (xy^2 - y^2 - 3x - 3y) + (6xy - 9x - 27) \geq 0,$$

which is true since  $x, y \geq 3$ . ■

**51** Let  $a, b, c > 0$ . Prove the inequality

$$(a+b)^2 + (a+b+4c)^2 \geq \frac{100abc}{a+b+c}.$$

*Solution* Since  $AM \geq GM$  we have

$$\begin{aligned} (a+b)^2 + (a+b+4c)^2 &= (a+b)^2 + (a+2c+b+2c)^2 \\ &\geq 4ab + (2\sqrt{2ac} + 2\sqrt{2bc})^2, \quad \text{i.e.} \end{aligned}$$

$$(a+b)^2 + (a+b+4c)^2 \geq 4ab + 8ac + 8bc + 16c\sqrt{ab}.$$

Now

$$\begin{aligned} &\frac{(a+b)^2 + (a+b+4c)^2}{abc} (a+b+c) \\ &\geq \frac{4ab + 8ac + 8bc + 16c\sqrt{ab}}{abc} (a+b+c) \\ &= \left( \frac{4}{c} + \frac{8}{b} + \frac{8}{a} + \frac{16}{\sqrt{ab}} \right) (a+b+c) \\ &= 8 \left( \frac{1}{2c} + \frac{1}{b} + \frac{1}{a} + \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{ab}} \right) \left( \frac{a}{2} + \frac{a}{2} + \frac{b}{2} + \frac{b}{2} + c \right). \end{aligned}$$

Using the last inequality and  $AM \geq GM$  once more we obtain

$$\frac{(a+b)^2 + (a+b+4c)^2}{abc} (a+b+c) \geq 8 \cdot 5 \sqrt[5]{\frac{1}{2a^2b^2c}} \cdot 5 \sqrt[5]{\frac{a^2b^2c}{16}} = 100,$$

i.e.

$$(a+b)^2 + (a+b+4c)^2 \geq \frac{100abc}{a+b+c}.$$

Equality occurs if and only if  $a = b = 2c$ . ■

**52** Let  $a, b, c > 0$  such that  $abc = 1$ . Prove the inequality

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ac}{1+a} \geq 3.$$

*Solution* Since  $abc = 1$  we have

$$\frac{1+ab}{1+a} = \frac{abc+ab}{1+a} = \frac{ab(c+1)}{a+1}$$

and similarly

$$\frac{1+bc}{1+b} = \frac{bc(a+1)}{b+1} \quad \text{and} \quad \frac{1+ca}{1+c} = \frac{ca(b+1)}{c+1}.$$

Now by  $AM \geq GM$  we obtain

$$\begin{aligned} \frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ca}{1+c} &= \frac{ab(c+1)}{a+1} + \frac{bc(a+1)}{b+1} + \frac{ca(b+1)}{c+1} \\ &\geq 3\sqrt[3]{\frac{ab(c+1)}{a+1} \cdot \frac{bc(a+1)}{b+1} \cdot \frac{ca(b+1)}{c+1}} \\ &= 3\sqrt[3]{(abc)^2} = 3. \end{aligned}$$

Equality occurs iff  $a = b = c = 1$ . ■

**53** Let  $a, b, c$  be real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \geq 16.$$

*Solution 1* We have

$$\begin{aligned} &\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \\ &= a^2 + b^2 + c^2 + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + 2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \\ &= a^2 + \frac{1}{a^2} + b^2 + \frac{1}{b^2} + c^2 + \frac{1}{c^2} + 2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \end{aligned}$$

$$\begin{aligned}
&= a^2 + \frac{ab+bc+ca}{a^2} + b^2 + \frac{ab+bc+ca}{b^2} + c^2 + \frac{ab+bc+ca}{c^2} \\
&\quad + 2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \\
&= (a^2 + b^2 + c^2) + \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) + 3\left(\frac{c}{a} + \frac{a}{b} + \frac{b}{c}\right) + \left(\frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2}\right) \\
&\geq ab + bc + ca + 3 + 9 + 3 = 1 + 3 + 9 + 3 = 16.
\end{aligned}$$

Clearly, equality occurs iff  $a = b = c = 1/\sqrt{3}$ . ■

*Solution 2* By well-known inequality  $x^2 + y^2 + z^2 \geq xy + yz + zx$  we have

$$\begin{aligned}
\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 &\geq \left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right) + \left(b + \frac{1}{c}\right)\left(c + \frac{1}{a}\right) \\
&\quad + \left(c + \frac{1}{a}\right)\left(a + \frac{1}{b}\right),
\end{aligned}$$

i.e.

$$\begin{aligned}
\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 &\geq ab + bc + ca + \frac{a}{c} + \frac{b}{a} + \frac{c}{b} + 3 \\
&\quad + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}. \tag{1}
\end{aligned}$$

Using  $AM \geq GM$  and  $AM \geq HM$  we get  $\frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq 3\sqrt[3]{\frac{a}{c} \cdot \frac{b}{a} \cdot \frac{c}{b}} = 3$  and  $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{9}{ab+bc+ca} = \frac{9}{1} = 9$ , respectively.

By last two inequalities and (1) we obtain

$$\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \geq 1 + 3 + 3 + 9 = 16,$$

as required. ■

*Solution 3* By  $QM \geq AM$  we have

$$\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \geq \frac{(a+b+c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}{3}. \tag{1}$$

By well-known  $(a+b+c)^2 \geq 3(ab+bc+ca)$  and  $ab+bc+ca=1$  we obtain

$$a+b+c \geq \sqrt{3}. \tag{2}$$

According to  $AM \geq GM$  we have

$$1 = ab + bc + ca \geq 3\sqrt[3]{(abc)^2},$$

i.e.

$$\frac{1}{\sqrt[3]{abc}} \geq \sqrt{3}.$$

By  $AM \geq GM$  and previous inequality we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{3}{\sqrt[3]{abc}} \geq 3\sqrt{3}. \quad (3)$$

Finally by (1), (2) and (3) we get

$$\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \geq \frac{(\sqrt{3} + 3\sqrt{3})^2}{3} = 16,$$

as required. Equality occurs iff  $a = b = c = 1/\sqrt{3}$ . ■

**54** Let  $a, b, c$  be positive real numbers such that  $abc \geq 1$ . Prove the inequality

$$a + b + c \geq \frac{1+a}{1+b} + \frac{1+b}{1+c} + \frac{1+c}{1+a}.$$

*Solution* We have

$$\begin{aligned} & a + b + c - \frac{1+a}{1+b} - \frac{1+b}{1+c} - \frac{1+c}{1+a} \\ &= \left(1 + a - \frac{1+a}{1+b}\right) + \left(1 + b - \frac{1+b}{1+c}\right) + \left(1 + c - \frac{1+c}{1+a}\right) - 3 \\ &= (1+a)\left(1 - \frac{1}{1+b}\right) + (1+b)\left(1 - \frac{1}{1+c}\right) + (1+c)\left(1 - \frac{1}{1+a}\right) - 3 \\ &= \frac{(1+a)b}{1+b} + \frac{(1+b)c}{1+c} + \frac{(1+c)a}{1+a} - 3 \\ &\geq 3\sqrt[3]{\frac{(1+a)b}{1+b} \cdot \frac{(1+b)c}{1+c} \cdot \frac{(1+c)a}{1+a}} - 3 \\ &= 3\sqrt[3]{abc} - 3 \geq 0 \quad (abc \geq 1). \end{aligned}$$

Equality occurs iff  $a = b = c = 1$ . ■

**55** Let  $a, b \in \mathbb{R}^+$ . Prove the inequality

$$\left(a^2 + b + \frac{3}{4}\right)\left(b^2 + a + \frac{3}{4}\right) \geq \left(2a + \frac{1}{2}\right)\left(2b + \frac{1}{2}\right).$$



**Solution** For any  $x \in \mathbb{R}$  we have  $x^2 + \frac{1}{4} \geq x$ .

So it follows that

$$\begin{aligned} \left(a^2 + b + \frac{3}{4}\right)\left(b^2 + a + \frac{3}{4}\right) &\geq \left(a + b + \frac{1}{2}\right)\left(a + b + \frac{1}{2}\right) = \left(a + b + \frac{1}{2}\right)^2 \\ &= \left(\frac{2a + 2b + 1}{2}\right)^2 = \left(\frac{2a + \frac{1}{2} + 2b + \frac{1}{2}}{2}\right)^2 \\ &\stackrel{A \geq G}{\geq} \left(2a + \frac{1}{2}\right)\left(2b + \frac{1}{2}\right). \quad \blacksquare \end{aligned}$$

**56** Let  $a, b, c \in \mathbb{R}^+$  such that  $abc = 1$ . Prove the inequality

$$\frac{a}{a^2 + 2} + \frac{b}{b^2 + 2} + \frac{c}{c^2 + 2} \leq 1.$$

**Solution** By the well-known inequality  $x^2 + 1 \geq 2x$ ,  $\forall x \in \mathbb{R}$ , we have

$$\begin{aligned} \frac{a}{a^2 + 2} + \frac{b}{b^2 + 2} + \frac{c}{c^2 + 2} &= \frac{a}{a^2 + 1 + 1} + \frac{b}{b^2 + 1 + 1} + \frac{c}{c^2 + 1 + 1} \\ &\leq \frac{a}{2a + 1} + \frac{b}{2b + 1} + \frac{c}{2c + 1} \\ &= \frac{1}{2 + \frac{1}{a}} + \frac{1}{2 + \frac{1}{b}} + \frac{1}{2 + \frac{1}{c}} = A. \end{aligned}$$

The inequality  $A \leq 1$  is equivalent to

$$\begin{aligned} \left(2 + \frac{1}{b}\right)\left(2 + \frac{1}{c}\right) + \left(2 + \frac{1}{a}\right)\left(2 + \frac{1}{c}\right) + \left(2 + \frac{1}{a}\right)\left(2 + \frac{1}{b}\right) \\ \leq \left(2 + \frac{1}{a}\right)\left(2 + \frac{1}{b}\right)\left(2 + \frac{1}{c}\right), \end{aligned}$$

i.e.

$$4 \leq \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} + \frac{1}{abc}, \quad \text{i.e.} \quad 3 \leq \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc},$$

which is true since  $\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} \geq 3\sqrt[3]{\left(\frac{1}{abc}\right)^3} = 3$ .

Equality occurs iff  $a = b = c = 1$ . ■

**57** Let  $x, y, z > 0$  be real numbers such that  $x + y + z = xyz$ . Prove the inequality

$$(x - 1)(y - 1)(z - 1) \leq 6\sqrt{3} - 10.$$

**Solution** Since  $x < xyz$  we have  $yz > 1$  and analogously  $xz > 1$  and  $xy > 1$ . At most one of  $x, y, z$  can be less than 1.

Let  $x \leq 1, y \geq 1, z \geq 1$ . Then we have  $(x-1)(y-1)(z-1) \leq 0$ , so the given inequality holds.

So it's enough to consider the case when  $x \geq 1, y \geq 1, z \geq 1$ .

Let  $x-1=a, y-1=b, z-1=c$ .

Then  $a, b, c$  are non-negative and since  $x=a+1, y=b+1, z=c+1$  we obtain

$$\begin{aligned} a+1+b+1+c+1 &= (a+1)(b+1)(c+1), \quad \text{i.e.} \\ abc+ab+bc+ca &= 2. \end{aligned} \quad (1)$$

Let  $x = \sqrt[3]{abc}$ , so we have

$$ab+bc+ca \geq 3\sqrt[3]{(abc)^2} = 3x^2. \quad (2)$$

Combine (1) and (2) we have

$$x^3 + 3x^2 \leq abc + ab + bc + ca = 2 \Leftrightarrow (x+1)(x^2 + 2x - 2) \leq 0,$$

so we must have  $x^2 + 2x - 2 \leq 0$  and we easily deduce that  $x \leq \sqrt{3} - 1$ , i.e. we get

$$x^3 \leq (\sqrt{3} - 1)^3 = 6\sqrt{3} - 10$$

and we are done. ■

**58** Let  $a, b, c \in (1, 2)$  be real numbers. Prove the inequality

$$\frac{b\sqrt{a}}{4b\sqrt{c} - c\sqrt{a}} + \frac{c\sqrt{b}}{4c\sqrt{a} - a\sqrt{b}} + \frac{a\sqrt{c}}{4a\sqrt{b} - b\sqrt{c}} \geq 1.$$

*Solution* Since  $a, b, c \in (1, 2)$  we have

$$4b\sqrt{c} - c\sqrt{a} > 4\sqrt{c} - 2\sqrt{c} = 2\sqrt{c} > 0.$$

Analogously we get  $4c\sqrt{a} - a\sqrt{b} > 0$  and  $4a\sqrt{b} - b\sqrt{c} > 0$ .

We'll prove that

$$\frac{b\sqrt{a}}{4b\sqrt{c} - c\sqrt{a}} \geq \frac{a}{a+b+c}. \quad (1)$$

Since  $4b\sqrt{c} - c\sqrt{a} > 0$  inequality (1) is

$$\begin{aligned} b(a+b+c) &\geq \sqrt{a}(4b\sqrt{c} - c\sqrt{a}) \\ \Leftrightarrow (a+b)(b+c) &\geq 4b\sqrt{ac}, \end{aligned}$$

which is clearly true ( $AM \geq GM$ ).

Similarly we deduce that

$$\frac{c\sqrt{b}}{4c\sqrt{a} - a\sqrt{b}} \geq \frac{b}{a+b+c} \quad (2)$$

and

$$\frac{a\sqrt{c}}{4a\sqrt{b} - b\sqrt{c}} \geq \frac{c}{a+b+c}. \quad (3)$$

Adding (1), (2) and (3) we get the required result. ■

**59** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 3$ . Prove the inequality

$$\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)} \geq 3\sqrt{2abc}.$$

*Solution* We have

$$\sqrt{ab+ac} \geq \frac{\sqrt{2}}{2}(\sqrt{ab} + \sqrt{ac}).$$

Analogously

$$\sqrt{bc+ba} \geq \frac{\sqrt{2}}{2}(\sqrt{bc} + \sqrt{ba}) \quad \text{and} \quad \sqrt{ca+cb} \geq \frac{\sqrt{2}}{2}(\sqrt{ca} + \sqrt{cb}).$$

So it suffices to show that

$$\sqrt{2}(\sqrt{ab} + \sqrt{ac} + \sqrt{bc}) \geq 3\sqrt{2abc},$$

i.e.

$$\sqrt{ab} + \sqrt{ac} + \sqrt{bc} \geq 3\sqrt{abc}. \quad (1)$$

By  $AM \geq GM$  we have

$$\sqrt{ab} + \sqrt{ac} + \sqrt{bc} \geq 3\sqrt[3]{abc} \geq 3\sqrt{abc}$$

where the last inequality is true since

$$\sqrt[3]{abc} \leq \frac{a+b+c}{3} = 1, \quad \text{i.e.} \quad abc \leq 1. \quad \blacksquare$$

**60** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\sqrt{a+bc} + \sqrt{b+ca} + \sqrt{c+ab} \leq 2.$$

*Solution* Since  $a + b + c = 1$  we have  $a + bc = a(a + b + c) + bc = (a + b)(a + c)$   
i.e.

$$\sqrt{a+bc} = \sqrt{(a+b)(a+c)} \leq \frac{(a+b) + (a+c)}{2} = \frac{2a+b+c}{2}.$$

Similarly we obtain

$$\sqrt{b+ca} \leq \frac{2b+c+a}{2} \quad \text{and} \quad \sqrt{c+ab} \leq \frac{2c+a+b}{2}.$$

After adding the last three inequalities, we obtain

$$\begin{aligned}\sqrt{a+bc} + \sqrt{b+ca} + \sqrt{c+ab} &\leq \frac{2a+b+c}{2} + \frac{2b+c+a}{2} + \frac{2c+a+b}{2} \\ &= 2(a+b+c) = 2.\end{aligned}$$

Equality occurs iff  $a = b = c = 1/3$ . ■

**61** Let  $a, b, c$  be positive real numbers such that  $a + b + c + 1 = 4abc$ . Prove that

$$\frac{b^2 + c^2}{a} + \frac{c^2 + a^2}{b} + \frac{a^2 + b^2}{c} \geq 2(ab + bc + ca).$$

*Solution* By the well-known inequalities:

$$x^2 + y^2 \geq 2xy \quad \text{and} \quad 3(x^2 + y^2 + z^2) \geq (x + y + z)^2,$$

we obtain

$$\begin{aligned}\frac{b^2 + c^2}{a} + \frac{c^2 + a^2}{b} + \frac{a^2 + b^2}{c} \\ \geq \frac{2bc}{a} + \frac{2ca}{b} + \frac{2ab}{c} = \frac{2((bc)^2 + (ca)^2 + (ab)^2)}{abc} \geq \frac{2(bc + ca + ab)^2}{3abc}. \quad (1)\end{aligned}$$

We have

$$(ab + bc + ca)^2 \geq 3((ab)(bc) + (bc)(ca) + (ca)(ab)) = 3abc(a + b + c),$$

i.e.

$$ab + bc + ca \geq \sqrt{3abc(a + b + c)}. \quad (2)$$

Also

$$4abc = a + b + c + 1 \geq 4\sqrt[4]{abc}$$

i.e.

$$abc \geq 1. \quad (3)$$

Therefore

$$a + b + c = 4abc - 1 = 3abc + abc - 1 \stackrel{(3)}{\geq} 3abc. \quad (4)$$

By (1), (2) and (4) we obtain

$$\begin{aligned}\frac{b^2 + c^2}{a} + \frac{c^2 + a^2}{b} + \frac{a^2 + b^2}{c} \\ \geq \frac{2(bc + ca + ab)^2}{3abc} \geq \frac{2(ab + bc + ca)\sqrt{3abc(a + b + c)}}{3abc} \\ \geq \frac{2(ab + bc + ca)\sqrt{(3abc)^2}}{3abc} = 2(ab + bc + ca).\end{aligned}$$
■

**62** Let  $a, b, c \in (-1, 1)$  be real numbers such that  $ab + bc + ac = 1$ . Prove the inequality

$$6\sqrt[3]{(1-a^2)(1-b^2)(1-c^2)} \leq 1 + (a+b+c)^2.$$

*Solution* Since  $a, b, c \in (-1, 1)$  we have  $1 - a^2, 1 - b^2, 1 - c^2 > 0$ .

By  $AM \geq GM$  we get

$$\begin{aligned} 6\sqrt[3]{(1-a^2)(1-b^2)(1-c^2)} &= 2 \cdot 3\sqrt[3]{(1-a^2)(1-b^2)(1-c^2)} \\ &\leq 2(1-a^2 + 1-b^2 + 1-c^2) \\ &= 2(3 - (a^2 + b^2 + c^2)) \\ &= 6 - 2(a^2 + b^2 + c^2). \end{aligned}$$

We'll show that

$$6 - 2(a^2 + b^2 + c^2) \leq 1 + (a+b+c)^2.$$

This inequality is equivalent to

$$6 - 2(a^2 + b^2 + c^2) \leq 1 + a^2 + b^2 + c^2 + 2$$

i.e.

$$3 \leq 3(a^2 + b^2 + c^2)$$

i.e.

$$a^2 + b^2 + c^2 \geq 1,$$

which is true since  $a^2 + b^2 + c^2 \geq ab + bc + ac = 1$ .

Equality holds iff  $a = b = c = \pm \frac{1}{\sqrt{3}}$ . ■

**63** Let  $a, b, c, d$  be positive real numbers such that  $a^2 + b^2 + c^2 + d^2 = 1$ . Prove the inequality

$$\sqrt{1-a} + \sqrt{1-b} + \sqrt{1-c} + \sqrt{1-d} \geq \sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}.$$

*Solution* First we'll show that

$$a + b + c + d \leq 2. \tag{1}$$

We have

$$\frac{a+b+c+d}{4} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}} = \frac{1}{2}$$

i.e.

$$a + b + c + d \leq 2.$$

Furthermore

$$\sqrt{1-a} - \sqrt{a} = \frac{(\sqrt{1-a} - \sqrt{a})(\sqrt{1-a} + \sqrt{a})}{\sqrt{1-a} + \sqrt{a}} = \frac{1-2a}{\sqrt{1-a} + \sqrt{a}}. \quad (2)$$

By  $AM \leq QM$  we have

$$\frac{\sqrt{1-a} + \sqrt{a}}{2} \leq \sqrt{\frac{1-a+a}{2}} = \frac{1}{\sqrt{2}}, \quad \text{i.e.} \quad \frac{1}{\sqrt{1-a} + \sqrt{a}} \geq \frac{1}{\sqrt{2}}. \quad (3)$$

Using (2) and (3) we deduce

$$\sqrt{1-a} - \sqrt{a} \geq \frac{1-2a}{\sqrt{2}}.$$

Similarly

$$\begin{aligned} \sqrt{1-b} - \sqrt{b} &\geq \frac{1-2b}{\sqrt{2}}, & \sqrt{1-c} - \sqrt{c} &\geq \frac{1-2c}{\sqrt{2}} \quad \text{and} \\ \sqrt{1-d} - \sqrt{d} &\geq \frac{1-2d}{\sqrt{2}}. \end{aligned}$$

So it follows that

$$\begin{aligned} &\sqrt{1-a} - \sqrt{a} + \sqrt{1-b} - \sqrt{b} + \sqrt{1-c} - \sqrt{c} + \sqrt{1-d} - \sqrt{d} \\ &\geq \frac{4 - 2(a+b+c+d)}{\sqrt{2}} \stackrel{(1)}{\geq} 0, \end{aligned}$$

as required. ■

**64** Let  $x, y, z$  be positive real numbers such that  $xyz = 1$ . Prove the inequality

$$\frac{1}{(x+1)^2 + y^2 + 1} + \frac{1}{(y+1)^2 + z^2 + 1} + \frac{1}{(z+1)^2 + x^2 + 1} \leq \frac{1}{2}.$$

*Solution* We have

$$\frac{1}{(x+1)^2 + y^2 + 1} = \frac{1}{2 + x^2 + y^2 + 2x} \leq \frac{1}{2(1+x+xy)}.$$

Similarly

$$\frac{1}{(y+1)^2 + z^2 + 1} \leq \frac{1}{2(1+y+yz)} \quad \text{and} \quad \frac{1}{(z+1)^2 + x^2 + 1} \leq \frac{1}{2(1+z+zx)}.$$

So we have

$$\begin{aligned} & \frac{1}{(x+1)^2 + y^2 + 1} + \frac{1}{(y+1)^2 + z^2 + 1} + \frac{1}{(z+1)^2 + x^2 + 1} \\ & \leq \frac{1}{2} \left( \frac{1}{1+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx} \right). \end{aligned}$$

We'll show that

$$\frac{1}{1+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx} = 1,$$

from which we'll deduce the required result.

We have

$$\begin{aligned} & \frac{1}{1+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx} \\ & = \frac{xyz}{xyz+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx} \\ & = \frac{yz}{yz+1+y} + \frac{1}{1+y+yz} + \frac{y}{y+yz+1} \\ & = \frac{1+y+yz}{1+y+yz} = 1, \end{aligned}$$

as required. ■

**65** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\sqrt{\frac{a^3}{a^3 + (b+c)^3}} + \sqrt{\frac{a^3}{a^3 + (b+c)^3}} + \sqrt{\frac{a^3}{a^3 + (b+c)^3}} \geq 1.$$

*Solution* We'll prove that for any  $x, y, z \in \mathbb{R}^+$  we have

$$\sqrt{\frac{x^3}{x^3 + (y+z)^3}} \geq \frac{x^2}{x^2 + y^2 + z^2}. \quad (1)$$

We have

$$\begin{aligned} & \sqrt{\frac{x^3}{x^3 + (y+z)^3}} \geq \frac{x^2}{x^2 + y^2 + z^2} \\ \Leftrightarrow & \frac{x^3}{x^3 + (y+z)^3} \geq \frac{x^4}{(x^2 + y^2 + z^2)^2} \\ \Leftrightarrow & 2x^2(y^2 + z^2) + (y^2 + z^2)^2 \geq x(y+z)^3. \quad (2) \end{aligned}$$

By  $AM \leq QM$  we have

$$2(y^2 + z^2) \geq (y + z)^2,$$

i.e.

$$8(y^2 + z^2)^3 \geq (y + z)^6.$$

Using  $AM \geq GM$  and the previous result we get

$$2x^2(y^2 + z^2) + (y^2 + z^2)^2 \geq 2\sqrt{2x^2(y^2 + z^2)^3} \geq 2\sqrt{\frac{2x^2(y + z)^6}{8}} = x(y + z)^3,$$

so we prove (2), i.e. (1).

By (1) we have

$$\begin{aligned} & \sqrt{\frac{a^3}{a^3 + (b + c)^3}} + \sqrt{\frac{a^3}{a^3 + (b + c)^3}} + \sqrt{\frac{a^3}{a^3 + (b + c)^3}} \\ & \geq \frac{a^2}{a^2 + b^2 + c^2} + \frac{b^2}{a^2 + b^2 + c^2} + \frac{c^2}{a^2 + b^2 + c^2} = 1. \quad \blacksquare \end{aligned}$$

**66** Let  $x, y, z \in \mathbb{R}^+$ . Prove the inequality

$$(x + y + z)^2(xy + yz + zx)^2 \leq 3(x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2).$$

*Solution* We have

$$x^2 + xy + y^2 = \frac{3}{4}(x + y)^2 + \frac{1}{4}(x - y)^2 \geq \frac{3}{4}(x + y)^2,$$

similarly

$$y^2 + yz + z^2 \geq \frac{3}{4}(y + z)^2 \quad \text{and} \quad z^2 + zx + x^2 \geq \frac{3}{4}(z + x)^2.$$

Hence

$$\begin{aligned} 3(x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2) & \geq 3\left(\frac{3}{4}\right)^3 (x + y)^2(y + z)^2(z + x)^2 \\ & = \frac{81}{64}((x + y)(y + z)(z + x))^2. \end{aligned}$$

We'll show that

$$\frac{81}{64}((x + y)(y + z)(z + x))^2 \geq (x + y + z)^2(xy + yz + zx)^2,$$

i.e.

$$\frac{9}{8}(x + y)(y + z)(z + x) \geq (x + y + z)(xy + yz + zx),$$



i.e.

$$9(x+y)(y+z)(z+x) \geq 8(x+y+z)(xy+yz+zx), \quad (1)$$

from which we'll obtain the desired inequality.

Let's note that

$$(x+y)(y+z)(z+x) = (x+y+z)(xy+yz+zx) - xyz.$$

Now by (1) we get

$$9(x+y)(y+z)(z+x) \geq 8((x+y)(y+z)(z+x) + xyz),$$

i.e.

$$(x+y)(y+z)(z+x) \geq 8xyz,$$

which is clearly true since

$$x+y \geq 2\sqrt{xy}, \quad y+z \geq 2\sqrt{yz}, \quad z+x \geq 2\sqrt{zx}.$$

Equality occurs if and only if  $x = y = z$ . ■

**67** Let  $a, b, c$  be real numbers such that  $a + b + c = 3$ . Prove the inequality

$$2(a^2b^2 + b^2c^2 + c^2a^2) + 3 \leq 3(a^2 + b^2 + c^2).$$

*Solution* Without loss of generality we may assume  $a \geq b \geq c$ .

Let's denote  $u = \frac{a+b}{2}$  and  $v = \frac{a-b}{2}$ .

We easily obtain  $a = u + v$  and  $b = u - v$ .

We have  $ab = u^2 - v^2 \geq c^2$  which implies  $2u^2 - 2c^2 - v^2 \geq 0$ .

Now we have

$$\begin{aligned} a^2b^2 + b^2c^2 + c^2a^2 &= c^2(a^2 + b^2) + a^2b^2 = c^2(2u^2 + 2v^2) + (u^2 - v^2)^2 \\ &= -v^2(2u^2 - 2c^2 - v^2) + u^4 + 2c^2u^2 \leq u^4 + 2c^2u^2. \end{aligned} \quad (1)$$

Also

$$a^2 + b^2 + c^2 = 2u^2 + 2v^2 + c^2 \geq 2u^2 + c^2. \quad (2)$$

We'll show that

$$2(u^4 + 2c^2u^2) + 3 \leq 3(2u^2 + c^2). \quad (3)$$

From  $a + b + c = 3$  we have  $c = 3 - 2u$ .

Now inequality (3) is equivalent to

$$\begin{aligned} 2u^4 + 4(3 - 2u)^2u^2 + 3 &\leq 6u^2 + 3(3 - 2u)^2 \\ \Leftrightarrow 3u^4 - 8u^3 + 3u^2 + 6u - 4 &\leq 0 \quad \Leftrightarrow (u - 1)^2(3u^2 - 2u - 4) \leq 0. \end{aligned}$$

Since  $2u \leq 3$  we easily deduce that  $3u^2 - 2u - 4 \leq 0$ . So inequality (3) holds.

Combining (1), (2) and (3) we obtain the required result.

Equality holds if and only if  $a = b = c = 1$ . ■

**68** Let  $a, b, c, d$  be positive real numbers. Prove the inequality

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \geq 0.$$

*Solution* Applying  $AM \geq HM$  we have

$$\begin{aligned} & \frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \\ &= \frac{a+c}{b+c} + \frac{b+d}{c+d} + \frac{c+a}{d+a} + \frac{d+b}{a+b} - 4 \\ &= (a+c) \left( \frac{1}{b+c} + \frac{1}{d+a} \right) + (b+d) \left( \frac{1}{c+d} + \frac{1}{a+b} \right) - 4 \\ &\geq \frac{4(a+c)}{a+b+c+d} + \frac{4(b+d)}{a+b+c+d} - 4 = 0. \end{aligned}$$
■

**69** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 1$ . Prove the inequality

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{9}{4}.$$

*Solution* We'll use the following well known inequalities:

For any  $a, b, c > 0$  we have  $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$  (Nesbit's) and for any  $x, y, z \geq 0$  we have

$$x^2 + y^2 + z^2 \geq \frac{(x+y+z)^2}{3}.$$

Now we obtain

$$\begin{aligned} \frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} &= \frac{a(a+b+c)}{(b+c)^2} + \frac{b(a+b+c)}{(c+a)^2} + \frac{c(a+b+c)}{(a+b)^2} \\ &= \left( \frac{a}{b+c} \right)^2 + \left( \frac{b}{c+a} \right)^2 + \left( \frac{c}{a+b} \right)^2 + \frac{a}{b+c} \\ &\quad + \frac{b}{c+a} + \frac{c}{a+b}. \end{aligned}$$

Using previous well-known inequalities we have

$$\begin{aligned}
 & \frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \\
 & \geq \frac{1}{3} \left( \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right)^2 + \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \\
 & \geq \frac{1}{3} \left( \frac{3}{2} \right)^2 + \frac{3}{2} = \frac{9}{4}.
 \end{aligned}$$

■

**70** Let  $a, b, c \in \mathbb{R}^+$  such that  $abc = 1$ . Prove the inequality

$$\frac{a^3c}{(b+c)(c+a)} + \frac{b^3a}{(c+a)(a+b)} + \frac{c^3b}{(a+b)(b+c)} \geq \frac{3}{4}.$$

*Solution* Clearing denominators gives us

$$\begin{aligned}
 & 4(a^4c + b^4a + c^4a + a^3cb + b^3ac + c^3ba) \\
 & \geq 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b + 2abc),
 \end{aligned}$$

i.e.

$$4(a^4c + b^4a + c^4a + a^2 + b^2 + c^2) \geq 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b + 2).$$

By  $AM \geq GM$  and  $abc = 1$  we have

$$\begin{aligned}
 & 4(a^4c + b^4a + c^4a + a^2 + b^2 + c^2) \\
 & = (a^4c + a^2 + b^4a) + (b^4a + b^2 + c^4b) + (c^4b + c^2 + a^4c) + (a^4c + a^2 + c^2) \\
 & \quad + (b^4a + b^2 + a^2) + (c^4b + c^2 + b^2) + (a^4c + b^4a + c^4b) + (a^2 + b^2 + c^2) \\
 & \geq 3\sqrt[3]{a^6b^3} + 3\sqrt[3]{b^6c^3} + 3\sqrt[3]{c^6a^3} + 3\sqrt[3]{a^6c^3} + 3\sqrt[3]{c^6b^3} + 3\sqrt[3]{a^5b^5c^5} \\
 & \quad + 3\sqrt[3]{a^2b^2c^2} \\
 & = 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b + 2),
 \end{aligned}$$

and we are done. ■

**71** Let  $a, b, c > 0$  be real numbers such that  $abc = 1$ . Prove that

$$(a+b)(b+c)(c+a) \geq 4(a+b+c-1).$$

*Solution* Using the identity

$$(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - 1,$$

the given inequality becomes

$$ab + bc + ca + \frac{3}{a + b + c} \geq 4.$$

By  $AM \geq GM$  we have

$$ab + bc + ca + \frac{3}{a + b + c} = \frac{3(ab + bc + ca)}{3} + \frac{3}{a + b + c} \geq 4\sqrt[4]{\frac{(ab + bc + ca)^3}{9(a + b + c)}}.$$

So it's enough to show that

$$(ab + bc + ca)^3 \geq 9(a + b + c). \quad (1)$$

By  $AM \geq GM$  and  $abc = 1$  we get

$$ab + bc + ca \geq 3\sqrt[3]{(abc)^2} = 3. \quad (2)$$

Furthermore, since  $(x + y + z)^2 \geq 3(xy + yz + zx)$ , we deduce

$$(ab + bc + ca)^2 \geq 3((ab)(bc) + (bc)(ca) + (ca)(ab)) = 3(a + b + c). \quad (3)$$

By (2) and (3) we obtain  $(ab + bc + ca)^3 \geq 9(a + b + c)$ , i.e. (1) is true. ■

**72** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$1 + \frac{3}{a + b + c} \geq \frac{6}{ab + bc + ca}.$$

*Solution* Let  $x = \frac{1}{a}$ ,  $y = \frac{1}{b}$ ,  $z = \frac{1}{c}$ . Then clearly  $xyz = 1$ .

The given inequality becomes

$$1 + \frac{3}{xy + yz + zx} \geq \frac{6}{x + y + z}.$$

Using the well-known inequality  $(x + y + z)^2 \geq 3(xy + yz + zx)$  we deduce

$$1 + \frac{3}{xy + yz + zx} \geq 1 + \frac{9}{(x + y + z)^2}.$$

So it's enough to prove that

$$1 + \frac{9}{(x + y + z)^2} \geq \frac{6}{x + y + z}.$$

The last inequality is equivalent to  $(1 - \frac{3}{x+y+z})^2 \geq 0$ , and clearly holds. ■

**73** Let  $x, y, z$  be positive real numbers such that  $x^2 + y^2 + z^2 = xyz$ . Prove the following inequalities:

- $1^\circ \quad xyz \geq 27$   
 $2^\circ \quad xy + yz + zx \geq 27$   
 $3^\circ \quad x + y + z \geq 9$   
 $4^\circ \quad xy + yz + zx \geq 2(x + y + z) + 9.$

*Solution*

$1^\circ$  Using  $AM \geq GM$  we get

$$xyz = x^2 + y^2 + z^2 \geq 3\sqrt[3]{(xyz)^2}, \quad \text{i.e.} \quad (xyz)^3 \geq 27(xyz)^2,$$

which implies

$$xyz \geq 27.$$

$2^\circ$  By  $AM \geq GM$  we get  $xy + yz + zx \geq 3\sqrt[3]{(xyz)^2} \geq 3\sqrt[3]{27^2} = 27.$

$3^\circ$  By  $AM \geq GM$  and  $1^\circ$  we get  $x + y + z \geq 3\sqrt[3]{xyz} \geq 3\sqrt[3]{27} = 9.$

$4^\circ$  Note that  $x^2 + y^2 + z^2 = xyz$  implies  $x^2 < xyz$ , i.e.  $x < yz$ ; analogously  $y < zx$  and  $z < xy$ .

So  $xy < yz \cdot zx$ , i.e.  $z^2 > 1$ , from which we deduce that  $z > 1$ ; analogously  $x > 1$  and  $y > 1$ . So all three numbers are greater than 1.

Let's denote  $a = x - 1, b = y - 1, c = z - 1$ . Then  $a, b, c > 0$  and clearly  $x = a + 1, y = b + 1, z = c + 1$ .

Now the initial condition  $x^2 + y^2 + z^2 = xyz$  becomes

$$a^2 + b^2 + c^2 + a + b + c + 2 = abc + ab + bc + ca. \quad (1)$$

If we set  $q = ab + bc + ca$  we have

$$a^2 + b^2 + c^2 \geq q, \quad a + b + c \geq \sqrt{3q} \quad \text{and} \quad abc \leq \left(\frac{q}{3}\right)^{3/2} = \frac{(3q)^{3/2}}{27}.$$

Finally by (1) and the last three inequalities we obtain

$$q + \sqrt{3q} + 2 \leq a^2 + b^2 + c^2 + a + b + c + 2 = abc + ab + bc + ca \leq \frac{(3q)^{3/2}}{27} + q,$$

i.e.

$$\sqrt{3q} + 2 \leq \frac{(3q)^{3/2}}{27}. \quad (2)$$

Denote  $\sqrt{3q} = A$ . Then inequality (2) is equivalent to

$$A + 2 \leq \frac{A^3}{27} \quad \Leftrightarrow \quad (A - 6)(A + 3)^2 \geq 0,$$

from which we deduce that we must have  $\sqrt{3q} = A \geq 6$ , i.e.  $q \geq 12$ .

Hence

$$ab + bc + ca \geq 12 \Leftrightarrow (x-1)(y-1) + (y-1)(z-1) + (z-1)(x-1) \geq 12,$$

from which we obtain  $xy + yz + zx \geq 2(x + y + z) + 9$ , and we are done. ■

**74** Let  $a, b, c$  be real numbers such that  $a^3 + b^3 + c^3 - 3abc = 1$ . Prove the inequality

$$a^2 + b^2 + c^2 \geq 1.$$

*Solution* Observe that

$$\begin{aligned} 1 &= a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= \frac{(a+b+c)}{2}((a-b)^2 + (b-c)^2 + (c-a)^2). \end{aligned}$$

Since  $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$  we must have  $a+b+c > 0$ .

According to

$$(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) = 1$$

we deduce

$$(a+b+c)\left(a^2 + b^2 + c^2 - \frac{(a+b+c)^2 - a^2 - b^2 - c^2}{2}\right) = 1$$

and easily find

$$a^2 + b^2 + c^2 = \frac{1}{3}\left((a+b+c)^2 + \frac{2}{a+b+c}\right).$$

Since  $a+b+c > 0$  we may use  $AM \geq GM$  as follows

$$a^2 + b^2 + c^2 = \frac{1}{3}\left((a+b+c)^2 + \frac{1}{a+b+c} + \frac{1}{a+b+c}\right) \geq 1,$$

as required.

Equality occurs iff  $a+b+c = 1$ . ■

**75** Let  $a, b, c, d \in \mathbb{R}^+$  such that  $\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+d^4} = 1$ . Prove that

$$abcd \geq 3.$$

*Solution* We'll use the following substitutions

$$\frac{1}{1+a^4} = x, \quad \frac{1}{1+b^4} = y, \quad \frac{1}{1+c^4} = z, \quad \frac{1}{1+d^4} = t.$$

Then we obtain  $x + y + z + t = 1$  and  $a^4 = \frac{1-x}{x}$ ,  $b^4 = \frac{1-y}{y}$ ,  $c^4 = \frac{1-z}{z}$ ,  $d^4 = \frac{1-t}{t}$ .

We need to show that

$$a^4 b^4 c^4 d^4 \geq 81,$$

i.e.

$$\frac{1-x}{x} \cdot \frac{1-y}{y} \cdot \frac{1-z}{z} \cdot \frac{1-t}{t} \geq 81.$$

Applying  $AM \geq GM$  we have

$$\begin{aligned} \frac{1-x}{x} \cdot \frac{1-y}{y} \cdot \frac{1-z}{z} \cdot \frac{1-t}{t} &= \frac{y+z+t}{x} \cdot \frac{x+z+t}{y} \cdot \frac{x+y+t}{z} \cdot \frac{x+y+z}{t} \\ &\geq \frac{3\sqrt[3]{yzt}}{x} \cdot \frac{3\sqrt[3]{xzt}}{y} \cdot \frac{3\sqrt[3]{xyt}}{z} \cdot \frac{3\sqrt[3]{xyz}}{t} = 81, \end{aligned}$$

as desired. ■

**76** Let  $a, b, c$  be non-negative real numbers. Prove the inequality

$$\sqrt{\frac{ab+bc+ca}{3}} \leq \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}.$$

*Solution* The given inequality is homogenous, so we may assume that  $ab+bc+ca=3$ .

Then clearly

$$(a+b+c)^2 \geq 3(ab+bc+ca) = 9, \quad \text{i.e.} \quad a+b+c \geq 3$$

and

$$1 = \frac{ab+bc+ca}{3} \geq \sqrt[3]{(abc)^2}, \quad \text{i.e.} \quad abc \leq 1.$$

So we need to prove that

$$\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}} \geq 1, \quad \text{i.e.} \quad (a+b)(b+c)(c+a) \geq 8.$$

We have

$$\begin{aligned} (a+b)(b+c)(c+a) &= (a+b+c)(ab+bc+ca) - abc \\ &= 3(a+b+c) - abc \geq 9 - 1 = 8, \end{aligned}$$

and we are done. ■

Equality holds iff  $a=b=c$ .

**77** Let  $a, b, c, d$  be positive real numbers such that  $a+b+c+d=1$ . Prove that

$$16(abc + bcd + cda + dab) \leq 1.$$

*Solution* We'll show that

$$16(abc + bcd + cda + dab) \leq (a + b + c + d)^3.$$

Applying  $AM \geq GM$  gives us

$$\begin{aligned} 16(abc + bcd + cda + dab) &= 16ab(c + d) + 16cd(a + b) \\ &\leq 4(a + b)^2(c + d) + 4(c + d)^2(a + b) \\ &= 4(c + d)(a + b)(a + b + c + d) \\ &\leq (a + b + c + d)^3. \end{aligned}$$

It is obvious that equality holds if and only if  $a = b = c = d = 1/4$ . ■

**78** Let  $a, b, c, d, e$  be positive real numbers such that  $a + b + c + d + e = 5$ . Prove the inequality

$$abc + bcd + cde + dea + eab \leq 5.$$

*Solution* Without loss of generality, we may assume that  $e = \min\{a, b, c, d, e\}$ .

By  $AM \geq GM$ , we have

$$\begin{aligned} abc + bcd + cde + dea + eab &= e(a + c)(b + d) + bc(a + d - e) \\ &\leq e \left( \frac{a + c + b + d}{2} \right)^2 + \left( \frac{b + c + a + d - e}{3} \right)^3 \\ &= \frac{e(5 - e)^2}{4} + \frac{(5 - 2e)^3}{27}. \end{aligned}$$

So it suffices to prove that

$$\frac{e(5 - e)^2}{4} + \frac{(5 - 2e)^3}{27} \leq 5,$$

which can be rewrite as  $(e - 1)^2(e + 8) \geq 0$ , which is obviously true.

Equality holds if and only if  $a = b = c = d = e = 1$ . ■

**79** Let  $a, b, c > 0$  be real numbers. Prove the inequality

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a + b}{b + c} + \frac{b + c}{c + a} + 1.$$

*Solution* Let  $x = \frac{a}{b}, y = \frac{c}{b}$ .

Then we get

$$\frac{c}{a} = \frac{y}{x}, \quad \frac{a + b}{b + c} = \frac{x + 1}{y + 1}, \quad \frac{b + c}{c + a} = \frac{y + 1}{x + y},$$



and the given inequality becomes

$$x^3y^2 + x^2 + x + y^3 + y^2 \geq x^2y + 2xy + 2xy^2. \quad (1)$$

Using  $AM \geq GM$  we obtain

$$\frac{x^3y^2 + x}{2} \geq x^2y, \quad \frac{x^3y^2 + x + y^3 + y^2}{2} \geq 2xy^2 \quad \text{and} \quad x^2 + y^2 \geq 2xy.$$

After adding the last three inequalities we obtain inequality (1).

Equality occurs iff  $x = y = 1$ , i.e. iff  $a = b = c$ . ■

**80** Let  $a, b, c > 0$  be real numbers such that  $abc = 1$ . Prove the inequality

$$\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right) \geq 2(1 + a + b + c).$$

*Solution* The given inequality is equivalent to

$$\frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} \geq 2(a + b + c).$$

Furthermore

$$\frac{a}{b} + \frac{a}{c} + 1 = \frac{a}{b} + \frac{a}{c} + abc \geq 3\sqrt[3]{a^3} = 3a.$$

Analogously

$$\frac{b}{a} + \frac{b}{c} + 1 \geq 3b \quad \text{and} \quad \frac{c}{a} + \frac{c}{b} + 1 \geq 3c.$$

So

$$\frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} + 3 \geq 3(a + b + c). \quad (1)$$

It is enough to show that  $a + b + c \geq 3$ .

We have  $a + b + c \geq 3\sqrt[3]{abc} = 3$ , and finally from (1) we obtain

$$\frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} + 3 \geq 2(a + b + c) + (a + b + c) \geq 2(a + b + c) + 3,$$

i.e.

$$\frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} \geq 2(a + b + c).$$

Equality holds iff  $a = b = c = 1$ . ■

**81** Let  $a, b, c$  be positive real numbers such that  $a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ . Prove the inequality

$$a + b + c \geq \frac{3}{a + b + c} + \frac{2}{abc}.$$

*Solution* By  $AM \geq HM$  we get

$$a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a + b + c},$$

i.e.

$$\frac{a + b + c}{3} \geq \frac{3}{a + b + c}. \quad (1)$$

We will prove that

$$\frac{2(a + b + c)}{3} \geq \frac{2}{abc}, \quad (2)$$

i.e.

$$a + b + c \geq \frac{3}{abc}.$$

Using the well-known inequality  $(xy + yz + zx)^2 \geq 3(xy + yz + zx)$  we obtain

$$(a + b + c)^2 \geq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 \geq 3\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) = 3\frac{a + b + c}{abc},$$

i.e.

$$a + b + c \geq \frac{3}{abc}.$$

After adding (1) and (2) we get the required inequality. ■

**82** Let  $a, b, c, d$  be positive real numbers such that  $abcd = 1$ . Prove the inequality

$$\frac{1 + ab}{1 + a} + \frac{1 + bc}{1 + b} + \frac{1 + cd}{1 + c} + \frac{1 + da}{1 + d} \geq 4.$$

*Solution* Clearly  $cd = \frac{1}{ab}$  and  $ad = \frac{1}{bc}$ .

Now we have

$$\begin{aligned} A &= \frac{1 + ab}{1 + a} + \frac{1 + bc}{1 + b} + \frac{1 + cd}{1 + c} + \frac{1 + da}{1 + d} \\ &= \frac{1 + ab}{1 + a} + \frac{1 + bc}{1 + b} + \frac{1 + 1/ab}{1 + c} + \frac{1 + 1/bc}{1 + d} \\ &= (1 + ab)\left(\frac{1}{1 + a} + \frac{1}{ab + abc}\right) + (1 + bc)\left(\frac{1}{1 + b} + \frac{1}{bc + bcd}\right). \quad (1) \end{aligned}$$

By  $AM \geq HM$  and (1) we deduce

$$\begin{aligned}
 A &= (1+ab)\left(\frac{1}{1+a} + \frac{1}{ab+abc}\right) + (1+bc)\left(\frac{1}{1+b} + \frac{1}{bc+bcd}\right) \\
 &\geq \frac{4(1+ab)}{1+a+ab+abc} + \frac{4(1+bc)}{1+b+bc+bcd} \\
 &= 4\left(\frac{1+ab}{1+a+ab+abc} + \frac{1+bc}{1+b+bc+bcd}\right) \\
 &= 4\left(\frac{1+ab}{1+a+ab+abc} + \frac{a+abc}{a+ab+abc+abcd}\right). \tag{2}
 \end{aligned}$$

Since  $abcd = 1$  from (2) we obtain  $A \geq 4$ , as required.  $\blacksquare$

**83** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}.$$

*Solution* Applying  $AM \geq GM$  we have

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq 3\sqrt[3]{\frac{1}{abc(a+b)(b+c)(c+a)}} \tag{1}$$

and

$$a+b+c \geq 3\sqrt[3]{abc}, \quad \text{i.e.} \quad \frac{1}{\sqrt[3]{abc}} \geq \frac{3}{a+b+c}. \tag{2}$$

Furthermore

$$a+b+c = \frac{1}{2}((a+b) + (b+c) + (c+a)) \geq \frac{3}{2}\sqrt[3]{(a+b)(b+c)(c+a)},$$

i.e.

$$\frac{1}{\sqrt[3]{(a+b)(b+c)(c+a)}} \geq \frac{3}{2(a+b+c)}. \tag{3}$$

Combining (2), (3) and (1) we get

$$\begin{aligned}
 \frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} &\geq 3 \cdot \sqrt[3]{\frac{1}{abc(a+b)(b+c)(c+a)}} \\
 &\geq 3 \cdot \frac{3}{a+b+c} \cdot \frac{3}{2(a+b+c)} = \frac{27}{2(a+b+c)^2}. \quad \blacksquare
 \end{aligned}$$

**84** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$\frac{a^2}{b^2 - 2b + 3} + \frac{b^2}{c^2 - 2c + 3} + \frac{c^2}{a^2 - 2a + 3} \geq \frac{3}{2}.$$

*Solution* Since  $a + b + c = 3$  by  $QM \geq AM$  we have

$$(b - 1)^2 = ((1 - a) + (1 - c))^2 \leq 2((a - 1)^2 + (c - 1)^2).$$

Hence

$$(b - 1)^2 \leq \frac{2}{3}((a - 1)^2 + (b - 1)^2 + (c - 1)^2) = \frac{2}{3}(a^2 + b^2 + c^2 - 3).$$

So we have

$$b^2 - 2b + 3 = (b - 1)^2 + 2 \leq \frac{2}{3}(a^2 + b^2 + c^2 - 3) + 2 = \frac{2}{3}(a^2 + b^2 + c^2),$$

which implies

$$\frac{a^2}{b^2 - 2b + 3} \geq \frac{a^2}{\frac{2}{3}(a^2 + b^2 + c^2)} = \frac{3a^2}{2(a^2 + b^2 + c^2)}.$$

Similarly we get

$$\frac{b^2}{c^2 - 2c + 3} \geq \frac{3b^2}{2(a^2 + b^2 + c^2)} \quad \text{and} \quad \frac{c^2}{a^2 - 2a + 3} \geq \frac{3c^2}{2(a^2 + b^2 + c^2)}.$$

By adding the last three inequalities we obtain the required inequality. ■

**85** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 3$ . Prove the inequality

$$\frac{1}{1 + a^2(b + c)} + \frac{1}{1 + b^2(c + a)} + \frac{1}{1 + c^2(a + b)} \leq \frac{1}{abc}.$$

*Solution* Observe that

$$\frac{1}{1 + a^2(b + c)} = \frac{1}{1 + a(ab + ac)} = \frac{1}{1 + a(3 - bc)} = \frac{1}{3a + 1 - abc}.$$

By  $AM \geq GM$  we get

$$1 = \frac{ab + bc + ca}{3} \geq \sqrt[3]{(abc)^2}.$$

Thus

$$abc \leq 1.$$

Therefore

$$\frac{1}{1+a^2(b+c)} = \frac{1}{3a+1-abc} \leq \frac{1}{3a}.$$

Similarly,

$$\frac{1}{1+b^2(c+a)} \leq \frac{1}{3b} \quad \text{and} \quad \frac{1}{1+c^2(a+b)} \leq \frac{1}{3c}.$$

Now we have

$$\begin{aligned} & \frac{1}{1+a^2(b+c)} + \frac{1}{1+b^2(c+a)} + \frac{1}{1+c^2(a+b)} \\ & \leq \frac{1}{3} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = \frac{1}{3} \left( \frac{ab+bc+ca}{abc} \right) = \frac{1}{abc}. \end{aligned}$$

Equality holds iff  $a = b = c = 1$ . ■

**86** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$a\sqrt[3]{1+b-c} + b\sqrt[3]{1+c-a} + c\sqrt[3]{1+a-b} \leq 1.$$

*Solution* Note that  $1+b-c = a+b+c+b-c = a+2b \geq 0$ .

Now by  $GM \leq AM$  we have

$$a\sqrt[3]{1+b-c} \leq a \frac{1+b-c+1+1}{3} = a + \frac{a(b-c)}{3}.$$

Similarly

$$b\sqrt[3]{1+c-a} \leq b + \frac{b(c-a)}{3} \quad \text{and} \quad c\sqrt[3]{1+a-b} \leq c + \frac{c(a-b)}{3}.$$

Adding these three inequalities we get

$$a\sqrt[3]{1+b-c} + b\sqrt[3]{1+c-a} + c\sqrt[3]{1+a-b} \leq a + b + c = 1.$$

Equality occurs iff  $a = b = c = 1/3$ . ■

**87** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 1$ . Prove the inequality

$$\frac{1-2ab}{c} + \frac{1-2bc}{a} + \frac{1-2ca}{b} \geq 7.$$

*Solution* We have

$$\begin{aligned} & \frac{1-2ab}{c} + \frac{1-2bc}{a} + \frac{1-2ca}{b} \\ & = \frac{(a+b+c)^2 - 2ab}{c} + \frac{(a+b+c)^2 - 2bc}{a} + \frac{(a+b+c)^2 - 2ca}{b} \end{aligned}$$

$$\begin{aligned}
&= \frac{a^2 + b^2 + c^2 + 2bc + 2ac}{c} + \frac{a^2 + b^2 + c^2 + 2ac + 2ab}{a} \\
&\quad + \frac{a^2 + b^2 + c^2 + 2ab + 2bc}{b} \\
&= (a^2 + b^2 + c^2) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + 4(a + b + c) \\
&= (a^2 + b^2 + c^2) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + 4. \tag{1}
\end{aligned}$$

By  $QM \geq AM$  we get

$$a^2 + b^2 + c^2 \geq \frac{(a + b + c)^2}{3} = \frac{1}{3} \quad \text{and} \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a + b + c} = 9.$$

Finally, from previous inequalities and (1) we obtain

$$\frac{1 - 2ab}{c} + \frac{1 - 2bc}{a} + \frac{1 - 2ca}{b} = (a^2 + b^2 + c^2) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + 4 \geq \frac{9}{3} + 4 = 7. \quad \blacksquare$$

**88** Let  $a, b, c$  be non-negative real numbers such that  $a^2 + b^2 + c^2 = 1$ . Prove the inequality

$$\frac{1 - ab}{7 - 3ac} + \frac{1 - ab}{7 - 3ac} + \frac{1 - ab}{7 - 3ac} \geq \frac{1}{3}.$$

*Solution* First we'll show that

$$\frac{1}{7 - 3ab} + \frac{1}{7 - 3bc} + \frac{1}{7 - 3ca} \leq \frac{1}{2}. \tag{1}$$

By  $AM \geq HM$  we have

$$\frac{1}{7 - 3ab} = \frac{1}{3(1 - ab) + 2 + 2} \leq \frac{1}{9} \left( \frac{1}{3(1 - ab)} + 1 \right).$$

Similarly we get

$$\frac{1}{7 - 3bc} \leq \frac{1}{9} \left( \frac{1}{3(1 - bc)} + 1 \right) \quad \text{and} \quad \frac{1}{7 - 3ca} \leq \frac{1}{9} \left( \frac{1}{3(1 - ca)} + 1 \right).$$

So it follows that

$$\frac{1}{7 - 3ab} + \frac{1}{7 - 3bc} + \frac{1}{7 - 3ca} \leq \frac{1}{27} \left( \frac{1}{1 - ab} + \frac{1}{1 - bc} + \frac{1}{1 - ca} \right) + \frac{1}{3}. \tag{2}$$

Recalling the well-known *Vasile Cirtoaje's* inequality

$$\frac{1}{1 - ab} + \frac{1}{1 - bc} + \frac{1}{1 - ca} \leq \frac{9}{2},$$

by (2) we obtain

$$\frac{1}{7-3ab} + \frac{1}{7-3bc} + \frac{1}{7-3ca} \leq \frac{1}{2}.$$

Since  $a^2 + b^2 + c^2 = 1$  we have  $a, b, c \leq 1$  and then clearly

$$7-3ab, 7-3ab, 7-3ab > 0,$$

so by  $AM \geq GM$  we have

$$\frac{7-3ab}{7-3ac} + \frac{7-3ab}{7-3ac} + \frac{7-3ab}{7-3ac} \geq 3. \quad (3)$$

Finally by (2) and (3) we have

$$\begin{aligned} & \frac{3-3ab}{7-3ac} + \frac{3-3ab}{7-3ac} + \frac{3-3ab}{7-3ac} \\ &= \left( \frac{7-3ab}{7-3ac} + \frac{7-3ab}{7-3ac} + \frac{7-3ab}{7-3ac} \right) - 4 \left( \frac{1}{7-3ab} + \frac{1}{7-3bc} + \frac{1}{7-3ca} \right) \\ &\geq 3 - 2 = 1, \end{aligned}$$

i.e.

$$\frac{1-ab}{7-3ac} + \frac{1-ab}{7-3ac} + \frac{1-ab}{7-3ac} \geq \frac{1}{3},$$

as required. ■

**89** Let  $x, y, z \in \mathbb{R}^+$  such that  $x + y + z = 1$ . Prove the inequality

$$\frac{xy}{\sqrt{\frac{1}{3} + z^2}} + \frac{zx}{\sqrt{\frac{1}{3} + y^2}} + \frac{yz}{\sqrt{\frac{1}{3} + x^2}} \leq \frac{1}{2}.$$

*Solution* We have

$$\begin{aligned} \frac{1}{3} + x^2 &= \frac{1}{3}(x+y+z)^2 + x^2 = \frac{x^2 + y^2 + z^2 + 2(xy + yz + zx)}{3} + x^2 \\ &\geq \frac{xy + yz + zx + 2(xy + yz + zx)}{3} + x^2 = xy + yz + zx + x^2 \\ &= (x+y)(x+z). \end{aligned}$$

Now we get

$$\frac{yz}{\sqrt{\frac{1}{3} + x^2}} \leq \frac{yz}{\sqrt{(x+y)(x+z)}} \stackrel{HM \leq GM}{\leq} \frac{yz}{2} \left( \frac{1}{x+y} + \frac{1}{x+z} \right). \quad (1)$$

Analogously

$$\frac{xy}{\sqrt{\frac{1}{3} + z^2}} \leq \frac{xy}{2} \left( \frac{1}{z+x} + \frac{1}{z+y} \right) \quad (2)$$

and

$$\frac{zx}{\sqrt{\frac{1}{3} + y^2}} \leq \frac{zx}{2} \left( \frac{1}{y+z} + \frac{1}{y+x} \right). \quad (3)$$

Adding (1), (2) and (3) we obtain

$$\begin{aligned} L &\leq \frac{xy}{2} \left( \frac{1}{z+x} + \frac{1}{z+y} \right) + \frac{zx}{2} \left( \frac{1}{y+z} + \frac{1}{y+x} \right) + \frac{yz}{2} \left( \frac{1}{x+y} + \frac{1}{x+z} \right) \\ &= \frac{1}{2} \left( \frac{xy + yz}{x+z} + \frac{xy + zx}{y+z} + \frac{yz + zx}{y+x} \right) = \frac{x+y+z}{2} = \frac{1}{2}. \quad \blacksquare \end{aligned}$$

**90** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\frac{a-bc}{a+bc} + \frac{b-ca}{b+ca} + \frac{c-ab}{c+ab} \leq \frac{3}{2}.$$

*Solution* Note that

$$1 - \frac{a-bc}{a+bc} = \frac{2bc}{1-b-c+bc} = \frac{2bc}{(1-b)(1-c)} = \frac{2bc}{(c+a)(a+b)},$$

i.e.

$$\frac{a-bc}{a+bc} = 1 - \frac{2bc}{(c+a)(a+b)}.$$

Similarly we get

$$\frac{b-ca}{b+ca} = 1 - \frac{2ca}{(c+b)(b+a)} \quad \text{and} \quad \frac{c-ab}{c+ab} = 1 - \frac{2ab}{(b+c)(c+a)}.$$

Now the given inequality becomes

$$1 - \frac{2bc}{(c+a)(a+b)} + 1 - \frac{2ca}{(c+b)(b+a)} + 1 - \frac{2ab}{(b+c)(c+a)} \leq \frac{3}{2}$$

or

$$\frac{2bc}{(c+a)(a+b)} + \frac{2ca}{(c+b)(b+a)} + \frac{2ab}{(b+c)(c+a)} \geq \frac{3}{2}.$$

After expanding we get the equivalent form as follows

$$4(bc(b+c) + ca(c+a) + ab(a+b)) \geq 3(a+b)(b+c)(c+a),$$



i.e.

$$ab + bc + ac \geq 9abc, \quad \text{i.e.} \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9,$$

which is true since

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c} = 9 \quad (AM \geq HM).$$

Equality occurs iff  $a = b = c = 1/3$ . ■

**91** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\sqrt{\frac{a+b}{a+1}} + \sqrt{\frac{b+c}{c+1}} + \sqrt{\frac{c+a}{a+1}} \geq 3.$$

*Solution* By  $AM \geq GM$  we get

$$\sqrt{\frac{a+b}{a+1}} + \sqrt{\frac{b+c}{c+1}} + \sqrt{\frac{c+a}{a+1}} \geq 3 \sqrt[6]{\frac{(a+b)(b+c)(c+a)}{(a+1)(b+1)(c+1)}}.$$

So it suffices to prove that

$$\frac{(a+b)(b+c)(c+a)}{(a+1)(b+1)(c+1)} \geq 1,$$

i.e.

$$(a+b)(b+c)(c+a) \geq (a+1)(b+1)(c+1).$$

Since  $abc = 1$  we need to prove that

$$ab(a+b) + bc(b+c) + ca(c+a) \geq a+b+c+ab+bc+ca. \quad (1)$$

According to  $AM \geq GM$  we have

$$\begin{aligned} & 2(ab(a+b) + bc(b+c) + ca(c+a)) + (ab+bc+ca) \\ &= \sum_{\text{cyc}} (a^2b + a^2b + a^2c + a^2c + bc) \geq 5 \sum_{\text{cyc}} a = 5(a+b+c) \end{aligned} \quad (2)$$

and

$$\begin{aligned} & 2(ab(a+b) + bc(b+c) + ca(c+a)) + (a+b+c) \\ &= \sum_{\text{cyc}} (a^2b + a^2b + b^2a + b^2a + c) \geq 5 \sum_{\text{cyc}} ab = 5(ab+bc+ca). \end{aligned} \quad (3)$$

After adding (2) and (3) we obtain

$$4(ab(a+b) + bc(b+c) + ca(c+a)) + (ab + bc + ca) + (a+b+c) \\ \geq 5(ab + bc + ca) + 5(a+b+c).$$

Hence we have proved (1), as required. Equality holds iff  $a = b = c = 1$ . ■

**92** Let  $x, y, z \geq 0$  be real numbers such that  $xy + yz + zx = 1$ . Prove the inequality

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} \leq \frac{3\sqrt{3}}{4}.$$

*Solution* We have

$$1 + x^2 = xy + yz + zx + x^2 = (x+y)(x+z).$$

Analogously we obtain

$$1 + y^2 = (y+x)(y+z) \quad \text{and} \quad 1 + z^2 = (z+x)(z+y).$$

Therefore

$$\begin{aligned} \frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} &= \frac{x}{(x+y)(x+z)} + \frac{y}{(y+x)(y+z)} + \frac{z}{(z+x)(z+y)} \\ &= \frac{x(y+z) + y(x+z) + z(x+y)}{(x+y)(y+z)(z+x)} \\ &= \frac{2}{(x+y)(y+z)(z+x)}. \end{aligned} \tag{1}$$

It is easy to show that

$$(x+y)(y+z)(z+x) = x+y+z - xyz. \tag{2}$$

Due to the well-known inequality  $(x+y+z)^2 \geq 3(xy+yz+zx)$  we obtain

$$(x+y+z)^2 \geq 3(xy+yz+zx) = 3, \quad \text{i.e.} \quad x+y+z \geq \sqrt{3}. \tag{3}$$

Applying  $AM \geq GM$  it follows that

$$xy + yz + zx \geq 3\sqrt[3]{(xyz)^2},$$

i.e.

$$\frac{1}{27} \geq (xyz)^2 \Leftrightarrow \frac{1}{3\sqrt{3}} \geq xyz. \tag{4}$$

Using (3) and (4) we obtain

$$x + y + z - xyz \geq \sqrt{3} - \frac{1}{3\sqrt{3}} = \frac{8}{3\sqrt{3}}. \quad (5)$$

Finally using (1), (2) and (5) we get

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} = \frac{2}{(x+y)(y+z)(z+x)} = \frac{2}{x+y+z-xyz} \leq \frac{3\sqrt{3}}{4}.$$

Equality occurs iff  $x = y = z = \frac{1}{\sqrt{3}}$ . ■

**93** Let  $a, b, c$  be non-negative real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \geq \frac{3\sqrt{3}}{\sqrt{3}+1}.$$

*Solution* After some algebraic calculations we get

$$\begin{aligned} \frac{4+2(a+b+c)}{2+a+b+c+abc} &\geq \frac{3\sqrt{3}}{\sqrt{3}+1} \\ \Leftrightarrow 2(2+a+b+c)(\sqrt{3}+1) &\geq 3\sqrt{3}(2+a+b+c+abc) \\ \Leftrightarrow 2+(a+b+c) &\geq \frac{3\sqrt{3}}{2-\sqrt{3}}abc, \end{aligned}$$

i.e.

$$2+(a+b+c) \geq 3\sqrt{3}(2+\sqrt{3})abc. \quad (1)$$

Applying  $AM \geq GM$  we obtain

$$1 = ab + bc + ca \geq 3\sqrt[3]{(abc)^2},$$

i.e.

$$\frac{1}{3\sqrt{3}} \geq abc. \quad (2)$$

Also we have

$$a+b+c \geq \sqrt{3}. \quad (3)$$

Using (2) and (3) we get

$$2+(a+b+c) \geq 2+\sqrt{3} \geq 3\sqrt{3}(2+\sqrt{3})abc,$$

i.e. we have shown inequality (1), as desired.

Equality holds if and only if  $a = b = c = 1/\sqrt{3}$ . ■

**94** Let  $a, b, c$  be non-negative real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$\frac{a^2}{1+a} + \frac{b^2}{1+b} + \frac{c^2}{1+c} \geq \frac{\sqrt{3}}{\sqrt{3}+1}.$$

*Solution* Using  $\frac{x^2}{1+x} = x - 1 + \frac{1}{1+x}$  we have

$$\frac{a^2}{1+a} + \frac{b^2}{1+b} + \frac{c^2}{1+c} = a + b + c - 3 + \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}.$$

Now using the result from Problem 89 and the inequality  $a + b + c \geq \sqrt{3}$  we obtain

$$\begin{aligned} \frac{a^2}{1+a} + \frac{b^2}{1+b} + \frac{c^2}{1+c} &= a + b + c - 3 + \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \\ &\geq \sqrt{3} - 3 + \frac{3\sqrt{3}}{\sqrt{3}+1} = \frac{\sqrt{3}}{\sqrt{3}+1}. \end{aligned}$$

Equality occurs if and only if  $a = b = c = 1/\sqrt{3}$ . ■

**95** Let  $a, b, c \in \mathbb{R}^+$  such that  $(a+b)(b+c)(c+a) = 8$ . Prove the inequality

$$\frac{a+b+c}{3} \geq \sqrt[27]{\frac{a^3+b^3+c^3}{3}}.$$

*Solution* We have

$$\begin{aligned} (a+b+c)^3 &= a^3 + b^3 + c^3 + 3(a+b)(b+c)(c+a) \\ &= a^3 + b^3 + c^3 + 24 = a^3 + b^3 + c^3 + \underbrace{3 + \dots + 3}_8 \\ &\geq 9\sqrt[9]{(a^3+b^3+c^3)3^8} \\ \Leftrightarrow \left(\frac{a+b+c}{3}\right)^3 &\geq \sqrt[9]{\frac{a^3+b^3+c^3}{3}}, \quad \text{i.e.} \quad \frac{a+b+c}{3} \geq \sqrt[27]{\frac{a^3+b^3+c^3}{3}}. \end{aligned}$$
■

**96** Find the maximum value of  $\frac{x^4-x^2}{x^6+2x^3-1}$ , where  $x \in \mathbb{R}, x > 1$ .

*Solution* We have

$$\frac{x^4-x^2}{x^6+2x^3-1} = \frac{x-\frac{1}{x}}{x^3+2-\frac{1}{x^3}} = \frac{x-\frac{1}{x}}{(x-\frac{1}{x})^3+2+3(x-\frac{1}{x})}. \quad (1)$$

We'll show that

$$\left(x - \frac{1}{x}\right)^3 + 2 \geq 3\left(x - \frac{1}{x}\right).$$

Since  $x > 1$  we have  $1 > \frac{1}{x}$ , i.e.  $x - \frac{1}{x} > 0$ .

From  $AM \geq GM$  we get

$$\left(x - \frac{1}{x}\right)^3 + 2 = \left(x - \frac{1}{x}\right)^3 + 1 + 1 \geq 3\sqrt[3]{\left(x - \frac{1}{x}\right)^3 \cdot 1 \cdot 1} = 3\left(x - \frac{1}{x}\right).$$

Now in (1) we obtain

$$\begin{aligned} \frac{x^4 - x^2}{x^6 + 2x^3 - 1} &= \frac{x - \frac{1}{x}}{x^3 + 2 - \frac{1}{x^3}} = \frac{x - \frac{1}{x}}{(x - \frac{1}{x})^3 + 2 + 3(x - \frac{1}{x})} \leq \frac{x - \frac{1}{x}}{3(x - \frac{1}{x}) + 3(x - \frac{1}{x})} \\ &= \frac{1}{6}. \end{aligned} \quad \blacksquare$$

**97** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a + \sqrt{ab} + \sqrt[3]{abc}}{3} \leq \sqrt[3]{a \cdot \frac{a+b}{2} \cdot \frac{a+b+c}{3}}.$$

*Solution* Applying  $AM \geq GM$  we get

$$\sqrt[3]{ab \cdot \frac{a+b}{2}} \geq \sqrt[3]{ab \cdot \sqrt{ab}} = \sqrt{ab}.$$

So

$$a + \sqrt{ab} + \sqrt[3]{abc} \leq a + \sqrt[3]{ab \cdot \frac{a+b}{2}} + \sqrt[3]{abc}.$$

Now, it is enough to show that

$$a + \sqrt[3]{ab \cdot \frac{a+b}{2}} + \sqrt[3]{abc} \leq 3\sqrt[3]{a \cdot \frac{a+b}{2} \cdot \frac{a+b+c}{3}}.$$

Another application of  $AM \geq GM$  gives us

$$\sqrt[3]{1 \cdot \frac{2a}{a+b} \cdot \frac{3a}{a+b+c}} \leq \frac{1 + \frac{2a}{a+b} + \frac{3a}{a+b+c}}{3}, \quad \sqrt[3]{1 \cdot 1 \cdot \frac{3b}{a+b+c}} \leq \frac{2 + \frac{3b}{a+b+c}}{3}$$

and

$$\sqrt[3]{1 \cdot \frac{2b}{a+b} \cdot \frac{3c}{a+b+c}} \leq \frac{1 + \frac{2b}{a+b} + \frac{3c}{a+b+c}}{3}.$$

Adding, we obtain

$$\sqrt[3]{\frac{2a}{a+b} \cdot \frac{3a}{a+b+c}} + \sqrt[3]{\frac{3b}{a+b+c}} + \sqrt[3]{\frac{2b}{a+b} \cdot \frac{3c}{a+b+c}} \leq 3,$$

i.e.

$$\sqrt[3]{\frac{1}{a} \cdot \frac{2}{a+b} \cdot \frac{3}{a+b+c}} \left( a + \sqrt[3]{ab \cdot \frac{a+b}{2}} + \sqrt[3]{abc} \right) \leq 3,$$

i.e.

$$a + \sqrt[3]{ab \cdot \frac{a+b}{2}} + \sqrt[3]{abc} \leq 3 \sqrt[3]{a \cdot \frac{a+b}{2} \cdot \frac{a+b+c}{3}}. \quad \blacksquare$$

**98** Let  $a, b, c$  be positive real numbers such that  $abc(a+b+c) = 3$ . Prove the inequality

$$(a+b)(b+c)(c+a) \geq 8.$$

*Solution* We have

$$\begin{aligned} A &= (a+b)(b+c)(c+a) = (ab+ac+b^2+bc)(c+a) \\ &= (b(a+b+c)+ac)(c+a) = \left( \frac{3}{ac} + ac \right) (c+a). \end{aligned}$$

By  $AM \geq GM$  we obtain

$$\begin{aligned} A &= \left( \frac{3}{ac} + ac \right) (c+a) = \left( \frac{1}{ac} + \frac{1}{ac} + \frac{1}{ac} + ac \right) (c+a) \\ &\geq 4 \sqrt[4]{\frac{ac}{(ac)^3}} \cdot 2\sqrt{ac} = 4 \frac{1}{\sqrt{ac}} \cdot 2\sqrt{ac} = 8. \end{aligned}$$

Equality occurs iff  $a = c$  and  $\frac{1}{ac} = ac$ , i.e.  $a = c = 1$ , and then we easily get  $b = 1$ . ■

**99** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\sqrt{\frac{2a}{b+c}} + \sqrt{\frac{2b}{c+a}} + \sqrt{\frac{2c}{a+b}} \leq 3.$$

*Solution* Applying  $AM \geq HM$  we get

$$\sqrt{1 \cdot \frac{2a}{b+c}} \leq \frac{2}{1 + \frac{b+c}{2a}} = \frac{4a}{2a+b+c}.$$

Analogously we obtain

$$\sqrt{\frac{2b}{c+a}} \leq \frac{4b}{a+2b+c} \quad \text{and} \quad \sqrt{\frac{2c}{a+b}} \leq \frac{4c}{a+b+2c}.$$

So it is enough to prove that

$$4\left(\frac{a}{2a+b+c} + \frac{b}{a+2b+c} + \frac{c}{a+b+2c}\right) \leq 3,$$

i.e.

$$\frac{a}{2a+b+c} + \frac{b}{a+2b+c} + \frac{c}{a+b+2c} \leq \frac{3}{4}. \quad (1)$$

Since the last inequality is homogeneous we can assume that  $a+b+c=1$ .

Now inequality (1) becomes

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} \leq \frac{3}{4}, \quad \text{i.e.} \quad 5(ab+bc+ca) + 9abc \leq 2. \quad (2)$$

By the well-known inequality  $3(ab+bc+ac) \leq (a+b+c)^2$  and  $AM \geq GM$  we obtain  $ab+bc+ac \leq \frac{1}{3}$  and  $abc \leq \frac{1}{27}$ . Now it is quite easy to prove inequality (2), as desired. ■

**100** Let  $a, b, c \in \mathbb{R}^+$  such that  $ab+bc+ca=1$ . Prove the inequality

$$\frac{1}{a(a+b)} + \frac{1}{b(b+c)} + \frac{1}{c(c+a)} \geq \frac{9}{2}.$$

*Solution* The given inequality is equivalent to

$$\frac{c(a+b)+ab}{a(a+b)} + \frac{a(b+c)+bc}{b(b+c)} + \frac{b(c+a)+ac}{c(c+a)} \geq \frac{9}{2},$$

i.e.

$$\begin{aligned} \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a} &\geq \frac{9}{2} \\ \Leftrightarrow \frac{a+b}{b} + \frac{b+c}{c} + \frac{c+a}{a} + \frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a} &\geq \frac{15}{2}. \end{aligned} \quad (1)$$

We have

$$\begin{aligned} \frac{a+b}{b} + \frac{b+c}{c} + \frac{c+a}{a} + \frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a} \\ = \frac{a+b}{4b} + \frac{b+c}{4c} + \frac{c+a}{4a} + \frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a} \end{aligned}$$

$$\begin{aligned}
& + \frac{3}{4} \left( \frac{a+b}{b} + \frac{b+c}{c} + \frac{c+a}{a} \right) \\
& \geq 6 \sqrt[6]{\frac{a+b}{4b} \cdot \frac{b+c}{4c} \cdot \frac{c+a}{4a} \cdot \frac{b}{a+b} \cdot \frac{c}{b+c} \cdot \frac{a}{c+a}} + \frac{3}{4} \left( \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 3 \right) \\
& \geq 3 + \frac{3}{4} \left( 3 \sqrt[3]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}} + 3 \right) = 3 + \frac{18}{4} = \frac{15}{2},
\end{aligned}$$

as required. ■

**101** Let  $0 \leq a \leq b \leq c \leq 1$  be real numbers. Prove that

$$a^2(b-c) + b^2(c-b) + c^2(1-c) \leq \frac{108}{529}.$$

*Solution* Using  $AM \geq GM$  we have

$$\begin{aligned}
a^2(b-c) + b^2(c-b) + c^2(1-c) & \leq 0 + \frac{1}{2}(b \cdot b \cdot (2c - 2b)) + c^2(1-c) \\
& \leq \frac{1}{2} \left( \frac{b+b+2c-2b}{3} \right)^3 + c^2(1-c) \\
& = c^2 \left( \frac{4c}{27} + 1 - c \right) = c^2 \left( 1 - \frac{23c}{27} \right) \\
& = \left( \frac{54}{23} \right)^2 \left( \frac{23c}{54} \right) \left( \frac{23c}{54} \right) \left( 1 - \frac{23c}{27} \right) \\
& \leq \left( \frac{54}{23} \right)^2 \left( \frac{1}{3} \right)^3 = \frac{108}{529}.
\end{aligned}$$
■

**102** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 1$ . Prove the inequality

$$S = a^4b + b^4c + c^4a \leq \frac{256}{3125}.$$

*Solution* Without loss of generality we can assume that  $a = \max\{a, b, c\}$ .

So it follows that

$$b^4c \leq a^3bc \quad \text{and} \quad c^4a \leq c^2a^3 \leq ca^4.$$

Since  $\frac{3c}{4} \geq \frac{c}{2}$  we obtain

$$\begin{aligned}
S & = a^4b + b^4c + \frac{c^4a}{2} + \frac{c^4a}{2} \leq a^4b + a^3bc + \frac{ca^4}{2} + \frac{c^2a^3}{2} \\
& = a^3b(a+c) + \frac{a^3c}{2}(a+c) = a^3(a+c) \left( b + \frac{c}{2} \right) \leq a^3(a+c) \left( b + \frac{3c}{4} \right). \quad (1)
\end{aligned}$$



Now using (1) and  $AM \geq GM$  we get

$$\begin{aligned} S &\leq a^3(a+c) \left(b + \frac{3c}{4}\right) = 4^4 \cdot \frac{a}{4} \cdot \frac{a}{4} \cdot \frac{a}{4} \cdot \frac{a+c}{4} \cdot \left(b + \frac{3c}{4}\right) \\ &\leq 4^4 \left(\frac{\frac{a}{4} + \frac{a}{4} + \frac{a}{4} + \frac{a+c}{4} + (b + \frac{3c}{4})}{5}\right)^5 = 4^4 \left(\frac{a+b+c}{5}\right)^5 = \frac{256}{3125}. \quad \blacksquare \end{aligned}$$

**103** Let  $a, b, c > 0$  be real numbers. Prove the inequality

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

*Solution* Let  $\frac{a}{b} = x$ ,  $\frac{b}{c} = y$ ,  $\frac{c}{a} = z$ . Then it is clear that  $xyz = 1$ , and the given inequality becomes

$$x^2 + y^2 + z^2 \geq x + y + z.$$

From  $QM \geq AM$  we have

$$\sqrt{\frac{x^2 + y^2 + z^2}{3}} \geq \frac{x + y + z}{3},$$

i.e.

$$x^2 + y^2 + z^2 \geq \frac{(x + y + z)^2}{3} \geq \frac{3\sqrt[3]{xyz}(x + y + z)}{3} = x + y + z. \quad \blacksquare$$

**104** Prove that for all positive real numbers  $a, b, c$  we have

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq a + b + c.$$

*Solution* Using  $AM \geq GM$  we get

$$\frac{a^3}{b^2} + 2b = \frac{a^3}{b^2} + b + b \geq 3\sqrt[3]{\frac{a^3}{b^2} \cdot b \cdot b} = 3a.$$

Analogously we have

$$\frac{b^3}{c^2} + 2c \geq 3b \quad \text{and} \quad \frac{c^3}{a^2} + 2a \geq 3c.$$

Adding these three inequalities we obtain

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} + 2(a + b + c) \geq 3(a + b + c),$$

as required. Equality holds iff  $a = b = c$ . \blacksquare

**105** Prove that for all positive real numbers  $a, b, c$  we have

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}.$$

*Solution* Using  $AM \geq GM$  we get

$$\frac{a^3}{b^2} + a \geq 2\sqrt{\frac{a^3}{b^2} \cdot a} = 2\frac{a^2}{b}.$$

Analogously we have

$$\frac{b^3}{c^2} + b \geq 2\frac{b^2}{c} \quad \text{and} \quad \frac{c^3}{a^2} + c \geq 2\frac{c^2}{a}.$$

Adding these three inequalities we obtain

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} + (a + b + c) \geq 2\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right). \quad (1)$$

According to Exercise 2.12 (Chap. 2) we have that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c. \quad (2)$$

Now using (1) and (2) we obtain

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} + (a + b + c) \geq 2\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right) \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + (a + b + c),$$

and equality holds iff  $a = b = c$ . ■

**106** Prove that for all positive real numbers  $a, b, c$  we have

$$\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} \geq ab + bc + ca.$$

*Solution* Using  $AM \geq GM$  we get

$$\frac{a^3}{b} + \frac{b^3}{c} + bc \geq 3\sqrt[3]{\frac{a^3}{b} \cdot \frac{b^3}{c} \cdot bc} = 3ab.$$

Analogously we have

$$\frac{b^3}{c} + \frac{c^3}{a} + ca \geq 3bc \quad \text{and} \quad \frac{c^3}{a} + \frac{a^3}{b} + ab \geq 3ca.$$

Adding these three inequalities we obtain

$$2\left(\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a}\right) + ab + bc + ca \geq 3(ab + bc + ca),$$

from which follows the desired inequality. Equality holds iff  $a = b = c$ . ■

**107** Prove that for all positive real numbers  $a, b, c$  we have

$$\frac{a^5}{b^3} + \frac{b^5}{c^3} + \frac{c^5}{a^3} \geq a^2 + b^2 + c^2.$$

*Solution* Using  $AM \geq GM$  we get

$$2\frac{a^5}{b^3} + 3b^2 = \frac{a^5}{b^3} + \frac{a^5}{b^3} + b^2 + b^2 + b^2 \geq 5\sqrt[5]{\frac{a^5}{b^3} \cdot \frac{a^5}{b^3} \cdot b^2 \cdot b^2 \cdot b^2} = 5a^2.$$

Analogously we have

$$2\frac{b^5}{c^3} + 3c^2 \geq 5b^2 \quad \text{and} \quad 2\frac{c^5}{a^3} + 3a^2 \geq 5c^2.$$

Adding these three inequalities we obtain

$$2\left(\frac{a^5}{b^3} + \frac{b^5}{c^3} + \frac{c^5}{a^3}\right) + 3(a^2 + b^2 + c^2) \geq 5(a^2 + b^2 + c^2),$$

i.e.

$$\frac{a^5}{b^3} + \frac{b^5}{c^3} + \frac{c^5}{a^3} \geq a^2 + b^2 + c^2.$$

Equality holds iff  $a = b = c$ . ■

**108** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 3$ . Prove the inequality

$$\frac{a^3}{b(2c+a)} + \frac{b^3}{c(2a+b)} + \frac{c^3}{a(2b+c)} \geq 1.$$

*Solution* We'll show that

$$\frac{a^3}{b(2c+a)} + \frac{b^3}{c(2a+b)} + \frac{c^3}{a(2b+c)} \geq \frac{a+b+c}{3},$$

from which, with the initial condition, will follow the desired inequality.

Using  $AM \geq GM$  we get

$$\frac{9a^3}{b(2c+a)} + 3b + (2c+a) \geq 3\sqrt[3]{\frac{9a^3}{b(2c+a)} \cdot 3b \cdot (2c+a)} = 9a.$$

Analogously we have

$$\frac{9b^3}{c(2a+b)} + 3c + (2a+b) \geq 3b \quad \text{and} \quad \frac{9c^3}{a(2b+c)} + 3a + (2b+c) \geq 3c.$$

Adding the last three inequalities we obtain

$$9\left(\frac{a^3}{b(2c+a)} + \frac{b^3}{c(2a+b)} + \frac{c^3}{a(2b+c)}\right) + 6(a+b+c) \geq 9(a+b+c),$$

i.e.

$$\frac{a^3}{b(2c+a)} + \frac{b^3}{c(2a+b)} + \frac{c^3}{a(2b+c)} \geq \frac{a+b+c}{3} = \frac{3}{3} = 1. \quad \blacksquare$$

**109** Let  $a, b, c \in \mathbb{R}^+$  and  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{a^3}{b+2c} + \frac{b^3}{c+2a} + \frac{c^3}{a+2b} \geq 1.$$

*Solution* We'll prove that

$$\frac{a^3}{b+2c} + \frac{b^3}{c+2a} + \frac{c^3}{a+2b} \geq \frac{a^2 + b^2 + c^2}{3},$$

from which since  $a^2 + b^2 + c^2 = 3$ , we'll obtain the required result.

Applying  $AM \geq GM$  we get

$$\frac{9a^3}{b+2c} + a(b+2c) \geq 2\sqrt{\frac{9a^3}{b+2c} \cdot a \cdot (b+2c)} = 6a^2.$$

Analogously we deduce

$$\frac{9b^3}{c+2a} + b(c+2a) \geq 6b^2 \quad \text{and} \quad \frac{9c^3}{a+2b} + c(a+2b) \geq 6c^2.$$

Adding the last three inequalities we obtain

$$9\left(\frac{a^3}{b+2c} + \frac{b^3}{c+2a} + \frac{c^3}{a+2b}\right) + 3(ab+bc+ca) \geq 6(a^2+b^2+c^2),$$

i.e.

$$\frac{a^3}{b+2c} + \frac{b^3}{c+2a} + \frac{c^3}{a+2b} \geq \frac{6(a^2+b^2+c^2) - 3(ab+bc+ca)}{9}. \quad (1)$$

Using the well-known inequality

$$a^2 + b^2 + c^2 \geq ab + bc + ca,$$

according to (1) we obtain

$$\frac{a^3}{b+2c} + \frac{b^3}{c+2a} + \frac{c^3}{a+2b} \geq \frac{3(a^2+b^2+c^2)}{9} = \frac{a^2+b^2+c^2}{3} = \frac{3}{3} = 1. \quad \blacksquare$$

**110** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{1}{a^3+2} + \frac{1}{b^3+2} + \frac{1}{c^3+2} \geq 1.$$

*Solution* We have

$$\frac{1}{a^3+2} = \frac{1}{2} \left( 1 - \frac{a^3}{a^3+2} \right) = \frac{1}{2} \left( 1 - \frac{a^3}{a^3+1+1} \right) \geq \frac{1}{2} \left( 1 - \frac{a^3}{3a} \right) = \frac{1}{2} \left( 1 - \frac{a^2}{3} \right).$$

Therefore

$$\begin{aligned} \frac{1}{a^3+2} + \frac{1}{b^3+2} + \frac{1}{c^3+2} &\geq \frac{1}{2} \left( 1 - \frac{a^2}{3} \right) + \frac{1}{2} \left( 1 - \frac{b^2}{3} \right) + \frac{1}{2} \left( 1 - \frac{c^2}{3} \right) \\ &= \frac{3}{2} - \frac{a^2+b^2+c^2}{6} = 1. \end{aligned}$$

Equality holds iff  $a = b = c = 1$ .  $\blacksquare$

**111** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 1$ . Prove the inequality

$$\frac{a^3}{a^2+b^2} + \frac{b^3}{b^2+c^2} + \frac{c^3}{c^2+a^2} \geq \frac{1}{2}.$$

*Solution* Clearly we have

$$\frac{a^2+b^2}{2} \geq ab \quad \text{i.e.} \quad \frac{ab}{a^2+b^2} \leq \frac{1}{2}.$$

Therefore

$$\frac{a^3}{a^2+b^2} = a - b \frac{ab}{a^2+b^2} \geq a - \frac{b}{2}.$$

Analogously

$$\frac{b^3}{b^2+c^2} \geq b - \frac{c}{2} \quad \text{and} \quad \frac{c^3}{c^2+a^2} \geq c - \frac{a}{2}.$$

After adding these and using  $a + b + c = 1$  we obtain

$$\frac{a^3}{a^2+b^2} + \frac{b^3}{b^2+c^2} + \frac{c^3}{c^2+a^2} \geq a + b + c - \frac{a+b+c}{2} = \frac{a+b+c}{2} = \frac{1}{2}. \quad \blacksquare$$

**112** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$\frac{1}{1+2a^2b} + \frac{1}{1+2b^2c} + \frac{1}{1+2c^2a} \geq 1.$$

*Solution* Note that

$$\begin{aligned} \frac{1}{1+2a^2b} &= 1 - \frac{2a^2b}{1+2a^2b} = 1 - \frac{2a^2b}{1+a^2b+a^2b} \geq 1 - \frac{2a^2b}{3\sqrt[3]{a^4b^2}} \\ &= 1 - \frac{2\sqrt[3]{a^2b}}{3} \geq 1 - \frac{2(2a+b)}{9}. \end{aligned}$$

After adding these inequalities for all variables we get

$$\frac{1}{1+2a^2b} + \frac{1}{1+2b^2c} + \frac{1}{1+2c^2a} \geq 3 - \frac{6(a+b+c)}{9} = 3 - 2 = 1,$$

as required.

Equality holds iff  $a = b = c = 1$ . ■

**113** Let  $a, b, c, d$  be positive real numbers such that  $a + b + c + d = 4$ . Prove the inequality

$$\frac{a}{1+b^2c} + \frac{b}{1+c^2d} + \frac{c}{1+d^2a} + \frac{d}{1+a^2b} \geq 2.$$

*Solution* Applying  $AM \geq GM$  we have

$$\begin{aligned} \frac{a}{1+b^2c} &= a - \frac{ab^2c}{1+b^2c} \geq a - \frac{ab^2c}{2b\sqrt{c}} = a - \frac{ab\sqrt{c}}{2} \geq a - \frac{b\sqrt{a \cdot ac}}{2} \\ &\geq a - \frac{b(a+ac)}{4}, \end{aligned}$$

i.e.

$$\frac{a}{1+b^2c} \geq a - \frac{1}{4}(ab + abc).$$

Analogously we obtain

$$\begin{aligned} \frac{b}{1+c^2d} &\geq b - \frac{1}{4}(bc + bcd), & \frac{c}{1+d^2a} &\geq c - \frac{1}{4}(cd + cda), \\ \frac{d}{1+a^2b} &\geq d - \frac{1}{4}(da + dab). \end{aligned}$$

Adding these three inequalities we obtain

$$\begin{aligned} \frac{a}{1+b^2c} + \frac{b}{1+c^2d} + \frac{c}{1+d^2a} + \frac{d}{1+a^2b} \\ \geq (a+b+c+d) - \frac{1}{4}(ab+bc+cd+da+abc+bcd+cda+dab). \end{aligned} \quad (1)$$

One more use of  $AM \geq GM$  give us

$$ab+bc+cd+da \leq \frac{1}{4}(a+b+c+d)^2 = 4 \quad (2)$$

and

$$abc+bcd+cda+dab \leq \frac{1}{16}(a+b+c+d)^3 = 4. \quad (3)$$

From (1), (2) and (3) it follows that

$$\frac{a}{1+b^2c} + \frac{b}{1+c^2d} + \frac{c}{1+d^2a} + \frac{d}{1+a^2b} \geq 4 - 2 = 2,$$

as desired.

Equality holds if and only if  $a=b=c=d=1$ . ■

**114** Let  $a, b, c, d$  be positive real numbers. Prove the inequality

$$\frac{a^3}{a^2+b^2} + \frac{b^3}{b^2+c^2} + \frac{c^3}{c^2+d^2} + \frac{d^3}{d^2+a^2} \geq \frac{a+b+c+d}{2}.$$

*Solution* Using  $AM \geq GM$  we get

$$\frac{a^3}{a^2+b^2} = a - \frac{ab^2}{a^2+b^2} \geq a - \frac{ab^2}{2ab} = a - \frac{b}{2}.$$

Analogously

$$\frac{b^3}{b^2+c^2} \geq b - \frac{c}{2}, \quad \frac{c^3}{c^2+d^2} \geq c - \frac{d}{2}, \quad \frac{d^3}{d^2+a^2} \geq d - \frac{a}{2}.$$

Adding these inequalities give us the required inequality. ■

**115** Let  $a, b, c$  be positive real numbers such that  $a+b+c=3$ . Prove the inequality

$$\frac{a^2}{a+2b^2} + \frac{b^2}{b+2c^2} + \frac{c^2}{c+2a^2} \geq 1.$$

*Solution* Applying  $AM \geq GM$  we get

$$\frac{a^2}{a+2b^2} = a - \frac{2ab^2}{a+2b^2} \geq a - \frac{2ab^2}{3\sqrt[3]{ab^4}} = a - \frac{2(ab)^{2/3}}{3}.$$

Analogously we obtain

$$\frac{b^2}{b+2c^2} \geq b - \frac{2(bc)^{2/3}}{3} \quad \text{and} \quad \frac{c^2}{c+2a^2} \geq c - \frac{2(ca)^{2/3}}{3}.$$

Adding these three inequalities gives us

$$\frac{a^2}{a+2b^2} + \frac{b^2}{b+2c^2} + \frac{c^2}{c+2a^2} \geq (a+b+c) - \frac{2}{3}((ab)^{2/3} + (bc)^{2/3} + (ca)^{2/3}).$$

So it is enough to show that

$$(a+b+c) - \frac{2}{3}((ab)^{2/3} + (bc)^{2/3} + (ca)^{2/3}) \geq 1,$$

i.e.

$$(ab)^{2/3} + (bc)^{2/3} + (ca)^{2/3} \leq 3. \quad (1)$$

Applying  $AM \geq GM$  we get

$$\begin{aligned} (ab)^{2/3} + (bc)^{2/3} + (ca)^{2/3} &\leq \frac{(a+ab+b) + (b+bc+c) + (c+ca+a)}{3} \\ &= \frac{2(a+b+c) + (ab+bc+ca)}{3} \\ &\leq \frac{2(a+b+c) + (a+b+c)^2/3}{3} = \frac{2 \cdot 3 + 3^2/3}{3} = 3, \end{aligned}$$

i.e. we have proved (1), and we are done.

Equality holds iff  $a = b = c = 1$ . ■

**116** Let  $a, b, c$  be positive real numbers such that  $a+b+c=3$ . Prove the inequality

$$\frac{a^2}{a+2b^3} + \frac{b^2}{b+2c^3} + \frac{c^2}{c+2a^3} \geq 1.$$

*Solution* Applying  $AM \geq GM$  gives us

$$\frac{a^2}{a+2b^3} = a - \frac{2ab^3}{a+2b^3} \geq a - \frac{2ab^3}{3\sqrt[3]{ab^4}} = a - \frac{2ba^{2/3}}{3}.$$

Analogously

$$\frac{b^2}{b+2c^3} \geq b - \frac{2cb^{2/3}}{3} \quad \text{and} \quad \frac{c^2}{c+2a^3} \geq c - \frac{2ac^{2/3}}{3}.$$

Adding these three inequalities implies

$$\frac{a^2}{a+2b^3} + \frac{b^2}{b+2c^3} + \frac{c^2}{c+2a^3} \geq (a+b+c) - \frac{2}{3}(ba^{2/3} + cb^{2/3} + ac^{2/3}).$$



So it is enough to prove that

$$(a + b + c) - \frac{2}{3}(ba^{2/3} + cb^{2/3} + ac^{2/3}) \geq 1,$$

i.e.

$$ba^{2/3} + cb^{2/3} + ac^{2/3} \leq 3. \quad (1)$$

After another application of  $AM \geq GM$  we get

$$\begin{aligned} ba^{2/3} + cb^{2/3} + ac^{2/3} &\leq \frac{b(2a+1) + c(2b+1) + a(2c+1)}{3} \\ &= \frac{a+b+c + 2(ab+bc+ca)}{3} \\ &\leq \frac{(a+b+c) + (a+b+c)^2/3}{3} = \frac{3 + 2 \cdot 3^2/3}{3} = 3, \end{aligned}$$

i.e. we have proved (1), and we are done.

Equality holds iff  $a = b = c = 1$ . ■

**117** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Find the minimum value of the expression

$$a + b + c + \frac{16}{a + b + c}.$$

*Solution* By the inequality  $AM \geq GM$  we get

$$a + b + c + \frac{16}{a + b + c} \geq 2\sqrt{16} = 8,$$

with equality if and only if  $a + b + c = \frac{16}{a+b+c}$  from which we deduce that  $a + b + c = 4$  and then

$$16 = (a + b + c)^2 \leq 3(a^2 + b^2 + c^2) = 9,$$

a contradiction.

We estimate that the minimal value occurs when  $a = b = c$ , i.e.  $a = b = c = 1$ .

Let  $a + b + c = \frac{\alpha}{a+b+c}$ . Thus  $\alpha = 9$  at the point of incidence  $a = b = c = 1$ .

Therefore let us rewrite the given expression as follows

$$a + b + c + \frac{9}{a + b + c} + \frac{7}{a + b + c}. \quad (1)$$

Applying  $AM \geq GM$  and  $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$  we have

$$a + b + c + \frac{9}{a + b + c} \geq 2\sqrt{9} = 6 \quad (2)$$

and

$$\frac{1}{a+b+c} \geq \frac{1}{\sqrt{3(a^2+b^2+c^2)}} = \frac{1}{3}. \quad (3)$$

By (1), (2) and (3) we obtain

$$a+b+c + \frac{16}{a+b+c} = a+b+c + \frac{9}{a+b+c} + \frac{7}{a+b+c} \geq 6 + \frac{7}{3} = \frac{25}{3},$$

with equality if and only if  $a = b = c = 1$ . ■

**118** Let  $a, b, c \geq 0$  be real numbers such that  $a^2 + b^2 + c^2 = 1$ . Find the minimal value of the expression

$$A = a + b + c + \frac{1}{abc}.$$

*Solution* By  $AM \geq GM$  we obtain

$$A = a + b + c + \frac{1}{abc} \geq 4\sqrt[4]{abc \cdot \frac{1}{abc}} = 4,$$

with equality iff  $a = b = c = \frac{1}{abc}$ , i.e.  $a = b = c = 1$ .

Thus  $a^2 + b^2 + c^2 = 3 \neq 1$ , a contradiction.

Since  $A$  is a symmetrical expression in  $a, b$  and  $c$ , we estimate that  $\min A$  occurs at the incidence point  $a = b = c$ , i.e.  $a = b = c = 1/\sqrt{3}$ .

Hence at the incidence point we have  $a = b = c = \frac{1}{abc} = \frac{1}{\sqrt{3}}$ , and it follows that  $\alpha = \frac{1}{a^2bc} = \frac{1}{(1/\sqrt{3})^4} = 9$ .

Therefore

$$\begin{aligned} A &= a + b + c + \frac{1}{abc} = a + b + c + \frac{1}{9abc} + \frac{8}{9abc} \\ &\geq 4\sqrt[4]{abc \cdot \frac{1}{9abc}} + \frac{8}{9abc} = 4\sqrt[4]{\frac{1}{9}} + \frac{8}{9abc}. \end{aligned} \quad (1)$$

By  $QM \geq GM$  we obtain

$$\sqrt{\frac{a^2 + b^2 + c^2}{3}} \geq 3\sqrt[3]{abc}, \quad \text{i.e.} \quad \sqrt{\frac{1}{3}} \geq 3\sqrt[3]{abc}.$$

Hence

$$\frac{1}{abc} \geq 3\sqrt{3}. \quad (2)$$

By (1) and (2) we get

$$A \geq \frac{4}{\sqrt{3}} + 3\sqrt{3} \cdot \frac{8}{9} = 4\sqrt{3}.$$

So  $\min A = 4\sqrt{3}$ , and it occurs iff  $a = b = c = 1/\sqrt{3}$ . ■

**119** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 6$ . Prove the inequality

$$\sqrt[3]{ab+bc} + \sqrt[3]{bc+ca} + \sqrt[3]{ca+ab} + \sqrt[3]{\frac{9}{4}(a^2+b^2+c^2)} \leq 9.$$

*Solution* Analogously as in the first solution of Exercise 5.13 we obtain that

$$\sqrt[3]{ab+bc} + \sqrt[3]{bc+ca} + \sqrt[3]{ca+ab} \leq \frac{1}{4} \left( \frac{2(ab+bc+ca) + 48}{3} \right). \quad (1)$$

At the point of incidence  $a = b = c = 2$  we have  $a^2 + b^2 + c^2 = 12$ .

Therefore by  $AM \geq GM$  we have

$$\begin{aligned} \sqrt[3]{\frac{9}{4}(a^2+b^2+c^2)} &= \sqrt[3]{\frac{9(a^2+b^2+c^2) \cdot 12 \cdot 12}{4 \cdot 12 \cdot 12}} = \frac{1}{4} \sqrt[3]{(a^2+b^2+c^2) \cdot 12 \cdot 12} \\ &\leq \frac{1}{4} \left( \frac{a^2+b^2+c^2+24}{3} \right). \end{aligned} \quad (2)$$

By (1) and (2) we obtain

$$\begin{aligned} &\sqrt[3]{ab+bc} + \sqrt[3]{bc+ca} + \sqrt[3]{ca+ab} + \sqrt[3]{\frac{9}{4}(a^2+b^2+c^2)} \\ &\leq \frac{1}{4} \left( \frac{2(ab+bc+ca) + 48}{3} \right) + \frac{1}{4} \left( \frac{a^2+b^2+c^2+24}{3} \right) \\ &= \frac{1}{12} ((a+b+c)^2 + 72) = \frac{6^2 + 72}{12} = 9, \end{aligned}$$

as required.

Equality occurs if and only if  $a = b = c = 2$ . ■

**120** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + 2b + 3c \geq 20$ . Prove the inequality

$$S = a + b + c + \frac{3}{a} + \frac{9}{2b} + \frac{4}{c} \geq 13.$$

*Solution*  $S = 13$  at the point  $a = 2, b = 3, c = 4$ .

Using  $AM \geq GM$  we get

$$a + \frac{4}{a} \geq 2\sqrt{a \cdot \frac{4}{a}} = 4, \quad b + \frac{9}{b} \geq 2\sqrt{b \cdot \frac{9}{b}} = 6, \quad c + \frac{16}{c} \geq 2\sqrt{c \cdot \frac{16}{c}} = 8,$$

i.e.

$$\frac{3}{4} \left( a + \frac{4}{a} \right) \geq 3, \quad \frac{1}{2} \left( b + \frac{9}{b} \right) \geq 3 \quad \text{and} \quad \frac{1}{4} \left( c + \frac{16}{c} \right) \geq 2.$$

Adding the last three inequalities we have

$$\frac{3}{4}a + \frac{1}{2}b + \frac{1}{4}c + \frac{3}{a} + \frac{9}{2b} + \frac{4}{c} \geq 8. \quad (1)$$

Using  $a + 2b + 3c \geq 20$  we obtain

$$\frac{1}{4}a + \frac{1}{2}b + \frac{3}{4}c \geq 5. \quad (2)$$

Finally, after adding (1) and (2) we get

$$a + b + c + \frac{3}{a} + \frac{9}{2b} + \frac{4}{c} \geq 13,$$

as desired. ■

**121** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$S = 30a + 3b^2 + \frac{2c^3}{9} + 36\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) \geq 84.$$

*Solution*  $S = 84$  at the point  $a = 1, b = 2, c = 3$ .

From  $AM \geq GM$  we obtain

$$\begin{aligned} 2 \cdot a + \frac{b^2}{4} + 2 \cdot \frac{2}{ab} &= a + a + \frac{b^2}{4} + \frac{2}{ab} + \frac{2}{ab} \geq 5 \sqrt[5]{a^2 \cdot \frac{b^2}{4} \cdot \left(\frac{2}{ab}\right)^2} = 5, \\ 3 \cdot \frac{b^2}{4} + 2 \cdot \frac{c^3}{27} + 6 \cdot \frac{6}{bc} &\geq 11 \sqrt[11]{\left(\frac{b^2}{4}\right)^3 \cdot \left(\frac{c^3}{27}\right)^2 \cdot \left(\frac{6}{bc}\right)^6} = 11, \\ \frac{c^3}{27} + 3 \cdot a + 3 \cdot \frac{3}{ca} &\geq 7 \sqrt[7]{\frac{c^3}{27} \cdot a^3 \cdot \left(\frac{3}{ca}\right)^3} = 7, \end{aligned}$$

i.e.

$$9\left(2a + \frac{b^2}{4} + \frac{4}{ab}\right) \geq 45, \quad \frac{3b^2}{4} + \frac{2c^3}{27} + \frac{36}{bc} \geq 11, \quad 4\left(\frac{c^3}{27} + 3a + \frac{9}{ca}\right) \geq 28.$$

After adding these three inequalities we get

$$30a + 3b^2 + \frac{2c^3}{9} + 36\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) \geq 84. \quad \blacksquare$$

**122** Let  $a, b, c \in \mathbb{R}^+$  such that  $ac \geq 12$  and  $bc \geq 8$ . Prove the inequality

$$S = a + b + c + 2\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) + \frac{8}{abc} \geq \frac{121}{12}.$$

*Solution*  $S = \frac{121}{12}$  at the point  $a = 3, b = 2, c = 4$ .

Use of  $AM \geq GM$  gives us

$$\begin{aligned} \frac{a}{3} + \frac{b}{2} + \frac{6}{ab} &\geq 3, & \frac{b}{2} + \frac{c}{4} + \frac{8}{bc} &\geq 3, & \frac{c}{4} + \frac{a}{3} + \frac{12}{ca} &\geq 3 \quad \text{and} \\ \frac{a}{3} + \frac{b}{2} + \frac{c}{4} + \frac{24}{abc} &\geq 4, \end{aligned}$$

i.e.

$$\begin{aligned} \frac{a}{3} + \frac{b}{2} + \frac{6}{ab} &\geq 3, & 4\left(\frac{b}{2} + \frac{c}{4} + \frac{8}{bc}\right) &\geq 12, & 7\left(\frac{c}{4} + \frac{a}{3} + \frac{12}{ca}\right) &\geq 21, \\ \frac{a}{3} + \frac{b}{2} + \frac{c}{4} + \frac{24}{abc} &\geq 4. \end{aligned}$$

After adding these three inequalities we get

$$3(a + b + c) + \frac{6}{ab} + \frac{32}{bc} + \frac{84}{ca} + \frac{24}{abc} \geq 40. \quad (1)$$

Also, since  $ac \geq 12$  and  $bc \geq 8$  we obtain

$$\frac{1}{ac} \leq \frac{1}{12} \quad \text{and} \quad \frac{1}{bc} \leq \frac{1}{8},$$

so from (1) it follows that

$$40 \leq 3S + \frac{26}{bc} + \frac{78}{ca} \leq 3S + \frac{26}{12} + \frac{78}{8}, \quad \text{i.e.} \quad S \geq \frac{121}{12}. \quad \blacksquare$$

**123** Let  $a, b, c, d > 0$  be real numbers. Determine the minimal value of the expression

$$A = \left(1 + \frac{2a}{3b}\right) \left(1 + \frac{2b}{3c}\right) \left(1 + \frac{2c}{3d}\right) \left(1 + \frac{2d}{3a}\right).$$

*Solution* By  $AM \geq GM$  we get

$$A \geq 2\sqrt{\frac{2a}{3b}} \cdot 2\sqrt{\frac{2b}{3c}} \cdot 2\sqrt{\frac{2c}{3d}} \cdot 2\sqrt{\frac{2d}{3a}} = 8,$$

with equality if and only if  $\frac{2a}{3b} = \frac{2b}{3c} = \frac{2c}{3d} = \frac{2d}{3a} = 1$ .

Hence  $2(a + b + c + d) = 3(a + b + c + d)$ , i.e.  $2 = 3$ , which is impossible.

Since  $A$  is a symmetrical expression in  $a, b, c$  and  $d$ , the minimum (maximum) occurs at the incidence point  $a = b = c = d > 0$ , and then

$$A = \left(1 + \frac{2}{3}\right)^4 = \frac{625}{81}.$$

We have

$$1 + \frac{2a}{3b} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{a}{3b} + \frac{a}{3b} \geq 5\sqrt[5]{\left(\frac{1}{3}\right)^3 \left(\frac{a}{3b}\right)^2} = \frac{5}{3} \left(\frac{a}{b}\right)^{2/5}.$$

Similarly we get

$$1 + \frac{2b}{3c} \geq \frac{5}{3} \left(\frac{b}{c}\right)^{2/5}, \quad 1 + \frac{2c}{3d} \geq \frac{5}{3} \left(\frac{c}{d}\right)^{2/5} \quad \text{and} \quad 1 + \frac{2d}{3a} \geq \frac{5}{3} \left(\frac{d}{a}\right)^{2/5}.$$

If we multiply the above inequalities we obtain  $A \geq \frac{625}{81}$ .

Equality holds if and only if  $a = b = c = d > 0$ . ■

**124** Let  $a, b, c > 0$  be real numbers such that  $a^2 + b^2 + c^2 = 12$ . Determine the maximal value of the expression

$$A = a\sqrt[3]{b^2 + c^2} + b\sqrt[3]{c^2 + a^2} + c\sqrt[3]{a^2 + b^2}.$$

*Solution* Since  $A$  is a symmetrical expression with respect to  $a, b$  and  $c$ ,  $\max A$  occurs when  $a = b = c > 0$ , i.e.  $a = b = c = 2$ .

Hence

$$2a^2 = 2b^2 = 2c^2 = 8$$

and

$$b^2 + c^2 = c^2 + a^2 = a^2 + b^2 = 8.$$

By  $AM \geq GM$  we have

$$\begin{aligned} a\sqrt[3]{b^2 + c^2} &= \sqrt[3]{a^3(b^2 + c^2)} = \sqrt[6]{a^6(b^2 + c^2)^2} = \frac{1}{2} \sqrt[6]{(2a^2)^3 \cdot (b^2 + c^2)^2 \cdot 8} \\ &= \frac{1}{2} \sqrt[6]{8(2a^2)(2a^2)(2a^2)(b^2 + c^2)(b^2 + c^2)} \\ &\leq \frac{1}{2} \cdot \frac{8 + 6a^2 + 2(b^2 + c^2)}{6} = \frac{4 + 3a^2 + b^2 + c^2}{6}. \end{aligned}$$

Similarly

$$b\sqrt[3]{c^2 + a^2} \leq \frac{4 + a^2 + 3b^2 + c^2}{6} \quad \text{and} \quad c\sqrt[3]{a^2 + b^2} \leq \frac{4 + a^2 + b^2 + 3c^2}{6}.$$

After adding the last three inequalities we get

$$A \leq \frac{12 + 5(a^2 + b^2 + c^2)}{6} = \frac{12 + 5 \cdot 12}{6} = 12,$$

with equality if and only if  $a = b = c = 2$ . ■

**125** Let  $a, b, c \geq 0$  such that  $a + b + c = 3$ . Prove the inequality

$$(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \leq 12.$$

*Solution* Without loss of generality we may assume that  $a \geq b \geq c \geq 0$ .

So it follows that

$$0 \leq b^2 - bc + c^2 \leq b^2 \quad \text{and} \quad 0 \leq c^2 - ca + a^2 \leq a^2,$$

i.e. we obtain

$$(b^2 - bc + c^2)(c^2 - ca + a^2) \leq a^2 b^2.$$

Now we have

$$\begin{aligned} & (a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \\ & \leq a^2 b^2 (a^2 - ab + b^2) \\ & = \frac{4}{9} \cdot \frac{3ab}{2} \cdot \frac{3ab}{2} \cdot (a^2 - ab + b^2) \leq \frac{4}{9} \cdot \left( \frac{1}{3} \left( \frac{3ab}{2} + \frac{3ab}{2} + (a^2 - ab + b^2) \right) \right)^3 \\ & = \frac{4}{9} \left( \frac{(a+b)^2}{3} \right)^3 \leq \frac{4}{9} \left( \frac{(a+b+c)^2}{3} \right)^3 = \frac{4}{9} \left( \frac{3^2}{3} \right)^3 = 12. \quad \blacksquare \end{aligned}$$

**126** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

*Solution* For every positive real number  $x$ , we have that  $x^2 - 1$  and  $x^3 - 1$  have the same signs, and because of this  $x^5 - x^3 - x^2 + 1 = (x^2 - 1)(x^3 - 1) \geq 0$ , i.e. we obtain

$$x^5 - x^2 + 3 \geq x^3 + 2.$$

Now we get

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a^3 + 2)(b^3 + 2)(c^3 + 2).$$

So it is enough to show that

$$(a^3 + 2)(b^3 + 2)(c^3 + 2) \geq (a + b + c)^3. \quad (1)$$

After a little algebra we obtain that (1) is equivalent to

$$\begin{aligned} & a^3 b^3 c^3 + 3(a^3 + b^3 + c^3) + 2(a^3 b^3 + b^3 c^3 + c^3 a^3) + 8 \\ & \geq 3(a^2 b + b^2 a + b^2 c + c^2 b + c^2 a + a^2 c) + 6abc. \end{aligned} \quad (2)$$

Using  $AM \geq GM$  we can easily obtain the following inequalities

$$\begin{aligned} a^3 + a^3b^3 + 1 &\geq 3a^2b, & a^3 + a^3c^3 + 1 &\geq 3a^2c, & b^3 + a^3b^3 + 1 &\geq 3b^2a, \\ b^3 + b^3c^3 + 1 &\geq 3b^2c, & c^3 + c^3a^3 + 1 &\geq 3c^2a, & c^3 + c^3b^3 + 1 &\geq 3c^2b, \\ a^3b^3c^3 + a^3 + b^3 + c^3 + 1 + 1 &\geq 6abc. \end{aligned}$$

After adding the previous inequalities we obtain inequality (2), as desired.  $\blacksquare$

**127** Let  $x, y, z \in \mathbb{R}^+$  such that  $x + y + z = 1$ . Prove the inequality

$$\frac{xy}{\sqrt{1+z^2}} + \frac{zx}{\sqrt{1+y^2}} + \frac{yz}{\sqrt{1+x^2}} \leq \frac{1}{\sqrt{10}}.$$

*Solution* We have

$$\begin{aligned} \sqrt{1+z^2} &= \sqrt{9 \cdot \left(\frac{1}{3}\right)^2 + z^2} = \sqrt{\underbrace{\frac{1}{3^2} + \dots + \frac{1}{3^2}}_9 + z^2} \stackrel{K \geq A}{\geq} \frac{1}{\sqrt{10}} \underbrace{\left(\frac{1}{3} + \dots + \frac{1}{3} + z\right)}_9 \\ &= \frac{3+z}{\sqrt{10}}, \end{aligned}$$

i.e. we obtain that

$$\frac{xy}{\sqrt{1+z^2}} \leq \sqrt{10} \frac{xy}{3+z}.$$

Analogously we obtain

$$\frac{yz}{\sqrt{1+x^2}} \leq \sqrt{10} \frac{yz}{3+x} \quad \text{and} \quad \frac{zx}{\sqrt{1+y^2}} \leq \sqrt{10} \frac{zx}{3+y}.$$

So it is enough to prove that

$$\sqrt{10} \left( \frac{xy}{3+z} + \frac{zx}{3+y} + \frac{yz}{3+x} \right) \leq \frac{1}{\sqrt{10}},$$

i.e.

$$\frac{xy}{3+z} + \frac{zx}{3+y} + \frac{yz}{3+x} \leq \frac{1}{10}. \quad (1)$$

Let  $a = 3 + x, b = 3 + y, c = 3 + z$ .

Then clearly  $a + b + c = 10$ .

Inequality (1) is equivalent to

$$\frac{(a-3)(b-3)}{c} + \frac{(c-3)(a-3)}{b} + \frac{(b-3)(c-3)}{a} \leq \frac{1}{10},$$



i.e.

$$\begin{aligned}
 & \frac{ab - 3(a + b) + 9}{c} + \frac{ca - 3(c + a) + 9}{b} + \frac{bc - 3(b + c) + 9}{a} \leq \frac{1}{10} \\
 \Leftrightarrow & \frac{ab + 3c - 21}{c} + \frac{ca + 3b - 21}{b} + \frac{bc + 3a - 21}{a} \leq \frac{1}{10} \\
 \Leftrightarrow & \frac{ab - 21}{c} + \frac{ca - 21}{b} + \frac{bc - 21}{a} \leq -\frac{89}{10}.
 \end{aligned}$$

After clearing denominators, we obtain

$$\begin{aligned}
 & 21(a^3(b + c) + b^3(a + c) + c^3(b + a)) + 16(a^2bc + b^2ac + c^2ab) \\
 & \geq 58(a^2b^2 + b^2c^2 + c^2a^2) \\
 \Leftrightarrow & (21ab - 8c^2)(a - b)^2 + (21bc - 8a^2)(b - c)^2 \\
 & + (21ca - 8b^2)(c - a)^2 \geq 0,
 \end{aligned}$$

which is true since  $a, b, c \in (3, 4)$ , i.e.

$$21ab - 8c^2 \geq 21 \cdot 3 \cdot 3 - 8 \cdot 4^2 = 61 > 0.$$

In the same way we find that  $21bc - 8a^2 > 0$  and  $21ca - 8b^2 > 0$ . ■

**128** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$(a + b + c)^6 \geq 27(a^2 + b^2 + c^2)(ab + bc + ca)^2.$$

*Solution* Denote  $x = a + b + c$ ,  $y = ab + bc + ca$ .

Then we have

$$\begin{aligned}
 & x^6 \geq 27(x^2 - 2y)y^2 \\
 \Leftrightarrow & x^6 \geq 27x^2y^2 - 54y^3 \\
 \Leftrightarrow & (x^2 - 3y)(x^4 + 3x^2y - 18y^2) \geq 0,
 \end{aligned}$$

which is true, since

$$\begin{aligned}
 & x^2 = (a + b + c)^2 \geq 3(ab + bc + ca) = 3y, \quad x^4 \geq 9y^2 \quad \text{and} \\
 & 3x^2y \geq 3 \cdot 3y \cdot y = 9y^2,
 \end{aligned}$$

i.e. we have

$$x^2 - 3y \geq 0 \quad \text{and} \quad x^4 + 3x^2y - 18y^2 \geq 9y^2 + 9y^2 - 18y^2 = 0. \quad \blacksquare$$

**129** Let  $a, b, c \in [1, 2]$  be real numbers. Prove the inequality

$$a^3 + b^3 + c^3 \leq 5abc.$$

*Solution* Without loss of generality we may assume that  $a \geq b \geq c$ .

Then since  $a, b, c \in [1, 2]$  we have

$$b^2 + b + 1 \leq a^2 + a + 1 \leq 2a + a + 1 \leq 5a \quad \text{and}$$

$$c^2 + c + 1 \leq a^2 + a + 1 \leq 5a \leq 5ab.$$

Because of the previous inequalities it follows that:

$$a^3 + 2 \leq 5a \quad \Leftrightarrow \quad (a - 2)(a^2 + 2a - 1) \leq 0, \quad (1)$$

$$5a + b^3 \leq 5ab + 1 \quad \Leftrightarrow \quad (b - 1)(b^2 + b + 1 - 5a) \leq 0, \quad (2)$$

$$5ab + c^3 \leq 5abc + 1 \quad \Leftrightarrow \quad (c - 1)(c^2 + c + 1 - 5ab) \leq 0. \quad (3)$$

Adding (1), (2) and (3) gives us the desired inequality.

Equality holds iff  $a = 2, b = c = 1$ . ■

**130** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 3$ . Prove the inequality

$$(a^7 - a^4 + 3)(b^5 - b^2 + 3)(c^4 - c + 3) \geq 27.$$

*Solution* For any real number  $x$ , the numbers  $x - 1, x^2 - 1, x^3 - 1$  and  $x^4 - 1$ , are of the same sign.

Therefore

$$(x - 1)(x^3 - 1) \geq 0, \quad (x^2 - 1)(x^3 - 1) \geq 0 \quad \text{and} \quad (x^3 - 1)(x^4 - 1) \geq 0,$$

i.e.

$$c^4 - c^3 - c + 1 \geq 0, \quad (1)$$

$$b^5 - b^3 - b^2 + 1 \geq 0, \quad (2)$$

$$a^7 - a^4 - a^3 + 1 \geq 0. \quad (3)$$

By (1), (2) and (3) we have

$$a^7 - a^4 + 3 \geq a^3 + 2, \quad b^5 - b^2 + 3 \geq b^3 + 2 \quad \text{and} \quad c^4 - c + 3 \geq c^3 + 2.$$

After multiplying these inequalities it follows that

$$(a^7 - a^4 + 3)(b^5 - b^2 + 3)(c^4 - c + 3) \geq (a^3 + 2)(b^3 + 2)(c^3 + 2). \quad (4)$$

Analogously as in Problem 126, we can prove that

$$(a^3 + 2)(b^3 + 2)(c^3 + 2) \geq (a + b + c)^3. \quad (5)$$

By the obvious inequality  $(a + b + c)^2 \geq 3(ab + bc + ca)$ , since  $ab + bc + ca = 3$  we deduce that

$$a + b + c \geq 3. \quad (6)$$

Finally from (4), (5) and (6) we obtain the required inequality.

Equality occurs iff  $a = b = c = 1$ . ■

**131** Let  $a, b, c \in [1, 2]$  be real numbers. Prove the inequality

$$(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \leq 10.$$

*Solution* The given inequality is equivalent to

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \leq 7. \quad (1)$$

Without loss of generality we may assume that  $a \geq b \geq c$ .

Then, since  $(a - b)(b - c) \geq 0$  we deduce that

$$ab + bc \geq b^2 + ac, \quad \text{i.e.} \quad \frac{a}{c} + 1 \geq \frac{a}{b} + \frac{b}{c}.$$

Analogously as  $ab + bc \geq b^2 + ac$  we have  $\frac{c}{a} + 1 \geq \frac{c}{b} + \frac{b}{a}$ .

Now we obtain

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{b} + \frac{b}{a} \leq \frac{a}{c} + \frac{c}{a} + 2.$$

So

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \leq 2 + 2 \left( \frac{a}{c} + \frac{c}{a} \right). \quad (2)$$

Let  $x = \frac{a}{c}$ . Then  $2 \geq x \geq 1$ , i.e. we have that  $(x - 2)(x - 1) \leq 0$ , from which we deduce that

$$x + \frac{1}{x} \leq \frac{5}{2}. \quad (3)$$

Finally using (2) and (3) we obtain inequality (1).

Equality occurs iff  $a = b = 2, c = 1$  or  $a = 2, b = c = 1$ . ■

**132** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 1$ . Prove the inequality

$$10(a^3 + b^3 + c^3) - 9(a^5 + b^5 + c^5) \geq 1.$$

**Solution** Denote  $L = 10(a^3 + b^3 + c^3) - 9(a^5 + b^5 + c^5)$ .

Let  $x = a + b + c = 1$ ,  $y = ab + bc + ca$ ,  $z = abc$ .

Then

$$\begin{aligned} 10(a^3 + b^3 + c^3) &= 10((a + b + c)^3 - 3(a + b + c)(ab + bc + ca) + 3abc) \\ &= 10 - 30y + 30z \end{aligned}$$

and

$$\begin{aligned} 9(a^5 + b^5 + c^5) &= 9(x^5 - 5x^3y + 5xy^2 + 5x^2z - 5yz) \\ &= 9(1 - 5y + 5y^2 + 5z - 5yz) \\ &= 9 - 45y + 45y^2 + 45z - 45yz. \end{aligned}$$

We have

$$L \geq 1 \quad \Leftrightarrow \quad 10 - 30y + 30z - 9 + 45y - 45y^2 - 45z + 45yz \geq 1,$$

i.e.

$$1 + 15y - 15z - 45y^2 + 45yz \geq 1,$$

i.e.

$$y - z - 3y(y - z) \geq 0 \quad \Leftrightarrow \quad (1 - 3y)(y - z) \geq 0. \quad (1)$$

Furthermore,

$$y = ab + bc + ca \leq \frac{(a + b + c)^2}{3} = \frac{1}{3}, \quad \text{i.e.} \quad 1 - 3y \geq 0$$

and

$$y = ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2} = 3\sqrt[3]{z^2} > z.$$

The last inequality is true since

$$z = abc \leq \left(\frac{a + b + c}{3}\right)^3 = 1 < 27.$$

From the previous two inequalities we get inequality (1), as desired. ■

**133** Let  $n \in \mathbb{N}$  and  $x_1, x_2, \dots, x_n \in (0, \pi)$ . Find maximum value of the expression

$$\sin x_1 \cos x_2 + \sin x_2 \cos x_3 + \dots + \sin x_n \cos x_1.$$

**Solution** It's clear that for all real numbers  $a, b$  we have  $a^2 + b^2 \geq 2ab$ . So we obtain

$$\begin{aligned} &\sin x_1 \cos x_2 + \sin x_2 \cos x_3 + \dots + \sin x_n \cos x_1 \\ &\leq \frac{\sin^2 x_1 + \cos^2 x_2}{2} + \frac{\sin^2 x_2 + \cos^2 x_3}{2} + \dots + \frac{\sin^2 x_n + \cos^2 x_1}{2} = \frac{n}{2}. \end{aligned}$$

Equality occurs iff  $x_1 = x_2 = \dots = x_n = \frac{\pi}{4}$ . ■

**134** Let  $\alpha_i \in [\frac{\pi}{4}, \frac{5\pi}{4}]$ , for  $i = 1, 2, \dots, n$ . Prove the inequality

$$\left( \sin \alpha_1 + \sin \alpha_2 + \dots + \sin \alpha_n + \frac{1}{4} \right)^2 \geq (\cos \alpha_1 + \cos \alpha_2 + \dots + \cos \alpha_n).$$

*Solution* Let  $S = \sin \alpha_1 + \sin \alpha_2 + \dots + \sin \alpha_n$ .

We have

$$\left( S + \frac{1}{4} \right)^2 = S^2 + \frac{S}{2} + \frac{1}{16} = S^2 - \frac{S}{2} + \frac{1}{16} + S = \left( S - \frac{1}{4} \right)^2 + S \geq S. \quad (1)$$

Since  $\alpha_i \in [\frac{\pi}{4}, \frac{5\pi}{4}]$  we deduce that

$$\sin \alpha_i \geq \cos \alpha_i, \quad \text{for all } i = 1, 2, \dots, n. \quad (2)$$

Using (1) and (2) we obtain the required inequality. ■

**135** Let  $a_1, a_2, \dots, a_n$ ;  $a_{n+1} = a_1, a_{n+2} = a_2$  be positive real numbers. Prove the inequality

$$\sum_{i=1}^n \frac{a_i - a_{i+2}}{a_{i+1} + a_{i+2}} \geq 0.$$

*Solution* Applying  $AM \geq GM$  we have

$$\begin{aligned} \sum_{i=1}^n \frac{a_i + a_{i+1}}{a_{i+1} + a_{i+2}} &= \frac{a_1 + a_2}{a_2 + a_3} + \frac{a_2 + a_3}{a_3 + a_4} + \dots + \frac{a_{n-1} + a_n}{a_n + a_1} + \frac{a_n + a_1}{a_1 + a_2} \\ &\geq n \sqrt[n]{\frac{a_1 + a_2}{a_2 + a_3} \cdot \frac{a_2 + a_3}{a_3 + a_4} \dots \frac{a_{n-1} + a_n}{a_n + a_1} \cdot \frac{a_n + a_1}{a_1 + a_2}} = n. \end{aligned} \quad (1)$$

So

$$\begin{aligned} \sum_{i=1}^n \frac{a_{i+1}}{a_{i+1} + a_{i+2}} &= \sum_{i=1}^n \frac{a_{i+1} + a_{i+2}}{a_{i+1} + a_{i+2}} - \sum_{i=1}^n \frac{a_{i+2}}{a_{i+1} + a_{i+2}} \\ &= n - \sum_{i=1}^n \frac{a_{i+2}}{a_{i+1} + a_{i+2}} \\ &\stackrel{(1)}{\leq} \sum_{i=1}^n \frac{a_i + a_{i+1}}{a_{i+1} + a_{i+2}} - \sum_{i=1}^n \frac{a_{i+2}}{a_{i+1} + a_{i+2}}, \end{aligned}$$

from where it follows that

$$\sum_{i=1}^n \frac{a_i - a_{i+2}}{a_{i+1} + a_{i+2}} \geq 0. \quad \blacksquare$$

**136** Let  $n \geq 2$ ,  $n \in \mathbb{N}$  and  $x_1, x_2, \dots, x_n$  be positive real numbers such that

$$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + \cdots + \frac{1}{x_n + 1998} = \frac{1}{1998}.$$

Prove the inequality

$$\sqrt[n]{x_1 x_2 \cdots x_n} \geq 1998(n-1).$$

*Solution* After setting  $\frac{1998}{x_i + 1998} = a_i$ , for  $i = 1, 2, \dots, n$ , the identity

$$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + \cdots + \frac{1}{x_n + 1998} = \frac{1}{1998}$$

becomes

$$a_1 + a_2 + \cdots + a_n = 1.$$

We need to show that

$$\left(\frac{1}{a_1} - 1\right)\left(\frac{1}{a_2} - 1\right) \cdots \left(\frac{1}{a_n} - 1\right) \geq (n-1)^n. \quad (1)$$

We have

$$\begin{aligned} \frac{1}{a_i} - 1 &= \frac{1 - a_i}{a_i} = \frac{a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n}{a_i} \\ &\geq (n-1) \sqrt[n-1]{\frac{a_1 \cdots a_{i-1} a_{i+1} \cdots a_n}{a_i^{n-1}}}. \end{aligned}$$

Multiplying these inequalities for  $i = 1, 2, \dots, n$  we obtain (1), as desired. ■

**137** Let  $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ . Prove the inequality

$$\sum_{k=1}^n k a_k \leq \binom{n}{2} + \sum_{k=1}^n a_k^k.$$

*Solution* For  $1 \leq k \leq n$  we have

$$a_k^k + (k-1) = a_k^k + \underbrace{1 + 1 + \cdots + 1}_{k-1} \geq k \sqrt[k]{a_k^k} = k a_k.$$

After adding these inequalities, for  $1 \leq k \leq n$  we get

$$\sum_{k=1}^n k a_k \leq \sum_{k=1}^n a_k^k + \sum_{k=1}^n (k-1) = \sum_{k=1}^n a_k^k + \frac{n(n-1)}{2} = \sum_{k=1}^n a_k^k + \binom{n}{2}. \quad \blacksquare$$

**138** Let  $a_1, a_2, \dots, a_n$  be positive real numbers such that  $a_1 + a_2 + \dots + a_n = n$ . Prove that for every natural number  $k$  the following inequality holds

$$a_1^k + a_2^k + \dots + a_n^k \geq a_1^{k-1} + a_2^{k-1} + \dots + a_n^{k-1}.$$

*Solution* Using  $AM \geq GM$  we get

$$(k-1)a_i^k + 1 = \underbrace{a_i^k + a_i^k + \dots + a_i^k}_{k-1} + 1 \geq k \sqrt[k]{a_i^{k(k-1)}} = k a_i^{k-1}$$

and if we add these inequalities for  $i = 1, 2, \dots, n$  we obtain

$$(k-1)(a_1^k + a_2^k + \dots + a_n^k) + n \geq k(a_1^{k-1} + a_2^{k-1} + \dots + a_n^{k-1}). \quad (1)$$

We'll show that

$$a_1^{k-1} + a_2^{k-1} + \dots + a_n^{k-1} \geq n. \quad (2)$$

One more application of  $AM \geq GM$  gives us

$$a_i^{k-1} + (k-2) = a_i^{k-1} + \underbrace{1 + \dots + 1}_{k-2} \geq (k-1) \sqrt[k-1]{a_i^{k-1}} = (k-1)a_i$$

and adding the previous inequalities for  $i = 1, 2, \dots, n$  we get

$$(a_1^{k-1} + a_2^{k-1} + \dots + a_n^{k-1}) + n(k-2) \geq (k-1)(a_1 + a_2 + \dots + a_n) = n(k-1),$$

from which we deduce

$$a_1^{k-1} + a_2^{k-1} + \dots + a_n^{k-1} \geq n.$$

So we are done with (2).

Now from (1) and (2) we obtain

$$\begin{aligned} (k-1)(a_1^k + a_2^k + \dots + a_n^k) + n &\geq k(a_1^{k-1} + a_2^{k-1} + \dots + a_n^{k-1}) \\ \Leftrightarrow (k-1)(a_1^k + a_2^k + \dots + a_n^k) + n &\geq (k-1)(a_1^{k-1} + a_2^{k-1} + \dots + a_n^{k-1}) + (a_1^{k-1} + a_2^{k-1} + \dots + a_n^{k-1}) \\ &\geq (k-1)(a_1^{k-1} + a_2^{k-1} + \dots + a_n^{k-1}) + n \\ \Leftrightarrow a_1^k + a_2^k + \dots + a_n^k &\geq a_1^{k-1} + a_2^{k-1} + \dots + a_n^{k-1}, \end{aligned}$$

as desired.

Equality holds iff  $a_1 = a_2 = \dots = a_n = 1$ . ■

*Remark* The given inequality immediately follows by *Chebyshev's inequality*.

**139** Let  $a, b, c, d$  be positive real numbers. Prove the inequality

$$\left(\frac{a}{a+b}\right)^5 + \left(\frac{b}{b+c}\right)^5 + \left(\frac{c}{c+d}\right)^5 + \left(\frac{d}{d+a}\right)^5 \geq \frac{1}{8}.$$

*Solution 1* Let  $x = b/a, y = c/b, z = d/c$  and  $t = a/d$ .

Then it is clear that  $xyz t = 1$ , and the given inequality becomes

$$A = \left(\frac{1}{1+x}\right)^5 + \left(\frac{1}{1+y}\right)^5 + \left(\frac{1}{1+z}\right)^5 + \left(\frac{1}{1+t}\right)^5 \geq \frac{1}{8}. \quad (1)$$

By the inequality  $AM \geq GM$  we have

$$2\left(\frac{1}{1+x}\right)^5 + \frac{3}{32} = \left(\frac{1}{1+x}\right)^5 + \left(\frac{1}{1+x}\right)^5 + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} \geq \frac{5}{8}\left(\frac{1}{1+x}\right)^2,$$

i.e.

$$2\left(\frac{1}{1+x}\right)^5 + \frac{3}{32} \geq \frac{5}{8}\left(\frac{1}{1+x}\right)^2.$$

So it follows that

$$2A + \frac{12}{32} \geq \frac{5}{8}\left(\left(\frac{1}{1+x}\right)^2 + \left(\frac{1}{1+y}\right)^2 + \left(\frac{1}{1+z}\right)^2 + \left(\frac{1}{1+t}\right)^2\right). \quad (2)$$

We'll prove that for all positive real numbers  $x$  and  $y$  the following inequality holds

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} \geq \frac{1}{1+xy}.$$

We have

$$\begin{aligned} \frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} - \frac{1}{1+xy} &= \frac{xy(x^2+y^2) - x^2y^2 - 2xy + 1}{(1+x)^2(1+y)^2(1+xy)} \\ &= \frac{xy(x-y)^2 + (xy-1)^2}{(1+x)^2(1+y)^2(1+xy)} \geq 0. \end{aligned}$$

Now according to the previous inequality and the condition  $xyz t = 1$ , we deduce

$$\begin{aligned} &\left(\frac{1}{1+x}\right)^2 + \left(\frac{1}{1+y}\right)^2 + \left(\frac{1}{1+z}\right)^2 + \left(\frac{1}{1+t}\right)^2 \\ &\geq \frac{1}{1+xy} + \frac{1}{1+zt} = \frac{1}{1+xy} + \frac{1}{1+1/xy} = 1. \end{aligned} \quad (3)$$

By (2) and (3) we get

$$2A + \frac{12}{32} \geq \frac{5}{8}, \quad \text{i.e.} \quad A \geq \frac{1}{8}.$$

Equality occurs iff  $x = y = z = t = 1$ , i.e.  $a = b = c = d$ . ■



**140** Let  $x_1, x_2, \dots, x_n$  be positive real numbers not greater than 1. Prove the inequality

$$(1+x_1)^{\frac{1}{x_2}}(1+x_2)^{\frac{1}{x_3}} \cdots (1+x_n)^{\frac{1}{x_1}} \geq 2^n.$$

*Solution* From  $0 < x_1, x_2, \dots, x_n \leq 1$  it follows that

$$\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n} \geq 1.$$

By Corollary 4.7, (Chap. 4) we have that for every  $x > -1$  and  $\alpha \in [1, \infty)$ , the following inequality

$$(1+x)^\alpha \geq 1+x\alpha$$

holds.

Hence we get

$$(1+x_1)^{\frac{1}{x_2}}(1+x_2)^{\frac{1}{x_3}} \cdots (1+x_n)^{\frac{1}{x_1}} \geq \left(1+\frac{x_1}{x_2}\right)\left(1+\frac{x_2}{x_3}\right) \cdots \left(1+\frac{x_n}{x_1}\right). \quad (1)$$

Furthermore, applying  $AM \geq GM$  we get

$$\left(1+\frac{x_1}{x_2}\right)\left(1+\frac{x_2}{x_3}\right) \cdots \left(1+\frac{x_n}{x_1}\right) \geq 2\sqrt{\frac{x_1}{x_2}} \cdot 2\sqrt{\frac{x_2}{x_3}} \cdots 2\sqrt{\frac{x_n}{x_1}} = 2^n. \quad (2)$$

By (1) and (2) we obtain

$$(1+x_1)^{\frac{1}{x_2}}(1+x_2)^{\frac{1}{x_3}} \cdots (1+x_n)^{\frac{1}{x_1}} \geq 2^n.$$

Equality occurs iff  $x_1 = x_2 = \cdots = x_n = 1$ . ■

**141** Let  $x_1, x_2, \dots, x_n$  be non-negative real numbers such that  $x_1 + x_2 + \cdots + x_n \leq \frac{1}{2}$ . Prove the inequality

$$(1-x_1)(1-x_2) \cdots (1-x_n) \geq \frac{1}{2}.$$

*Solution* From  $x_1 + x_2 + \cdots + x_n \leq \frac{1}{2}$  and the fact that  $x_1, x_2, \dots, x_n$  are non-negative we deduce that

$$0 \leq x_i \leq \frac{1}{2} < 1, \quad \text{i.e.} \quad -x_i > -1, \quad \text{for all } i = 1, 2, \dots, n,$$

and it's clear that all  $-x_i$  are of the same sign.

Applying *Bernoulli's inequality* we obtain

$$\begin{aligned} (1-x_1)(1-x_2) \cdots (1-x_n) &= (1+(-x_1))(1+(-x_2)) \cdots (1+(-x_n)) \\ &\geq 1+(-x_1-x_2-\cdots-x_n) \\ &= 1-(x_1+x_2+\cdots+x_n) \geq 1-\frac{1}{2} = \frac{1}{2}. \end{aligned} \quad \blacksquare$$

**142** Let  $a, b, c \in \mathbb{R}^+$  such that  $abc = 1$ . Prove the inequality

$$\frac{1}{a^3 + b^3 + 1} + \frac{1}{b^3 + c^3 + 1} + \frac{1}{c^3 + a^3 + 1} \leq 1.$$

*Solution* We have

$$\frac{1}{a^3 + b^3 + 1} = \frac{1}{(a+b)((a-b)^2 + ab) + 1} \leq \frac{1}{(a+b)ab + 1},$$

and since  $ab = \frac{1}{c}$  we deduce

$$\frac{1}{a^3 + b^3 + 1} \leq \frac{1}{(a+b)ab + 1} = \frac{c}{a + b + c}.$$

Similarly

$$\frac{1}{b^3 + c^3 + 1} \leq \frac{a}{a + b + c} \quad \text{and} \quad \frac{1}{c^3 + a^3 + 1} \leq \frac{b}{a + b + c}.$$

Adding the last three inequalities we obtain the required inequality.

Equality holds if and only if  $a = b = c = 1$ . ■

**143** Let  $0 \leq a, b, c \leq 1$ . Prove the inequality

$$\frac{c}{7 + a^3 + b^3} + \frac{b}{7 + c^3 + a^3} + \frac{a}{7 + b^3 + c^3} \leq \frac{1}{3}.$$

*Solution* Since  $0 \leq a, b, c \leq 1$  it follows that  $0 \leq a^3, b^3, c^3 \leq 1$ , so we have

$$\begin{aligned} & \frac{c}{7 + a^3 + b^3} + \frac{b}{7 + c^3 + a^3} + \frac{a}{7 + b^3 + c^3} \\ & \leq \frac{c}{6 + a^3 + b^3 + c^3} + \frac{b}{6 + c^3 + a^3 + b^3} + \frac{a}{6 + b^3 + c^3 + a^3} \\ & = \frac{a + b + c}{6 + a^3 + b^3 + c^3}. \end{aligned}$$

It suffices to prove that

$$3(a + b + c) \leq 6 + a^3 + b^3 + c^3,$$

which is true since  $t^3 - 3t + 2 = (t - 1)^2(t + 2) \geq 0$ , for  $0 \leq t \leq 1$ . ■

**144** Let  $a, b, c \in \mathbb{R}^+$  such that  $abc = 1$ . Prove the inequality

$$\frac{ab}{a^5 + ab + b^5} + \frac{bc}{b^5 + bc + c^5} + \frac{ca}{c^5 + ca + a^5} \leq 1.$$

*Solution* Since

$$a^4 - a^3b - ab^3 + b^4 = a^3(a - b) - b^3(a - b) = (a - b)^2(a^2 - ab + b^2) \geq 0,$$

we have

$$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4) \geq (a + b)a^2b^2.$$

So

$$\frac{ab}{a^5 + ab + b^5} \leq \frac{ab}{(a + b)a^2b^2 + ab} = \frac{abc^2}{(a + b)a^2b^2c^2 + abc^2} = \frac{c}{a + b + c}. \quad (1)$$

Analogously

$$\frac{bc}{b^5 + bc + c^5} \leq \frac{a}{a + b + c} \quad (2)$$

and

$$\frac{ca}{c^5 + ca + a^5} \leq \frac{b}{a + b + c}. \quad (3)$$

Adding (1), (2) and (3) gives us the required inequality. ■

**145** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 3$ . Prove the inequality

$$\frac{a^3}{a^2 + ab + b^2} + \frac{b^3}{b^2 + bc + c^2} + \frac{c^3}{c^2 + ca + a^2} \geq 1.$$

*Solution* We'll show that

$$A = \frac{a^3}{a^2 + ab + b^2} + \frac{b^3}{b^2 + bc + c^2} + \frac{c^3}{c^2 + ca + a^2} \geq \frac{a + b + c}{3}.$$

For every  $x, y \in \mathbb{R}^+$  we have  $\frac{x^3 + y^3}{x^2 + xy + y^2} \geq \frac{x + y}{3}$ , in which equality occurs iff  $x = y$ .

(This inequality follows from the obvious inequality  $2(x + y)(x - y)^2 \geq 0$ .)

On the other hand, we have

$$\begin{aligned} A &= \frac{a^3}{a^2 + ab + b^2} + \frac{b^3}{b^2 + bc + c^2} + \frac{c^3}{c^2 + ca + a^2} = \frac{b^3}{a^2 + ab + b^2} \\ &\quad + \frac{c^3}{b^2 + bc + c^2} + \frac{a^3}{c^2 + ca + a^2}, \end{aligned}$$

so

$$2A = \frac{a^3 + b^3}{a^2 + ab + b^2} + \frac{b^3 + c^3}{b^2 + bc + c^2} + \frac{c^3 + a^3}{c^2 + ca + a^2} \geq \frac{a + b}{3} + \frac{b + c}{3} + \frac{c + a}{3},$$

i.e.

$$A \geq \frac{a+b+c}{3} = 1.$$

Equality occurs if and only if  $a = b = c = 1/3$ . ■

**146** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3abc$ . Prove the inequality

$$\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} \geq \frac{9}{a+b+c}.$$

*Solution* The given inequality is equivalent to

$$(a^3 + b^3 + c^3)(a + b + c) \geq 9a^2b^2c^2.$$

Applying the *Cauchy–Schwarz inequality* we have

$$(a^3 + b^3 + c^3)(a + b + c) \geq (a^2 + b^2 + c^2)^2.$$

Since  $a^2 + b^2 + c^2 = 3abc$  we obtain

$$(a^3 + b^3 + c^3)(a + b + c) \geq (a^2 + b^2 + c^2)^2 = (3abc)^2 = 9a^2b^2c^2.$$

Equality holds if and only if  $a = b = c = 1$ . ■

**147** Let  $a, b, c, x, y, z$  be positive real number, and let  $a + b = 3$ . Prove the inequality

$$\frac{x}{ay + bz} + \frac{y}{az + bx} + \frac{z}{ax + by} \geq 1.$$

*Solution* We'll show that

$$\frac{x}{ay + bz} + \frac{y}{az + bx} + \frac{z}{ax + by} \geq \frac{3}{a+b},$$

and combining with  $a + b = 3$  will give us the required inequality.

Applying the *Cauchy–Schwarz inequality* we have

$$\begin{aligned} & \frac{x}{ay + bz} + \frac{y}{az + bx} + \frac{z}{ax + by} \\ &= \frac{x^2}{axy + bxz} + \frac{y^2}{ayz + bxy} + \frac{z^2}{axz + byz} \geq \frac{(x + y + z)^2}{(a+b)(xy + yz + zx)} \\ &\geq \frac{3}{a+b} = 1. \end{aligned} \quad \blacksquare$$

**148** Let  $x, y, z > 0$  be real numbers. Prove the inequality

$$\frac{x}{x+2y+3z} + \frac{y}{y+2z+3x} + \frac{z}{z+2x+3y} \geq \frac{1}{2}.$$

*Solution* The *Cauchy–Schwarz inequality* gives us

$$\begin{aligned} \frac{x^2}{x^2+2xy+3xz} + \frac{y^2}{y^2+2yz+3xy} + \frac{z^2}{z^2+2xz+3yz} \\ \geq \frac{(x+y+z)^2}{x^2+y^2+z^2+5(xy+yz+zx)}. \end{aligned}$$

It suffices to prove that

$$2(x+y+z)^2 \geq x^2+y^2+z^2+5(xy+yz+zx),$$

which is exactly  $x^2+y^2+z^2 \geq xy+yz+zx$ , and clearly holds. ■

**149** Let  $a, b, c, d \in \mathbb{R}^+$ . Prove the inequality

$$\frac{c}{a+3b} + \frac{d}{b+3c} + \frac{a}{c+3d} + \frac{b}{d+3a} \geq 1.$$

*Solution* Let  $L = \frac{c^2}{ac+3bc} + \frac{d^2}{bd+3cd} + \frac{a^2}{ca+3da} + \frac{b^2}{bd+3ab}$ .

Applying the *Cauchy–Schwarz inequality* we get

$$\begin{aligned} ((ac+3bc) + (bd+3cd) + (ca+3da) + (bd+3ab)) \cdot L &\geq (a+b+c+d)^2 \\ \Leftrightarrow L &\geq \frac{(a+b+c+d)^2}{2ac+2bd+3bc+3cd+3ad+3ab}. \end{aligned}$$

It suffices to prove that

$$\begin{aligned} (a+b+c+d)^2 &\geq 2ac+2bd+3bc+3cd+3ad+3ab \\ \Leftrightarrow (a-b)^2 + (a-d)^2 + (b-c)^2 + (c-d)^2 &\geq 0, \end{aligned}$$

which is clearly true.

Equality holds iff  $a = b = c = d$ . ■

**150** Let  $a, b, c, d, e$  be positive real numbers. Prove the inequality

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+e} + \frac{d}{e+a} + \frac{e}{a+b} \geq \frac{5}{2}.$$

*Solution* Applying the *Cauchy–Schwarz inequality* we have

$$\begin{aligned} & \frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+e} + \frac{d}{e+a} + \frac{e}{a+b} \\ &= \frac{a^2}{ab+ac} + \frac{b^2}{bc+bd} + \frac{c^2}{cd+ce} + \frac{d^2}{de+ad} + \frac{e^2}{ae+be} \\ &\geq \frac{(a+b+c+d+e)^2}{ab+ac+ad+ae+bc+bd+be+cd+ce+de}. \end{aligned}$$

So it suffices to show that

$$\frac{(a+b+c+d+e)^2}{ab+ac+ad+ae+bc+bd+be+cd+ce+de} \geq \frac{5}{2},$$

which clearly holds (Why?). ■

**151** Prove that for all positive real numbers  $a, b, c$  the following inequality holds

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{a^2+b^2+c^2}{a+b+c}.$$

*Solution* Applying the *Cauchy–Schwarz inequality* we have

$$\begin{aligned} A &= \frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \\ &= \frac{a^4}{a(a^2+ab+b^2)} + \frac{b^4}{b(b^2+bc+c^2)} + \frac{c^4}{c(c^2+ca+a^2)} \\ &\geq \frac{(a^2+b^2+c^2)^2}{(a(a^2+ab+b^2)+b(b^2+bc+c^2)+c(c^2+ca+a^2))}. \end{aligned}$$

So it suffices to prove that

$$(a+b+c)(a^2+b^2+c^2) \geq a(a^2+ab+b^2) + b(b^2+bc+c^2) + c(c^2+ca+a^2),$$

which is true. ■

**152** Let  $a, b, c$  be positive real numbers such that  $ab+bc+ca=1$ . Prove the inequality

$$\frac{1}{4a^2-bc+1} + \frac{1}{4b^2-ca+1} + \frac{1}{4c^2-ab+1} \geq \frac{3}{2}.$$

*Solution* Since  $1-bc=ac+ab$ ,  $1-ca=ab+bc$  and  $1-ab=ac+bc$ , the given inequality can be rewritten as

$$\frac{1}{a(4a+b+c)} + \frac{1}{b(4b+c+a)} + \frac{1}{c(4c+a+b)} \geq \frac{3}{2}.$$

By the *Cauchy–Schwarz inequality* we get

$$\begin{aligned} & \left( \frac{1}{a(4a+b+c)} + \frac{1}{b(4b+c+a)} + \frac{1}{c(4c+a+b)} \right) \\ & \quad \times \left( \frac{4a+b+c}{a} + \frac{4b+c+a}{b} + \frac{4c+a+b}{c} \right) \\ & \geq \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^2 = \frac{1}{a^2b^2c^2}. \end{aligned}$$

So it suffices to prove that

$$\frac{2}{3a^2b^2c^2} \geq \frac{4a+b+c}{a} + \frac{4b+c+a}{b} + \frac{4c+a+b}{c}. \quad (1)$$

We have

$$\begin{aligned} & \frac{4a+b+c}{a} + \frac{4b+c+a}{b} + \frac{4c+a+b}{c} \\ & = 9 + \frac{a+b+c}{a} + \frac{b+c+a}{b} + \frac{c+a+b}{c} \\ & = 9 + (a+b+c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \\ & = 9 + \frac{(a+b+c)(ab+bc+ca)}{abc} \\ & = 9 + \frac{a+b+c}{abc}, \end{aligned}$$

so inequality (1) becomes

$$\frac{2}{3a^2b^2c^2} \geq 9 + \frac{a+b+c}{abc}, \quad \text{i.e.} \quad 27a^2b^2c^2 + 3abc(a+b+c) \leq 2. \quad (2)$$

By  $AM \geq GM$  we have

$$1 = ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2}, \quad \text{i.e.} \quad 27a^2b^2c^2 \leq 1. \quad (3)$$

By the well-known inequality  $(x+y+z)^2 \geq 3(xy+yz+zx)$  we get

$$3abc(a+b+c) \leq (ab+bc+ca)^2 = 1. \quad (4)$$

Finally by (3) and (4) we get inequality (2), as required.

Equality occurs iff  $a = b = c = \frac{1}{\sqrt{3}}$ . ■

**153** Let  $a, b, c$  be positive real numbers such that

$$\frac{1}{a^2 + b^2 + 1} + \frac{1}{b^2 + c^2 + 1} + \frac{1}{c^2 + a^2 + 1} \geq 1.$$

Prove the inequality

$$ab + bc + ca \leq 3.$$

*Solution* Using the *Cauchy–Schwarz inequality* gives us

$$(a^2 + b^2 + 1)(1 + 1 + c^2) \geq (a + b + c)^2, \quad \text{i.e.} \quad \frac{1}{a^2 + b^2 + 1} \leq \frac{2 + c^2}{(a + b + c)^2}.$$

Analogous we obtain

$$\frac{1}{b^2 + c^2 + 1} \leq \frac{2 + a^2}{(a + b + c)^2} \quad \text{and} \quad \frac{1}{c^2 + a^2 + 1} \leq \frac{2 + b^2}{(a + b + c)^2}.$$

So we have

$$1 \leq \frac{1}{a^2 + b^2 + 1} + \frac{1}{b^2 + c^2 + 1} + \frac{1}{c^2 + a^2 + 1} \leq \frac{6 + a^2 + b^2 + c^2}{(a + b + c)^2},$$

i.e.

$$6 + a^2 + b^2 + c^2 \geq (a + b + c)^2, \quad \text{i.e.} \quad ab + bc + ca \leq 3. \quad \blacksquare$$

**154** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 1/3$ . Prove the inequality

$$\frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \geq \frac{1}{a + b + c}.$$

*Solution* Applying the *Cauchy–Schwarz inequality* we have

$$\begin{aligned} & \frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \\ &= \frac{a^2}{a^3 - abc + a} + \frac{b^2}{b^3 - abc + b} + \frac{c^2}{c^3 - abc + c} \\ &\geq \frac{(a + b + c)^2}{a^3 + b^3 + c^3 + a + b + c - 3abc}. \end{aligned}$$

Furthermore, since

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= (a + b + c)(a^2 + b^2 + c^2 - 1/3), \end{aligned}$$



we obtain

$$\begin{aligned}
 \frac{(a+b+c)^2}{a^3+b^3+c^3+a+b+c-3abc} &= \frac{(a+b+c)^2}{(a+b+c)(a^2+b^2+c^2+1-1/3)} \\
 &= \frac{a+b+c}{a^2+b^2+c^2+2/3} \\
 &= \frac{a+b+c}{a^2+b^2+c^2+2(ab+bc+ca)} \\
 &= \frac{1}{a+b+c},
 \end{aligned}$$

as required. ■

**155** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a^3}{a^3+b^3+abc} + \frac{b^3}{b^3+c^3+abc} + \frac{c^3}{c^3+a^3+abc} \geq 1.$$

*Solution* Let  $x = \frac{b}{a}, y = \frac{c}{b}, z = \frac{a}{c}$ . Then clearly  $xyz = 1$ .

Therefore

$$\begin{aligned}
 \frac{a^3}{a^3+b^3+abc} &= \frac{1}{1+x^3+\frac{x}{z}} = \frac{1}{1+x^3+x^2y} = \frac{xyz}{xyz+x^3+x^2y} \\
 &= \frac{yz}{yz+x^2+xy}.
 \end{aligned}$$

Similarly we deduce

$$\frac{b^3}{b^3+c^3+abc} = \frac{xz}{xz+y^2+zy} \quad \text{and} \quad \frac{c^3}{c^3+a^3+abc} = \frac{xy}{xy+z^2+xz}.$$

So it suffices to prove that

$$\frac{yz}{yz+x^2+xy} + \frac{xz}{xz+y^2+zy} + \frac{xy}{xy+z^2+xz} \geq 1.$$

According to the *Cauchy–Schwarz inequality* (Corollary 4.3, Chap. 4) we have

$$\begin{aligned}
 &\frac{yz}{yz+x^2+xy} + \frac{xz}{xz+y^2+zy} + \frac{xy}{xy+z^2+xz} \\
 &\geq \frac{(xy+yz+zx)^2}{yz(yz+x^2+xy) + xz(xz+y^2+zy) + xy(xy+z^2+xz)}.
 \end{aligned}$$

We need to prove that

$$(xy+yz+zx)^2 \geq yz(yz+x^2+xy) + xz(xz+y^2+zy) + xy(xy+z^2+xz),$$

which is in fact an equality.

Equality holds iff  $x = y = z$ , i.e.  $a = b = c$ . ■

**156** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{a}{a^2 + 2b + 3} + \frac{b}{b^2 + 2c + 3} + \frac{c}{c^2 + 2a + 3} \leq \frac{1}{2}.$$

*Solution* Clearly  $x^2 + 1 \geq 2x$ , for every real  $x$ , and therefore

$$\begin{aligned} \frac{a}{a^2 + 2b + 3} + \frac{b}{b^2 + 2c + 3} + \frac{c}{c^2 + 2a + 3} \\ \leq \frac{a}{2(a + b + 1)} + \frac{b}{2(b + c + 1)} + \frac{c}{2(c + a + 1)}. \end{aligned}$$

So it remains to prove that

$$\frac{a}{a + b + 1} + \frac{b}{b + c + 1} + \frac{c}{c + a + 1} \leq 1. \quad (1)$$

Inequality (1) is equivalent to

$$\frac{b + 1}{a + b + 1} + \frac{c + 1}{b + c + 1} + \frac{a + 1}{c + a + 1} \geq 2.$$

According to the *Cauchy–Schwarz inequality* (Corollary 4.3) we have

$$\begin{aligned} \frac{b + 1}{a + b + 1} + \frac{c + 1}{b + c + 1} + \frac{a + 1}{c + a + 1} \\ \geq \frac{(a + b + c + 3)^2}{(b + 1)(a + b + 1) + (c + 1)(b + c + 1) + (a + 1)(c + a + 1)} = 2. \end{aligned}$$

Equality holds iff  $a = b = c = 1$ . ■

**157** Let  $a, b, c, d > 1$  be real numbers. Prove the inequality

$$\sqrt{a-1} + \sqrt{b-1} + \sqrt{c-1} + \sqrt{d-1} \leq \sqrt{(ab+1)(cd+1)}.$$

*Solution* We'll prove that for every  $x, y \in \mathbb{R}^+$  we have  $\sqrt{x-1} + \sqrt{y-1} \leq \sqrt{xy}$ .

Applying the *Cauchy–Schwarz inequality* for  $a_1 = \sqrt{x-1}, a_2 = 1; b_1 = 1, b_2 = \sqrt{y-1}$  gives us

$$(\sqrt{x-1} + \sqrt{y-1})^2 \leq xy, \quad \text{i.e.} \quad \sqrt{x-1} + \sqrt{y-1} \leq \sqrt{xy}.$$

Now we easily deduce that

$$\sqrt{a-1} + \sqrt{b-1} + \sqrt{c-1} + \sqrt{d-1} \leq \sqrt{ab} + \sqrt{cd} \leq \sqrt{(ab+1)(cd+1)}. \quad \blacksquare$$

**158** Let  $a_1, a_2, \dots, a_n \in \mathbb{R}^+$  such that  $a_1 a_2 \cdots a_n = 1$ . Prove the inequality

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} \leq a_1 + a_2 + \cdots + a_n.$$

*Solution* Applying  $AM \geq GM$  we obtain

$$\frac{\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}}{n} \geq \sqrt[n]{\sqrt{a_1}\sqrt{a_2}\cdots\sqrt{a_n}} = 1$$

i.e.

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} \geq n. \quad (1)$$

Now we'll use the *Cauchy–Schwarz inequality*.

We have

$$(\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n})^2 \leq (a_1 + a_2 + \cdots + a_n)(1 + 1 + \cdots + 1),$$

i.e.

$$(\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n})^2 \leq n(a_1 + a_2 + \cdots + a_n). \quad (2)$$

Using (1) and (2) gives us

$$\begin{aligned} (\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n})^2 &\leq n(a_1 + a_2 + \cdots + a_n) \leq (\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}) \\ &\quad \times (a_1 + a_2 + \cdots + a_n) \end{aligned}$$

i.e.

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} \leq a_1 + a_2 + \cdots + a_n,$$

as required. ■

**159** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$a\sqrt{b} + b\sqrt{c} + c\sqrt{a} \leq \frac{1}{\sqrt{3}}.$$

*Solution* Applying the *Cauchy–Schwarz inequality* we have

$$(a\sqrt{b} + b\sqrt{c} + c\sqrt{a})^2 \leq (a^2 + b^2 + c^2)(a + b + c) = a^2 + b^2 + c^2. \quad (1)$$

One more use of the *Cauchy–Schwarz inequality* for

$$\begin{aligned} A_1 &= \sqrt{a}, & A_2 &= \sqrt{b}, & A_3 &= \sqrt{c} & \text{and} \\ B_1 &= \sqrt{ab}, & B_2 &= \sqrt{bc}, & B_3 &= \sqrt{ca} \end{aligned}$$

gives us

$$(a\sqrt{b} + b\sqrt{c} + c\sqrt{a})^2 \leq (a + b + c)(ab + bc + ca) = ab + bc + ca,$$

i.e.

$$2(a\sqrt{b} + b\sqrt{c} + c\sqrt{a})^2 \leq 2(ab + bc + ca). \quad (2)$$

By adding (1) and (2) we get

$$3(a\sqrt{b} + b\sqrt{c} + c\sqrt{a})^2 \leq a^2 + b^2 + c^2 + 2(ab + bc + ca),$$

i.e.

$$3(a\sqrt{b} + b\sqrt{c} + c\sqrt{a})^2 \leq (a + b + c)^2 = 1,$$

i.e.

$$a\sqrt{b} + b\sqrt{c} + c\sqrt{a} \leq \frac{1}{\sqrt{3}}. \quad \blacksquare$$

**160** Let  $a, b, c \in (0, 1)$  be real numbers. Prove the inequality

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1.$$

*Solution 1* For  $x \in (0, 1)$  we have  $\sqrt{x} < \sqrt[3]{x}$ .

So

$$\sqrt{abc} < \sqrt[3]{abc} \quad \text{and} \quad \sqrt{(1-a)(1-b)(1-c)} < \sqrt[3]{(1-a)(1-b)(1-c)}. \quad (1)$$

Using (1) and  $AM \geq GM$  gives us

$$\begin{aligned} \sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} &< \sqrt[3]{abc} + \sqrt[3]{(1-a)(1-b)(1-c)} \\ &\leq \frac{a+b+c}{3} + \frac{1-a+1-b+1-c}{3} = 1. \quad \blacksquare \end{aligned}$$

*Solution 2* Since  $a, b, c \in (0, 1)$  we obtain

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < \sqrt{b}\sqrt{c} + \sqrt{1-b}\sqrt{1-c}. \quad (1)$$

Using the *Cauchy-Schwarz inequality* we have

$$\sqrt{b}\sqrt{c} + \sqrt{1-b}\sqrt{1-c} \leq \sqrt{(b+1-b)^2(c+1-c)^2} = 1. \quad (2)$$

From (1) and (2), we obtain the required inequality.  $\blacksquare$

**161** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$\frac{a^3 + 2}{b + 2} + \frac{b^3 + 2}{c + 2} + \frac{c^3 + 2}{a + 2} \geq 3.$$

*Solution* By  $AM \geq GM$  we have

$$\frac{a^3 + 2}{b + 2} = \frac{a^3 + 1 + 1}{b + 2} \geq \frac{3\sqrt[3]{a^3 \cdot 1 \cdot 1}}{b + 2} = \frac{3a}{b + 2}.$$

Similarly we get

$$\frac{b^3+2}{c+2} \geq \frac{3b}{c+2} \quad \text{and} \quad \frac{c^3+2}{a+2} \geq \frac{3c}{a+2}.$$

Therefore

$$\frac{a^3+2}{b+2} + \frac{b^3+2}{c+2} + \frac{c^3+2}{a+2} \geq 3 \left( \frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} \right). \quad (1)$$

Applying the *Cauchy-Schwarz inequality* (Corollary 4.3) we obtain

$$\begin{aligned} \frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} &= \frac{a^2}{a(b+2)} + \frac{b^2}{b(c+2)} + \frac{c^2}{c(a+2)} \\ &\geq \frac{(a+b+c)^2}{a(b+2) + b(c+2) + c(a+2)} \\ &= \frac{(a+b+c)^2}{ab+bc+ca+2(a+b+c)}. \end{aligned} \quad (2)$$

Since  $(a+b+c)^2 \geq 3(ab+bc+ca)$  we deduce that

$$\frac{1}{ab+bc+ca} \geq \frac{3}{(a+b+c)^2}. \quad (3)$$

From (2) and (3) we get

$$\begin{aligned} \frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} &\geq \frac{(a+b+c)^2}{ab+bc+ca+2(a+b+c)} \\ &\geq \frac{(a+b+c)^2}{(a+b+c)^2/3+2(a+b+c)} \\ &= \frac{3(a+b+c)^2}{(a+b+c)^2+6(a+b+c)} = \frac{3(a+b+c)}{(a+b+c)+6}. \end{aligned} \quad (4)$$

Finally by (1), (4) and since  $a+b+c=3$  we obtain

$$A \geq 3 \left( \frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} \right) \geq \frac{9(a+b+c)}{(a+b+c)+6} = \frac{27}{9} = 3,$$

as required. Equality occurs iff  $a=b=c=1$ . ■

**162** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{1}{2-a} + \frac{1}{2-b} + \frac{1}{2-c} \geq 3.$$

*Solution* Rewrite the given inequality as follows

$$\frac{a}{2-a} + \frac{b}{2-b} + \frac{c}{2-c} \geq 3,$$

i.e.

$$\frac{a^2}{2a-a^2} + \frac{b^2}{2b-b^2} + \frac{c^2}{2c-c^2} \geq 3.$$

Clearly  $a, b, c \in (0, \sqrt{3})$ , so  $2a - a^2, 2b - b^2, 2c - c^2 > 0$ .

Now by the *Cauchy-Schwarz inequality* (Corollary 4.3) we obtain

$$\begin{aligned} \frac{a^2}{2a-a^2} + \frac{b^2}{2b-b^2} + \frac{c^2}{2c-c^2} &\geq \frac{(a+b+c)^2}{2(a+b+c) - (a^2+b^2+c^2)} \\ &= \frac{9}{2(a+b+c) - 3}. \end{aligned}$$

So it remains to prove that

$$\frac{(a+b+c)^2}{2(a+b+c) - 3} \geq 3,$$

which is equivalent to  $(a+b+c-3)^2 \geq 0$ , and clearly holds.

Equality holds iff  $a = b = c = 1$ . ■

**163** Let  $a, b, c$  be positive real numbers such that  $abc = 8$ . Prove the inequality

$$\frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \leq 0.$$

*Solution* Rewrite the given inequality as follows

$$\frac{a+1-3}{a+1} + \frac{b+1-3}{b+1} + \frac{c+1-3}{c+1} \leq 0$$

or

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \geq 1.$$

Let  $a = \frac{2x}{y}, b = \frac{2y}{z}, c = \frac{2z}{x}$ .

Then

$$\begin{aligned} \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} &= \frac{1}{\frac{2x}{y}+1} + \frac{1}{\frac{2y}{z}+1} + \frac{1}{\frac{2z}{x}+1} \\ &= \frac{y}{2x+y} + \frac{z}{2y+z} + \frac{x}{2z+x} \end{aligned}$$

$$\begin{aligned}
&= \frac{y^2}{2xy + y^2} + \frac{z^2}{2yz + z^2} + \frac{x^2}{2zx + x^2} \\
&\geq \frac{(x + y + z)^2}{2xy + y^2 + 2yz + z^2 + 2zx + x^2} = 1.
\end{aligned}$$

In the last step we used the *Cauchy–Schwarz inequality* (Corollary 4.3). ■

**164** Let  $a, b, c \in \mathbb{R}^+$  such that  $a^2 + b^2 + c^2 = 1$ . Prove the inequality

$$a + b + c - 2abc \leq \sqrt{2}.$$

*Solution* Since  $a^2 + b^2 + c^2 = 1$  and  $a^2 \geq 0$  it follows that  $b^2 + c^2 \leq 1$ , i.e.  $2bc \leq 1$ .

Applying the *Cauchy–Schwarz inequality* we have

$$\begin{aligned}
a + b + c - 2abc &= a(1 - 2bc) + (b + c) \cdot 1 \leq \sqrt{a^2 + (b + c)^2} \sqrt{(1 - 2bc)^2 + 1} \\
&= \sqrt{(a^2 + b^2 + c^2 + 2bc)(2 - 4bc + 4b^2c^2)} \\
&= \sqrt{(1 + 2bc)(2 - 4bc + 4b^2c^2)}.
\end{aligned}$$

So it suffices to show that

$$(1 + 2bc)(2 - 4bc + 4b^2c^2) \leq 2.$$

We have

$$2 - (1 + 2bc)(2 - 4bc + 4b^2c^2) = 4b^2c^2(1 - 2bc) \geq 0. \quad \blacksquare$$

**165** Let  $x, y, z \in \mathbb{R}^+$  such that  $x^2 + y^2 + z^2 = 2$ . Prove the inequality

$$x + y + z \leq 2 + xyz.$$

*Solution 1* Let  $x = a\sqrt{2}$ ,  $y = b\sqrt{2}$ ,  $z = c\sqrt{2}$ . Then  $a^2 + b^2 + c^2 = 1$  and the given inequality becomes  $a + b + c - 2abc \leq \sqrt{2}$ , which is true (Problem 127). ■

*Solution 2* The given inequality becomes

$$x(1 - yz) + y + z \leq 2.$$

Using the *Cauchy–Schwarz inequality* we get

$$\begin{aligned}
(x(1 - yz) + (y + z) \cdot 1)^2 &\leq (x^2 + (y + z)^2)((1 - yz)^2 + 1^2) \\
\Leftrightarrow (x + y + z - xyz)^2 &\leq (x^2 + y^2 + z^2 + 2yz)(2 - 2yz + y^2z^2) \\
\Leftrightarrow (x + y + z - xyz)^2 &\leq 2(1 + yz)(2 - 2yz + y^2z^2).
\end{aligned}$$

So it suffices to show that

$$2(1 + yz)(2 - 2yz + y^2z^2) \leq 4,$$

i.e.

$$(1 + yz)(2 - 2yz + y^2z^2) \leq 2 \quad \Leftrightarrow \quad y^3z^3 \leq y^2z^2,$$

i.e.

$$yz \leq 1.$$

The last inequality is true since  $2yz \leq y^2 + z^2 \leq x^2 + y^2 + z^2 = 2$ . ■

**166** Let  $x, y, z > -1$  be real numbers. Prove the inequality

$$\frac{1 + x^2}{1 + y + z^2} + \frac{1 + y^2}{1 + z + x^2} + \frac{1 + z^2}{1 + x + y^2} \geq 2.$$

*Solution* Notice that  $\frac{1+y^2}{2} \geq y$  and  $1 + y + z^2 > 0$ .

So

$$\frac{1 + x^2}{1 + y + z^2} \geq \frac{1 + x^2}{1 + \frac{1+y^2}{2} + z^2} = \frac{2(1 + x^2)}{2(1 + z^2) + 1 + y^2}.$$

Analogously

$$\frac{1 + y^2}{1 + z + x^2} \geq \frac{2(1 + y^2)}{2(1 + x^2) + 1 + z^2} \quad \text{and} \quad \frac{1 + z^2}{1 + x + y^2} \geq \frac{2(1 + z^2)}{2(1 + y^2) + 1 + x^2}.$$

It suffices to show that

$$\frac{2(1 + x^2)}{2(1 + z^2) + 1 + y^2} + \frac{2(1 + y^2)}{2(1 + x^2) + 1 + z^2} + \frac{2(1 + z^2)}{2(1 + y^2) + 1 + x^2} \geq 2.$$

Let  $1 + x^2 = a$ ,  $1 + y^2 = b$ ,  $1 + z^2 = c$ , i.e. we need to show that

$$\frac{a}{2c + b} + \frac{b}{2a + c} + \frac{c}{2b + a} \geq 1.$$

Applying the *Cauchy-Schwarz inequality* we obtain

$$3 \left( \frac{a^2}{2ca + ab} + \frac{b^2}{2ab + bc} + \frac{c^2}{2bc + ca} \right) (ab + bc + ca) \geq (a + b + c)^2$$

i.e.

$$\frac{a}{2c + b} + \frac{b}{2a + c} + \frac{c}{2b + a} \geq \frac{(a + b + c)^2}{3(ab + bc + ca)} \geq 1,$$

as required. ■



**167** Let  $a, b, c, d$  be positive real numbers such that  $abcd = 1$ . Prove the inequality

$$(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) \geq (a + b + c + d)^2.$$

*Solution* Since  $abcd = 1$ , there are two numbers  $x, y$  among  $a, b, c, d$ , such that  $x, y \geq 1$  or  $x, y \leq 1$ . Without loss of generality we may suppose that they are  $b$  and  $d$ . Then clearly  $(b - 1)(d - 1) \geq 0$ , i.e.  $bd + 1 \geq b + d$ .

According to the *Cauchy-Schwarz inequality* and the previous note, we obtain

$$\begin{aligned} (1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) &= (1 + a^2 + b^2 + a^2b^2)(c^2 + 1 + d^2 + c^2d^2) \\ &\geq (c + a + bd + 1)^2 \geq (a + b + c + d)^2. \end{aligned}$$

Equality holds iff  $a = b = c = d = 1$ . ■

**168** Let  $a, b, c, d \in \mathbb{R}^+$  such that  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 4$ . Prove the inequality

$$\sqrt[3]{\frac{a^3 + b^3}{2}} + \sqrt[3]{\frac{b^3 + c^3}{2}} + \sqrt[3]{\frac{c^3 + d^3}{2}} + \sqrt[3]{\frac{d^3 + a^3}{2}} \leq 2(a + b + c + d) - 4.$$

*Solution*

**Lemma 21.2** If  $x, y \in \mathbb{R}^+$  then  $\sqrt[3]{\frac{x^3 + y^3}{2}} \leq \frac{x^2 + y^2}{x + y}$ .

*Proof* The given inequality is equivalent to  $(x - y)^4(x^2 + xy + y^2) \geq 0$ . □

So it follows that

$$\begin{aligned} &\sqrt[3]{\frac{a^3 + b^3}{2}} + \sqrt[3]{\frac{b^3 + c^3}{2}} + \sqrt[3]{\frac{c^3 + d^3}{2}} + \sqrt[3]{\frac{d^3 + a^3}{2}} \\ &\leq \frac{a^2 + b^2}{a + b} + \frac{b^2 + c^2}{b + c} + \frac{c^2 + d^2}{c + d} + \frac{d^2 + a^2}{d + a}. \end{aligned}$$

Furthermore, we have

$$(a + b) - \frac{a^2 + b^2}{a + b} = \frac{2ab}{a + b}.$$

So

$$L \leq (a + b) - \frac{2ab}{a + b} + (b + c) - \frac{2bc}{b + c} + (c + d) - \frac{2cd}{c + d} + (d + a) - \frac{2da}{d + a},$$

and it is sufficient to prove that

$$\frac{ab}{a + b} + \frac{bc}{b + c} + \frac{cd}{c + d} + \frac{da}{d + a} \geq 2.$$

Applying the *Cauchy–Schwarz inequality* we obtain

$$\left( \frac{ab}{a+b} + \frac{bc}{b+c} + \frac{cd}{c+d} + \frac{da}{d+a} \right) \cdot \left( 2 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \right) \geq 4^2,$$

i.e.

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{cd}{c+d} + \frac{da}{d+a} \geq \frac{16}{8} = 2,$$

as required. ■

**169** Let  $x, y, z \in [-1, 1]$  be real numbers such that  $x + y + z + xyz = 0$ . Prove the inequality

$$\sqrt{x+1} + \sqrt{y+1} + \sqrt{z+1} \leq 3.$$

*Solution* Applying the *Cauchy–Schwarz inequality* we have

$$\sqrt{x+1} + \sqrt{y+1} + \sqrt{z+1} \leq \sqrt{3(x+y+z+3)}.$$

If  $x + y + z \leq 0$  then  $\sqrt{x+1} + \sqrt{y+1} + \sqrt{z+1} \leq 3$ , and the given inequality clearly holds.

So let us assume that  $x + y + z > 0$ . Then we have  $xyz = -(x + y + z) < 0$ . Without loss of generality we may assume that  $z < 0$  and then it's clear that  $x, y \in (0, 1]$ .

Applying once more, the *Cauchy–Schwarz inequality* we obtain

$$\sqrt{x+1} + \sqrt{y+1} + \sqrt{z+1} \leq \sqrt{2x+2y+4} + \sqrt{z+1}.$$

So it suffices to show that

$$\sqrt{2x+2y+4} + \sqrt{z+1} \leq 3.$$

We have

$$\begin{aligned} \sqrt{2x+2y+4} + \sqrt{z+1} \leq 3 &\Leftrightarrow \sqrt{2x+2y+4} - 2 \leq 1 - \sqrt{z+1} \\ \Leftrightarrow \frac{2(x+y)}{\sqrt{2x+2y+4} + 2} &\leq \frac{-z}{\sqrt{z+1} + 1} \\ \Leftrightarrow \frac{-2z(1+xy)}{\sqrt{2x+2y+4} + 2} &\leq \frac{-z}{\sqrt{z+1} + 1} \\ \Leftrightarrow 2(1+xy)(1 + \sqrt{1+z}) &\leq \sqrt{2x+2y+4} + 2 \\ \Leftrightarrow 2xy + 2(1+xy)\sqrt{1+z} &\leq \sqrt{2x+2y+4}. \end{aligned} \tag{1}$$

We can easily deduce that  $1+z = \frac{(1-x)(1-y)}{1+xy}$ , and then inequality (1) is equivalent to

$$xy + \sqrt{(1-x)(1-y)(1+xy)} \leq \sqrt{1 + \frac{x+y}{2}}.$$

Finally, using the *Cauchy–Schwarz inequality* we obtain

$$\begin{aligned}
 xy + \sqrt{(1-x)(1-y)(1+xy)} &= \sqrt{x}\sqrt{xy^2} + \sqrt{1-x}\sqrt{1+xy-y-xy^2} \\
 &\leq \sqrt{(x+1-x)(xy^2+1+xy-y-xy^2)} \\
 &= \sqrt{1+y(1-x)} \leq 1 \leq \sqrt{1+\frac{x+y}{2}},
 \end{aligned}$$

as desired. ■

**170** Let  $a, b, c > 0$  be positive real numbers such that  $a + b + c = abc$ . Prove the inequality

$$ab + bc + ca \geq 3 + \sqrt{a^2 + 1} + \sqrt{b^2 + 1} + \sqrt{c^2 + 1}.$$

*Solution* First we'll show that

$$a^2b^2 + b^2c^2 + c^2a^2 \geq a^2b^2c^2. \quad (1)$$

We have

$$(ab)^2 + (bc)^2 + (ca)^2 \geq (ab)(bc) + (bc)(ca) + (ca)(ab) = abc(a + b + c)$$

i.e.

$$\begin{aligned}
 \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} &= \frac{(ab)^2 + (bc)^2 + (ca)^2}{(abc)^2} \geq \frac{a + b + c}{abc} = 1 \\
 \Leftrightarrow a^2b^2 + b^2c^2 + c^2a^2 &\geq a^2b^2c^2.
 \end{aligned}$$

Furthermore

$$\begin{aligned}
 (ab + bc + ca)^2 &= a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a + b + c) \\
 &\stackrel{(1)}{\geq} a^2b^2c^2 + 2abc(a + b + c) = 3(a + b + c)^2.
 \end{aligned} \quad (2)$$

So

$$\begin{aligned}
 (ab + bc + ca - 3)^2 &= (ab + bc + ca)^2 - 6(ab + bc + ca) + 9 \\
 &\stackrel{(2)}{\geq} 3(a + b + c)^2 - 6(ab + bc + ca) + 9 \\
 &= 3(a^2 + b^2 + c^2) + 9,
 \end{aligned}$$

i.e.

$$ab + bc + ca \geq 3 + \sqrt{3(a^2 + b^2 + c^2) + 9}. \quad (3)$$

Applying the *Cauchy–Schwarz inequality* we have

$$\begin{aligned} 3(a^2 + b^2 + c^2) + 9 &= 3((a^2 + 1) + (b^2 + 1) + (c^2 + 1)) \\ &\geq (\sqrt{a^2 + 1} + \sqrt{b^2 + 1} + \sqrt{c^2 + 1})^2, \end{aligned}$$

i.e.

$$\sqrt{3(a^2 + b^2 + c^2) + 9} \geq \sqrt{a^2 + 1} + \sqrt{b^2 + 1} + \sqrt{c^2 + 1}. \quad (4)$$

Using (3) and (4) we obtain

$$ab + bc + ca \geq 3 + \sqrt{3(a^2 + b^2 + c^2) + 9} \geq 3 + \sqrt{a^2 + 1} + \sqrt{b^2 + 1} + \sqrt{c^2 + 1},$$

as required. ■

**171** Let  $a, b, c, x, y, z$  be positive real numbers such that  $ax + by + cz = xyz$ . Prove the inequality

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} < x + y + z.$$

*Solution* We have  $\frac{a}{yz} + \frac{b}{xz} + \frac{c}{xy} = 1$ .

Let

$$u = \frac{a}{yz}, \quad v = \frac{b}{xz}, \quad w = \frac{c}{xy}.$$

We need to show that

$$\sqrt{z(yu + xv)} + \sqrt{x(zv + yw)} + \sqrt{y(xw + zu)} < x + y + z,$$

where  $u + v + w = 1$ .

Applying the *Cauchy–Schwarz inequality* we obtain

$$\begin{aligned} &(\sqrt{z(yu + xv)} + \sqrt{x(zv + yw)} + \sqrt{y(xw + zu)})^2 \\ &\leq (x + y + z)(yu + xv + zv + yw + xw + zu). \end{aligned}$$

Also we have

$$\begin{aligned} yu + xv + zv + yw + xw + zu &= x(1 - u) + y(1 - v) + z(1 - w) \\ &= x + y + z - (xu + yv + zw) < x + y + z. \end{aligned}$$

Now we obtain

$$(\sqrt{z(yu + xv)} + \sqrt{x(zv + yw)} + \sqrt{y(xw + zu)})^2 < (x + y + z)^2,$$

i.e.

$$\sqrt{z(yu + xv)} + \sqrt{x(zv + yw)} + \sqrt{y(xw + zu)} < x + y + z. \quad \blacksquare$$

**172** Let  $a, b, c$  be non-negative real numbers such that  $a^2 + b^2 + c^2 = 1$ . Prove the inequality

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \geq \frac{3}{4}(a\sqrt{a} + b\sqrt{b} + c\sqrt{c})^2.$$

*Solution* We'll use the *Cauchy-Schwarz inequality*, i.e.

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \geq (a_1b_1 + a_2b_2 + a_3b_3)^2. \quad (1)$$

Let

$$a_1 = \sqrt{a^2(b^2+1)}, \quad a_2 = \sqrt{b^2(c^2+1)}, \quad a_3 = \sqrt{c^2(a^2+1)} \quad \text{and}$$

$$b_1 = \sqrt{\frac{a}{b^2+1}}, \quad b_2 = \sqrt{\frac{b}{c^2+1}}, \quad b_3 = \sqrt{\frac{c}{a^2+1}}.$$

Then using (1) we get

$$\begin{aligned} & (a^2(b^2+1) + b^2(c^2+1) + c^2(a^2+1)) \left( \frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \right) \\ & \geq (a\sqrt{a} + b\sqrt{b} + c\sqrt{c})^2, \end{aligned}$$

i.e.

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \geq \frac{(a\sqrt{a} + b\sqrt{b} + c\sqrt{c})^2}{a^2(b^2+1) + b^2(c^2+1) + c^2(a^2+1)}.$$

So it suffices to show that

$$a^2(b^2+1) + b^2(c^2+1) + c^2(a^2+1) \leq \frac{4}{3}.$$

From the obvious inequality  $(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 \geq 0$  we deduce that

$$a^2b^2 + b^2c^2 + c^2a^2 \leq a^4 + b^4 + c^4. \quad (2)$$

Now we have

$$\begin{aligned} & a^2(b^2+1) + b^2(c^2+1) + c^2(a^2+1) \\ &= a^2 + b^2 + c^2 + a^2b^2 + b^2c^2 + c^2a^2 \\ &= a^2 + b^2 + c^2 + \frac{3(a^2b^2 + b^2c^2 + c^2a^2)}{3} \\ &\stackrel{(2)}{\leq} a^2 + b^2 + c^2 + \frac{2(a^2b^2 + b^2c^2 + c^2a^2) + a^4 + b^4 + c^4}{3} \\ &= a^2 + b^2 + c^2 + \frac{(a^2 + b^2 + c^2)^2}{3} = 1 + \frac{1}{3} = \frac{4}{3}, \end{aligned}$$

as required.

Equality occurs iff  $a = b = c = 1/\sqrt{3}$ . ■

**173** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}.$$

*Solution* Applying the *Cauchy–Schwarz inequality* gives us

$$(a+b+c) \left( \frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \right) \geq \left( \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right)^2. \quad (1)$$

Recalling *Nesbitt's inequality* we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}. \quad (2)$$

From (1) and (2) we obtain the required inequality. ■

**174** Let  $x \geq y \geq z > 0$  be real numbers. Prove the inequality

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2.$$

*Solution* Applying the *Cauchy–Schwarz inequality* we obtain

$$\left( \frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \right) \left( \frac{x^2z}{y} + \frac{y^2x}{z} + \frac{z^2y}{x} \right) \geq (x^2 + y^2 + z^2)^2. \quad (1)$$

We'll prove that

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq \frac{x^2z}{y} + \frac{y^2x}{z} + \frac{z^2y}{x}. \quad (2)$$

From

$$\begin{aligned} & \left( \frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \right) - \left( \frac{x^2z}{y} + \frac{y^2x}{z} + \frac{z^2y}{x} \right) \\ &= \frac{(xy + yz + zx)(x - y)(x - z)(y - z)}{xyz} \geq 0, \end{aligned}$$

we deduce that

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq \frac{x^2z}{y} + \frac{y^2x}{z} + \frac{z^2y}{x}.$$

Combining (1) and (2) give us

$$\begin{aligned} \left( \frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \right)^2 &\geq \left( \frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \right) \left( \frac{x^2z}{y} + \frac{y^2x}{z} + \frac{z^2y}{x} \right) \\ &\geq (x^2 + y^2 + z^2)^2, \end{aligned}$$

i.e.

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2.$$

Equality occurs if and only if  $x = y = z$ . ■

**175** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} \leq 1.$$

*Solution* The given inequality can be rewritten as

$$1 - \frac{2}{2+a} + 1 - \frac{2}{2+b} + 1 - \frac{2}{2+c} \geq 1,$$

which is equivalent with

$$\frac{a}{2+a} + \frac{b}{2+b} + \frac{c}{2+c} \geq 1. \quad (1)$$

Let  $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$ .

Inequality (1) becomes

$$\frac{x}{x+2y} + \frac{y}{y+2z} + \frac{z}{z+2x} \geq 1. \quad (2)$$

Applying the *Cauchy–Schwarz inequality* we have

$$\begin{aligned} \frac{x}{x+2y} + \frac{y}{y+2z} + \frac{z}{z+2x} &= \frac{x^2}{x^2+2xy} + \frac{y^2}{y^2+2yz} + \frac{z^2}{z^2+2zx} \\ &\geq \frac{(x+y+z)^2}{x^2+y^2+z^2+2xy+2yz+2zx} = 1. \end{aligned}$$

So we have proved (2) and we are done.

Equality occurs iff  $x = y = z$ , i.e.  $a = b = c = 1$ . ■

**176** Let  $a, b, c$  be positive real numbers such that  $abc \geq 1$ . Prove the inequality

$$\frac{1}{a^4+b^3+c^2} + \frac{1}{b^4+c^3+a^2} + \frac{1}{c^4+a^3+b^2} \leq 1.$$

*Solution* By the *Cauchy–Schwarz inequality* we have

$$\frac{1}{a^4 + b^3 + c^2} = \frac{1 + b + c^2}{(a^4 + b^3 + c^2)(1 + b + c^2)} \leq \frac{1 + b + c^2}{(a^2 + b^2 + c^2)^2}.$$

Similarly we get

$$\frac{1}{b^4 + c^3 + a^2} \leq \frac{1 + c + a^2}{(a^2 + b^2 + c^2)^2} \quad \text{and} \quad \frac{1}{c^4 + a^3 + b^2} \leq \frac{1 + a + b^2}{(a^2 + b^2 + c^2)^2}.$$

It follows that

$$\frac{1}{a^4 + b^3 + c^2} + \frac{1}{b^4 + c^3 + a^2} + \frac{1}{c^4 + a^3 + b^2} \leq \frac{a^2 + b^2 + c^2 + a + b + c + 3}{(a^2 + b^2 + c^2)^2}.$$

So it remains to prove that

$$\frac{a^2 + b^2 + c^2 + a + b + c + 3}{(a^2 + b^2 + c^2)^2} \leq 1.$$

By  $AM \geq GM$  we have  $a + b + c \geq 3$  and  $a^2 + b^2 + c^2 \geq 3$ .

Consider the well-known inequality  $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$ .

Then we obtain

$$\begin{aligned} \frac{a^2 + b^2 + c^2 + a + b + c + 3}{(a^2 + b^2 + c^2)^2} &\leq \frac{a^2 + b^2 + c^2 + \frac{(a+b+c)^2}{3} + \frac{(a+b+c)^2}{3}}{(a^2 + b^2 + c^2)^2} \\ &\leq \frac{a^2 + b^2 + c^2 + (a^2 + b^2 + c^2) + (a^2 + b^2 + c^2)}{(a^2 + b^2 + c^2)^2} \\ &= \frac{3}{a^2 + b^2 + c^2} \leq 1, \end{aligned}$$

as required. ■

**177** Let  $a, b, c, d$  be positive real numbers such that  $abcd = 1$ . Prove the inequality

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+d)} + \frac{1}{d(1+a)} \geq 2.$$

*Solution* With the substitutions  $a = \frac{x}{y}, b = \frac{y}{x}, c = \frac{z}{t}, d = \frac{t}{z}$ , the given inequality becomes

$$\frac{x}{z+t} + \frac{y}{x+t} + \frac{z}{x+y} + \frac{t}{z+y} \geq 2.$$



By the *Cauchy–Schwarz inequality* we have

$$\begin{aligned}\frac{x}{z+t} + \frac{y}{x+t} + \frac{z}{x+y} + \frac{t}{z+y} &= \frac{x^2}{xz+xt} + \frac{y^2}{yx+yt} + \frac{z^2}{zx+zy} + \frac{t^2}{tz+ty} \\ &\geq \frac{(x+y+z+t)^2}{2xz+2yt+xt+yx+zy+tz}.\end{aligned}$$

Hence it suffices to prove that

$$\frac{(x+y+z+t)^2}{2xz+2yt+xt+yx+zy+tz} \geq 2,$$

which is equivalent to

$$(x-z)^2 + (y-t)^2 \geq 0.$$

Equality occurs iff  $x = z$ ,  $y = t$ , i.e.  $a = c = 1/b = 1/d$ . ■

**178** Let  $a, b, c$  be non-negative real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\frac{ab}{c+1} + \frac{bc}{a+1} + \frac{ca}{b+1} \leq \frac{1}{4}.$$

*Solution* If one of  $a, b, c$  is equal to zero then it is easy to show that the given inequality is true. Equality in this case occurs iff one of  $a, b, c$  is zero, and the other two numbers are equal to  $1/2$ .

Because of this we can assume that  $a, b, c \in \mathbb{R}^+$ .

From  $a + b + c = 1$  it follows that at least one of the numbers  $a, b, c$  is less than  $4/9$ . In the opposite case, if all of them are greater than  $4/9$ , we will have

$$a + b + c > 3 \cdot \frac{4}{9} = \frac{4}{3} > 1,$$

a contradiction.

So we can assume that

$$c < 4/9. \tag{1}$$

$$\text{Let } A = \frac{ab}{c+1} + \frac{bc}{a+1} + \frac{ca}{b+1}.$$

Then

$$A = abc \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{a+1} - \frac{1}{b+1} - \frac{1}{c+1} \right). \tag{2}$$

Since

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = \frac{1}{4}((a+1) + (b+1) + (c+1)) \left( \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \right),$$

applying the *Cauchy–Schwarz inequality* we have

$$\begin{aligned}
 & \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \\
 &= \frac{1}{4}((a+1) + (b+1) + (c+1)) \left( \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \right) \\
 &\geq \frac{1}{4}(1+1+1)^2 = \frac{9}{4}.
 \end{aligned} \tag{3}$$

Now using (2) and (3) we obtain

$$\begin{aligned}
 A &= abc \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \left( \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \right) \right) \\
 &\leq abc \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{9}{4} \right) = ab + ac + bc - \frac{9abc}{4}.
 \end{aligned} \tag{4}$$

On the other hand, we have

$$(1-c)^2 = (a+b)^2 \geq 4ab,$$

i.e.

$$ab \leq \frac{(1-c)^2}{4}, \tag{5}$$

and using (4) we get

$$\begin{aligned}
 A - \frac{1}{4} &\leq ab + ac + bc - \frac{9abc}{4} - \frac{1}{4} = ab + c(a+b) - \frac{9abc}{4} - \frac{1}{4} \\
 &= ab \left( 1 - \frac{9c}{4} \right) + c(1-c) - \frac{1}{4} \\
 &\stackrel{(1), (5)}{\leq} \frac{(1-c)^2}{4} \left( 1 - \frac{9c}{4} \right) + c(1-c) - \frac{1}{4} \\
 &= \frac{1}{16}(-9c^3 + 6c^2 - c) = \frac{-c}{16}(9c^2 - 6c + 1) = \frac{-c(3c-1)^2}{16} \leq 0,
 \end{aligned}$$

as required.

Equality occurs iff  $a = b = c = 1/3$ . ■

**179** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{1}{(a+1)^2(b+c)} + \frac{1}{(b+1)^2(c+a)} + \frac{1}{(c+1)^2(a+b)} \leq \frac{3}{8}.$$

*Solution* Let  $a = x^2$ ,  $b = y^2$ ,  $c = z^2$ . The given inequality becomes

$$\frac{1}{(x^2 + 1)^2(y^2 + z^2)} + \frac{1}{(y^2 + 1)^2(z^2 + x^2)} + \frac{1}{(z^2 + 1)^2(x^2 + y^2)} \leq \frac{3}{8}.$$

By the *Cauchy–Schwarz inequality* we have

$$\sqrt{(x^2 + 1)(y^2 + z^2)} \geq xy + z = \frac{1}{z} + z = \frac{z^2 + 1}{z}$$

and

$$\sqrt{(x^2 + 1)(z^2 + y^2)} \geq xz + y = \frac{1}{y} + y = \frac{y^2 + 1}{y}.$$

Multiplying these two inequalities we get

$$(x^2 + 1)(z^2 + y^2) \geq \frac{(y^2 + 1)(z^2 + 1)}{yz}$$

i.e.

$$(x^2 + 1)^2(z^2 + y^2) \geq \frac{(x^2 + 1)(y^2 + 1)(z^2 + 1)}{yz}.$$

Hence

$$\frac{1}{(x^2 + 1)^2(y^2 + z^2)} \leq \frac{yz}{(x^2 + 1)(y^2 + 1)(z^2 + 1)}.$$

Similarly we obtain

$$\frac{1}{(y^2 + 1)^2(z^2 + x^2)} \leq \frac{zx}{(x^2 + 1)(y^2 + 1)(z^2 + 1)}$$

and

$$\frac{1}{(z^2 + 1)^2(x^2 + y^2)} \leq \frac{xy}{(x^2 + 1)(y^2 + 1)(z^2 + 1)}.$$

We have

$$\begin{aligned} & \frac{1}{(x^2 + 1)^2(y^2 + z^2)} + \frac{1}{(y^2 + 1)^2(z^2 + x^2)} + \frac{1}{(z^2 + 1)^2(x^2 + y^2)} \\ & \leq \frac{xy + yz + zx}{(x^2 + 1)(y^2 + 1)(z^2 + 1)}, \end{aligned}$$

and it suffices to prove that

$$\frac{xy + yz + zx}{(x^2 + 1)(y^2 + 1)(z^2 + 1)} \leq \frac{3}{8}$$

i.e.

$$(x^2 + 1)(y^2 + 1)(z^2 + 1) \geq \frac{8}{3}(xy + yz + zx).$$

By the *Cauchy–Schwarz inequality* we have

$$\begin{aligned} \sqrt{(x^2 + 1)(1 + y^2)} &\geq x + y, & \sqrt{(z^2 + 1)(1 + x^2)} &\geq z + x \quad \text{and} \\ \sqrt{(y^2 + 1)(1 + z^2)} &\geq y + z. \end{aligned}$$

Multiplying these three inequalities gives us

$$(x^2 + 1)(y^2 + 1)(z^2 + 1) \geq (x + y)(y + z)(z + x). \quad (1)$$

By the well-known inequality

$$(x + y)(y + z)(z + x) \geq \frac{8}{9}(x + y + z)(xy + yz + zx),$$

and the  $AM \geq GM$  we obtain

$$(x + y)(y + z)(z + x) \geq \frac{8}{9}(x + y + z)(xy + yz + zx) \geq \frac{8}{3}(xy + yz + zx). \quad (2)$$

By (1) and (2) we obtain

$$(x^2 + 1)(y^2 + 1)(z^2 + 1) \geq (x + y)(y + z)(z + x) \geq \frac{8}{3}(xy + yz + zx),$$

as required.

Equality occurs iff  $x = y = z = 1$  i.e.  $a = b = c = 1$ . ■

**180** Let  $x, y, z$  be positive real numbers. Prove the inequality

$$xy(x + y - z) + yz(y + z - x) + zx(z + x - y) \geq \sqrt{3(x^3y^3 + y^3z^3 + z^3x^3)}.$$

*Solution* Notice that

$$\begin{aligned} &xy(x + y - z) + yz(y + z - x) + zx(z + x - y) \\ &= \frac{x(y^3 + z^3)}{y + z} + \frac{y(z^3 + x^3)}{z + x} + \frac{z(x^3 + y^3)}{x + y}. \end{aligned}$$

Let  $a = x^3, b = y^3, c = z^3$ .

Using Corollary 4.5 (Chap. 4) and the previous identity we obtain

$$\begin{aligned}
 & xy(x+y-z) + yz(y+z-x) + zx(z+x-y) \\
 &= \frac{x(y^3+z^3)}{y+z} + \frac{y(z^3+x^3)}{z+x} + \frac{z(x^3+y^3)}{x+y} \\
 &= \frac{x}{y+z}(b+c) + \frac{y}{z+x}(c+a) + \frac{z}{x+y}(a+b) \\
 &\geq \sqrt{3(ab+bc+ca)} = \sqrt{3(x^3y^3+y^3z^3+z^3x^3)},
 \end{aligned}$$

as required. ■

**181** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{ab(a^3+b^3)}{a^2+b^2} + \frac{bc(b^3+c^3)}{b^2+c^2} + \frac{ca(c^3+a^3)}{c^2+a^2} \geq \sqrt{3abc(a^3+b^3+c^3)}.$$

*Solution* Let

$$\begin{aligned}
 x &= \frac{1}{c^2}, & y &= \frac{1}{b^2}, & z &= \frac{1}{a^2} \quad \text{and} \\
 A &= \frac{a^2b^2}{c}, & B &= \frac{b^2c^2}{a}, & C &= \frac{a^2c^2}{b}.
 \end{aligned}$$

We have

$$\frac{x}{y+z}(B+C) = \frac{ab(a^3+b^3)}{a^2+b^2}, \quad \frac{y}{z+x}(C+A) = \frac{bc(b^3+c^3)}{b^2+c^2}$$

and

$$\frac{z}{x+y}(A+B) = \frac{ca(c^3+a^3)}{c^2+a^2}.$$

Using Corollary 4.5 (Chap. 4) and the previous identities we obtain

$$\begin{aligned}
 & \frac{ab(a^3+b^3)}{a^2+b^2} + \frac{bc(b^3+c^3)}{b^2+c^2} + \frac{ca(c^3+a^3)}{c^2+a^2} \\
 &= \frac{x}{y+z}(B+C) + \frac{y}{z+x}(C+A) + \frac{z}{x+y}(A+B) \\
 &\geq \sqrt{3(AB+BC+CA)} = \sqrt{3abc(a^3+b^3+c^3)}.
 \end{aligned}$$
■

**182** Let  $a, b, c$  be positive real numbers. Prove the inequality.

$$ab \frac{a+c}{b+c} + bc \frac{b+a}{c+a} + ca \frac{c+b}{a+b} \geq \sqrt{3abc(a+b+c)}.$$

**Solution** Let  $x = \frac{1}{bc}$ ,  $y = \frac{1}{ac}$ ,  $z = \frac{1}{ab}$  and  $A = ac$ ,  $B = ab$ ,  $C = bc$ .

We have

$$\begin{aligned}\frac{x}{y+z}(B+C) &= ab \frac{a+c}{b+c}, & \frac{y}{z+x}(C+A) &= bc \frac{b+a}{c+a} \quad \text{and} \\ \frac{z}{x+y}(A+B) &= ca \frac{c+b}{a+b}.\end{aligned}$$

Using Corollary 4.5 (Chap. 4) and the previous identities we obtain

$$\begin{aligned}ab \frac{a+c}{b+c} + bc \frac{b+a}{c+a} + ca \frac{c+b}{a+b} \\ = \frac{x}{y+z}(B+C) + \frac{y}{z+x}(C+A) + \frac{z}{x+y}(A+B) \\ \geq \sqrt{3(AB+BC+CA)} = \sqrt{3abc(a+b+c)}.\end{aligned}$$

■

**183** Let  $a, b, c$  and  $x, y, z$  be positive real numbers. Prove the inequality

$$a(y+z) + b(z+x) + c(x+y) \geq 2\sqrt{(xy+yz+zx)(ab+bc+ca)}.$$

**Solution** Since the given inequality is homogenous we may assume that  $x+y+z=1$ .

Now the given inequality can be written as follows

$$2\sqrt{(xy+yz+zx)(ab+bc+ca)} + ax + by + cz \leq a + b + c.$$

Applying the *Cauch-Schwarz inequality* twice we have

$$\begin{aligned}ax + by + cz + 2\sqrt{(xy+yz+zx)(ab+bc+ca)} \\ \leq \sqrt{a^2+b^2+c^2} \cdot \sqrt{x^2+y^2+z^2} + \sqrt{2(xy+yz+zx)} \cdot \sqrt{2(ab+bc+ca)} \\ \leq \sqrt{a^2+b^2+c^2+2(ab+bc+ca)} \cdot \sqrt{x^2+y^2+z^2+2(xy+yz+zx)} \\ = a + b + c.\end{aligned}$$

■

**184** Let  $a, b, c$  be positive real numbers such that  $abc \geq 1$ . Prove the inequality

$$a^3 + b^3 + c^3 \geq ab + bc + ca.$$

**Solution** By *Chebyshev's inequality* it is easy to obtain

$$3(a^3 + b^3 + c^3) \geq (a+b+c)(a^2 + b^2 + c^2). \quad (1)$$

Now by  $AM \geq GM$  we have

$$a + b + c \geq 3\sqrt[3]{abc} \geq 3$$

and clearly

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

So by (1) we obtain

$$a^3 + b^3 + c^3 \geq \frac{(a+b+c)(a^2+b^2+c^2)}{3} \geq \frac{3(ab+bc+ca)}{3} = ab+bc+ca. \quad \blacksquare$$

**185** Let  $a, b, c > 0$  be real numbers such that  $a^{2/3} + b^{2/3} + c^{2/3} = 3$ . Prove the inequality

$$a^2 + b^2 + c^2 \geq a^{4/3} + b^{4/3} + c^{4/3}.$$

*Solution* After setting  $a^{1/3} = x, b^{1/3} = y, c^{1/3} = z$  the initial condition becomes

$$x^2 + y^2 + z^2 = 3, \quad (1)$$

and the given inequality is equivalent to

$$x^6 + y^6 + z^6 \geq x^4 + y^4 + z^4.$$

Assume that  $x^2 \leq y^2 \leq z^2$ . Then it is clear that  $x^4 \leq y^4 \leq z^4$ .

Applying *Chebyshev's inequality* we get

$$(x^2 + y^2 + z^2)(x^4 + y^4 + z^4) \leq 3(x^6 + y^6 + z^6),$$

and using (1) we obtain  $x^6 + y^6 + z^6 \geq x^4 + y^4 + z^4$ , as required.  $\blacksquare$

**186** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$\frac{1}{c^2 + a + b} + \frac{1}{a^2 + b + c} + \frac{1}{b^2 + c + a} \leq 1.$$

*Solution* Observe that

$$\frac{1}{a^2 + b + c} - \frac{1}{3} = \frac{1}{a^2 - a + 3} - \frac{1}{3} = \frac{a(1-a)}{3(a^2 - a + 3)}.$$

Analogously

$$\frac{1}{b^2 + c + a} - \frac{1}{3} = \frac{b(1-b)}{3(b^2 - b + 3)} \quad \text{and} \quad \frac{1}{c^2 + a + b} - \frac{1}{3} = \frac{c(1-c)}{3(c^2 - c + 3)}.$$

Now the given inequality is equivalent to

$$\frac{a(a-1)}{a^2 - a + 3} + \frac{b(b-1)}{b^2 - b + 3} + \frac{c(c-1)}{c^2 - c + 3} \geq 0$$

i.e.

$$\frac{a-1}{a-1+3/a} + \frac{b-1}{b-1+3/b} + \frac{c-1}{c-1+3/c} \geq 0.$$

Without loss of generality we may assume that  $a \geq b \geq c$ .

Then clearly  $a - 1 \geq b - 1 \geq c - 1$  and since  $a + b + c = 3$  it follows that  $ab, bc, ca \leq 3$ . Now we can easily show that

$$\frac{1}{a-1+3/a} \geq \frac{1}{b-1+3/b} \geq \frac{1}{c-1+3/c}.$$

Applying *Chebishev's inequality* we obtain

$$(a-1+b-1+c-1) \left( \frac{1}{a-1+3/a} + \frac{1}{b-1+3/b} + \frac{1}{c-1+3/c} \right) \leq 3A$$

i.e.

$$A \geq 0.$$

Equality occurs iff  $a = b = c = 1$ . ■

**187** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$\frac{2a^2}{b+c} + \frac{2b^2}{c+a} + \frac{2c^2}{a+b} \geq a+b+c.$$

*Solution* Without loss of generality we can assume that  $a \geq b \geq c$ .

Then clearly

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}.$$

By *Chebishev's inequality* we have

$$\frac{2a^2}{b+c} + \frac{2b^2}{c+a} + \frac{2c^2}{a+b} \geq \frac{2}{3}(a^2+b^2+c^2) \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right). \quad (1)$$

Applying  $QM \geq AM$  we deduce

$$\sqrt{\frac{a^2+b^2+c^2}{3}} \geq \frac{a+b+c}{3}, \quad \text{i.e.} \quad \frac{a^2+b^2+c^2}{3} \geq \left( \frac{a+b+c}{3} \right)^2.$$

By (1) and the previous inequality it follows that

$$\frac{2a^2}{b+c} + \frac{2b^2}{c+a} + \frac{2c^2}{a+b} \geq \frac{2}{9}(a+b+c)^2 \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right). \quad (2)$$

Applying  $AM \geq HM$  we deduce

$$\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \geq \frac{9}{(b+c) + (c+a) + (a+b)} = \frac{9}{2(a+b+c)}.$$

Finally from the previous inequality and (2), we get required result. ■



**188** Let  $a, b, c$  be positive real numbers such that  $abc = 2$ . Prove the inequality.

$$a^3 + b^3 + c^3 \geq a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}.$$

*Solution* Applying the *Cauchy-Schwarz inequality* we get

$$a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b} \leq \sqrt{2(a^2 + b^2 + c^2)(a+b+c)}. \quad (1)$$

Using *Chebyshev's inequality* we get

$$\sqrt{2(a^2 + b^2 + c^2)(a+b+c)} \leq \sqrt{6(a^3 + b^3 + c^3)}. \quad (2)$$

Also from  $AM \geq GM$  we have

$$a^3 + b^3 + c^3 \geq 3abc = 6. \quad (3)$$

Combining (1), (2) and (3) we have

$$a^3 + b^3 + c^3 \geq \sqrt{6(a^3 + b^3 + c^3)} \geq a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}.$$

Equality holds iff  $a = b = c = \sqrt[3]{2}$ . ■

**189** Let  $a_1, a_2, \dots, a_n$  be positive real numbers. Prove the inequality

$$\frac{1}{\frac{1}{1+a_1} + \frac{1}{1+a_2} + \dots + \frac{1}{1+a_n}} - \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} \geq \frac{1}{n}.$$

*Solution* We can assume that  $a_1 \geq a_2 \geq \dots \geq a_n$ .

If we take  $x_i = \frac{1}{a_i}$ ,  $y_i = \frac{1}{a_i+1}$  for  $i = 1, 2, \dots, n$  then

$$\frac{1}{a_1} \leq \frac{1}{a_2} \leq \dots \leq \frac{1}{a_n} \quad \text{and} \quad \frac{1}{a_1+1} \leq \frac{1}{a_2+1} \leq \dots \leq \frac{1}{a_n+1}.$$

Also we have that

$$x_i y_i = \frac{1}{a_i(a_i+1)} = \frac{1}{a_i} - \frac{1}{a_i+1} = x_i - y_i.$$

So we can use *Chebyshev's inequality*, i.e. we have

$$\begin{aligned} \sum_{i=1}^n \frac{1}{a_i} \cdot \sum_{i=1}^n \frac{1}{a_i+1} &\leq n \cdot \sum_{i=1}^n \frac{1}{a_i(a_i+1)} = n \cdot \sum_{i=1}^n \frac{1}{a_i} - \frac{1}{a_i+1} \\ &= n \cdot \left( \sum_{i=1}^n \frac{1}{a_i} - \sum_{i=1}^n \frac{1}{a_i+1} \right). \end{aligned}$$

Now we easily obtain

$$\frac{1}{\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n}} - \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \geq \frac{1}{n}.$$

Equality holds iff  $a_1 = a_2 = \cdots = a_n$ . ■

**190** Let  $a, b, c, d \in \mathbb{R}^+$  such that  $ab + bc + cd + da = 1$ . Prove the inequality

$$\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{b+d+a} + \frac{d^3}{b+c+a} \geq \frac{1}{3}.$$

*Solution* Let  $a + b + c + d = s$ . Then the given inequality is equivalent to

$$A = \frac{a^3}{s-a} + \frac{b^3}{s-b} + \frac{c^3}{s-c} + \frac{d^3}{s-d} \geq \frac{1}{3}. \quad (1)$$

Let us assume  $a \geq b \geq c \geq d$ . Then

$$a^3 \geq b^3 \geq c^3 \geq d^3 \quad \text{and} \quad \frac{1}{s-a} \geq \frac{1}{s-b} \geq \frac{1}{s-c} \geq \frac{1}{s-d}.$$

Applying *Chebyshev's inequality* we get

$$\begin{aligned} & (a^3 + b^3 + c^3 + d^3) \left( \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{1}{s-d} \right) \\ & \leq 4 \left( \frac{a^3}{s-a} + \frac{b^3}{s-b} + \frac{c^3}{s-c} + \frac{d^3}{s-d} \right) \end{aligned}$$

i.e.

$$4A \geq (a^3 + b^3 + c^3 + d^3) \left( \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{1}{s-d} \right). \quad (2)$$

Since  $a \geq b \geq c \geq d$  it follows that  $a^2 \geq b^2 \geq c^2 \geq d^2$ , and one more application of *Chebyshev's inequality* gives us

$$(a^2 + b^2 + c^2 + d^2)(a + b + c + d) \leq 4(a^3 + b^3 + c^3 + d^3),$$

i.e.

$$a^3 + b^3 + c^3 + d^3 \geq \frac{(a^2 + b^2 + c^2 + d^2)(a + b + c + d)}{4}. \quad (3)$$

Furthermore

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= \frac{a^2 + b^2}{2} + \frac{b^2 + c^2}{2} + \frac{c^2 + d^2}{2} + \frac{d^2 + a^2}{2} \\ &\geq ab + bc + cd + da = 1. \end{aligned}$$

So in (3) we deduce

$$a^3 + b^3 + c^3 + d^3 \geq \frac{a+b+c+d}{4} \quad (4)$$

and clearly we have

$$a+b+c+d = \frac{(s-a) + (s-b) + (s-c) + (s-d)}{3}. \quad (5)$$

Now from (4) and (5) we obtain

$$a^3 + b^3 + c^3 + d^3 \geq \frac{(s-a) + (s-b) + (s-c) + (s-d)}{12}. \quad (6)$$

Using (2), (6) and  $AM \geq HM$  we have

$$\begin{aligned} 4A &\geq \left( \frac{(s-a) + (s-b) + (s-c) + (s-d)}{12} \right) \left( \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{1}{s-d} \right) \\ &\geq \frac{16}{12} = \frac{4}{3}, \end{aligned}$$

i.e. it follows that  $A \geq \frac{1}{3}$ , as required. ■

**191** Let  $\alpha, x, y, z$  be positive real numbers such that  $xyz = 1$  and  $\alpha \geq 1$ . Prove the inequality

$$\frac{x^\alpha}{y+z} + \frac{y^\alpha}{z+x} + \frac{z^\alpha}{x+y} \geq \frac{3}{2}.$$

*Solution* Without loss of generality we may assume that  $x \geq y \geq z$ .

Then

$$\frac{x}{y+z} \geq \frac{y}{z+x} \geq \frac{z}{x+y} \quad \text{and} \quad x^{\alpha-1} \geq y^{\alpha-1} \geq z^{\alpha-1}.$$

Applying *Chebyshev's inequality* we have

$$(x^{\alpha-1} + y^{\alpha-1} + z^{\alpha-1}) \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) \leq 3 \left( \frac{x^\alpha}{y+z} + \frac{y^\alpha}{z+x} + \frac{z^\alpha}{x+y} \right). \quad (1)$$

Recalling  $AM \geq GM$  we get

$$x^{\alpha-1} + y^{\alpha-1} + z^{\alpha-1} \geq 3\sqrt[3]{(xyz)^{\alpha-1}} = 3. \quad (2)$$

*Nesbitt's inequality* gives us

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}. \quad (3)$$

Finally using (1), (2) and (3) we obtain

$$\begin{aligned} 3\left(\frac{x^\alpha}{y+z} + \frac{y^\alpha}{z+x} + \frac{z^\alpha}{x+y}\right) &\geq (x^{\alpha-1} + y^{\alpha-1} + z^{\alpha-1})\left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right) \\ &= 3 \cdot \frac{3}{2} \end{aligned}$$

i.e.

$$\frac{x^\alpha}{y+z} + \frac{y^\alpha}{z+x} + \frac{z^\alpha}{x+y} \geq \frac{3}{2}. \quad \blacksquare$$

**192** Let  $x_1, x_2, \dots, x_n$  be positive real numbers such that

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \dots + \frac{1}{1+x_n} = 1.$$

Prove the inequality

$$\frac{\sqrt{x_1} + \sqrt{x_2} + \dots + \sqrt{x_n}}{n-1} \geq \frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \dots + \frac{1}{\sqrt{x_n}}.$$

*Solution* Let  $\frac{1}{1+x_i} = a_i$ , for  $i = 1, 2, \dots, n$ .

Clearly  $\sum_{i=1}^n a_i = 1$  and the given inequality becomes

$$\begin{aligned} \sum_{i=1}^n \sqrt{\frac{1-a_i}{a_i}} &\geq (n-1) \sum_{i=1}^n \sqrt{\frac{a_i}{1-a_i}} \Leftrightarrow \sum_{i=1}^n \sqrt{\frac{1}{a_i(1-a_i)}} \geq n \sum_{i=1}^n \sqrt{\frac{a_i}{1-a_i}} \\ &\Leftrightarrow n \sum_{i=1}^n \sqrt{\frac{a_i}{1-a_i}} \leq \left(\sum_{i=1}^n a_i\right) \left(\sum_{i=1}^n \sqrt{\frac{1}{a_i(1-a_i)}}\right). \end{aligned}$$

The last inequality is true according to *Chebyshev's inequality* applied to the sequences

$$(a_1, a_2, \dots, a_n) \quad \text{and} \quad \left(\frac{1}{\sqrt{a_1(1-a_1)}}, \frac{1}{\sqrt{a_2(1-a_2)}}, \dots, \frac{1}{\sqrt{a_n(1-a_n)}}\right). \quad \blacksquare$$

**193** Let  $x_1, x_2, \dots, x_n > 0$  be real numbers. Prove the inequality

$$x_1^{x_1} x_2^{x_2} \dots x_n^{x_n} \geq (x_1 x_2 \dots x_n)^{\frac{x_1 + x_2 + \dots + x_n}{n}}.$$

*Solution* If we take the logarithm of both sides the given inequality becomes:

$$x_1 \ln x_1 + x_2 \ln x_2 + \dots + x_n \ln x_n \geq \frac{x_1 + x_2 + \dots + x_n}{n} (\ln x_1 + \ln x_2 + \dots + \ln x_n). \quad (1)$$

We may assume that  $x_1 \geq x_2 \geq \dots \geq x_n$ , then  $\ln x_1 \geq \ln x_2 \geq \dots \geq \ln x_n$ .

Applying *Chebyshev's inequality* we get

$$(x_1 + x_2 + \cdots + x_n)(\ln x_1 + \ln x_2 + \cdots + \ln x_n) \leq n(x_1 \ln x_1 + x_2 \ln x_2 + \cdots + x_n \ln x_n),$$

i.e.

$$x_1 \ln x_1 + x_2 \ln x_2 + \cdots + x_n \ln x_n \geq \frac{x_1 + x_2 + \cdots + x_n}{n} (\ln x_1 + \ln x_2 + \cdots + \ln x_n). \quad \blacksquare$$

**194** Let  $a, b, c > 0$  be real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\frac{a^2 + b}{b + c} + \frac{b^2 + c}{c + a} + \frac{c^2 + a}{a + b} \geq 2.$$

*Solution 1* Applying the *Cauchy–Schwarz inequality* for the sequences

$$a_1 = \sqrt{\frac{a^2 + b}{b + c}}, \quad a_2 = \sqrt{\frac{b^2 + c}{c + a}}, \quad a_3 = \sqrt{\frac{c^2 + a}{a + b}}$$

and

$$b_1 = \sqrt{(a^2 + b)(b + c)}, \quad b_2 = \sqrt{(b^2 + c)(c + a)}, \quad b_3 = \sqrt{(c^2 + a)(a + b)}$$

we obtain

$$\frac{a^2 + b}{b + c} + \frac{b^2 + c}{c + a} + \frac{c^2 + a}{a + b} \geq \frac{(a^2 + b^2 + c^2 + 1)^2}{(a^2 + b)(b + c) + (b^2 + c)(c + a) + (c^2 + a)(a + b)}.$$

So it suffices to show that

$$\frac{(a^2 + b^2 + c^2 + 1)^2}{(a^2 + b)(b + c) + (b^2 + c)(c + a) + (c^2 + a)(a + b)} \geq 2.$$

We have

$$\begin{aligned} & \frac{(a^2 + b^2 + c^2 + 1)^2}{(a^2 + b)(b + c) + (b^2 + c)(c + a) + (c^2 + a)(a + b)} \geq 2 \\ \Leftrightarrow & (a^2 + b^2 + c^2 + 1)^2 \geq 2((a^2 + b)(b + c) + (b^2 + c)(c + a) \\ & \quad + (c^2 + a)(a + b)) \\ \Leftrightarrow & 1 + (a^2 + b^2 + c^2)^2 \geq 2(a^2(b + c) + b^2(c + a) + c^2(a + b)) \\ & \quad + 2(ab + bc + ca) \\ \Leftrightarrow & 1 + (a^2 + b^2 + c^2)^2 \geq 2(a^2(1 - a) + b^2(1 - b) + c^2(1 - c)) \\ & \quad + 2(ab + bc + ca) \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow 1 + (a^2 + b^2 + c^2)^2 \geq 2(a^2 + b^2 + c^2 - a^3 - b^3 - c^3) \\
&\quad + 2(ab + bc + ca) \\
&\Leftrightarrow (a^2 + b^2 + c^2)^2 + 2(a^3 + b^3 + c^3) \geq 2(a^2 + b^2 + c^2 + ab + bc + ca) - 1 \\
&\Leftrightarrow (a^2 + b^2 + c^2)^2 + 2(a^3 + b^3 + c^3) \geq 2(a(1 - c) + b(1 - a) \\
&\quad + c(1 - b)) - 1 \\
&\Leftrightarrow (a^2 + b^2 + c^2)^2 + 2(a^3 + b^3 + c^3) \geq 1 - 2(ab + bc + ca) \\
&\Leftrightarrow (a^2 + b^2 + c^2)^2 + 2(a^3 + b^3 + c^3) \geq (a + b + c)^2 - 2(ab + bc + ca) \\
&\quad = a^2 + b^2 + c^2.
\end{aligned}$$

So we need to show that

$$(a^2 + b^2 + c^2)^2 + 2(a^3 + b^3 + c^3) \geq a^2 + b^2 + c^2. \quad (1)$$

By *Chebyshev's inequality* we deduce

$$(a + b + c)(a^2 + b^2 + c^2) \leq 3(a^3 + b^3 + c^3), \quad \text{i.e.} \quad a^3 + b^3 + c^3 \geq \frac{a^2 + b^2 + c^2}{3},$$

and clearly  $(a^2 + b^2 + c^2)^2 \geq \frac{a^2 + b^2 + c^2}{3}$ .

Adding these inequalities gives us inequality (1). ■

*Solution 2* Take  $a + b + c = p = 1$ ,  $ab + bc + ca = q$ ,  $abc = r$  and use the method from Chap. 14. ■

**195** Let  $a, b, c > 1$  be positive real numbers such that  $\frac{1}{a^2-1} + \frac{1}{b^2-1} + \frac{1}{c^2-1} = 1$ . Prove the inequality

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \leq 1.$$

*Solution* Without loss of generality we may assume that  $a \geq b \geq c$ . Then we have

$$\frac{a-2}{a+1} \geq \frac{b-2}{b+1} \geq \frac{c-2}{c+1} \quad \text{and} \quad \frac{a+2}{a-1} \leq \frac{b+2}{b-1} \leq \frac{c+2}{c-1}.$$

Now by *Chebyshev's inequality* we get

$$\begin{aligned}
3 \left( \frac{a^2-4}{a^2-1} + \frac{b^2-4}{b^2-1} + \frac{c^2-4}{c^2-1} \right) &\leq \left( \frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \right) \\
&\quad \times \left( \frac{a+2}{a-1} + \frac{b+2}{b-1} + \frac{c+2}{c-1} \right).
\end{aligned}$$

Since

$$\frac{a^2-4}{a^2-1} + \frac{b^2-4}{b^2-1} + \frac{c^2-4}{c^2-1} = 3 - 3\left(\frac{1}{a^2-1} + \frac{1}{b^2-1} + \frac{1}{c^2-1}\right) = 0$$

and

$$\frac{a+2}{a-1} + \frac{b+2}{b-1} + \frac{c+2}{c-1} > 0$$

we must have

$$\frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \geq 0,$$

which is equivalent to  $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \leq 1$ , as required.

Equality holds iff  $a=b=c=2$ . ■

**196** Let  $a, b, c, d$  be positive real numbers such that  $a^2 + b^2 + c^2 + d^2 = 4$ . Prove the inequality

$$\frac{1}{5-a} + \frac{1}{5-b} + \frac{1}{5-c} + \frac{1}{5-d} \leq 1.$$

*Solution* The given inequality is equivalent to

$$\frac{1}{5-a} - \frac{1}{4} + \frac{1}{5-b} - \frac{1}{4} + \frac{1}{5-c} - \frac{1}{4} + \frac{1}{5-d} - \frac{1}{4} \leq 0,$$

i.e.

$$\frac{a-1}{5-a} + \frac{b-1}{5-b} + \frac{c-1}{5-c} + \frac{d-1}{5-d} \leq 0.$$

Without loss of generality we may assume that  $a \geq b \geq c \geq d$ .

Then we have  $a^2 - 1 \geq b^2 - 1 \geq c^2 - 1 \geq d^2 - 1$ .

We'll show that  $\frac{1}{4a-a^2+5} \leq \frac{1}{4b-b^2+5}$ .

We have

$$4a - a^2 + 5 \geq 4b - b^2 + 5 \Leftrightarrow a + b \leq 4,$$

which is obviously true since  $a^2 + b^2 \leq 4$ .

So we have

$$\frac{1}{4a - a^2 + 5} \leq \frac{1}{4b - b^2 + 5} \leq \frac{1}{4c - c^2 + 5} \leq \frac{1}{4d - d^2 + 5}.$$

Now by *Chebyshev's inequality* we obtain

$$\begin{aligned} & 3\left(\frac{a^2-1}{4a-a^2+5} + \frac{b^2-1}{4b-b^2+5} + \frac{a^2-1}{4c-c^2+5} + \frac{a^2-1}{4d-d^2+5}\right) \\ & \leq \sum_{\text{cyc}} (a^2-1) \sum_{\text{cyc}} \left(\frac{1}{4a-a^2+5}\right) = 0. \end{aligned}$$

Thus

$$\begin{aligned} 0 &\geq \frac{a^2 - 1}{4a - a^2 + 5} + \frac{b^2 - 1}{4b - b^2 + 5} + \frac{a^2 - 1}{4c - c^2 + 5} + \frac{a^2 - 1}{4d - d^2 + 5} \\ &= \frac{a - 1}{5 - a} + \frac{b - 1}{5 - b} + \frac{c - 1}{5 - c} + \frac{d - 1}{5 - d}, \end{aligned}$$

as required.

Equality holds iff  $a = b = c = d = 1$ . ■

**197** Let  $a, b, c, d \in \mathbb{R}$  such that  $\frac{1}{4+a} + \frac{1}{4+b} + \frac{1}{4+c} + \frac{1}{4+d} + \frac{1}{4+e} = 1$ . Prove the inequality

$$\frac{a}{4+a^2} + \frac{b}{4+b^2} + \frac{c}{4+c^2} + \frac{d}{4+d^2} + \frac{e}{4+e^2} \leq 1.$$

*Solution* We have

$$\begin{aligned} &\frac{1-a}{4+a} + \frac{1-b}{4+b} + \frac{1-c}{4+c} + \frac{1-d}{4+d} + \frac{1-e}{4+e} \\ &= \frac{5-(4+a)}{4+a} + \frac{5-(4+b)}{4+b} + \frac{5-(4+c)}{4+c} + \frac{5-(4+d)}{4+d} + \frac{5-(4+e)}{4+e} \\ &= 5 - 5 = 0. \end{aligned}$$

We'll prove that

$$\begin{aligned} \frac{a}{4+a^2} + \frac{b}{4+b^2} + \frac{c}{4+c^2} + \frac{d}{4+d^2} + \frac{e}{4+e^2} &\leq \frac{1}{4+a} + \frac{1}{4+b} + \frac{1}{4+c} + \frac{1}{4+d} \\ &\quad + \frac{1}{4+e}. \end{aligned} \tag{1}$$

Inequality (1) is equivalent to

$$\begin{aligned} &\frac{1-a}{(4+a)(4+a^2)} + \frac{1-b}{(4+b)(4+b^2)} + \frac{1-c}{(4+c)(4+c^2)} + \frac{1-d}{(4+d)(4+d^2)} \\ &\quad + \frac{1-e}{(4+e)(4+e^2)} \geq 0. \end{aligned} \tag{2}$$

Without loss of generality we may assume that  $a \geq b \geq c \geq d \geq e$ , and then we easily deduce that

$$\begin{aligned} \frac{1-a}{4+a} &\leq \frac{1-b}{4+b} \leq \frac{1-c}{4+c} \leq \frac{1-d}{4+d} \leq \frac{1-e}{4+e} \quad \text{and} \\ \frac{1}{4+a^2} &\leq \frac{1}{4+b^2} \leq \frac{1}{4+c^2} \leq \frac{1}{4+d^2} \leq \frac{1}{4+e^2}. \end{aligned}$$



So by *Chebyshev's inequality* we get

$$5 \sum_{\text{sym}} \frac{1-a}{(4+a)(4+a^2)} \geq \sum_{\text{sym}} \frac{1-a}{4+a} \cdot \sum_{\text{sym}} \frac{1}{4+a^2} = 0,$$

which means that inequality (2) holds, i.e. inequality (1) is true and since  $\frac{1}{4+a} + \frac{1}{4+b} + \frac{1}{4+c} + \frac{1}{4+d} + \frac{1}{4+e} = 1$  we obtain the required result.

Equality occurs iff  $a = b = c = d = e = 1$ . ■

**198** Let  $a, b, c$  be real numbers different from 1, such that  $a + b + c = 1$ . Prove the inequality

$$\frac{1+a^2}{1-a^2} + \frac{1+b^2}{1-b^2} + \frac{1+c^2}{1-c^2} \geq \frac{15}{4}.$$

*Solution* Since  $a, b, c > 0, a \neq 1, b \neq 1, c \neq 1$  and  $a + b + c = 1$  it follows that  $0 < a, b, c < 1$ .

The given inequality is symmetric, so without loss of generality we may assume that  $a \leq b \leq c$ .

Then we have

$$1 + a^2 \leq 1 + b^2 \leq 1 + c^2 \quad \text{and} \quad 1 - c^2 \leq 1 - b^2 \leq 1 - a^2.$$

Hence

$$\frac{1}{1-a^2} \leq \frac{1}{1-b^2} \leq \frac{1}{1-c^2}.$$

Now by *Chebyshev's inequality* we have

$$\begin{aligned} A &= \frac{1+a^2}{1-a^2} + \frac{1+b^2}{1-b^2} + \frac{1+c^2}{1-c^2} \\ &\geq \frac{1}{3}(1+a^2+1+b^2+1+c^2) \left( \frac{1}{1-a^2} + \frac{1}{1-b^2} + \frac{1}{1-c^2} \right), \end{aligned}$$

i.e.

$$A \geq \frac{(a^2+b^2+c^2+3)}{3} \left( \frac{1}{1-a^2} + \frac{1}{1-b^2} + \frac{1}{1-c^2} \right). \quad (1)$$

Also we have the well-known inequality

$$a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3} = \frac{1}{3}.$$

Therefore by (1) we obtain

$$A \geq \frac{(1/3+3)}{3} \left( \frac{1}{1-a^2} + \frac{1}{1-b^2} + \frac{1}{1-c^2} \right) = \frac{10}{9} \left( \frac{1}{1-a^2} + \frac{1}{1-b^2} + \frac{1}{1-c^2} \right). \quad (2)$$

Since  $1 - a^2, 1 - b^2, 1 - c^2 > 0$ , by using  $AM \geq HM$  we deduce

$$\frac{1}{1-a^2} + \frac{1}{1-b^2} + \frac{1}{1-c^2} \geq \frac{9}{3-(a^2+b^2+c^2)} \geq \frac{9}{3-1/3} = \frac{27}{8}. \quad (3)$$

Finally from (2) and (3) we get

$$A \geq \frac{10}{9} \left( \frac{1}{1-a^2} + \frac{1}{1-b^2} + \frac{1}{1-c^2} \right) \geq \frac{10}{9} \cdot \frac{27}{8} = \frac{15}{4},$$

with equality iff  $a = b = c = 1/3$ . ■

**199** Let  $x, y, z > 0$ , such that  $xyz = 1$ . Prove the inequality

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

*Solution* Let  $x \geq y \geq z$ . Then

$$x^3 \geq y^3 \geq z^3 \quad \text{and} \quad \frac{1}{(1+y)(1+z)} \geq \frac{1}{(1+z)(1+x)} \geq \frac{1}{(1+x)(1+y)}.$$

Applying *Chebishev's inequality* we get

$$\begin{aligned} 3S &= 3 \left( \frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \right) \\ &\geq (x^3 + y^3 + z^3) \left( \frac{1}{(1+y)(1+z)} + \frac{1}{(1+z)(1+x)} + \frac{1}{(1+x)(1+y)} \right) \\ &= (x^3 + y^3 + z^3) \left( \frac{(1+x) + (1+y) + (1+z)}{(1+x)(1+y)(1+z)} \right) \\ &= (x^3 + y^3 + z^3) \left( \frac{3+x+y+z}{(1+x)(1+y)(1+z)} \right), \end{aligned}$$

i.e.

$$S \geq \left( \frac{x^3 + y^3 + z^3}{3} \right) \left( \frac{3+x+y+z}{(1+x)(1+y)(1+z)} \right). \quad (1)$$

Let  $\frac{x+y+z}{3} = a$ . Then we have

$$\frac{x^3 + y^3 + z^3}{3} \geq \left( \frac{x+y+z}{3} \right)^3 = a^3 \quad \text{and} \quad 3a \geq 3\sqrt[3]{xyz} = 3, \quad \text{i.e.} \quad a \geq 1.$$

From  $AM \geq GM$  we get

$$(1+x)(1+y)(1+z) \leq \left( \frac{3+x+y+z}{3} \right)^3 = (1+a)^3.$$

So by (1) we obtain

$$S \geq \left( \frac{x^3 + y^3 + z^3}{3} \right) \left( \frac{3 + x + y + z}{(1+x)(1+y)(1+z)} \right) \geq a^3 \left( \frac{6}{(1+a)^3} \right).$$

Hence it suffices to show that

$$\frac{6a^3}{(1+a)^3} \geq \frac{3}{4},$$

i.e.

$$6 \left( 1 - \frac{1}{1+a} \right)^3 = \frac{6a^3}{(1+a)^3} \geq \frac{3}{4}.$$

Since  $a \geq 1$ , and the function  $f(x) = 6(1 - \frac{1}{1+x})^3$  increases on  $[1, \infty]$  (why?), it follows that  $f(a) \geq f(1) = \frac{3}{4}$ , as required. ■

**200** Let  $a, b, c, d > 0$  be real numbers. Prove the inequality

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geq \frac{2}{3}.$$

*Solution* Let

$$A = b+2c+3d, \quad B = c+2d+3a, \quad C = d+2a+3b, \quad D = a+2b+3c.$$

By the *Cauchy–Schwarz inequality* we have

$$\begin{aligned} \left( \frac{a}{A} + \frac{b}{B} + \frac{c}{C} + \frac{d}{D} \right) (aA + bB + cC + dD) &\geq (a+b+c+d)^2 \\ \Leftrightarrow \frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \\ &\geq \frac{(a+b+c+d)^2}{aA + bB + cC + dD}. \end{aligned} \tag{1}$$

Furthermore

$$aA + bB + cC + dD = 4(ab + ac + ad + bc + bd + cd),$$

and (1) becomes

$$\begin{aligned} \frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \\ \geq \frac{(a+b+c+d)^2}{4(ab + ac + ad + bc + bd + cd)}. \end{aligned}$$

So it suffices to prove that

$$\frac{(a+b+c+d)^2}{4(ab+ac+ad+bc+bd+cd)} \geq \frac{2}{3},$$

i.e.

$$3(a+b+c+d)^2 \geq 8(ab+ac+ad+bc+bd+cd). \quad (2)$$

We'll use *Maclaurin's theorem*.

We have

$$p_2 = \frac{c_2}{6} = \frac{ab+ac+ad+bc+bd+cd}{6}, \quad \text{i.e.}$$

$$ab+ac+ad+bc+bd+cd = 6p_2$$

and

$$p_1 = \frac{c_1}{4} = \frac{a+b+c+d}{4}, \quad \text{i.e.} \quad a+b+c+d = 4p_1.$$

Now inequality (2) is equivalent to  $48p_1^2 \geq 48p_2$ , i.e.  $p_1 \geq p_2^{1/2}$ , which is true due to *Maclaurin's theorem*. ■

**201** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq a+b+c.$$

*Solution* Assume  $a \geq b \geq c$ . Then clearly  $a^2 \geq b^2 \geq c^2$  and  $\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$ .

According to the *rearrangement inequality* we have

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{b^2}{b+c} + \frac{c^2}{c+a} + \frac{a^2}{a+b},$$

i.e.

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq \frac{b^2+bc}{b+c} + \frac{c^2+ca}{c+a} + \frac{a^2+ab}{a+b} = a+b+c.$$

Equality occurs iff  $a = b = c$ . ■

**202** Let  $a, b > 0, n \in \mathbb{N}$ . Prove the inequality

$$\left(1 + \frac{a}{b}\right)^n + \left(1 + \frac{b}{a}\right)^n \geq 2^{n+1}.$$

*Solution* We'll use the fact that the function  $f(x) = x^n$  is concave on  $(0, \infty)$ .

So according to *Jensen's inequality* we have

$$\frac{x^n + y^n}{2} \geq \left( \frac{x + y}{2} \right)^n.$$

*Remark* Note that this is a *power mean inequality*.

Now we have

$$\frac{1}{2} \left( \left( 1 + \frac{a}{b} \right)^n + \left( 1 + \frac{b}{a} \right)^n \right) \geq \left( \frac{1 + a/b + 1 + b/a}{2} \right)^n = \left( \frac{2 + a/b + b/a}{2} \right)^n. \quad (1)$$

Using  $\frac{a}{b} + \frac{b}{a} \geq 2$  and (1) we deduce

$$\left( 1 + \frac{a}{b} \right)^n + \left( 1 + \frac{b}{a} \right)^n \geq 2 \left( \frac{2+2}{2} \right)^n = 2^{n+1}. \quad \blacksquare$$

**203** Let  $a, b, c > 0$  be real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\left( a + \frac{1}{a} \right)^2 + \left( b + \frac{1}{b} \right)^2 + \left( c + \frac{1}{c} \right)^2 \geq \frac{100}{3}.$$

*Solution* The function  $f(x) = x^2$  is convex on  $(0, \infty)$ .

So according to *Jensen's inequality* we have

$$\frac{1}{3} \left( \left( a + \frac{1}{a} \right)^2 + \left( b + \frac{1}{b} \right)^2 + \left( c + \frac{1}{c} \right)^2 \right) \geq \left( \frac{1}{3} \left( a + \frac{1}{a} + b + \frac{1}{b} + c + \frac{1}{c} \right) \right)^2,$$

i.e.

$$\begin{aligned} \left( a + \frac{1}{a} \right)^2 + \left( b + \frac{1}{b} \right)^2 + \left( c + \frac{1}{c} \right)^2 &\geq \frac{1}{3} \left( a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^2 \\ &\geq \frac{1}{3} (1 + 9)^2 = \frac{100}{3}. \quad \blacksquare \end{aligned}$$

**204** Let  $x, y, z > 0$  be real numbers. Prove the inequality

$$\frac{x}{2x + y + z} + \frac{y}{x + 2y + z} + \frac{z}{x + y + 2z} \leq \frac{3}{4}.$$

*Solution* Let  $s = x + y + z$ .

The given inequality becomes

$$\frac{x}{s+x} + \frac{y}{s+y} + \frac{z}{s+z} \leq \frac{3}{4}.$$

Consider the function  $f : (0, +\infty) \rightarrow (0, +\infty)$ , defined by  $f(a) = \frac{a}{s+a}$ .

We can easily show that  $f''(a) \leq 0$ , for every  $a \in \mathbb{R}^+$ , i.e.  $f$  is concave on  $\mathbb{R}^+$ . By *Jensen's inequality* we have

$$\frac{f(x) + f(y) + f(z)}{3} \leq f\left(\frac{x + y + z}{3}\right),$$

i.e.

$$\begin{aligned} \frac{x}{s+x} + \frac{y}{s+y} + \frac{z}{s+z} &= f(x) + f(y) + f(z) \\ &\leq 3f\left(\frac{x+y+z}{3}\right) = 3f\left(\frac{s}{3}\right) = \frac{s/3}{s+s/3} = \frac{3}{4}, \end{aligned}$$

as required. ■

**205** Let  $a, b, c, d > 0$  be real numbers such that  $a \leq 1, a + b \leq 5, a + b + c \leq 14, a + b + c + d \leq 30$ . Prove that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d} \leq 10.$$

*Solution* The function  $f : (0, +\infty) \rightarrow (0, +\infty)$  defined by  $f(x) = \sqrt{x}$  is concave on  $(0, +\infty)$ , so by *Jensen's inequality*, for

$$n = 4, \quad \alpha_1 = \frac{1}{10}, \quad \alpha_2 = \frac{2}{10}, \quad \alpha_3 = \frac{3}{10}, \quad \alpha_4 = \frac{4}{10}$$

we get

$$\frac{1}{10}\sqrt{a} + \frac{2}{10}\sqrt{\frac{b}{4}} + \frac{3}{10}\sqrt{\frac{c}{9}} + \frac{4}{10}\sqrt{\frac{d}{16}} \leq \sqrt{\frac{a}{10} + \frac{b}{20} + \frac{c}{30} + \frac{d}{40}},$$

i.e.

$$\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d} \leq 10\sqrt{\frac{12a + 6b + 4c + 3d}{120}}. \quad (1)$$

On the other hand, we have

$$\begin{aligned} 12a + 6b + 4c + 3d &= 3(a + b + c + d) + (a + b + c) + 2(a + b) + 6a \\ &\leq 3 \cdot 30 + 14 + 2 \cdot 5 + 6 \cdot 1 = 120. \end{aligned}$$

By (1) and the last inequality we obtain the required result. ■

**206** Let  $a, b, c, d$  be positive real numbers such that  $a + b + c + d = 4$ . Prove the inequality

$$\frac{a}{b^2 + b} + \frac{b}{c^2 + c} + \frac{c}{d^2 + d} + \frac{d}{a^2 + a} \geq \frac{8}{(a+c)(b+d)}.$$

*Solution* Denote  $A = \frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{d^2+d} + \frac{d}{a^2+a}$ .

Consider the function  $f(x) = \frac{1}{x(x+1)}$ . Then  $f$  is convex for  $x > 0$ .

According to *Jensen's inequality*, we have

$$\frac{a}{4} \cdot f(b) + \frac{b}{4} \cdot f(c) + \frac{c}{4} \cdot f(d) + \frac{d}{4} \cdot f(a) \geq f\left(\frac{ab+bc+cd+da}{4}\right),$$

i.e.

$$A \geq \frac{64}{(ab+bc+cd+da)^2 + 4(ab+bc+cd+da)}.$$

So it remains to prove that

$$\frac{64}{(ab+bc+cd+da)^2 + 4(ab+bc+cd+da)} \geq \frac{8}{(a+c)(b+d)},$$

i.e.

$$ab+bc+cd+da \leq 4,$$

i.e.

$$(a-b+c-d)^2 \geq 0,$$

which is obviously true. Equality holds iff  $a=b=c=d=1$ . ■

**207** Let  $x_1, x_2, \dots, x_n > 0$  and  $n \in \mathbb{N}, n > 1$ , such that  $x_1 + x_2 + \dots + x_n = 1$ . Prove the inequality

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \dots + \frac{x_n}{\sqrt{1-x_n}} \geq \frac{\sqrt{x_1} + \sqrt{x_2} + \dots + \sqrt{x_n}}{\sqrt{n-1}}.$$

*Solution* The function  $f(x) = \frac{x}{\sqrt{1-x}}$  is convex on  $(0, \infty)$ . (Why?)

Hence by *Jensen's inequality* we have

$$\begin{aligned} \frac{1}{n} \left( \frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \dots + \frac{x_n}{\sqrt{1-x_n}} \right) &\geq \left( \frac{\frac{x_1+x_2+\dots+x_n}{n}}{\sqrt{1-\frac{x_1+x_2+\dots+x_n}{n}}} \right) \\ &= \frac{\frac{1}{n}}{\sqrt{1-\frac{1}{n}}} = \frac{1}{\sqrt{n(n-1)}}. \end{aligned}$$

It follows that

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \dots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}. \quad (1)$$

By  $QM \geq AM$  we have

$$\frac{\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n}}{n} \leq \sqrt{\frac{x_1 + x_2 + \cdots + x_n}{n}} = \frac{1}{\sqrt{n}},$$

i.e.

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \leq \sqrt{n}. \quad (2)$$

By (1) and (2) we deduce

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}} \geq \frac{\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n}}{\sqrt{n-1}},$$

as required. ■

**208** Let  $n \in \mathbb{N}, n \geq 2$ . Determine the minimal value of

$$\frac{x_1^5}{x_2 + x_3 + \cdots + x_n} + \frac{x_2^5}{x_1 + x_3 + \cdots + x_n} + \cdots + \frac{x_n^5}{x_1 + x_2 + \cdots + x_{n-1}},$$

where  $x_1, x_2, \dots, x_n \in \mathbb{R}^+$  such that  $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ .

*Solution* Let  $S = x_1 + x_2 + \cdots + x_n$ . We may assume that  $x_1 \geq x_2 \geq \cdots \geq x_n$ .

$$\text{Let } A = \frac{x_1^5}{S-x_1} + \frac{x_2^5}{S-x_2} + \cdots + \frac{x_n^5}{S-x_n} = \sum_{i=1}^n \frac{x_i^5}{S-x_i}.$$

Since

$$x_1^4 \geq x_2^4 \geq \cdots \geq x_n^4 \quad \text{and} \quad \frac{x_1}{S-x_1} \geq \frac{x_2}{S-x_2} \geq \cdots \geq \frac{x_n}{S-x_n}$$

we can use *Chebishev's inequality*.

$$\text{We have } A = \sum_{i=1}^n x_i^4 \frac{x_i}{S-x_i}.$$

So

$$A = \sum_{i=1}^n x_i^4 \frac{x_i}{S-x_i} \geq n \sum_{i=1}^n x_i^4 \cdot \sum_{i=1}^n \frac{x_i}{S-x_i}. \quad (1)$$

By  $QM \geq AM$  we have

$$\sqrt{\frac{\sum_{i=1}^n x_i^4}{n}} \geq \frac{\sum_{i=1}^n x_i^2}{n}, \quad \text{i.e.} \quad \sum_{i=1}^n x_i^4 \geq \frac{n}{n^2} \sum_{i=1}^n x_i^2 = \frac{1}{n}. \quad (2)$$

The function  $f(x) = \frac{x}{S-x}$  is convex.

So by *Jensen's inequality* we have

$$f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \leq \frac{1}{n} \sum_{i=1}^n f(x_i),$$



i.e.

$$\frac{1}{n} \sum_{i=1}^n \frac{x_i}{S - x_i} \geq \frac{\frac{x_1 + x_2 + \dots + x_n}{n}}{x_1 + x_2 + \dots + x_n - \frac{x_1 + x_2 + \dots + x_n}{n}} = \frac{\frac{1}{n}}{1 - \frac{1}{n}} = \frac{1}{n-1},$$

from which it follows that

$$\sum_{i=1}^n \frac{x_i}{S - x_i} \geq \frac{n}{n-1}. \quad (3)$$

Finally using (2), (3) and (1) we obtain

$$A \geq n \cdot \frac{1}{n} \cdot \frac{n}{n-1} = \frac{n}{n-1}.$$

Equality occurs if and only if  $x_1 = x_2 = \dots = x_n = 1/\sqrt{n}$ . ■

**209** Let  $P, L, R$  denote the area, perimeter and circumradius of  $\triangle ABC$ , respectively. Determine the maximum value of the expression  $\frac{LP}{R^3}$ .

*Solution* We have

$$\frac{LP}{R^3} = \frac{(a+b+c)abc}{R^3 4R} = \frac{2R(\sin \alpha + \sin \beta + \sin \gamma) 8R^3 \sin \alpha \sin \beta \sin \gamma}{4R^4},$$

i.e.

$$\frac{LP}{R^3} = 4(\sin \alpha + \sin \beta + \sin \gamma) \sin \alpha \sin \beta \sin \gamma. \quad (1)$$

By  $AM \geq GM$  we have

$$\sin \alpha \sin \beta \sin \gamma \leq \left( \frac{\sin \alpha + \sin \beta + \sin \gamma}{3} \right)^3.$$

So by (1) we get

$$\frac{LP}{R^3} \leq \frac{4(\sin \alpha + \sin \beta + \sin \gamma)^4}{27}. \quad (2)$$

The function  $f(x) = -\sin x$  is convex on  $[0, \pi]$ , so by *Jensen's inequality* we have

$$\frac{\sin \alpha + \sin \beta + \sin \gamma}{3} \leq \sin \left( \frac{\alpha + \beta + \gamma}{3} \right) = \frac{\sqrt{3}}{2}.$$

Finally from (2) we obtain

$$\frac{LP}{R^3} \leq \frac{4}{27} \left( \frac{3\sqrt{3}}{2} \right)^4 = \frac{27}{4}.$$

Equality occurs iff  $a = b = c$ . ■

**210** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = abc$ . Prove the inequality

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2}.$$

*Solution 1* After taking  $a = \tan \alpha, b = \tan \beta, c = \tan \gamma$  where  $\alpha, \beta, \gamma \in (0, \pi/2)$ , the given inequality becomes

$$\frac{1}{\sqrt{1 + \frac{\sin^2 \alpha}{\cos^2 \alpha}}} + \frac{1}{\sqrt{1 + \frac{\sin^2 \beta}{\cos^2 \beta}}} + \frac{1}{\sqrt{1 + \frac{\sin^2 \gamma}{\cos^2 \gamma}}} \leq \frac{3}{2},$$

i.e.

$$\cos \alpha + \cos \beta + \cos \gamma \leq \frac{3}{2}.$$

Also

$$\begin{aligned} \tan(\alpha + \beta + \gamma) &= \frac{\tan \alpha + \tan \beta + \tan \gamma - \tan \alpha \tan \beta \tan \gamma}{1 - \tan \alpha \tan \beta - \tan \beta \tan \gamma - \tan \gamma \tan \alpha} \\ &= \frac{a + b + c - abc}{1 - ab - bc - ca} = 0, \end{aligned}$$

which means  $\alpha + \beta + \gamma = \pi$ .

The function  $f(x) = -\cos x$  is convex on  $[0, \pi/2]$ .

So by *Jensen's inequality* we have

$$\frac{\cos \alpha + \cos \beta + \cos \gamma}{3} \leq \cos \frac{\alpha + \beta + \gamma}{3} = \frac{1}{2},$$

i.e. we get

$$\cos \alpha + \cos \beta + \cos \gamma \leq \frac{3}{2},$$

as required. ■

*Solution 2* Let  $a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}$ .

The constraint  $a + b + c = abc$  becomes  $xy + yz + zx = 1$ , and the given inequality becomes equivalent to

$$\frac{x}{\sqrt{x^2 + 1}} + \frac{y}{\sqrt{y^2 + 1}} + \frac{z}{\sqrt{z^2 + 1}} \leq \frac{3}{2},$$

i.e.

$$\frac{x}{\sqrt{x^2 + xy + yz + zx}} + \frac{y}{\sqrt{y^2 + xy + yz + zx}} + \frac{z}{\sqrt{z^2 + xy + yz + zx}} \leq \frac{3}{2},$$

i.e.

$$\frac{x}{\sqrt{(x+y)(x+z)}} + \frac{y}{\sqrt{(y+z)(y+x)}} + \frac{z}{\sqrt{(z+x)(z+y)}} \leq \frac{3}{2}. \quad (1)$$

By  $AM \geq GM$  we have

$$\begin{aligned} \frac{x}{\sqrt{(x+y)(x+z)}} &= \frac{x\sqrt{(x+y)(x+z)}}{(x+y)(x+z)} \leq \frac{x((x+y) + (x+z))}{2(x+y)(x+z)} \\ &= \frac{1}{2} \left( \frac{x}{x+y} + \frac{x}{x+z} \right). \end{aligned}$$

Analogously we get

$$\begin{aligned} \frac{y}{\sqrt{(y+z)(y+x)}} &\leq \frac{1}{2} \left( \frac{y}{y+z} + \frac{y}{y+x} \right) \quad \text{and} \\ \frac{z}{\sqrt{(z+x)(z+y)}} &\leq \frac{1}{2} \left( \frac{z}{z+x} + \frac{z}{z+y} \right). \end{aligned}$$

Adding these three inequalities we get inequality (1). ■

**211** Let  $a, b, c \in \mathbb{R}$  such that  $abc + a + c = b$ . Prove the inequality

$$\frac{2}{a^2+1} - \frac{2}{b^2+1} + \frac{3}{c^2+1} \leq \frac{10}{3}.$$

*Solution* The given condition is equivalent to  $b = \frac{a+c}{1-ac}$ .

This suggests the substitutions:

$$a = \tan \alpha, \quad b = \tan \beta, \quad c = \tan \gamma,$$

where  $\tan \beta = \tan(\alpha + \gamma)$  and  $\alpha, \beta, \gamma \in (-\pi/2, \pi/2)$ , so we have

$$\begin{aligned} A &= \frac{2}{a^2+1} - \frac{2}{b^2+1} + \frac{3}{c^2+1} = \frac{2}{\tan^2 \alpha + 1} - \frac{2}{\tan^2(\alpha + \gamma) + 1} + \frac{3}{\tan^2 \gamma + 1} \\ &= 2 \cos^2 \alpha - 2 \cos^2(\alpha + \gamma) + 3 \cos^2 \gamma \\ &= (2 \cos^2 \alpha - 1) - (2 \cos^2(\alpha + \gamma) - 1) + 3 \cos^2 \gamma \\ &= \cos 2\alpha - \cos(2\alpha + 2\gamma) + 3 \cos^2 \gamma \\ &= 2 \sin(2\alpha + \gamma) \sin \gamma + 3 \cos^2 \gamma. \end{aligned}$$

Let  $x = |\sin \gamma|$ . Then we have

$$A \leq 2x + 3(1 - x^2) = -3x^2 + 2x + 3 = -3 \left( x - \frac{1}{3} \right)^2 + \frac{10}{3} \leq \frac{10}{3}.$$

Equality holds if and only if  $\sin(2\alpha + \gamma) = 1$  and  $\sin \gamma = \frac{1}{3}$ , from which we deduce  $(a, b, c) = (\sqrt{2}/2, \sqrt{2}, \sqrt{2}/4)$ . ■

**212** Let  $x, y, z > 1$  be real numbers such that  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$ . Prove the inequality

$$\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1} \leq \sqrt{x+y+z}.$$

*Solution 1* Let  $x = a + 1, y = b + 1, z = c + 1$ , and clearly  $a, b$  and  $c$  are positive real numbers.

The initial condition  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$  becomes  $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$ , i.e.

$$ab + bc + ca + 2abc = 1. \quad (1)$$

We need to show that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \leq \sqrt{a+b+c+3}. \quad (2)$$

After squaring inequality (2) we get

$$a + b + c + 2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ca} \leq a + b + c + 3$$

or

$$2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ca} \leq 3,$$

i.e.

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \leq \frac{3}{2}. \quad (3)$$

Identity (1) is equivalent to

$$(\sqrt{ab})^2 + (\sqrt{bc})^2 + (\sqrt{ca})^2 + 2(\sqrt{ab} \cdot \sqrt{bc} \cdot \sqrt{ca}) = 1,$$

so due to Case 7 (Chap. 8) we may take

$$\sqrt{ab} = \sin \frac{\alpha}{2}, \quad \sqrt{bc} = \sin \frac{\beta}{2}, \quad \sqrt{ca} = \sin \frac{\gamma}{2},$$

where  $\alpha, \beta, \gamma \in (0, \pi)$  and  $\alpha + \beta + \gamma = \pi$ .

Now inequality (3) is equivalent to

$$\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \leq \frac{3}{2},$$

where  $\alpha, \beta, \gamma \in (0, \pi), \alpha + \beta + \gamma = \pi$ , which is true by  $N_3$  (Chap. 8). ■

*Solution 2* Applying the *Cauchy-Schwarz inequality* we have

$$\left( \frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z} \right) (x+y+z) \geq (\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1})^2.$$

Also

$$\frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z} = 3 - \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) = 1.$$

So

$$x + y + z \geq (\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1})^2,$$

i.e.

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Equality occurs iff  $\frac{x-1}{x^2} = \frac{y-1}{y^2} = \frac{z-1}{z^2}$  and  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$ , i.e.  $x = y = z = 3/2$ . ■

**213** Let  $a, b, c$  be positive real numbers such that  $a+b+c=1$ . Prove the inequality

$$\sqrt{\frac{1}{a}-1}\sqrt{\frac{1}{b}-1} + \sqrt{\frac{1}{b}-1}\sqrt{\frac{1}{c}-1} + \sqrt{\frac{1}{c}-1}\sqrt{\frac{1}{a}-1} \geq 6.$$

*Solution* Let  $a = xy, b = yz, c = zx$ . Then  $xy + yz + zx = 1$  and due to *Case 3* (Chap. 8) we may take

$$x = \tan \frac{\alpha}{2}, \quad y = \tan \frac{\beta}{2}, \quad z = \tan \frac{\gamma}{2},$$

where  $\alpha, \beta, \gamma \in (0, \pi)$  and  $\alpha + \beta + \gamma = \pi$ .

We have

$$\begin{aligned} \sqrt{\frac{1}{a}-1}\sqrt{\frac{1}{b}-1} &= \sqrt{\frac{(1-a)(1-b)}{ab}} = \sqrt{\frac{(1-xy)(1-yz)}{xy^2z}} \\ &= \sqrt{\frac{(yz+zx)(zx+xy)}{xy^2z}} = \sqrt{\frac{(y+x)(z+y)}{y^2}} = \frac{\sqrt{1+y^2}}{y} \\ &= \frac{\sqrt{1+\tan^2 \frac{\beta}{2}}}{\tan \frac{\beta}{2}} = \frac{1}{\sin \frac{\beta}{2}}. \end{aligned}$$

Similarly we obtain

$$\sqrt{\frac{1}{b}-1}\sqrt{\frac{1}{c}-1} = \frac{1}{\sin \frac{\gamma}{2}} \quad \text{and} \quad \sqrt{\frac{1}{c}-1}\sqrt{\frac{1}{a}-1} = \frac{1}{\sin \frac{\alpha}{2}}.$$

Now the given inequality becomes

$$\frac{1}{\sin \frac{\alpha}{2}} + \frac{1}{\sin \frac{\beta}{2}} + \frac{1}{\sin \frac{\gamma}{2}} \geq 6.$$

By  $AM \geq HM$  we have

$$\frac{1}{\sin \frac{\alpha}{2}} + \frac{1}{\sin \frac{\beta}{2}} + \frac{1}{\sin \frac{\gamma}{2}} \geq \frac{9}{\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2}}.$$

So we need to prove that  $\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \leq \frac{3}{2}$  which is true according to  $N_3$  (Chap. 8).

Equality occurs if and only if  $\alpha = \beta = \gamma = \pi/3$ , i.e.  $a = b = c = \frac{1}{3}$ . ■

**214** Let  $a, b, c$  be positive real numbers such that  $a + b + c + 1 = 4abc$ . Prove the inequalities

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3 \geq \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}.$$

*Solution* We have

$$a + b + c + 1 = 4abc$$

$$\Leftrightarrow \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} + \frac{1}{abc} = 4$$

$$\Leftrightarrow \frac{1}{(2\sqrt{ab})^2} + \frac{1}{(2\sqrt{bc})^2} + \frac{1}{(2\sqrt{ca})^2} + \frac{2}{(2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca})} = 1.$$

Due to Case 7 (Chap. 8) we can make the substitutions

$$\frac{1}{2\sqrt{bc}} = \sin \frac{\alpha}{2}, \quad \frac{1}{2\sqrt{ca}} = \sin \frac{\beta}{2}, \quad \frac{1}{2\sqrt{ab}} = \sin \frac{\gamma}{2}, \quad (1)$$

where  $\alpha, \beta, \gamma \in (0, \pi)$  and  $\alpha + \beta + \gamma = \pi$ .

From (1) we easily obtain

$$\frac{1}{a} = \frac{2 \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\sin \frac{\alpha}{2}}, \quad \frac{1}{b} = \frac{2 \sin \frac{\gamma}{2} \sin \frac{\alpha}{2}}{\sin \frac{\beta}{2}} \quad \text{and} \quad \frac{1}{c} = \frac{2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\sin \frac{\gamma}{2}}. \quad (2)$$

Now the given inequality becomes

$$2 \sin \frac{\alpha}{2} + 2 \sin \frac{\beta}{2} + 2 \sin \frac{\gamma}{2} \leq 3,$$

i.e.

$$\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \leq \frac{3}{2},$$

where  $\alpha, \beta, \gamma \in (0, \pi)$  and  $\alpha + \beta + \gamma = \pi$ , which clearly holds due to  $N_3$ .

We need to show the left inequality which, due to (2) is equivalent to

$$\frac{2 \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\sin \frac{\alpha}{2}} + \frac{2 \sin \frac{\gamma}{2} \sin \frac{\alpha}{2}}{\sin \frac{\beta}{2}} + \frac{2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\sin \frac{\gamma}{2}} \geq 3. \quad (3)$$

Let  $a, b, c$  be the lengths of the sides of the triangle with angles  $\alpha, \beta$  and  $\gamma$ , let  $s$  be its semi-perimeter, and let  $x = s - a, y = s - b, z = s - c$ .

Then due to Case 9 (Chap. 8) inequality (3) is equivalent to

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2},$$

i.e. we obtain the famous *Nesbitt's inequality*, which clearly holds. And we are done. ■

**215** Let  $a, b, c$  be non-negative real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$\frac{a}{1+a^2} + \frac{b}{1+b^2} + \frac{c}{1+c^2} \leq \frac{3\sqrt{3}}{4}.$$

*Solution* Since  $ab + bc + ca = 1$  (Case 3, Chap. 8) we take:

$$a = \tan \frac{\alpha}{2}, \quad b = \tan \frac{\beta}{2}, \quad c = \tan \frac{\gamma}{2},$$

where  $\alpha, \beta, \gamma \in (0, \pi)$  and  $\alpha + \beta + \gamma = \pi$ .

So we have

$$\frac{a}{1+a^2} + \frac{b}{1+b^2} + \frac{c}{1+c^2} = \frac{1}{2}(\sin \alpha + \sin \beta + \sin \gamma),$$

and the given inequality becomes

$$\frac{1}{2}(\sin \alpha + \sin \beta + \sin \gamma) \leq \frac{3\sqrt{3}}{4},$$

i.e.

$$\sin \alpha + \sin \beta + \sin \gamma \leq \frac{3\sqrt{3}}{2},$$

which is true according to  $N_1$  (Chap. 8).

Equality occurs if and only if  $a = b = c = 1/\sqrt{3}$ . ■

*Remark* This is the same problem as Problem 92.

**216** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$\sqrt{\frac{ab}{c+ab}} + \sqrt{\frac{bc}{a+bc}} + \sqrt{\frac{ca}{b+ca}} \leq \frac{3}{2}.$$

*Solution* We have

$$\begin{aligned} (c+a)(c+b) &= c^2 + ca + cb + ab = c^2 + c(a+b) + ab = c^2 + c(1-c) + ab \\ &= c + ab. \end{aligned}$$

Analogously we get

$$(a+b)(a+c) = a+bc \quad \text{and} \quad (b+c)(b+a) = b+ca.$$

Now the given inequality becomes

$$\sqrt{\frac{ab}{(c+a)(c+b)}} + \sqrt{\frac{bc}{(a+b)(a+c)}} + \sqrt{\frac{ca}{(b+c)(b+a)}} \leq \frac{3}{2}.$$

According to Case 9 (Chap. 8) it suffices to show that

$$\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \leq \frac{3}{2},$$

where  $\alpha, \beta, \gamma \in (0, \pi)$  and  $\alpha + \beta + \gamma = \pi$ , which is true due to  $N_3$  (Chap. 8). ■

**217** Let  $a, b, c > 0$  be real numbers such that  $(a+b)(b+c)(c+a) = 1$ . Prove the inequality

$$ab + bc + ca \leq \frac{3}{4}.$$

*Solution* We homogenize as follows

$$(ab + bc + ca)^3 \leq \frac{27}{64}(a+b)^2(b+c)^2(c+a)^2. \quad (1)$$

Since inequality (1) is homogenous, we may assume that  $ab + bc + ca = 1$ .

Now, by Case 3 (Chap. 8) we can use the substitutions

$$a = \tan \frac{\alpha}{2}, \quad b = \tan \frac{\beta}{2}, \quad c = \tan \frac{\gamma}{2},$$

where  $\alpha, \beta, \gamma \in (0, \pi)$  and  $\alpha + \beta + \gamma = \pi$ .

Then

$$a + b = \tan \frac{\alpha}{2} + \tan \frac{\beta}{2} = \frac{\sin \frac{\alpha}{2} \cos \frac{\beta}{2} + \cos \frac{\alpha}{2} \sin \frac{\beta}{2}}{\cos \frac{\alpha}{2} \cos \frac{\beta}{2}} = \frac{\sin \frac{\alpha+\beta}{2}}{\cos \frac{\alpha}{2} \cos \frac{\beta}{2}} = \frac{\cos \frac{\gamma}{2}}{\cos \frac{\alpha}{2} \cos \frac{\beta}{2}}.$$

Similarly

$$b + c = \frac{\cos \frac{\alpha}{2}}{\cos \frac{\beta}{2} \cos \frac{\gamma}{2}} \quad \text{and} \quad c + a = \frac{\cos \frac{\beta}{2}}{\cos \frac{\gamma}{2} \cos \frac{\alpha}{2}},$$

i.e. we obtain

$$(a+b)(b+c)(c+a) = \frac{1}{\cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}}.$$



Therefore inequality (1) becomes

$$\frac{1}{\cos^2 \frac{\alpha}{2} \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2}} \geq \frac{64}{27}, \quad \text{i.e.} \quad \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \leq \frac{3\sqrt{3}}{8},$$

which is true due to  $N_8$  (Chap. 8). So we are done.  $\blacksquare$

**218** Let  $a, b, c \geq 0$  be real numbers such that  $a^2 + b^2 + c^2 + abc = 4$ . Prove the inequality

$$0 \leq ab + bc + ca - abc \leq 2.$$

*Solution* Observe that if  $a, b, c > 1$  then  $a^2 + b^2 + c^2 + abc > 4$ .

Therefore at least one number from  $a, b$  and  $c$  must be less than or equal to 1.

Without loss of generality assume that  $a \leq 1$ .

Then we have

$$ab + bc + ca - abc \geq bc - abc = bc(1 - a) \geq 0.$$

So we have proved the left inequality.

Let  $a = 2x, b = 2y, c = 2z$ .

Then the condition  $a^2 + b^2 + c^2 + abc = 4$  becomes

$$x^2 + y^2 + z^2 + 2xyz = 1 \tag{1}$$

and the given inequality becomes

$$2xy + 2yz + 2zx - 4xyz \leq 1. \tag{2}$$

By (1) and Case 8 (Chap. 8) we can take

$$x = \cos \alpha, \quad y = \cos \beta, \quad z = \cos \gamma,$$

where  $\alpha, \beta, \gamma \in [0, \pi/2]$  and  $\alpha + \beta + \gamma = \pi$ .

Therefore inequality (2) becomes

$$2 \cos \alpha \cos \beta + 2 \cos \beta \cos \gamma + 2 \cos \gamma \cos \alpha - 4 \cos \alpha \cos \beta \cos \gamma \leq 1,$$

i.e.

$$\cos \alpha \cos \beta + \cos \beta \cos \gamma + \cos \gamma \cos \alpha - 2 \cos \alpha \cos \beta \cos \gamma \leq \frac{1}{2}. \tag{3}$$

Clearly at least one of the angles  $\alpha, \beta$  and  $\gamma$  is less than or equal to  $\pi/3$ .

Without loss of generality, we may assume  $\alpha \leq \pi/3$  and it follows that  $\cos \alpha \geq \frac{1}{2}$ .

We have

$$\begin{aligned} & \cos \alpha \cos \beta + \cos \beta \cos \gamma + \cos \gamma \cos \alpha - 2 \cos \alpha \cos \beta \cos \gamma \\ &= \cos \alpha (\cos \beta + \cos \gamma) + \cos \beta \cos \gamma (1 - 2 \cos \alpha). \end{aligned} \tag{4}$$

By  $N_5$  (Chap. 8) we have that

$$\cos \alpha + \cos \beta + \cos \gamma \leq \frac{3}{2}, \quad \text{i.e.} \quad \cos \beta + \cos \gamma \leq \frac{3}{2} - \cos \alpha. \quad (5)$$

Also

$$2 \cos \beta \cos \gamma = \cos(\beta - \gamma) + \cos(\beta + \gamma) \leq 1 + \cos(\beta + \gamma) = 1 - \cos \alpha. \quad (6)$$

By (4), (5) and (6) we obtain

$$\begin{aligned} & \cos \alpha \cos \beta + \cos \beta \cos \gamma + \cos \gamma \cos \alpha - 2 \cos \alpha \cos \beta \cos \gamma \\ &= \cos \alpha (\cos \beta + \cos \gamma) + \cos \beta \cos \gamma (1 - 2 \cos \alpha) \\ &\leq \cos \alpha \left( \frac{3}{2} - \cos \alpha \right) + \frac{1 - \cos \alpha}{2} (1 - 2 \cos \alpha) = 2, \end{aligned}$$

as required. ■

**219** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$a^2 + b^2 + c^2 + 2abc + 3 \geq (1+a)(1+b)(1+c).$$

*Solution* The given inequality is equivalent to

$$a^2 + b^2 + c^2 + abc + 2 \geq a + b + c + ab + bc + ac.$$

Recall the *Turkevicius inequality*:

For any positive real numbers  $x, y, z, t$  we have

$$x^4 + y^4 + z^4 + t^4 + 2xyz t \geq x^2 y^2 + y^2 z^2 + z^2 t^2 + t^2 x^2 + x^2 z^2 + y^2 t^2.$$

If we set  $a = x^2, b = y^2, c = z^2, t = 1$  we deduce

$$a^2 + b^2 + c^2 + 2\sqrt{abc} + 1 \geq a + b + c + ab + bc + ac. \quad (1)$$

Since  $AM \geq GM$  we get

$$2\sqrt{abc} \leq abc + 1. \quad (2)$$

From (1) and (2) we obtain

$$a^2 + b^2 + c^2 + abc + 2 \geq a^2 + b^2 + c^2 + 2\sqrt{abc} + 1 \geq a + b + c + ab + bc + ac. \quad \blacksquare$$

**220** Let  $a, b, c$  be real numbers. Prove the inequality

$$\sqrt{a^2 + (1-b)^2} + \sqrt{b^2 + (1-c)^2} + \sqrt{c^2 + (1-a)^2} \geq \frac{3\sqrt{2}}{2}.$$

*Solution* By Minkowski's inequality we have

$$\begin{aligned} & \sqrt{a^2 + (1-b)^2} + \sqrt{b^2 + (1-c)^2} + \sqrt{c^2 + (1-a)^2} \\ & \geq \sqrt{(a+b+c)^2 + (3-a-b-c)^2} = \sqrt{2\left(a+b+c-\frac{3}{2}\right)^2 + \frac{9}{2}} \geq \frac{3\sqrt{2}}{2}. \quad \blacksquare \end{aligned}$$

**221** Let  $a_1, a_2, \dots, a_n \in \mathbb{R}^+$  such that  $\sum_{i=1}^n a_i^3 = 3$  and  $\sum_{i=1}^n a_i^5 = 5$ . Prove the inequality

$$\sum_{i=1}^n a_i > \frac{3}{2}.$$

*Solution* We'll use Hölder's inequality:

If  $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n \in \mathbb{R}^+$  and  $p, q \in (0, 1)$ ,  $1/p + 1/q = 1$  then we have

$$\sum_{i=1}^n a_i b_i \leq \left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n b_i^q \right)^{\frac{1}{q}}.$$

We have

$$\sum_{i=1}^n a_i^3 = \sum_{i=1}^n a_i a_i^2 \leq \left( \sum_{i=1}^n a_i^{5/3} \right)^{3/5} \left( \sum_{i=1}^n (a_i^2)^{5/2} \right)^{2/5},$$

i.e.

$$3 \leq \left( \sum_{i=1}^n a_i^{5/3} \right)^{3/5} \cdot 5^{2/5} \quad \text{i.e.} \quad \frac{3}{5^{2/5}} \leq \left( \sum_{i=1}^n a_i^{5/3} \right)^{3/5}. \quad (1)$$

We'll show that

$$\sum_{i=1}^n a_i^{5/3} \leq \left( \sum_{i=1}^n a_i \right)^{5/3}.$$

Let  $S = \sum_{i=1}^n a_i$ .

Since  $0 < \frac{a_i}{S} \leq 1$  and  $\frac{5}{3} > 1$  we have that  $(\frac{a_i}{S})^{5/3} \leq \frac{a_i}{S} = 1$  from which we deduce

$$\sum_{i=1}^n \left( \frac{a_i}{S} \right)^{5/3} \leq \sum_{i=1}^n \frac{a_i}{S} = 1.$$

So

$$\sum_{i=1}^n a_i^{5/3} \leq S^{5/3} = \left( \sum_{i=1}^n a_i \right)^{5/3},$$

since  $2^5 > 5^2$ ,  $2 > 5^{2/5}$  and by (1) we obtain the required inequality. ■

**222** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 3$ . Prove the inequality

$$(1 + a^2)(1 + b^2)(1 + c^2) \geq 8.$$

*Solution* By Hölder's inequality we have

$$(a^2b^2 + a^2 + b^2 + 1)(b^2 + c^2 + b^2c^2 + 1)(a^2 + a^2c^2 + c^2 + 1) \geq (1 + ab + bc + ca)^3$$

i.e.

$$(1 + a^2)^2(1 + b^2)^2(1 + c^2)^2 \geq 2^6,$$

as required. ■

**223** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 1.$$

*Solution* We'll show stronger inequality, i.e.

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq (ab + bc + ca)^3.$$

By Hölder's inequality we have

$$\begin{aligned} & (a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \\ &= (ab + a^2 + b^2)(b^2 + c^2 + bc)(a^2 + ca + c^2) \geq (ab + bc + ca)^3, \end{aligned}$$

as required. ■

**224** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{a}{\sqrt{7 + b^2 + c^2}} + \frac{b}{\sqrt{7 + c^2 + a^2}} + \frac{c}{\sqrt{7 + a^2 + b^2}} \geq 1.$$

*Solution* Denote

$$A = \frac{a}{\sqrt{7 + b^2 + c^2}} + \frac{b}{\sqrt{7 + c^2 + a^2}} + \frac{c}{\sqrt{7 + a^2 + b^2}}$$

and

$$B = a(7 + b^2 + c^2) + b(7 + c^2 + a^2) + c(7 + a^2 + b^2).$$

By Hölder's inequality we have

$$A^2B \geq (a + b + c)^3. \quad (1)$$

Furthermore

$$\begin{aligned} B &= 7(a+b+c) + (a+b+c)(ab+bc+ca) - 3 \\ &\leq 7(a+b+c) + \frac{(a+b+c)^3}{3} - 3 \leq (a+b+c)^3 \end{aligned} \quad (2)$$

and by (1) and (2) we obtain

$$A^2 \geq \frac{(a+b+c)^3}{B} \geq 1, \quad \text{i.e.} \quad A \geq 1,$$

as required. ■

**225** Let  $a_1, a_2, \dots, a_n$  be positive real numbers such that  $a_1 + a_2 + \dots + a_n = 1$ . Prove the inequality

$$\frac{a_1}{\sqrt{1-a_1}} + \frac{a_2}{\sqrt{1-a_2}} + \dots + \frac{a_n}{\sqrt{1-a_n}} \geq \sqrt{\frac{n}{n-1}}.$$

*Solution* Let us denote

$$\begin{aligned} A &= \frac{a_1}{\sqrt{1-a_1}} + \frac{a_2}{\sqrt{1-a_2}} + \dots + \frac{a_n}{\sqrt{1-a_n}}, \\ B &= a_1(1-a_1) + a_2(1-a_2) + \dots + a_n(1-a_n). \end{aligned}$$

By Hölder's inequality we have

$$A^2 B \geq (a_1 + a_2 + \dots + a_n)^3 = 1. \quad (1)$$

Applying  $QM \geq AM$  we deduce

$$B = 1 - (a_1^2 + a_2^2 + \dots + a_n^2) \leq 1 - \frac{(a_1 + a_2 + \dots + a_n)^2}{n} = \frac{n-1}{n}. \quad (2)$$

By (1) and (2) we obtain

$$\frac{n-1}{n} \cdot A^2 \geq A^2 B \geq 1, \quad \text{i.e.} \quad A \geq \sqrt{\frac{n}{n-1}}.$$

Equality holds iff  $a_i = \frac{1}{n}$ , for every  $i = 1, 2, \dots, n$ . ■

**226** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a}{\sqrt{2b^2 + 2c^2 - a^2}} + \frac{b}{\sqrt{2c^2 + 2a^2 - b^2}} + \frac{c}{\sqrt{2a^2 + 2b^2 - c^2}} \geq \sqrt{3}.$$

*Solution* Denote

$$A = \frac{a}{\sqrt{2b^2 + 2c^2 - a^2}} + \frac{b}{\sqrt{2c^2 + 2a^2 - b^2}} + \frac{c}{\sqrt{2a^2 + 2b^2 - c^2}}$$

and

$$\begin{aligned} B &= a(2b^2 + 2c^2 - a^2) + b(2c^2 + 2a^2 - b^2) + c(2a^2 + 2b^2 - c^2) \\ &= 2ab(a + b) + 2bc(b + c) + 2ca(c + a) - a^3 - b^3 - c^3. \end{aligned}$$

By Hölder's inequality we have

$$A^2 B \geq (a + b + c)^3. \quad (1)$$

We'll show that

$$(a + b + c)^3 \geq 3B, \quad (2)$$

and then by (1) we'll obtain the required inequality.

Inequality (2) is equivalent to

$$4(a^3 + b^3 + c^3) + 6abc \geq 4(ab(a + b) + bc(b + c) + ca(c + a)). \quad (3)$$

The following inequalities are true:

$$\begin{aligned} 3((a^3 + b^3 + c^3) + 3abc) &\geq 4(ab(a + b) + bc(b + c) + ca(c + a)) \quad (\text{Schur}), \\ a^3 + b^3 + c^3 &\geq 3abc \quad (AM \geq GM). \end{aligned}$$

Adding the last two inequalities we obtain inequality (3).

Equality occurs iff  $a = b = c$ . ■

**227** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca \geq 3$ . Prove the inequality

$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \geq \frac{3}{\sqrt{2}}.$$

*Solution* By Hölder's inequality we have

$$\left( \frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \right)^{2/3} (a(a+b) + b(b+c) + c(c+a))^{1/3} \geq a + b + c,$$

i.e.

$$\left( \frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \right)^2 \geq \frac{(a + b + c)^3}{a^2 + b^2 + c^2 + ab + bc + ca}.$$

It is enough to show that

$$\frac{(a + b + c)^3}{a^2 + b^2 + c^2 + ab + bc + ca} \geq \frac{9}{2},$$

i.e.

$$2(a+b+c)^3 \geq 9(a^2+b^2+c^2+ab+bc+ca). \quad (1)$$

Let  $p = a + b + c$  and  $q = ab + bc + ca$ .

Using the initial condition we have  $q \geq 3$ , and then inequality (1) is equivalent to

$$2p^3 \geq 9(p^2 - 2q + q) \quad \text{or} \quad 2p^3 + 9q \geq 9p^2.$$

Applying  $AM \geq GM$  we obtain

$$2p^3 + 9q \geq 2p^3 + 27 = p^3 + p^3 + 27 \geq 3\sqrt[3]{27p^6} = 9p^2,$$

as required. ■

**228** Let  $a, b, c \geq 1$  be real numbers such that  $a + b + c = 2abc$ . Prove the inequality

$$\sqrt[3]{(a+b+c)^2} \geq \sqrt[3]{ab-1} + \sqrt[3]{bc-1} + \sqrt[3]{ca-1}.$$

*Solution* By the initial condition we have

$$a + b + c = 2abc \quad \text{or} \quad \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = 2 \quad \Leftrightarrow \quad \frac{ab-1}{ab} + \frac{bc-1}{bc} + \frac{ca-1}{ca} = 1.$$

By Hölder's inequality for triples

$$(a, b, c), (b, c, a), \left( \frac{ab-1}{ab}, \frac{bc-1}{bc}, \frac{ca-1}{ca} \right)$$

we obtain

$$\begin{aligned} (a+b+c)^{1/3} (b+c+a)^{1/3} \left( \frac{ab-1}{ab} + \frac{bc-1}{bc} + \frac{ca-1}{ca} \right)^{1/3} \\ \geq (ab-1)^{1/3} + (bc-1)^{1/3} + (ca-1)^{1/3}. \end{aligned}$$

Since

$$\frac{ab-1}{ab} + \frac{bc-1}{bc} + \frac{ca-1}{ca} = 1$$

we get

$$\sqrt[3]{(a+b+c)^2} \geq \sqrt[3]{ab-1} + \sqrt[3]{bc-1} + \sqrt[3]{ca-1}. \quad \blacksquare$$

**229** Let  $t_a, t_b, t_c$  be the lengths of the medians, and  $a, b, c$  be the lengths of the sides of a given triangle. Prove the inequality

$$t_a t_b + t_b t_c + t_c t_a < \frac{5}{4}(ab + bc + ca).$$

*Solution* We can easily show the inequalities

$$t_a < \frac{b+c}{2}, \quad t_b < \frac{a+c}{2}, \quad t_c < \frac{b+a}{2}.$$

After adding these we get

$$t_a + t_b + t_c < a + b + c. \quad (1)$$

By squaring (1) we deduce

$$t_a^2 + t_b^2 + t_c^2 + 2(t_at_b + t_bt_c + t_ct_a) < a^2 + b^2 + c^2 + 2(ab + bc + ca). \quad (2)$$

On the other hand, we have

$$t_a^2 = \frac{2(b^2 + c^2) - a^2}{4}, \quad t_b^2 = \frac{2(a^2 + c^2) - b^2}{4}, \quad t_c^2 = \frac{2(b^2 + a^2) - c^2}{4}$$

so

$$t_a^2 + t_b^2 + t_c^2 = \frac{3}{4}(a^2 + b^2 + c^2).$$

Now using the previous result and (2) we get

$$t_at_b + t_bt_c + t_ct_a < \frac{1}{8}(a^2 + b^2 + c^2) + (ab + bc + ca). \quad (3)$$

Also we have  $a^2 + b^2 + c^2 < 2(ab + bc + ca)$ , since

$$a^2 + b^2 + c^2 - 2(ab + bc + ca) = a(a - b - c) + b(b - a - c) + c(c - a - b) < 0.$$

Finally by (3) and the previous inequality we obtain

$$t_at_b + t_bt_c + t_ct_a < \frac{5}{4}(ab + bc + ca). \quad \blacksquare$$

**230** Let  $a, b, c$  and  $t_a, t_b, t_c$  be the lengths of the sides and lengths of the medians of an arbitrary triangle, respectively. Prove the inequality

$$at_a + bt_b + ct_c \leq \frac{\sqrt{3}}{2}(a^2 + b^2 + c^2).$$

*Solution* By the *Cauchy–Schwarz inequality* we have

$$(a^2 + b^2 + c^2)(t_a^2 + t_b^2 + t_c^2) \geq (at_a + bt_b + ct_c)^2. \quad (1)$$

Also

$$t_a^2 + t_b^2 + t_c^2 = \frac{3}{4}(a^2 + b^2 + c^2). \quad (2)$$



From (1) and (2) we get

$$(at_a + bt_b + ct_c)^2 \leq \frac{3}{4}(a^2 + b^2 + c^2)^2 \quad \text{i.e.} \quad at_a + bt_b + ct_c \leq \frac{\sqrt{3}}{2}(a^2 + b^2 + c^2),$$

as required. ■

**231** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove the inequality

$$\sqrt{a+b-c} + \sqrt{c+a-b} + \sqrt{b+c-a} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

*Solution* We'll use *Ravi's substitutions*, i.e. let  $a = x + y, b = y + z, c = z + x$ , where  $x, y, z \in \mathbb{R}^+$ .

Now the given inequality is equivalent to

$$\sqrt{2x} + \sqrt{2y} + \sqrt{2z} \leq \sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x}.$$

By  $QM \geq AM$  we have  $\sqrt{\frac{x+y}{2}} \geq \frac{\sqrt{x} + \sqrt{y}}{2}$ , from which we deduce that

$$\sqrt{x+y} \geq \frac{\sqrt{x} + \sqrt{y}}{\sqrt{2}}.$$

Analogously we get

$$\sqrt{y+z} \geq \frac{\sqrt{y} + \sqrt{z}}{\sqrt{2}} \quad \text{and} \quad \sqrt{z+x} \geq \frac{\sqrt{z} + \sqrt{x}}{\sqrt{2}}.$$

After adding these three inequalities we obtain

$$\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x} \geq 2\frac{\sqrt{x}}{\sqrt{2}} + 2\frac{\sqrt{y}}{\sqrt{2}} + 2\frac{\sqrt{z}}{\sqrt{2}},$$

i.e.

$$\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x} \geq \sqrt{2x} + \sqrt{2y} + \sqrt{2z},$$

as required. ■

**232** Let  $P$  be the area of the triangle with side lengths  $a, b$  and  $c$ , and  $T$  be the area of the triangle with side lengths  $a+b, b+c$  and  $c+a$ . Prove that  $T \geq 4P$ .

*Solution* We have

$$P^2 = s(s-a)(s-b)(s-c), \quad \text{where } s = \frac{a+b+c}{2},$$

i.e.

$$16P^2 = (a+b+c)(a+b-c)(a+c-b)(b+c-a).$$

Let  $s_1$  be the semi-perimeter of the triangle with side lengths  $a+b, a+c, b+c$ .

Then

$$s_1 = \frac{a+b+a+c+b+c}{2} = a+b+c = 2s.$$

So we get

$$\begin{aligned} T^2 &= s_1(s_1 - (a+b))(s_1 - (a+c))(s_1 - (b+c)) \\ &= 2s(2s - (a+b))(2s - (a+c))(2s - (b+c)) = abc(a+b+c). \end{aligned}$$

It suffices to show that  $T^2 \geq 16P^2$  i.e.

$$abc(a+b+c) \geq (a+b+c)(a+b-c)(a+c-b)(b+c-a).$$

We have

$$a^2 \geq a^2 - (b-c)^2 = (a-b+c)(a+b-c) = (a+c-b)(a+b-c).$$

Analogously

$$b^2 \geq (a+b-c)(b+c-a) \quad \text{and} \quad c^2 \geq (b+c-a)(a+c-b).$$

If we multiply the last three inequalities (Can we do this?) we obtain

$$a^2b^2c^2 \geq (a+b-c)^2(a+c-b)^2(b+c-a)^2,$$

i.e.

$$abc \geq (a+b-c)(a+c-b)(b+c-a),$$

as required.

Equality occurs iff  $a = b = c$ . ■

**233** Let  $a, b, c$  be the lengths of the sides of a triangle, such that  $a + b + c = 3$ . Prove the inequality

$$a^2 + b^2 + c^2 + \frac{4abc}{3} \geq \frac{13}{3}.$$

*Solution* Let  $a = x + y, b = y + z$  and  $c = z + x$ .

So we have  $x + y + z = \frac{3}{2}$  and since  $AM \geq GM$  we get  $xyz \leq \left(\frac{x+y+z}{3}\right)^3 = \frac{1}{8}$ .

Now we obtain

$$\begin{aligned}
 & a^2 + b^2 + c^2 + \frac{4abc}{3} \\
 &= \frac{(a^2 + b^2 + c^2)(a + b + c) + 4abc}{3} \\
 &= \frac{2((x + y)^2 + (y + z)^2 + (z + x)^2)(x + y + z) + 4(x + y)(y + z)(z + x)}{3} \\
 &= \frac{4}{3}((x + y + z)^3 - xyz) \geq \frac{4}{3} \left( \left( \frac{3}{2} \right)^3 - \frac{1}{8} \right) = \frac{13}{3}.
 \end{aligned}$$

Equality occurs iff  $x = y = z$ , i.e.  $a = b = c = 1$ . ■

**234** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove that

$$\sqrt[3]{\frac{a^3 + b^3 + c^3 + 3abc}{2}} \geq \max\{a, b, c\}.$$

*Solution* Without loss of generality we may assume that  $a \geq b \geq c$ .

We need to show that

$$\sqrt[3]{\frac{a^3 + b^3 + c^3 + 3abc}{2}} \geq a,$$

i.e.

$$-a^3 + b^3 + c^3 + 3abc \geq 0.$$

Since

$$\begin{aligned}
 -a^3 + b^3 + c^3 + 3abc &= (-a)^3 + b^3 + c^3 - 3(-a)bc \\
 &= \frac{1}{2}(-a + b + c)((a + b)^2 + (a + c)^2 + (b - c)^2),
 \end{aligned}$$

and since  $b + c > a$  we obtain

$$-a^3 + b^3 + c^3 + 3abc \geq 0,$$

as required. ■

**235** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove the inequality

$$abc < a^2(s - a) + b^2(s - a) + c^2(s - a) \leq \frac{3}{2}abc.$$

*Solution* Since

$$2(a^2(s-a) + b^2(s-a) + c^2(s-a)) = a^2b + a^2c + b^2a + b^2c + c^2a + c^2b \\ - (a^3 + b^3 + c^3)$$

and

$$(b+c-a)(c+a-b)(a+b-c) \\ = a^2b + a^2c + b^2a + b^2c + c^2a + c^2b - (a^3 + b^3 + c^3) - 2abc$$

we have

$$2(a^2(s-a) + b^2(s-a) + c^2(s-a)) = (b+c-a)(c+a-b)(a+b-c) + 2abc.$$

Hence

$$a^2(s-a) + b^2(s-a) + c^2(s-a) = \frac{(b+c-a)(c+a-b)(a+b-c)}{2} + abc \\ > abc.$$

Recalling the well-known inequality

$$(b+c-a)(c+a-b)(a+b-c) \leq abc,$$

we have

$$a^2(s-a) + b^2(s-a) + c^2(s-a) = \frac{(b+c-a)(c+a-b)(a+b-c)}{2} + abc \\ \leq \frac{3}{2}abc.$$

Equality holds if and only if the triangle is equilateral. ■

**236** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove that

$$\frac{1}{\sqrt{a} + \sqrt{b} - \sqrt{c}} + \frac{1}{\sqrt{b} + \sqrt{c} - \sqrt{a}} + \frac{1}{\sqrt{c} + \sqrt{a} - \sqrt{b}} \geq \frac{3(\sqrt{a} + \sqrt{b} + \sqrt{c})}{a + b + c}.$$

*Solution* Firstly it is easy to show that if there exists a triangle with lengths sides  $a, b, c$  then there also exists a triangle with length sides  $\sqrt{a}, \sqrt{b}, \sqrt{c}$ .

Furthermore

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 = a + b + c + 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}) \leq 3(a + b + c)$$

i.e.

$$\frac{1}{\sqrt{a} + \sqrt{b} + \sqrt{c}} \geq \frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{3(a + b + c)}. \quad (1)$$

Applying  $AM \geq HM$  we deduce

$$\frac{3}{\frac{1}{\sqrt{a}+\sqrt{b}-\sqrt{c}} + \frac{1}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{1}{\sqrt{c}+\sqrt{a}-\sqrt{b}}} \leq \frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{3},$$

i.e.

$$\frac{1}{\sqrt{a} + \sqrt{b} - \sqrt{c}} + \frac{1}{\sqrt{b} + \sqrt{c} - \sqrt{a}} + \frac{1}{\sqrt{c} + \sqrt{a} - \sqrt{b}} \geq \frac{9}{\sqrt{a} + \sqrt{b} + \sqrt{c}}. \quad (2)$$

By (1) and (2) we get the required inequality. ■

**237** Let  $a, b, c$  be the lengths of the sides of a triangle with area  $P$ . Prove that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}P.$$

*Solution* After setting  $a = x + y, b = y + z, c = z + x$  where  $x, y, z > 0$ , the given inequality becomes

$$((x + y)^2 + (y + z)^2 + (z + x)^2)^2 \geq 48xyz(x + y + z).$$

From  $AM \geq GM$  we have

$$((x + y)^2 + (y + z)^2 + (z + x)^2)^2 \geq (4xy + 4yz + 4zx)^2 = 16(xy + yz + zx)^2. \quad (1)$$

Since for every  $p, q, r \in \mathbb{R}$  we have  $(p + q + r)^2 \geq 3(pq + qr + rp)$ , by (1) we get

$$\begin{aligned} & ((x + y)^2 + (y + z)^2 + (z + x)^2)^2 \\ & \geq 16(xy + yz + zx)^2 \\ & \geq 16 \cdot 3((xy)(yz) + (yz)(zx) + (zx)(xy)) = 48xyz(x + y + z), \end{aligned}$$

as required.

Equality holds iff  $x = y = z$ , i.e. iff  $a = b = c$ . ■

**238 (Hadwinger–Finsler)** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove the inequality

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}P + (a - b)^2 + (b - c)^2 + (c - a)^2.$$

*Solution 1* The given inequality is equivalent to

$$2(ab + bc + ca) - (a^2 + b^2 + c^2) \geq 4\sqrt{3}P.$$

We'll use *Ravi's substitutions*, i.e.  $a = x + y, b = y + z, c = z + x$ , where  $x, y, z > 0$ .

Then the previous inequality becomes

$$xy + yz + zx \geq \sqrt{3xyz(x + y + z)},$$

which is true due to

$$(xy + yz + zx)^2 - 3xyz(x + y + z) = \frac{(xy - yz)^2 + (yz - zx)^2 + (zx - xy)^2}{2}.$$

Clearly equality holds iff  $x = y = z$ , i.e. iff  $a = b = c$ . ■

*Solution 2* The given inequality can be rewritten as

$$2(ab + bc + ca) \geq 4\sqrt{3}P + a^2 + b^2 + c^2. \quad (1)$$

Using  $\frac{ab \sin \gamma}{2} = \frac{ac \sin \beta}{2} = \frac{bc \sin \alpha}{2} = P$  it follows that

$$ab = \frac{2P}{\sin \gamma}, \quad ac = \frac{2P}{\sin \beta}, \quad bc = \frac{2P}{\sin \alpha}.$$

From

$$\cot \alpha = \frac{\cos \alpha}{\sin \alpha} = \frac{\frac{b^2 + c^2 - a^2}{2bc}}{\frac{a}{2R}} = \frac{R}{abc}(b^2 + c^2 - a^2)$$

we get

$$\cot \alpha + \cot \beta + \cot \gamma = \frac{R}{abc}(a^2 + b^2 + c^2),$$

i.e.

$$a^2 + b^2 + c^2 = 4P(\cot \alpha + \cot \beta + \cot \gamma),$$

and inequality (1) becomes

$$4P\left(\frac{1}{\sin \alpha} + \frac{1}{\sin \gamma} + \frac{1}{\sin \beta}\right) \geq 4\sqrt{3}P + 4P(\cot \alpha + \cot \beta + \cot \gamma),$$

i.e.

$$\begin{aligned} &\left(\frac{1}{\sin \alpha} - \cot \alpha\right) + \left(\frac{1}{\sin \beta} - \cot \beta\right) + \left(\frac{1}{\sin \gamma} - \cot \gamma\right) \geq \sqrt{3} \\ \Leftrightarrow &\frac{1 - \cos \alpha}{\sin \alpha} + \frac{1 - \cos \beta}{\sin \beta} + \frac{1 - \cos \gamma}{\sin \gamma} \geq \sqrt{3}. \end{aligned} \quad (2)$$

But  $1 - \cos \alpha = 2 \sin^2 \frac{\alpha}{2}$  and  $\sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$ , so we have

$$\frac{1 - \cos \alpha}{\sin \alpha} = \frac{2 \sin^2 \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} = \tan \frac{\alpha}{2}.$$

Now inequality (2) is equivalent to

$$\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} \geq \sqrt{3},$$

which is true, since  $\tan x$  is convex on  $(0, \pi/2)$  (*Jensen's inequality*). ■

**239** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove that

$$\frac{1}{8abc + (a+b-c)^3} + \frac{1}{8abc + (b+c-a)^3} + \frac{1}{8abc + (c+a-b)^3} \leq \frac{1}{3abc}.$$

*Solution* The given inequality is equivalent to

$$\begin{aligned} & \frac{1}{8abc} - \frac{1}{8abc + (a+b-c)^3} + \frac{1}{8abc} - \frac{1}{8abc + (b+c-a)^3} + \frac{1}{8abc} \\ & \quad - \frac{1}{8abc + (c+a-b)^3} \\ & \geq \frac{3}{8abc} - \frac{1}{3abc}, \end{aligned}$$

i.e.

$$\frac{(a+b-c)^3}{8abc + (a+b-c)^3} + \frac{(b+c-a)^3}{8abc + (b+c-a)^3} + \frac{(c+a-b)^3}{8abc + (c+a-b)^3} \geq \frac{1}{3}. \quad (1) \quad \blacksquare$$

**Lemma 21.3** Let  $a, b, c, x, y, z \in \mathbb{R}^+$ . Then

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}.$$

*Proof* We'll use the *generalized Hölder inequality*, i.e.

If  $(a_i), (b_i), (c_i), i = 1, 2, \dots, n$ , are positive real numbers and  $p, q, r$  are such that  $p + q + r = 1$ , then

$$\left( \sum_{i=1}^n a_i \right)^p \cdot \left( \sum_{i=1}^n b_i \right)^q \cdot \left( \sum_{i=1}^n c_i \right)^r \geq \sum_{i=1}^n a_i^p b_i^q c_i^r.$$

For  $n = 3, p = q = r = 1/3$  and

$$\begin{aligned} a_1 = a_2 = a_3 = 1; \quad b_1 = x, \quad b_2 = y, \quad b_3 = z; \\ c_1 = \frac{a^3}{x}, \quad c_2 = \frac{b^3}{y}, \quad c_3 = \frac{c^3}{z} \end{aligned}$$

we get

$$\begin{aligned} & \sqrt[3]{(1+1+1)(x+y+z) \left( \frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \right)} \\ & \geq \sqrt[3]{1 \cdot x \cdot \frac{a^3}{x}} + \sqrt[3]{1 \cdot y \cdot \frac{b^3}{y}} + \sqrt[3]{1 \cdot z \cdot \frac{c^3}{z}}, \end{aligned}$$

i.e.

$$3(x+y+z)\left(\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z}\right) \geq (a+b+c)^3 \Leftrightarrow \frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}.$$

According to (1) and Lemma 21.3, we have

$$\begin{aligned} & \frac{(a+b-c)^3}{8abc + (a+b-c)^3} + \frac{(b+c-a)^3}{8abc + (b+c-a)^3} + \frac{(c+a-b)^3}{8abc + (c+a-b)^3} \\ & \geq \frac{(a+b-c+b+c-a+c+a-b)^3}{3(24abc + (a+b-c)^3 + (b+c-a)^3 + (c+a-b)^3)} = \frac{1}{3}. \quad \square \end{aligned}$$

**240** In the triangle  $ABC$ ,  $\overline{AC}^2$  is the arithmetic mean of  $\overline{BC}^2$  and  $\overline{AB}^2$ . Prove that

$$\cot^2 \beta \geq \cot \alpha \cdot \cot \gamma.$$

*Solution* Let  $\overline{BC} = a$ ,  $\overline{AC} = b$ ,  $\overline{AB} = c$ . Then we have  $2b^2 = a^2 + c^2$ . By the law of sines and cosines we have

$$\begin{aligned} \cot \beta &= \frac{\cos \beta}{\sin \beta} = \frac{\frac{a^2+c^2-b^2}{2ac}}{\frac{b}{2R}} = \frac{(a^2+c^2-b^2)R}{abc}, \\ \cot \alpha &= \frac{(b^2+c^2-a^2)R}{abc} \quad \text{and} \quad \cot \gamma = \frac{(b^2+a^2-c^2)R}{abc}. \end{aligned}$$

So we need to prove that

$$\frac{(b^2+c^2-a^2)R}{abc} \cdot \frac{(b^2+a^2-c^2)R}{abc} \leq \frac{(a^2+c^2-b^2)^2 R^2}{(abc)^2},$$

i.e.

$$(b^2+c^2-a^2) \cdot (b^2+a^2-c^2) \leq (a^2+c^2-b^2)^2.$$

Applying  $AM \geq GM$  we have

$$(b^2+c^2-a^2) \cdot (b^2+a^2-c^2) \leq \left( \frac{b^2+c^2-a^2+b^2+a^2-c^2}{2} \right)^2,$$

i.e.

$$(b^2+c^2-a^2) \cdot (b^2+a^2-c^2) \leq (b^2)^2 = (2b^2-b^2)^2 = (a^2+c^2-b^2)^2,$$

as required.

Equality occurs iff  $a = b = c$ . ■

**241** Let  $d_1, d_2$  and  $d_3$  be the distances from an arbitrary point to the sides  $BC, CA, AB$ , respectively, of the triangle  $ABC$ . Prove the inequality



$$\frac{9}{4}(d_1^2 + d_2^2 + d_3^2) \geq \left(\frac{P}{R}\right)^2.$$

*Solution* We have  $P = \frac{ad_1 + bd_2 + cd_3}{2}$ , i.e.

$$P^2 = \frac{1}{4}(ad_1 + bd_2 + cd_3)^2. \quad (1)$$

By the *Cauchy–Schwarz inequality* we have

$$(ad_1 + bd_2 + cd_3)^2 \leq (a^2 + b^2 + c^2)(d_1^2 + d_2^2 + d_3^2). \quad (2)$$

Also

$$a^2 + b^2 + c^2 \leq 9R^2. \quad (3)$$

Finally by (1), (2) and (3) we obtain the required inequality.

Equality holds iff the triangle is equilateral and the given point is the center of the triangle. ■

**242** Let  $a, b, c$  be the side lengths, and  $h_a, h_b, h_c$  be the lengths of the altitudes (respectively) of a given triangle. Prove the inequality

$$\frac{h_a + h_b + h_c}{a + b + c} \leq \frac{\sqrt{3}}{2}.$$

*Solution* We have

$$(a + b + c)^2 \geq 3(ab + bc + ca) = \frac{3abc}{2P}(h_a + h_b + h_c) = 6R \cdot (h_a + h_b + h_c). \quad (1)$$

Recall the well-known inequality  $a^2 + b^2 + c^2 \leq 9R^2$ .

Then we have

$$(a + b + c)^2 \leq 3(a^2 + b^2 + c^2) \leq 27R^2, \quad \text{i.e.} \quad a + b + c \leq 3\sqrt{3}R. \quad (2)$$

Now by (1) and (2) we get

$$(a + b + c)^2 \geq 6 \frac{(a + b + c)}{3\sqrt{3}} \cdot (h_a + h_b + h_c), \quad \text{i.e.} \quad \frac{h_a + h_b + h_c}{a + b + c} \leq \frac{\sqrt{3}}{2}.$$

Equality occurs iff  $a = b = c$ . ■

**243** Let  $O$  be an arbitrary point in the interior of  $\triangle ABC$ . Let  $x, y$  and  $z$  be the distances from  $O$  to the sides  $BC, CA, AB$ , respectively, and let  $R$  be the circumradius of the triangle  $\triangle ABC$ . Prove the inequality

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \leq 3\sqrt{\frac{R}{2}}.$$

*Solution* Let  $\overline{BC} = a$ ,  $\overline{CA} = b$ ,  $\overline{AB} = c$ .

By the *Cauchy-Schwarz inequality* we have

$$(\sqrt{x} + \sqrt{y} + \sqrt{z})^2 \leq (ax + by + cz) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

Since  $ax + by + cz = 2P$  and  $P = \frac{abc}{4R}$  we have

$$(\sqrt{x} + \sqrt{y} + \sqrt{z})^2 \leq 2P \cdot \frac{ab + bc + ca}{abc} = \frac{ab + bc + ca}{2R}. \quad (1)$$

Also we have

$$ab + bc + ca \leq a^2 + b^2 + c^2 \leq 9R^2. \quad (2)$$

By (1) and (2) it follows that

$$(\sqrt{x} + \sqrt{y} + \sqrt{z})^2 \leq \frac{9}{2}R, \quad \text{i.e.} \quad \sqrt{x} + \sqrt{y} + \sqrt{z} \leq 3\sqrt{\frac{R}{2}}.$$

Equality holds iff the triangle is equilateral. ■

**244** Let  $D$ ,  $E$  and  $F$  be the feet of the altitudes of the triangle  $ABC$  dropped from the vertices  $A$ ,  $B$  and  $C$ , respectively. Prove the inequality

$$\left( \frac{\overline{EF}}{a} \right)^2 + \left( \frac{\overline{FD}}{b} \right)^2 + \left( \frac{\overline{DE}}{c} \right)^2 \geq \frac{3}{4}.$$

*Solution* Clearly  $\overline{EF} = a \cos \alpha$ ,  $\overline{FD} = b \cos \beta$ ,  $\overline{DE} = c \cos \gamma$ , and the given inequality becomes

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma \geq \frac{3}{4},$$

which is true according to  $N_{11}$  (Chap. 8). ■

**245** Let  $a$ ,  $b$ ,  $c$  be the side-lengths and  $h_a$ ,  $h_b$ ,  $h_c$  be the lengths of the respective altitudes, and  $s$  be the semi-perimeter of a given triangle. Prove the inequality

$$\frac{h_a}{a} + \frac{h_b}{b} + \frac{h_c}{c} \leq \frac{s}{2r}.$$

*Solution* From  $\sqrt{(s-b)(s-c)} \leq \frac{s-b+s-c}{2} = \frac{a}{2}$  (equality holds iff  $b = c$ ), we have

$$\frac{1}{a} \leq \frac{1}{2\sqrt{(s-b)(s-c)}}.$$

Hence

$$\frac{h_a}{a} = \frac{2P}{a^2} \leq \frac{P}{2(s-b)(s-c)}.$$

Analogously we get

$$\frac{h_b}{b} \leq \frac{P}{2(s-c)(s-a)} \quad \text{and} \quad \frac{h_c}{c} \leq \frac{P}{2(s-a)(s-b)}.$$

Hence

$$\begin{aligned} \frac{h_a}{a} + \frac{h_b}{b} + \frac{h_c}{c} &\leq \frac{P}{2} \left( \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)} + \frac{1}{(s-a)(s-b)} \right) \\ &= \frac{sP}{2(s-a)(s-b)(s-c)} = \frac{s^2P}{2P^2} = \frac{s^2}{2P} = \frac{s^2}{2sr} = \frac{s}{2r}. \end{aligned}$$

Equality occurs iff the triangle is equilateral. ■

**246** Let  $a, b, c$  be the side lengths, and  $h_a, h_b, h_c$  be the altitudes, respectively, of a triangle. Prove the inequality

$$\frac{a^2}{h_b^2 + h_c^2} + \frac{b^2}{h_a^2 + h_c^2} + \frac{c^2}{h_a^2 + h_b^2} \geq 2.$$

*Solution* We have

$$\begin{aligned} \frac{a^2}{h_b^2 + h_c^2} + \frac{b^2}{h_a^2 + h_c^2} + \frac{c^2}{h_a^2 + h_b^2} &= \frac{a^2b^2c^2}{4P^2(b^2 + c^2)} + \frac{a^2b^2c^2}{4P^2(a^2 + c^2)} + \frac{a^2b^2c^2}{4P^2(a^2 + b^2)} \\ &= \frac{a^2b^2c^2}{4P^2} \left( \frac{1}{b^2 + c^2} + \frac{1}{a^2 + c^2} + \frac{1}{a^2 + b^2} \right). \end{aligned}$$

Also

$$a^2b^2c^2 = 16P^2R^2$$

and

$$\frac{1}{b^2 + c^2} + \frac{1}{a^2 + c^2} + \frac{1}{a^2 + b^2} \geq \frac{9}{2(a^2 + b^2 + c^2)} \quad (\text{since } AM \geq HM).$$

Therefore

$$\frac{a^2}{h_b^2 + h_c^2} + \frac{b^2}{h_a^2 + h_c^2} + \frac{c^2}{h_a^2 + h_b^2} \geq \frac{16P^2R^2}{4P^2} \cdot \frac{9}{2(a^2 + b^2 + c^2)} = \frac{18R^2}{a^2 + b^2 + c^2} \geq 2,$$

where the last inequality is true since  $a^2 + b^2 + c^2 \leq 9R^2$ .

Equality holds iff the triangle is equilateral. ■

**247** Let  $a, b, c$  be the side lengths,  $h_a, h_b, h_c$  be the altitudes, respectively and  $r$  be the inradius of a triangle. Prove the inequality

$$\frac{1}{h_a - 2r} + \frac{1}{h_b - 2r} + \frac{1}{h_c - 2r} \geq \frac{3}{r}.$$

*Solution* By  $\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$  we obtain

$$\frac{h_a - 2r}{h_a} + \frac{h_b - 2r}{h_b} + \frac{h_c - 2r}{h_c} = 1.$$

Applying  $AM \geq HM$  we get

$$\left( \frac{h_a - 2r}{h_a} + \frac{h_b - 2r}{h_b} + \frac{h_c - 2r}{h_c} \right) \left( \frac{h_a}{h_a - 2r} + \frac{h_b}{h_b - 2r} + \frac{h_c}{h_c - 2r} \right) \geq 9,$$

i.e.

$$\frac{h_a}{h_a - 2r} + \frac{h_b}{h_b - 2r} + \frac{h_c}{h_c - 2r} \geq 9.$$

Therefore

$$\begin{aligned} & \frac{2r}{h_a - 2r} + \frac{2r}{h_b - 2r} + \frac{2r}{h_c - 2r} \\ &= \frac{h_a - (h_a - 2r)}{h_a - 2r} + \frac{h_b - (h_b - 2r)}{h_b - 2r} + \frac{h_c - (h_c - 2r)}{h_c - 2r} \\ &= \frac{h_a}{h_a - 2r} + \frac{h_b}{h_b - 2r} + \frac{h_c}{h_c - 2r} - 3 \geq 9 - 3 = 6, \end{aligned}$$

i.e.

$$\frac{1}{h_a - 2r} + \frac{1}{h_b - 2r} + \frac{1}{h_c - 2r} \geq \frac{3}{r}. \quad \blacksquare$$

**248** Let  $a, b, c; l_\alpha, l_\beta, l_\gamma$  be the lengths of the sides and the bisectors of the respective angles. Let  $s$  be the semi-perimeter and  $r$  denote the inradius of a given triangle. Prove the inequality

$$\frac{l_\alpha}{a} + \frac{l_\beta}{b} + \frac{l_\gamma}{c} \leq \frac{s}{2r}.$$

*Solution* The following identities hold:

$$l_\alpha = \frac{2\sqrt{bc}}{b+c} \sqrt{s(s-a)}, \quad l_\beta = \frac{2\sqrt{ca}}{c+a} \sqrt{s(s-b)} \quad \text{and} \quad l_\gamma = \frac{2\sqrt{ab}}{a+b} \sqrt{s(s-c)}.$$

From the obvious inequality  $\frac{2\sqrt{xy}}{x+y} \leq 1$  and the previous identities we obtain that

$$l_\alpha \leq \sqrt{s(s-a)}, \quad l_\beta \leq \sqrt{s(s-b)} \quad \text{and} \quad l_\gamma \leq \sqrt{s(s-c)}. \quad (1)$$

Also

$$h_a \leq l_\alpha, \quad h_b \leq l_\beta \quad \text{and} \quad h_c \leq l_\gamma. \quad (2)$$

So we have

$$\begin{aligned}
 \frac{l_\alpha}{a} + \frac{l_\beta}{b} + \frac{l_\gamma}{c} &= \frac{l_\alpha h_a}{2P} + \frac{l_\beta h_b}{2P} + \frac{l_\gamma h_c}{2P} \stackrel{(2)}{\leq} \frac{l_\alpha^2 + l_\beta^2 + l_\gamma^2}{2P} \\
 &\stackrel{(1)}{\leq} \frac{s(s-a) + s(s-b) + s(s-c)}{2P} \\
 &= \frac{3s^2 - s(a+b+c)}{2rs} = \frac{3s^2 - 2s^2}{2rs} = \frac{s^2}{2rs} = \frac{s}{2r}.
 \end{aligned}$$

Equality occurs iff the triangle is equilateral. ■

**249** Let  $a, b, c; l_\alpha, l_\beta, l_\gamma$  be the lengths of the sides and of the bisectors of respective angles. Let  $R$  and  $r$  be the circumradius and inradius, respectively, of a given triangle. Prove the inequality

$$18r^2\sqrt{3} \leq al_\alpha + bl_\beta + cl_\gamma < 9R^2.$$

*Solution* We have

$$a^2 \geq a^2 - (b-c)^2 = (a+b-c)(a+c-b) = 4(s-c)(s-b).$$

Hence

$$a \geq 2\sqrt{(s-c)(s-b)},$$

with equality if and only if  $b = c$ .

Since  $l_\alpha = 2\sqrt{bc} \frac{\sqrt{s(s-a)}}{b+c}$  and by the previous inequality we get

$$al_\alpha \geq \frac{4\sqrt{bc}}{b+c} \sqrt{s(s-a)(s-c)(s-b)} = \frac{4\sqrt{bc}}{b+c} P.$$

Analogously we obtain

$$bl_\beta \geq \frac{4\sqrt{ac}}{a+c} P \quad \text{and} \quad cl_\gamma \geq \frac{4\sqrt{ab}}{a+b} P.$$

Therefore

$$al_\alpha + bl_\beta + cl_\gamma \geq 4P \left( \frac{4\sqrt{bc}}{b+c} + \frac{4\sqrt{ac}}{a+c} + \frac{4\sqrt{ab}}{a+b} \right). \quad (1)$$

By  $AM \geq GM$  we have

$$\frac{4\sqrt{bc}}{b+c} + \frac{4\sqrt{ac}}{a+c} + \frac{4\sqrt{ab}}{a+b} \geq 3 \sqrt[3]{\frac{abc}{(a+b)(b+c)(c+a)}}. \quad (2)$$

Also we have

$$4s = (a+b) + (b+c) + (c+a) \geq 3\sqrt[3]{(a+b)(b+c)(c+a)}.$$

Hence

$$\sqrt[3]{\frac{1}{(a+b)(b+c)(c+a)}} \geq \frac{3}{4s}. \quad (3)$$

By (1), (2) and (3) we obtain

$$al_\alpha + bl_\beta + cl_\gamma \geq \frac{9P}{s} \sqrt[3]{abc} = \frac{9sr}{s} \sqrt[3]{4PR} = 9r \sqrt[3]{4srR}. \quad (4)$$

According to Exercise 13.2 (Chap. 3) we have that  $s \geq 3r\sqrt{3}$ , and clearly  $R \geq 2r$ .

Now by (4) we get

$$al_\alpha + bl_\beta + cl_\gamma \geq 9r \sqrt[3]{4srR} \geq 9r \sqrt[3]{24r^3\sqrt{3}} = 18r^2\sqrt{3}.$$

Equality occurs iff  $a = b = c$ .

We need to show the right-hand side inequality.

We have

$$\sqrt{s(s-a)} \leq \frac{s+s-a}{2} = \frac{b+c}{2}.$$

Note that we have a strict inequality since  $s \neq s-a$ .

Now we have

$$l_\alpha = 2\sqrt{bc} \frac{\sqrt{s(s-a)}}{b+c} < \sqrt{bc} \leq \frac{b+c}{2}, \quad \text{i.e.} \quad al_\alpha < a \frac{b+c}{2}.$$

Analogously we obtain

$$bl_\beta < b \frac{a+c}{2} \quad \text{and} \quad cl_\gamma < c \frac{a+b}{2}.$$

So

$$al_\alpha + bl_\beta + cl_\gamma < ab + bc + ca. \quad (5)$$

If we consider the well-known inequalities

$$ab + bc + ca \leq a^2 + b^2 + c^2 \quad \text{and} \quad a^2 + b^2 + c^2 \leq 9R^2,$$

from (5) we obtain the required inequality. ■

**250** Let  $a, b, c$  be the lengths of the sides of triangle, with circumradius  $r = 1/2$ . Prove the inequality

$$\frac{a^4}{b+c-a} + \frac{b^4}{a+c-b} + \frac{c^4}{a+b-c} \geq 9\sqrt{3}.$$

*Solution* Let  $s$  be the semi-perimeter of the given triangle. The given inequality becomes

$$A = \frac{a^4}{2(s-a)} + \frac{b^4}{2(s-b)} + \frac{c^4}{2(s-c)} \geq 9\sqrt{3}.$$

By the *Cauchy-Schwarz inequality* we obtain

$$\begin{aligned} A \cdot (2(s-a) + 2(s-b) + 2(s-c)) &\geq (a^2 + b^2 + c^2)^2 \\ \Leftrightarrow 2s \cdot A &\geq (a^2 + b^2 + c^2)^2, \end{aligned}$$

i.e.

$$A \geq \frac{(a^2 + b^2 + c^2)^2}{a + b + c}. \quad (1)$$

Applying  $QM \geq AM$  we deduce

$$\frac{a^2 + b^2 + c^2}{3} \geq \left( \frac{a + b + c}{3} \right)^2, \quad \text{i.e.} \quad a^2 + b^2 + c^2 \geq \frac{(a + b + c)^2}{3}.$$

Then by (1) we get

$$A \geq \frac{(a^2 + b^2 + c^2)^2}{a + b + c} \geq \frac{(a + b + c)^4}{9(a + b + c)} = \frac{(a + b + c)^3}{9}. \quad (2)$$

Let's introduce *Ravi's substitutions*, i.e. let us take  $a = x + y$ ,  $b = y + z$ ,  $c = z + x$ . Then clearly  $s = \frac{a+b+c}{2} = x + y + z$ .

By *Heron's formula* we obtain

$$P^2 = s(s-a)(s-b)(s-c) = xyz(x+y+z). \quad (3)$$

Also

$$P^2 = s^2 r^2 = \frac{(x+y+z)^2}{4}. \quad (4)$$

By (3) and (4) we get

$$x + y + z = 4xyz. \quad (5)$$

Since  $AM \geq GM$  and using (5) we obtain

$$\left( \frac{x+y+z}{3} \right)^3 \geq xyz = \frac{x+y+z}{4},$$

i.e.

$$x + y + z \geq \frac{3\sqrt{3}}{2}.$$

Thus

$$a + b + c = 2(x + y + z) \geq 3\sqrt{3}. \quad (6)$$

Finally according to (2) and (6) it follows that

$$A \geq \frac{(a + b + c)^3}{9} = \frac{(3\sqrt{3})^3}{9} \geq 9\sqrt{3}.$$

Equality occurs if and only if the triangle is equilateral with side equal to  $\sqrt{3}$ . ■

**251** Let  $a, b, c$  be the side-lengths of a triangle. Prove the inequality

$$\frac{a}{3a - b + c} + \frac{b}{3b - c + a} + \frac{c}{3c - a + b} \geq 1.$$

*Solution* We have

$$\begin{aligned} & \frac{4a}{3a - b + c} + \frac{4b}{3b - c + a} + \frac{4c}{3c - a + b} \\ &= 3 + \frac{a + b - c}{3a - b + c} + \frac{b + c - a}{3b - c + a} + \frac{c + a - b}{3c - a + b}. \end{aligned}$$

So it remains to show that

$$\frac{a + b - c}{3a - b + c} + \frac{b + c - a}{3b - c + a} + \frac{c + a - b}{3c - a + b} \geq 1.$$

By the *Cauchy–Schwarz inequality* (Corollary 4.3) we have

$$\begin{aligned} & \frac{a + b - c}{3a - b + c} + \frac{b + c - a}{3b - c + a} + \frac{c + a - b}{3c - a + b} \\ &= \frac{(a + b - c)^2}{(3a - b + c)(a + b - c)} + \frac{(b + c - a)^2}{(3b - c + a)(b + c - a)} \\ & \quad + \frac{(c + a - b)^2}{(3c - a + b)(c + a - b)} \\ &\geq \frac{(a + b + c)^2}{(3a - b + c)(a + b - c) + (3b - c + a)(b + c - a) + (3c - a + b)(c + a - b)} \\ &= 1, \end{aligned}$$

as required.

Equality holds iff  $a = b = c = 1$ . ■

**252** Let  $h_a, h_b$  and  $h_c$  be the lengths of the altitudes, and  $R$  and  $r$  be the circumradius and inradius, respectively, of a given triangle. Prove the inequality

$$h_a + h_b + h_c \leq 2R + 5r.$$



*Solution*

**Lemma 21.4** *In an arbitrary triangle we have*

$$ab + bc + ca = r^2 + s^2 + 4rR \quad \text{and} \quad a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2).$$

*Proof* We have

$$\begin{aligned} r^2 + s^2 + 4rR &= \frac{P^2}{s^2} + s^2 + \frac{abc}{P} \cdot \frac{P}{s} = \frac{(s-a)(s-b)(s-c)}{s} + s^2 + \frac{abc}{s} \\ &= \frac{s^3 - as^2 - bs^2 - cs^2 + abs + bcs + cas - abc + s^3 + abc}{s} \\ &= 2s^2 - s(a+b+c) + ab + bc + ca \\ &= 2s^2 - 2s^2 + ab + bc + ca = ab + bc + ca. \end{aligned}$$

Hence

$$ab + bc + ca = r^2 + s^2 + 4rR. \quad (1)$$

Now by (1) we have

$$\begin{aligned} ab + bc + ca &= r^2 + s^2 + 4rR = \frac{1}{2} \left( 2r^2 + 8rR + \frac{(a+b+c)^2}{2} \right) \\ &= \frac{1}{2} \left( 2r^2 + 8rR + \frac{a^2 + b^2 + c^2}{2} \right) + \frac{ab + bc + ca}{2}, \end{aligned}$$

from which it follows that

$$ab + bc + ca = 2r^2 + 8rR + \frac{a^2 + b^2 + c^2}{2}. \quad (2)$$

Now (1) and (2) yields

$$a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2). \quad (3)$$

□

Without proof we will give the following lemma (the proof can be found in [6]).

**Lemma 21.5** *In an arbitrary triangle we have*

$$s^2 \leq 4R^2 + 4Rr + 3r^2. \quad (4)$$

**Lemma 21.6** *In an arbitrary triangle we have  $a^2 + b^2 + c^2 \leq 8R^2 + 4r^2$ .*

*Proof* From (3) and (4) we have

$$a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2) \leq 2(4R^2 + 4Rr + 3r^2 - 4Rr - r^2) = 8R^2 + 4r^2.$$

Hence

$$a^2 + b^2 + c^2 \leq 8R^2 + 4r^2. \quad (5)$$

□

Now let us consider our problem.

We have

$$\begin{aligned} 2R(h_a + h_b + h_c) &= 2R\left(\frac{2P}{a} + \frac{2P}{b} + \frac{2P}{c}\right) = 4PR \frac{ab + bc + ca}{abc} \\ &= ab + bc + ca \\ &\stackrel{(2)}{=} 2r^2 + 8rR + \frac{a^2 + b^2 + c^2}{2} \\ &\stackrel{(4)}{\leq} 2r^2 + 8rR + 4R^2 + 2r^2 \\ \Leftrightarrow R(h_a + h_b + h_c) &\leq 2R^2 + 4Rr + 2r^2 \leq 2R^2 + 4Rr + Rr \leq R(2R + 5r). \end{aligned}$$

Hence

$$h_a + h_b + h_c \leq 2R + 5r.$$

Equality occurs iff  $a = b = c$ . ■

**253** Let  $a, b, c$  be the side-lengths, and  $\alpha, \beta$  and  $\gamma$  be the angles of a given triangle, respectively. Prove the inequality

$$a\left(\frac{1}{\beta} + \frac{1}{\gamma}\right) + b\left(\frac{1}{\gamma} + \frac{1}{\alpha}\right) + c\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \geq 2\left(\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}\right).$$

*Solution* If  $a \geq b$  then  $\alpha \geq \beta$  and analogously if  $a \leq b$  then we have  $\alpha \leq \beta$ .

So we have  $(a - b)(\alpha - \beta) \geq 0$ , i.e. we have

$$a\alpha + b\beta \geq a\beta + b\alpha$$

i.e.

$$\frac{a}{\beta} + \frac{b}{\alpha} \geq \frac{a}{\alpha} + \frac{b}{\beta}. \quad (1)$$

Analogously we have

$$\frac{a}{\gamma} + \frac{c}{\alpha} \geq \frac{a}{\alpha} + \frac{c}{\gamma} \quad (2)$$

and

$$\frac{c}{\beta} + \frac{b}{\gamma} \geq \frac{c}{\beta} + \frac{b}{\gamma}. \quad (3)$$

Adding (1), (2) and (3) we obtain the required inequality.

Equality occurs iff  $a = b = c$ , i.e. if the triangle is equilateral. ■

**254** Let  $a, b, c$  be the lengths of the sides of a given triangle, and  $\alpha, \beta, \gamma$  be the respective angles (in radians). Prove the inequalities

$$1^\circ \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \geq \frac{9}{\pi}.$$

$$2^\circ \frac{b+c-a}{\alpha} + \frac{c+a-b}{\beta} + \frac{a+b-c}{\gamma} \geq \frac{6s}{\pi}, \text{ where } s = \frac{a+b+c}{2}.$$

$$3^\circ \frac{b+c-a}{a\alpha} + \frac{c+a-b}{b\beta} + \frac{a+b-c}{c\gamma} \geq \frac{9}{\pi}.$$

*Solution*  $1^\circ$  Since  $AM \geq HM$  we have

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \geq \frac{9}{\alpha + \beta + \gamma} = \frac{9}{\pi}.$$

$2^\circ$  Let  $x = b + c - a$ ,  $y = c + a - b$  and  $z = a + b - c$ .

Without loss the generality we may assume that  $a \leq b \leq c$ . Then clearly  $\alpha \leq \beta \leq \gamma$ .

Also  $x \geq y \geq z$  and  $\frac{1}{\alpha} \geq \frac{1}{\beta} \geq \frac{1}{\gamma}$ .

*Chebyshev's inequality* gives us

$$(x + y + z) \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right) \leq 3 \left( \frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} \right)$$

i.e.

$$\begin{aligned} \left( \frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} \right) &\geq \frac{1}{3} (x + y + z) \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right) \\ &\geq \frac{1}{3} \cdot \frac{9(x + y + z)}{\alpha + \beta + \gamma} = \frac{6s}{\pi}. \end{aligned}$$

$3^\circ$  Let  $x = \frac{b+c-a}{a}$ ,  $y = \frac{c+a-b}{b}$  and  $z = \frac{a+b-c}{c}$ .

Without loss the generality we may assume  $a \leq b \leq c$ . Then  $\alpha \leq \beta \leq \gamma$ .

Also  $x \geq y \geq z$  and  $\frac{1}{\alpha} \geq \frac{1}{\beta} \geq \frac{1}{\gamma}$ .

*Chebyshev's inequality* gives us

$$\begin{aligned} \left( \frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} \right) &\geq \frac{1}{3} (x + y + z) \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right) \\ &\geq \frac{1}{3} \left( \frac{b+c-a}{a} + \frac{c+a-b}{b} + \frac{a+b-c}{c} \right) \cdot \frac{9}{\pi} \\ &= \frac{3}{\pi} \left( \frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b} - 3 \right) \geq \frac{3}{\pi} (2 + 2 + 2 - 3) = \frac{9}{\pi}. \quad \blacksquare \end{aligned}$$

**255** Let  $X$  be an arbitrary interior point of a given regular  $n$ -gon with side-length  $a$ . Let  $h_1, h_2, \dots, h_n$  be the distances from  $X$  to the sides of the  $n$ -gon. Prove that

$$\frac{1}{h_1} + \frac{1}{h_2} + \dots + \frac{1}{h_n} > \frac{2\pi}{a}.$$

*Solution* Let  $S$  be the area of the given  $n$ -gon, and let  $r$  be the inradius of its inscribed circle.

Then  $S = \frac{nar}{2}$ .

On the other hand, we have

$$S = \frac{1}{2}a(h_1 + h_2 + \cdots + h_n).$$

Applying  $AM \geq HM$  we have

$$\frac{n}{\frac{1}{h_1} + \frac{1}{h_2} + \cdots + \frac{1}{h_n}} \leq \frac{h_1 + h_2 + \cdots + h_n}{n} = \frac{2S}{na} = r,$$

i.e.

$$\frac{1}{h_1} + \frac{1}{h_2} + \cdots + \frac{1}{h_n} \geq \frac{n}{r}. \quad (1)$$

The perimeter of the  $n$ -gon is larger than the perimeter of its inscribed circle, so we have

$$na > 2\pi r, \quad \text{i.e.} \quad \frac{n}{r} > \frac{2\pi}{a}.$$

Now by (1) we obtain

$$\frac{1}{h_1} + \frac{1}{h_2} + \cdots + \frac{1}{h_n} \geq \frac{n}{r} > \frac{2\pi}{a}. \quad \blacksquare$$

**256** Prove that among the lengths of the sides of an arbitrary  $n$ -gon ( $n \geq 3$ ), there always exist two of them (let's denote them by  $b$  and  $c$ ), such that  $1 \leq \frac{b}{c} < 2$ .

*Solution* Let  $a_1, a_2, \dots, a_n$  be the lengths of the sides of the given  $n$ -gon.

Without loss of generality we may assume that  $a_1 \geq a_2 \geq \cdots \geq a_n$ .

Suppose that such a side does not exist, i.e. let us suppose that for any two sides  $b$  and  $c$  we have  $\frac{b}{c} \geq 2$  ( $b > c$ ), i.e. let us suppose that for every  $i \in \{1, 2, \dots, n-1\}$  we have  $\frac{a_i}{a_{i+1}} \geq 2$ .

So it follows that

$$a_2 \leq \frac{a_1}{2}, \quad a_3 \leq \frac{a_2}{2} \leq \frac{a_1}{4}, \dots, \quad a_n \leq \frac{a_{n-1}}{2} \leq \frac{a_1}{2^{n-1}}.$$

If we add these inequalities we obtain

$$a_2 + \cdots + a_n \leq a_1 \left( \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \right) = a_1 \left( 1 - \frac{1}{2^{n-1}} \right) < a_1,$$

which is impossible (why?). \blacksquare

**257** Let  $a_1, a_2, a_3, a_4$  be the lengths of the sides, and  $s$  be the semi-perimeter of an arbitrary quadrilateral. Prove that

$$\sum_{i=1}^4 \frac{1}{s+a_i} \leq \frac{2}{9} \sum_{1 \leq i < j \leq 4} \frac{1}{\sqrt{(s-a_i)(s-a_j)}}.$$

*Solution* From  $AM \geq GM$  we have

$$\begin{aligned} \frac{2}{9} \sum_{1 \leq i < j \leq 4} \frac{1}{\sqrt{(s-a_i)(s-a_j)}} &\geq \frac{2}{9} \cdot 2 \sum_{1 \leq i < j \leq 4} \frac{1}{(s-a_i) + (s-a_j)} \\ &= \frac{4}{9} \sum_{1 \leq i < j \leq 4} \frac{1}{a_i + a_j}. \end{aligned} \quad (1)$$

Let  $a_1 = a, a_2 = b, a_3 = c, a_4 = d$ .

We'll show that

$$\begin{aligned} &\frac{2}{9} \left( \frac{1}{a+b} + \frac{1}{a+c} + \frac{1}{a+d} + \frac{1}{b+c} + \frac{1}{b+d} + \frac{1}{c+d} \right) \\ &\geq \frac{1}{3a+b+c+d} + \frac{1}{a+3b+c+d} + \frac{1}{a+b+3c+d} + \frac{1}{a+b+c+3d}. \end{aligned}$$

From  $AM \geq HM$  we deduce

$$\left( \frac{1}{a+b} + \frac{1}{a+c} + \frac{1}{a+d} \right) ((a+b) + (a+c) + (a+d)) \geq 9,$$

i.e.

$$\frac{1}{9} \left( \frac{1}{a+b} + \frac{1}{a+c} + \frac{1}{a+d} \right) \geq \frac{1}{3a+b+c+d}.$$

Similarly we obtain

$$\begin{aligned} \frac{1}{9} \left( \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{b+d} \right) &\geq \frac{1}{a+3b+c+d}, \\ \frac{1}{9} \left( \frac{1}{a+c} + \frac{1}{b+c} + \frac{1}{c+d} \right) &\geq \frac{1}{a+b+3c+d}, \\ \frac{1}{9} \left( \frac{1}{a+d} + \frac{1}{b+d} + \frac{1}{c+d} \right) &\geq \frac{1}{a+b+c+3d}. \end{aligned}$$

Adding these inequalities we get

$$\begin{aligned} & \frac{2}{9} \left( \frac{1}{a+b} + \frac{1}{a+c} + \frac{1}{a+d} + \frac{1}{b+c} + \frac{1}{b+d} + \frac{1}{c+d} \right) \\ & \geq \frac{1}{3a+b+c+d} + \frac{1}{a+3b+c+d} + \frac{1}{a+b+3c+d} + \frac{1}{a+b+c+3d} \\ & = \frac{1}{2} \left( \frac{1}{s+a} + \frac{1}{s+b} + \frac{1}{s+c} + \frac{1}{s+d} \right), \end{aligned}$$

i.e.

$$\frac{4}{9} \sum_{1 \leq i < j \leq 4} \frac{1}{a_i + a_j} \geq \sum_{i=1}^4 \frac{1}{s + a_i}. \quad (2)$$

From (1) and (2) we obtain the given inequality.

Equality holds iff  $a = b = c = d$ . ■

**258** Let  $n \in \mathbb{N}$ , and  $\alpha, \beta, \gamma$  be the angles of a given triangle. Prove the inequality

$$\cot^n \frac{\alpha}{2} + \cot^n \frac{\beta}{2} + \cot^n \frac{\gamma}{2} \geq 3^{\frac{n+2}{2}}.$$

*Solution* We use the identity

$$\cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} = \cot \frac{\alpha}{2} \cdot \cot \frac{\beta}{2} \cdot \cot \frac{\gamma}{2}.$$

Since  $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2} \in (0, \pi/2)$  it follows that  $\cot \frac{\alpha}{2}, \cot \frac{\beta}{2}, \cot \frac{\gamma}{2} \geq 0$ .

Applying  $AM \geq GM$  we have

$$\cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} \geq 3 \sqrt[3]{\cot \frac{\alpha}{2} \cdot \cot \frac{\beta}{2} \cdot \cot \frac{\gamma}{2}}$$

or

$$\cot \frac{\alpha}{2} \cdot \cot \frac{\beta}{2} \cdot \cot \frac{\gamma}{2} \geq 3 \sqrt[3]{\cot \frac{\alpha}{2} \cdot \cot \frac{\beta}{2} \cdot \cot \frac{\gamma}{2}},$$

i.e.

$$\cot \frac{\alpha}{2} \cdot \cot \frac{\beta}{2} \cdot \cot \frac{\gamma}{2} \geq 3^{3/2}. \quad (1)$$

Furthermore, using the *power mean inequality* we get

$$\cot^n \frac{\alpha}{2} + \cot^n \frac{\beta}{2} + \cot^n \frac{\gamma}{2} \geq 3 \left( \cot \frac{\alpha}{2} \cdot \cot \frac{\beta}{2} \cdot \cot \frac{\gamma}{2} \right)^{n/3}.$$

Now from the previous inequality and (1) we obtain

$$\cot^n \frac{\alpha}{2} + \cot^n \frac{\beta}{2} + \cot^n \frac{\gamma}{2} \geq 3^{\frac{n+2}{2}}.$$

Equality occurs iff  $\alpha = \beta = \gamma = \pi/3$ . ■

**259** Let  $\alpha, \beta, \gamma$  be the angles of an arbitrary acute triangle. Prove that

$$2(\sin \alpha + \sin \beta + \sin \gamma) > 3(\cos \alpha + \cos \beta + \cos \gamma).$$

*Solution* Clearly  $\alpha + \beta > \frac{\pi}{2}$ .

Since  $\sin x$  is an increasing function on  $[0, \pi/2]$  we have

$$\sin \alpha > \sin\left(\frac{\pi}{2} - \beta\right) = \cos \beta. \quad (1)$$

Analogously

$$\sin \beta > \sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha. \quad (2)$$

Now (1) and (2) give us

$$1 - \cos \beta > 1 - \sin \alpha \quad \text{and} \quad 1 - \cos \alpha > 1 - \sin \beta.$$

If we multiply these inequalities we get

$$(1 - \cos \beta)(1 - \cos \alpha) > (1 - \sin \alpha)(1 - \sin \beta)$$

or

$$1 - \cos \beta - \cos \alpha + \cos \alpha \cos \beta > 1 - \sin \beta - \sin \alpha + \sin \alpha \sin \beta$$

or

$$\begin{aligned} \sin \alpha + \sin \beta &> \cos \alpha + \cos \beta - \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ &= \cos \alpha + \cos \beta - \cos(\alpha + \beta) = \cos \alpha + \cos \beta + \cos \gamma. \end{aligned}$$

Analogously we obtain

$$\sin \beta + \sin \gamma > \cos \alpha + \cos \beta + \cos \gamma \quad \text{and} \quad \sin \gamma + \sin \alpha > \cos \alpha + \cos \beta + \cos \gamma.$$

After adding these inequalities we get

$$2(\sin \alpha + \sin \beta + \sin \gamma) > 3(\cos \alpha + \cos \beta + \cos \gamma),$$

as required. ■

**260** Let  $\alpha, \beta, \gamma$  be the angles of a triangle. Prove the inequality

$$\sin \alpha + \sin \beta + \sin \gamma \geq \sin 2\alpha + \sin 2\beta + \sin 2\gamma.$$

*Solution* Applying the sine law we obtain

$$\sin \alpha + \sin \beta + \sin \gamma = \frac{a+b+c}{2R} = \frac{P}{rR}.$$

Also

$$\begin{aligned} \sin 2\alpha + \sin 2\beta + \sin 2\gamma &= 2(\sin \alpha \cos \alpha + \sin \beta \cos \beta + \sin \gamma \cos \gamma) \\ &= \frac{1}{R}(a \cos \alpha + b \cos \beta + c \cos \gamma). \end{aligned}$$

Since

$$a \cos \alpha + b \cos \beta + c \cos \gamma = \frac{2P}{R}$$

we have

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = \frac{2P}{R^2}.$$

Therefore

$$\frac{\sin \alpha + \sin \beta + \sin \gamma}{\sin 2\alpha + \sin 2\beta + \sin 2\gamma} = \frac{R}{2r} \geq 1.$$

Equality holds if and only if the triangle is equilateral. ■

**261** Let  $\alpha, \beta, \gamma$  be the angles of a triangle. Prove the inequality

$$\cos \alpha + \sqrt{2}(\cos \beta + \cos \gamma) \leq 2.$$

*Solution* Since  $\alpha + \beta + \gamma = \pi$ , we have

$$\begin{aligned} \cos \alpha + \sqrt{2}(\cos \beta + \cos \gamma) &= \cos \alpha + 2\sqrt{2} \cos \frac{\beta + \gamma}{2} \cos \frac{\beta - \gamma}{2} \\ &= \cos \alpha + 2\sqrt{2} \sin \frac{\alpha}{2} \cos \frac{\beta - \gamma}{2} \\ &\leq \cos \alpha + 2\sqrt{2} \sin \frac{\alpha}{2} = 2 - 2 \left( \sin \frac{\alpha}{2} - \frac{\sqrt{2}}{2} \right)^2 \leq 2. \end{aligned}$$

Equality holds if and only if  $\alpha = \pi/2, \beta = \gamma$ . ■

**262** Let  $\alpha, \beta, \gamma$  be the angles of a triangle and let  $t$  be a real number. Prove the inequality

$$\cos \alpha + t(\cos \beta + \cos \gamma) \leq 1 + \frac{t^2}{2}.$$



*Solution* For any three real numbers  $\beta, \gamma, t$ , the following inequality holds:

$$(\cos \beta + \cos \gamma - t)^2 + (\sin \beta - \sin \gamma)^2 \geq 0,$$

i.e.

$$-\cos(\beta + \gamma) + t(\cos \beta + \cos \gamma) \leq 1 + \frac{t^2}{2}.$$

Since  $\alpha + \beta + \gamma = \pi$  we have

$$\cos \alpha + t(\cos \beta + \cos \gamma) \leq 1 + \frac{t^2}{2}.$$

Equality occurs iff  $0 < t < 2$ ,  $\cos \alpha = 1 - \frac{t^2}{2}$ ,  $\cos \beta = \cos \gamma$ . ■

**263** Let  $0 \leq \alpha, \beta, \gamma \leq 90^\circ$  such that  $\sin \alpha + \sin \beta + \sin \gamma = 1$ . Prove the inequality

$$\tan^2 \alpha + \tan^2 \beta + \tan^2 \gamma \geq \frac{3}{8}.$$

*Solution* We have

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1.$$

The given inequality becomes

$$\frac{1}{\cos^2 \alpha} + \frac{1}{\cos^2 \beta} + \frac{1}{\cos^2 \gamma} \geq \frac{3}{8} + 3 = \frac{27}{8}.$$

Applying  $AM \geq HM$  we get

$$\frac{3}{\frac{1}{\cos^2 \alpha} + \frac{1}{\cos^2 \beta} + \frac{1}{\cos^2 \gamma}} \leq \frac{\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma}{3} = 1 - \frac{\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma}{3}, \quad (1)$$

and since  $\sin x \geq 0$  for  $x \in [0, \pi]$  we have

$$\sqrt{\frac{\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma}{3}} \geq \frac{\sin \alpha + \sin \beta + \sin \gamma}{3} = \frac{1}{3},$$

i.e.

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma \geq \frac{1}{3}.$$

So in (1) we obtain

$$\frac{3}{\frac{1}{\cos^2 \alpha} + \frac{1}{\cos^2 \beta} + \frac{1}{\cos^2 \gamma}} = 1 - \frac{\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma}{3} \leq 1 - \frac{1}{9} = \frac{8}{9},$$

i.e.

$$\frac{1}{\cos^2 \alpha} + \frac{1}{\cos^2 \beta} + \frac{1}{\cos^2 \gamma} \geq \frac{27}{8},$$

as required. ■

**264** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$(1 + a + a^2)(1 + b + b^2)(1 + c + c^2) \geq 9(ab + bc + ca).$$

*Solution* Let us denote  $x = a + b + c = 3$ ,  $y = ab + bc + ca$ ,  $z = abc$ .

Now the given inequality can be rewritten as

$$z^2 - 2z - 2xz + z(x + y) + x^2 + x + y^2 - y + 3xy + 1 \geq 9y,$$

i.e.

$$(z - 1)^2 - (z - 1)(x - y) + (x - y)^2 \geq 0,$$

which is obviously true. Equality holds iff  $a = b = c = 1$ . ■

**265** Let  $a, b, c > 0$  such that  $a + b + c = 1$ . Prove the inequality

$$6(a^3 + b^3 + c^3) + 1 \geq 5(a^2 + b^2 + c^2).$$

*Solution* Let  $a + b + c = p = 1$ ,  $ab + bc + ca = q$ ,  $abc = r$ .

By  $I_1$  and  $I_2$  (Chap. 14) we have

$$a^3 + b^3 + c^3 = p(p^2 - 3q) + 3r = 1 - 3q + 3r$$

and

$$a^2 + b^2 + c^2 = p^2 - 2q = 1 - 2q.$$

Now the given inequality becomes

$$18r + 1 - 2q - 6q + 1 \geq 0,$$

i.e.

$$9r + 1 \geq 4q$$

which is true due to  $N_1$  (Chap. 14). ■

**266** Let  $x, y, z \in \mathbb{R}^+$  such that  $x + y + z = 1$ . Prove the inequality

$$(1 - x^2)^2 + (1 - y^2)^2 + (1 - z^2)^2 \leq (1 + x)(1 + y)(1 + z).$$

*Solution* Let  $p = x + y + z = 1$ ,  $q = xy + yz + zx$ ,  $r = xyz$ .

The given inequality is equivalent to

$$3 - 2(x^2 + y^2 + z^2) + x^4 + y^4 + z^4 \leq (1+x)(1+y)(1+z).$$

By  $I_1$ ,  $I_4$  and  $I_9$  (Chap. 14) we have

$$x^2 + y^2 + z^2 = p^2 - 2q = 1 - 2q,$$

$$x^4 + y^4 + z^4 = (p^2 - 2q)^2 - 2(q^2 - 2pr) = (1 - 2q)^2 - 2(q^2 - 2r),$$

$$(1+x)(1+y)(1+z) = 1 + p + q + r = 2 + q + r.$$

So we need to show that

$$3 - 2(1 - 2q) + (1 - 2q)^2 - 2(q^2 - 2r) \leq 2 + q + r,$$

i.e.

$$3 - 2 + 4q + 1 - 4q + 4q^2 - 2q^2 + 4r \leq 2 + q + r$$

$$\Leftrightarrow 2q^2 - q + 3r \leq 0.$$

By  $N_1$  and  $N_3$  (Chap. 14) we have

$$3q \leq p^2 = 1, \quad \text{i.e.} \quad q \leq \frac{1}{3}, \quad (1)$$

and

$$pq \geq 9r, \quad \text{i.e.} \quad q \geq 9r, \quad \text{i.e.} \quad r \leq \frac{q}{9}. \quad (2)$$

By (2) we have

$$2q^2 - q + 3r \leq 2q^2 - q + 3\frac{q}{9} = 2q\left(q - \frac{1}{3}\right) \leq 0.$$

The last inequality is true due to (1) and the fact that  $q \geq 0$ , so we are done.  $\blacksquare$

**267** Let  $x, y, z$  be non-negative real numbers such that  $x^2 + y^2 + z^2 = 1$ . Prove the inequality

$$(1 - xy)(1 - yz)(1 - zx) \geq \frac{8}{27}.$$

*Solution* Let  $p = x + y + z$ ,  $q = xy + yz + zx$ ,  $r = xyz$ . Clearly  $p, q, r \geq 0$ .

Then  $x^2 + y^2 + z^2 = p^2 - 2q$ , and the constraint becomes

$$p^2 - 2q = 1. \quad (1)$$

We can easily show that

$$(1 - xy)(1 - yz)(1 - zx) = 1 - q + pr - r^2.$$

Now the given inequality becomes

$$1 - q + pr - r^2 \geq \frac{8}{27}. \quad (2)$$

By  $N_1 : p^3 - 4pq + 9r \geq 0$  and (1), we have

$$\begin{aligned} p(p^2 - 4q) + 9r &\geq 0 \\ \Leftrightarrow p(1 - 2q) + 9r &\geq 0 \\ \Leftrightarrow 9r &\geq p(2q - 1). \end{aligned} \quad (3)$$

By  $N_4 : p^2 \geq 3q$  and  $p^2 - 2q = 1$  we obtain

$$2q + 1 \geq 3q, \quad \text{i.e. } q \leq 1. \quad (4)$$

From (4) and  $N_3 : pq - 9r \geq 0$  we obtain

$$p \geq pq \geq 9r \quad \Leftrightarrow \quad 9p - 9r \geq 8p \quad \Leftrightarrow \quad p - r \geq \frac{8}{9}p,$$

from which we deduce

$$r(p - r) \geq \frac{8}{9}pr \stackrel{(3)}{\geq} \frac{8}{9}p \frac{p(2q - 1)}{9} = \frac{8p^2(2q - 1)}{81} \stackrel{(1)}{=} \frac{8(2q + 1)(2q - 1)}{81}. \quad (5)$$

Now we have

$$1 - q + pr - r^2 = 1 - q + r(p - r) \geq 1 - q + \frac{8(2q + 1)(2q - 1)}{81}. \quad (6)$$

By (2) and (6), we have that it suffices to show that

$$1 - q + \frac{8(2q + 1)(2q - 1)}{81} \geq \frac{8}{27},$$

which is equivalent to

$$(1 - q)(49 - 32q) \geq 0,$$

which clearly holds, due to (4). ■

**268** Let  $a, b, c \in \mathbb{R}^+$  such that  $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$ . Prove the inequalities:

$$\begin{aligned} 1^\circ \quad & \frac{1}{8a^2+1} + \frac{1}{8b^2+1} + \frac{1}{8c^2+1} \geq 1 \\ 2^\circ \quad & \frac{1}{4ab+1} + \frac{1}{4bc+1} + \frac{1}{4ca+1} \geq \frac{3}{2} \end{aligned}$$

*Solution* Let  $p = a + b + c$ ,  $q = ab + bc + ca$ ,  $r = abc$ .

From  $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$  we deduce

$$(a+1)(b+1) + (b+1)(c+1) + (c+1)(a+1) = 2(a+1)(b+1)(c+1). \quad (1)$$

According to  $I_9$  and  $I_{10}$  (Chap. 14), (1) is equivalent to

$$3 + 2p + q = 2(1 + p + q + r)$$

i.e.

$$q + 2r = 1. \quad (2)$$

1° We easily get that

$$\begin{aligned} (8a^2 + 1)(8b^2 + 1) + (8b^2 + 1)(8c^2 + 1) + (8c^2 + 1)(8a^2 + 1) \\ = 64(q^2 - 2pr) + 16(p^2 - 2q) + 3 \end{aligned}$$

and

$$(8a^2 + 1)(8b^2 + 1)(8c^2 + 1) = 512r^2 + 64(q^2 - 2pr) + 8(p^2 - 2q) + 1.$$

So inequality 1° becomes

$$64(q^2 - 2pr) + 16(p^2 - 2q) + 3 \geq 512r^2 + 64(q^2 - 2pr) + 8(p^2 - 2q) + 1,$$

i.e.

$$8(p^2 - 2q) + 2 \geq 512r^2. \quad (3)$$

Using that  $q^3 \geq 27r^2$  and  $q = 1 - 2r$  we get

$$\begin{aligned} (1 - 2r)^3 \geq 27r^2 &\Leftrightarrow 8r^3 + 15r^2 + 6r - 1 \leq 0 \\ &\Leftrightarrow (8r - 1)(r^2 + 2r + 1) \leq 0, \end{aligned}$$

from where we deduce that

$$8r - 1 \leq 0, \quad \text{i.e.} \quad r \leq \frac{1}{8}. \quad (4)$$

Since  $AM \geq HM$  we have

$$((a+1) + (b+1) + (c+1)) \left( \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \right) \geq 9$$

or

$$2(a + b + c + 3) \geq 9,$$

i.e.

$$p = a + b + c \geq \frac{3}{2}. \quad (5)$$

From  $N_1 : p^2 \geq 3q$  (Chap. 14) it follows that

$$\frac{p^2}{3} \geq q. \quad (6)$$

By (5) and (6) we have

$$8(p^2 - 2q) + 2 \geq 8\left(p^2 - 2\frac{p^2}{3}\right) + 2 = \frac{8}{3}p^2 + 2 \geq \frac{8}{3} \cdot \frac{9}{4} + 2 = 8. \quad (7)$$

From (3) and (7) we have that it suffices to show that

$$8 \geq 512r^2$$

or

$$r \leq \frac{1}{8},$$

which is true according to (4). And we are done.

2° We have

$$(4ab + 1)(4bc + 1) + (4bc + 1)(4ca + 1) + (4ca + 1)(4ab + 1) = 64pr + 8q + 3$$

and

$$(4ab + 1)(4bc + 1)(4ca + 1) = 64r^2 + 16pr + 4q + 1.$$

We need to show that

$$64pr + 8q + 3 \geq \frac{3}{2}(64r^2 + 16pr + 4q + 1)$$

or

$$32pr + 16q + 6 \geq 192r^2 + 48pr + 12q + 3,$$

i.e.

$$192r^2 + 16pr - 4q - 3 \leq 0. \quad (8)$$

By  $N_7 : q^2 \geq 3pr$  (Chap. 14), it follows that  $pr \leq \frac{q^2}{3}$ .

Now since  $q = 1 - 2r$  we get

$$\begin{aligned} 192r^2 + 16pr - 4q - 3 &\leq 192r^2 + 16\frac{q^2}{3} - 4q - 3 \\ &= 192r^2 + 16\frac{(1-2r)^2}{3} - 4(1-2r) - 3 \end{aligned}$$

$$\begin{aligned}
&= \frac{5}{3}(128r^2 - 8r - 1) \\
&= \frac{5}{3} \cdot 128 \left(r - \frac{1}{8}\right) \left(r + \frac{1}{16}\right). \quad (9)
\end{aligned}$$

From (9) and  $r \leq \frac{1}{8}$  it follows that  $192r^2 + 16pr - 4q - 3 \leq 0$ , which means that inequality (8), i.e. inequality  $2^\circ$ , holds. ■

**269** Let  $a, b, c > 0$  be real numbers such that  $ab + bc + ca = 1$ . Prove the inequality

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} - \frac{1}{a+b+c} \geq 2.$$

*Solution* Let  $p = a + b + c$ ,  $q = ab + bc + ca = 1$ ,  $r = abc$ .

The given inequality is equivalent to

$$\frac{(a+b)(b+c) + (b+c)(c+a) + (c+a)(a+b)}{(a+b)(b+c)(c+a)} - \frac{1}{a+b+c} \geq 2. \quad (1)$$

By  $I_5$ ,  $I_6$  (Chap. 14) and (1) we have that it is enough to prove that

$$\frac{p^2 + q}{pq - r} - \frac{1}{p} \geq 2,$$

i.e.

$$\frac{p^2 + 1}{p - r} - \frac{1}{p} \geq 2,$$

which is equivalent as follows

$$\begin{aligned}
p^3 + p - p + r &\geq 2p^2 - 2pr \\
\Leftrightarrow p^3 - 2p^2 + 2pr + r &\geq 0 \\
\Leftrightarrow p^3 - 2p^2 + r(2p + 1) &\geq 0. \quad (2)
\end{aligned}$$

Let

$$f(p) = p^3 - 2p^2 + r(2p + 1). \quad (3)$$

From  $N_4 : p^2 \geq 3q = 3$  (Chap. 14) it follows that  $p \geq \sqrt{3}$ .

If  $p \geq 2$  then clearly  $f(p) \geq 0$ .

Let  $\sqrt{3} \leq p < 2$ .

By  $N_1 : p^3 - 4pq + 9r \geq 0$  we have

$$p^3 - 4p + 9r \geq 0, \quad \text{i.e.} \quad r \geq \frac{4p - p^3}{9}. \quad (4)$$

By (3) and (4) we obtain

$$\begin{aligned} f(p) &= p^3 - 2p^2 + r(2p + 1) \geq p^3 - 2p^2 + \left(\frac{4p - p^3}{9}\right)(2p + 1) \\ &= -2p(p - 2)(p - 1)^2 \geq 0. \end{aligned}$$

The last inequality holds, since  $p < 2$ . So we have proved (2), and we are done. ■

**270** Let  $a, b, c \geq 0$  be real numbers. Prove the inequality

$$\frac{ab + 4bc + ca}{a^2 + bc} + \frac{bc + 4ca + ab}{b^2 + ca} + \frac{ca + 4ab + bc}{c^2 + ab} \geq 6.$$

*Solution* Let  $p = a + b + c$ ,  $q = ab + bc + ca$ ,  $r = abc$ .

Since the given inequality is homogenous we may assume that  $p = 1$ .

After elementary algebraic operations we can easily rewrite the given inequality in the form

$$7pq - 12r^2 \geq 4q^3 - q^2. \quad (1)$$

By  $N_1 : p^3 - 4pq + 9r \geq 0$  (Chap. 14) we have  $9r \geq 4q - 1$  and clearly  $0 \leq q \leq \frac{1}{3}$ .  
So

$$9rq^2 \geq q^2(4q - 1) \Leftrightarrow \frac{9rq}{3} \geq q^2(4q - 1) \Leftrightarrow 3rq \geq q^2(4q - 1). \quad (2)$$

From  $N_3 : pq - 9r \geq 0$  (Chap. 14) it follows that  $q \geq 9r$ , i.e. we have

$$4rq \geq 36r^2 \geq 12r^2. \quad (3)$$

By (2) and (3) we obtain

$$7pq - 12r^2 = 3rq + 4rq - 12r^2 \geq 3rq \geq q^2(4q - 1),$$

i.e. inequality (1) holds, as required. ■

**271** Let  $a, b, c$  be positive real numbers such that  $a + b + c + 1 = 4abc$ . Prove the inequality

$$\frac{1}{a^4 + b + c} + \frac{1}{b^4 + c + a} + \frac{1}{c^4 + a + b} \leq \frac{3}{a + b + c}.$$

*Solution* By the *Cauchy-Schwarz inequality* we have

$$\frac{1}{a^4 + b + c} = \frac{1 + b^3 + c^3}{(a^4 + b + c)(1 + b^3 + c^3)} \leq \frac{1 + b^3 + c^3}{(a^2 + b^2 + c^2)^2}.$$



Similarly we get

$$\frac{1}{b^4 + c + a} \leq \frac{1 + c^3 + b^3}{(a^2 + b^2 + c^2)^2} \quad \text{and} \quad \frac{1}{c^4 + a + b} \leq \frac{1 + a^3 + b^3}{(a^2 + b^2 + c^2)^2}.$$

After adding the last three inequalities we obtain

$$\frac{1}{a^4 + b + c} + \frac{1}{b^4 + c + a} + \frac{1}{c^4 + a + b} \leq \frac{3 + 2(a^3 + b^3 + c^3)}{(a^2 + b^2 + c^2)^2},$$

so it suffices to prove that

$$\frac{3 + 2(a^3 + b^3 + c^3)}{(a^2 + b^2 + c^2)^2} \leq \frac{3}{a + b + c},$$

i.e.

$$3(a^2 + b^2 + c^2)^2 \geq (a + b + c)(3 + 2(a^3 + b^3 + c^3)).$$

Let  $a + b + c = p$ ,  $ab + bc + ca = q$  and  $abc = r$ .

Then since  $a + b + c + 1 = 4abc$ , by  $AM \geq GM$  it follows that

$$4r = a + b + c + 1 \geq 4\sqrt[4]{r}, \quad \text{i.e.} \quad r \geq 1.$$

Now we have

$$\begin{aligned} A &= 3(a^2 + b^2 + c^2)^2 - (a + b + c)(3 + 2(a^3 + b^3 + c^3)) \\ &= 3(p^2 - 2q)^2 - p(3 + 2p(p^2 - 3q) + 6r) \\ &= 3(p^2 - 2q)^2 - 3p - 2p^2(p^2 - 3q) - 6pr \\ &= 3p^4 - 12p^2q + 12q^2 - 3p - 2p^4 + 6p^2q - 6pr \\ &= p^4 - 6p^2q + 12q^2 - 3p - 6pr \\ &= (p^2 - 3q)^2 + q^2 - 3p + 2(q^2 - 3pr). \end{aligned}$$

Since  $r \geq 1$  we have  $q^2 - 3p \geq q^2 - 3pr$  and it follows that

$$A = (p^2 - 3q)^2 + q^2 - 3p + 2(q^2 - 3pr) \geq (p^2 - 3q)^2 + 3(q^2 - 3pr).$$

According to  $N_7 : q^2 - 3pr \geq 0$  we deduce that

$$A \geq (p^2 - 3q)^2 + 3(q^2 - 3pr) \geq 0,$$

as required. ■

**272** Let  $x, y, z > 0$  be real numbers such that  $x + y + z = 1$ . Prove the inequality

$$(x^2 + y^2)(y^2 + z^2)(z^2 + x^2) \leq \frac{1}{32}.$$

*Solution* Let  $p = x + y + z = 1$ ,  $q = xy + yz + zx$ ,  $r = xyz$ .

Then we have

$$\begin{aligned} x^2 + y^2 &= (x + y)^2 - 2xy = (1 - z)^2 - 2xy = 1 - 2z + z^2 - 2xy \\ &= 1 - z - z(1 - z) - 2xy = 1 - z - z(x + y) - 2xy = 1 - z - q - xy. \end{aligned}$$

Analogously we deduce

$$y^2 + z^2 = 1 - x - q - yz \quad \text{and} \quad z^2 + x^2 = 1 - y - q - zx.$$

So the given inequality becomes

$$(1 - z - q - xy)(1 - x - q - yz)(1 - y - q - zx) \leq \frac{1}{32}. \quad (1)$$

After algebraic transformations we find that inequality (1) is equivalent to

$$q^2 - 2q^3 - r(2 + r - 4q) \leq \frac{1}{32}. \quad (2)$$

Assume that  $q \leq \frac{1}{4}$ .

Using  $N_1 : p^3 - 4pq + 9r \geq 0$  (Chap. 14), it follows that

$$9r \geq 4q - 1, \quad \text{i.e.} \quad r \geq \frac{4q - 1}{9},$$

and clearly  $q \leq \frac{1}{3}$ .

It follows that

$$2 + r - 4q \geq 2 + \frac{4q - 1}{9} - 4q = \frac{17 - 32q}{9} \geq \frac{17 - \frac{32}{3}}{9} > 0.$$

So we have

$$\begin{aligned} q^2 - 2q^3 - r(2 + r - 4q) &\leq q^2 - 2q^3 = q^2(1 - 2q) \\ &= \frac{q}{2} \cdot 2q(1 - 2q) \leq \frac{q}{2} \left( \frac{2q + (1 - 2q)}{2} \right)^2 = \frac{q}{8} \leq \frac{1}{32}, \end{aligned}$$

i.e. inequality (2) holds for  $q \leq \frac{1}{4}$ .

We need just to consider the case when  $q > \frac{1}{4}$ .

Let

$$f(r) = q^2 - 2q^3 - r(2 + r - 4q). \quad (3)$$

Clearly  $r \geq \frac{4q-1}{9}$ .

Using  $N_3 : pq - 9r \geq 0$  (Chap. 14) it follows that  $9r \leq q$ , i.e.  $r \leq \frac{q}{9}$ .

We have

$$f'(r) = 4q - 2 - 2r \leq \frac{4}{3} - 2 - 2r \leq 0.$$

This means that  $f$  is a strictly decreasing function on  $(\frac{4q-1}{9}, \frac{q}{9})$ , from which it follows that

$$f(r) \leq f\left(\frac{4q-1}{9}\right) = q^2 - 2q^3 - \frac{1}{81}(4q-1)(17-32q),$$

i.e.

$$f(r) \leq \frac{81(q^2 - 2q^3) - (4q-1)(17-32q)}{81}. \quad (4)$$

Let

$$g(q) = 81(q^2 - 2q^3) - (4q-1)(17-32q). \quad (5)$$

Then

$$g'(q) = -486q^2 + 418q - 100.$$

Since  $\frac{1}{4} < q \leq \frac{1}{3}$ , we get

$$g'(q) = -486q^2 + 418q - 100 < \frac{-486}{16} + \frac{418}{3} - 100 < 0.$$

So  $g$  decreases on  $(1/4, 1/3)$ , i.e. we have

$$g(q) < g\left(\frac{1}{4}\right) = \frac{81}{32}. \quad (6)$$

Finally by (3), (4), (5) and (6) we obtain

$$\begin{aligned} q^2 - 2q^3 - r(2+r-4q) = f(r) &\leq f\left(\frac{4q-1}{9}\right) \\ &= \frac{81(q^2 - 2q^3) - (4q-1)(17-32q)}{81} \\ &= \frac{g(q)}{81} < \frac{g(\frac{1}{4})}{81} = \frac{\frac{81}{32}}{81} = \frac{1}{32}, \end{aligned}$$

as required. ■

**273** Let  $x, y, z \in \mathbb{R}^+$  such that  $x + y + z = 1$ . Prove the inequalities:

$$1 \leq \frac{x}{1-yz} + \frac{y}{1-zx} + \frac{z}{1-xy} \leq \frac{9}{8}.$$

*Solution* Let  $p = x + y + z = 1$ ,  $q = xy + yz + zx$ ,  $r = xyz$ .

We have

$$\begin{aligned}
 & x(1-zx)(1-xy) + y(1-yz)(1-xy) + z(1-zx)(1-yz) \\
 &= x(1-xy-zx+x^2yz) + y(1-xy-yz+y^2xz) + z(1-zx-zy+z^2xy) \\
 &= x+y+z-x^2(y+z)-y^2(z+x)-z^2(x+y)+x^3yz+y^3zx+z^3xy \\
 &= p-x^2(p-x)-y^2(p-y)-z^2(p-z)+xyz(x^2+y^2+z^2) \\
 &= p-(p-xyz)(x^2+y^2+z^2)+x^3+y^3+z^3 \\
 &= p-(p-r)(p^2-2q)+p(p^2-3q)+3r \\
 &= 1-(1-r)(1-2q)+1-3q+3r; \tag{1}
 \end{aligned}$$

also we have

$$\begin{aligned}
 (1-xy)(1-yz)(1-zx) &= (1-xy-yz+y^2xz)(1-zx) \\
 &= 1-zx-xy-yz+x^2yz+y^2zx+z^2xy-x^2y^2z^2 \\
 &= 1-q+pr-r^2=1-q+r-r^2. \tag{2}
 \end{aligned}$$

By (1) and (2) we have that the left inequality is equivalent to

$$1-q+r-r^2 \leq 1-(1-r)(1-2q)+1-3q+3r \Leftrightarrow r-2q+3 \geq 0. \tag{3}$$

Using  $N_5 : p^3 \geq 27r$  (Chap. 14), it follows that  $r \leq \frac{1}{27}$ .

Also by  $N_1 : p^3 - 4pq + 9r \geq 0$  (Chap. 14), we have  $q \leq \frac{9r+1}{4}$ .

Now we deduce

$$\begin{aligned}
 r-2q+3 &\geq r-2\frac{9r+1}{4}+3 = \frac{4r-18r-2+12}{4} = \frac{10-14r}{4} = \frac{5-7r}{2} \\
 &\geq \frac{5-\frac{7}{27}}{2} > 0,
 \end{aligned}$$

i.e. inequality (3) holds.

We need to show the right side inequality from (1), which, using identities (1) and (2) is

$$9r^2 + 23r + q - 16qr \leq 1. \tag{4}$$

Let us denote  $f(r) = 9r^2 + r(23 - 16q) + q$ .

By  $N_7 : q^2 \geq 3pr = 3r$  (Chap. 14), it follows that

$$r \leq \frac{q^2}{3}, \quad \text{i.e.} \quad 0 \leq r \leq \frac{q^2}{3}.$$

We have

$$f'(r) = 18r + 23 - 16q. \tag{5}$$

Using  $N_4 : p^2 \geq 3q$  (Chap. 14), it follows that  $q \leq \frac{1}{3}$ .

By (5) we have

$$f'(r) = 18r + 23 - 16q \geq 18r + 23 - \frac{16}{3} > 0,$$

i.e.  $f$  increases on  $(0, \frac{q^2}{3})$ , where  $q \leq \frac{1}{3}$ .

So we obtain  $f(r) \leq f(\frac{q^2}{3})$ .

It suffices to show that  $f(\frac{q^2}{3}) \leq 1$ .

We have  $f(\frac{q^2}{3}) = q^4 - \frac{16}{3}q^3 + \frac{23}{3}q^2 + q$ .

Now we get

$$\begin{aligned} f\left(\frac{q^2}{3}\right) &\leq 1 \\ \Leftrightarrow \quad q^4 - \frac{16}{3}q^3 + \frac{23}{3}q^2 + q - 1 &\leq 0 \\ \Leftrightarrow \quad (3q - 1)(q^3 - 5q^2 + 6q + 3) &\leq 0 \\ \Leftrightarrow \quad (3q - 1)(q(q - 2)(q - 3) + 3) &\leq 0, \end{aligned}$$

which clearly holds since  $0 \leq q \leq \frac{1}{3}$ . This complete the proof. ■

**274** Let  $x, y, z \in \mathbb{R}^+$ , such that  $xyz = 1$ . Prove the inequality

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} + \frac{1}{(1+z)^2} + \frac{2}{(1+x)(1+y)(1+z)} \geq 1.$$

*Solution* Let  $x + y + z = p$ ,  $xy + yz + zx = q$  and  $xyz = r = 1$ .

The given inequality becomes

$$\begin{aligned} (1+x)^2(1+y)^2 + (1+y)^2(1+z)^2 + (1+z)^2(1+x)^2 + 2(1+x)(1+y)(1+z) \\ \geq (1+x)^2(1+y)^2(1+z)^2. \end{aligned} \tag{1}$$

By  $I_9$  and  $I_{11}$  (Chap. 14), we have

$$(1+x)(1+y)(1+z) = 1 + p + q + r = 2 + p + q$$

and

$$\begin{aligned} (1+x)^2(1+y)^2 + (1+y)^2(1+z)^2 + (1+z)^2(1+x)^2 \\ = (3+2p+q)^2 - 2(3+p)(1+p+q+r) \\ = (3+2p+q)^2 - 2(3+p)(2+p+q). \end{aligned}$$

So inequality (1) becomes

$$(3 + 2p + q)^2 - 2(3 + p)(2 + p + q) + 2(2 + p + q) \geq (2 + p + q)^2$$

$$\Leftrightarrow p^2 \geq 2q + 3.$$

According to  $N_6 : q^3 \geq 27r^2 = 27$  (Chap. 14), it follows that

$$q \geq 3. \quad (2)$$

By  $N_4 : p^2 \geq 3q$  (Chap. 14), we obtain

$$p^2 \geq 3q = 2q + q \stackrel{(2)}{\geq} 2q + 3,$$

as required. ■

**275** Let  $a, b, c \geq 0$  such that  $a + b + c = 1$ . Prove the inequalities:

- 1°  $ab + bc + ca \leq a^3 + b^3 + c^3 + 6abc$   
 2°  $a^3 + b^3 + c^3 + 6abc \leq a^2 + b^2 + c^2$   
 3°  $a^2 + b^2 + c^2 \leq 2(a^3 + b^3 + c^3) + 3abc$ .

*Solution* Let  $p = a + b + c = 1, q = ab + bc + ca, r = abc$ .

1° Using  $I_2 : a^3 + b^3 + c^3 = p(p^2 - 3q) + 3r = 1 - 3q + 3r$  we have that inequality 1° is equivalent to

$$q \leq 1 - 3q + 3r + 6r \Leftrightarrow 9r + 1 \geq 4q,$$

which is true since  $N_1$  (Chap. 14).

2° Using  $I_1 : a^2 + b^2 + c^2 = p^2 - 2q = 1 - 2q$  we get the equivalent form

$$1 - 3q + 9r \leq 1 - 2q \Leftrightarrow 9r \leq q$$

which is true since  $N_3$  (Chap. 14).

3° The given inequality is equivalent to

$$1 - 2q \leq 2(1 - 3q + 3r) + 3r \Leftrightarrow 4q \leq 1 + 9r,$$

which is true since  $N_1$  (Chap. 14). ■

**276** Let  $x, y, z \geq 0$  be real numbers such that  $xy + yz + zx + xyz = 4$ . Prove the inequality

$$3(x^2 + y^2 + z^2) + xyz \geq 10.$$

*Solution* Let  $p = x + y + z = 1, q = xy + yz + zx, r = xyz$ .

The given inequality becomes

$$3(p^2 - 2q) + r \geq 10, \quad \text{with constraint } q + r = 4.$$

So it is enough to show that

$$3p^2 - 6q + 4 - q \geq 10, \quad \text{i.e.} \quad 3p^2 - 7q - 6 \geq 0. \quad (1)$$

Applying  $N_1 : p^3 - 4pq + 9r \geq 0$  (Chap. 14), and since  $q + r = 4$  we deduce

$$p^3 - 4pq + 9(4 - q) \geq 0, \quad \text{i.e.} \quad q \leq \frac{p^3 + 36}{4p + 9}.$$

So

$$3p^2 - 7q - 6 \geq 3p^2 - 7 \frac{p^3 + 36}{4p + 9} - 6 = \frac{(p - 3)(5p^2 + 42p + 102)}{4p + 9}. \quad (2)$$

Applying  $AM \geq GM$  we obtain

$$\begin{aligned} 4 &= xy + yz + zx + xyz \geq 4\sqrt[4]{(xyz)^3} \\ &\Leftrightarrow 1 \geq xyz. \end{aligned} \quad (3)$$

Also  $(x + y + z)^2 \geq 3(xy + yz + zx)$ , so we deduce

$$p = x + y + z \geq \sqrt{3(4 - xyz)} \stackrel{(3)}{\geq} \sqrt{3(4 - 1)} = 3.$$

Finally by using (2) we obtain that  $3p^2 - 7q - 6 \geq 0$ , i.e. inequality (1) holds. ■

**277** Let  $a, b, c \in \mathbb{R}^+$ . Prove the inequality

$$x^4(y + z) + y^4(z + x) + z^4(x + y) \leq \frac{1}{12}(x + y + z)^5.$$

*Solution* Let  $p = a + b + c$ ,  $q = ab + bc + ca$ ,  $r = abc$ .

Since the given inequality is homogenous, without loss of generality we may assume that  $p = 1$ .

We have

$$\begin{aligned} x^4(y + z) + y^4(z + x) + z^4(x + y) &= x^3(xy + xz) + y^3(yz + yx) + z^3(zx + zy) \\ &= x^3(q - yz) + y^3(q - zx) + z^3(q - xy) \\ &= q(x^3 + y^3 + z^3) - xyz(x^2 + y^2 + z^2) \\ &= q(p(p^2 - 3q) + 3r) - r(p^2 - 2q) \\ &= q(1 - 3q + 3r) - r(1 - 2q) \\ &= q(1 - 3q) + r(5q - 1). \end{aligned}$$

Now the given inequality becomes

$$q(1 - 3q) + r(5q - 1) \leq \frac{1}{12}. \quad (1)$$

From  $3q \leq p^2$  it follows that

$$q \leq \frac{1}{3}. \quad (2)$$

If  $q \leq \frac{1}{5}$  then  $r(5q - 1) \leq 0$ , so we have

$$q(1 - 3q) + r(5q - 1) \leq q(1 - 3q) = \frac{1}{3}(1 - 3q) \cdot 3q \stackrel{G \leq A}{\leq} \frac{1}{3} \left( \frac{(1 - 3q) + 3q}{2} \right)^2 = \frac{1}{12},$$

i.e. inequality (1) holds.

Let

$$q > \frac{1}{5}, \quad (3)$$

i.e. let  $q \in (1/5, 1/3]$  and denote

$$f(q) = q(1 - 3q) + 5rq - r.$$

Then

$$f'(q) = 1 - 6q + 5r. \quad (4)$$

Using  $N_3 : pq \geq 9r$  (Chap. 14), we get

$$q \geq 9r. \quad (5)$$

Now according to (3), (4) and (5) we deduce

$$f'(q) = 1 - 6q + 5r \leq 1 - 6q + \frac{5}{9}q = 1 - \frac{49}{9}q < 1 - \frac{49}{9} \cdot \frac{1}{5} < 0,$$

i.e.  $f$  is strictly decreasing on  $q \in (1/5, 1/3]$ , so it follows that  $f(q) < f(\frac{1}{5})$ , i.e. we deduce that

$$q(1 - 3q) + r(5q - 1) < \frac{1}{5} \left( 1 - \frac{3}{5} \right) + r \left( 5 \frac{1}{5} - 1 \right) = \frac{2}{25} < \frac{1}{12},$$

as required. ■

**278** Let  $a, b, c \in \mathbb{R}^+$  such that  $a + b + c = 1$ . Prove the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 48(ab + bc + ca) \geq 25.$$



*Solution* Setting  $ab + bc + ca = \frac{1-q^2}{3} \geq 0$ ,  $q \geq 0$ , it follows that  $q \in [0, 1]$ .

We have

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 48(ab + bc + ca) &= \frac{ab + bc + ca}{abc} + 48(ab + bc + ca) \\ &= \frac{1-q^2}{3r} + 16(1-q^2). \end{aligned}$$

So it suffices to show that

$$\frac{1-q^2}{3r} + 16(1-q^2) \geq 25.$$

Due to Theorem 15.1 (Chap. 15) we have

$$\begin{aligned} \frac{1-q^2}{3r} + 16(1-q^2) &\geq 27 \frac{1-q^2}{3(1-q)^2(1+2q)} + 16(1-q^2) \\ &= 9 \frac{1+q}{(1-q)(1+2q)} + 16(1-q^2) \\ &= \frac{2q^2(4q-1)^2}{(1-q)(1+2q)} + 25 \geq 25. \end{aligned}$$

Equality occurs if and only if  $(a, b, c) = (1/3, 1/3, 1/3)$  or  $(a, b, c) = (1/2, 1/4, 1/4)$  (up to permutation). ■

**279** Let  $a, b, c$  be non-negative real numbers such that  $a + b + c = 2$ . Prove the inequality

$$a^4 + b^4 + c^4 + abc \geq a^3 + b^3 + c^3.$$

*Solution* Applying *Schur's inequality* (fourth degree) we have that

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq a^3(b + c) + b^3(c + a) + c^3(a + b),$$

i.e.

$$2(a^4 + b^4 + c^4) + abc(a + b + c) \geq (a^3 + b^3 + c^3)(a + b + c)$$

from which, using the initial condition, we obtain the result as required.

Equality holds iff  $a = b = c = 2/3$  or  $a = b = 1, c = 0$  (over all permutations). ■

**280** Let  $a, b, c$  be non-negative real numbers. Prove the inequality

$$2(a^2 + b^2 + c^2) + abc + 8 \geq 5(a + b + c).$$

*Solution* We'll use *Schur's inequality*, i.e.

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x+y) + yz(y+z) + zx(z+x), \quad \text{for all } x, y, z \geq 0.$$

By  $AM \geq GM$  and  $QM \geq AM$  we have

$$\begin{aligned} & 6(2(a^2 + b^2 + c^2) + abc + 8 - 5(a + b + c)) \\ &= 12(a^2 + b^2 + c^2) + 6abc + 48 - 30(a + b + c) \\ &= 12(a^2 + b^2 + c^2) + 3(2abc + 1) + 45 - 30(a + b + c) \\ &\geq 12(a^2 + b^2 + c^2) + 9\sqrt[3]{(abc)^2} + 45 - 5((a + b + c)^2 + 9) \\ &= \frac{9abc}{\sqrt[3]{abc}} + 3(a^2 + b^2 + c^2) - 6(ab + bc + ca) \\ &\quad + 2((a - b)^2 + (b - c)^2 + (c - a)^2) \\ &\geq \frac{9abc}{\sqrt[3]{abc}} + 3(a^2 + b^2 + c^2) - 6(ab + bc + ca) \\ &\geq \frac{27abc}{a + b + c} + 3(a + b + c)^2 - 12(ab + bc + ca) \\ &= \frac{3}{a + b + c} (9abc + (a + b + c)^3 - 4(ab + bc + ca)(a + b + c)) \\ &= \frac{3}{a + b + c} (a^3 + b^3 + c^3 + 3abc - ab(a + b) + bc(b + c) + ca(c + a)) \geq 0. \end{aligned}$$

And we are done. Equality holds iff  $a = b = c = 1$ . ■

**281** Let  $a, b, c$  be non-negative real numbers. Prove the inequality

$$a^3 + b^3 + c^3 + 4(a + b + c) + 9abc \geq 8(ab + bc + ca).$$

*Solution* We'll use *Schur's inequality*, i.e. for all  $a, b, c \geq 0$  we have

$$a^3 + b^3 + c^3 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2).$$

By  $AM \geq GM$  we have

$$4(a + b + c) + \frac{4(ab + bc + ca)^2}{(a + b + c)} \geq 8(ab + bc + ca).$$

So it suffices to prove that

$$a^3 + b^3 + c^3 + 9abc \geq \frac{4(ab + bc + ca)^2}{(a + b + c)}.$$

The previous inequality is equivalent to

$$a^4 + b^4 + c^4 + abc(a + b + c) + ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) \geq 4(a^2b^2 + b^2c^2 + c^2a^2).$$

Applying *Schur's inequality* and  $AM \geq GM$  we obtain

$$\begin{aligned} a^4 + b^4 + c^4 + abc(a + b + c) + ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) \\ \geq 2(ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2)) \\ \geq 2(ab(2ab) + bc(2bc) + ca(2ca)) = 4(a^2b^2 + b^2c^2 + c^2a^2), \end{aligned}$$

as required.

Equality holds iff  $a = b = c = 1$  or  $a = b = 2, c = 0$  (up to permutation). ■

**282** Let  $a, b, c$  be non-negative real numbers. Prove the inequality

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq a + b + c.$$

*Solution* Applying the *Cauchy-Schwarz inequality* (Corollary 4.3) we deduce

$$\begin{aligned} \frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \\ = \frac{a^4}{a(b^2 - bc + c^2)} + \frac{b^4}{b(c^2 - ca + a^2)} + \frac{c^4}{c(a^2 - ab + b^2)} \\ \geq \frac{(a^2 + b^2 + c^2)^2}{a(b^2 - bc + c^2) + b(c^2 - ca + a^2) + c(a^2 - ab + b^2)}. \end{aligned}$$

So it suffices to prove that

$$(a^2 + b^2 + c^2)^2 \geq (a(b^2 - bc + c^2) + b(c^2 - ca + a^2) + c(a^2 - ab + b^2))(a + b + c).$$

The previous inequality is equivalent to

$$\begin{aligned} a^4 + b^4 + c^4 + 2(a^2b^2 + b^2c^2 + c^2a^2) \\ \geq (a + b + c)(a^2(b + c) + b^2(c + a) + c^2(a + b)) - 3abc(a + b + c) \end{aligned}$$

or

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq a^3(b + c) + b^3(c + a) + c^3(a + b),$$

and it is *Schur's inequality* (fourth degree).

Equality holds iff  $a = b = c$  or  $a = b, c = 0$  (up to permutation). ■

**283** Let  $a, b, c$  be non-negative real numbers such that  $a + b + c = 2$ . Prove the inequality

$$a^3 + b^3 + c^3 + \frac{15abc}{4} \geq 2.$$

*Solution* Applying *Schur's inequality* we have that the following inequality holds

$$a^3 + b^3 + c^3 + \frac{15abc}{4} \geq \frac{(a+b+c)^3}{4},$$

from which we obtain the required inequality. Equality holds iff  $a = b = c = 2/3$  or  $a = b = 1, c = 0$  (over all permutations). ■

**284** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{a^2 + bc}{a^2(b+c)} + \frac{b^2 + ca}{b^2(c+a)} + \frac{c^2 + ab}{c^2(a+b)} \geq ab + bc + ca.$$

*Solution* We'll show that

$$\frac{a^2 + bc}{a^2(b+c)} + \frac{b^2 + ca}{b^2(c+a)} + \frac{c^2 + ab}{c^2(a+b)} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}. \quad (1)$$

We have

$$\frac{a^2 + bc}{a^2(b+c)} - \frac{1}{a} = \frac{(a-b)(a-c)}{a^2(b+c)}.$$

Analogously we deduce

$$\frac{b^2 + ca}{b^2(c+a)} - \frac{1}{b} = \frac{(b-c)(b-a)}{b^2(c+a)} \quad \text{and} \quad \frac{c^2 + ab}{c^2(a+b)} - \frac{1}{c} = \frac{(c-a)(c-b)}{c^2(a+b)}.$$

Applying the previous identities and Corollary 12.1 from *Schur's inequality* we obtain (1). From (1) and  $abc = 1$  we obtain the required inequality.

Equality holds iff  $a = b = c = 1$ . ■

**285** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$\frac{a^3 + abc}{(b+c)^2} + \frac{b^3 + abc}{(c+a)^2} + \frac{c^3 + abc}{(a+b)^2} \geq \frac{3}{2}.$$

*Solution* We'll show that

$$\frac{a^3 + abc}{(b+c)^2} + \frac{b^3 + abc}{(c+a)^2} + \frac{c^3 + abc}{(a+b)^2} \geq \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}. \quad (1)$$

We have

$$\frac{a^3 + abc}{(b+c)^2} - \frac{a^2}{b+c} = \frac{a}{(b+c)^2} (a-b)(a-c);$$

analogously we get the other two identities.

Now (1) is equivalent to

$$\frac{a}{(b+c)^2}(a-b)(a-c) + \frac{b}{(c+a)^2}(b-c)(b-a) + \frac{c}{(a+b)^2}(c-a)(c-b) \geq 0. \quad (2)$$

Assume that  $a \geq b \geq c$ .

Then we easily deduce that  $\frac{a}{(b+c)^2} \geq \frac{b}{(c+a)^2} \geq \frac{c}{(a+b)^2}$ , and the correctness of (2) will follow from Corollary 12.1 of *Schur's inequality*.

Furthermore, we'll show that

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{\sqrt{3(a^2+b^2+c^2)}}{2}. \quad (3)$$

Assume that  $a \geq b \geq c$ . Then

$$a^2 \geq b^2 \geq c^2 \quad \text{and} \quad \frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}.$$

Applying *Chebyshev's inequality* and  $AM \geq HM$  we get

$$\begin{aligned} \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} &\geq \frac{1}{3}(a^2+b^2+c^2) \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) \\ &\geq \frac{1}{3}(a^2+b^2+c^2) \frac{9}{2(a+b+c)} \\ &\geq \frac{3(a^2+b^2+c^2)}{2\sqrt{3(a^2+b^2+c^2)}} = \frac{\sqrt{3(a^2+b^2+c^2)}}{2}. \end{aligned}$$

So inequality (3) is proved.

By (1), (3) and the initial condition we obtain

$$\begin{aligned} \frac{a^3+abc}{(b+c)^2} + \frac{b^3+abc}{(c+a)^2} + \frac{c^3+abc}{(a+b)^2} &\geq \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{\sqrt{3(a^2+b^2+c^2)}}{2} \\ &= \frac{3}{2}. \end{aligned}$$

Equality holds iff  $a = b = c = 1$ . ■

**286** Let  $a, b, c$  be positive real numbers such that  $a^4 + b^4 + c^4 = 3$ . Prove the inequality

$$\frac{1}{4-ab} + \frac{1}{4-bc} + \frac{1}{4-ca} \leq 1.$$

*Solution 1* After clearing denominators the given inequality becomes

$$48 - 8 \sum_{sym} ab + abc \sum_{sym} a \leq 64 - 16 \sum_{sym} ab + 4abc \sum_{sym} a - a^2 b^2 c^2,$$

i.e.

$$16 + 3abc(a + b + c) \geq a^2b^2c^2 + 8(ab + bc + ca). \quad (1)$$

Applying *Schur's inequality* we have that

$$(a^3 + b^3 + c^3 + 3abc)(a + b + c) \geq (ab(a + b) + bc(b + c) + ca(c + a))(a + b + c),$$

and since  $a^4 + b^4 + c^4 = 3$  we deduce

$$3 + 3abc(a + b + c) \geq (ab + ac)^2 + (ac + bc)^2 + (bc + ab)^2. \quad (2)$$

Using  $AM \geq GM$  we get

$$(ab + ac)^2 + (ac + bc)^2 + (bc + ab)^2 + 12 \geq 8(ab + bc + ca). \quad (3)$$

Now from (2) and (3) we deduce

$$15 + 3abc(a + b + c) \geq 8(ab + bc + ca). \quad (4)$$

Once more we apply  $AM \geq GM$ , and we get

$$3 = a^4 + b^4 + c^4 \geq 3\sqrt[3]{(abc)^4}, \quad \text{i.e.} \quad 1 \geq abc$$

or

$$1 \geq a^2b^2c^2. \quad (5)$$

Finally using (4) and (5) we get inequality (1).

Equality holds iff  $a = b = c = 1$ . ■

*Solution 2* Let  $x = ab$ ,  $y = bc$  and  $z = ac$ . The given inequality is equivalent to

$$\frac{1-x}{4-x} + \frac{1-y}{4-y} + \frac{1-z}{4-z} \geq 0$$

or

$$\frac{1-x^2}{4+3x-x^2} + \frac{1-y^2}{4+3y-y^2} + \frac{1-z^2}{4+3z+z^2} \geq 0.$$

Notice that

$$x^2 + y^2 + z^2 = (ab)^2 + (bc)^2 + (ca)^2 \leq a^4 + b^4 + c^4 = 3.$$

Assume that  $x \geq y \geq z$ . Then clearly

$$1 - x^2 \leq 1 - y^2 \leq 1 - z^2 \quad \text{and} \quad \frac{1}{4+3x-x^2} \leq \frac{1}{4+3y-y^2} \leq \frac{1}{4+3z+z^2}.$$

Therefore by *Chebyshev's inequality* we obtain

$$\begin{aligned} & 3 \left( \frac{1-x^2}{4+3x-x^2} + \frac{1-y^2}{4+3y-y^2} + \frac{1-z^2}{4+3z+z^2} \right) \\ & \geq (1-x^2 + 1-y^2 + 1-z^2) \left( \frac{1}{4+3x-x^2} + \frac{1}{4+3y-y^2} + \frac{1}{4+3z+z^2} \right) \\ & \geq 0, \end{aligned}$$

as required.

Equality occurs iff  $a = b = c = 1$ . ■

**287** Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 3$ . Prove the inequality

$$(a^3 - a + 5)(b^5 - b^3 + 5)(c^7 - c^5 + 5) \geq 125.$$

*Solution* For any real number  $x$ , the numbers  $x - 1$ ,  $x^2 - 1$ ,  $x^3 - 1$  and  $x^5 - 1$  are of the same sign.

Therefore

$$(x - 1)(x^2 - 1) \geq 0, \quad (x^2 - 1)(x^3 - 1) \geq 0 \quad \text{and} \quad (x^2 - 1)(x^5 - 1) \geq 0,$$

i.e.

$$a^3 - a^2 - a + 1 \geq 0,$$

$$b^5 - b^3 - b^2 + 1 \geq 0,$$

$$c^7 - c^5 - c^2 + 1 \geq 0.$$

So it follows that

$$a^3 - a + 5 \geq a^2 + 4, \quad b^5 - b^3 + 5 \geq b^2 + 4 \quad \text{and} \quad c^7 - c^5 + 5 \geq c^2 + 4.$$

Multiplying these inequalities gives us

$$(a^3 - a + 5)(b^5 - b^3 + 5)(c^7 - c^5 + 5) \geq (a^2 + 4)(b^2 + 4)(c^2 + 4). \quad (1)$$

We'll prove that

$$(a^2 + 4)(b^2 + 4)(c^2 + 4) \geq 25(ab + bc + ca + 2). \quad (2)$$

We have

$$\begin{aligned} & (a^2 + 4)(b^2 + 4)(c^2 + 4) \\ & = a^2 b^2 c^2 + 4(a^2 b^2 + b^2 c^2 + c^2 a^2) + 16(a^2 + b^2 + c^2) + 64 \end{aligned}$$

$$\begin{aligned}
&= a^2b^2c^2 + (a^2 + b^2 + c^2) + 2 + 4(a^2b^2 + b^2c^2 + c^2a^2 + 3) \\
&\quad + 15(a^2 + b^2 + c^2) + 50.
\end{aligned} \tag{3}$$

By the obvious inequalities

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0 \quad \text{and} \quad (ab-1)^2 + (bc-1)^2 + (ca-1)^2 \geq 0$$

we obtain

$$a^2 + b^2 + c^2 \geq ab + bc + ca, \tag{4}$$

$$a^2b^2 + b^2c^2 + c^2a^2 + 3 \geq 2(ab + bc + ca). \tag{5}$$

We'll prove that

$$a^2b^2c^2 + (a^2 + b^2 + c^2) + 2 \geq 2(ab + bc + ca). \tag{6}$$

**Lemma 21.7** *Let  $x, y, z > 0$ . Then*

$$3xyz + x^3 + y^3 + z^3 \geq 2((xy)^{3/2} + (yz)^{3/2} + (zx)^{3/2}).$$

*Proof* By Schur's inequality and  $AM \geq GM$  we have

$$\begin{aligned}
x^3 + y^3 + z^3 + 3xyz &\geq (x^2y + y^2x) + (z^2y + y^2z) + (x^2z + z^2x) \\
&\geq 2((xy)^{3/2} + (yz)^{3/2} + (zx)^{3/2}). \quad \square
\end{aligned}$$

By Lemma 21.7 for  $x = a^{2/3}, y = b^{2/3}, z = c^{2/3}$  we deduce

$$3(abc)^{2/3} + a^2 + b^2 + c^2 \geq 2(ab + bc + ca).$$

Therefore it suffices to prove that

$$a^2b^2c^2 + 2 \geq 3(abc)^{2/3},$$

which follows immediately by  $AM \geq GM$ .

Thus we have proved inequality (6).

Now by (3), (4), (5) and (6) we obtain inequality (2).

Finally by (1), (2) and since  $ab + bc + ca = 3$  we obtain the required inequality.

Equality occurs if and only if  $a = b = c = 1$ . ■

**288** Let  $x, y, z$  be positive real numbers. Prove the inequality

$$\frac{1}{x^2 + xy + y^2} + \frac{1}{y^2 + yz + z^2} + \frac{1}{z^2 + zx + x^2} \geq \frac{9}{(x + y + z)^2}.$$



**Solution** It is true that  $x^2 + xy + y^2 = (x + y + z)^2 - (xy + yz + zx) - (x + y + z)z$ .

Now we have

$$\frac{(x + y + z)^2}{x^2 + xy + y^2} = \frac{1}{1 - \frac{xy + yz + zx}{(x + y + z)^2} - \frac{z}{x + y + z}},$$

i.e.

$$\frac{(x + y + z)^2}{x^2 + xy + y^2} = \frac{1}{1 - (ab + bc + ca) - c}$$

where  $a = \frac{x}{x+y+z}$ ,  $b = \frac{y}{x+y+z}$ ,  $c = \frac{z}{x+y+z}$ .

The given inequality can be written in the form

$$\frac{1}{1-d-c} + \frac{1}{1-d-b} + \frac{1}{1-d-a} \geq 9 \quad (1)$$

where  $a, b, c$  are positive real numbers such that

$$a + b + c = 1 \quad \text{and} \quad d = ab + bc + ca.$$

After clearing the denominators, inequality (1) becomes

$$9d^3 - 6d^2 - 3d + 1 + 9abc \geq 0 \quad \text{or} \quad d(3d - 1)^2 + (1 - 4d + 9abc) \geq 0,$$

which is true since  $1 - 4d + 9abc \geq 0$  (the last inequality is a direct consequences of *Schur's inequality*). ■

**289** Let  $x, y, z$  be positive real numbers such that  $xyz = x + y + z + 2$ . Prove the inequalities

$$1^\circ \quad xy + yz + zx \geq 2(x + y + z)$$

$$2^\circ \quad \sqrt{x} + \sqrt{y} + \sqrt{z} \leq \frac{3\sqrt{xyz}}{2}.$$

**Solution**  $1^\circ$  The identity  $xyz = x + y + z + 2$  can be rewritten as

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1.$$

Let's denote  $\frac{1}{1+x} = a$ ,  $\frac{1}{1+y} = b$ ,  $\frac{1}{1+z} = c$ .

Then

$$a + b + c = 1 \quad \text{and} \quad x = \frac{b+c}{a}, \quad y = \frac{c+a}{b}, \quad z = \frac{a+b}{c}.$$

Now we have

$$\begin{aligned}
 xy + yz + zx &\geq 2(x + y + z) \\
 \Leftrightarrow \frac{b+c}{a} \cdot \frac{c+a}{b} + \frac{c+a}{b} \cdot \frac{a+b}{c} + \frac{a+b}{c} \cdot \frac{b+c}{a} \\
 &\geq 2\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right) \\
 \Leftrightarrow a^3 + b^3 + c^3 + 3abc &\geq ab(a+b) + bc(b+c) + ca(c+a),
 \end{aligned}$$

which clearly holds (*Schur's inequality*).

2° The given inequality is equivalent to

$$\begin{aligned}
 \frac{1}{\sqrt{yz}} + \frac{1}{\sqrt{zx}} + \frac{1}{\sqrt{xy}} &\leq \frac{3}{2} \\
 \Leftrightarrow \sqrt{\frac{a}{b+c} \cdot \frac{b}{c+a}} + \sqrt{\frac{b}{c+a} \cdot \frac{c}{a+b}} + \sqrt{\frac{c}{a+b} \cdot \frac{a}{b+c}} &\leq \frac{3}{2}. \quad (1)
 \end{aligned}$$

Using  $AM \geq GM$  we have

$$\begin{aligned}
 \sqrt{\frac{a}{b+c} \cdot \frac{b}{c+a}} &\leq \frac{1}{2} \left( \frac{a}{a+c} + \frac{b}{c+b} \right), \\
 \sqrt{\frac{b}{c+a} \cdot \frac{c}{a+b}} &\leq \frac{1}{2} \left( \frac{b}{a+b} + \frac{c}{c+a} \right) \quad \text{and} \\
 \sqrt{\frac{c}{a+b} \cdot \frac{a}{b+c}} &\leq \frac{1}{2} \left( \frac{c}{b+c} + \frac{a}{a+b} \right).
 \end{aligned}$$

Adding the last three inequalities we obtain inequality (1), as required. ■

**290** Let  $x, y, z$  be positive real numbers. Prove the inequality

$$8(x^3 + y^3 + z^3) \geq (x+y)^3 + (y+z)^3 + (z+x)^3.$$

*Solution 1* The given inequality is equivalent to

$$\begin{aligned}
 2(x^3 + y^3 + z^3) &\geq x^2y + x^2z + y^2x + y^2z + z^2x + z^2y \\
 \Leftrightarrow T[3, 0, 0] &\geq T[2, 1, 0], \quad (1)
 \end{aligned}$$

which obviously holds according to *Muirhead's inequality*. ■

*Solution 2* Let  $p = x + y + z, q = xy + yz + zx, r = xyz$ .

Since the given inequality is homogenous we may assume that  $p = 1$ .

Using  $I_2$  we get

$$x^3 + y^3 + z^3 = p(p^2 - 3q) + 3r = 1 - 3q + 3r$$

and

$$\begin{aligned}
 x^2y + x^2z + y^2x + y^2z + z^2x + z^2y &= xy(x+y) + yz(y+z) + zx(z+x) \\
 &= xy(1-z) + yz(1-x) + zx(1-y) \\
 &= xy + yz + zx - 3xyz = q - 3r.
 \end{aligned}$$

Now inequality (1) becomes

$$2(1 - 3q + 3r) \geq q - 3r \quad \Leftrightarrow \quad 2 + 9r \geq 7q,$$

which is true according to  $N_8$ , and we are done. ■

*Solution 3* We can easily deduce that

$$4(x^3 + y^3) - (x + y)^3 = 3(x + y)(x - y)^2 \geq 0, \quad \text{i.e.} \quad 4(x^3 + y^3) \geq (x + y)^3.$$

Analogously we get

$$4(y^3 + z^3) \geq (y + z)^3 \quad \text{and} \quad 4(z^3 + x^3) \geq (z + x)^3.$$

Adding these three inequalities we obtain the result. ■

*Solution 4* According to *Jensen's inequality* for the convex function  $f(x) = x^3$ , we obtain

$$\begin{aligned}
 \frac{1}{2}f(x) + \frac{1}{2}f(y) &\geq f\left(\frac{x+y}{2}\right) \quad \text{or} \quad \frac{x^3 + y^3}{2} \geq \left(\frac{x+y}{2}\right)^3 \\
 \Leftrightarrow \quad 4(x^3 + y^3) &\geq (x + y)^3.
 \end{aligned}$$

Now the solution follows as in the previous solution. ■

**291** Let  $a, b, c$  be non-negative real numbers. Prove the inequality

$$a^3 + b^3 + c^3 + abc \geq \frac{1}{7}(a + b + c)^3.$$

*Solution* We have

$$\begin{aligned}
 (a + b + c)^3 &= a^3 + b^3 + c^3 + 3(a^2(b+c) + b^2(c+a) + c^2(a+b)) + 6abc \\
 &= \frac{T[3, 0, 0]}{2} + 3T[2, 1, 0] + T[1, 1, 1]
 \end{aligned}$$

and

$$a^3 + b^3 + c^3 + abc = \frac{T[3, 0, 0]}{2} + \frac{T[1, 1, 1]}{6}.$$

So we need to prove that

$$7\left(\frac{T[3, 0, 0]}{2} + \frac{T[1, 1, 1]}{6}\right) \geq \frac{T[3, 0, 0]}{2} + 3T[2, 1, 0] + T[1, 1, 1],$$

i.e.

$$3T[3, 0, 0] + \frac{T[1, 1, 1]}{6} \geq 3T[2, 1, 0],$$

which is true according to  $T[3, 0, 0] \geq T[2, 1, 0]$  and  $T[1, 1, 1] \geq 0$  (*Muirhead's theorem*). ■

**292** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ . Prove the inequality

$$a^2 + b^2 + c^2 + 3abc \geq \frac{4}{9}.$$

*Solution* We will normalize as follows

$$9(a + b + c)(a^2 + b^2 + c^2) + 27abc \geq 4(a + b + c)^3$$

which is equivalent to

$$5(a^3 + b^3 + c^3) + 3abc \geq 3(ab(a + b) + bc(b + c) + ca(c + a)). \quad (1)$$

According to *Schur's inequality* we have that

$$a^3 + b^3 + c^3 + 3abc \geq ab(a + b) + bc(b + c) + ca(c + a) \quad (2)$$

and by *Muirhead's theorem* we have that

$$2T[3, 0, 0] \geq 2T[2, 1, 0],$$

i.e.

$$4(a^3 + b^3 + c^3) \geq 2(ab(a + b) + bc(b + c) + ca(c + a)). \quad (3)$$

Adding these two inequalities gives us inequality (1). ■

**293** Let  $a_1, a_2, \dots, a_n$  be positive real numbers. Prove the inequality

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \leq \left(1 + \frac{a_1^2}{a_2}\right) \left(1 + \frac{a_2^2}{a_3}\right) \cdots \left(1 + \frac{a_n^2}{a_1}\right).$$

*Solution* Let  $x_i = \ln a_i$ , then given inequality becomes

$$(1 + e^{x_1})(1 + e^{x_2}) \cdots (1 + e^{x_n}) \leq (1 + e^{2x_1 - x_2})(1 + e^{2x_2 - x_3}) \cdots (1 + e^{2x_n - x_1}).$$

After taking logarithm on the both sides we obtain

$$\ln(1 + e^{x_1}) + \cdots + \ln(1 + e^{x_n}) \leq \ln(1 + e^{2x_1 - x_2}) + \cdots + \ln(1 + e^{2x_n - x_1}).$$

Let consider the sequences  $a : 2x_1 - x_2, 2x_2 - x_3, \dots, 2x_n - x_1$  and  $b : x_1, x_2, \dots, x_n$ .

Since  $f(x) = \ln(1 + e^x)$  is convex function on  $\mathbb{R}$  by *Karamata's inequality* it suffices to prove that  $a$  (ordered in some way) majorizes the sequences  $b$  (ordered in some way), which can be done exactly as in Exercise 12.13, and therefore is left to the reader. ■

**294** Let  $a, b, c, d$  be positive real numbers such that  $abcd = 1$ . Prove the inequality

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1.$$

*Solution 1* First we'll show that for all real numbers  $x$  and  $y$  the following inequality holds

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} \geq \frac{1}{1+xy}.$$

We have

$$\begin{aligned} & \frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} - \frac{1}{1+xy} \\ &= \frac{xy(x^2 + y^2) - x^2y^2 - 2xy + 1}{(1+x)^2(1+y)^2(1+xy)} = \frac{xy(x-y)^2 + (xy-1)^2}{(1+x)^2(1+y)^2(1+xy)} \geq 0. \end{aligned}$$

Now we obtain

$$\begin{aligned} & \frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \\ & \geq \frac{1}{1+ab} + \frac{1}{1+cd} = \frac{1}{1+ab} + \frac{1}{1+1/ab} \\ &= \frac{1}{1+ab} + \frac{ab}{1+ab} = 1. \end{aligned}$$

Equality holds iff  $a = b = c = d = 1$ . ■

*Solution 2* Let

$$\begin{aligned} f(a, b, c, d) &= \frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \quad \text{and} \\ g(a, b, c, d) &= abcd - 1. \end{aligned}$$

Define

$$L = f - \lambda g = \frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} - \lambda(abcd - 1).$$

For the first partial derivatives we have

$$\begin{aligned}\frac{\partial L}{\partial a} &= \frac{-4}{(1+a)^2} - \frac{\lambda}{a} = 0, & \text{i.e. } \lambda &= \frac{-4a}{(1+a)^2}, \\ \frac{\partial L}{\partial b} &= \frac{-4}{(1+b)^2} - \frac{\lambda}{b} = 0, & \text{i.e. } \lambda &= \frac{-4b}{(1+b)^2}, \\ \frac{\partial L}{\partial c} &= \frac{-4}{(1+c)^2} - \frac{\lambda}{c} = 0, & \text{i.e. } \lambda &= \frac{-4c}{(1+c)^2}, \\ \frac{\partial L}{\partial d} &= \frac{-4}{(1+d)^2} - \frac{\lambda}{d} = 0, & \text{i.e. } \lambda &= \frac{-4d}{(1+d)^2}.\end{aligned}$$

So we have  $\frac{-4a}{(1+a)^2} = \frac{-4b}{(1+b)^2} = \frac{-4c}{(1+c)^2} = \frac{-4d}{(1+d)^2} = \lambda$ , from which we get the following system of equations:

$$\begin{aligned}(a-b)(1-ab) &= 0, & (a-c)(1-ac) &= 0, & (a-d)(1-ad) &= 0, \\ (b-c)(1-bc) &= 0, & (b-d)(1-bd) &= 0, & (c-d)(1-cd) &= 0.\end{aligned}$$

Solving this system we get that we must have  $a = b = c = d$ , and using  $abcd = 1$  it follows that  $a = b = c = d = 1$  and then we have

$$f(1, 1, 1, 1) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1.$$

Since  $f(1, 1, 1/2, 2) = \frac{1}{4} + \frac{1}{4} + \frac{1}{9} + \frac{4}{9} = \frac{1}{2} + \frac{5}{9} > 1$ , by *Lagrange's theorem* we conclude that  $f(a, b, c, d) \geq 1$ , as required. ■

**295** Let  $a, b, c, d \geq 0$  be real numbers such that  $a + b + c + d = 4$ . Prove the inequality

$$abc + bcd + cda + dab + (abc)^2 + (bcd)^2 + (cda)^2 + (dab)^2 \leq 8.$$

*Solution* Let us denote

$$f(a, b, c, d) = abc + bcd + cda + dab + (abc)^2 + (bcd)^2 + (cda)^2 + (dab)^2.$$

Because of symmetry we may assume that  $a \geq b \geq c \geq d$ .

We have

$$\begin{aligned}& f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right) - f(a, b, c, d) \\ &= \left(\frac{a-c}{2}\right)^2 \left((b+d) + \left(\left(\frac{a+c}{2}\right)^2 + ac\right)(b^2 + d^2) - 2b^2d^2\right) \\ &\geq \left(\frac{a-c}{2}\right)^2 (4abcd - 2b^2d^2) \geq 0 \quad (abcd \geq b^2d^2).\end{aligned}$$

So

$$f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right) \geq f(a, b, c, d).$$

According to the *SMV theorem* it suffices to show that

$$f(t, t, t, d) \leq 8$$

where  $3t + d = 4$  and clearly  $0 \leq t \leq \frac{4}{3}$ .

We have

$$\begin{aligned} f(t, t, t, d) \leq 8 &\Leftrightarrow t^3 + 3t^2(3 - 3t) + 3t^4(4 - 3t)^2 + t^6 \leq 8 \\ &\Leftrightarrow (t - 1)^2(28t^4 - 16t^3 - 12t^2 - 8) \leq 0. \end{aligned}$$

So it is enough to show that  $28t^4 - 16t^3 - 12t^2 - 8 \leq 0$ , which is easy to prove for  $0 \leq t \leq \frac{4}{3}$ .

Equality holds iff  $a = b = c = d = 1$ . ■

**296** Let  $a, b, c, d \geq 0$  such that  $a + b + c + d = 1$ . Prove the inequality

$$a^4 + b^4 + c^4 + d^4 + \frac{148}{27}abcd \geq \frac{1}{27}.$$

*Solution* Denote  $f(a, b, c, d) = a^4 + b^4 + c^4 + d^4 + \frac{148}{27}abcd - \frac{1}{27}$ .

Since the given inequality is symmetric we may assume that  $a \geq b \geq c \geq d$ .

We have

$$f(a, b, c, d) - f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right) = \left(\frac{7}{8}(a-c)^2 + 3ac - \frac{37}{27}bd\right)(a-b)^2.$$

Since  $ac \geq bd$  it follows that

$$f(a, b, c, d) - f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right) \geq 0,$$

i.e.

$$f(a, b, c, d) \geq f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right).$$

According to the *SMV theorem* it suffices to show that

$$f(t, t, t, d) \geq 0, \quad \text{where } t = \frac{1-d}{3}.$$

We have

$$f(t, t, t, d) = \frac{(1-d)^4}{27} + d^4 + \frac{148d(1-d)^3}{729} - \frac{1}{27} = \frac{2d(4d-1)^2(19d+20)}{729} \geq 0.$$

Equality occurs if and only if  $a = b = c = d = 1/4$  or  $a = b = c = 1/3, d = 0$  (up to permutation). ■

**297** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$a^2b^2 + b^2c^2 + c^2a^2 \leq a + b + c.$$

*Solution* Without loss of generality we may assume that  $a \leq b \leq c$ . Then clearly  $a \leq 1$  and  $b^2 + c^2 \geq 2$ , from which it follows that  $b + c \geq \sqrt{2}$ .

Let  $f(a, b, c) = a + b + c - a^2b^2 - b^2c^2 - c^2a^2$ . Then we have

$$\begin{aligned} f(a, b, c) - f\left(a, \sqrt{\frac{b^2 + c^2}{2}}, \sqrt{\frac{b^2 + c^2}{2}}\right) \\ = (b - c)^2 \left( \frac{(b + c)^2}{4} - \frac{1}{b + c + \sqrt{2(b^2 + c^2)}} \right) \geq \left( \frac{2}{4} - \frac{1}{2 + \sqrt{2}} \right) (b - c)^2 \geq 0. \end{aligned}$$

Thus

$$f(a, b, c) \geq f\left(a, \sqrt{\frac{b^2 + c^2}{2}}, \sqrt{\frac{b^2 + c^2}{2}}\right).$$

By the *SMV theorem* it suffices to prove that  $f(a, t, t) \geq 0$ , when  $a^2 + 2t^2 = 3$ .

We have

$$\begin{aligned} f(a, t, t) &\geq 0 \\ \Leftrightarrow a + \sqrt{2(3 - a^2)} &\geq a^2(3 - a^2) + \frac{1}{4}(3 - a^2)^2 \\ \Leftrightarrow (a - 1)^2 \left( \frac{3}{4}(a + 1)^2 - \frac{3}{3 - a + \sqrt{2(3 - a^2)}} \right) &\geq 0. \end{aligned} \quad (1)$$

Since  $a \leq 1$  it follows that

$$\frac{3}{3 - a + \sqrt{2(3 - a^2)}} \leq \frac{3}{4} \leq \frac{3}{4}(a + 1)^2.$$

Therefore inequality (1) is true, and we are done.

Equality occurs iff  $a = b = c = 1$ . ■

**298** Let  $a, b, c, d \geq 0$  be real numbers such that  $a + b + c + d = 4$ . Prove the inequality

$$(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) \geq (1 + a)(1 + b)(1 + c)(1 + d).$$



*Solution* Let

$$f(a, b, c, d) = (1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) - (1 + a)(1 + b)(1 + c)(1 + d),$$

and assume that  $a \leq b \leq c \leq d$  (symmetry).

We'll show that

$$f(a, b, c, d) \geq f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right).$$

Clearly

$$a + c \leq 2, \tag{1}$$

so it follows that

$$\begin{aligned} f(a, b, c, d) - f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right) \\ = (1 + b^2)(1 + d^2) \left( (1 + a^2)(1 + c^2) - \left(1 + \left(\frac{a+c}{2}\right)^2\right)^2 \right) \\ + (1 + b)(1 + d) \left( \left(1 + \frac{a+c}{2}\right)^2 - (1 + a)(1 + c) \right). \end{aligned}$$

Since

$$(1 + a^2)(1 + c^2) - \left(1 + \left(\frac{a+c}{2}\right)^2\right)^2 = (a - c)^2 \left(\frac{1}{2} - \frac{(a+c)^2 + 4ac}{16}\right) \geq 0$$

(this inequality follows by (1)) and by  $AM \geq GM$  it follows that

$$(1 + a)(1 + c) \leq \left(1 + \frac{a+c}{2}\right)^2.$$

So

$$\begin{aligned} f(a, b, c, d) - f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right) &\geq 0, \quad \text{i.e.} \\ f(a, b, c, d) &\geq f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right). \end{aligned}$$

According to the *SMV theorem* it suffices to show that

$$f(t, t, t, d) \geq 0$$

where  $3t + d = 4$  i.e.  $d = 4 - 3t$ .

We have

$$\begin{aligned}
 f(t, t, t, d) &= (1+t^2)^3(1+(4-3t)^2) - (1+t)^3(5-3t) \\
 &= 9t^8 - 24t^7 + 44t^6 - 72t^5 + 81t^4 - 68t^3 - 54t^2 - 36t + 12 \\
 &= (t-1)^2(9t^6 - 6t^5 + 23t^4 - 20t^3 + 18t^2 - 12t + 12) \\
 &= (t-1)^2(t^4(3t-1)^2 + 2t^4 + 5t^2(2t-1)^2 + 10t^2 + 3(t-2)^2) \geq 0.
 \end{aligned}$$

Equality holds if and only if  $a = b = c = d = 1$ . ■

**299** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{6}{a+b+c} \geq 5.$$

*Solution* Without loss of generality we may assume that  $a \geq b \geq c$ .

$$\text{Let } f(a, b, c) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{6}{a+b+c}.$$

We'll prove that

$$f(a, b, c) \geq f(a, \sqrt{bc}, \sqrt{bc}).$$

We have

$$\begin{aligned}
 f(a, b, c) &\geq f(a, \sqrt{bc}, \sqrt{bc}) \\
 \Leftrightarrow \quad \frac{1}{b} + \frac{1}{c} + \frac{6}{a+b+c} &\geq \frac{2}{\sqrt{bc}} + \frac{6}{a+2\sqrt{bc}} \\
 \Leftrightarrow \quad c(a+b+c)(a+2\sqrt{bc}) + b(a+b+c)(a+2\sqrt{bc}) + 6bc(a+2\sqrt{bc}) \\
 &\geq 2\sqrt{bc}(a+b+c)(a+2\sqrt{bc}) + 6bc(a+b+c) \\
 \Leftrightarrow \quad (\sqrt{b} - \sqrt{c})^2((a+b+c)(a+2\sqrt{bc}) - 6bc) &\geq 0. \tag{1}
 \end{aligned}$$

Since  $a \geq b \geq c$  we have  $a \geq \frac{b+c}{2} \geq \sqrt{bc}$ .

Thus

$$(a+b+c)(a+2\sqrt{bc}) \geq (\sqrt{bc} + 2\sqrt{bc})(\sqrt{bc} + 2\sqrt{bc}) = 9bc \geq 6bc.$$

So due to (1) and the last inequality we have

$$f(a, b, c) \geq f(a, \sqrt{bc}, \sqrt{bc}).$$

According to the *SMV theorem* we need to prove that  $f(a, t, t) \geq 5$ , with  $at^2 = 1$ .

We have

$$f(a, t, t) \geq 5 \quad \Leftrightarrow \quad \frac{1}{a} + \frac{2}{t} + \frac{6}{a+2t} \geq 5$$

which is equivalent to

$$(t-1)^2(2t^4 + 4t^3 - 4t^2 - t + 2) \geq 0,$$

which is true since  $2t^4 + 4t^3 - 4t^2 - t + 2 > 0$  for  $t > 0$ . ■

**300** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$12\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 4(a^3 + b^3 + c^3) + 21.$$

*Solution* Without loss of generality we may assume that  $a \leq b \leq c$ .

Let

$$f(a, b, c) = 12\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 4(a^3 + b^3 + c^3).$$

Then we have

$$\begin{aligned} f(a, b, c) - f\left(\frac{a+b}{2}, \frac{a+b}{2}, c\right) &= 12\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 4(a^3 + b^3 + c^3) - 12\left(\frac{4}{a+b} + \frac{1}{c}\right) + (a+b)^3 + 4c^3 \\ &= 12\left(\frac{1}{a} + \frac{1}{b} - \frac{4}{a+b}\right) + (a+b)^3 - 4(a^3 + b^3) \\ &= 3(a-b)^2\left(\frac{4}{ab(a+b)} - (a+b)\right). \end{aligned} \tag{1}$$

Since  $a \leq b \leq c$  we must have  $a + b \leq 2$ , and clearly  $c \geq 1$ .

By the  $AM \geq GM$  we have

$$ab(a+b)^2 \leq \frac{(a+b)^4}{4} \leq 4, \quad \text{i.e.} \quad \frac{4}{ab(a+b)} - (a+b) \geq 0.$$

Hence by (1) we deduce that

$$f(a, b, c) - f\left(\frac{a+b}{2}, \frac{a+b}{2}, c\right) \geq 0, \quad \text{i.e.} \quad f(a, b, c) \geq f\left(\frac{a+b}{2}, \frac{a+b}{2}, c\right).$$

So according to the *SMV theorem* it suffices to prove that  $f(t, t, c) \geq 21$ , when  $2t + c = 3, c \geq t$ .

We have

$$\begin{aligned} f(t, t, c) &\geq 21 \\ \Leftrightarrow 12\left(\frac{2}{t} + \frac{1}{c}\right) + (2t)^3 - 4c^3 &\geq 21 \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow 12\left(\frac{4}{2t} + \frac{1}{c}\right) + (2t)^3 - 4c^3 \geq 21 \\
&\Leftrightarrow 12\left(\frac{4}{3-c} + \frac{1}{c}\right) + (3-c)^3 - 4c^3 \geq 21 \\
&\Leftrightarrow c^5 - 18c^3 + 48c^2 - 36c + 12 \geq 0 \\
&\Leftrightarrow (c-2)^2(c-1)(c^2+3c-3) \geq 0
\end{aligned}$$

which is true since  $c \geq 1$ .

Equality occurs iff  $(a, b, c) = (2, 1/2, 1/2)$ . ■

**301** Let  $a, b, c, d$  be non-negative real numbers such that  $a + b + c + d + e = 5$ . Prove the inequality

$$4(a^2 + b^2 + c^2 + d^2 + e^2) + 5abcd \geq 25.$$

*Solution* Without loss of generality we may assume that  $a \geq b \geq c \geq d \geq e$ .

Let us denote

$$f(a, b, c, d, e) = 4(a^2 + b^2 + c^2 + d^2 + e^2) + 5abcd.$$

Then we easily deduce that

$$f(a, b, c, d, e) - f\left(\frac{a+d}{2}, b, c, \frac{a+d}{2}, e\right) = \frac{(a-d)^2}{4}(8-5bce). \quad (1)$$

Since  $a \geq b \geq c \geq d \geq e$ , we have

$$3\sqrt[3]{bce} \leq b + c + e \leq \frac{3(a+b+c+d+e)}{5} = 3.$$

Thus it follows that  $bce \leq 1$ .

Now, by (1) and the last inequality we get

$$\begin{aligned}
f(a, b, c, d, e) - f\left(\frac{a+d}{2}, b, c, \frac{a+d}{2}, e\right) &= \frac{(a-d)^2}{4}(8-5bce) \\
&\geq \frac{(a-d)^2}{4}(8-5) \geq 0,
\end{aligned}$$

i.e.

$$f(a, b, c, d, e) \geq f\left(\frac{a+d}{2}, b, c, \frac{a+d}{2}, e\right).$$

According to the *SMV theorem* it remains to prove that  $f(t, t, t, t, e) \geq 25$ , under the condition  $4t + e = 5$ .

Clearly  $4t \leq 5$ .

We have

$$\begin{aligned}
 f(t, t, t, t, e) &\geq 25 \\
 \Leftrightarrow 4(t^2 + e^2) + 5t^4e &\geq 25 \\
 \Leftrightarrow 4t^2 + 4(5 - 4t)^2 + 5t^4(5 - 4t) - 25 &\geq 0 \\
 \Leftrightarrow (5 - 4t)(t - 1)^2(t^2 + 2t + 3) &\geq 0,
 \end{aligned}$$

which is true.

Equality occurs if and only if  $a = b = c = d = e = 1$  or  $a = b = c = d = 5/4, e = 0$  (up to permutation). ■

**302** Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove the inequality

$$\frac{1}{2 + a^2 + b^2} + \frac{1}{2 + b^2 + c^2} + \frac{1}{2 + c^2 + a^2} \leq \frac{3}{4}.$$

*Solution* Without loss of generality we may assume that  $a \geq b \geq c$ .

$$\text{Let } f(a, b, c) = \frac{1}{2 + a^2 + b^2} + \frac{1}{2 + b^2 + c^2} + \frac{1}{2 + c^2 + a^2}.$$

We have

$$\begin{aligned}
 &f\left(a, \frac{b+c}{2}, \frac{b+c}{2}\right) - f(a, b, c) \\
 &= \left(b^2 + c^2 - \frac{(b+c)^2}{2}\right) \\
 &\quad \times \left(\frac{1}{(b^2 + c^2 + 2)(2 + \frac{(b+c)^2}{2})} - \frac{1}{(4 + 2a^2 + b^2 + c^2)(4 + 2a^2 + \frac{(b+c)^2}{2})}\right).
 \end{aligned}$$

Since

$$\begin{aligned}
 b^2 + c^2 &\geq \frac{(b+c)^2}{2}, & 4 + 2a^2 + b^2 + c^2 &\geq b^2 + c^2 + 2 \quad \text{and} \\
 4 + 2a^2 + \frac{(b+c)^2}{2} &\geq 2 + \frac{(b+c)^2}{2}
 \end{aligned}$$

we have

$$\begin{aligned}
 f\left(a, \frac{b+c}{2}, \frac{b+c}{2}\right) - f(a, b, c) &\geq 0, \quad \text{i.e.} \\
 f\left(a, \frac{b+c}{2}, \frac{b+c}{2}\right) &\geq f(a, b, c).
 \end{aligned}$$

According to *SMV theorem* it suffices to prove that  $f(a, t, t) \leq \frac{3}{4}$ , when  $a + 2t = 3$ .

We have

$$\begin{aligned}
 f(a, t, t) &\leq \frac{3}{4} \\
 \Leftrightarrow \frac{2}{2+a^2+t^2} + \frac{1}{2+2t^2} &\leq \frac{3}{4} \\
 \Leftrightarrow \frac{8}{8+4a^2+(2t)^2} + \frac{2}{4+(2t)^2} &\leq \frac{3}{4} \\
 \Leftrightarrow \frac{8}{8+4a^2+(3-a)^2} + \frac{2}{4+(3-a)^2} &\leq \frac{3}{4},
 \end{aligned}$$

which can be easily transformed to  $(a-1)^2(15a^2-78a+111) \geq 0$ , and clearly holds.

Equality holds iff  $a = b = c = 1$ . ■

**303** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove the inequality

$$ab + bc + ca \leq abc + 2.$$

*Solution* Without loss of generality we may assume that  $a \geq b \geq c$ .

Let  $f(a, b, c) = ab + bc + ca - abc$ .

We have

$$\begin{aligned}
 f(a, b, c) - f\left(\sqrt{\frac{a^2+b^2}{2}}, \sqrt{\frac{a^2+b^2}{2}}, c\right) \\
 &= ab + bc + ca - abc - \frac{a^2+b^2}{2} - 2c\sqrt{\frac{a^2+b^2}{2}} + c\frac{a^2+b^2}{2} \\
 &= \left(ab - \frac{a^2+b^2}{2}\right) + c((a+b) - \sqrt{2(a^2+b^2)}) - c\left(ab - \frac{a^2+b^2}{2}\right) \\
 &= \frac{-(a-b)^2}{2} - \frac{c(a-b)^2}{(a+b) + \sqrt{2(a^2+b^2)}} + \frac{c(a-b)^2}{2} \\
 &= (a-b)^2 \left(\frac{c}{2} - \frac{1}{2} - \frac{c}{(a+b) + \sqrt{2(a^2+b^2)}}\right). \tag{1}
 \end{aligned}$$

Notice that since  $a \geq b \geq c$  we must have  $c^2 \leq 1$ , i.e.  $c \leq 1$  and  $a^2 + b^2 \geq 2$ .

By  $AM \leq QM$  we have

$$\begin{aligned}
 \frac{c}{2} - \frac{1}{2} - \frac{c}{(a+b) + \sqrt{2(a^2+b^2)}} &\leq \frac{c}{2} - \frac{1}{2} - \frac{c}{2\sqrt{2} \cdot (a^2+b^2)} \\
 &\leq \frac{c}{2} - \frac{1}{2} - \frac{c}{2\sqrt{2} \cdot (a^2+b^2+c^2)}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{c}{2} - \frac{1}{2} - \frac{c}{2\sqrt{6}} \leq \frac{1}{2} - \frac{1}{2} - \frac{c}{2\sqrt{6}} \\
&\leq -\frac{c}{2\sqrt{6}} \leq 0.
\end{aligned}$$

Hence by (1) we get that

$$f(a, b, c) - f\left(\sqrt{\frac{a^2 + b^2}{2}}, \sqrt{\frac{a^2 + b^2}{2}}, c\right) \leq 0,$$

i.e.

$$f(a, b, c) \leq f\left(\sqrt{\frac{a^2 + b^2}{2}}, \sqrt{\frac{a^2 + b^2}{2}}, c\right).$$

According to the *SMV theorem* we need to prove that  $f(t, t, c) \leq 2$ , when  $2t^2 + c^2 = 3$ .

We have

$$\begin{aligned}
f(t, t, c) \leq 2 &\Leftrightarrow t^2 + 2ct - t^2c \leq 2 \\
&\Leftrightarrow 2t^2 + 4ct \leq 2t^2c + 4 \Leftrightarrow 4ct \leq 2t^2c + 3 - 2t^2 + 1 \\
&\Leftrightarrow 4ct \leq 2t^2c + c^2 + 1,
\end{aligned}$$

which is true due to  $AM \geq GM$ , i.e.

$$2t^2c + c^2 + 1 = t^2c + c^2 + 1 + t^2c \geq 4\sqrt{t^2c \cdot c^2 \cdot t^2c} = 4ct. \quad \blacksquare$$

**304** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{a+c}{a+b}.$$

*Solution* Without loss of generality we may assume that  $c = \min\{a, b, c\}$ .

Notice that for  $x, y, z > 0$  we have

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} - 3 = \frac{1}{xy}(x-y)^2 + \frac{1}{xz}(x-z)(y-z).$$

Now we obtain

$$\begin{aligned}
\frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 3 &\geq \frac{c+a}{c+b} + \frac{b+c}{b+a} + \frac{a+c}{a+b} - 3 \\
&\Leftrightarrow \frac{1}{ab}(a-b)^2 + \frac{1}{ac}(a-c)(b-c) \\
&\geq \frac{1}{(a+c)(b+c)}(a-b)^2 + \frac{1}{(a+c)(a+b)}(b-c)(a-c)
\end{aligned}$$

$$\Leftrightarrow \left( \frac{1}{ab} - \frac{1}{(a+c)(b+c)} \right) (a-b)^2 + \left( \frac{1}{ac} - \frac{1}{(a+c)(a+b)} \right) \\ \times (a-c)(b-c) \geq 0.$$

The last inequality is true, since:

$$c = \min\{a, b, c\}, \quad \frac{1}{ac} - \frac{1}{(a+c)(a+b)} > 0 \quad \text{and} \quad \frac{1}{ab} - \frac{1}{(a+c)(b+c)} > 0. \quad \blacksquare$$

**305** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

*Solution* We have

$$\begin{aligned} \frac{a^2}{b^2 + c^2} - \frac{a}{b+c} &= \frac{ab(a-b) + ac(a-c)}{(b^2 + c^2)(b+c)}, \\ \frac{b^2}{c^2 + a^2} - \frac{b}{c+a} &= \frac{bc(b-c) + ab(b-a)}{(c^2 + a^2)(c+a)} \quad \text{and} \\ \frac{c^2}{a^2 + b^2} - \frac{c}{a+b} &= \frac{ac(c-a) + bc(c-b)}{(b^2 + a^2)(b+a)}. \end{aligned}$$

Now we obtain

$$\begin{aligned} &\frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} - \left( \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right) \\ &= \frac{ab(a-b) + ac(a-c)}{(b^2 + c^2)(b+c)} + \frac{bc(b-c) + ab(b-a)}{(c^2 + a^2)(c+a)} + \frac{ac(c-a) + bc(c-b)}{(b^2 + a^2)(b+a)} \\ &= (a^2 + b^2 + c^2 + ab + bc + ca) \cdot \sum \frac{ab(a-b)^2}{(b+c)(c+a)(b^2 + c^2)(c^2 + a^2)} \geq 0. \quad \blacksquare \end{aligned}$$

**306** Let  $a, b, c$  be positive real numbers such that  $a \geq b \geq c$ . Prove the inequality

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

*Solution* We have

$$\begin{aligned} &a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \\ &= a^2b(a-b) + b^2c(b-c) + c^2a(c-a) - ab^2(a-b) - ab^2(b-c) \\ &\quad - ab^2(c-a) \end{aligned}$$



$$\begin{aligned}
&= (a^2b(a-b) - ab^2(a-b)) + (b^2c(b-c) - ab^2(b-c)) \\
&\quad + (c^2a(c-a) - ab^2(c-a)) \\
&= ab(a-b)^2 + (ab+ac-b^2)(a-c)(b-c).
\end{aligned}$$

So we need to show that

$$ab(a-b)^2 + (ab+ac-b^2)(a-c)(b-c) \geq 0,$$

which clearly holds since  $a \geq b \geq c$ . ■

**307** Let  $a, b, c$  be the lengths of the sides of a triangle. Prove the inequality

$$\frac{(b+c)^2}{a^2+bc} + \frac{(c+a)^2}{b^2+ca} + \frac{(a+b)^2}{c^2+ab} \geq 6.$$

*Solution* We have

$$\begin{aligned}
&\frac{(b+c)^2}{a^2+bc} - 2 + \frac{(c+a)^2}{b^2+ca} - 2 + \frac{(a+b)^2}{c^2+ab} - 2 \geq 0 \\
&\Leftrightarrow \frac{b^2+c^2-2a^2}{a^2+bc} + \frac{c^2+a^2-2b^2}{b^2+ca} + \frac{a^2+b^2-2c^2}{c^2+ab} \geq 0 \\
&\Leftrightarrow \left( \frac{b^2-a^2}{a^2+bc} + \frac{a^2-b^2}{b^2+ca} \right) + \left( \frac{c^2-a^2}{a^2+bc} + \frac{a^2-c^2}{c^2+ab} \right) \\
&\quad + \left( \frac{c^2-b^2}{b^2+ca} + \frac{b^2-c^2}{c^2+ab} \right) \geq 0 \\
&\Leftrightarrow \frac{(b-a)^2(a+b)(a+b-c)}{(a^2+bc)(b^2+ca)} + \frac{(c-a)^2(c+a)(c+a-b)}{(a^2+bc)(c^2+ab)} \\
&\quad + \frac{(b-c)^2(b+c)(b+c-a)}{(b^2+ca)(c^2+ab)} \geq 0,
\end{aligned}$$

which is clearly true. ■

**308** Let  $a, b, c$  be positive real numbers. Prove the inequality

$$\frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b} + 3 \frac{ab+bc+ca}{(a+b+c)^2} \geq 4.$$

*Solution* Without loss of generality we may assume that  $c = \min\{a, b, c\}$ .

Now we have

$$\begin{aligned}
\frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b} - 3 &= \frac{1}{(a+c)(b+c)}(a-b)^2 \\
&\quad + \frac{1}{(a+b)(b+c)}(a-c)(b-c)
\end{aligned}$$

and

$$3 \frac{ab + bc + ca}{(a + b + c)^2} - 1 = -\frac{1}{(a + b + c)^2} (a - b)^2 - \frac{1}{(a + b + c)^2} (a - c)(b - c).$$

The given inequality becomes

$$M(a - b)^2 + N(a - c)(b - c) \geq 0, \quad (1)$$

where  $M = \frac{1}{(a+c)(b+c)} - \frac{1}{(a+b+c)^2}$  and  $N = \frac{1}{(a+b)(b+c)} - \frac{1}{(a+b+c)^2}$ .

We can easily prove that  $M, N \geq 0$ , and since  $c = \min\{a, b, c\}$  we get inequality (1). ■

**309** Let  $a, b, c$  be real numbers. Prove the inequality

$$3(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \geq a^3b^3 + b^3c^3 + c^3a^3.$$

*Solution* It is enough to consider the case when  $a, b, c \geq 0$ .

We have

$$\begin{aligned} (a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) &= \sum_{\text{sym}} a^4b^2 - \sum_{\text{cyc}} a^3b^3 - \sum_{\text{cyc}} a^4bc \\ &\quad + a^2b^2c^2. \end{aligned}$$

The given inequality is equivalent to

$$3 \sum_{\text{sym}} a^4b^2 - 4 \sum_{\text{cyc}} a^3b^3 - 3 \sum_{\text{cyc}} a^4bc + 3a^2b^2c^2 \geq 0,$$

which is equivalent to

$$\sum_{\text{cyc}} (2c^4 + 3a^2b^2 - abc(a + b + c))(a - b)^2 \geq 0. \quad (1)$$

Assume  $a \geq b \geq c$  and denote

$$S_a = 2a^4 + 3b^2c^2 - abc(a + b + c),$$

$$S_b = 2b^4 + 3a^2c^2 - abc(a + b + c)$$

and

$$S_c = 2c^4 + 3a^2b^2 - abc(a + b + c).$$

We have

$$S_a = 2a^4 + 3b^2c^2 - abc(a + b + c) \geq a^4 + 2a^2bc - abc(a + b + c) \geq 0,$$

$$S_c = 2c^4 + 3a^2b^2 - abc(a + b + c) \geq 3a^2b^2 - abc(a + b + c) \geq 0,$$

$$S_a + 2S_b = 2a^4 + 3b^2c^2 + 4b^4 + 6a^2c^2 - 3abc(a + b + c) \\ \geq a^4 + 2a^2bc + 8b^2ca - 3abc(a + b + c) \geq 0$$

and

$$S_c + 2S_b = 2c^4 + 3a^2b^2 + 4b^4 + 6a^2c^2 - 3abc(a + b + c) \\ \geq (3a^2b^2 + 3a^2c^2) + 3a^2c^2 - 3abc(a + b + c) \geq 0.$$

(Since the given inequality is cyclic if we assume that  $a \leq b \leq c$  similarly we can show that  $S_a, S_c, S_a + 2S_b, S_c + 2S_b \geq 0$ .)

According to the *SOS theorem* we obtain that inequality (1) holds, as required.

Equality holds iff  $a = b = c$ . ■

**310** Let  $a, b, c, d \in \mathbb{R}^+$  such that  $a + b + c + d + abcd = 5$ . Prove the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \geq 4.$$

*Solution* We'll use *Lagrange's theorem*.

Let

$$f(a, b, c, d) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \quad \text{and} \quad g(a, b, c, d) = a + b + c + d + abcd - 5 = 0.$$

We define

$$L = f - \lambda g = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} - \lambda(a + b + c + d + abcd - 5).$$

For the first partial derivatives we get

$$\begin{aligned} \frac{\partial L}{\partial a} &= -\frac{1}{a^2} - \lambda(1 + bcd) = 0, & \frac{\partial L}{\partial b} &= -\frac{1}{b^2} - \lambda(1 + acd) = 0, \\ \frac{\partial L}{\partial c} &= -\frac{1}{c^2} - \lambda(1 + abd) = 0, & \frac{\partial L}{\partial d} &= -\frac{1}{d^2} - \lambda(1 + abc) = 0. \end{aligned}$$

So

$$\lambda = -\frac{1}{a^2(1 + bcd)} = -\frac{1}{b^2(1 + acd)} = -\frac{1}{c^2(1 + abd)} = -\frac{1}{d^2(1 + abc)}.$$

From the first two equations we deduce

$$a^2(1 + bcd) = b^2(1 + acd), \quad \text{i.e.} \quad (a - b)(a + b + abcd) = 0.$$

Since  $a + b + abcd > 0$  we must have  $a = b$ .

Analogously we deduce that  $a = c = d$ , i.e.  $a = b = c = d$ .

Using  $a + b + c + d + abcd = 5$  we get

$$a^4 + 4a - 5 = 0 \quad \Leftrightarrow \quad (a - 1)(a^3 + a^2 + a + 5) = 0,$$

and it follows that we must have  $a = 1$ .

So  $a = b = c = d = 1$ .

Finally we have  $f(1, 1, 1, 1) = 1 + 1 + 1 + 1 = 4$ , and we are done. ■



# Index of Problems

## Chapter 1

- 1.8 BMO 2001
- 1.17 Russia MO 2002

## Chapter 2

- 2.4 Viorel Vâjăitu, Alexandra Zaharescu, *Gazeta Matematică*
- 2.15 Ireland MO 2000

## Chapter 3

- 3.6 IMO, shortlist 1969 (Romania)
- 3.8 India MO 2003

## Chapter 4

- 4.1 France MO 1996
- 4.3 South Africa MO 1995
- 4.7 Crux Mathematicorum
- 4.8 Sefket Arslanagic
- 4.10 Art of problem solving
- 4.11 Art of problem solving
- 4.13 Andrei Ciupan, Romania 2007
- 4.15 Crux Mathematicorum
- 4.20 Pham Kim Hung
- Corollary 4.5: Walther Janous

## Chapter 5

- 5.3 IMO, shortlist 1974 (Finland)
- 5.10 Zdravko Cvetkovski
- 5.13 Zdravko Cvetkovski
- 5.14 Zdravko Cvetkovski
- 5.15 Zdravko Cvetkovski

## Chapter 6

- 6.2 IMO 1975
- 6.4 IMO 1964 (Hungary)
- 6.6 IMO 1995
- 6.10 Song Yoon Kim

## Chapter 8

- 8.1 Darij Grinberg
- 8.2 Poland MO 1999
- 8.3 Calin Popa
- 8.4 Walther Janous, Crux Mathematicorum
- 8.6 APMO 2004

## Chapter 9

- 9.1 Singapore MO 2002
- 9.2 Sefket Arslanagic
- 9.6 Le Viet Thai
- 9.7 Pham Kim Hung
- 9.8 Pham Kim Hung
- 9.10 Walther Janous, Crux Mathematicorum

## Chapter 10

- 10.5 Zdravko Cvetkovski
- 10.6 Zdravko Cvetkovski
- 10.7 Zdravko Cvetkovski
- 10.11 Nguyen Manh Dung

## Chapter 12

- 12.5 Darij Grinberg
- 12.7 APMO 2004
- 12.10 IMO 1984
- 12.11 IMO 1995

## Chapter 14

- 14.2 Iran MO 1996
- 14.6 United Kingdom 1999

## Chapter 15

- 15.2 Vietnam TST 1996
- 15.3 Vietnam 2002
- 15.4 Darij Grinberg

## Chapter 17

- 17.2 IMO 2005

## Chapter 18

18.3 Pham Kim Hung

18.5 Nguyen Minh Duc, IMO shortlist 1993

## Chapter 20

6 Titu Andreescu, TST 2001 USA

9 Russia 2002

12 Czech and Slovak Republics 2005

14 Walther Janous, Crux Mathematicorum

16 Vasile Cîrtoaje, Gazeta Matematică

20 Titu Andreescu, Gabriel Dospinescu

22 Art of problem solving

23 Vasile Cîrtoaje

28 Art of problem solving

33 Art of problem solving

36 Belarus 1996

37 Art of problem solving

40 Zdravko Cvetkovski

41 Zdravko Cvetkovski

42 Zdravko Cvetkovski

45 Mircea Lascu, Gazeta Matematică

48 Art of problem solving

49 IMO 2000, Titu Andreescu

50 Bulgaria, 1997

56 Baltic Way, 2005

57 Gabriel Dospinescu, Marian Tetiva

58 Adrian Zahariuc

68 Vasile Cîrtoaje

71 MOSP 2001

72 Vasile Cîrtoaje, Mircea Lascu, Junior TST 2003 Romania

73 Marian Tetiva

75 Latvia 2002

81 Peru 2007

82 Romania 2003

84 Art of problem solving

86 Japan 2005

88 Art of problem solving

90 Canada 2008

91 Mathlinks contest

97 Kiran Kedlaya

98 Zdravko Cvetkovski, shortlist JBMO 2010

99 Vasile Cîrtoaje, Gazeta Matematică

105 JBMO 2002, shortlist

110 Pham Kim Hung

113 Pham Kim Hung



- 115 Pham Kim Hung
- 117 Zdravko Cvetkovski
- 118 Macedonia MO 1999
- 119 Zdravko Cvetkovski
- 125 Pham Kim Hung
- 126 Titu Andreescu, USAMO 2004
- 129 MYM, 2001
- 131 Vietnam
- 136 Vietnam, 1998
- 139 Zdravko Cvetkovski, BMO shortlist 2010
- 144 IMO, shortlist 1996 (Slovenia)
- 154 Nguyen Van Thach
- 156 Pham Kim Hung
- 160 Dinu Șerbanescu, Junior TST 2002, Romania
- 161 Zdravko Cvetkovski, Macedonia MO 2010
- 162 Pham Kim Hung
- 163 Romania 2008
- 165 IMO, shortlist 1987
- 166 JBMO, 2003
- 167 Pham Kim Hung
- 170 Florina Cârlan, Marian Tetiva
- 173 Darij Grinberg
- 176 Art of problem solving
- 179 Art of problem solving
- 184 Ukraine MO 2004
- 188 JBMO 2002, shortlist
- 190 IMO, shortlist 1990 (Thailand)
- 191 Titu Andreescu, Mircea Lascu
- 195 Poru Loh, Crux Mathematicorum
- 196 Pham Kim Hung
- 197 Moldova TST 2005
- 198 IMO shortlist 1994
- 199 IMO shortlist 1998 (Russia)
- 200 IMO shortlist 1993 (USA)
- 202 IMO, shortlist 1968 (Poland)
- 207 China MO 1996
- 212 IMO, shortlist 1992 (Great Britain)
- 213 A. Teplinsky, Ukraine MO 2005
- 214 Daniel Campos Salas, Mathematical Reflections 2007
- 217 Cezar Lupu, Romania MO 2005
- 218 Titu Andreescu, USAMO 2001
- 219 Gabriel Dospinescu, Mircea Lascu, Marian Tetiva
- 237 IMO 1961 (Poland)
- 250 Zdravko Cvetkovski
- 257 Calin Popescu, Romania 2004

- 264 Pham Kim Hung
- 265 Mihai Piticari, Dan Popescu
- 271 Art of problem solving
- 280 Tran Nam Dung
- 281 Le Trung Kien
- 287 Zdravko Cvetkovski
- 288 Vasile Cîrtoaje, Gazeta Matematică
- 292 Serbia 2008
- 294 Vasile Cîrtoaje, Gazeta Matematică
- 298 Pham Kim Hung
- 300 Art of problem solving
- 302 Art of problem solving
- 303 Art of problem solving
- 304 India 2002
- 305 Vasile Cîrtoaje, Gazeta Matematică



# Abbreviations

**APMO** Asian-Pacific Mathematical Olympiad  
**BMO** Balkan Mathematical Olympiad  
**IMO** International Mathematical Olympiad  
**JBMO** Junior Balkan Mathematical Olympiad  
**MYM** Mathematics and Youth magazine, Vietnam  
**MO** Mathematical Olympiad  
**MOSP** Mathematical Olympiad Summer Program  
**USAMO** United States of America Mathematical Olympiad



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