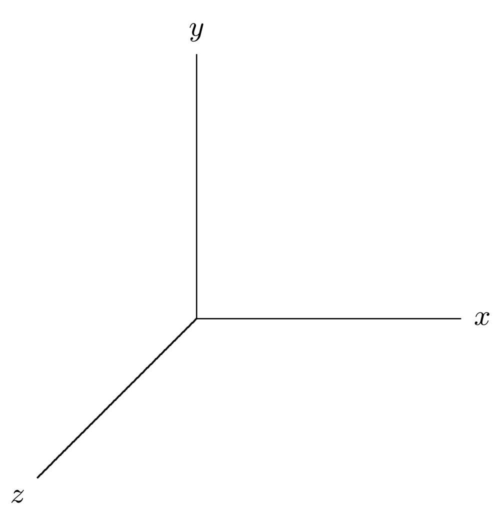
12

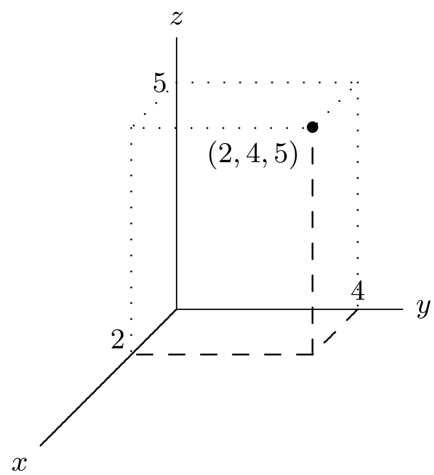
Three Dimensions

12.1 The Coordinate System

So far we have been investigating functions of the form $y = f(x)$, with one independent and one dependent variable. Such functions can be represented in two dimensions, using two numerical axes that allow us to identify every point in the plane with two numbers. We now want to talk about three-dimensional space; to identify every point in three dimensions we require three numerical values. The obvious way to make this association is to add one new axis, perpendicular to the x and y axes we already understand. We could, for example, add a third axis, the z axis, with the positive z axis coming straight out of the page, and the negative z axis going out the back of the page. This is difficult to work with on a printed page, so more often we draw a view of the three axes from an angle:



You must then imagine that the z axis is perpendicular to the other two. Just as we have investigated functions of the form $y = f(x)$ in two dimensions, we will investigate three dimensions largely by considering functions; now the functions will (typically) have the form $z = f(x, y)$. Because we are used to having the result of a function graphed in the vertical direction, it is somewhat easier to maintain that convention in three dimensions. To accomplish this, we normally rotate the axes so that z points up; the result is then:



Note that if you imagine looking down from above, along the z axis, the positive z axis will come straight toward you, the positive y axis will point up, and the positive x axis will point to your right, as usual. Any point in space is identified by providing the three coordinates of the point, as shown; naturally, we list the coordinates in the order (x, y, z). One useful way to think of this is to use the x and y coordinates to identify a point in the x-y plane, then move straight up (or down) a distance given by the z coordinate.

It is now fairly simple to understand some "shapes" in three dimensions that correspond to simple conditions on the coordinates. In two dimensions the equation $x = 1$ describes the vertical line through $(1, 0)$. In three dimensions, it still describes all points with x-coordinate 1, but this is now a plane, as in figure 12.1.1.

Recall the very useful distance formula in two dimensions: the distance between points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$; this comes directly from the Pythagorean theorem. What is the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in three dimensions? Geometrically, we want the length of the long diagonal labeled c in the "box" in figure 12.1.2. Since a, b, c form a right triangle, $a^2 + b^2 = c^2$. b is the vertical distance between (x_1, y_1, z_1) and (x_2, y_2, z_2), so $b = |z_1 - z_2|$. The length a runs parallel to the x-y plane, so it is simply the distance between (x_1, y_1) and (x_2, y_2), that is, $a^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$. Now we see that $c^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$ and $c = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

It is sometimes useful to give names to points, for example we might let $P_1 = (x_1, y_1, z_1)$, or more concisely we might refer to the point $P_1(x_1, y_1, z_1)$, and subsequently

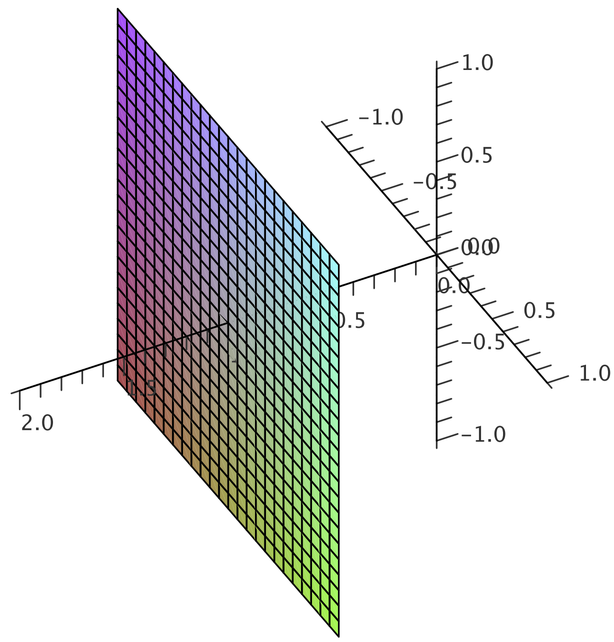


Figure 12.1.1 The plane $x = 1$. (AP)

use just P_1. Distance between two points in either two or three dimensions is sometimes denoted by d, so for example the formula for the distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ might be expressed as

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

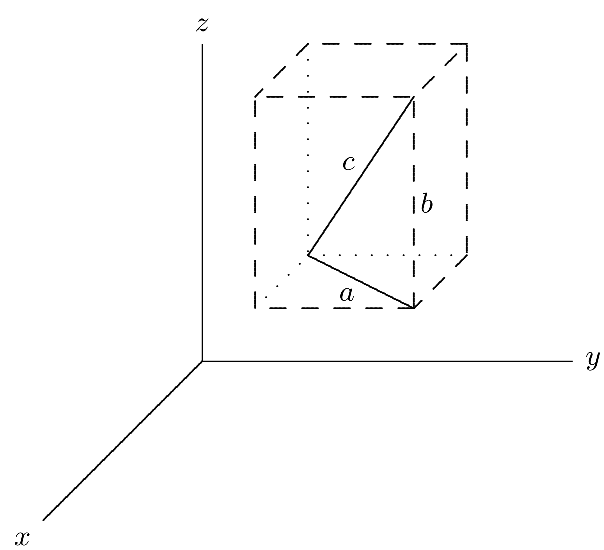


Figure 12.1.2 Distance in three dimensions.

In two dimensions, the distance formula immediately gives us the equation of a circle: the circle of radius r and center at (h, k) consists of all points (x, y) at distance r from

(h,k), so the equation is $r = \sqrt{(x-h)^2 + (y-k)^2}$ or $r^2 = (x-h)^2 + (y-k)^2$. Now we can get the similar equation $r^2 = (x-h)^2 + (y-k)^2 + (z-l)^2$, which describes all points (x,y,z) at distance r from (h,k,l), namely, the sphere with radius r and center (h,k,l).

Exercises 12.1.

1. Sketch the location of the points $(1,1,0)$, $(2,3,-1)$, and $(-1,2,3)$ on a single set of axes.

2. Describe geometrically the set of points (x,y,z) that satisfy $z = 4$.

3. Describe geometrically the set of points (x,y,z) that satisfy $y = -3$.

4. Describe geometrically the set of points (x,y,z) that satisfy $x + y = 2$.

5. The equation $x + y + z = 1$ describes some collection of points in $\mathbb{R}^3$. Describe and sketch the points that satisfy $x + y + z = 1$ and are in the x-y plane, in the x-z plane, and in the y-z plane.

6. Find the lengths of the sides of the triangle with vertices $(1,0,1)$, $(2,2,-1)$, and $(-3,2,-2)$. ⇒

7. Find the lengths of the sides of the triangle with vertices $(2,2,3)$, $(8,6,5)$, and $(-1,0,2)$. Why do the results tell you that this isn't really a triangle? ⇒

8. Find an equation of the sphere with center at $(1,1,1)$ and radius 2. ⇒

9. Find an equation of the sphere with center at $(2,-1,3)$ and radius 5. ⇒

10. Find an equation of the sphere with center $(3,-2,1)$ and that goes through the point $(4,2,5)$. ⇒

11. Find an equation of the sphere with center at $(2,1,-1)$ and radius 4. Find an equation for the intersection of this sphere with the y-z plane; describe this intersection geometrically. ⇒

12. Consider the sphere of radius 5 centered at $(2,3,4)$. What is the intersection of this sphere with each of the coordinate planes?

13. Show that for all values of θ and ϕ, the point $(a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi)$ lies on the sphere given by $x^2 + y^2 + z^2 = a^2$.

14. Prove that the midpoint of the line segment connecting (x_1,y_1,z_1) to (x_2,y_2,z_2) is at $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

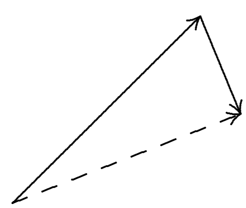
15. Any three points $P_1(x_1,y_1,z_1)$, $P_2(x_2,y_2,z_2)$, $P_3(x_3,y_3,z_3)$, lie in a plane and form a triangle. The **triangle inequality** says that $d(P_1,P_3) \le d(P_1,P_2) + d(P_2,P_3)$. Prove the triangle inequality using either algebra (messy) or the law of cosines (less messy).

16. Is it possible for a plane to intersect a sphere in exactly two points? Exactly one point? Explain.

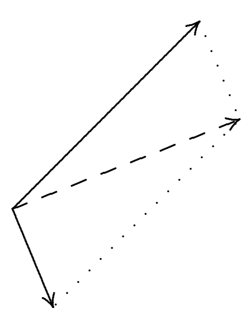
12.2 Vectors

A vector is a quantity consisting of a non-negative magnitude and a direction. We could represent a vector in two dimensions as (m,θ), where m is the magnitude and θ is the direction, measured as an angle from some agreed upon direction. For example, we might think of the vector $(5, 45^\circ)$ as representing "5 km toward the northeast"; that is, this vector might be a **displacement vector**, indicating, say, that your grandfather walked 5 kilometers toward the northeast to school in the snow. On the other hand, the same vector could represent a velocity, indicating that your grandfather walked at 5 km/hr toward the northeast. What the vector does not indicate is where this walk occurred: a vector represents a magnitude and a direction, but not a location. Pictorially it is useful to represent a vector as an arrow; the direction of the vector, naturally, is the direction in which the arrow points; the magnitude of the vector is reflected in the length of the arrow.

It turns out that many, many quantities behave as vectors, e.g., displacement, velocity, acceleration, force. Already we can get some idea of their usefulness using displacement vectors. Suppose that your grandfather walked 5 km NE and then 2 km SSE; if the terrain allows, and perhaps armed with a compass, how could your grandfather have walked directly to his destination? We can use vectors (and a bit of geometry) to answer this question. We begin by noting that since vectors do not include a specification of position, we can "place" them anywhere that is convenient. So we can picture your grandfather's journey as two displacement vectors drawn head to tail:



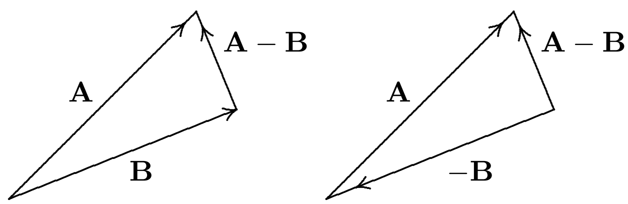
The displacement vector for the shortcut route is the vector drawn with a dashed line, from the tail of the first to the head of the second. With a little trigonometry, we can compute that the third vector has magnitude approximately 4.62 and direction 21.43°, so walking 4.62 km in the direction 21.43° north of east (approximately ENE) would get your grandfather to school. This sort of calculation is so common, we dignify it with a name: we say that the third vector is the **sum** of the other two vectors. There is another common way to picture the sum of two vectors. Put the vectors tail to tail and then complete the parallelogram they indicate; the sum of the two vectors is the diagonal of the parallelogram:



This is a more natural representation in some circumstances. For example, if the two original vectors represent forces acting on an object, the sum of the two vectors is the net or effective force on the object, and it is nice to draw all three with their tails at the location of the object.

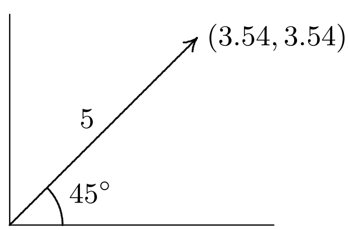
We also define **scalar multiplication** for vectors: if $\mathbf{A}$ is a vector (m, θ) and $a \geq 0$ is a real number, the vector $a\mathbf{A}$ is (am, θ), namely, it points in the same direction but has a times the magnitude. If $a < 0$, $a\mathbf{A}$ is $(|a|m, \theta + \pi)$, with $|a|$ times the magnitude and pointing in the opposite direction (unless we specify otherwise, angles are measured in radians).

Now we can understand subtraction of vectors: $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}$:

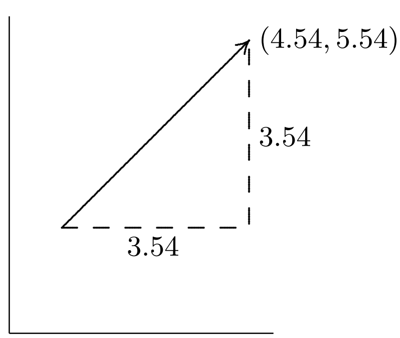


Note that as you would expect, $\mathbf{B} + (\mathbf{A} - \mathbf{B}) = \mathbf{A}$.

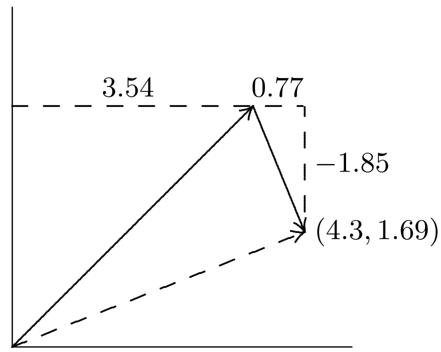
We can represent a vector in ways other than (m, θ), and in fact (m, θ) is not generally used at all. How else could we describe a particular vector? Consider again the vector $(5, 45°)$. Let's draw it again, but impose a coordinate system. If we put the tail of the arrow at the origin, the head of the arrow ends up at the point $(5/\sqrt{2}, 5/\sqrt{2}) \approx (3.54, 3.54)$.



In this picture the coordinates $(3.54, 3.54)$ identify the head of the arrow, provided we know that the tail of the arrow has been placed at $(0, 0)$. Then in fact the vector can always be identified as $(3.54, 3.54)$, no matter where it is placed; we just have to remember that the numbers 3.54 must be interpreted as a *change* from the position of the tail, not as the actual coordinates of the arrow head; to emphasize this we will write $\langle 3.54, 3.54 \rangle$ to mean the vector and $(3.54, 3.54)$ to mean the point. Then if the vector $\langle 3.54, 3.54 \rangle$ is drawn with its tail at $(1, 2)$ it looks like this:



Consider again the two part trip: 5 km NE and then 2 km SSE. The vector representing the first part of the trip is $\langle 5/\sqrt{2}, 5/\sqrt{2}\rangle$, and the second part of the trip is represented by $\langle 2\cos(-3\pi/8), 2\sin(-3\pi/8)\rangle \approx \langle 0.77, -1.85\rangle$. We can represent the sum of these with the usual head to tail picture:



It is clear from the picture that the coordinates of the destination point are $(5/\sqrt{2} + 2\cos(-3\pi/8), 5/\sqrt{2} + 2\sin(-3\pi/8))$ or approximately $(4.3, 1.69)$, so the sum of the two vectors is $\langle 5/\sqrt{2} + 2\cos(-3\pi/8), 5/\sqrt{2} + 2\sin(-3\pi/8)\rangle \approx \langle 4.3, 1.69\rangle$. Adding the two vectors is easier in this form than in the (m, θ) form, provided that we're willing to have the answer in this form as well.

It is easy to see that scalar multiplication and vector subtraction are also easy to compute in this form: $a\langle v, w\rangle = \langle av, aw\rangle$ and $\langle v_1, w_1\rangle - \langle v_2, w_2\rangle = \langle v_1 - v_2, w_1 - w_2\rangle$. What about the magnitude? The magnitude of the vector $\langle v, w\rangle$ is still the length of the corresponding arrow representation; this is the distance from the origin to the point (v, w), namely, the distance from the tail to the head of the arrow. We know how to compute distances, so the magnitude of the vector is simply $\sqrt{v^2 + w^2}$, which we also denote with absolute value bars: $|\langle v, w\rangle| = \sqrt{v^2 + w^2}$.

In three dimensions, vectors are still quantities consisting of a magnitude and a direction, but of course there are many more possible directions. It's not clear how we might represent the direction explicitly, but the coordinate version of vectors makes just as much sense in three dimensions as in two. By $\langle 1, 2, 3\rangle$ we mean the vector whose head is at $(1, 2, 3)$ if its tail is at the origin. As before, we can place the vector anywhere we want; if it has its tail at $(4, 5, 6)$ then its head is at $(5, 7, 9)$. It remains true that arithmetic is easy to do with vectors in this form:

$$a\langle v_1, v_2, v_3\rangle = \langle av_1, av_2, av_3\rangle$$

$$\langle v_1, v_2, v_3\rangle + \langle w_1, w_2, w_3\rangle = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3\rangle$$

$$\langle v_1, v_2, v_3\rangle - \langle w_1, w_2, w_3\rangle = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3\rangle$$

The magnitude of the vector is again the distance from the origin to the head of the arrow, or $|\langle v_1, v_2, v_3\rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

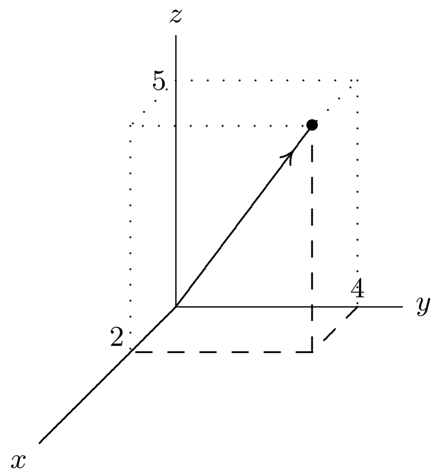


Figure 12.2.1 The vector $\langle 2,4,5\rangle$ with its tail at the origin.

Three particularly simple vectors turn out to be quite useful: $\mathbf{i}=\langle 1,0,0\rangle$, $\mathbf{j}=\langle 0,1,0\rangle$, and $\mathbf{k}=\langle 0,0,1\rangle$. These play much the same role for vectors that the axes play for points. In particular, notice that

$$\begin{aligned}\langle v_1,v_2,v_3\rangle &= \langle v_1,0,0\rangle+\langle 0,v_2,0\rangle+\langle 0,0,v_3\rangle\\ &= v_1\langle 1,0,0\rangle+v_2\langle 0,1,0\rangle+v_3\langle 0,0,1\rangle\\ &= v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}\end{aligned}$$

We will frequently want to produce a vector that points from one point to another. That is, if P and Q are points, we seek the vector $\mathbf{x}$ such that when the tail of $\mathbf{x}$ is placed at P, its head is at Q; we refer to this vector as $\overrightarrow{PQ}$. If we know the coordinates of P and Q, the coordinates of the vector are easy to find.

EXAMPLE 12.2.1 Suppose $P=(1,-2,4)$ and $Q=(-2,1,3)$. The vector $\overrightarrow{PQ}$ is $\langle -2-1,1--2,3-4\rangle=\langle -3,3,-1\rangle$ and $\overrightarrow{QP}=\langle 3,-3,1\rangle$. □

Exercises 12.2.

1. Draw the vector $\langle 3,-1\rangle$ with its tail at the origin.
2. Draw the vector $\langle 3,-1,2\rangle$ with its tail at the origin.
3. Let $\mathbf{A}$ be the vector with tail at the origin and head at $(1,2)$; let $\mathbf{B}$ be the vector with tail at the origin and head at $(3,1)$. Draw $\mathbf{A}$ and $\mathbf{B}$ and a vector $\mathbf{C}$ with tail at $(1,2)$ and head at $(3,1)$. Draw $\mathbf{C}$ with its tail at the origin.
4. Let $\mathbf{A}$ be the vector with tail at the origin and head at $(-1,2)$; let $\mathbf{B}$ be the vector with tail at the origin and head at $(3,3)$. Draw $\mathbf{A}$ and $\mathbf{B}$ and a vector $\mathbf{C}$ with tail at $(-1,2)$ and head at $(3,3)$. Draw $\mathbf{C}$ with its tail at the origin.

5. Let **A** be the vector with tail at the origin and head at $(5, 2)$; let **B** be the vector with tail at the origin and head at $(1, 5)$. Draw **A** and **B** and a vector **C** with tail at $(5, 2)$ and head at $(1, 5)$. Draw **C** with its tail at the origin.

6. Find $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ and $-2\mathbf{v}$ for $\mathbf{v}=\langle 1,3\rangle$ and $\mathbf{w}=\langle -1,-5\rangle$. ⇒

7. Find $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ and $-2\mathbf{v}$ for $\mathbf{v}=\langle 1,2,3\rangle$ and $\mathbf{w}=\langle -1,2,-3\rangle$. ⇒

8. Find $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ and $-2\mathbf{v}$ for $\mathbf{v}=\langle 1,0,1\rangle$ and $\mathbf{w}=\langle -1,-2,2\rangle$. ⇒

9. Find $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ and $-2\mathbf{v}$ for $\mathbf{v}=\langle 1,-1,1\rangle$ and $\mathbf{w}=\langle 0,0,3\rangle$. ⇒

10. Find $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ and $-2\mathbf{v}$ for $\mathbf{v}=\langle 3,2,1\rangle$ and $\mathbf{w}=\langle -1,-1,-1\rangle$. ⇒

11. Let $P = (4, 5, 6)$, $Q = (1, 2, -5)$. Find $\overrightarrow{PQ}$. Find a vector with the same direction as $\overrightarrow{PQ}$ but with length 1. Find a vector with the same direction as $\overrightarrow{PQ}$ but with length 4. ⇒

12. If A, B, and C are three points, find $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}$. ⇒

13. Consider the 12 vectors that have their tails at the center of a clock and their respective heads at each of the 12 digits. What is the sum of these vectors? What if we remove the vector corresponding to 4 o'clock? What if, instead, all vectors have their tails at 12 o'clock, and their heads on the remaining digits? ⇒

14. Let **a** and **b** be nonzero vectors in two dimensions that are not parallel or anti-parallel. Show, algebraically, that if **c** is any two dimensional vector, there are scalars s and t such that $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$.

15. Does the statement in the previous exercise hold if the vectors **a**, **b**, and **c** are three dimensional vectors? Explain.

12.3 The Dot Product

Here's a question whose answer turns out to be very useful: Given two vectors, what is the angle between them?

It may not be immediately clear that the question makes sense, but it's not hard to turn it into a question that does. Since vectors have no position, we are as usual free to place vectors wherever we like. If the two vectors are placed tail-to-tail, there is now a reasonable interpretation of the question: we seek the measure of the smallest angle between the two vectors, in the plane in which they lie. Figure 12.3.1 illustrates the situation.

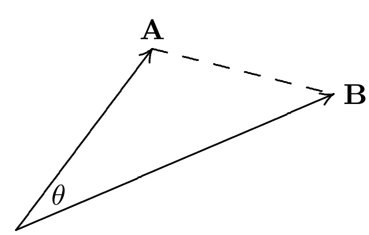


Figure 12.3.1 The angle between vectors **A** and **B**.

Since the angle θ lies in a triangle, we can compute it using a bit of trigonometry, namely, the law of cosines. The lengths of the sides of the triangle in figure 12.3.1 are $|\mathbf{A}|$,

$|\mathbf{B}|$, and $|\mathbf{A}-\mathbf{B}|$. Let $\mathbf{A}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{B}=\langle b_1,b_2,b_3\rangle$; then

$$\begin{aligned}
|\mathbf{A}-\mathbf{B}|^2 &= |\mathbf{A}|^2+|\mathbf{B}|^2-2|\mathbf{A}||\mathbf{B}|\cos\theta\\
2|\mathbf{A}||\mathbf{B}|\cos\theta &= |\mathbf{A}|^2+|\mathbf{B}|^2-|\mathbf{A}-\mathbf{B}|^2\\
&= a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^2-(a_1-b_1)^2-(a_2-b_2)^2-(a_3-b_3)^2\\
&= a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^2\\
&\qquad -(a_1^2-2a_1b_1+b_1^2)-(a_2^2-2a_2b_2+b_2^2)-(a_3^2-2a_3b_3+b_3^2)\\
&= 2a_1b_1+2a_2b_2+2a_3b_3\\
|\mathbf{A}||\mathbf{B}|\cos\theta &= a_1b_1+a_2b_2+a_3b_3\\
\cos\theta &= (a_1b_1+a_2b_2+a_3b_3)/(|\mathbf{A}||\mathbf{B}|)
\end{aligned}$$

So a bit of simple arithmetic with the coordinates of $\mathbf{A}$ and $\mathbf{B}$ allows us to compute the cosine of the angle between them. If necessary we can use the arccosine to get θ, but in many problems $\cos\theta$ turns out to be all we really need.

The numerator of the fraction that gives us $\cos\theta$ turns up a lot, so we give it a name and more compact notation: we call it the **dot product**, and write it as

$$\mathbf{A}\cdot\mathbf{B}=a_1b_1+a_2b_2+a_3b_3.$$

This is the same symbol we use for ordinary multiplication, but there should never be any confusion; you can tell from context whether we are "multiplying" vectors or numbers. (We might also use the dot for scalar multiplication: $a\cdot\mathbf{V}=a\mathbf{V}$; again, it is clear what is meant from context.)

EXAMPLE 12.3.1 Find the angle between the vectors $\mathbf{A}=\langle 1,2,1\rangle$ and $\mathbf{B}=\langle 3,1,-5\rangle$. We know that $\cos\theta=\mathbf{A}\cdot\mathbf{B}/(|\mathbf{A}||\mathbf{B}|)=(1\cdot 3+2\cdot 1+1\cdot(-5))/(|\mathbf{A}||\mathbf{B}|)=0$, so $\theta=\pi/2$, that is, the vectors are perpendicular. □

EXAMPLE 12.3.2 Find the angle between the vectors $\mathbf{A}=\langle 3,3,0\rangle$ and $\mathbf{B}=\langle 1,0,0\rangle$. We compute

$$\begin{aligned}
\cos\theta &= (3\cdot 1+3\cdot 0+0\cdot 0)/(\sqrt{9+9+0}\sqrt{1+0+0})\\
&= 3/\sqrt{18}=1/\sqrt{2}
\end{aligned}$$

so $\theta=\pi/4$. □

EXAMPLE 12.3.3 Some special cases are worth looking at: Find the angles between $\mathbf{A}$ and $\mathbf{A}$; $\mathbf{A}$ and $-\mathbf{A}$; $\mathbf{A}$ and $\mathbf{0}=\langle 0,0,0\rangle$.

$\cos\theta = \mathbf{A}\cdot\mathbf{A}/(|\mathbf{A}||\mathbf{A}|) = (a_1^2+a_2^2+a_3^2)/(\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{a_1^2+a_2^2+a_3^2}) = 1$, so the angle between $\mathbf{A}$ and itself is zero, which of course is correct.

$\cos\theta = \mathbf{A}\cdot -\mathbf{A}/(|\mathbf{A}||-\mathbf{A}|) = (-a_1^2-a_2^2-a_3^2)/(\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{a_1^2+a_2^2+a_3^2}) = -1$, so the angle is π, that is, the vectors point in opposite directions, as of course we already knew.

$\cos\theta = \mathbf{A}\cdot\mathbf{0}/(|\mathbf{A}||\mathbf{0}|) = (0+0+0)/(\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{0^2+0^2+0^2})$, which is undefined. On the other hand, note that since $\mathbf{A}\cdot\mathbf{0} = 0$ it looks at first as if $\cos\theta$ will be zero, which as we have seen means that vectors are perpendicular; only when we notice that the denominator is also zero do we run into trouble. One way to "fix" this is to adopt the convention that the zero vector $\mathbf{0}$ is perpendicular to all vectors; then we can say in general that if $\mathbf{A}\cdot\mathbf{B} = 0$, $\mathbf{A}$ and $\mathbf{B}$ are perpendicular. □

Generalizing the examples, note the following useful facts:

1. If $\mathbf{A}$ is parallel or anti-parallel to $\mathbf{B}$ then $\mathbf{A}\cdot\mathbf{B}/(|\mathbf{A}||\mathbf{B}|) = \pm 1$, and conversely, if $\mathbf{A}\cdot\mathbf{B}/(|\mathbf{A}||\mathbf{B}|) = 1$, $\mathbf{A}$ and $\mathbf{B}$ are parallel, while if $\mathbf{A}\cdot\mathbf{B}/(|\mathbf{A}||\mathbf{B}|) = -1$, $\mathbf{A}$ and $\mathbf{B}$ are anti-parallel. (Vectors are parallel if they point in the same direction, anti-parallel if they point in opposite directions.)
2. If $\mathbf{A}$ is perpendicular to $\mathbf{B}$ then $\mathbf{A}\cdot\mathbf{B}/(|\mathbf{A}||\mathbf{B}|) = 0$, and conversely if $\mathbf{A}\cdot\mathbf{B}/(|\mathbf{A}||\mathbf{B}|) = 0$ then $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

Given two vectors, it is often useful to find the **projection** of one vector onto the other, because this turns out to have important meaning in many circumstances. More precisely, given $\mathbf{A}$ and $\mathbf{B}$, we seek a vector parallel to $\mathbf{B}$ but with length determined by $\mathbf{A}$ in a natural way, as shown in figure 12.3.2. $\mathbf{V}$ is chosen so that the triangle formed by $\mathbf{A}$, $\mathbf{V}$, and $\mathbf{A}-\mathbf{V}$ is a right triangle.

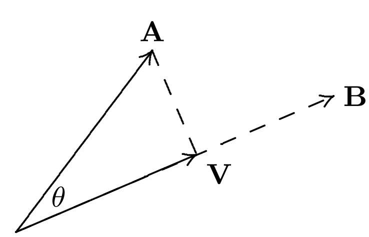


Figure 12.3.2 $\mathbf{V}$ is the projection of $\mathbf{A}$ onto $\mathbf{B}$.

Using a little trigonometry, we see that

$$|\mathbf{V}| = |\mathbf{A}|\cos\theta = |\mathbf{A}|\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|};$$

this is sometimes called the **scalar projection of A onto B**. To get **V** itself, we multiply this length by a vector of length one parallel to **B**:

$$\mathbf{V} = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|}\frac{\mathbf{B}}{|\mathbf{B}|} = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|^2}\mathbf{B}.$$

Be sure that you understand why $\mathbf{B}/|\mathbf{B}|$ is a vector of length one (also called a **unit vector**) parallel to **B**.

The discussion so far implicitly assumed that $0 \le \theta \le \pi/2$. If $\pi/2 < \theta \le \pi$, the picture is like figure 12.3.3. In this case $\mathbf{A}\cdot\mathbf{B}$ is negative, so the vector

$$\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|^2}\mathbf{B}$$

is anti-parallel to **B**, and its length is

$$\left|\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|}\right|.$$

So in general, the scalar projection of **A** onto **B** may be positive or negative. If it is negative, it means that the projection vector is anti-parallel to **B** and that the length of the projection vector is the absolute value of the scalar projection. Of course, you can also compute the length of the projection vector as usual, by applying the distance formula to the vector.

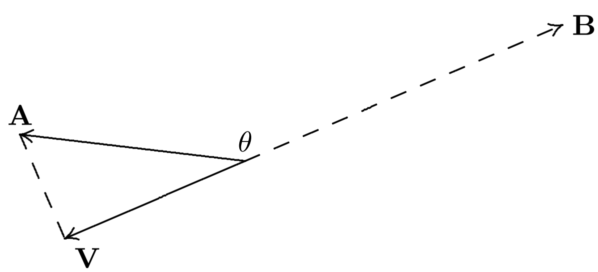


Figure 12.3.3 **V** is the projection of **A** onto **B**.

Note that the phrase "projection onto **B**" is a bit misleading if taken literally; all that **B** provides is a direction; the length of **B** has no impact on the final vector. In figure 12.3.4, for example, **B** is shorter than the projection vector, but this is perfectly acceptable.

Figure 12.3.4 **V** is the projection of **A** onto **B**.

EXAMPLE 12.3.4 Physical force is a vector quantity. It is often necessary to compute the "component" of a force acting in a different direction than the force is being applied. For example, suppose a ten pound weight is resting on an inclined plane—a pitched roof, for example. Gravity exerts a force of ten pounds on the object, directed straight down. It is useful to think of the component of this force directed down and parallel to the roof, and the component down and directly into the roof. These forces are the projections of the force vector onto vectors parallel and perpendicular to the roof. Suppose the roof is tilted at a 30° angle, as in figure 12.3.5. A vector parallel to the roof is $\langle -\sqrt{3}, -1\rangle$, and a vector perpendicular to the roof is $\langle 1, -\sqrt{3}\rangle$. The force vector is $\mathbf{F} = \langle 0, -10\rangle$. The component of the force directed down the roof is then

$$\mathbf{F}_1 = \frac{\mathbf{F}\cdot\langle -\sqrt{3}, -1\rangle}{|\langle -\sqrt{3}, -1\rangle|^2}\langle -\sqrt{3}, -1\rangle = \frac{10}{2}\frac{\langle -\sqrt{3}, -1\rangle}{2} = \langle -5\sqrt{3}/2, -5/2\rangle$$

with length 5. The component of the force directed into the roof is

$$\mathbf{F}_2 = \frac{\mathbf{F}\cdot\langle 1, -\sqrt{3}\rangle}{|\langle 1, -\sqrt{3}\rangle|^2}\langle 1, -\sqrt{3}\rangle = \frac{10\sqrt{3}}{2}\frac{\langle 1, -\sqrt{3}\rangle}{2} = \langle 5\sqrt{3}/2, -15/2\rangle$$

with length $5\sqrt{3}$. Thus, a force of 5 pounds is pulling the object down the roof, while a force of $5\sqrt{3}$ pounds is pulling the object into the roof. □

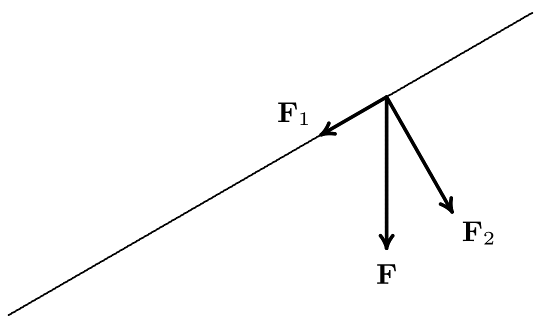


Figure 12.3.5 Components of a force.

The dot product has some familiar-looking properties that will be useful later, so we list them here. These may be proved by writing the vectors in coordinate form and then performing the indicated calculations; subsequently it can be easier to use the properties instead of calculating with coordinates.

THEOREM 12.3.5 If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and a is a real number, then

1. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$

2. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

4. $(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (a\mathbf{v})$ ■

Exercises 12.3.

1. Find $\langle 1,1,1\rangle \cdot \langle 2,-3,4\rangle$. ⇒

2. Find $\langle 1,2,0\rangle \cdot \langle 0,0,57\rangle$. ⇒

3. Find $\langle 3,2,1\rangle \cdot \langle 0,1,0\rangle$. ⇒

4. Find $\langle -1,-2,5\rangle \cdot \langle 1,0,-1\rangle$. ⇒

5. Find $\langle 3,4,6\rangle \cdot \langle 2,3,4\rangle$. ⇒

6. Find the cosine of the angle between $\langle 1,2,3\rangle$ and $\langle 1,1,1\rangle$; use a calculator if necessary to find the angle. ⇒

7. Find the cosine of the angle between $\langle -1,-2,-3\rangle$ and $\langle 5,0,2\rangle$; use a calculator if necessary to find the angle. ⇒

8. Find the cosine of the angle between $\langle 47,100,0\rangle$ and $\langle 0,0,5\rangle$; use a calculator if necessary to find the angle. ⇒

9. Find the cosine of the angle between $\langle 1,0,1\rangle$ and $\langle 0,1,1\rangle$; use a calculator if necessary to find the angle. ⇒

10. Find the cosine of the angle between $\langle 2,0,0\rangle$ and $\langle -1,1,-1\rangle$; use a calculator if necessary to find the angle. ⇒

11. Find the angle between the diagonal of a cube and one of the edges adjacent to the diagonal. ⇒

12. Find the scalar and vector projections of $\langle 1,2,3\rangle$ onto $\langle 1,2,0\rangle$. ⇒

13. Find the scalar and vector projections of $\langle 1,1,1\rangle$ onto $\langle 3,2,1\rangle$. ⇒

14. A force of 10 pounds is applied to a wagon, directed at an angle of 30°. Find the component of this force pulling the wagon straight up, and the component pulling it horizontally along the ground. ⇒

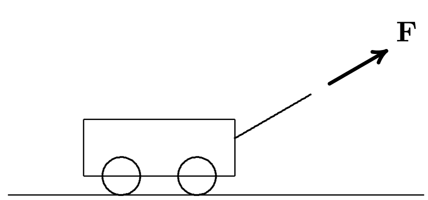


Figure 12.3.6 Pulling a wagon.

15. A force of 15 pounds is applied to a wagon, directed at an angle of 45°. Find the component of this force pulling the wagon straight up, and the component pulling it horizontally along the ground. ⇒

16. Use the dot product to find a non-zero vector $\mathbf{w}$ perpendicular to both $\mathbf{u} = \langle 1,2,-3\rangle$ and $\mathbf{v} = \langle 2,0,1\rangle$. ⇒

17. Let $\mathbf{x} = \langle 1, 1, 0\rangle$ and $\mathbf{y} = \langle 2, 4, 2\rangle$. Find a unit vector that is perpendicular to both $\mathbf{x}$ and $\mathbf{y}$. ⇒

18. Do the three points $(1, 2, 0)$, $(-2, 1, 1)$, and $(0, 3, -1)$ form a right triangle? ⇒

19. Do the three points $(1, 1, 1)$, $(2, 3, 2)$, and $(5, 0, -1)$ form a right triangle? ⇒

20. Show that $|\mathbf{A}\cdot\mathbf{B}| \le |\mathbf{A}||\mathbf{B}|$

21. Let $\mathbf{x}$ and $\mathbf{y}$ be perpendicular vectors. Use Theorem 12.3.5 to prove that $|\mathbf{x}|^2+|\mathbf{y}|^2 = |\mathbf{x}+\mathbf{y}|^2$. What is this result better known as?

22. Prove that the diagonals of a rhombus intersect at right angles.

23. Suppose that $\mathbf{z} = |\mathbf{x}|\mathbf{y} + |\mathbf{y}|\mathbf{x}$ where $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are all nonzero vectors. Prove that $\mathbf{z}$ bisects the angle between $\mathbf{x}$ and $\mathbf{y}$.

24. Prove Theorem 12.3.5.

12.4 The Cross Product

Another useful operation: Given two vectors, find a third vector perpendicular to the first two. There are of course an infinite number of such vectors of different lengths. Nevertheless, let us find one. Suppose $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3\rangle$. We want to find a vector $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ with $\mathbf{v}\cdot\mathbf{A} = \mathbf{v}\cdot\mathbf{B} = 0$, or

$$a_1v_1 + a_2v_2 + a_3v_3 = 0,$$
$$b_1v_1 + b_2v_2 + b_3v_3 = 0.$$

Multiply the first equation by b_3 and the second by a_3 and subtract to get

$$\begin{aligned} b_3a_1v_1 + b_3a_2v_2 + b_3a_3v_3 &= 0\\ a_3b_1v_1 + a_3b_2v_2 + a_3b_3v_3 &= 0\\ (a_1b_3 - b_1a_3)v_1 + (a_2b_3 - b_2a_3)v_2 &= 0 \end{aligned}$$

Of course, this equation in two variables has many solutions; a particularly easy one to see is $v_1 = a_2b_3 - b_2a_3$, $v_2 = b_1a_3 - a_1b_3$. Substituting back into either of the original equations and solving for v_3 gives $v_3 = a_1b_2 - b_1a_2$.

This particular answer to the problem turns out to have some nice properties, and it is dignified with a name: the **cross product**:

$$\mathbf{A}\times\mathbf{B} = \langle a_2b_3 - b_2a_3, b_1a_3 - a_1b_3, a_1b_2 - b_1a_2\rangle.$$

While there is a nice pattern to this vector, it can be a bit difficult to memorize; here is a convenient mnemonic. The determinant of a two by two matrix is

$$\begin{vmatrix} a & b\\ c & d\end{vmatrix} = ad - cb.$$

This is extended to the determinant of a three by three matrix:

$$\begin{aligned}\begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} &= x\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - y\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + z\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= x(a_2b_3 - b_2a_3) - y(a_1b_3 - b_1a_3) + z(a_1b_2 - b_1a_2) \\ &= x(a_2b_3 - b_2a_3) + y(b_1a_3 - a_1b_3) + z(a_1b_2 - b_1a_2).\end{aligned}$$

Each of the two by two matrices is formed by deleting the top row and one column of the three by three matrix; the subtraction of the middle term must also be memorized. This is not the place to extol the uses of the determinant; suffice it to say that determinants are extraordinarily useful and important. Here we want to use it merely as a mnemonic device. You will have noticed that the three expressions in parentheses on the last line are precisely the three coordinates of the cross product; replacing x, y, z by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ gives us

$$\begin{aligned}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} &= (a_2b_3 - b_2a_3)\mathbf{i} - (a_1b_3 - b_1a_3)\mathbf{j} + (a_1b_2 - b_1a_2)\mathbf{k} \\ &= (a_2b_3 - b_2a_3)\mathbf{i} + (b_1a_3 - a_1b_3)\mathbf{j} + (a_1b_2 - b_1a_2)\mathbf{k} \\ &= \langle a_2b_3 - b_2a_3, b_1a_3 - a_1b_3, a_1b_2 - b_1a_2\rangle \\ &= \mathbf{A}\times\mathbf{B}.\end{aligned}$$

Given $\mathbf{A}$ and $\mathbf{B}$, there are typically two possible directions and an infinite number of magnitudes that will give a vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. As we have picked a particular one, we should investigate the magnitude and direction.

We know how to compute the magnitude of $\mathbf{A}\times\mathbf{B}$; it's a bit messy but not difficult. It is somewhat easier to work initially with the square of the magnitude, so as to avoid the square root:

$$\begin{aligned}|\mathbf{A}\times\mathbf{B}|^2 &= (a_2b_3 - b_2a_3)^2 + (b_1a_3 - a_1b_3)^2 + (a_1b_2 - b_1a_2)^2 \\ &= a_2^2b_3^2 - 2a_2b_3b_2a_3 + b_2^2a_3^2 + b_1^2a_3^2 - 2b_1a_3a_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1b_2b_1a_2 + b_1^2a_2^2\end{aligned}$$

While it is far from obvious, this nasty looking expression can be simplified:

$$\begin{aligned}|\mathbf{A}\times\mathbf{B}|^2 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= |\mathbf{A}|^2|\mathbf{B}|^2 - (\mathbf{A}\cdot\mathbf{B})^2 \\ &= |\mathbf{A}|^2|\mathbf{B}|^2 - |\mathbf{A}|^2|\mathbf{B}|^2\cos^2\theta \\ &= |\mathbf{A}|^2|\mathbf{B}|^2(1 - \cos^2\theta) \\ &= |\mathbf{A}|^2|\mathbf{B}|^2\sin^2\theta \\ |\mathbf{A}\times\mathbf{B}| &= |\mathbf{A}||\mathbf{B}|\sin\theta\end{aligned}$$

The magnitude of $\mathbf{A}\times\mathbf{B}$ is thus very similar to the dot product. In particular, notice that if $\mathbf{A}$ is parallel to $\mathbf{B}$, the angle between them is zero, so $\sin\theta = 0$, so $|\mathbf{A}\times\mathbf{B}| = 0$, and likewise if they are anti-parallel, $\sin\theta = 0$, and $|\mathbf{A}\times\mathbf{B}| = 0$. Conversely, if $|\mathbf{A}\times\mathbf{B}| = 0$ and $|\mathbf{A}|$ and $|\mathbf{B}|$ are not zero, it must be that $\sin\theta = 0$, so $\mathbf{A}$ is parallel or anti-parallel to $\mathbf{B}$.

Here is a curious fact about this quantity that turns out to be quite useful later on: Given two vectors, we can put them tail to tail and form a parallelogram, as in figure 12.4.1. The height of the parallelogram, h, is $|\mathbf{A}|\sin\theta$, and the base is $|\mathbf{B}|$, so the area of the parallelogram is $|\mathbf{A}||\mathbf{B}|\sin\theta$, exactly the magnitude of $|\mathbf{A}\times\mathbf{B}|$.

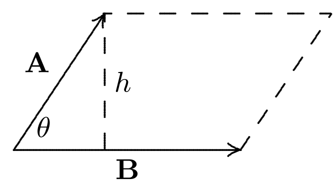


Figure 12.4.1 A parallelogram.

What about the direction of the cross product? Remarkably, there is a simple rule that describes the direction. Let's look at a simple example: Let $\mathbf{A} = \langle a, 0, 0\rangle$, $\mathbf{B} = \langle b, c, 0\rangle$. If the vectors are placed with tails at the origin, $\mathbf{A}$ lies along the x-axis and $\mathbf{B}$ lies in the x-y plane, so we know the cross product will point either up or down. The cross product is

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & 0 & 0 \\ b & c & 0 \end{vmatrix} = \langle 0, 0, ac\rangle.$$

As predicted, this is a vector pointing up or down, depending on the sign of ac. Suppose that $a > 0$, so the sign depends only on c: if $c > 0$, $ac > 0$ and the vector points up; if $c < 0$, the vector points down. On the other hand, if $a < 0$ and $c > 0$, the vector points down, while if $a < 0$ and $c < 0$, the vector points up. Here is how to interpret these facts with a single rule: Imagine rotating vector $\mathbf{A}$ until it points in the same direction as $\mathbf{B}$; there are two ways to do this—use the rotation that goes through the smaller angle. If $a > 0$ and $c > 0$, or $a < 0$ and $c < 0$, the rotation will be counter-clockwise when viewed from above; in the other two cases, $\mathbf{A}$ must be rotated clockwise to reach $\mathbf{B}$. The rule is: counter-clockwise means up, clockwise means down. If $\mathbf{A}$ and $\mathbf{B}$ are any vectors in the x-y plane, the same rule applies—$\mathbf{A}$ need not be parallel to the x-axis.

Although it is somewhat difficult computationally to see how this plays out for any two starting vectors, the rule is essentially the same. Place $\mathbf{A}$ and $\mathbf{B}$ tail to tail. The plane in which $\mathbf{A}$ and $\mathbf{B}$ lie may be viewed from two sides; view it from the side for which $\mathbf{A}$ must rotate counter-clockwise to reach $\mathbf{B}$; then the vector $\mathbf{A}\times\mathbf{B}$ points toward you.

This rule is usually called the **right hand rule**. Imagine placing the heel of your right hand at the point where the tails are joined, so that your slightly curled fingers indicate the direction of rotation from $\mathbf{A}$ to $\mathbf{B}$. Then your thumb points in the direction of the cross product $\mathbf{A}\times\mathbf{B}$.

One immediate consequence of these facts is that $\mathbf{A}\times\mathbf{B}\neq\mathbf{B}\times\mathbf{A}$, because the two cross products point in the opposite direction. On the other hand, since

$$|\mathbf{A}\times\mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\theta = |\mathbf{B}||\mathbf{A}|\sin\theta = |\mathbf{B}\times\mathbf{A}|,$$

the lengths of the two cross products are equal, so we know that $\mathbf{A}\times\mathbf{B} = -(\mathbf{B}\times\mathbf{A})$.

The cross product has some familiar-looking properties that will be useful later, so we list them here. As with the dot product, these can be proved by performing the appropriate calculations on coordinates, after which we may sometimes avoid such calculations by using the properties.

THEOREM 12.4.1 If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and a is a real number, then

1. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$
2. $(\mathbf{v}+\mathbf{w})\times\mathbf{u} = \mathbf{v}\times\mathbf{u}+\mathbf{w}\times\mathbf{u}$
3. $(a\mathbf{u})\times\mathbf{v} = a(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times(a\mathbf{v})$
4. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$
5. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ ■

Exercises 12.4.

1. Find the cross product of $\langle 1,1,1\rangle$ and $\langle 1,2,3\rangle$. ⇒
2. Find the cross product of $\langle 1,0,2\rangle$ and $\langle -1,-2,4\rangle$. ⇒
3. Find the cross product of $\langle -2,1,3\rangle$ and $\langle 5,2,-1\rangle$. ⇒
4. Find the cross product of $\langle 1,0,0\rangle$ and $\langle 0,0,1\rangle$. ⇒
5. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are separated by an angle of $\pi/6$, and $|\mathbf{u}|=2$ and $|\mathbf{v}|=3$. Find $|\mathbf{u}\times\mathbf{v}|$. ⇒
6. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are separated by an angle of $\pi/4$, and $|\mathbf{u}|=3$ and $|\mathbf{v}|=7$. Find $|\mathbf{u}\times\mathbf{v}|$. ⇒
7. Find the area of the parallelogram with vertices $(0,0)$, $(1,2)$, $(3,7)$, and $(2,5)$. ⇒
8. Find and explain the value of $(\mathbf{i}\times\mathbf{j})\times\mathbf{k}$ and $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j})$.
9. Prove that for all vectors $\mathbf{u}$ and $\mathbf{v}$, $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v}=0$.
10. Prove Theorem 12.4.1.
11. Define the triple product of three vectors, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, to be the scalar $\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})$. Show that three vectors lie in the same plane if and only if their triple product is zero. Verify that $\langle 1,5,-2\rangle$, $\langle 4,3,0\rangle$ and $\langle 6,13,-4\rangle$ are coplanar.

12.5 Lines and Planes

Lines and planes are perhaps the simplest of curves and surfaces in three dimensional space. They also will prove important as we seek to understand more complicated curves and surfaces.

The equation of a line in two dimensions is $ax + by = c$; it is reasonable to expect that a line in three dimensions is given by $ax + by + cz = d$; reasonable, but wrong—it turns out that this is the equation of a plane.

A plane does not have an obvious "direction" as does a line. It is possible to associate a plane with a direction in a very useful way, however: there are exactly two directions perpendicular to a plane. Any vector with one of these two directions is called **normal** to the plane. So while there are many normal vectors to a given plane, they are all parallel or anti-parallel to each other.

Suppose two points (v_1, v_2, v_3) and (w_1, w_2, w_3) are in a plane; then the vector $\langle w_1 - v_1, w_2 - v_2, w_3 - v_3\rangle$ is parallel to the plane; in particular, if this vector is placed with its tail at (v_1, v_2, v_3) then its head is at (w_1, w_2, w_3) and it lies in the plane. As a result, any vector perpendicular to the plane is perpendicular to $\langle w_1 - v_1, w_2 - v_2, w_3 - v_3\rangle$. In fact, it is easy to see that the plane consists of *precisely* those points (w_1, w_2, w_3) for which $\langle w_1 - v_1, w_2 - v_2, w_3 - v_3\rangle$ is perpendicular to a normal to the plane, as indicated in figure 12.5.1. Turning this around, suppose we know that $\langle a, b, c\rangle$ is normal to a plane containing the point (v_1, v_2, v_3). Then (x, y, z) is in the plane if and only if $\langle a, b, c\rangle$ is perpendicular to $\langle x - v_1, y - v_2, z - v_3\rangle$. In turn, we know that this is true precisely when $\langle a, b, c\rangle \cdot \langle x - v_1, y - v_2, z - v_3\rangle = 0$. That is, (x, y, z) is in the plane if and only if

$$\begin{aligned}
\langle a, b, c\rangle \cdot \langle x - v_1, y - v_2, z - v_3\rangle &= 0\\
a(x - v_1) + b(y - v_2) + c(z - v_3) &= 0\\
ax + by + cz - av_1 - bv_2 - cv_3 &= 0\\
ax + by + cz &= av_1 + bv_2 + cv_3.
\end{aligned}$$

Working backwards, note that if (x, y, z) is a point satisfying $ax + by + cz = d$ then

$$\begin{aligned}
ax + by + cz &= d\\
ax + by + cz - d &= 0\\
a(x - d/a) + b(y - 0) + c(z - 0) &= 0\\
\langle a, b, c\rangle \cdot \langle x - d/a, y, z\rangle &= 0.
\end{aligned}$$

Namely, $\langle a, b, c\rangle$ is perpendicular to the vector with tail at $(d/a, 0, 0)$ and head at (x, y, z). This means that the points (x, y, z) that satisfy the equation $ax + by + cz = d$ form a

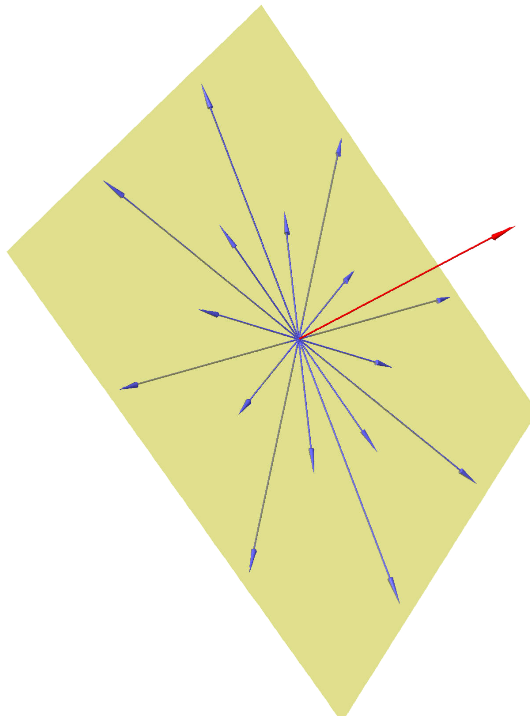


Figure 12.5.1 A plane defined via vectors perpendicular to a normal. (AP)

plane perpendicular to $\langle a,b,c\rangle$. (This doesn't work if $a=0$, but in that case we can use b or c in the role of a. That is, either $a(x-0)+b(y-d/b)+c(z-0)=0$ or $a(x-0)+b(y-0)+c(z-d/c)=0$.)

Thus, given a vector $\langle a,b,c\rangle$ we know that all planes perpendicular to this vector have the form $ax+by+cz=d$, and any surface of this form is a plane perpendicular to $\langle a,b,c\rangle$.

EXAMPLE 12.5.1 Find an equation for the plane perpendicular to $\langle 1,2,3\rangle$ and containing the point $(5,0,7)$.

Using the derivation above, the plane is $1x+2y+3z=1\cdot 5+2\cdot 0+3\cdot 7=26$. Alternately, we know that the plane is $x+2y+3z=d$, and to find d we may substitute the known point on the plane to get $5+2\cdot 0+3\cdot 7=d$, so $d=26$. □

EXAMPLE 12.5.2 Find a vector normal to the plane $2x-3y+z=15$.

One example is $\langle 2,-3,1\rangle$. Any vector parallel or anti-parallel to this works as well, so for example $-2\langle 2,-3,1\rangle=\langle -4,6,-2\rangle$ is also normal to the plane. □

We will frequently need to find an equation for a plane given certain information about the plane. While there may occasionally be slightly shorter ways to get to the desired result, it is always possible, and usually advisable, to use the given information to find a normal to the plane and a point on the plane, and then to find the equation as above.

EXAMPLE 12.5.3 The planes $x - z = 1$ and $y + 2z = 3$ intersect in a line. Find a third plane that contains this line and is perpendicular to the plane $x + y - 2z = 1$.

First, we note that two planes are perpendicular if and only if their normal vectors are perpendicular. Thus, we seek a vector $\langle a, b, c\rangle$ that is perpendicular to $\langle 1, 1, -2\rangle$. In addition, since the desired plane is to contain a certain line, $\langle a, b, c\rangle$ must be perpendicular to any vector parallel to this line. Since $\langle a, b, c\rangle$ must be perpendicular to two vectors, we may find it by computing the cross product of the two. So we need a vector parallel to the line of intersection of the given planes. For this, it suffices to know two points on the line. To find two points on this line, we must find two points that are simultaneously on the two planes, $x - z = 1$ and $y + 2z = 3$. Any point on both planes will satisfy $x - z = 1$ and $y + 2z = 3$. It is easy to find values for x and z satisfying the first, such as $x = 1, z = 0$ and $x = 2, z = 1$. Then we can find corresponding values for y using the second equation, namely $y = 3$ and $y = 1$, so $(1, 3, 0)$ and $(2, 1, 1)$ are both on the line of intersection because both are on both planes. Now $\langle 2-1, 1-3, 1-0\rangle = \langle 1, -2, 1\rangle$ is parallel to the line. Finally, we may choose $\langle a, b, c\rangle = \langle 1, 1, -2\rangle \times \langle 1, -2, 1\rangle = \langle -3, -3, -3\rangle$. While this vector will do perfectly well, any vector parallel or anti-parallel to it will work as well, so for example we might choose $\langle 1, 1, 1\rangle$ which is anti-parallel to it.

Now we know that $\langle 1, 1, 1\rangle$ is normal to the desired plane and $(2, 1, 1)$ is a point on the plane. Therefore an equation of the plane is $x + y + z = 4$. As a quick check, since $(1, 3, 0)$ is also on the line, it should be on the plane; since $1 + 3 + 0 = 4$, we see that this is indeed the case.

Note that had we used $\langle -3, -3, -3\rangle$ as the normal, we would have discovered the equation $-3x - 3y - 3z = -12$, then we might well have noticed that we could divide both sides by -3 to get the equivalent $x + y + z = 4$. □

So we now understand equations of planes; let us turn to lines. Unfortunately, it turns out to be quite inconvenient to represent a typical line with a single equation; we need to approach lines in a different way.

Unlike a plane, a line in three dimensions does have an obvious direction, namely, the direction of any vector parallel to it. In fact a line can be defined and uniquely identified by providing one point on the line and a vector parallel to the line (in one of two possible directions). That is, the line consists of exactly those points we can reach by starting at the point and going for some distance in the direction of the vector. Let's see how we can translate this into more mathematical language.

Suppose a line contains the point (v_1, v_2, v_3) and is parallel to the vector $\langle a, b, c\rangle$. If we place the vector $\langle v_1, v_2, v_3\rangle$ with its tail at the origin and its head at (v_1, v_2, v_3), and if we place the vector $\langle a, b, c\rangle$ with its tail at (v_1, v_2, v_3), then the head of $\langle a, b, c\rangle$ is at a point on the line. We can get to *any* point on the line by doing the same thing, except using $t\langle a, b, c\rangle$ in place of $\langle a, b, c\rangle$, where t is some real number. Because of the way vector

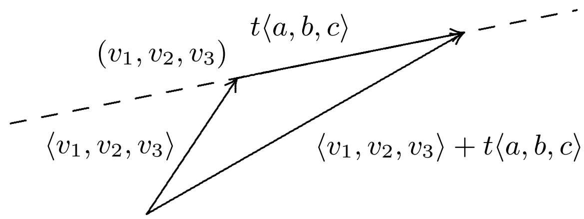


Figure 12.5.2 Vector form of a line.

addition works, the point at the head of the vector $t\langle a, b, c\rangle$ is the point at the head of the vector $\langle v_1, v_2, v_3\rangle + t\langle a, b, c\rangle$, namely $(v_1 + ta, v_2 + tb, v_3 + tc)$; see figure 12.5.2.

In other words, as t runs through all possible real values, the vector $\langle v_1, v_2, v_3\rangle + t\langle a, b, c\rangle$ points to every point on the line when its tail is placed at the origin. Another common way to write this is as a set of **parametric equations**:

$$x = v_1 + ta \qquad y = v_2 + tb \qquad z = v_3 + tc.$$

It is occasionally useful to use this form of a line even in two dimensions; a vector form for a line in the x-y plane is $\langle v_1, v_2\rangle + t\langle a, b\rangle$, which is the same as $\langle v_1, v_2, 0\rangle + t\langle a, b, 0\rangle$.

EXAMPLE 12.5.4 Find a vector expression for the line through $(6, 1, -3)$ and $(2, 4, 5)$. To get a vector parallel to the line we subtract $\langle 6, 1, -3\rangle - \langle 2, 4, 5\rangle = \langle 4, -3, -8\rangle$. The line is then given by $\langle 2, 4, 5\rangle + t\langle 4, -3, -8\rangle$; there are of course many other possibilities, such as $\langle 6, 1, -3\rangle + t\langle 4, -3, -8\rangle$. □

EXAMPLE 12.5.5 Determine whether the lines $\langle 1, 1, 1\rangle + t\langle 1, 2, -1\rangle$ and $\langle 3, 2, 1\rangle + t\langle -1, -5, 3\rangle$ are parallel, intersect, or neither.

In two dimensions, two lines either intersect or are parallel; in three dimensions, lines that do not intersect might not be parallel. In this case, since the direction vectors for the lines are not parallel or anti-parallel we know the lines are not parallel. If they intersect, there must be two values a and b so that $\langle 1, 1, 1\rangle + a\langle 1, 2, -1\rangle = \langle 3, 2, 1\rangle + b\langle -1, -5, 3\rangle$, that is,

$$1 + a = 3 - b$$
$$1 + 2a = 2 - 5b$$
$$1 - a = 1 + 3b$$

This gives three equations in two unknowns, so there may or may not be a solution in general. In this case, it is easy to discover that $a = 3$ and $b = -1$ satisfies all three equations, so the lines do intersect at the point $(4, 7, -2)$. □

EXAMPLE 12.5.6 Find the distance from the point $(1, 2, 3)$ to the plane $2x - y + 3z = 5$. The distance from a point P to a plane is the shortest distance from P to any point on the plane; this is the distance measured from P perpendicular to the plane; see figure 12.5.3. This distance is the absolute value of the scalar projection of $\overrightarrow{QP}$ onto a normal vector $\mathbf{n}$, where Q is any point on the plane. It is easy to find a point on the plane, say $(1,0,1)$. Thus the distance is

$$\frac{\overrightarrow{QP}\cdot\mathbf{n}}{|\mathbf{n}|}=\frac{\langle 0,2,2\rangle\cdot\langle 2,-1,3\rangle}{|\langle 2,-1,3\rangle|}=\frac{4}{\sqrt{14}}.$$

□

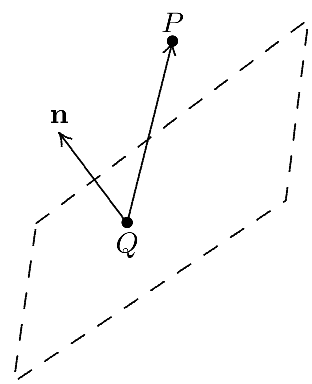


Figure 12.5.3 Distance from a point to a plane.

EXAMPLE 12.5.7 Find the distance from the point $(-1,2,1)$ to the line $\langle 1,1,1\rangle + t\langle 2,3,-1\rangle$. Again we want the distance measured perpendicular to the line, as indicated in figure 12.5.4. The desired distance is

$$|\overrightarrow{QP}|\sin\theta=\frac{|\overrightarrow{QP}\times\mathbf{A}|}{|\mathbf{A}|},$$

where $\mathbf{A}$ is any vector parallel to the line. From the equation of the line, we can use $Q=(1,1,1)$ and $\mathbf{A}=\langle 2,3,-1\rangle$, so the distance is

$$\frac{|\langle -2,1,0\rangle\times\langle 2,3,-1\rangle|}{\sqrt{14}}=\frac{|\langle -1,-2,-8\rangle|}{\sqrt{14}}=\frac{\sqrt{69}}{\sqrt{14}}.$$

□

Exercises 12.5.

1. Find an equation of the plane containing $(6,2,1)$ and perpendicular to $\langle 1,1,1\rangle$. ⇒

2. Find an equation of the plane containing $(-1,2,-3)$ and perpendicular to $\langle 4,5,-1\rangle$. ⇒

3. Find an equation of the plane containing $(1,2,-3)$, $(0,1,-2)$ and $(1,2,-2)$. ⇒

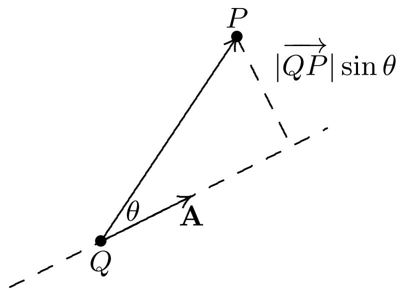


Figure 12.5.4 Distance from a point to a line.

4. Find an equation of the plane containing $(1,0,0)$, $(4,2,0)$ and $(3,2,1)$. ⇒

5. Find an equation of the plane containing $(1,0,0)$ and the line $\langle 1,0,2\rangle + t\langle 3,2,1\rangle$. ⇒

6. Find an equation of the plane containing the line of intersection of $x+y+z=1$ and $x-y+2z=2$, and perpendicular to the x-y plane. ⇒

7. Find an equation of the line through $(1,0,3)$ and $(1,2,4)$. ⇒

8. Find an equation of the line through $(1,0,3)$ and perpendicular to the plane $x+2y-z=1$. ⇒

9. Find an equation of the line through the origin and perpendicular to the plane $x+y-z=2$. ⇒

10. Find a and c so that $(a,1,c)$ is on the line through $(0,2,3)$ and $(2,7,5)$. ⇒

11. Explain how to discover the solution in example 12.5.5.

12. Determine whether the lines $\langle 1,3,-1\rangle + t\langle 1,1,0\rangle$ and $\langle 0,0,0\rangle + t\langle 1,4,5\rangle$ are parallel, intersect, or neither. ⇒

13. Determine whether the lines $\langle 1,0,2\rangle + t\langle -1,-1,2\rangle$ and $\langle 4,4,2\rangle + t\langle 2,2,-4\rangle$ are parallel, intersect, or neither. ⇒

14. Determine whether the lines $\langle 1,2,-1\rangle + t\langle 1,2,3\rangle$ and $\langle 1,0,1\rangle + t\langle 2/3,2,4/3\rangle$ are parallel, intersect, or neither. ⇒

15. Determine whether the lines $\langle 1,1,2\rangle + t\langle 1,2,-3\rangle$ and $\langle 2,3,-1\rangle + t\langle 2,4,-6\rangle$ are parallel, intersect, or neither. ⇒

16. Find a unit normal vector to each of the coordinate planes.

17. Show that $\langle 2,1,3\rangle + t\langle 1,1,2\rangle$ and $\langle 3,2,5\rangle + s\langle 2,2,4\rangle$ are the same line.

18. Give a prose description for each of the following processes:

- **a.** Given two distinct points, find the line that goes through them.
- **b.** Given three points (not all on the same line), find the plane that goes through them. Why do we need the caveat that not all points be on the same line?
- **c.** Given a line and a point not on the line, find the plane that contains them both.
- **d.** Given a plane and a point not on the plane, find the line that is perpendicular to the plane through the given point.

19. Find the distance from $(2,2,2)$ to $x+y+z=-1$. ⇒

20. Find the distance from $(2,-1,-1)$ to $2x-3y+z=2$. ⇒

21. Find the distance from $(2,-1,1)$ to $\langle 2,2,0\rangle + t\langle 1,2,3\rangle$. ⇒

22. Find the distance from $(1,0,1)$ to $\langle 3,2,1\rangle + t\langle 2,-1,-2\rangle$. ⇒

23. Find the cosine of the angle between the planes $x+y+z=2$ and $x+2y+3z=8$. ⇒

24. Find the cosine of the angle between the planes $x-y+2z=2$ and $3x-2y+z=5$. ⇒

12.6 Other Coordinate Systems

Coordinate systems are tools that let us use algebraic methods to understand geometry. While the **rectangular** (also called **Cartesian**) coordinates that we have been discussing are the most common, some problems are easier to analyze in alternate coordinate systems.

A coordinate system is a scheme that allows us to identify any point in the plane or in three-dimensional space by a set of numbers. In rectangular coordinates these numbers are interpreted, roughly speaking, as the lengths of the sides of a rectangular "box."

In two dimensions you may already be familiar with an alternative, called **polar coordinates**. In this system, each point in the plane is identified by a pair of numbers (r,θ). The number θ measures the angle between the positive x-axis and a vector with tail at the origin and head at the point, as shown in figure 12.6.1; the number r measures the distance from the origin to the point. Either of these may be negative; a negative θ indicates the angle is measured clockwise from the positive x-axis instead of counter-clockwise, and a negative r indicates the point at distance $|r|$ in the opposite of the direction given by θ. Figure 12.6.1 also shows the point with rectangular coordinates $(1,\sqrt{3})$ and polar coordinates $(2,\pi/3)$, 2 units from the origin and $\pi/3$ radians from the positive x-axis.

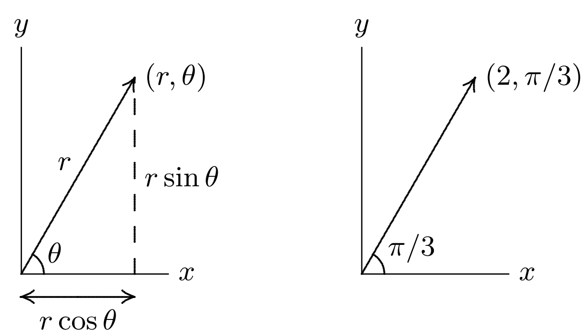


Figure 12.6.1 Polar coordinates: the general case and the point with rectangular coordinates $(1,\sqrt{3})$.

We can extend polar coordinates to three dimensions simply by adding a z coordinate; this is called **cylindrical coordinates**. Each point in three-dimensional space is represented by three coordinates (r,θ,z) in the obvious way: this point is z units above or below the point (r,θ) in the x-y plane, as shown in figure 12.6.2. The point with rectangular coordinates $(1,\sqrt{3},3)$ and cylindrical coordinates $(2,\pi/3,3)$ is also indicated in figure 12.6.2.

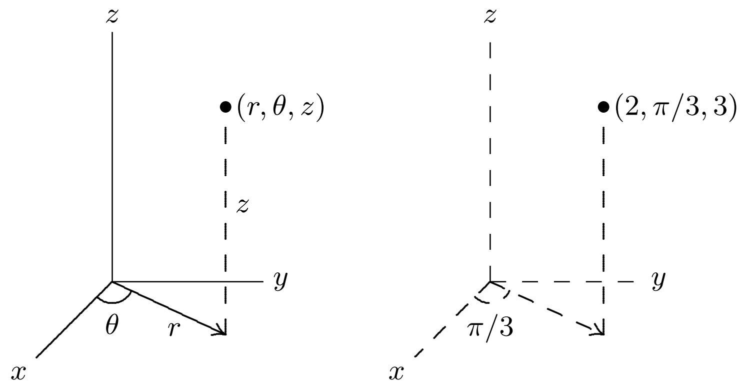


Figure 12.6.2 Cylindrical coordinates: the general case and the point with rectangular coordinates $(1, \sqrt{3}, 3)$.

Some figures with relatively complicated equations in rectangular coordinates will be represented by simpler equations in cylindrical coordinates. For example, the cylinder in figure 12.6.3 has equation $x^2 + y^2 = 4$ in rectangular coordinates, but equation $r = 2$ in cylindrical coordinates.

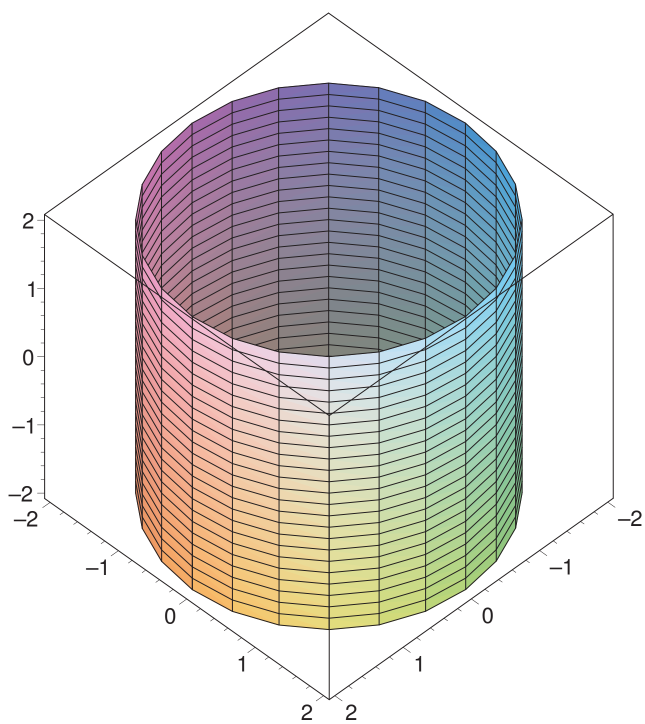


Figure 12.6.3 The cylinder $r = 2$.

Given a point (r, θ) in polar coordinates, it is easy to see (as in figure 12.6.1) that the rectangular coordinates of the same point are $(r\cos\theta, r\sin\theta)$, and so the point (r, θ, z) in cylindrical coordinates is $(r\cos\theta, r\sin\theta, z)$ in rectangular coordinates. This means it is usually easy to convert any equation from rectangular to cylindrical coordinates: simply substitute

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and leave z alone. For example, starting with $x^2 + y^2 = 4$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ gives

$$\begin{aligned} r^2\cos^2\theta + r^2\sin^2\theta &= 4 \\ r^2(\cos^2\theta + \sin^2\theta) &= 4 \\ r^2 &= 4 \\ r &= 2. \end{aligned}$$

Of course, it's easy to see directly that this defines a cylinder as mentioned above.

Cylindrical coordinates are an obvious extension of polar coordinates to three dimensions, but the use of the z coordinate means they are not as closely analogous to polar coordinates as another standard coordinate system. In polar coordinates, we identify a point by a direction and distance from the origin; in three dimensions we can do the same thing, in a variety of ways. The question is: how do we represent a direction? One way is to give the angle of rotation, θ, from the positive x axis, just as in cylindrical coordinates, and also an angle of rotation, ϕ, from the positive z axis. Roughly speaking, θ is like longitude and ϕ is like latitude. (Earth longitude is measured as a positive or negative angle from the prime meridian, and is always between 0 and 180 degrees, east or west; θ can be any positive or negative angle, and we use radians except in informal circumstances. Earth latitude is measured north or south from the equator; ϕ is measured from the north pole down.) This system is called **spherical coordinates**; the coordinates are listed in the order (ρ, θ, ϕ), where ρ is the distance from the origin, and like r in cylindrical coordinates it may be negative. The general case and an example are pictured in figure 12.6.4; the length marked r is the r of cylindrical coordinates.

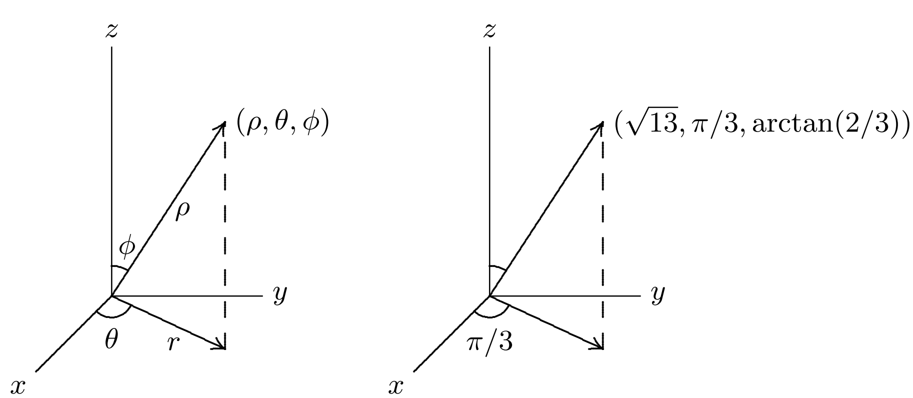


Figure 12.6.4 Spherical coordinates: the general case and the point with rectangular coordinates $(1, \sqrt{3}, 3)$.

As with cylindrical coordinates, we can easily convert equations in rectangular coordinates to the equivalent in spherical coordinates, though it is a bit more difficult to discover the proper substitutions. Figure 12.6.5 shows the typical point in spherical coordinates from figure 12.6.4, viewed now so that the arrow marked r in the original graph appears as the horizontal "axis" in the left hand graph. From this diagram it is easy to see that the z coordinate is $\rho\cos\phi$, and that $r = \rho\sin\phi$, as shown. Thus, in converting from rectangular to spherical coordinates we will replace z by $\rho\cos\phi$. To see the substitutions for x and y we now view the same point from above, as shown in the right hand graph. The hypotenuse of the triangle in the right hand graph is $r = \rho\sin\phi$, so the sides of the triangle, as shown, are $x = r\cos\theta = \rho\sin\phi\cos\theta$ and $y = r\sin\theta = \rho\sin\phi\sin\theta$. So the upshot is that to convert from rectangular to spherical coordinates, we make these substitutions:

$$
\begin{aligned}
x &= \rho\sin\phi\cos\theta \\
y &= \rho\sin\phi\sin\theta \\
z &= \rho\cos\phi.
\end{aligned}
$$

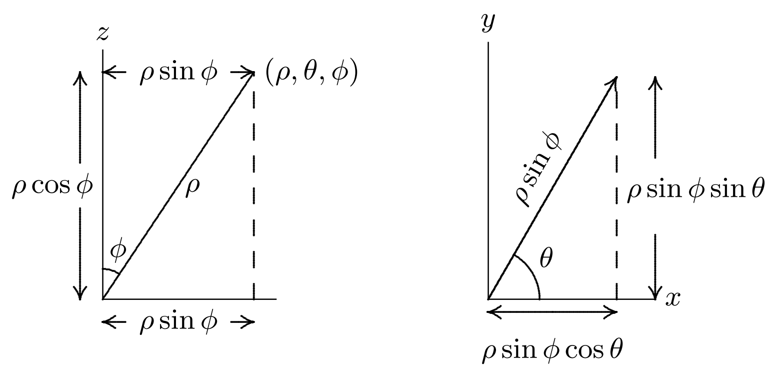


Figure 12.6.5 Converting from rectangular to spherical coordinates.

EXAMPLE 12.6.1 As the cylinder had a simple equation in cylindrical coordinates, so does the sphere in spherical coordinates: $\rho = 2$ is the sphere of radius 2. If we start with the Cartesian equation of the sphere and substitute, we get the spherical equation:

$$
\begin{aligned}
x^2 + y^2 + z^2 &= 2^2 \\
\rho^2 \sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi &= 2^2 \\
\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) + \rho^2\cos^2\phi &= 2^2 \\
\rho^2\sin^2\phi + \rho^2\cos^2\phi &= 2^2 \\
\rho^2(\sin^2\phi + \cos^2\phi) &= 2^2 \\
\rho^2 &= 2^2 \\
\rho &= 2
\end{aligned}
$$

□

EXAMPLE 12.6.2 Find an equation for the cylinder $x^2 + y^2 = 4$ in spherical coordinates.

Proceeding as in the previous example:

$$
\begin{aligned}
x^2 + y^2 &= 4 \\
\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta &= 4 \\
\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) &= 4 \\
\rho^2\sin^2\phi &= 4 \\
\rho\sin\phi &= 2 \\
\rho &= \frac{2}{\sin\phi}
\end{aligned}
$$

□

Exercises 12.6.

1. Convert the following points in rectangular coordinates to cylindrical and spherical coordinates:

- **a.** $(1, 1, 1)$
- **b.** $(7, -7, 5)$
- **c.** $(\cos(1), \sin(1), 1)$
- **d.** $(0, 0, -\pi)$ ⇒

2. Find an equation for the sphere $x^2 + y^2 + z^2 = 4$ in cylindrical coordinates. ⇒

3. Find an equation for the y-z plane in cylindrical coordinates. ⇒

4. Find an equation equivalent to $x^2 + y^2 + 2z^2 + 2z - 5 = 0$ in cylindrical coordinates. ⇒

5. Suppose the curve $z = e^{-x^2}$ in the x-z plane is rotated around the z axis. Find an equation for the resulting surface in cylindrical coordinates. ⇒

6. Suppose the curve $z = x$ in the x-z plane is rotated around the z axis. Find an equation for the resulting surface in cylindrical coordinates. ⇒

7. Find an equation for the plane $y = 0$ in spherical coordinates. ⇒

8. Find an equation for the plane $z = 1$ in spherical coordinates. ⇒

9. Find an equation for the sphere with radius 1 and center at $(0, 1, 0)$ in spherical coordinates. ⇒

10. Find an equation for the cylinder $x^2 + y^2 = 9$ in spherical coordinates. ⇒

11. Suppose the curve $z = x$ in the x-z plane is rotated around the z axis. Find an equation for the resulting surface in spherical coordinates. ⇒

12. Plot the polar equations $r = \sin(\theta)$ and $r = \cos(\theta)$ and comment on their similarities. (If you get stuck on how to plot these, you can multiply both sides of each equation by r and convert back to rectangular coordinates).

13. Extend exercises 6 and 11 by rotating the curve $z = mx$ around the z axis and converting to both cylindrical and spherical coordinates. ⇒

14. Convert the spherical formula $\rho = \sin\theta \sin\phi$ to rectangular coordinates and describe the surface defined by the formula (Hint: Multiply both sides by ρ.) ⇒

15. We can describe points in the first octant by $x > 0$, $y > 0$ and $z > 0$. Give similar inequalities for the first octant in cylindrical and spherical coordinates. ⇒

13

Vector Functions

13.1 Space Curves

We have already seen that a convenient way to describe a line in three dimensions is to provide a vector that "points to" every point on the line as a parameter t varies, like

$$\langle 1,2,3\rangle + t\langle 1,-2,2\rangle = \langle 1+t, 2-2t, 3+2t\rangle.$$

Except that this gives a particularly simple geometric object, there is nothing special about the individual functions of t that make up the coordinates of this vector—any vector with a parameter, like $\langle f(t), g(t), h(t)\rangle$, will describe some curve in three dimensions as t varies through all possible values.

EXAMPLE 13.1.1 Describe the curves $\langle \cos t, \sin t, 0\rangle$, $\langle \cos t, \sin t, t\rangle$, and $\langle \cos t, \sin t, 2t\rangle$.

As t varies, the first two coordinates in all three functions trace out the points on the unit circle, starting with $(1,0)$ when $t = 0$ and proceeding counter-clockwise around the circle as t increases. In the first case, the z coordinate is always 0, so this describes precisely the unit circle in the x-y plane. In the second case, the x and y coordinates still describe a circle, but now the z coordinate varies, so that the height of the curve matches the value of t. When $t = \pi$, for example, the resulting vector is $\langle -1, 0, \pi\rangle$. A bit of thought should convince you that the result is a helix. In the third vector, the z coordinate varies twice as fast as the parameter t, so we get a stretched out helix. Both are shown in figure 13.1.1. On the left is the first helix, shown for t between 0 and 4π; on the right is the second helix, shown for t between 0 and 2π. Both start and end at the same point, but the first helix takes two full "turns" to get there, because its z coordinate grows more slowly. □

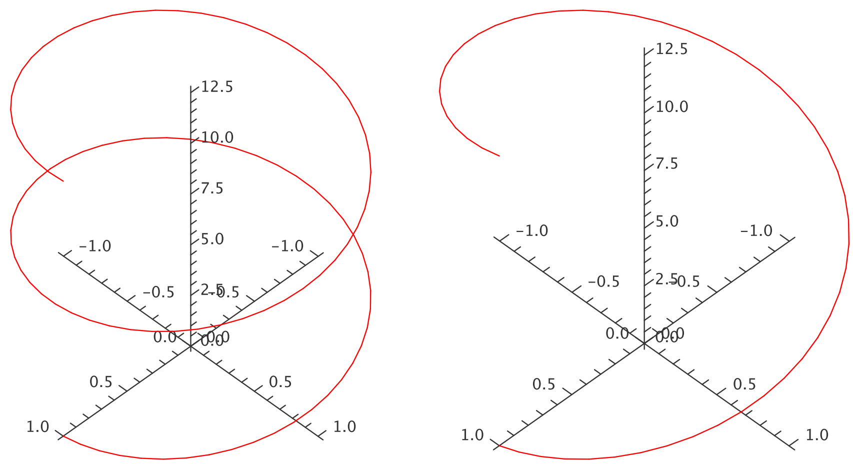


Figure 13.1.1 Two helixes. (AP)

A vector expression of the form $\langle f(t), g(t), h(t)\rangle$ is called a **vector function**; it is a function from the real numbers $\mathbb{R}$ to the set of all three-dimensional vectors. We can alternately think of it as three separate functions, $x = f(t)$, $y = g(t)$, and $z = h(t)$, that describe points in space. In this case we usually refer to the set of equations as **parametric equations** for the curve, just as for a line. While the parameter t in a vector function might represent any one of a number of physical quantities, or be simply a "pure number", it is often convenient and useful to think of t as representing time. The vector function then tells you where in space a particular object is at any time.

Vector functions can be difficult to understand, that is, difficult to picture. When available, computer software can be very helpful. When working by hand, one useful approach is to consider the "projections" of the curve onto the three standard coordinate planes. We have already done this in part: in example 13.1.1 we noted that all three curves project to a circle in the x-y plane, since $\langle \cos t, \sin t\rangle$ is a two dimensional vector function for the unit circle.

EXAMPLE 13.1.2 Graph the projections of $\langle \cos t, \sin t, 2t\rangle$ onto the x-z plane and the y-z plane. The two dimensional vector function for the projection onto the x-z plane is $\langle \cos t, 2t\rangle$, or in parametric form, $x = \cos t$, $z = 2t$. By eliminating t we get the equation $x = \cos(z/2)$, the familiar curve shown on the left in figure 13.1.2. For the projection onto the y-z plane, we start with the vector function $\langle \sin t, 2t\rangle$, which is the same as $y = \sin t$, $z = 2t$. Eliminating t gives $y = \sin(z/2)$, as shown on the right in figure 13.1.2. □

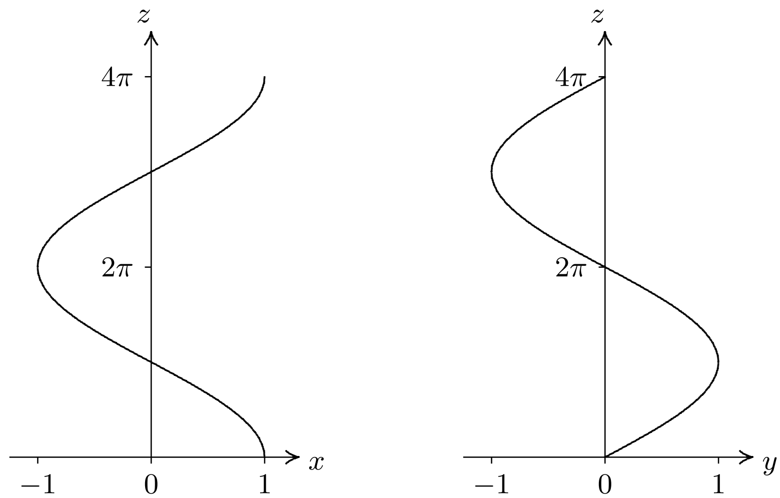


Figure 13.1.2 The projections of $\langle \cos t, \sin t, 2t\rangle$ onto the x-z and y-z planes.

Exercises 13.1.

1. Describe the curve $\mathbf{r} = \langle \sin t, \cos t, \cos 8t\rangle$.
2. Describe the curve $\mathbf{r} = \langle t\cos t, t\sin t, t\rangle$.
3. Describe the curve $\mathbf{r} = \langle t, t^2, \cos t\rangle$.
4. Describe the curve $\mathbf{r} = \langle \cos(20t)\sqrt{1-t^2}, \sin(20t)\sqrt{1-t^2}, t\rangle$
5. Find a vector function for the curve of intersection of $x^2+y^2=9$ and $y+z=2$. ⇒
6. A bug is crawling outward along the spoke of a wheel that lies along a radius of the wheel. The bug is crawling at 1 unit per second and the wheel is rotating at 1 radian per second. Suppose the wheel lies in the y-z plane with center at the origin, and at time $t=0$ the spoke lies along the positive y axis and the bug is at the origin. Find a vector function $\mathbf{r}(t)$ for the position of the bug at time t. ⇒
7. What is the difference between the parametric curves $f(t) = \langle t, t, t^2\rangle$, $g(t) = \langle t^2, t^2, t^4\rangle$, and $h(t) = \langle \sin(t), \sin(t), \sin^2(t)\rangle$as t runs over all real numbers?
8. Plot each of the curves below in 2 dimensions, projected onto each of the three standard planes (the x-y, x-z, and y-z planes).
   a. $f(t) = \langle t, t^3, t^2\rangle$, t ranges over all real numbers
   b. $f(t) = \langle t^2, t-1, t^2+5\rangle$ for $0 \le t \le 3$
9. Given points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, give parametric equations for the line *segment* connecting A and B. Be sure to give appropriate t values.
10. With a parametric plot and a set of t values, we can associate a ‘direction’. For example, the curve $\langle \cos t, \sin t\rangle$ is the unit circle traced counterclockwise. How can we amend a set of given parametric equations and t values to get the same curve, only traced backwards?

13.2 Calculus with vector functions

A vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is a function of one variable—that is, there is only one “input” value. What makes vector functions more complicated than the functions

$y = f(x)$ that we studied in the first part of this book is of course that the "output" values are now three-dimensional vectors instead of simply numbers. It is natural to wonder if there is a corresponding notion of derivative for vector functions. In the simpler case of a function $y = s(t)$, in which t represents time and $s(t)$ is position on a line, we have seen that the derivative $s'(t)$ represents velocity; we might hope that in a similar way the derivative of a vector function would tell us something about the velocity of an object moving in three dimensions.

One way to approach the question of the derivative for vector functions is to write down an expression that is analogous to the derivative we already understand, and see if we can make sense of it. This gives us

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\langle f(t+\Delta t) - f(t), g(t+\Delta t) - g(t), h(t+\Delta t) - h(t)\rangle}{\Delta t} \\
&= \lim_{\Delta t \to 0} \langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \rangle \\
&= \langle f'(t), g'(t), h'(t)\rangle,
\end{aligned}$$

if we say that what we mean by the limit of a vector is the vector of the individual coordinate limits. So starting with a familiar expression for what appears to be a derivative, we find that we can make good computational sense out of it—but what does it actually mean?

We know how to interpret $\mathbf{r}(t+\Delta t)$ and $\mathbf{r}(t)$—they are vectors that point to locations in space; if t is time, we can think of these points as positions of a moving object at times that are Δt apart. We also know what $\Delta\mathbf{r} = \mathbf{r}(t+\Delta t) - \mathbf{r}(t)$ means—it is a vector that points from the head of $\mathbf{r}(t)$ to the head of $\mathbf{r}(t+\Delta t)$, assuming both have their tails at the origin. So when Δt is small, $\Delta\mathbf{r}$ is a tiny vector pointing from one point on the path of the object to a nearby point. As Δt gets close to 0, this vector points in a direction that is closer and closer to the direction in which the object is moving; geometrically, it approaches a vector tangent to the path of the object at a particular point.

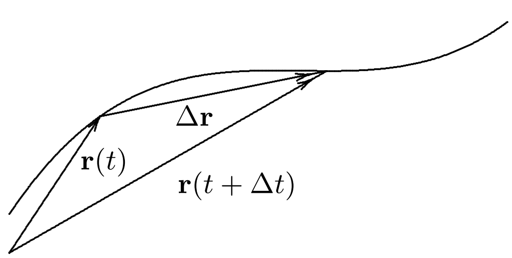


Figure 13.2.1 Approximating the derivative.

Unfortunately, the vector $\Delta\mathbf{r}$ approaches 0 in length; the vector $\langle 0,0,0\rangle$ is not very informative. By dividing by Δt, when it is small, we effectively keep magnifying the length of $\Delta\mathbf{r}$ so that in the limit it doesn't disappear. Thus the limiting vector $\langle f'(t), g'(t), h'(t)\rangle$ will (usually) be a good, non-zero vector that is tangent to the curve.

What about the length of this vector? It's nice that we've kept it away from zero, but what does it measure, if anything? Consider the length of one of the vectors that approaches the tangent vector:

$$\left|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\right| = \frac{|\mathbf{r}(t+\Delta t)-\mathbf{r}(t)|}{|\Delta t|}$$

The numerator is the length of the vector that points from one position of the object to a "nearby" position; this length is approximately the distance traveled by the object between times t and $t+\Delta t$. Dividing this distance by the length of time it takes to travel that distance gives the average speed. As Δt approaches zero, this average speed approaches the actual, instantaneous speed of the object at time t.

So by performing an "obvious" calculation to get something that looks like the derivative of $\mathbf{r}(t)$, we get precisely what we would want from such a derivative: the vector $\mathbf{r}'(t)$ points in the direction of travel of the object and its length tells us the speed of travel. In the case that t is time, then, we call $\mathbf{v}(t) = \mathbf{r}'(t)$ the velocity vector. Even if t is not time, $\mathbf{r}'(t)$ is useful—it is a vector tangent to the curve.

EXAMPLE 13.2.1 We have seen that $\mathbf{r} = \langle \cos t, \sin t, t\rangle$ is a helix. We compute $\mathbf{r}' = \langle -\sin t, \cos t, 1\rangle$, and $|\mathbf{r}'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. So thinking of this as a description of a moving object, its speed is always $\sqrt{2}$; see figure 13.2.2. □

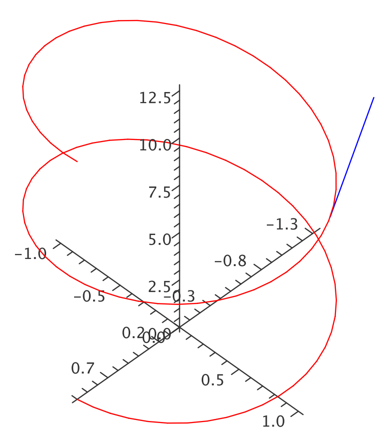


Figure 13.2.2 A tangent vector on the helix. (AP)

EXAMPLE 13.2.2 The velocity vector for $\langle \cos t, \sin t, \cos t\rangle$ is $\langle -\sin t, \cos t, -\sin t\rangle$. As before, the first two coordinates mean that from above this curve looks like a circle. The z coordinate is now also periodic, so that as the object moves around the curve its height oscillates up and down. In fact it turns out that the curve is a tilted ellipse, as shown in figure 13.2.3. □

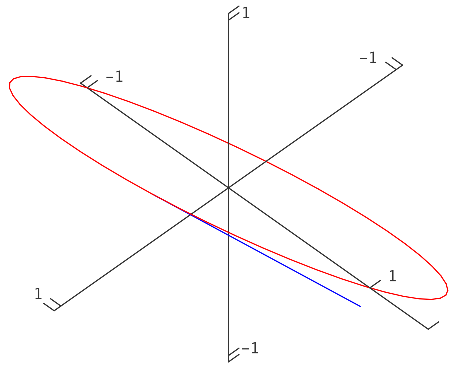


Figure 13.2.3 The ellipse $\mathbf{r} = \langle \cos t, \sin t, \cos t\rangle$. (AP)

EXAMPLE 13.2.3 The velocity vector for $\langle \cos t, \sin t, \cos 2t\rangle$ is $\langle -\sin t, \cos t, -2\sin 2t\rangle$. The z coordinate is now oscillating twice as fast as in the previous example, so the graph is not surprising; see figure 13.2.4. □

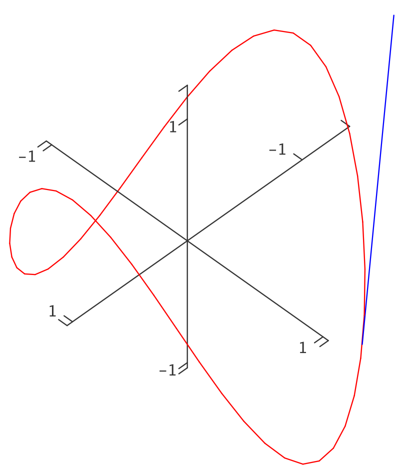


Figure 13.2.4 $\langle \cos t, \sin t, \cos 2t\rangle$. (AP)

EXAMPLE 13.2.4 Find the angle between the curves $\langle t, 1-t, 3+t^2\rangle$ and $\langle 3-t, t-2, t^2\rangle$ where they meet.

The angle between two curves at a point is the angle between their tangent vectors—any tangent vectors will do, so we can use the derivatives. We need to find the point of intersection, evaluate the two derivatives there, and finally find the angle between them.

To find the point of intersection, we need to solve the equations

$$\begin{aligned} t &= 3-u \\ 1-t &= u-2 \\ 3+t^2 &= u^2 \end{aligned}$$

Solving either of the first two equations for u and substituting in the third gives $3+t^2 = (3-t)^2$, which means $t = 1$. This together with $u = 2$ satisfies all three equations. Thus the two curves meet at $(1, 0, 4)$, the first when $t = 1$ and the second when $t = 2$.

The derivatives are $\langle 1, -1, 2t\rangle$ and $\langle -1, 1, 2t\rangle$; at the intersection point these are $\langle 1, -1, 2\rangle$ and $\langle -1, 1, 4\rangle$. The cosine of the angle between them is then

$$\cos\theta = \frac{-1-1+8}{\sqrt{6}\sqrt{18}} = \frac{1}{\sqrt{3}},$$

so $\theta = \arccos(1/\sqrt{3}) \approx 0.96$. □

The derivatives of vector functions obey some familiar looking rules, which we will occasionally need.

THEOREM 13.2.5 Suppose $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are differentiable functions, $f(t)$ is a differentiable function, and a is a real number.

a. $\frac{d}{dt} a\mathbf{r}(t) = a\mathbf{r}'(t)$

b. $\frac{d}{dt}(\mathbf{r}(t) + \mathbf{s}(t)) = \mathbf{r}'(t) + \mathbf{s}'(t)$

c. $\frac{d}{dt} f(t)\mathbf{r}(t) = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$

d. $\frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{s}(t)) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$

e. $\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{s}(t)) = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$

f. $\frac{d}{dt}\mathbf{r}(f(t)) = \mathbf{r}'(f(t))f'(t)$

■

Note that because the cross product is not commutative you must remember to do the three cross products in formula (e) in the correct order.

When the derivative of a function $f(t)$ is zero, we know that the function has a horizontal tangent line, and may have a local maximum or minimum point. If $\mathbf{r}'(t) = \mathbf{0}$, the geometric interpretation is quite different, though the interpretation in terms of motion is similar. Certainly we know that the object has speed zero at such a point, and it may thus be abruptly changing direction. In three dimensions there are many ways to change direction; geometrically this often means the curve has a cusp or a point, as in the path of a ball that bounces off the floor or a wall.

EXAMPLE 13.2.6 Suppose that $\mathbf{r}(t) = \langle 1 + t^3, t^2, 1\rangle$, so $\mathbf{r}'(t) = \langle 3t^2, 2t, 0\rangle$. This is $\mathbf{0}$ at $t = 0$, and there is indeed a cusp at the point $(1, 0, 1)$, as shown in figure 13.2.5. □

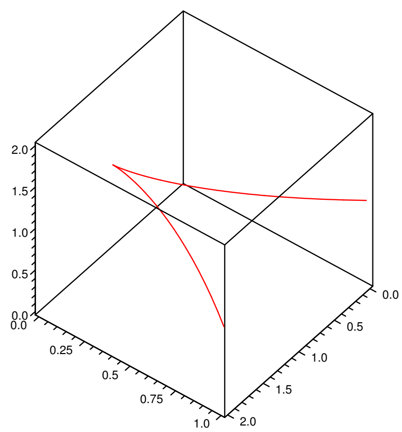


Figure 13.2.5 $\langle 1 + t^3, t^2, 1\rangle$ has a cusp at $\langle 1, 0, 1\rangle$. (AP)

Sometimes we will be interested in the direction of $\mathbf{r}'$ but not its length. In some cases, we can still work with $\mathbf{r}'$, as when we find the angle between two curves. On other occasions it will be useful to work with a unit vector in the same direction as $\mathbf{r}'$; of course, we can compute such a vector by dividing $\mathbf{r}'$ by its own length. This standard unit tangent vector is usually denoted by $\mathbf{T}$:

$$\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}.$$

In a sense, when we computed the angle between two tangent vectors we have already made use of the unit tangent, since

$$\cos\theta = \frac{\mathbf{r}' \cdot \mathbf{s}'}{|\mathbf{r}'||\mathbf{s}'|} = \frac{\mathbf{r}'}{|\mathbf{r}'|} \cdot \frac{\mathbf{s}'}{|\mathbf{s}'|}$$

Now that we know how to make sense of $\mathbf{r}'$, we immediately know what an antiderivative must be, namely

$$\int \mathbf{r}(t)\,dt = \langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \rangle,$$

if $\mathbf{r} = \langle f(t), g(t), h(t)\rangle$. What about definite integrals? Suppose that $\mathbf{v}(t)$ gives the velocity of an object at time t. Then $\mathbf{v}(t)\Delta t$ is a vector that approximates the displacement of the object over the time Δt: $\mathbf{v}(t)\Delta t$ points in the direction of travel, and $|\mathbf{v}(t)\Delta t| = |\mathbf{v}(t)||\Delta t|$ is the speed of the object times Δt, which is approximately the distance traveled. Thus, if we sum many such tiny vectors:

$$\sum_{i=0}^{n-1} \mathbf{v}(t_i)\Delta t$$

we get an approximation to the displacement vector over the time interval $[t_0, t_n]$. If we take the limit we get the exact value of the displacement vector:

$$\lim \sum_{i=0}^{n-1} \mathbf{v}(t_i)\Delta t = \int_{t_0}^{t_n} \mathbf{v}(t)\,dt = \mathbf{r}(t_n) - \mathbf{r}(t_0).$$

Denote $\mathbf{r}(t_0)$ by $\mathbf{r}_0$. Then given the velocity vector we can compute the vector function $\mathbf{r}$ giving the location of the object:

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}(u)\,du.$$

EXAMPLE 13.2.7 An object moves with velocity vector $\langle \cos t, \sin t, \cos t\rangle$, starting at $(1, 1, 1)$ at time 0. Find the function $\mathbf{r}$ giving its location.

$$\begin{aligned} \mathbf{r}(t) &= \langle 1,1,1\rangle + \int_0^t \langle \cos u, \sin u, \cos u\rangle\,du \\ &= \langle 1,1,1\rangle + \langle \sin u, -\cos u, \sin u\rangle\big|_0^t \\ &= \langle 1,1,1\rangle + \langle \sin t, -\cos t, \sin t\rangle - \langle 0,-1,0\rangle \\ &= \langle 1+\sin t, 2-\cos t, 1+\sin t\rangle \end{aligned}$$

See figure 13.2.6. □

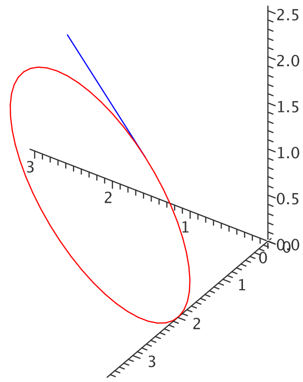


Figure 13.2.6 Path of the object with its initial velocity vector. (AP)

Exercises 13.2.

1. Find $\mathbf{r}'$ and $\mathbf{T}$ for $\mathbf{r} = \langle t^2, 1, t\rangle$. ⇒
2. Find $\mathbf{r}'$ and $\mathbf{T}$ for $\mathbf{r} = \langle \cos t, \sin 2t, t^2\rangle$. ⇒
3. Find $\mathbf{r}'$ and $\mathbf{T}$ for $\mathbf{r} = \langle \cos(e^t), \sin(e^t), \sin t\rangle$. ⇒
4. Find a vector function for the line tangent to the helix $\langle \cos t, \sin t, t\rangle$ when $t = \pi/4$. ⇒
5. Find a vector function for the line tangent to $\langle \cos t, \sin t, \cos 4t\rangle$ when $t = \pi/3$. ⇒
6. Find the cosine of the angle between the curves $\langle 0, t^2, t\rangle$ and $\langle \cos(\pi t/2), \sin(\pi t/2), t\rangle$ where they intersect. ⇒
7. Find the cosine of the angle between the curves $\langle \cos t, -\sin(t)/4, \sin t\rangle$ and $\langle \cos t, \sin t, \sin(2t)\rangle$ where they intersect. ⇒
8. Suppose that $|\mathbf{r}(t)| = k$, for some constant k. This means that $\mathbf{r}$ describes some path on the sphere of radius k with center at the origin. Show that $\mathbf{r}$ is perpendicular to $\mathbf{r}'$ at every point. Hint: Use Theorem 13.2.5, part (d).
9. A bug is crawling along the spoke of a wheel that lies along a radius of the wheel. The bug is crawling at 1 unit per second and the wheel is rotating at 1 radian per second. Suppose the wheel lies in the y-z plane with center at the origin, and at time $t = 0$ the spoke lies along the positive y axis and the bug is at the origin. Find a vector function $\mathbf{r}(t)$ for the position of the bug at time t, the velocity vector $\mathbf{r}'(t)$, the unit tangent $\mathbf{T}(t)$, and the speed of the bug $|\mathbf{r}'(t)|$. ⇒
10. An object moves with velocity vector $\langle \cos t, \sin t, t\rangle$, starting at $\langle 0, 0, 0\rangle$ when $t = 0$. Find the function $\mathbf{r}$ giving its location. ⇒
11. The position function of a particle is given by $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t\rangle$, $t \geq 0$. When is the speed of the particle a minimum? ⇒
12. A particle moves so that its position is given by $\langle \cos t, \sin t, \cos(6t)\rangle$. Find the maximum and minimum speeds of the particle. ⇒
13. An object moves with velocity vector $\langle t, t^2, \cos t\rangle$, starting at $\langle 0, 0, 0\rangle$ when $t = 0$. Find the function $\mathbf{r}$ giving its location. ⇒

14. What is the physical interpretation of the dot product of two vector valued functions? What is the physical interpretation of the cross product of two vector valued functions?

15. Show, using the rules of cross products and differentiation, that

$$\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{r}'(t)) = \mathbf{r}(t) \times \mathbf{r}''(t).$$

16. Determine the point at which $\mathbf{f}(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{g}(t) = \langle \cos(t), \cos(2t), t+1 \rangle$ intersect, and find the angle between the curves at that point. (Hint: You'll need to set this one up like a line intersection problem, writing one in s and one in t.) If these two functions were the trajectories of two airplanes on the same scale of time, would the planes collide at their point of intersection? Explain. ⇒

17. Find the equation of the plane perpendicular to the curve $\mathbf{r}(t) = \langle 2\sin(3t), t, 2\cos(3t) \rangle$ at the point $(0, \pi, -2)$. ⇒

18. Find the equation of the plane perpendicular to $\langle \cos t, \sin t, \cos(6t) \rangle$ when $t = \pi/4$. ⇒

19. At what point on the curve $\mathbf{r}(t) = \langle t^3, 3t, t^4 \rangle$ is the plane perpendicular to the curve also parallel to the plane $6x + 6y - 8z = 1$? ⇒

20. Find the equation of the line tangent to $\langle \cos t, \sin t, \cos(6t) \rangle$ when $t = \pi/4$. ⇒

13.3 Arc length and curvature

Sometimes it is useful to compute the length of a curve in space; for example, if the curve represents the path of a moving object, the length of the curve between two points may be the distance traveled by the object between two times.

Recall that if the curve is given by the vector function $\mathbf{r}$ then the vector $\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ points from one position on the curve to another, as depicted in figure 13.2.1. If the points are close together, the length of $\Delta\mathbf{r}$ is close to the length of the curve between the two points. If we add up the lengths of many such tiny vectors, placed head to tail along a segment of the curve, we get an approximation to the length of the curve over that segment. In the limit, as usual, this sum turns into an integral that computes precisely the length of the curve. First, note that

$$|\Delta\mathbf{r}| = \frac{|\Delta\mathbf{r}|}{\Delta t}\,\Delta t \approx |\mathbf{r}'(t)|\,\Delta t,$$

when Δt is small. Then the length of the curve between $\mathbf{r}(a)$ and $\mathbf{r}(b)$ is

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} |\Delta\mathbf{r}| = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{|\Delta\mathbf{r}|}{\Delta t}\,\Delta t = \lim_{n\to\infty} \sum_{i=0}^{n-1} |\mathbf{r}'(t)|\,\Delta t = \int_a^b |\mathbf{r}'(t)|\,dt.$$

(Well, sometimes. This works if between a and b the segment of curve is traced out exactly once.)

EXAMPLE 13.3.1 Let's find the length of one turn of the helix $\mathbf{r} = \langle \cos t, \sin t, t\rangle$ (see figure 13.1.1). We compute $\mathbf{r}' = \langle -\sin t, \cos t, 1\rangle$ and $|\mathbf{r}'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$, so the length is

$$\int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\pi.$$

□

EXAMPLE 13.3.2 Suppose $y = \ln x$; what is the length of this curve between $x = 1$ and $x = \sqrt{3}$?

Although this problem does not appear to involve vectors or three dimensions, we can interpret it in those terms: let $\mathbf{r}(t) = \langle t, \ln t, 0\rangle$. This vector function traces out precisely $y = \ln x$ in the x-y plane. Then $\mathbf{r}'(t) = \langle 1, 1/t, 0\rangle$ and $|\mathbf{r}'(t)| = \sqrt{1 + 1/t^2}$ and the desired length is

$$\int_1^{\sqrt{3}} \sqrt{1 + \frac{1}{t^2}}\,dt = 2 - \sqrt{2} + \ln(\sqrt{2} + 1) - \frac{1}{2}\ln 3.$$

(This integral is a bit tricky, but requires only methods we have learned.) □

Notice that there is nothing special about $y = \ln x$, except that the resulting integral can be computed. In general, given any $y = f(x)$, we can think of this as the vector function $\mathbf{r}(t) = \langle t, f(t), 0\rangle$. Then $\mathbf{r}'(t) = \langle 1, f'(t), 0\rangle$ and $|\mathbf{r}'(t)| = \sqrt{1 + (f')^2}$. The length of the curve $y = f(x)$ between a and b is thus

$$\int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

Unfortunately, such integrals are often impossible to do exactly and must be approximated.

One useful application of arc length is the **arc length parameterization**. A vector function $\mathbf{r}(t)$ gives the position of a point in terms of the parameter t, which is often time, but need not be. Suppose s is the distance along the curve from some fixed starting point; if we use s for the variable, we get $\mathbf{r}(s)$, the position in space in terms of distance along the curve. We might still imagine that the curve represents the position of a moving object; now we get the position of the object as a function of how far the object has traveled.

EXAMPLE 13.3.3 Suppose $\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$. We know that this curve is a circle of radius 1. While t might represent time, it can also in this case represent the usual angle between the positive x-axis and $\mathbf{r}(t)$. The distance along the circle from $(1, 0, 0)$ to $(\cos t, \sin t, 0)$ is also t—this is the definition of radian measure. Thus, in this case $s = t$ and $\mathbf{r}(s) = \langle \cos s, \sin s, 0\rangle$. □

EXAMPLE 13.3.4 Suppose $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$. We know that this curve is a helix. The distance along the helix from $(1, 0, 0)$ to $(\cos t, \sin t, t)$ is

$$s = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{\cos^2 u + \sin^2 u + 1}\,du = \int_0^t \sqrt{2}\,du = \sqrt{2}t.$$

Thus, the value of t that gets us distance s along the helix is $t = s/\sqrt{2}$, and so the same curve is given by $\hat{\mathbf{r}}(s) = \langle \cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}\rangle$. □

In general, if we have a vector function $\mathbf{r}(t)$, to convert it to a vector function in terms of arc length we compute

$$s = \int_a^t |\mathbf{r}'(u)|\,du = f(t),$$

solve $s = f(t)$ for t, getting $t = g(s)$, and substitute this back into $\mathbf{r}(t)$ to get $\hat{\mathbf{r}}(s) = \mathbf{r}(g(s))$.

Suppose that t is time. By the Fundamental Theorem of Calculus, if we start with arc length

$$s(t) = \int_a^t |\mathbf{r}'(u)|\,du$$

and take the derivative, we get

$$s'(t) = |\mathbf{r}'(t)|.$$

Here $s'(t)$ is the rate at which the arc length is changing, and we have seen that $|\mathbf{r}'(t)|$ is the speed of a moving object; these are of course the same.

Suppose that $\mathbf{r}(s)$ is given in terms of arc length; what is $|\mathbf{r}'(s)|$? It is the rate at which arc length is changing *relative to arc length*; it must be 1! In the case of the helix, for example, the arc length parameterization is $\langle \cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}\rangle$, the derivative is $\langle -\sin(s/\sqrt{2})/\sqrt{2}, \cos(s/\sqrt{2})/\sqrt{2}, 1/\sqrt{2}\rangle$, and the length of this is

$$\sqrt{\frac{\sin^2(s/\sqrt{2})}{2} + \frac{\cos^2(s/\sqrt{2})}{2} + \frac{1}{2}} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1.$$

So in general, $\mathbf{r}'$ is a unit tangent vector.

Given a curve $\mathbf{r}(t)$, we would like to be able to measure, at various points, how sharply curved it is. Clearly this is related to how "fast" a tangent vector is changing direction, so a first guess might be that we can measure curvature with $|\mathbf{r}''(t)|$. A little thought shows that this is flawed; if we think of t as time, for example, we could be tracing out the curve more or less quickly as time passes. The second derivative $|\mathbf{r}''(t)|$ incorporates this notion of time, so it depends not simply on the geometric properties of the curve but on how quickly we move along the curve.

EXAMPLE 13.3.5 Consider $\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$ and $\mathbf{s}(t) = \langle \cos 2t, \sin 2t, 0\rangle$. Both of these vector functions represent the unit circle in the x-y plane, but if t is interpreted as time, the second describes an object moving twice as fast as the first. Computing the second derivatives, we find $|\mathbf{r}''(t)| = 1$, $|\mathbf{s}''(t)| = 4$. □

To remove the dependence on time, we use the arc length parameterization. If a curve is given by $\mathbf{r}(s)$, then the first derivative $\mathbf{r}'(s)$ is a unit vector, that is, $\mathbf{r}'(s) = \mathbf{T}(s)$. We now compute the second derivative $\mathbf{r}''(s) = \mathbf{T}'(s)$ and use $|\mathbf{T}'(s)|$ as the "official" measure of **curvature**, usually denoted κ.

EXAMPLE 13.3.6 We have seen that the arc length parameterization of a particular helix is $\mathbf{r}(s) = \langle \cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}\rangle$. Computing the second derivative gives $\mathbf{r}''(s) = \langle -\cos(s/\sqrt{2})/2, -\sin(s/\sqrt{2})/2, 0\rangle$ with length $1/2$. □

What if we are given a curve as a vector function $\mathbf{r}(t)$, where t is not arc length? We have seen that arc length can be difficult to compute; fortunately, we do not need to convert to the arc length parameterization to compute curvature. Instead, let us imagine that we have done this, so we have found $t = g(s)$ and then formed $\hat{\mathbf{r}}(s) = \mathbf{r}(g(s))$. The first derivative $\hat{\mathbf{r}}'(s)$ is a unit tangent vector, so it is the same as the unit tangent vector $\mathbf{T}(t) = \mathbf{T}(g(s))$. Taking the derivative of this we get

$$\frac{d}{ds}\mathbf{T}(g(s)) = \mathbf{T}'(g(s))g'(s) = \mathbf{T}'(t)\frac{dt}{ds}.$$

The curvature is the length of this vector:

$$\kappa = |\mathbf{T}'(t)||\frac{dt}{ds}| = \frac{|\mathbf{T}'(t)|}{|ds/dt|} = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

(Recall that we have seen that $ds/dt = |\mathbf{r}'(t)|$.) Thus we can compute the curvature by computing only derivatives with respect to t; we do not need to do the conversion to arc length.

EXAMPLE 13.3.7 Returning to the helix, suppose we start with the parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$. Then $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle$, $|\mathbf{r}'(t)| = \sqrt{2}$, and $\mathbf{T}(t) = \langle -\sin t, \cos t, 1\rangle/\sqrt{2}$. Then $\mathbf{T}'(t) = \langle -\cos t, -\sin t, 0\rangle/\sqrt{2}$ and $|\mathbf{T}'(t)| = 1/\sqrt{2}$. Finally, $\kappa = 1/\sqrt{2}/\sqrt{2} = 1/2$, as before. □

EXAMPLE 13.3.8 Consider this circle of radius a: $\mathbf{r}(t) = \langle a\cos t, a\sin t, 1\rangle$. Then $\mathbf{r}'(t) = \langle -a\sin t, a\cos t, 0\rangle$, $|\mathbf{r}'(t)| = a$, and $\mathbf{T}(t) = \langle -a\sin t, a\cos t, 0\rangle/a$. Now $\mathbf{T}'(t) = \langle -a\cos t, -a\sin t, 0\rangle/a$ and $|\mathbf{T}'(t)| = 1$. Finally, $\kappa = 1/a$: the curvature of a circle is everywhere the inverse of the radius. It is sometimes useful to think of curvature as describing what circle a curve most resembles at a point. The curvature of the helix in the previous example is 1/2; this means that a small piece of the helix looks very much like a circle of radius 2, as shown in figure 13.3.1. □

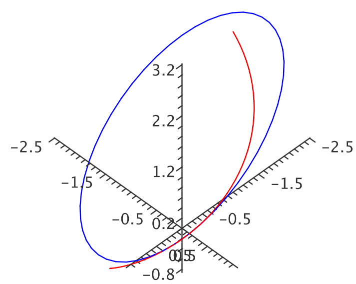


Figure 13.3.1 A circle with the same curvature as the helix. (AP)

EXAMPLE 13.3.9 Consider $\mathbf{r}(t) = \langle \cos t, \sin t, \cos 2t\rangle$, as shown in figure 13.2.4. $\mathbf{r}'(t) = \langle -\sin t, \cos t, -2\sin(2t)\rangle$ and $|\mathbf{r}'(t)| = \sqrt{1+4\sin^2(2t)}$, so

$$\mathbf{T}(t) = \left\langle \frac{-\sin t}{\sqrt{1+4\sin^2(2t)}}, \frac{\cos t}{\sqrt{1+4\sin^2(2t)}}, \frac{-2\sin 2t}{\sqrt{1+4\sin^2(2t)}} \right\rangle.$$

Computing the derivative of this and then the length of the resulting vector is possible but unpleasant. □

Fortunately, there is an alternate formula for the curvature that is often simpler than the one we have:

$$\kappa = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

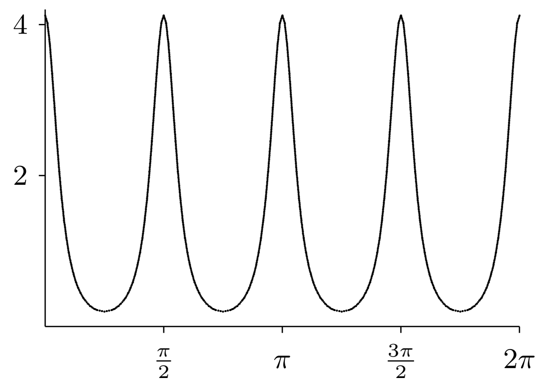
EXAMPLE 13.3.10 Returning to the previous example, we compute the second derivative $\mathbf{r}''(t) = \langle -\cos t, -\sin t, -4\cos(2t)\rangle$. Then the cross product $\mathbf{r}'(t)\times\mathbf{r}''(t)$ is

$$\langle -4\cos t\cos 2t - 2\sin t\sin 2t, 2\cos t\sin 2t - 4\sin t\cos 2t, 1\rangle.$$

Computing the length of this vector and dividing by $|\mathbf{r}'(t)|^3$ is still a bit tedious. With the aid of a computer we get

$$\kappa = \frac{\sqrt{48\cos^4 t - 48\cos^2 t + 17}}{(-16\cos^4 t + 16\cos^2 t + 1)^{3/2}}.$$

Graphing this we get



Compare this to figure 13.2.4—you may want to load the Java applet there so that you can see it from different angles. The highest curvature occurs where the curve has its highest and lowest points, and indeed in the picture these appear to be the most sharply curved portions of the curve, while the curve is almost a straight line midway between those points. □

Let's see why this alternate formula is correct. Starting with the definition of $\mathbf{T}$, $\mathbf{r}' = |\mathbf{r}'|\mathbf{T}$ so by the product rule $\mathbf{r}'' = |\mathbf{r}'|'\mathbf{T} + |\mathbf{r}'|\mathbf{T}'$. Then by Theorem 12.4.1 the cross product is

$$\begin{aligned} \mathbf{r}' \times \mathbf{r}'' &= |\mathbf{r}'|\mathbf{T} \times |\mathbf{r}'|'\mathbf{T} + |\mathbf{r}'|\mathbf{T} \times |\mathbf{r}'|\mathbf{T}' \\ &= |\mathbf{r}'||\mathbf{r}'|'(\mathbf{T} \times \mathbf{T}) + |\mathbf{r}'|^2(\mathbf{T} \times \mathbf{T}') \\ &= |\mathbf{r}'|^2(\mathbf{T} \times \mathbf{T}') \end{aligned}$$

because $\mathbf{T} \times \mathbf{T} = \mathbf{0}$, since $\mathbf{T}$ is parallel to itself. Then

$$\begin{aligned} |\mathbf{r}' \times \mathbf{r}''| &= |\mathbf{r}'|^2|\mathbf{T} \times \mathbf{T}'| \\ &= |\mathbf{r}'|^2|\mathbf{T}||\mathbf{T}'|\sin\theta \\ &= |\mathbf{r}'|^2|\mathbf{T}'| \end{aligned}$$

using exercise 8 in section 13.2 to see that $\theta = \pi/2$. Dividing both sides by $|\mathbf{r}'|^3$ then gives the desired formula.

We used the fact here that $\mathbf{T}'$ is perpendicular to $\mathbf{T}$; the vector $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$ is thus a unit vector perpendicular to $\mathbf{T}$, called the **unit normal** to the curve. Occasionally of use is the **unit binormal** $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, a unit vector perpendicular to both $\mathbf{T}$ and $\mathbf{N}$.

Exercises 13.3.

1. Find the length of $\langle 3\cos t, 2t, 3\sin t\rangle$, $t\in[0,2\pi]$. ⇒
2. Find the length of $\langle t^2, 2, t^3\rangle$, $t\in[0,1]$. ⇒
3. Find the length of $\langle t^2, \sin t, \cos t\rangle$, $t\in[0,1]$. ⇒
4. Find the length of the curve $y=x^{3/2}$, $x\in[1,9]$. ⇒
5. Set up an integral to compute the length of $\langle \cos t, \sin t, e^t\rangle$, $t\in[0,5]$. (It is tedious but not too difficult to compute this integral.) ⇒
6. Find the curvature of $\langle t, t^2, t\rangle$. ⇒
7. Find the curvature of $\langle t, t^2, t^2\rangle$. ⇒
8. Find the curvature of $\langle t, t^2, t^3\rangle$. ⇒
9. Find the curvature of $y=x^4$ at $(1,1)$. ⇒

13.4 Motion along a curve

We have already seen that if t is time and an object's location is given by $\mathbf{r}(t)$, then the derivative $\mathbf{r}'(t)$ is the velocity vector $\mathbf{v}(t)$. Just as $\mathbf{v}(t)$ is a vector describing how $\mathbf{r}(t)$ changes, so is $\mathbf{v}'(t)$ a vector describing how $\mathbf{v}(t)$ changes, namely, $\mathbf{a}(t)=\mathbf{v}'(t)=\mathbf{r}''(t)$ is the **acceleration vector**.

EXAMPLE 13.4.1 Suppose $\mathbf{r}(t)=\langle \cos t, \sin t, 1\rangle$. Then $\mathbf{v}(t)=\langle -\sin t, \cos t, 0\rangle$ and $\mathbf{a}(t)=\langle -\cos t, -\sin t, 0\rangle$. This describes the motion of an object traveling on a circle of radius 1, with constant z coordinate 1. The velocity vector is of course tangent to the curve; note that $\mathbf{a}\cdot\mathbf{v}=0$, so $\mathbf{v}$ and $\mathbf{a}$ are perpendicular. In fact, it is not hard to see that $\mathbf{a}$ points from the location of the object to the center of the circular path at $(0,0,1)$. □

Recall that the unit tangent vector is given by $\mathbf{T}(t)=\mathbf{v}(t)/|\mathbf{v}(t)|$, so $\mathbf{v}=|\mathbf{v}|\mathbf{T}$. If we take the derivative of both sides of this equation we get

$$\mathbf{a}=|\mathbf{v}|'\mathbf{T}+|\mathbf{v}|\mathbf{T}'. \tag{13.4.1}$$

Also recall the definition of the curvature, $\kappa=|\mathbf{T}'|/|\mathbf{v}|$, or $|\mathbf{T}'|=\kappa|\mathbf{v}|$. Finally, recall that we defined the unit normal vector as $\mathbf{N}=\mathbf{T}'/|\mathbf{T}'|$, so $\mathbf{T}'=|\mathbf{T}'|\mathbf{N}=\kappa|\mathbf{v}|\mathbf{N}$. Substituting into equation 13.4.1 we get

$$\mathbf{a}=|\mathbf{v}|'\mathbf{T}+\kappa|\mathbf{v}|^2\mathbf{N}. \tag{13.4.2}$$

The quantity $|\mathbf{v}(t)|$ is the speed of the object, often written as $v(t)$; $|\mathbf{v}(t)|'$ is the rate at which the speed is changing, or the scalar acceleration of the object, $a(t)$. Rewriting equation 13.4.2 with these gives us

$$\mathbf{a}=a\mathbf{T}+\kappa v^2\mathbf{N}=a_T\mathbf{T}+a_N\mathbf{N};$$

a_T is the **tangential component of acceleration** and a_N is the **normal component of acceleration**. We have already seen that a_T measures how the speed is changing; if you are riding in a vehicle with large a_T you will feel a force pulling you into your seat. The other component, a_N, measures how sharply your direction is changing *with respect to time*. So it naturally is related to how sharply the path is curved, measured by κ, and also to how fast you are going. Because a_N includes v^2, note that the effect of speed is magnified; doubling your speed around a curve quadruples the value of a_N. You feel the effect of this as a force pushing you toward the outside of the curve, the "centrifugal force."

In practice, if want a_N we would use the formula for κ:

$$a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}|\mathbf{r}'|^2 = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|}.$$

To compute a_T we can project $\mathbf{a}$ onto $\mathbf{v}$:

$$a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|} = \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|}.$$

EXAMPLE 13.4.2 Suppose $\mathbf{r} = \langle t, t^2, t^3 \rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N.

Taking derivatives we get $\mathbf{v} = \langle 1, 2t, 3t^2 \rangle$ and $\mathbf{a} = \langle 0, 2, 6t \rangle$. Then

$$a_T = \frac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4}} \quad \text{and} \quad a_N = \frac{\sqrt{4 + 36t^2 + 36t^4}}{\sqrt{1 + 4t^2 + 9t^4}}.$$

□

Exercises 13.4.

1. Let $\mathbf{r} = \langle \cos t, \sin t, t \rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N. ⇒
2. Let $\mathbf{r} = \langle \cos t, \sin t, t^2 \rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N. ⇒
3. Let $\mathbf{r} = \langle \cos t, \sin t, e^t \rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N. ⇒
4. Let $\mathbf{r} = \langle e^t, \sin t, e^t \rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N. ⇒
5. Suppose an object moves so that its acceleration is given by $\mathbf{a} = \langle -3\cos t, -2\sin t, 0 \rangle$. At time $t = 0$ the object is at $(3, 0, 0)$ and its velocity vector is $\langle 0, 2, 0 \rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the object. ⇒
6. Suppose an object moves so that its acceleration is given by $\mathbf{a} = \langle -3\cos t, -2\sin t, 0 \rangle$. At time $t = 0$ the object is at $(3, 0, 0)$ and its velocity vector is $\langle 0, 2.1, 0 \rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the object. ⇒
7. Suppose an object moves so that its acceleration is given by $\mathbf{a} = \langle -3\cos t, -2\sin t, 0 \rangle$. At time $t = 0$ the object is at $(3, 0, 0)$ and its velocity vector is $\langle 0, 2, 1 \rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the object. ⇒
8. Suppose an object moves so that its acceleration is given by $\mathbf{a} = \langle -3\cos t, -2\sin t, 0 \rangle$. At time $t = 0$ the object is at $(3, 0, 0)$ and its velocity vector is $\langle 0, 2.1, 1 \rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the object. ⇒

9. Describe a situation in which the normal component of acceleration is 0 and the tangential component of acceleration is non-zero. Is it possible for the tangential component of acceleration to be 0 while the normal component of acceleration is non-zero? Explain. Finally, is it possible for an object to move (not be stationary) so that both the tangential and normal components of acceleration are 0? Explain.

14

Partial Differentiation

14.1 Functions of Several Variables

In single-variable calculus we were concerned with functions that map the real numbers $\mathbb{R}$ to $\mathbb{R}$, sometimes called "real functions of one variable", meaning the "input" is a single real number and the "output" is likewise a single real number. In the last chapter we considered functions taking a real number to a vector, which may also be viewed as functions $f\colon \mathbb{R} \to \mathbb{R}^3$, that is, for each input value we get a position in space. Now we turn to functions of several variables, meaning several input variables, functions $f\colon \mathbb{R}^n \to \mathbb{R}$. We will deal primarily with $n = 2$ and to a lesser extent $n = 3$; in fact many of the techniques we discuss can be applied to larger values of n as well.

A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ maps a pair of values (x, y) to a single real number. The three-dimensional coordinate system we have already used is a convenient way to visualize such functions: above each point (x, y) in the x-y plane we graph the point (x, y, z), where of course $z = f(x, y)$.

EXAMPLE 14.1.1 Consider $f(x, y) = 3x + 4y - 5$. Writing this as $z = 3x + 4y - 5$ and then $3x + 4y - z = 5$ we recognize the equation of a plane. In the form $f(x, y) = 3x + 4y - 5$ the emphasis has shifted: we now think of x and y as independent variables and z as a variable dependent on them, but the geometry is unchanged. □

EXAMPLE 14.1.2 We have seen that $x^2 + y^2 + z^2 = 4$ represents a sphere of radius 2. We cannot write this in the form $f(x, y)$, since for each x and y in the disk $x^2 + y^2 < 4$ there are two corresponding points on the sphere. As with the equation of a circle, we can resolve this equation into two functions, $f(x,y) = \sqrt{4-x^2-y^2}$ and $f(x,y) = -\sqrt{4-x^2-y^2}$, representing the upper and lower hemispheres. Each of these is an example of a function with a restricted domain: only certain values of x and y make sense (namely, those for which $x^2+y^2 \le 4$) and the graphs of these functions are limited to a small region of the plane. □

EXAMPLE 14.1.3 Consider $f = \sqrt{x} + \sqrt{y}$. This function is defined only when both x and y are non-negative. When $y = 0$ we get $f(x,y) = \sqrt{x}$, the familiar square root function in the x-z plane, and when $x = 0$ we get the same curve in the y-z plane. Generally speaking, we see that starting from $f(0,0) = 0$ this function gets larger in every direction in roughly the same way that the square root function gets larger. For example, if we restrict attention to the line $x = y$, we get $f(x,y) = 2\sqrt{x}$ and along the line $y = 2x$ we have $f(x,y) = \sqrt{x} + \sqrt{2x} = (1+\sqrt{2})\sqrt{x}$. □

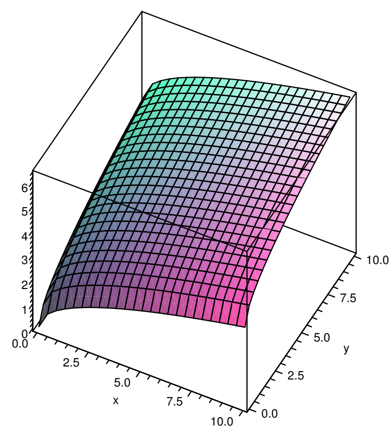


Figure 14.1.1 $f(x,y) = \sqrt{x} + \sqrt{y}$ (AP)

A computer program that plots such surfaces can be very useful, as it is often difficult to get a good idea of what they look like. Still, it is valuable to be able to visualize relatively simple surfaces without such aids. As in the previous example, it is often a good idea to examine the function on restricted subsets of the plane, especially lines. It can also be useful to identify those points (x,y) that share a common z-value.

EXAMPLE 14.1.4 Consider $f(x,y) = x^2 + y^2$. When $x = 0$ this becomes $f = y^2$, a parabola in the y-z plane; when $y = 0$ we get the "same" parabola $f = x^2$ in the x-z plane. Now consider the line $y = kx$. If we simply replace y by kx we get $f(x,y) = (1+k^2)x^2$ which is a parabola, but it does not really "represent" the cross-section along $y = kx$, because the cross-section has the line $y = kx$ where the horizontal axis should be. In order to pretend that this line is the horizontal axis, we need to write the function in terms of the distance from the origin, which is $\sqrt{x^2+y^2} = \sqrt{x^2+k^2x^2}$. Now $f(x,y) = x^2+k^2x^2 = (\sqrt{x^2+k^2x^2})^2$. So the cross-section is the "same" parabola as in the x-z and y-z planes, namely, the height is always the distance from the origin squared. This means that $f(x,y) = x^2+y^2$ can be formed by starting with $z = x^2$ and rotating this curve around the z axis.

Finally, picking a value $z = k$, at what points does $f(x,y) = k$? This means $x^2+y^2 = k$, which we recognize as the equation of a circle of radius $\sqrt{k}$. So the graph of $f(x,y)$ has parabolic cross-sections, and the same height everywhere on concentric circles with center at the origin. This fits with what we have already discovered. □

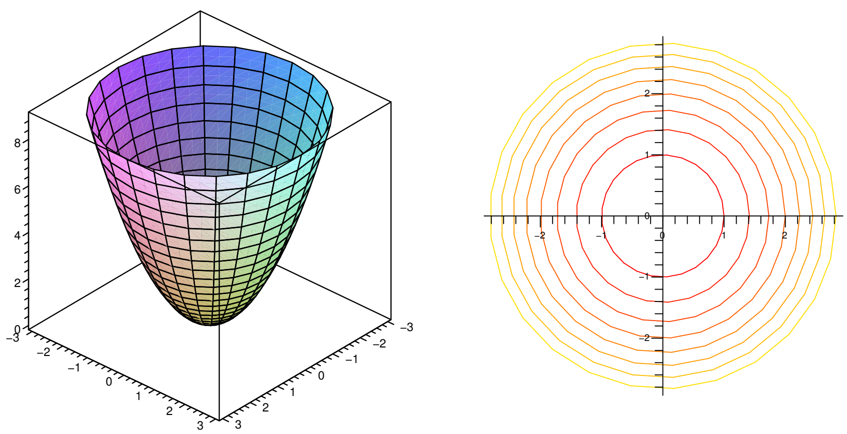


Figure 14.1.2 $f(x,y) = x^2+y^2$ (AP)

As in this example, the points (x,y) such that $f(x,y) = k$ usually form a curve, called a **level curve** of the function. A graph of some level curves can give a good idea of the shape of the surface; it looks much like a topographic map of the surface. In figure 14.1.2 both the surface and its associated level curves are shown. Note that, as with a topographic map, the heights corresponding to the level curves are evenly spaced, so that where curves are closer together the surface is steeper.

Functions $f\colon \mathbb{R}^n \to \mathbb{R}$ behave much like functions of two variables; we will on occasion discuss functions of three variables. The principal difficulty with such functions is visualizing them, as they do not "fit" in the three dimensions we are familiar with. For three variables there are various ways to interpret functions that make them easier to understand. For example, $f(x,y,z)$ could represent the temperature at the point (x,y,z), or the pressure, or the strength of a magnetic field. It remains useful to consider those points at which $f(x,y,z) = k$, where k is some constant value. If $f(x,y,z)$ is temperature, the set of points (x,y,z) such that $f(x,y,z) = k$ is the collection of points in space with temperature

k; in general this is called a **level set**; for three variables, a level set is typically a surface, called a **level surface**.

EXAMPLE 14.1.5 Suppose the temperature at (x, y, z) is $T(x, y, z) = e^{-(x^2+y^2+z^2)}$. This function has a maximum value of 1 at the origin, and tends to 0 in all directions. If k is positive and at most 1, the set of points for which $T(x, y, z) = k$ is those points satisfying $x^2 + y^2 + z^2 = -\ln k$, a sphere centered at the origin. The level surfaces are the concentric spheres centered at the origin. □

Exercises 14.1.

1. Let $f(x, y) = (x - y)^2$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface. ⇒

2. Let $f(x, y) = |x| + |y|$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface. ⇒

3. Let $f(x, y) = e^{-(x^2+y^2)} \sin(x^2 + y^2)$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface. ⇒

4. Let $f(x, y) = \sin(x - y)$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface. ⇒

5. Let $f(x, y) = (x^2 - y^2)^2$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface. ⇒

6. Find the domain of each of the following functions of two variables:

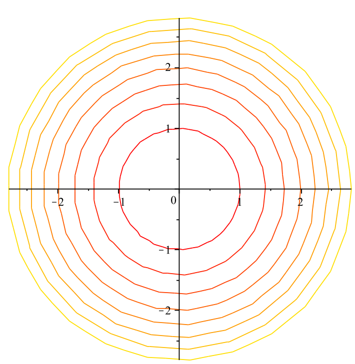
a. $\sqrt{9 - x^2} + \sqrt{y^2 - 4}$

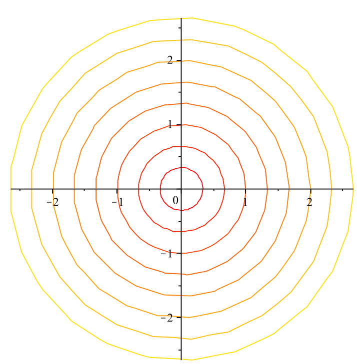
b. $\arcsin(x^2 + y^2 - 2)$

c. $\sqrt{16 - x^2 - 4y^2}$

⇒

7. Below are two sets of level curves. One is for a cone, one is for a paraboloid. Which is which? Explain.





14.2 Limits and Continuity

To develop calculus for functions of one variable, we needed to make sense of the concept of a limit, which we needed to understand continuous functions and to define the derivative. Limits involving functions of two variables can be considerably more difficult to deal with; fortunately, most of the functions we encounter are fairly easy to understand.

The potential difficulty is largely due to the fact that there are many ways to "approach" a point in the x-y plane. If we want to say that $\lim_{(x,y)\to(a,b)} f(x,y) = L$, we need to capture the idea that as (x,y) gets close to (a,b) then $f(x,y)$ gets close to L. For functions of one variable, $f(x)$, there are only two ways that x can approach a: from the left or right. But there are an infinite number of ways to approach (a,b): along any one of an infinite number of lines, or an infinite number of parabolas, or an infinite number of sine curves, and so on. We might hope that it's really not so bad—suppose, for example, that along every possible line through (a,b) the value of $f(x,y)$ gets close to L; surely this means that "$f(x,y)$ approaches L as (x,y) approaches (a,b)". Sadly, no.

EXAMPLE 14.2.1 Consider $f(x,y) = xy^2/(x^2+y^4)$. When $x=0$ or $y=0$, $f(x,y)$ is 0, so the limit of $f(x,y)$ approaching the origin along either the x or y axis is 0. Moreover, along the line $y=mx$, $f(x,y) = m^2x^3/(x^2+m^4x^4)$. As x approaches 0 this expression approaches 0 as well. So along every line through the origin $f(x,y)$ approaches 0. Now suppose we approach the origin along $x=y^2$. Then

$$f(x,y) = \frac{y^2y^2}{y^4+y^4} = \frac{y^4}{2y^4} = \frac{1}{2},$$

so the limit is 1/2. Looking at figure 14.2.1, it is apparent that there is a ridge above $x=y^2$. Approaching the origin along a straight line, we go over the ridge and then drop down toward 0, but approaching along the ridge the height is a constant 1/2. Thus, there is no limit at $(0,0)$. □

Fortunately, we can define the concept of limit without needing to specify how a particular point is approached—indeed, in definition 2.3.2, we didn't need the concept of "approach." Roughly, that definition says that when x is close to a then $f(x)$ is close to L; there is no mention of "how" we get close to a. We can adapt that definition to two variables quite easily:

DEFINITION 14.2.2 Limit Suppose $f(x,y)$ is a function. We say that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, $|f(x,y) - L| < \epsilon$. □

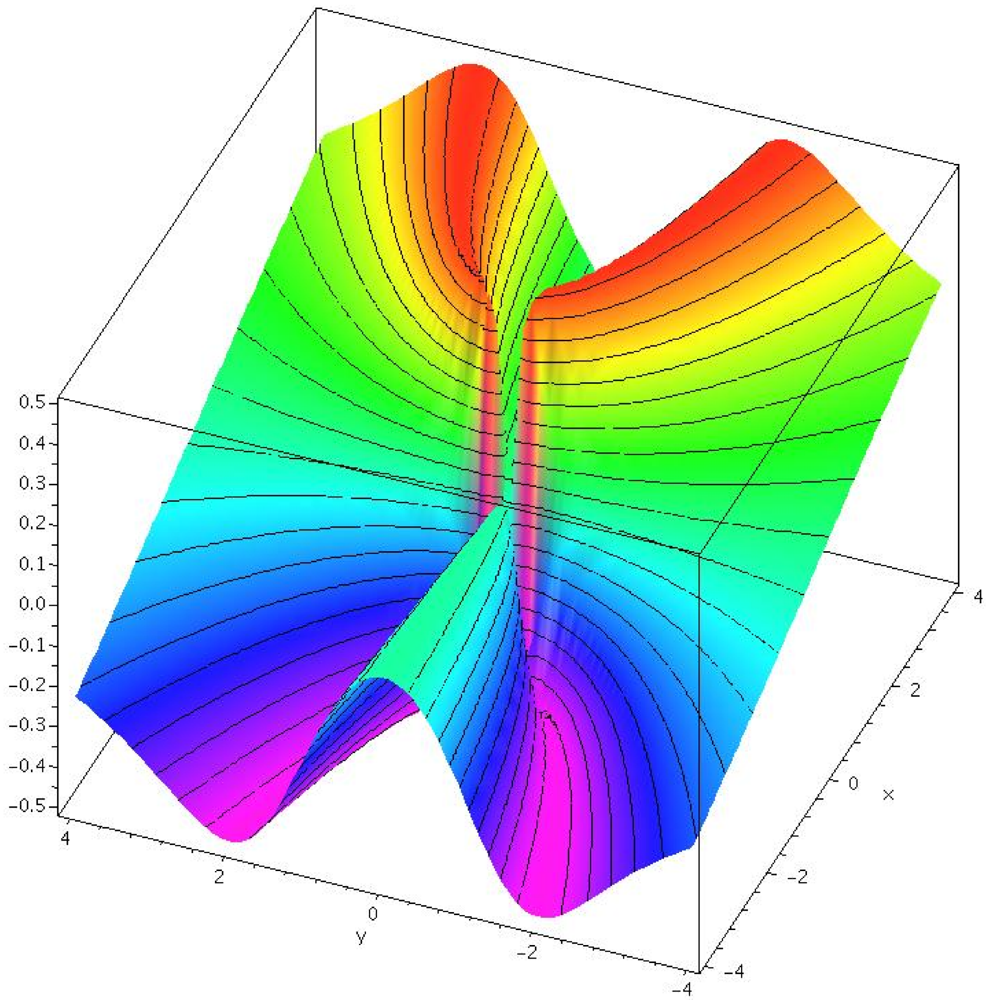


Figure 14.2.1 $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ (AP)

This says that we can make $|f(x,y)-L| < \epsilon$, no matter how small ϵ is, by making the distance from (x,y) to (a,b) "small enough".

EXAMPLE 14.2.3 We show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$. Suppose $\epsilon > 0$. Then

$$\left|\frac{3x^2y}{x^2+y^2}\right| = \frac{x^2}{x^2+y^2}3|y|.$$

Note that $x^2/(x^2+y^2) \le 1$ and $|y| = \sqrt{y^2} \le \sqrt{x^2+y^2} < \delta$. So

$$\frac{x^2}{x^2+y^2}3|y| < 1 \cdot 3 \cdot \delta.$$

We want to force this to be less than ϵ by picking δ "small enough." If we choose $\delta = \epsilon/3$ then

$$\left|\frac{3x^2y}{x^2+y^2}\right| < 1 \cdot 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

□

Recall that a function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$; roughly this says that there is no "hole" or "jump" at $x = a$. We can say exactly the same thing about a function of two variables.

DEFINITION 14.2.4 $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. □

EXAMPLE 14.2.5 The function $f(x,y) = 3x^2y/(x^2+y^2)$ is not continuous at $(0,0)$, because $f(0,0)$ is not defined. However, we know that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, so we can easily "fix" the problem, by extending the definition of f so that $f(0,0) = 0$. This surface is shown in figure 14.2.2. □

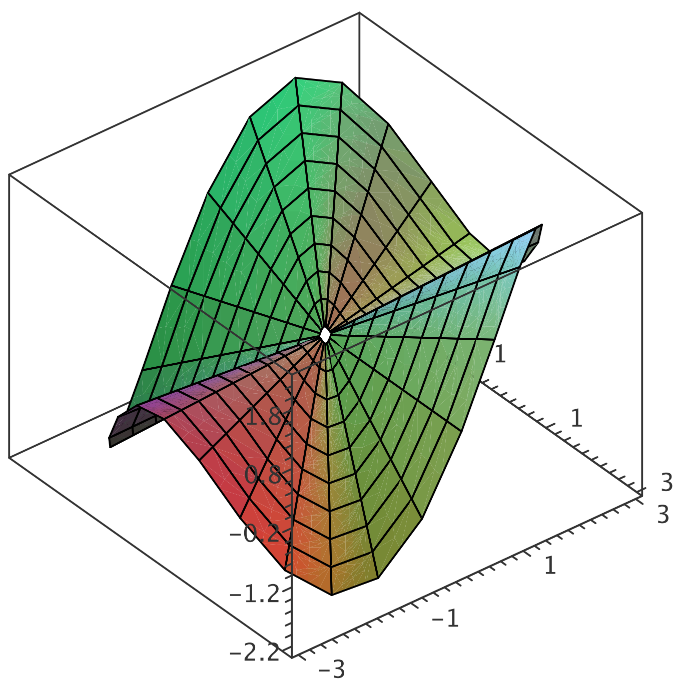


Figure 14.2.2 $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$ (AP)

Note that in contrast to this example we cannot fix example 14.2.1 at $(0,0)$ because the limit does not exist. No matter what value we try to assign to f at $(0,0)$ the surface will have a "jump" there.

Fortunately, the functions we will examine will typically be continuous almost everywhere. Usually this follows easily from the fact that closely related functions of one variable are continuous. As with single variable functions, two classes of common functions are particularly useful and easy to describe. A polynomial in two variables is a sum of terms of the form $ax^m y^n$, where a is a real number and m and n are non-negative integers. A rational function is a quotient of polynomials.

THEOREM 14.2.6 Polynomials are continuous everywhere. Rational functions are continuous everywhere they are defined. ■

Exercises 14.2.

Determine whether each limit exists. If it does, find the limit and prove that it is the limit; if it does not, explain how you know.

1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ ⇒

2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ ⇒

3. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+y^2}$ ⇒

4. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$ ⇒

5. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ ⇒

6. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{2x^2+y^2}}$ ⇒

7. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$ ⇒

8. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$ ⇒

9. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2}$ ⇒

10. $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{(x-1)^2+y^2}$ ⇒

11. $\displaystyle\lim_{(x,y)\to(1,-1)} 3x+4y$ ⇒

12. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^2+y^2}$ ⇒

13. Does the function $f(x,y) = \dfrac{x-y}{1+x+y}$ have any discontinuities? What about $f(x,y) = \dfrac{x-y}{1+x^2+y^2}$? Explain.

14.3 Partial Differentiation

When we first considered what the derivative of a vector function might mean, there was really not much difficulty in understanding either how such a thing might be computed or what it might measure. In the case of functions of two variables, things are a bit harder to understand. If we think of a function of two variables in terms of its graph, a surface, there is a more-or-less obvious derivative-like question we might ask, namely, how "steep" is the surface. But it's not clear that this has a simple answer, nor how we might proceed. We will start with what seem to be very small steps toward the goal; surprisingly, it turns out that these simple ideas hold the keys to a more general understanding.

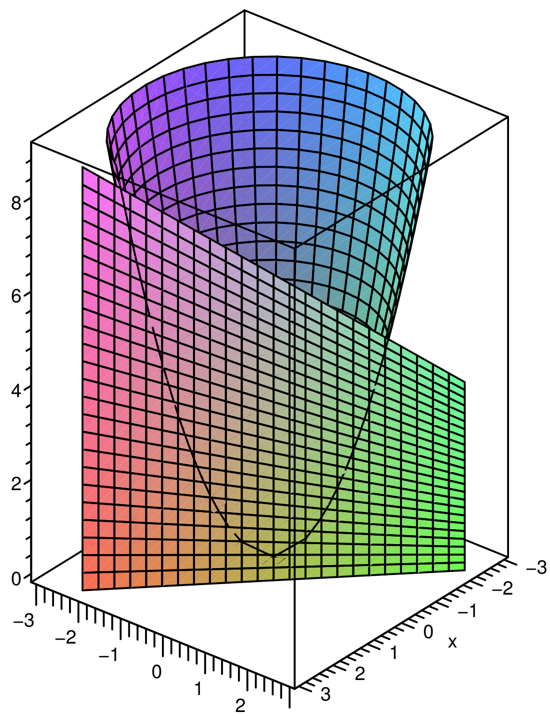


Figure 14.3.1 $f(x,y) = x^2 + y^2$, cut by the plane $x + y = 1$ (AP)

Imagine a particular point on a surface; what might we be able to say about how steep it is? We can limit the question to make it more familiar: how steep is the surface in a particular direction? What does this even mean? Here's one way to think of it: Suppose we're interested in the point (a,b,c). Pick a straight line in the x-y plane through the point $(a,b,0)$, then extend the line vertically into a plane. Look at the intersection of the plane with the surface. If we pay attention to just the plane, we see the chosen straight line where the x-axis would normally be, and the intersection with the surface shows up as a curve in the plane. Figure 14.3.1 shows the parabolic surface from figure 14.1.2, exposing its cross-section above the line $x + y = 1$.

In principle, this is a problem we know how to solve: find the slope of a curve in a plane. Let's start by looking at some particularly easy lines: those parallel to the x or y axis. Suppose we are interested in the cross-section of $f(x, y)$ above the line $y = b$. If we substitute b for y in $f(x, y)$, we get a function in one variable, describing the height of the cross-section as a function of x. Because $y = b$ is parallel to the x-axis, if we view it from a vantage point on the negative y-axis, we will see what appears to be simply an ordinary curve in the x-z plane.

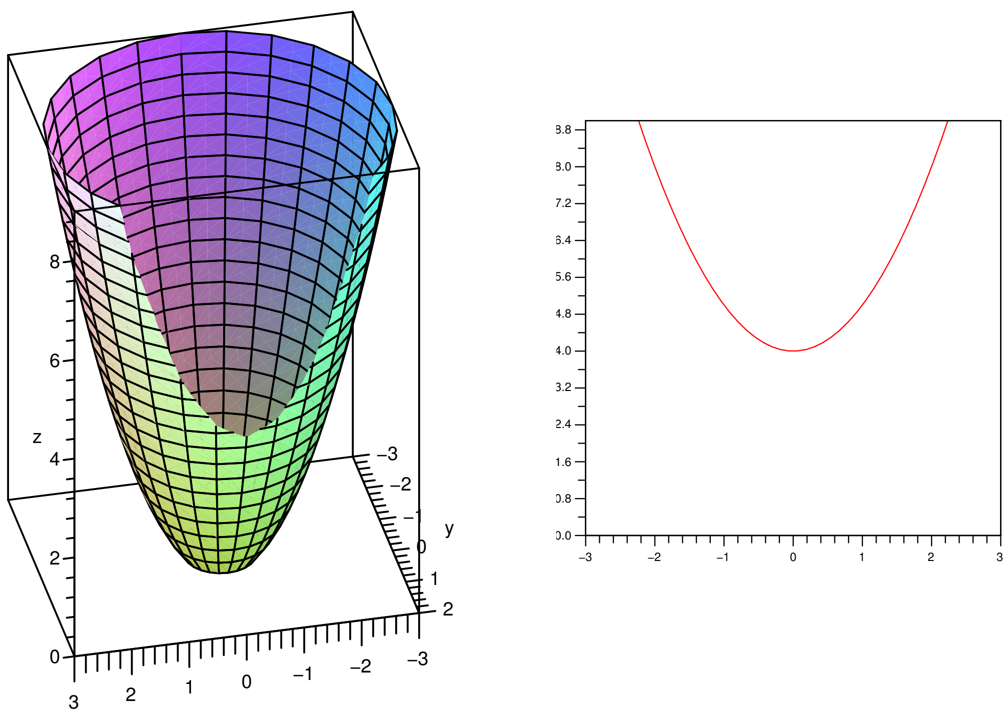


Figure 14.3.2 $f(x, y) = x^2 + y^2$, cut by the plane $y = 2$ (AP)

Consider again the parabolic surface $f(x, y) = x^2 + y^2$. The cross-section above the line $y = 2$ consists of all points $(x, 2, x^2 + 4)$. Looking at this cross-section from somewhere on the negative y axis, we see what appears to be just the curve $f(x) = x^2 + 4$. At any point on the cross-section, $(a, 2, a^2 + 4)$, the steepness of the surface *in the direction of the line* $y = 2$ is simply the slope of the curve $f(x) = x^2 + 4$, namely $2x$. Figure 14.3.2 shows the same parabolic surface as before, but now cut by the plane $y = 2$. The left graph shows the cut-off surface, the right shows just the cross-section, looking up from the negative y-axis toward the origin.

If, say, we're interested in the point $(-1, 2, 5)$ on the surface, then the slope in the direction of the line $y = 2$ is $2x = 2(-1) = -2$. This means that starting at $(-1, 2, 5)$ and moving on the surface, above the line $y = 2$, in the direction of increasing x values, the surface goes down; of course moving in the opposite direction, toward decreasing x values, the surface will rise.

If we're interested in some other line $y = k$, there is really no change in the computation. The equation of the cross-section above $y = k$ is $x^2 + k^2$ with derivative $2x$. We can save ourselves the effort, small as it is, of substituting k for y: all we are in effect doing is temporarily assuming that y is some constant. With this assumption, the derivative $\frac{d}{dx}(x^2 + y^2) = 2x$. To emphasize that we are only temporarily assuming y is constant, we use a slightly different notation: $\frac{\partial}{\partial x}(x^2 + y^2) = 2x$; the "$\partial$" reminds us that there are more variables than x, but that only x is being treated as a variable. We read the equation as "the partial derivative of $(x^2 + y^2)$ with respect to x is $2x$." A convenient alternate notation for the partial derivative of $f(x, y)$ with respect to x is is $f_x(x, y)$.

EXAMPLE 14.3.1 The partial derivative with respect to x of $x^3 + 3xy$ is $3x^2 + 3y$. Note that the partial derivative includes the variable y, unlike the example $x^2 + y^2$. It is somewhat unusual for the partial derivative to depend on a single variable; this example is more typical. □

Of course, we can do the same sort of calculation for lines parallel to the y-axis. We temporarily hold x constant, which gives us the equation of the cross-section above a line $x = k$. We can then compute the derivative with respect to y; this will measure the steepness of the curve in the y direction.

EXAMPLE 14.3.2 The partial derivative with respect to y of $f(x, y) = \sin(xy) + 3xy$ is

$$f_y(x, y) = \frac{\partial}{\partial y}\sin(xy) + 3xy = \cos(xy)\frac{\partial}{\partial y}(xy) + 3x = x\cos(xy) + 3x.$$

□

So far, using no new techniques, we have succeeded in measuring the slope of a surface in two quite special directions. For functions of one variable, the derivative is closely linked to the notion of tangent line. For surfaces, the analogous idea is the tangent plane—a plane that just touches a surface at a point, and has the same "steepness" as the surface in all directions. Even though we haven't yet figured out how to compute the slope in all directions, we have enough information to find tangent planes. Suppose we want the plane tangent to a surface at a particular point (a, b, c). If we compute the two partial derivatives of the function for that point, we get enough information to determine two lines tangent to the surface, both through (a, b, c) and both tangent to the surface in their

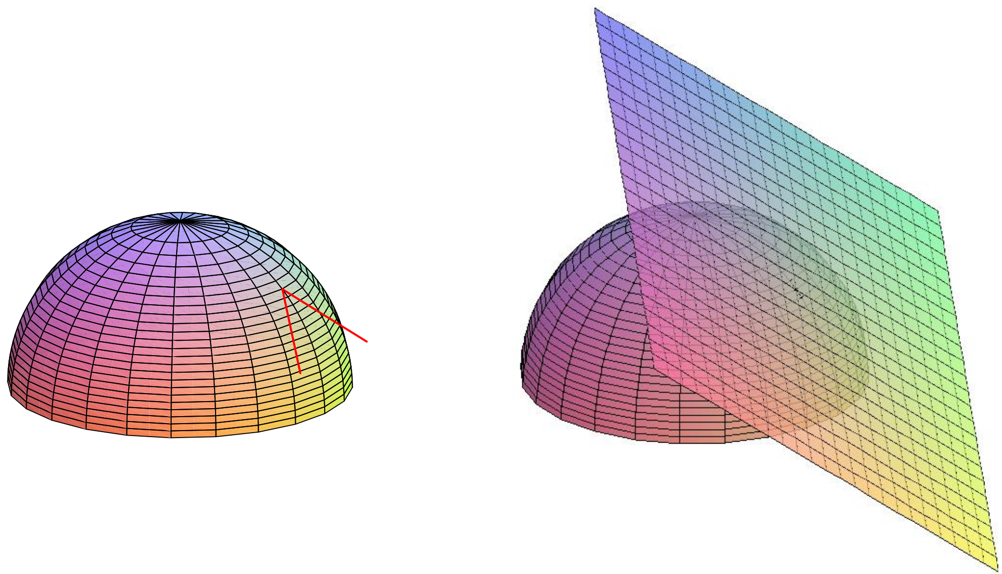


Figure 14.3.3 Tangent vectors and tangent plane. (AP)

respective directions. These two lines determine a plane, that is, there is exactly one plane containing the two lines: the tangent plane. Figure 14.3.3 shows (part of) two tangent lines at a point, and the tangent plane containing them.

How can we discover an equation for this tangent plane? We know a point on the plane, (a, b, c); we need a vector normal to the plane. If we can find two vectors, one parallel to each of the tangent lines we know how to find, then the cross product of these vectors will give the desired normal vector.

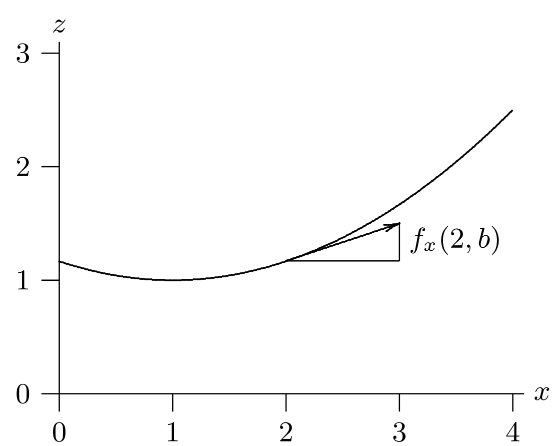


Figure 14.3.4 A tangent vector.

How can we find vectors parallel to the tangent lines? Consider first the line tangent to the surface above the line $y = b$. A vector $\langle u, v, w\rangle$ parallel to this tangent line must have y component $v = 0$, and we may as well take the x component to be $u = 1$. The ratio of the z component to the x component is the slope of the tangent line, precisely what we know how to compute. The slope of the tangent line is $f_x(a,b)$, so

$$f_x(a,b) = \frac{w}{u} = \frac{w}{1} = w.$$

In other words, a vector parallel to this tangent line is $\langle 1, 0, f_x(a,b)\rangle$, as shown in figure 14.3.4. If we repeat the reasoning for the tangent line above $x = a$, we get the vector $\langle 0, 1, f_y(a,b)\rangle$.

Now to find the desired normal vector we compute the cross product, $\langle 0, 1, f_y\rangle \times \langle 1, 0, f_x\rangle = \langle f_x, f_y, -1\rangle$. From our earlier discussion of planes, we can write down the equation we seek: $f_x(a,b)x + f_y(a,b)y - z = k$, and k as usual can be computed by substituting a known point: $f_x(a,b)(a) + f_y(a,b)(b) - c = k$. There are various more-or-less nice ways to write the result:

$$\begin{gathered}
f_x(a,b)x + f_y(a,b)y - z = f_x(a,b)a + f_y(a,b)b - c \\
f_x(a,b)x + f_y(a,b)y - f_x(a,b)a - f_y(a,b)b + c = z \\
f_x(a,b)(x-a) + f_y(a,b)(y-b) + c = z \\
f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b) = z
\end{gathered}$$

EXAMPLE 14.3.3 Find the plane tangent to $x^2 + y^2 + z^2 = 4$ at $(1, 1, \sqrt{2})$. This point is on the upper hemisphere, so we use $f(x,y) = \sqrt{4 - x^2 - y^2}$. Then $f_x(x,y) = -x(4 - x^2 - y^2)^{-1/2}$ and $f_y(x,y) = -y(4 - x^2 - y^2)^{-1/2}$, so $f_x(1,1) = f_y(1,1) = -1/\sqrt{2}$ and the equation of the plane is

$$z = -\frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1) + \sqrt{2}.$$

The hemisphere and this tangent plane are pictured in figure 14.3.3. □

So it appears that to find a tangent plane, we need only find two quite simple ordinary derivatives, namely f_x and f_y. This is true *if the tangent plane exists*. It is, unfortunately, not always the case that if f_x and f_y exist there is a tangent plane. Consider the function $xy^2/(x^2 + y^4)$ pictured in figure 14.2.1. This function has value 0 when $x = 0$ or $y = 0$, and we can "plug the hole" by agreeing that $f(0,0) = 0$. Now it's clear that $f_x(0,0) = f_y(0,0) = 0$, because in the x and y directions the surface is simply a horizontal line. But it's also clear from the picture that this surface does not have anything that deserves to be called a "tangent plane" at the origin, certainly not the x-y plane containing these two tangent lines.

When does a surface have a tangent plane at a particular point? What we really want from a tangent plane, as from a tangent line, is that the plane be a "good" approximation of the surface near the point. Here is how we can make this precise:

DEFINITION 14.3.4 Let $\Delta x = x - x_0$, $\Delta y = y - y_0$, and $\Delta z = z - z_0$ where $z_0 = f(x_0, y_0)$. The function $z = f(x, y)$ is differentiable at (x_0, y_0) if

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

and both ϵ_1 and ϵ_2 approach 0 as (x, y) approaches (x_0, y_0). □

This definition takes a bit of absorbing. Let's rewrite the central equation a bit:

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0) + \epsilon_1 \Delta x + \epsilon_2 \Delta y. \quad (14.3.1)$$

The first three terms on the right are the equation of the tangent plane, that is,

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

is the z-value of the point on the plane above (x, y). Equation 14.3.1 says that the z-value of a point on the surface is equal to the z-value of a point on the plane plus a "little bit," namely $\epsilon_1 \Delta x + \epsilon_2 \Delta y$. As (x, y) approaches (x_0, y_0), both Δx and Δy approach 0, so this little bit $\epsilon_1 \Delta x + \epsilon_2 \Delta y$ also approaches 0, and the z-values on the surface and the plane get close to each other. But that by itself is not very interesting: since the surface and the plane both contain the point (x_0, y_0, z_0), the z values will approach z_0 and hence get close to each other whether the tangent plane is "tangent" to the surface or not. The extra condition in the definition says that as (x, y) approaches (x_0, y_0), the ϵ values approach 0—this means that $\epsilon_1 \Delta x + \epsilon_2 \Delta y$ approaches 0 much, much faster, because $\epsilon_1 \Delta x$ is much smaller than either ϵ_1 or Δx. It is this extra condition that makes the plane a tangent plane.

We can see that the extra condition on ϵ_1 and ϵ_2 is just what is needed if we look at partial derivatives. Suppose we temporarily fix $y = y_0$, so $\Delta y = 0$. Then the equation from the definition becomes

$$\Delta z = f_x(x_0, y_0)\Delta x + \epsilon_1 \Delta x$$

or

$$\frac{\Delta z}{\Delta x} = f_x(x_0, y_0) + \epsilon_1.$$

Now taking the limit of the two sides as Δx approaches 0, the left side turns into the partial derivative of z with respect to x at (x_0, y_0), or in other words $f_x(x_0, y_0)$, and the right side does the same, because as (x, y) approaches (x_0, y_0), ϵ_1 approaches 0. Essentially the same calculation works for f_y.

Almost all of the functions we will encounter are differentiable at points we will be interested in, and often at all points. This is usually because the functions satisfy the hypotheses of this theorem.

THEOREM 14.3.5 If $f(x, y)$ and its partial derivatives are continuous at a point (x_0, y_0), then f is differentiable there. ■

Exercises 14.3.

1. Find f_x and f_y where $f(x, y) = \cos(x^2 y) + y^3$. ⇒

2. Find f_x and f_y where $f(x, y) = \dfrac{xy}{x^2 + y}$. ⇒

3. Find f_x and f_y where $f(x, y) = e^{x^2+y^2}$. ⇒

4. Find f_x and f_y where $f(x, y) = xy \ln(xy)$. ⇒

5. Find f_x and f_y where $f(x, y) = \sqrt{1 - x^2 - y^2}$. ⇒

6. Find f_x and f_y where $f(x, y) = x \tan(y)$. ⇒

7. Find f_x and f_y where $f(x, y) = \dfrac{1}{xy}$. ⇒

8. Find an equation for the plane tangent to $2x^2 + 3y^2 - z^2 = 4$ at $(1, 1, -1)$. ⇒

9. Find an equation for the plane tangent to $f(x, y) = \sin(xy)$ at $(\pi, 1/2, 1)$. ⇒

10. Find an equation for the plane tangent to $f(x, y) = x^2 + y^3$ at $(3, 1, 10)$. ⇒

11. Find an equation for the plane tangent to $f(x, y) = x \ln(xy)$ at $(2, 1/2, 0)$. ⇒

12. Find an equation for the line normal to $x^2 + 4y^2 = 2z$ at $(2, 1, 4)$. ⇒

13. Explain in your own words why, when taking a partial derivative of a function of multiple variables, we can treat the variables not being differentiated as constants.

14. Consider a differentiable function, $f(x, y)$. Give physical interpretations of the meanings of $f_x(a, b)$ and $f_y(a, b)$ as they relate to the graph of f.

15. In much the same way that we used the tangent line to approximate the value of a function from single variable calculus, we can use the tangent plane to approximate a function from multivariable calculus. Consider the tangent plane found in Exercise 11. Use this plane to approximate $f(1.98, 0.4)$.

16. Suppose that one of your colleagues has calculated the partial derivatives of a given function, and reported to you that $f_x(x, y) = 2x + 3y$ and that $f_y(x, y) = 4x + 6y$. Do you believe them? Why or why not? If not, what answer might you have accepted for f_y?

17. Suppose $f(t)$ and $g(t)$ are single variable differentiable functions. Find $\partial z/\partial x$ and $\partial z/\partial y$ for each of the following two variable functions.

a. $z = f(x)g(y)$

b. $z = f(xy)$

c. $z = f(x/y)$

14.4 The Chain Rule

Consider the surface $z = x^2y + xy^2$, and suppose that $x = 2 + t^4$ and $y = 1 - t^3$. We can think of the latter two equations as describing how x and y change relative to, say, time. Then

$$z = x^2y + xy^2 = (2 + t^4)^2(1 - t^3) + (2 + t^4)(1 - t^3)^2$$

tells us explicitly how the z coordinate of the corresponding point on the surface depends on t. If we want to know dz/dt we can compute it more or less directly—it's actually a bit simpler to use the chain rule:

$$\begin{aligned}
\frac{dz}{dt} &= x^2y' + 2xx'y + x2yy' + x'y^2 \\
&= (2xy + y^2)x' + (x^2 + 2xy)y' \\
&= (2(2 + t^4)(1 - t^3) + (1 - t^3)^2)(4t^3) + ((2 + t^4)^2 + 2(2 + t^4)(1 - t^3))(-3t^2)
\end{aligned}$$

If we look carefully at the middle step, $dz/dt = (2xy + y^2)x' + (x^2 + 2xy)y'$, we notice that $2xy + y^2$ is $\partial z/\partial x$, and $x^2 + 2xy$ is $\partial z/\partial y$. This turns out to be true in general, and gives us a new chain rule:

THEOREM 14.4.1 Suppose that $z = f(x, y)$, f is differentiable, $x = g(t)$, and $y = h(t)$. Assuming that the relevant derivatives exist,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Proof. If f is differentiable, then

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y,$$

where ϵ_1 and ϵ_2 approach 0 as (x, y) approaches (x_0, y_0). Then

$$\frac{\Delta z}{\Delta t} = f_x\frac{\Delta x}{\Delta t} + f_y\frac{\Delta y}{\Delta t} + \epsilon_1\frac{\Delta x}{\Delta t} + \epsilon_2\frac{\Delta y}{\Delta t}. \tag{14.4.1}$$

As Δt approaches 0, (x, y) approaches (x_0, y_0) and so

$$\begin{aligned}
\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t} &= \frac{dz}{dt} \\
\lim_{\Delta t\to 0}\epsilon_1\frac{\Delta x}{\Delta t} &= 0\cdot\frac{dx}{dt} \\
\lim_{\Delta t\to 0}\epsilon_2\frac{\Delta y}{\Delta t} &= 0\cdot\frac{dy}{dt}
\end{aligned}$$

and so taking the limit of (14.4.1) as Δt goes to 0 gives

$$\frac{dz}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt},$$

as desired. ∎

We can write the chain rule in way that is somewhat closer to the single variable chain rule:

$$\frac{df}{dt} = \langle f_x, f_y \rangle \cdot \langle x', y' \rangle,$$

or (roughly) the derivatives of the outside function "times" the derivatives of the inside functions. Not surprisingly, essentially the same chain rule works for functions of more than two variables, for example, given a function of three variables $f(x, y, z)$, where each of x, y and z is a function of t,

$$\frac{df}{dt} = \langle f_x, f_y, f_z \rangle \cdot \langle x', y', z' \rangle.$$

We can even extend the idea further. Suppose that $f(x, y)$ is a function and $x = g(s, t)$ and $y = h(s, t)$ are functions of two variables s and t. Then f is "really" a function of s and t as well, and

$$\frac{\partial f}{\partial s} = f_x g_s + f_y h_s \qquad \frac{\partial f}{\partial t} = f_x g_t + f_y h_t.$$

The natural extension of this to $f(x, y, z)$ works as well.

Recall that we used the ordinary chain rule to do implicit differentiation. We can do the same with the new chain rule.

EXAMPLE 14.4.2 $x^2 + y^2 + z^2 = 4$ defines a sphere, which is not a function of x and y, though it can be thought of as two functions, the top and bottom hemispheres. We can think of z as one of these two functions, so really $z = z(x, y)$, and we can think of x and y as particularly simple functions of x and y, and let $f(x, y, z) = x^2 + y^2 + z^2$. Since $f(x, y, z) = 4$, $\partial f/\partial x = 0$, but using the chain rule:

$$\begin{aligned} 0 = \frac{\partial f}{\partial x} &= f_x \frac{\partial x}{\partial x} + f_y \frac{\partial y}{\partial x} + f_z \frac{\partial z}{\partial x} \\ &= (2x)(1) + (2y)(0) + (2z)\frac{\partial z}{\partial x}, \end{aligned}$$

noting that since y is temporarily held constant its derivative $\partial y/\partial x = 0$. Now we can solve for $\partial z/\partial x$:

$$\frac{\partial z}{\partial x} = -\frac{2x}{2z} = -\frac{x}{z}.$$

In a similar manner we can compute $\partial z/\partial y$. □

Exercises 14.4.

1. Use the chain rule to compute dz/dt for $z = \sin(x^2 + y^2)$, $x = t^2 + 3$, $y = t^3$. ⇒

2. Use the chain rule to compute dz/dt for $z = x^2 y$, $x = \sin(t)$, $y = t^2 + 1$. ⇒

3. Use the chain rule to compute $\partial z/\partial s$ and $\partial z/\partial t$ for $z = x^2 y$, $x = \sin(st)$, $y = t^2 + s^2$. ⇒

4. Use the chain rule to compute $\partial z/\partial s$ and $\partial z/\partial t$ for $z = x^2 y^2$, $x = st$, $y = t^2 - s^2$. ⇒

5. Use the chain rule to compute $\partial z/\partial x$ and $\partial z/\partial y$ for $2x^2 + 3y^2 - 2z^2 = 9$. ⇒

6. Use the chain rule to compute $\partial z/\partial x$ and $\partial z/\partial y$ for $2x^2 + y^2 + z^2 = 9$. ⇒

7. Chemistry students will recognize the *ideal gas law*, given by $PV = nRT$ which relates the Pressure, Volume, and Temperature of n moles of gas. (R is the ideal gas constant). Thus, we can view pressure, volume, and temperature as variables, each one dependent on the other two.

a. If pressure of a gas is increasing at a rate of $0.2Pa/\text{min}$ and temperature is increasing at a rate of $1K/\text{min}$, how fast is the volume changing?

b. If the volume of a gas is decreasing at a rate of $0.3L/\text{min}$ and temperature is increasing at a rate of $.5K/\text{min}$, how fast is the pressure changing?

c. If the pressure of a gas is decreasing at a rate of $0.4Pa/\text{min}$ and the volume is increasing at a rate of $3L/\text{min}$, how fast is the temperature changing?

⇒

8. Verify the following identity in the case of the ideal gas law:

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$$

9. The previous exercise was a special case of the following fact, which you are to verify here: If $F(x, y, z)$ is a function of 3 variables, and the relation $F(x, y, z) = 0$ defines each of the variables in terms of the other two, namely $x = f(y, z)$, $y = g(x, z)$ and $z = h(x, y)$, then

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$$

14.5 Directional Derivatives

We still have not answered one of our first questions about the steepness of a surface: starting at a point on a surface given by $f(x, y)$, and walking in a particular direction, how steep is the surface? We are now ready to answer the question.

We already know roughly what has to be done: as shown in figure 14.3.1, we extend a line in the x-y plane to a vertical plane, and we then compute the slope of the curve that is the cross-section of the surface in that plane. The major stumbling block is that what appears in this plane to be the horizontal axis, namely the line in the x-y plane, is not an actual axis—we know nothing about the "units" along the axis. Our goal is to make this line into a t axis; then we need formulas to write x and y in terms of this new variable t; then we can write z in terms of t since we know z in terms of x and y; and finally we can simply take the derivative.

So we need to somehow "mark off" units on the line, and we need a convenient way to refer to the line in calculations. It turns out that we can accomplish both by using the vector form of a line. Suppose that $\mathbf{u}$ is a unit vector $\langle u_1, u_2\rangle$ in the direction of interest. A vector equation for the line through (x_0, y_0) in this direction is $\mathbf{v}(t) = \langle u_1 t + x_0, u_2 t + y_0\rangle$. The height of the surface above the point $(u_1 t+x_0, u_2 t+y_0)$ is $g(t) = f(u_1 t+x_0, u_2 t+y_0)$. Because $\mathbf{u}$ is a unit vector, the value of t is precisely the distance along the line from (x_0, y_0) to $(u_1 t + x_0, u_2 t + y_0)$; this means that the line is effectively a t axis, with origin at the point (x_0, y_0), so the slope we seek is

$$\begin{aligned} g'(0) &= \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle \cdot \langle u_1, u_2\rangle \\ &= \langle f_x, f_y\rangle \cdot \mathbf{u} \\ &= \nabla f \cdot \mathbf{u} \end{aligned}$$

Here we have used the chain rule and the derivatives $\frac{d}{dt}(u_1 t+x_0) = u_1$ and $\frac{d}{dt}(u_2 t+y_0) = u_2$. The vector $\langle f_x, f_y\rangle$ is very useful, so it has its own symbol, ∇f, pronounced "del f"; it is also called the **gradient** of f.

EXAMPLE 14.5.1 Find the slope of $z = x^2 + y^2$ at $(1, 2)$ in the direction of the vector $\langle 3, 4\rangle$.

We first compute the gradient at $(1, 2)$: $\nabla f = \langle 2x, 2y\rangle$, which is $\langle 2, 4\rangle$ at $(1, 2)$. A unit vector in the desired direction is $\langle 3/5, 4/5\rangle$, and the desired slope is then $\langle 2, 4\rangle \cdot \langle 3/5, 4/5\rangle = 6/5 + 16/5 = 22/5$. □

EXAMPLE 14.5.2 Find a tangent vector to $z = x^2 + y^2$ at $(1, 2)$ in the direction of the vector $\langle 3, 4\rangle$ and show that it is parallel to the tangent plane at that point.

Since $\langle 3/5, 4/5\rangle$ is a unit vector in the desired direction, we can easily expand it to a tangent vector simply by adding the third coordinate computed in the previous example: $\langle 3/5, 4/5, 22/5\rangle$. To see that this vector is parallel to the tangent plane, we can compute its dot product with a normal to the plane. We know that a normal to the tangent plane is

$$\langle f_x(1, 2), f_y(1, 2), -1\rangle = \langle 2, 4, -1\rangle,$$

and the dot product is $\langle 2, 4, -1\rangle \cdot \langle 3/5, 4/5, 22/5\rangle = 6/5 + 16/5 - 22/5 = 0$, so the two vectors are perpendicular. (Note that the vector normal to the surface, namely $\langle f_x, f_y, -1\rangle$, is simply the gradient with a -1 tacked on as the third component.) □

The slope of a surface given by $z = f(x, y)$ in the direction of a (two-dimensional) vector $\mathbf{u}$ is called the **directional derivative** of f, written $D_{\mathbf{u}} f$. The directional derivative immediately provides us with some additional information. We know that

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta$$

if $\mathbf{u}$ is a unit vector; θ is the angle between ∇f and $\mathbf{u}$. This tells us immediately that the largest value of $D_{\mathbf{u}} f$ occurs when $\cos\theta = 1$, namely, when $\theta = 0$, so ∇f is parallel to $\mathbf{u}$. In other words, the gradient ∇f points in the direction of steepest ascent of the surface, and $|\nabla f|$ is the slope in that direction. Likewise, the smallest value of $D_{\mathbf{u}} f$ occurs when $\cos\theta = -1$, namely, when $\theta = \pi$, so ∇f is anti-parallel to $\mathbf{u}$. In other words, $-\nabla f$ points in the direction of steepest descent of the surface, and $-|\nabla f|$ is the slope in that direction.

EXAMPLE 14.5.3 Investigate the direction of steepest ascent and descent for $z = x^2 + y^2$.

The gradient is $\langle 2x, 2y\rangle = 2\langle x, y\rangle$; this is a vector parallel to the vector $\langle x, y\rangle$, so the direction of steepest ascent is directly away from the origin, starting at the point (x, y). The direction of steepest descent is thus directly toward the origin from (x, y). Note that at $(0, 0)$ the gradient vector is $\langle 0, 0\rangle$, which has no direction, and it is clear from the plot of this surface that there is a minimum point at the origin, and tangent vectors in all directions are parallel to the x-y plane. □

If ∇f is perpendicular to $\mathbf{u}$, $D_{\mathbf{u}} f = |\nabla f| \cos(\pi/2) = 0$, since $\cos(\pi/2) = 0$. This means that in either of the two directions perpendicular to ∇f, the slope of the surface is 0; this implies that a vector in either of these directions is tangent to the level curve at that point. Starting with $\nabla f = \langle f_x, f_y\rangle$, it is easy to find a vector perpendicular to it: either $\langle f_y, -f_x\rangle$ or $\langle -f_y, f_x\rangle$ will work.

If $f(x, y, z)$ is a function of three variables, all the calculations proceed in essentially the same way. The rate at which f changes in a particular direction is $\nabla f \cdot \mathbf{u}$, where now $\nabla f = \langle f_x, f_y, f_z\rangle$ and $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ is a unit vector. Again ∇f points in the direction of maximum rate of increase, $-\nabla f$ points in the direction of maximum rate of decrease, and any vector perpendicular to ∇f is tangent to the level surface $f(x, y, z) = k$ at the point in question. Of course there are no longer just two such vectors; the vectors perpendicular to ∇f describe the tangent plane to the level surface, or in other words ∇f is a normal to the tangent plane.

EXAMPLE 14.5.4 Suppose the temperature at a point in space is given by $T(x, y, z) = T_0/(1 + x^2 + y^2 + z^2)$; at the origin the temperature in Kelvin is $T_0 > 0$, and it decreases in every direction from there. It might be, for example, that there is a source of heat at the origin, and as we get farther from the source, the temperature decreases. The gradient is

$$\begin{aligned}\nabla T &= \langle \frac{-2T_0 x}{(1+x^2+y^2+z^2)^2}, \frac{-2T_0 y}{(1+x^2+y^2+z^2)^2}, \frac{-2T_0 z}{(1+x^2+y^2+z^2)^2} \rangle \\ &= \frac{-2T_0}{(1+x^2+y^2+z^2)^2} \langle x, y, z \rangle.\end{aligned}$$

The gradient points directly at the origin from the point (x, y, z)—by moving directly toward the heat source, we increase the temperature as quickly as possible. □

EXAMPLE 14.5.5 Find the points on the surface defined by $x^2 + 2y^2 + 3z^2 = 1$ where the tangent plane is parallel to the plane defined by $3x - y + 3z = 1$.

Two planes are parallel if their normals are parallel or anti-parallel, so we want to find the points on the surface with normal parallel or anti-parallel to $\langle 3, -1, 3\rangle$. Let $f = x^2 + 2y^2 + 3z^2$; the gradient of f is normal to the level surface at every point, so we are looking for a gradient parallel or anti-parallel to $\langle 3, -1, 3\rangle$. The gradient is $\langle 2x, 4y, 6z\rangle$; if it is parallel or anti-parallel to $\langle 3, -1, 3\rangle$, then

$$\langle 2x, 4y, 6z\rangle = k\langle 3, -1, 3\rangle$$

for some k. This means we need a solution to the equations

$$2x = 3k \qquad 4y = -k \qquad 6z = 3k$$

but this is three equations in four unknowns—we need another equation. What we haven't used so far is that the points we seek are on the surface $x^2 + 2y^2 + 3z^2 = 1$; this is the fourth equation. If we solve the first three equations for x, y, and z and substitute into the fourth equation we get

$$\begin{aligned}1 &= \left(\frac{3k}{2}\right)^2 + 2\left(\frac{-k}{4}\right)^2 + 3\left(\frac{3k}{6}\right)^2 \\ &= \left(\frac{9}{4} + \frac{2}{16} + \frac{3}{4}\right)k^2 \\ &= \frac{25}{8}k^2\end{aligned}$$

so $k = \pm\frac{2\sqrt{2}}{5}$. The desired points are $\left(\frac{3\sqrt{2}}{5}, -\frac{\sqrt{2}}{10}, \frac{\sqrt{2}}{5}\right)$ and $\left(-\frac{3\sqrt{2}}{5}, \frac{\sqrt{2}}{10}, -\frac{\sqrt{2}}{5}\right)$. The ellipsoid and the three planes are shown in figure 14.5.1.

□

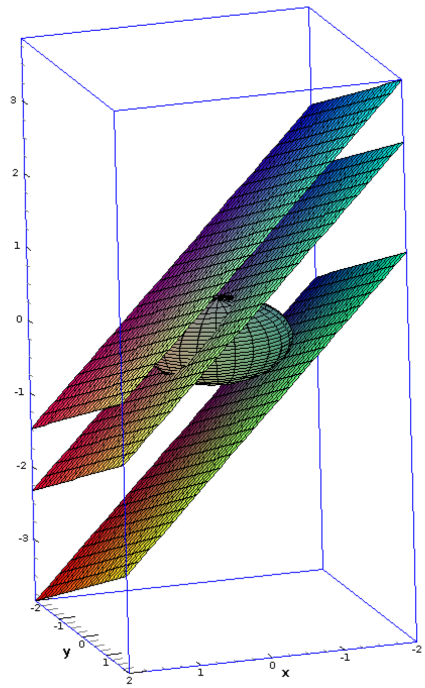


Figure 14.5.1 Ellipsoid with two tangent planes parallel to a given plane. (AP)

Exercises 14.5.

1. Find $D_{\mathbf{u}}f$ for $f = x^2 + xy + y^2$ in the direction of $\mathbf{v} = \langle 2, 1\rangle$ at the point $(1, 1)$. ⇒
2. Find $D_{\mathbf{u}}f$ for $f = \sin(xy)$ in the direction of $\mathbf{v} = \langle -1, 1\rangle$ at the point $(3, 1)$. ⇒
3. Find $D_{\mathbf{u}}f$ for $f = e^x \cos(y)$ in the direction 30 degrees from the positive x axis at the point $(1, \pi/4)$. ⇒
4. The temperature of a thin plate in the x-y plane is $T = x^2 + y^2$. How fast does temperature change at the point $(1, 5)$ moving in a direction 30 degrees from the positive x axis? ⇒
5. Suppose the density of a thin plate at (x, y) is $1/\sqrt{x^2 + y^2 + 1}$. Find the rate of change of the density at $(2, 1)$ in a direction $\pi/3$ radians from the positive x axis. ⇒
6. Suppose the electric potential at (x, y) is $\ln\sqrt{x^2 + y^2}$. Find the rate of change of the potential at $(3, 4)$ toward the origin and also in a direction at a right angle to the direction toward the origin. ⇒
7. A plane perpendicular to the x-y plane contains the point $(2, 1, 8)$ on the paraboloid $z = x^2 + 4y^2$. The cross-section of the paraboloid created by this plane has slope 0 at this point. Find an equation of the plane. ⇒
8. A plane perpendicular to the x-y plane contains the point $(3, 2, 2)$ on the paraboloid $36z = 4x^2 + 9y^2$. The cross-section of the paraboloid created by this plane has slope 0 at this point. Find an equation of the plane. ⇒
9. Suppose the temperature at (x, y, z) is given by $T = xy + \sin(yz)$. In what direction should you go from the point $(1, 1, 1)$ to decrease the temperature as quickly as possible? What is the rate of change of temperature in this direction? ⇒

10. Suppose the temperature at (x, y, z) is given by $T = xyz$. In what direction can you go from the point $(1, 1, 1)$ to maintain the same temperature? ⇒

11. Find an equation for the plane tangent to $x^2 - 3y^2 + z^2 = 7$ at $(1, 1, 3)$. ⇒

12. Find an equation for the plane tangent to $xyz = 6$ at $(1, 2, 3)$. ⇒

13. Find a vector function for the line normal to $x^2 + 2y^2 + 4z^2 = 26$ at $(2, -3, -1)$. ⇒

14. Find a vector function for the line normal to $x^2 + y^2 + 9z^2 = 56$ at $(4, 2, -2)$. ⇒

15. Find a vector function for the line normal to $x^2 + 5y^2 - z^2 = 0$ at $(4, 2, 6)$. ⇒

16. Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin(xy)$ at the point $(1, 0)$ has the value 1. ⇒

17. Show that the curve $\mathbf{r}(t) = \langle \ln(t), t\ln(t), t\rangle$ is tangent to the surface $xz^2 - yz + \cos(xy) = 1$ at the point $(0, 0, 1)$.

18. A bug is crawling on the surface of a hot plate, the temperature of which at the point x units to the right of the lower left corner and y units up from the lower left corner is given by $T(x, y) = 100 - x^2 - 3y^3$.

a. If the bug is at the point $(2, 1)$, in what direction should it move to cool off the fastest? How fast will the temperature drop in this direction?

b. If the bug is at the point $(1, 3)$, in what direction should it move in order to maintain its temperature?

⇒

19. The elevation on a portion of a hill is given by $f(x, y) = 100 - 4x^2 - 2y$. From the location above $(2, 1)$, in which direction will water run? ⇒

20. Suppose that $g(x, y) = y - x^2$. Find the gradient at the point $(-1, 3)$. Sketch the level curve to the graph of g when $g(x, y) = 2$, and plot both the tangent line and the gradient vector at the point $(-1, 3)$. (Make your sketch large). What do you notice, geometrically? ⇒

21. The gradient ∇f is a vector valued function of two variables. Prove the following gradient rules. Assume $f(x, y)$ and $g(x, y)$ are differentiable functions.

a. $\nabla(fg) = f\nabla(g) + g\nabla(f)$

b. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$

c. $\nabla((f(x, y))^n) = nf(x, y)^{n-1}\nabla f$

14.6 Higher order derivatives

In single variable calculus we saw that the second derivative is often useful: in appropriate circumstances it measures acceleration; it can be used to identify maximum and minimum points; it tells us something about how sharply curved a graph is. Not surprisingly, second derivatives are also useful in the multi-variable case, but again not surprisingly, things are a bit more complicated.

It's easy to see where some complication is going to come from: with two variables there are four possible second derivatives. To take a "derivative," we must take a partial derivative with respect to x or y, and there are four ways to do it: x then x, x then y, y then x, y then y.

EXAMPLE 14.6.1 Compute all four second derivatives of $f(x,y) = x^2y^2$.

Using an obvious notation, we get:

$$f_{xx} = 2y^2 \qquad f_{xy} = 4xy \qquad f_{yx} = 4xy \qquad f_{yy} = 2x^2.$$

□

You will have noticed that two of these are the same, the "mixed partials" computed by taking partial derivatives with respect to both variables in the two possible orders. This is not an accident—as long as the function is reasonably nice, this will always be true.

THEOREM 14.6.2 **Clairaut's Theorem** If the mixed partial derivatives are continuous, they are equal. ■

EXAMPLE 14.6.3 Compute the mixed partials of $f = xy/(x^2 + y^2)$.

$$f_x = \frac{y^3 - x^2y}{(x^2+y^2)^2} \qquad f_{xy} = -\frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^3}$$

We leave f_{yx} as an exercise. □

Exercises 14.6.

1. Let $f = xy/(x^2 + y^2)$; compute f_{xx}, f_{yx}, and f_{yy}. ⇒
2. Find all first and second partial derivatives of $x^3y^2 + y^5$. ⇒
3. Find all first and second partial derivatives of $4x^3 + xy^2 + 10$. ⇒
4. Find all first and second partial derivatives of $x \sin y$. ⇒
5. Find all first and second partial derivatives of $\sin(3x)\cos(2y)$. ⇒
6. Find all first and second partial derivatives of e^{x+y^2}. ⇒
7. Find all first and second partial derivatives of $\ln\sqrt{x^3 + y^4}$. ⇒
8. Find all first and second partial derivatives of z with respect to x and y if $x^2+4y^2+16z^2-64 = 0$. ⇒
9. Find all first and second partial derivatives of z with respect to x and y if $xy + yz + xz = 1$. ⇒
10. Let α and k be constants. Prove that the function $u(x,t) = e^{-\alpha^2k^2t}\sin(kx)$ is a solution to the heat equation $u_t = \alpha^2 u_{xx}$
11. Let a be a constant. Prove that $u = \sin(x-at) + \ln(x+at)$ is a solution to the wave equation $u_{tt} = a^2 u_{xx}$.
12. How many third-order derivatives does a function of 2 variables have? How many of these are distinct?

13. How many nth order derivatives does a function of 2 variables have? How many of these are distinct?

14.7 Maxima and minima

Suppose a surface given by $f(x,y)$ has a local maximum at (x_0,y_0,z_0); geometrically, this point on the surface looks like the top of a hill. If we look at the cross-section in the plane $y=y_0$, we will see a local maximum on the curve at (x_0,z_0), and we know from single-variable calculus that $\frac{\partial z}{\partial x}=0$ at this point. Likewise, in the plane $x=x_0$, $\frac{\partial z}{\partial y}=0$. So if there is a local maximum at (x_0,y_0,z_0), both partial derivatives at the point must be zero, and likewise for a local minimum. Thus, to find local maximum and minimum points, we need only consider those points at which both partial derivatives are 0. As in the single-variable case, it is possible for the derivatives to be 0 at a point that is neither a maximum or a minimum, so we need to test these points further.

You will recall that in the single variable case, we examined three methods to identify maximum and minimum points; the most useful is the second derivative test, though it does not always work. For functions of two variables there is also a second derivative test; again it is by far the most useful test, though it doesn't always work.

THEOREM 14.7.1 Suppose that the second partial derivatives of $f(x,y)$ are continuous near (x_0,y_0), and $f_x(x_0,y_0)=f_y(x_0,y_0)=0$. We denote by D the **discriminant** $D(x_0,y_0)=f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-f_{xy}(x_0,y_0)^2$. If $D>0$ and $f_{xx}(x_0,y_0)<0$ there is a local maximum at (x_0,y_0); if $D>0$ and $f_{xx}(x_0,y_0)>0$ there is a local minimum at (x_0,y_0); if $D<0$ there is neither a maximum nor a minimum at (x_0,y_0); if $D=0$, the test fails. ■

EXAMPLE 14.7.2 Verify that $f(x,y)=x^2+y^2$ has a minimum at $(0,0)$.

First, we compute all the needed derivatives:

$$f_x=2x \qquad f_y=2y \qquad f_{xx}=2 \qquad f_{yy}=2 \qquad f_{xy}=0.$$

The derivatives f_x and f_y are zero only at $(0,0)$. Applying the second derivative test there:

$$D(0,0)=f_{xx}(0,0)f_{yy}(0,0)-f_{xy}(0,0)^2=2\cdot 2-0=4>0$$

and

$$f_{xx}(0,0)=2>0,$$

so there is a local minimum at $(0,0)$, and there are no other possibilities. □

EXAMPLE 14.7.3 Find all local maxima and minima for $f(x,y)=x^2-y^2$.

The derivatives:

$$f_x = 2x \qquad f_y = -2y \qquad f_{xx} = 2 \qquad f_{yy} = -2 \qquad f_{xy} = 0.$$

Again there is a single critical point, at $(0,0)$, and

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 2 \cdot -2 - 0 = -4 < 0,$$

so there is neither a maximum nor minimum there, and so there are no local maxima or minima. The surface is shown in figure 14.7.1. □

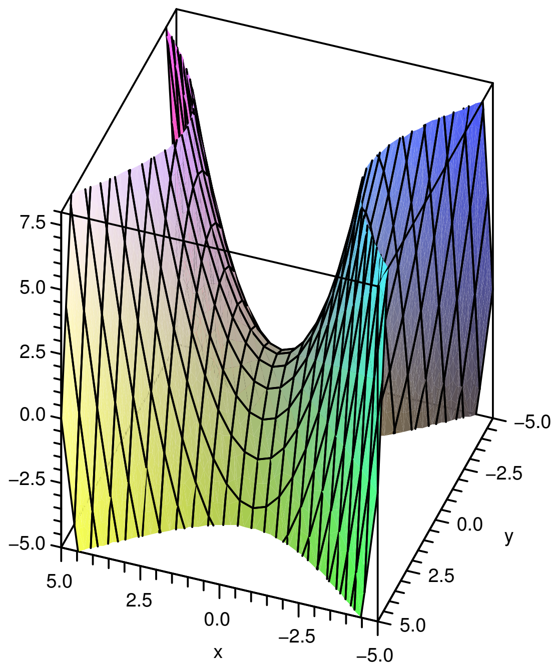


Figure 14.7.1 A saddle point, neither a maximum nor a minimum. (AP)

EXAMPLE 14.7.4 Find all local maxima and minima for $f(x,y) = x^4 + y^4$.

The derivatives:

$$f_x = 4x^3 \qquad f_y = 4y^3 \qquad f_{xx} = 12x^2 \qquad f_{yy} = 12y^2 \qquad f_{xy} = 0.$$

Again there is a single critical point, at $(0,0)$, and

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0 \cdot 0 - 0 = 0,$$

so we get no information. However, in this case it is easy to see that there is a minimum at $(0,0)$, because $f(0,0) = 0$ and at all other points $f(x,y) > 0$. □

EXAMPLE 14.7.5 Find all local maxima and minima for $f(x, y) = x^3 + y^3$.

The derivatives:

$$f_x = 3x^2 \qquad f_y = 3y^2 \qquad f_{xx} = 6x^2 \qquad f_{yy} = 6y^2 \qquad f_{xy} = 0.$$

Again there is a single critical point, at $(0, 0)$, and

$$D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 0 \cdot 0 - 0 = 0,$$

so we get no information. In this case, a little thought shows there is neither a maximum nor a minimum at $(0, 0)$: when x and y are both positive, $f(x, y) > 0$, and when x and y are both negative, $f(x, y) < 0$, and there are points of both kinds arbitrarily close to $(0, 0)$. Alternately, if we look at the cross-section when $y = 0$, we get $f(x, 0) = x^3$, which does not have either a maximum or minimum at $x = 0$. □

EXAMPLE 14.7.6 Suppose a box with no top is to hold a certain volume V. Find the dimensions for the box that result in the minimum surface area.

The area of the box is $A = 2hw + 2hl + lw$, and the volume is $V = lwh$, so we can write the area as a function of two variables,

$$A(l, w) = \frac{2V}{l} + \frac{2V}{w} + lw.$$

Then

$$A_l = -\frac{2V}{l^2} + w \quad \text{and} \quad A_w = -\frac{2V}{w^2} + l.$$

If we set these equal to zero and solve, we find $w = (2V)^{1/3}$ and $l = (2V)^{1/3}$, and the corresponding height is $h = V/(2V)^{2/3}$.

The second derivatives are

$$A_{ll} = \frac{4V}{l^3} \qquad A_{ww} = \frac{4V}{w^3} \qquad A_{lw} = 1,$$

so the discriminant is

$$D = \frac{4V}{l^3}\frac{4V}{w^3} - 1 = 4 - 1 = 3 > 0.$$

Since A_{ll} is 2, there is a local minimum at the critical point. Is this a global minimum? It is, but it is difficult to see this analytically; physically and graphically it is clear that there is a minimum, in which case it must be at the single critical point. This applet shows an example of such a graph. Note that we must choose a value for V in order to graph it. □

Recall that when we did single variable global maximum and minimum problems, the easiest cases were those for which the variable could be limited to a finite closed interval, for then we simply had to check all critical values and the endpoints. The previous example is difficult because there is no finite boundary to the domain of the problem—both w and l can be in $(0, \infty)$. As in the single variable case, the problem is often simpler when there is a finite boundary.

THEOREM 14.7.7 If $f(x, y)$ is continuous on a closed and bounded subset of $\mathbb{R}^2$, then it has both a maximum and minimum value. ■

As in the case of single variable functions, this means that the maximum and minimum values must occur at a critical point or on the boundary; in the two variable case, however, the boundary is a curve, not merely two endpoints.

EXAMPLE 14.7.8 The length of the diagonal of a box is to be 1 meter; find the maximum possible volume.

If the box is placed with one corner at the origin, and sides along the axes, the length of the diagonal is $\sqrt{x^2 + y^2 + z^2}$, and the volume is

$$V = xyz = xy\sqrt{1 - x^2 - y^2}.$$

Clearly, $x^2 + y^2 \le 1$, so the domain we are interested in is the quarter of the unit disk in the first quadrant. Computing derivatives:

$$V_x = \frac{y - 2yx^2 - y^3}{\sqrt{1 - x^2 - y^2}}$$

$$V_y = \frac{x - 2xy^2 - x^3}{\sqrt{1 - x^2 - y^2}}$$

If these are both 0, then $x = 0$ or $y = 0$, or $x = y = 1/\sqrt{3}$. The boundary of the domain is composed of three curves: $x = 0$ for $y \in [0, 1]$; $y = 0$ for $x \in [0, 1]$; and $x^2 + y^2 = 1$, where $x \ge 0$ and $y \ge 0$. In all three cases, the volume $xy\sqrt{1 - x^2 - y^2}$ is 0, so the maximum occurs at the only critical point $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. See figure 14.7.2. □

Exercises 14.7.

1. Find all local maximum and minimum points of $f = x^2 + 4y^2 - 2x + 8y - 1$. ⇒
2. Find all local maximum and minimum points of $f = x^2 - y^2 + 6x - 10y + 2$. ⇒
3. Find all local maximum and minimum points of $f = xy$. ⇒
4. Find all local maximum and minimum points of $f = 9 + 4x - y - 2x^2 - 3y^2$. ⇒

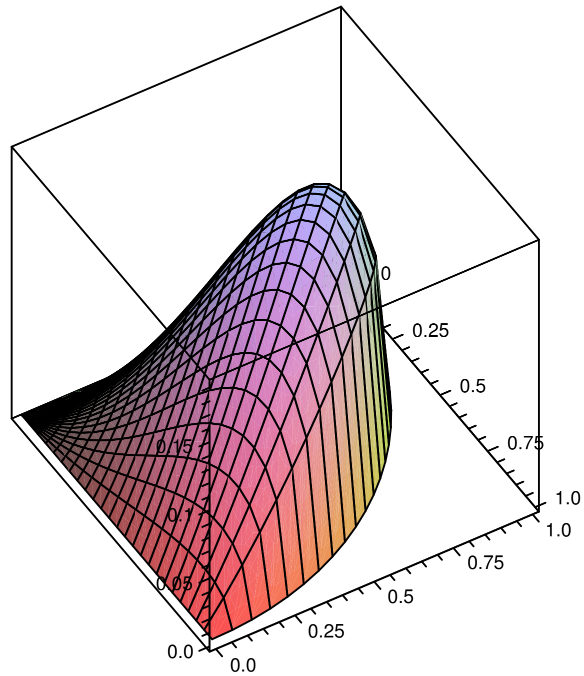


Figure 14.7.2 The volume of a box with fixed length diagonal.

5. Find all local maximum and minimum points of $f = x^2 + 4xy + y^2 - 6y + 1$. ⇒

6. Find all local maximum and minimum points of $f = x^2 - xy + 2y^2 - 5x + 6y - 9$. ⇒

7. Find the absolute maximum and minimum points of $f = x^2 + 3y - 3xy$ over the region bounded by $y = x$, $y = 0$, and $x = 2$. ⇒

8. A six-sided rectangular box is to hold 1/2 cubic meter; what shape should the box be to minimize surface area? ⇒

9. The post office will accept packages whose combined length and girth is at most 130 inches. (Girth is the maximum distance around the package perpendicular to the length; for a rectangular box, the length is the largest of the three dimensions.) What is the largest volume that can be sent in a rectangular box? ⇒

10. The bottom of a rectangular box costs twice as much per unit area as the sides and top. Find the shape for a given volume that will minimize cost. ⇒

11. Using the methods of this section, find the shortest distance from the origin to the plane $x + y + z = 10$. ⇒

12. Using the methods of this section, find the shortest distance from the point (x_0, y_0, z_0) to the plane $ax + by + cz = d$. You may assume that $c \neq 0$; use of Sage or similar software is recommended. ⇒

13. A trough is to be formed by bending up two sides of a long metal rectangle so that the cross-section of the trough is an isosceles trapezoid, as in figure 6.2.6. If the width of the metal sheet is 2 meters, how should it be bent to maximize the volume of the trough? ⇒

14. Given the three points $(1,4)$, $(5,2)$, and $(3,-2)$, $(x-1)^2+(y-4)^2+(x-5)^2+(y-2)^2+(x-3)^2+(y+2)^2$ is the sum of the squares of the distances from point (x,y) to the three points. Find x and y so that this quantity is minimized. ⇒

15. Suppose that $f(x,y)=x^2+y^2+kxy$. Find and classify the critical points, and discuss how they change when k takes on different values.

16. Find the shortest distance from the point $(0,b)$ to the parabola $y=x^2$. ⇒

17. Find the shortest distance from the point $(0,0,b)$ to the paraboloid $z=x^2+y^2$. ⇒

18. Consider the function $f(x,y)=x^3-3x^2y+y^3$.

a. Show that $(0,0)$ is the only critical point of f.

b. Show that the discriminant test is inconclusive for f.

c. Determine the cross-sections of f obtained by setting $y=kx$ for various values of k.

d. What kind of critical point is $(0,0)$?

19. Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid $2x^2+72y^2+18z^2=288$. ⇒

14.8 Lagrange Multipliers

Many applied max/min problems take the form of the last two examples: we want to find an extreme value of a function, like $V=xyz$, subject to a constraint, like $1=\sqrt{x^2+y^2+z^2}$. Often this can be done, as we have, by explicitly combining the equations and then finding critical points. There is another approach that is often convenient, the method of **Lagrange multipliers**.

It is somewhat easier to understand two variable problems, so we begin with one as an example. Suppose the perimeter of a rectangle is to be 100 units. Find the rectangle with largest area. This is a fairly straightforward problem from single variable calculus. We write down the two equations: $A=xy$, $P=100=2x+2y$, solve the second of these for y (or x), substitute into the first, and end up with a one-variable maximization problem. Let's now think of it differently: the equation $A=xy$ defines a surface, and the equation $100=2x+2y$ defines a curve (a line, in this case) in the x-y plane. If we graph both of these in the three-dimensional coordinate system, we can phrase the problem like this: what is the highest point on the surface above the line? The solution we already understand effectively produces the equation of the cross-section of the surface above the line and then treats it as a single variable problem. Instead, imagine that we draw the level curves (the contour lines) for the surface in the x-y plane, along with the line.

Imagine that the line represents a hiking trail and the contour lines are, as on a topographic map, the lines of constant altitude. How could you estimate, based on the graph, the high (or low) points on the path? As the path crosses contour lines, you know the path must be increasing or decreasing in elevation. At some point you will see the path just touch a contour line (tangent to it), and then begin to cross contours in the opposite

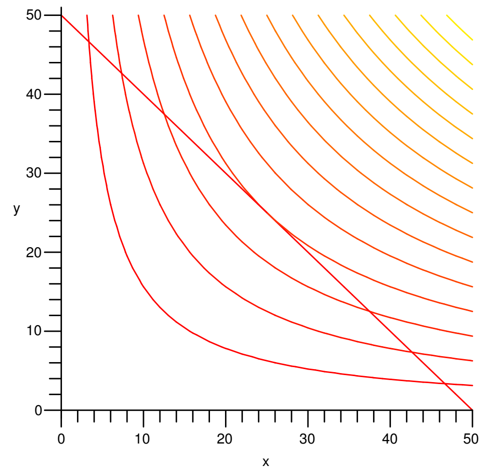


Figure 14.8.1 Constraint line with contour plot of the surface xy.

order—that point of tangency must be a maximum or minimum point. If we can identify all such points, we can then check them to see which gives the maximum and which the minimum value. As usual, we also need to check boundary points; in this problem, we know that x and y are positive, so we are interested in just the portion of the line in the first quadrant, as shown. The endpoints of the path, the two points on the axes, are not points of tangency, but they are the two places that the function xy is a minimum in the first quadrant.

How can we actually make use of this? At the points of tangency that we seek, the constraint curve (in this case the line) and the level curve have the same slope—their tangent lines are parallel. This also means that the constraint curve is perpendicular to the gradient vector of the function; going a bit further, if we can express the constraint curve itself as a level curve, then we seek the points at which the two level curves have parallel gradients. The curve $100 = 2x + 2y$ can be thought of as a level curve of the function $2x + 2y$; figure 14.8.2 shows both sets of level curves on a single graph. We are interested in those points where two level curves are tangent—but there are many such points, in fact an infinite number, as we've only shown a few of the level curves. All along the line $y = x$ are points at which two level curves are tangent. While this might seem to be a show-stopper, it is not.

The gradient of $2x + 2y$ is $\langle 2, 2\rangle$, and the gradient of xy is $\langle y, x\rangle$. They are parallel when $\langle 2, 2\rangle = \lambda\langle y, x\rangle$, that is, when $2 = \lambda y$ and $2 = \lambda x$. We have two equations in three unknowns, which typically results in many solutions (as we expected). A third equation will reduce the number of solutions; the third equation is the original constraint, $100 = 2x+2y$.

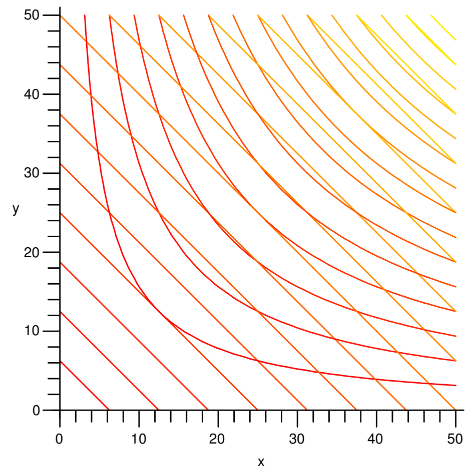


Figure 14.8.2 Contour plots for $2x + 2y$ and xy.

So we have the following system to solve:

$$2 = \lambda y \qquad 2 = \lambda x \qquad 100 = 2x + 2y.$$

In the first two equations, λ can't be 0, so we may divide by it to get $x = y = 2/\lambda$. Substituting into the third equation we get

$$2\frac{2}{\lambda} + 2\frac{2}{\lambda} = 100$$
$$\frac{8}{100} = \lambda$$

so $x = y = 25$. Note that we are not really interested in the value of λ—it is a clever tool, the Lagrange multiplier, introduced to solve the problem. In many cases, as here, it is easier to find λ than to find everything else without using λ.

The same method works for functions of three variables, except of course everything is one dimension higher: the function to be optimized is a function of three variables and the constraint represents a surface—for example, the function may represent temperature, and we may be interested in the maximum temperature on some surface, like a sphere. The points we seek are those at which the constraint surface is tangent to a level surface of the function. Once again, we consider the constraint surface to be a level surface of some function, and we look for points at which the two gradients are parallel, giving us three equations in four unknowns. The constraint provides a fourth equation.

EXAMPLE 14.8.1 Recall example 14.7.8: the diagonal of a box is 1, we seek to maximize the volume. The constraint is $1 = \sqrt{x^2 + y^2 + z^2}$, which is the same as $1 =$

$x^2 + y^2 + z^2$. The function to maximize is xyz. The two gradient vectors are $\langle 2x, 2y, 2z\rangle$ and $\langle yz, xz, xy\rangle$, so the equations to be solved are

$$\begin{aligned} yz &= 2x\lambda \\ xz &= 2y\lambda \\ xy &= 2z\lambda \\ 1 &= x^2 + y^2 + z^2 \end{aligned}$$

If $\lambda = 0$ then at least two of x, y, z must be 0, giving a volume of 0, which will not be the maximum. If we multiply the first two equations by x and y respectively, we get

$$\begin{aligned} xyz &= 2x^2\lambda \\ xyz &= 2y^2\lambda \end{aligned}$$

so $2x^2\lambda = 2y^2\lambda$ or $x^2 = y^2$; in the same way we can show $x^2 = z^2$. Hence the fourth equation becomes $1 = x^2 + x^2 + x^2$ or $x = 1/\sqrt{3}$, and so $x = y = z = 1/\sqrt{3}$ gives the maximum volume. This is of course the same answer we obtained previously. □

Another possibility is that we have a function of three variables, and we want to find a maximum or minimum value not on a surface but on a curve; often the curve is the intersection of two surfaces, so that we really have two constraint equations, say $g(x, y, z) = c_1$ and $h(x, y, z) = c_2$. It turns out that at points on the intersection of the surfaces where f has a maximum or minimum value,

$$\nabla f = \lambda \nabla g + \mu \nabla h.$$

As before, this gives us three equations, one for each component of the vectors, but now in five unknowns, x, y, z, λ, and μ. Since there are two constraint functions, we have a total of five equations in five unknowns, and so can usually find the solutions we need.

EXAMPLE 14.8.2 The plane $x + y - z = 1$ intersects the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse closest to and farthest from the origin.

We want the extreme values of $f = \sqrt{x^2 + y^2 + z^2}$ subject to the constraints $g = x^2 + y^2 = 1$ and $h = x + y - z = 1$. To simplify the algebra, we may use instead $f = x^2 + y^2 + z^2$, since this has a maximum or minimum value at exactly the points at which $\sqrt{x^2 + y^2 + z^2}$ does. The gradients are

$$\nabla f = \langle 2x, 2y, 2z\rangle \qquad \nabla g = \langle 2x, 2y, 0\rangle \qquad \nabla h = \langle 1, 1, -1\rangle,$$

so the equations we need to solve are

$$\begin{aligned}2x &= \lambda 2x + \mu\\ 2y &= \lambda 2y + \mu\\ 2z &= 0 - \mu\\ 1 &= x^2 + y^2\\ 1 &= x + y - z.\end{aligned}$$

Subtracting the first two we get $2y - 2x = \lambda(2y - 2x)$, so either $\lambda = 1$ or $x = y$. If $\lambda = 1$ then $\mu = 0$, so $z = 0$ and the last two equations are

$$1 = x^2 + y^2 \qquad \text{and} \qquad 1 = x + y.$$

Solving these gives $x = 1$, $y = 0$, or $x = 0$, $y = 1$, so the points of interest are $(1, 0, 0)$ and $(0, 1, 0)$, which are both distance 1 from the origin. If $x = y$, the fourth equation is $2x^2 = 1$, giving $x = y = \pm 1/\sqrt{2}$, and from the fifth equation we get $z = -1 \pm \sqrt{2}$. The distance from the origin to $(1/\sqrt{2}, 1/\sqrt{2}, -1 + \sqrt{2})$ is $\sqrt{4 - 2\sqrt{2}} \approx 1.08$ and the distance from the origin to $(-1/\sqrt{2}, -1/\sqrt{2}, -1 - \sqrt{2})$ is $\sqrt{4 + 2\sqrt{2}} \approx 2.6$. Thus, the points $(1, 0, 0)$ and $(0, 1, 0)$ are closest to the origin and $(-1/\sqrt{2}, -1/\sqrt{2}, -1 - \sqrt{2})$ is farthest from the origin. This applet shows the cylinder, the plane, the four points of interest, and the origin. □

Exercises 14.8.

1. A six-sided rectangular box is to hold 1/2 cubic meter; what shape should the box be to minimize surface area? ⇒
2. The post office will accept packages whose combined length and girth are at most 130 inches (girth is the maximum distance around the package perpendicular to the length). What is the largest volume that can be sent in a rectangular box? ⇒
3. The bottom of a rectangular box costs twice as much per unit area as the sides and top. Find the shape for a given volume that will minimize cost. ⇒
4. Using Lagrange multipliers, find the shortest distance from the point (x_0, y_0, z_0) to the plane $ax + by + cz = d$. ⇒
5. Find all points on the surface $xy - z^2 + 1 = 0$ that are closest to the origin. ⇒
6. The material for the bottom of an aquarium costs half as much as the high strength glass for the four sides. Find the shape of the cheapest aquarium that holds a given volume V. ⇒
7. The plane $x - y + z = 2$ intersects the cylinder $x^2 + y^2 = 4$ in an ellipse. Find the points on the ellipse closest to and farthest from the origin. ⇒
8. Find three positive numbers whose sum is 48 and whose product is as large as possible. ⇒
9. Find all points on the plane $x + y + z = 5$ in the first octant at which $f(x, y, z) = xy^2z^2$ has a maximum value. ⇒

10. Find the points on the surface $x^2 - yz = 5$ that are closest to the origin. ⇒

11. A manufacturer makes two models of an item, standard and deluxe. It costs \$40 to manufacture the standard model and \$60 for the deluxe. A market research firm estimates that if the standard model is priced at x dollars and the deluxe at y dollars, then the manufacturer will sell $500(y - x)$ of the standard items and $45,000 + 500(x - 2y)$ of the deluxe each year. How should the items be priced to maximize profit? ⇒

12. A length of sheet metal is to be made into a water trough by bending up two sides as shown in figure 14.8.3. Find x and ϕ so that the trapezoid–shaped cross section has maximum area, when the width of the metal sheet is 27 inches (that is, $2x + y = 27$). ⇒

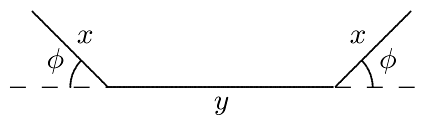


Figure 14.8.3 Cross-section of a trough.

13. Find the maximum and minimum values of $f(x, y, z) = 6x + 3y + 2z$ subject to the constraint $g(x, y, z) = 4x^2 + 2y^2 + z^2 - 70 = 0$. ⇒

14. Find the maximum and minimum values of $f(x, y) = e^{xy}$ subject to the constraint $g(x, y) = x^3 + y^3 - 16 = 0$. ⇒

15. Find the maximum and minimum values of $f(x, y) = xy + \sqrt{9 - x^2 - y^2}$ when $x^2 + y^2 \le 9$. ⇒

16. Find three real numbers whose sum is 9 and the sum of whose squares is a small as possible. ⇒

17. Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere. ⇒

15

Multiple Integration

15.1 Volume and Average Height

Consider a surface $f(x, y)$; you might temporarily think of this as representing physical topography—a hilly landscape, perhaps. What is the average height of the surface (or average altitude of the landscape) over some region?

As with most such problems, we start by thinking about how we might approximate the answer. Suppose the region is a rectangle, $[a, b] \times [c, d]$. We can divide the rectangle into a grid, m subdivisions in one direction and n in the other, as indicated in figure 15.1.1. We pick x values x_0, x_1,..., x_{m-1} in each subdivision in the x direction, and similarly in the y direction. At each of the points (x_i, y_j) in one of the smaller rectangles in the grid, we compute the height of the surface: $f(x_i, y_j)$. Now the average of these heights should be (depending on the fineness of the grid) close to the average height of the surface:

$$\frac{f(x_0, y_0) + f(x_1, y_0) + \cdots + f(x_0, y_1) + f(x_1, y_1) + \cdots + f(x_{m-1}, y_{n-1})}{mn}.$$

As both m and n go to infinity, we expect this approximation to converge to a fixed value, the actual average height of the surface. For reasonably nice functions this does indeed happen.

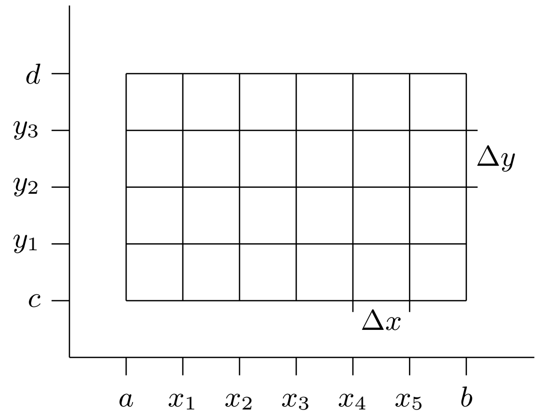


Figure 15.1.1 A rectangular subdivision of $[a,b]\times[c,d]$.

Using sigma notation, we can rewrite the approximation:

$$\begin{aligned}\frac{1}{mn}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j,y_i) &= \frac{1}{(b-a)(d-c)}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j,y_i)\frac{b-a}{m}\frac{d-c}{n}\\ &= \frac{1}{(b-a)(d-c)}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j,y_i)\Delta x\Delta y.\end{aligned}$$

The two parts of this product have useful meaning: $(b-a)(d-c)$ is of course the area of the rectangle, and the double sum adds up mn terms of the form $f(x_j,y_i)\Delta x\Delta y$, which is the height of the surface at a point times the area of one of the small rectangles into which we have divided the large rectangle. In short, each term $f(x_j,y_i)\Delta x\Delta y$ is the volume of a tall, thin, rectangular box, and is approximately the volume under the surface and above one of the small rectangles; see figure 15.1.2. When we add all of these up, we get an approximation to the volume under the surface and above the rectangle $R=[a,b]\times[c,d]$. When we take the limit as m and n go to infinity, the double sum becomes the actual volume under the surface, which we divide by $(b-a)(d-c)$ to get the average height.

Double sums like this come up in many applications, so in a way it is the most important part of this example; dividing by $(b-a)(d-c)$ is a simple extra step that allows the computation of an average. As we did in the single variable case, we introduce a special notation for the limit of such a double sum:

$$\lim_{m,n\to\infty}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j,y_i)\Delta x\Delta y = \iint\limits_R f(x,y)\,dx\,dy = \iint\limits_R f(x,y)\,dA,$$

the **double integral** of f over the region R. The notation dA indicates a small bit of area, without specifying any particular order for the variables x and y; it is shorter and

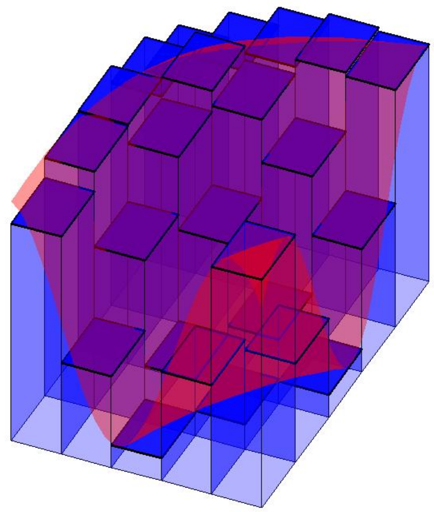


Figure 15.1.2 Approximating the volume under a surface.

more "generic" than writing $dx\,dy$. The average height of the surface in this notation is

$$\frac{1}{(b-a)(d-c)}\iint\limits_R f(x,y)\,dA.$$

The next question, of course, is: How do we compute these double integrals? You might think that we will need some two-dimensional version of the Fundamental Theorem of Calculus, but as it turns out we can get away with just the single variable version, applied twice.

Going back to the double sum, we can rewrite it to emphasize a particular order in which we want to add the terms:

$$\sum_{i=0}^{n-1}\left(\sum_{j=0}^{m-1} f(x_j,y_i)\Delta x\right)\Delta y.$$

In the sum in parentheses, only the value of x_j is changing; y_i is temporarily constant. As m goes to infinity, this sum has the right form to turn into an integral:

$$\lim_{m\to\infty}\sum_{j=0}^{m-1} f(x_j,y_i)\Delta x = \int_a^b f(x,y_i)\,dx.$$

So after we take the limit as m goes to infinity, the sum is

$$\sum_{i=0}^{n-1}\left(\int_a^b f(x,y_i)\,dx\right)\Delta y.$$

Of course, for different values of y_i this integral has different values; in other words, it is really a function applied to y_i:

$$G(y) = \int_a^b f(x,y)\,dx.$$

If we substitute back into the sum we get

$$\sum_{i=0}^{n-1} G(y_i)\Delta y.$$

This sum has a nice interpretation. The value $G(y_i)$ is the area of a cross section of the region under the surface $f(x,y)$, namely, when $y = y_i$. The quantity $G(y_i)\Delta y$ can be interpreted as the volume of a solid with face area $G(y_i)$ and thickness Δy. Think of the surface $f(x,y)$ as the top of a loaf of sliced bread. Each slice has a cross-sectional area and a thickness; $G(y_i)\Delta y$ corresponds to the volume of a single slice of bread. Adding these up approximates the total volume of the loaf. (This is very similar to the technique we used to compute volumes in section 9.3, except that there we need the cross-sections to be in some way "the same".) Figure 15.1.3 shows this "sliced loaf" approximation using the same surface as shown in figure 15.1.2. Nicely enough, this sum looks just like the sort of sum that turns into an integral, namely,

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=0}^{n-1} G(y_i)\Delta y &= \int_c^d G(y)\,dy\\ &= \int_c^d\int_a^b f(x,y)\,dx\,dy.\end{aligned}$$

Let's be clear about what this means: we first will compute the inner integral, temporarily treating y as a constant. We will do this by finding an anti-derivative with respect to x, then substituting $x = a$ and $x = b$ and subtracting, as usual. The result will be an expression with no x variable but some occurrences of y. Then the outer integral will be an ordinary one-variable problem, with y as the variable.

EXAMPLE 15.1.1 Figure 15.1.2 shows the function $\sin(xy)+6/5$ on $[0.5,3.5]\times[0.5,2.5]$. The volume under this surface is

$$\int_{0.5}^{2.5}\int_{0.5}^{3.5} \sin(xy) + \frac{6}{5}\,dx\,dy.$$

The inner integral is

$$\int_{0.5}^{3.5} \sin(xy) + \frac{6}{5}\,dx = \frac{-\cos(xy)}{y} + \frac{6x}{5}\Bigg|_{0.5}^{3.5} = \frac{-\cos(3.5y)}{y} + \frac{\cos(0.5y)}{y} + \frac{18}{5}.$$

Unfortunately, this gives a function for which we can't find a simple anti-derivative. To complete the problem we could use Sage or similar software to approximate the integral.

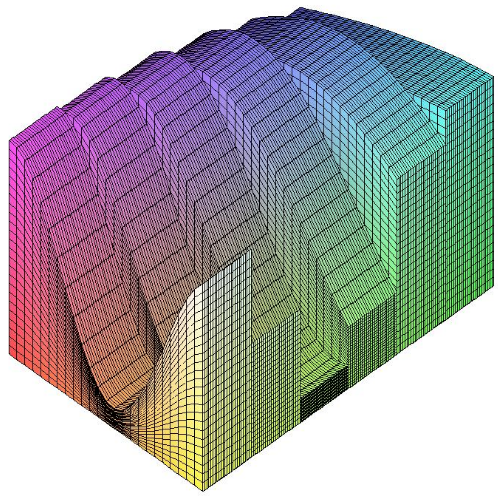


Figure 15.1.3 Approximating the volume under a surface with slices. (AP)

Doing this gives a volume of approximately 8.84, so the average height is approximately $8.84/6 \approx 1.47$. □

Because addition and multiplication are commutative and associative, we can rewrite the original double sum:

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j, y_i)\Delta x\Delta y = \sum_{j=0}^{m-1}\sum_{i=0}^{n-1} f(x_j, y_i)\Delta y\Delta x.$$

Now if we repeat the development above, the inner sum turns into an integral:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_j, y_i)\Delta y = \int_c^d f(x_j, y)\,dy,$$

and then the outer sum turns into an integral:

$$\lim_{m\to\infty}\sum_{j=0}^{m-1}\left(\int_c^d f(x_j, y)\,dy\right)\Delta x = \int_a^b\int_c^d f(x, y)\,dy\,dx.$$

In other words, we can compute the integrals in either order, first with respect to x then y, or vice versa. Thinking of the loaf of bread, this corresponds to slicing the loaf in a direction perpendicular to the first.

We haven't really proved that the value of a double integral is equal to the value of the corresponding two single integrals in either order of integration, but provided the function is reasonably nice, this is true; the result is called **Fubini's Theorem**.

EXAMPLE 15.1.2 We compute $\displaystyle\iint_R 1+(x-1)^2+4y^2\,dA$, where $R=[0,3]\times[0,2]$, in two ways.

First,

$$\begin{aligned}\int_0^3\int_0^2 1+(x-1)^2+4y^2\,dy\,dx &= \int_0^3 y+(x-1)^2y+\frac{4}{3}y^3\Big|_0^2\,dx\\ &= \int_0^3 2+2(x-1)^2+\frac{32}{3}\,dx\\ &= 2x+\frac{2}{3}(x-1)^3+\frac{32}{3}x\Big|_0^3\\ &= 6+\frac{2}{3}\cdot 8+\frac{32}{3}\cdot 3-(0-1\cdot\frac{2}{3}+0)\\ &= 44.\end{aligned}$$

In the other order:

$$\begin{aligned}\int_0^2\int_0^3 1+(x-1)^2+4y^2\,dx\,dy &= \int_0^2 x+\frac{(x-1)^3}{3}+4y^2x\Big|_0^3\,dy\\ &= \int_0^2 3+\frac{8}{3}+12y^2+\frac{1}{3}\,dy\\ &= 3y+\frac{8}{3}y+4y^3+\frac{1}{3}y\Big|_0^2\\ &= 6+\frac{16}{3}+32+\frac{2}{3}\\ &= 44.\end{aligned}$$

□

In this example there is no particular reason to favor one direction over the other; in some cases, one direction might be much easier than the other, so it's usually worth considering the two different possibilities.

Frequently we will be interested in a region that is not simply a rectangle. Let's compute the volume under the surface $x+2y^2$ above the region described by $0\le x\le 1$ and $0\le y\le x^2$, shown in figure 15.1.4.

In principle there is nothing more difficult about this problem. If we imagine the three-dimensional region under the surface and above the parabolic region as an oddly shaped loaf of bread, we can still slice it up, approximate the volume of each slice, and add these

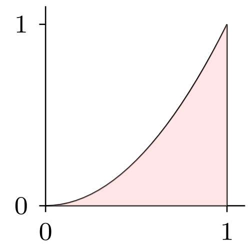


Figure 15.1.4 A parabolic region of integration.

volumes up. For example, if we slice perpendicular to the x axis at x_i, the thickness of a slice will be Δx and the area of the slice will be

$$\int_0^{x_i^2} x_i + 2y^2\,dy.$$

When we add these up and take the limit as Δx goes to 0, we get the double integral

$$\begin{aligned}\int_0^1 \int_0^{x^2} x + 2y^2\,dy\,dx &= \int_0^1 xy + \frac{2}{3}y^3\Big|_0^{x^2}\,dx\\ &= \int_0^1 x^3 + \frac{2}{3}x^6\,dx\\ &= \frac{x^4}{4} + \frac{2}{21}x^7\Big|_0^1\\ &= \frac{1}{4} + \frac{2}{21} = \frac{29}{84}.\end{aligned}$$

We could just as well slice the solid perpendicular to the y axis, in which case we get

$$\begin{aligned}\int_0^1 \int_{\sqrt{y}}^1 x + 2y^2\,dx\,dy &= \int_0^1 \frac{x^2}{2} + 2y^2x\Big|_{\sqrt{y}}^1\,dy\\ &= \int_0^1 \frac{1}{2} + 2y^2 - \frac{y}{2} - 2y^2\sqrt{y}\,dy\\ &= \frac{y}{2} + \frac{2}{3}y^3 - \frac{y^2}{4} - \frac{4}{7}y^{7/2}\Big|_0^1\\ &= \frac{1}{2} + \frac{2}{3} - \frac{1}{4} - \frac{4}{7} = \frac{29}{84}.\end{aligned}$$

What is the average height of the surface over this region? As before, it is the volume divided by the area of the base, but now we need to use integration to compute the area of the base, since it is not a simple rectangle. The area is

$$\int_0^1 x^2\,dx = \frac{1}{3},$$

so the average height is 29/28.

EXAMPLE 15.1.3 Find the volume under the surface $z = \sqrt{1-x^2}$ and above the triangle formed by $y = x$, $x = 1$, and the x-axis.

Let's consider the two possible ways to set this up:

$$\int_0^1 \int_0^x \sqrt{1-x^2}\,dy\,dx \qquad \text{or} \qquad \int_0^1 \int_y^1 \sqrt{1-x^2}\,dx\,dy.$$

Which appears easier? In the first, the first (inner) integral is easy, because we need an anti-derivative with respect to y, and the entire integrand $\sqrt{1-x^2}$ is constant with respect to y. Of course, the second integral may be more difficult. In the second, the first integral is mildly unpleasant—a trig substitution. So let's try the first one, since the first step is easy, and see where that leaves us.

$$\int_0^1 \int_0^x \sqrt{1-x^2}\,dy\,dx = \int_0^1 y\sqrt{1-x^2}\Big|_0^x\,dx = \int_0^1 x\sqrt{1-x^2}\,dx.$$

This is quite easy, since the substitution $u = 1-x^2$ works:

$$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}u^{3/2} = -\frac{1}{3}(1-x^2)^{3/2}.$$

Then

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{3}(1-x^2)^{3/2}\Big|_0^1 = \frac{1}{3}.$$

This is a good example of how the order of integration can affect the complexity of the problem. In this case it is possible to do the other order, but it is a bit messier. In some cases one order may lead to a very difficult or impossible integral; it's usually worth considering both possibilities before going very far. □

Exercises 15.1.

1. Compute $\displaystyle\int_0^2\int_0^4 1+x\,dy\,dx$. ⇒

2. Compute $\displaystyle\int_{-1}^1\int_0^2 x+y\,dy\,dx$. ⇒

3. Compute $\displaystyle\int_1^2\int_0^y xy\,dx\,dy$. ⇒

4. Compute $\displaystyle\int_0^1\int_{y^2/2}^{\sqrt{y}} dx\,dy$. ⇒

5. Compute $\displaystyle\int_1^2\int_1^x \frac{x^2}{y^2}\,dy\,dx$. ⇒

6. Compute $\displaystyle\int_0^1\int_0^{x^2} \frac{y}{e^x}\,dy\,dx$. ⇒

7. Compute $\displaystyle\int_0^{\sqrt{\pi/2}}\int_0^{x^2} x\cos y\,dy\,dx$. ⇒

8. Compute $\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta} r^2(\cos\theta-r)\,dr\,d\theta$. ⇒

9. Compute: $\displaystyle\int_0^1\int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy$. ⇒

10. Compute: $\displaystyle\int_0^1\int_{y^2}^1 y\sin(x^2)\,dx\,dy$. ⇒

11. Compute: $\displaystyle\int_0^1\int_{x^2}^1 x\sqrt{1+y^2}\,dy\,dx$ ⇒

12. Compute: $\displaystyle\int_0^1\int_0^y \frac{2}{\sqrt{1-x^2}}\,dx\,dy$ ⇒

13. Compute: $\displaystyle\int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy$ ⇒

14. Compute $\displaystyle\int_{-1}^1\int_0^{1-x^2} x^2-\sqrt{y}\,dy\,dx$. ⇒

15. Compute $\displaystyle\int_0^{\sqrt{2}/2}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} x\,dy\,dx$. ⇒

16. Evaluate $\displaystyle\iint x^2\,dA$ over the region in the first quadrant bounded by the hyperbola $xy=16$ and the lines $y=x$, $y=0$, and $x=8$. ⇒

17. Find the volume below $z=1-y$ above the region $-1\le x\le 1$, $0\le y\le 1-x^2$. ⇒

18. Find the volume bounded by $z=x^2+y^2$ and $z=4$. ⇒

19. Find the volume in the first octant bounded by $y^2=4-x$ and $y=2z$. ⇒

20. Find the volume in the first octant bounded by $y^2=4x$, $2x+y=4$, $z=y$, and $y=0$. ⇒

21. Find the volume in the first octant bounded by $x+y+z=9$, $2x+3y=18$, and $x+3y=9$. ⇒

22. Find the volume in the first octant bounded by $x^2+y^2=a^2$ and $z=x+y$. ⇒

23. Find the volume bounded by $4x^2+y^2=4z$ and $z=2$. ⇒

24. Find the volume bounded by $z=x^2+y^2$ and $z=y$. ⇒

25. Find the volume under the surface $z=xy$ above the triangle with vertices $(1,1,0)$, $(4,1,0)$, $(1,2,0)$. ⇒

26. Find the volume enclosed by $y=x^2$, $y=4$, $z=x^2$, $z=0$. ⇒

27. A swimming pool is circular with a 40 meter diameter. The depth is constant along east-west lines and increases linearly from 2 meters at the south end to 7 meters at the north end. Find the volume of the pool. ⇒

28. Find the average value of $f(x,y)=e^y\sqrt{x+e^y}$ on the rectangle with vertices $(0,0)$, $(4,0)$, $(4,1)$ and $(0,1)$. ⇒

29. Figure 15.1.5 shows a temperature map of Colorado. Use the data to estimate the average temperature in the state using 4, 16 and 25 subdivisions. Give both an upper and lower estimate. Why do we like Colorado for this problem? What other state(s) might we like?

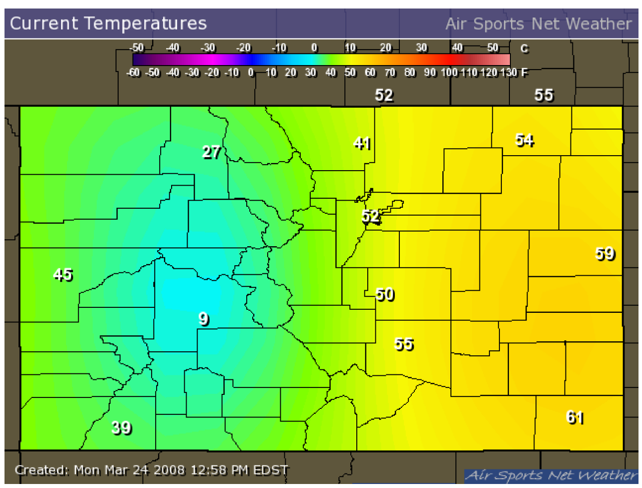


Figure 15.1.5 Colorado temperatures.

30. Three cylinders of radius 1 intersect at right angles at the origin, as shown in figure 15.1.6. Find the volume contained inside all three cylinders. ⇒

31. Prove that if $f(x,y)$ is integrable and if $g(x,y)=\int_a^x\int_b^y f(s,t)\,dt\,ds$ then $g_{xy}=g_{yx}=f(x,y)$.

32. Reverse the order of integration on each of the following integrals

a. $\displaystyle\int_0^9\int_0^{\sqrt{9-y}} f(x,y)\,dx\,dy$

b. $\displaystyle\int_1^2\int_0^{\ln x} f(x,y)\,dy\,dx$

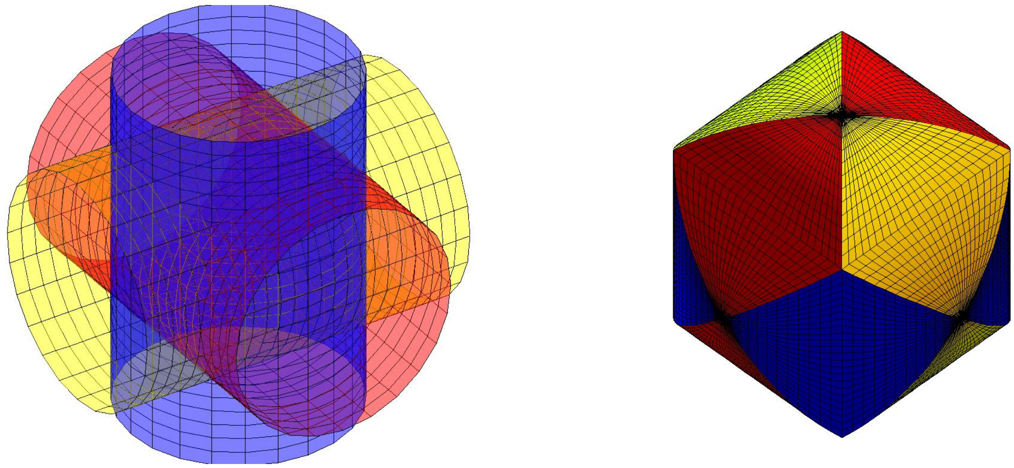


Figure 15.1.6 Intersection of three cylinders. (AP)

c. $\displaystyle\int_0^1 \int_{\arcsin y}^{\pi/2} f(x,y)\,dx\,dy$

d. $\displaystyle\int_0^1 \int_{4x}^{4} f(x,y)\,dy\,dx$

e. $\displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} f(x,y)\,dx\,dy$

33. What are the parallels between Fubini's Theorem and Clairaut's Theorem?

15.2 Double Integrals in Cylindrical Coordinates

Suppose we have a surface given in cylindrical coordinates as $z = f(r,\theta)$ and we wish to find the integral over some region. We could attempt to translate into rectangular coordinates and do the integration there, but it is often easier to stay in cylindrical coordinates.

How might we approximate the volume under such a surface in a way that uses cylindrical coordinates directly? The basic idea is the same as before: we divide the region into many small regions, multiply the area of each small region by the height of the surface somewhere in that little region, and add them up. What changes is the shape of the small regions; in order to have a nice representation in terms of r and θ, we use small pieces of ring-shaped areas, as shown in figure 15.2.1. Each small region is roughly rectangular, except that two sides are segments of a circle and the other two sides are not quite parallel. Near a point (r,θ), the length of either circular arc is about $r\Delta\theta$ and the length of each straight side is simply Δr. When Δr and $\Delta\theta$ are very small, the region is nearly a rectangle with area $r\Delta r\Delta\theta$, and the volume under the surface is approximately

$$\sum\sum f(r_i,\theta_j) r_i \Delta r \Delta\theta.$$

In the limit, this turns into a double integral

$$\int_{\theta_0}^{\theta_1}\int_{r_0}^{r_1} f(r,\theta) r\,dr\,d\theta.$$

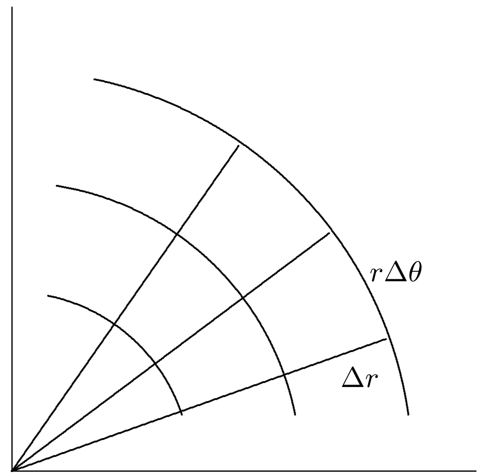


Figure 15.2.1 A cylindrical coordinates "grid".

EXAMPLE 15.2.1 Find the volume under $z = \sqrt{4-r^2}$ above the quarter circle bounded by the two axes and the circle $x^2+y^2=4$ in the first quadrant.

In terms of r and θ, this region is described by the restrictions $0 \le r \le 2$ and $0 \le \theta \le \pi/2$, so we have

$$\begin{aligned}\int_0^{\pi/2}\int_0^2 \sqrt{4-r^2}\, r\,dr\,d\theta &= \int_0^{\pi/2} -\frac{1}{3}(4-r^2)^{3/2}\Big|_0^2\,d\theta\\ &= \int_0^{\pi/2}\frac{8}{3}\,d\theta\\ &= \frac{4\pi}{3}.\end{aligned}$$

The surface is a portion of the sphere of radius 2 centered at the origin, in fact exactly one-eighth of the sphere. We know the formula for volume of a sphere is $(4/3)\pi r^3$, so the volume we have computed is $(1/8)(4/3)\pi 2^3 = (4/3)\pi$, in agreement with our answer. □

This example is much like a simple one in rectangular coordinates: the region of interest may be described exactly by a constant range for each of the variables. As with rectangular coordinates, we can adapt the method to deal with more complicated regions.

EXAMPLE 15.2.2 Find the volume under $z = \sqrt{4-r^2}$ above the region enclosed by the curve $r = 2\cos\theta$, $-\pi/2 \le \theta \le \pi/2$; see figure 15.2.2. The region is described in polar coordinates by the inequalities $-\pi/2 \le \theta \le \pi/2$ and $0 \le r \le 2\cos\theta$, so the double integral is

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \sqrt{4-r^2}\; r\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{2\cos\theta} \sqrt{4-r^2}\; r\,dr\,d\theta.$$

We can rewrite the integral as shown because of the symmetry of the volume; this avoids a complication during the evaluation. Proceeding:

$$\begin{aligned}
2\int_0^{\pi/2}\int_0^{2\cos\theta} \sqrt{4-r^2}\; r\,dr\,d\theta &= 2\int_0^{\pi/2} -\frac{1}{3}\,(4-r^2)^{3/2}\Big|_0^{2\cos\theta}\,d\theta \\
&= 2\int_0^{\pi/2} -\frac{8}{3}\sin^3\theta + \frac{8}{3}\,d\theta \\
&= 2\left(-\frac{8}{3}\frac{\cos^3\theta}{3} - \cos\theta + \frac{8}{3}\theta\right)\Big|_0^{\pi/2} \\
&= \frac{8}{3}\pi - \frac{32}{9}.
\end{aligned}$$

□

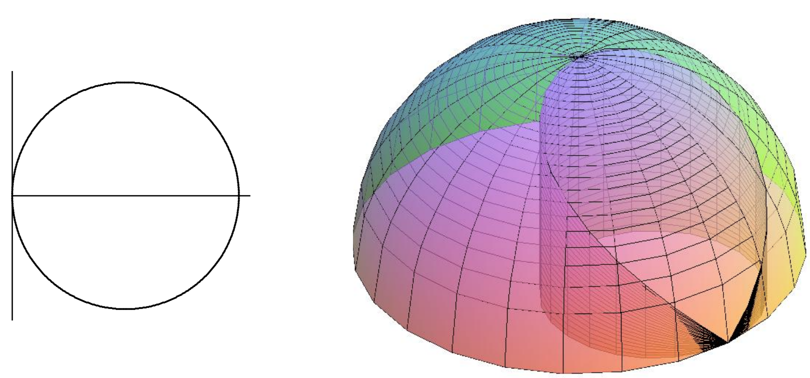


Figure 15.2.2 Volume over a region with non-constant limits.

You might have learned a formula for computing areas in polar coordinates. It is possible to compute areas as volumes, so that you need only remember one technique. Consider the surface $z = 1$, a horizontal plane. The volume under this surface and above a region in the x-y plane is simply $1 \cdot$ (area of the region), so computing the volume really just computes the area of the region.

EXAMPLE 15.2.3 Find the area outside the circle $r = 2$ and inside $r = 4\sin\theta$; see figure 15.2.3. The region is described by $\pi/6 \le \theta \le 5\pi/6$ and $2 \le r \le 4\sin\theta$, so the integral is

$$\int_{\pi/6}^{5\pi/6}\int_{2}^{4\sin\theta} 1\,r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} \left.\frac{1}{2}r^2\right|_{2}^{4\sin\theta} d\theta$$

$$= \int_{\pi/6}^{5\pi/6} 8\sin^2\theta - 2\,d\theta$$

$$= \frac{4}{3}\pi + 2\sqrt{3}.$$

□

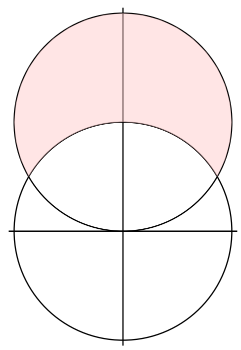


Figure 15.2.3 Finding area by computing volume.

Exercises 15.2.

1. Find the volume above the x-y plane, under the surface $r^2 = 2z$, and inside $r = 2$. ⇒

2. Find the volume inside both $r = 1$ and $r^2 + z^2 = 4$. ⇒

3. Find the volume below $z = \sqrt{1 - r^2}$ and above the top half of the cone $z = r$. ⇒

4. Find the volume below $z = r$, above the x-y plane, and inside $r = \cos\theta$. ⇒

5. Find the volume below $z = r$, above the x-y plane, and inside $r = 1 + \cos\theta$. ⇒

6. Find the volume between $x^2 + y^2 = z^2$ and $x^2 + y^2 = z$. ⇒

7. Find the area inside $r = 1 + \sin\theta$ and outside $r = 2\sin\theta$. ⇒

8. Find the area inside both $r = 2\sin\theta$ and $r = 2\cos\theta$. ⇒

9. Find the area inside the four-leaf rose $r = \cos(2\theta)$ and outside $r = 1/2$. ⇒

10. Find the area inside the cardioid $r = 2(1 + \cos\theta)$ and outside $r = 2$. ⇒

11. Find the area of one loop of the three-leaf rose $r = \cos(3\theta)$. ⇒

12. Compute $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2 + y^2)\,dy\,dx$ by converting to cylindrical coordinates. ⇒

13. Compute $\displaystyle\int_{0}^{a}\int_{-\sqrt{a^2-x^2}}^{0} x^2y\,dy\,dx$ by converting to cylindrical coordinates. ⇒

14. Find the volume under $z = y^2 + x + 2$ above the region $x^2 + y^2 \leq 4$ ⇒

15. Find the volume between $z = x^2y^3$ and $z = 1$ above the region $x^2 + y^2 \leq 1$ ⇒

16. Find the volume inside $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$. ⇒

17. Find the volume under $z = r$ above $r = 3 + \cos\theta$. ⇒

18. Figure 15.2.4 shows the plot of $r = 1 + 4\sin(5\theta)$.

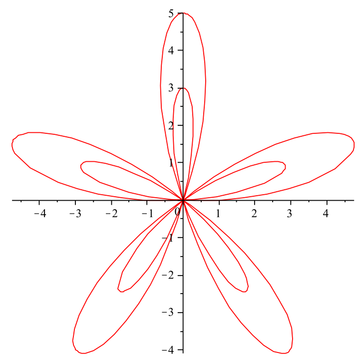


Figure 15.2.4 $r = 1 + 4\sin(5\theta)$

a. Describe the behavior of the graph in terms of the given equation. Specifically, explain maximum and minimum values, number of leaves, and the 'leaves within leaves'.

b. Give an integral or integrals to determine the area outside a smaller leaf but inside a larger leaf.

c. How would changing the value of a in the equation $r = 1 + a\cos(5\theta)$ change the relative sizes of the inner and outer leaves? Focus on values $a \geq 1$. (Hint: How would we change the maximum and minimum values?)

19. Consider the integral $\displaystyle\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dA$, where D is the unit disk centered at the origin. (See the graph here.)

a. Why might this integral be considered improper?

b. Calculate the value of the integral of the same function $1/\sqrt{x^2+y^2}$ over the annulus with outer radius 1 and inner radius δ.

c. Obtain a value for the integral on the whole disk by letting δ approach 0. ⇒

d. For which values λ can we replace the denominator with $(x^2+y^2)^\lambda$ in the original integral and still get a finite value for the improper integral?

15.3 Moment and Center of Mass

Using a single integral we were able to compute the center of mass for a one-dimensional object with variable density, and a two dimensional object with constant density. With a double integral we can handle two dimensions and variable density.

Just as before, the coordinates of the center of mass are

$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M},$$

where M is the total mass, M_y is the moment around the y-axis, and M_x is the moment around the x-axis. (You may want to review the concepts in section 9.6.)

The key to the computation, just as before, is the approximation of mass. In the two-dimensional case, we treat density σ as mass per square area, so when density is constant, mass is (density)(area). If we have a two-dimensional region with varying density given by $\sigma(x, y)$, and we divide the region into small subregions with area ΔA, then the mass of one subregion is approximately $\sigma(x_i, y_j)\Delta A$, the total mass is approximately the sum of many of these, and as usual the sum turns into an integral in the limit:

$$M = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \sigma(x, y)\, dy\, dx,$$

and similarly for computations in cylindrical coordinates. Then as before

$$M_x = \int_{x_0}^{x_1} \int_{y_0}^{y_1} y\sigma(x, y)\, dy\, dx$$

$$M_y = \int_{x_0}^{x_1} \int_{y_0}^{y_1} x\sigma(x, y)\, dy\, dx.$$

EXAMPLE 15.3.1 Find the center of mass of a thin, uniform plate whose shape is the region between $y = \cos x$ and the x-axis between $x = -\pi/2$ and $x = \pi/2$. Since the density is constant, we may take $\sigma(x, y) = 1$.

It is clear that $\bar{x} = 0$, but for practice let's compute it anyway. First we compute the mass:

$$M = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} 1\, dy\, dx = \int_{-\pi/2}^{\pi/2} \cos x\, dx = \sin x \Big|_{-\pi/2}^{\pi/2} = 2.$$

Next,

$$M_x = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} y\, dy\, dx = \int_{-\pi/2}^{\pi/2} \frac{1}{2} \cos^2 x\, dx = \frac{\pi}{4}.$$

Finally,

$$M_y = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} x\, dy\, dx = \int_{-\pi/2}^{\pi/2} x \cos x\, dx = 0.$$

So $\bar{x} = 0$ as expected, and $\bar{y} = \pi/4/2 = \pi/8$. This is the same problem as in example 9.6.4; it may be helpful to compare the two solutions. □

EXAMPLE 15.3.2 Find the center of mass of a two-dimensional plate that occupies the quarter circle $x^2+y^2 \le 1$ in the first quadrant and has density $k(x^2+y^2)$. It seems clear that because of the symmetry of both the region and the density function (both are important!), $\bar{x}=\bar{y}$. We'll do both to check our work.

Jumping right in:

$$M = \int_0^1 \int_0^{\sqrt{1-x^2}} k(x^2+y^2)\,dy\,dx = k\int_0^1 x^2\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3}\,dx.$$

This integral is something we can do, but it's a bit unpleasant. Since everything in sight is related to a circle, let's back up and try polar coordinates. Then $x^2+y^2=r^2$ and

$$M = \int_0^{\pi/2}\int_0^1 k(r^2)\,r\,dr\,d\theta = k\int_0^{\pi/2} \left.\frac{r^4}{4}\right|_0^1 d\theta = k\int_0^{\pi/2}\frac{1}{4}\,d\theta = k\frac{\pi}{8}.$$

Much better. Next, since $y = r\sin\theta$,

$$M_x = k\int_0^{\pi/2}\int_0^1 r^4\sin\theta\,dr\,d\theta = k\int_0^{\pi/2}\frac{1}{5}\sin\theta\,d\theta = k\left.-\frac{1}{5}\cos\theta\right|_0^{\pi/2} = \frac{k}{5}.$$

Similarly,

$$M_y = k\int_0^{\pi/2}\int_0^1 r^4\cos\theta\,dr\,d\theta = k\int_0^{\pi/2}\frac{1}{5}\cos\theta\,d\theta = k\left.\frac{1}{5}\sin\theta\right|_0^{\pi/2} = \frac{k}{5}.$$

Finally, $\bar{x}=\bar{y}=\dfrac{8}{5\pi}$. □

Exercises 15.3.

1. Find the center of mass of a two-dimensional plate that occupies the square $[0,1]\times[0,1]$ and has density function xy. ⇒
2. Find the center of mass of a two-dimensional plate that occupies the triangle $0 \le x \le 1$, $0 \le y \le x$, and has density function xy. ⇒
3. Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at $(0,0)$ and has density function y. ⇒
4. Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at $(0,0)$ and has density function x^2. ⇒
5. Find the center of mass of a two-dimensional plate that occupies the triangle formed by $x=2$, $y=x$, and $y=2x$ and has density function $2x$. ⇒
6. Find the center of mass of a two-dimensional plate that occupies the triangle formed by $x=0$, $y=x$, and $2x+y=6$ and has density function x^2. ⇒

7. Find the center of mass of a two-dimensional plate that occupies the region enclosed by the parabolas $x = y^2$, $y = x^2$ and has density function $\sqrt{x}$. ⇒

8. Find the centroid of the area in the first quadrant bounded by $x^2 - 8y + 4 = 0$, $x^2 = 4y$, and $x = 0$. (Recall that the centroid is the center of mass when the density is 1 everywhere.) ⇒

9. Find the centroid of one loop of the three-leaf rose $r = \cos(3\theta)$. (Recall that the centroid is the center of mass when the density is 1 everywhere, and that the mass in this case is the same as the area, which was the subject of exercise 11 in section 15.2.) The computations of the integrals for the moments M_x and M_y are elementary but quite long; Sage can help. ⇒

10. Find the center of mass of a two dimensional object that occupies the region $0 \le x \le \pi$, $0 \le y \le \sin x$, with density $\sigma = 1$. ⇒

11. A two-dimensional object has shape given by $r = 1 + \cos\theta$ and density $\sigma(r, \theta) = 2 + \cos\theta$. Set up the three integrals required to compute the center of mass. ⇒

12. A two-dimensional object has shape given by $r = \cos\theta$ and density $\sigma(r, \theta) = r + 1$. Set up the three integrals required to compute the center of mass. ⇒

13. A two-dimensional object sits inside $r = 1 + \cos\theta$ and outside $r = \cos\theta$, and has density 1 everywhere. Set up the integrals required to compute the center of mass. ⇒

15.4 Surface Area

We next seek to compute the area of a surface above (or below) a region in the x-y plane. How might we approximate this? We start, as usual, by dividing the region into a grid of small rectangles. We want to approximate the area of the surface above one of these small rectangles. The area is very close to the area of the tangent plane above the small rectangle. If the tangent plane just happened to be horizontal, of course the area would simply be the area of the rectangle. For a typical plane, however, the area is the area of a parallelogram, as indicated in figure 15.4.1. Note that the area of the parallelogram is obviously larger the more "tilted" the tangent plane. In the interactive figure you can see that viewed from above the four parallelograms exactly cover a rectangular region in the x-y plane.

Now recall a curious fact: the area of a parallelogram can be computed as the cross product of two vectors (page 315). We simply need to acquire two vectors, parallel to the sides of the parallelogram and with lengths to match. But this is easy: in the x direction we use the tangent vector we already know, namely $\langle 1, 0, f_x\rangle$ and multiply by Δx to shrink it to the right size: $\langle \Delta x, 0, f_x \Delta x\rangle$. In the y direction we do the same thing and get $\langle 0, \Delta y, f_y \Delta y\rangle$. The cross product of these vectors is $\langle f_x, f_y, -1\rangle\, \Delta x\, \Delta y$ with length $\sqrt{f_x^2 + f_y^2 + 1}\,\Delta x\,\Delta y$, the area of the parallelogram. Now we add these up and take the limit, to produce the integral

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} \sqrt{f_x^2 + f_y^2 + 1}\,dy\,dx.$$

As before, the limits need not be constant.

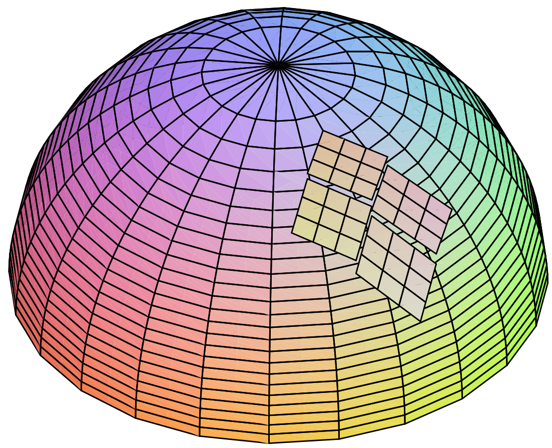


Figure 15.4.1 Small parallelograms at points of tangency. (AP)

EXAMPLE 15.4.1 We find the area of the hemisphere $z = \sqrt{1-x^2-y^2}$. We compute the derivatives

$$f_x = \frac{-x}{\sqrt{1-x^2-y^2}} \qquad f_x = \frac{-y}{\sqrt{1-x^2-y^2}},$$

and then the area is

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{\frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2} + 1}\,dy\,dx.$$

This is a bit on the messy side, but we can use polar coordinates:

$$\int_0^{2\pi}\int_0^1 \sqrt{\frac{1}{1-r^2}}\,r\,dr\,d\theta.$$

This integral is improper, since the function is undefined at the limit 1. We therefore compute

$$\lim_{a\to 1^-}\int_0^a \sqrt{\frac{1}{1-r^2}}\,r\,dr = \lim_{a\to 1^-} -\sqrt{1-a^2}+1 = 1,$$

using the substitution $u = 1-r^2$. Then the area is

$$\int_0^{2\pi} 1\,d\theta = 2\pi.$$

You may recall that the area of a sphere of radius r is $4\pi r^2$, so half the area of a unit sphere is $(1/2)4\pi = 2\pi$, in agreement with our answer. □

Exercises 15.4.

1. Find the area of the surface of a right circular cone of height h and base radius a. ⇒
2. Find the area of the portion of the plane $z = mx$ inside the cylinder $x^2 + y^2 = a^2$. ⇒
3. Find the area of the portion of the plane $x + y + z = 1$ in the first octant. ⇒
4. Find the area of the upper half of the cone $x^2 + y^2 = z^2$ inside the cylinder $x^2 + y^2 - 2x = 0$. ⇒
5. Find the area of the upper half of the cone $x^2 + y^2 = z^2$ above the interior of one loop of $r = \cos(2\theta)$. ⇒
6. Find the area of the upper hemisphere of $x^2 + y^2 + z^2 = 1$ above the interior of one loop of $r = \cos(2\theta)$. ⇒
7. The plane $ax + by + cz = d$ cuts a triangle in the first octant provided that a, b, c and d are all positive. Find the area of this triangle. ⇒
8. Find the area of the portion of the cone $x^2 + y^2 = 3z^2$ lying above the xy plane and inside the cylinder $x^2 + y^2 = 4y$. ⇒

15.5 Triple Integrals

It will come as no surprise that we can also do triple integrals—integrals over a three-dimensional region. The simplest application allows us to compute volumes in an alternate way.

To approximate a volume in three dimensions, we can divide the three-dimensional region into small rectangular boxes, each $\Delta x \times \Delta y \times \Delta z$ with volume $\Delta x \Delta y \Delta z$. Then we add them all up and take the limit, to get an integral:

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} dz\, dy\, dx.$$

If the limits are constant, we are simply computing the volume of a rectangular box.

EXAMPLE 15.5.1 We use an integral to compute the volume of the box with opposite corners at $(0, 0, 0)$ and $(1, 2, 3)$.

$$\int_0^1 \int_0^2 \int_0^3 dz\, dy\, dx = \int_0^1 \int_0^2 z|_0^3\; dy\, dx = \int_0^1 \int_0^2 3\, dy\, dx = \int_0^1 3y|_0^2\; dx = \int_0^1 6\, dx = 6.$$

□

Of course, this is more interesting and useful when the limits are not constant.

EXAMPLE 15.5.2 Find the volume of the tetrahedron with corners at $(0, 0, 0)$, $(0, 3, 0)$, $(2, 3, 0)$, and $(2, 3, 5)$.

The whole problem comes down to correctly describing the region by inequalities: $0 \le x \le 2$, $3x/2 \le y \le 3$, $0 \le z \le 5x/2$. The lower y limit comes from the equation of the line $y = 3x/2$ that forms one edge of the tetrahedron in the x-y plane; the upper z limit comes from the equation of the plane $z = 5x/2$ that forms the "upper" side of the tetrahedron; see figure 15.5.1. Now the volume is

$$\begin{aligned}
\int_0^2 \int_{3x/2}^3 \int_0^{5x/2} dz\,dy\,dx &= \int_0^2 \int_{3x/2}^3 z\Big|_0^{5x/2}\,dy\,dx \\
&= \int_0^2 \int_{3x/2}^3 \frac{5x}{2}\,dy\,dx \\
&= \int_0^2 \frac{5x}{2}y\Big|_{3x/2}^3\,dx \\
&= \int_0^2 \frac{15x}{2} - \frac{15x^2}{4}\,dx \\
&= \frac{15x^2}{4} - \frac{15x^3}{12}\Big|_0^2 \\
&= 15 - 10 = 5.
\end{aligned}$$

□

Pretty much just the way we did for two dimensions we can use triple integration to compute mass, center of mass, and various average quantities.

EXAMPLE 15.5.3 Suppose the temperature at a point is given by $T = xyz$. Find the average temperature in the cube with opposite corners at $(0,0,0)$ and $(2,2,2)$.

In two dimensions we add up the temperature at "each" point and divide by the area; here we add up the temperatures and divide by the volume, 8:

$$\begin{aligned}
\frac{1}{8}\int_0^2\int_0^2\int_0^2 xyz\,dz\,dy\,dx &= \frac{1}{8}\int_0^2\int_0^2 \frac{xyz^2}{2}\Big|_0^2\,dy\,dx = \frac{1}{16}\int_0^2\int_0^2 xy\,dy\,dx \\
&= \frac{1}{4}\int_0^2 \frac{xy^2}{2}\Big|_0^2\,dx = \frac{1}{8}\int_0^2 4x\,dx = \frac{1}{2}\,\frac{x^2}{2}\Big|_0^2 = 1.
\end{aligned}$$

□

EXAMPLE 15.5.4 Suppose the density of an object is given by xz, and the object occupies the tetrahedron with corners $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, and $(0,1,1)$. Find the mass and center of mass of the object.

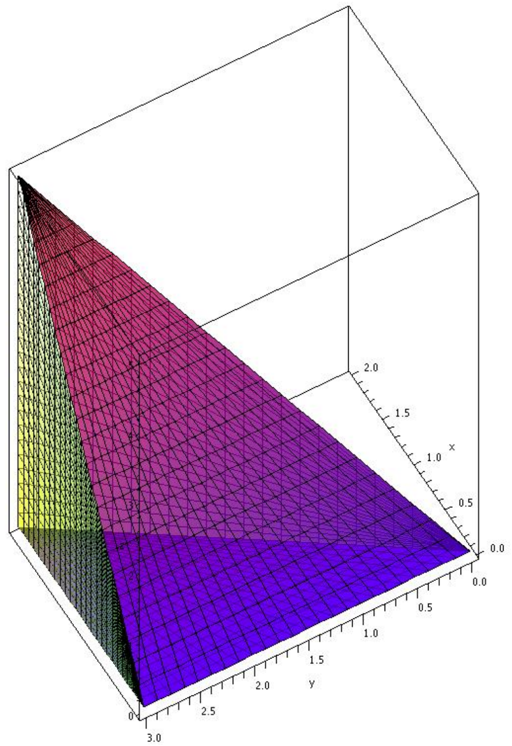


Figure 15.5.1 A tetrahedron. (AP)

As usual, the mass is the integral of density over the region:

$$M = \int_0^1 \int_x^1 \int_0^{y-x} xz\,dz\,dy\,dx = \int_0^1 \int_x^1 \frac{x(y-x)^2}{2}\,dy\,dx = \frac{1}{2}\int_0^1 \frac{x(1-x)^3}{3}\,dx$$
$$= \frac{1}{6}\int_0^1 x - 3x^2 + 3x^3 - x^4\,dx = \frac{1}{120}.$$

We compute moments as before, except now there is a third moment:

$$M_{xy} = \int_0^1 \int_x^1 \int_0^{y-x} xz^2\,dz\,dy\,dx = \frac{1}{360},$$
$$M_{xz} = \int_0^1 \int_x^1 \int_0^{y-x} xyz\,dz\,dy\,dx = \frac{1}{144},$$
$$M_{yz} = \int_0^1 \int_x^1 \int_0^{y-x} x^2z\,dz\,dy\,dx = \frac{1}{360}.$$

Finally, the coordinates of the center of mass are $\bar{x} = M_{yz}/M = 1/3$, $\bar{y} = M_{xz}/M = 5/6$, and $\bar{z} = M_{xy}/M = 1/3$. □

Exercises 15.5.

1. Evaluate $\displaystyle\int_0^1\int_0^x\int_0^{x+y} 2x+y-1\,dz\,dy\,dx$. ⇒

2. Evaluate $\displaystyle\int_0^2\int_{-1}^{x^2}\int_1^{y} xyz\,dz\,dy\,dx$. ⇒

3. Evaluate $\displaystyle\int_0^1\int_0^x\int_0^{\ln y} e^{x+y+z}\,dz\,dy\,dx$. ⇒

4. Evaluate $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta}\int_0^{r\cos\theta} r^2\,dz\,dr\,d\theta$. ⇒

5. Evaluate $\displaystyle\int_0^{\pi}\int_0^{\sin\theta}\int_0^{r\sin\theta} r\cos^2\theta\,dz\,dr\,d\theta$. ⇒

6. Evaluate $\displaystyle\int_0^1\int_0^{y^2}\int_0^{x+y} x\,dz\,dx\,dy$. ⇒

7. Evaluate $\displaystyle\int_1^2\int_y^{y^2}\int_0^{\ln(y+z)} e^x\,dx\,dz\,dy$. ⇒

8. Compute $\displaystyle\int_0^{\pi}\int_0^{\pi/2}\int_0^1 z\sin x+z\cos y\,dz\,dy\,dx$. ⇒

9. For each of the integrals in the previous exercises, give a description of the volume (both algebraic and geometric) that is the domain of integration.

10. Compute $\displaystyle\int\int\int x+y+z\,dV$ over the region $x^2+y^2+z^2\le 1$ in the first octant. ⇒

11. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one corner. ⇒

12. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one edge. ⇒

13. An object occupies the volume of the upper hemisphere of $x^2+y^2+z^2=4$ and has density z at (x,y,z). Find the center of mass. ⇒

14. An object occupies the volume of the pyramid with corners at $(1,1,0)$, $(1,-1,0)$, $(-1,-1,0)$, $(-1,1,0)$, and $(0,0,2)$ and has density x^2+y^2 at (x,y,z). Find the center of mass. ⇒

15. Verify the moments M_{xy}, M_{xz}, and M_{yz} of example 15.5.4 by evaluating the integrals.

16. Find the region E for which $\displaystyle\iiint_E (1-x^2-y^2-z^2)\,dV$ is a maximum.

15.6 Cylindrical and Spherical Coordinates

We have seen that sometimes double integrals are simplified by doing them in polar coordinates; not surprisingly, triple integrals are sometimes simpler in cylindrical coordinates or spherical coordinates. To set up integrals in polar coordinates, we had to understand the shape and area of a typical small region into which the region of integration was divided. We need to do the same thing here, for three dimensional regions.

The cylindrical coordinate system is the simplest, since it is just the polar coordinate system plus a z coordinate. A typical small unit of volume is the shape shown in figure 15.2.1 "fattened up" in the z direction, so its volume is $r\Delta r\Delta\theta\Delta z$, or in the limit, $r\,dr\,d\theta\,dz$.

EXAMPLE 15.6.1 Find the volume under $z=\sqrt{4-r^2}$ above the quarter circle inside $x^2+y^2=4$ in the first quadrant.

We could of course do this with a double integral, but we'll use a triple integral:

$$\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta=\int_0^{\pi/2}\int_0^2\sqrt{4-r^2}\,r\,dr\,d\theta=\frac{4\pi}{3}.$$

Compare this to example 15.2.1. □

EXAMPLE 15.6.2 An object occupies the space inside both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$, and has density x^2 at (x,y,z). Find the total mass.

We set this up in cylindrical coordinates, recalling that $x=r\cos\theta$:

$$\begin{aligned}\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3\cos^2(\theta)\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^1 2\sqrt{4-r^2}\,r^3\cos^2(\theta)\,dr\,d\theta\\ &=\int_0^{2\pi}\left(\frac{128}{15}-\frac{22}{5}\sqrt{3}\right)\cos^2(\theta)\,d\theta\\ &=\left(\frac{128}{15}-\frac{22}{5}\sqrt{3}\right)\pi\end{aligned}$$

□

Spherical coordinates are somewhat more difficult to understand. The small volume we want will be defined by $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$, as pictured in figure 15.6.1. To gain a better understanding, see the Java applet. The small volume is nearly box shaped, with 4 flat sides and two sides formed from bits of concentric spheres. When $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$ are all very small, the volume of this little region will be nearly the volume we get by treating it as a box. One dimension of the box is simply $\Delta\rho$, the change in distance from the origin. The other two dimensions are the lengths of small circular arcs, so they are $r\Delta\alpha$ for some suitable r and α, just as in the polar coordinates case.

The easiest of these to understand is the arc corresponding to a change in ϕ, which is nearly identical to the derivation for polar coordinates, as shown in the left graph in figure 15.6.2. In that graph we are looking "face on" at the side of the box we are interested in, so the small angle pictured is precisely $\Delta\phi$, the vertical axis really is the z axis, but the horizontal axis is *not* a real axis—it is just some line in the x-y plane. Because the

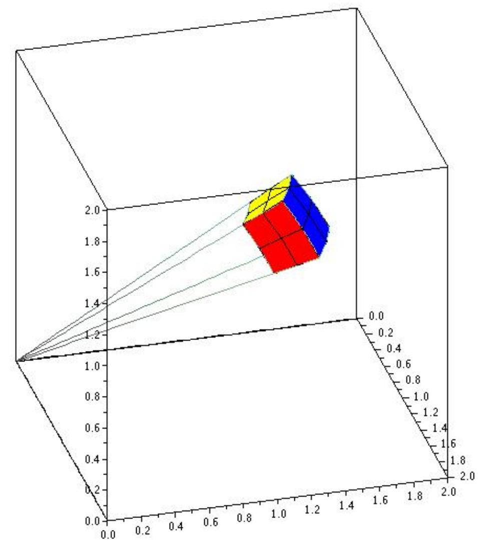


Figure 15.6.1 A small unit of volume for spherical coordinates. (AP)

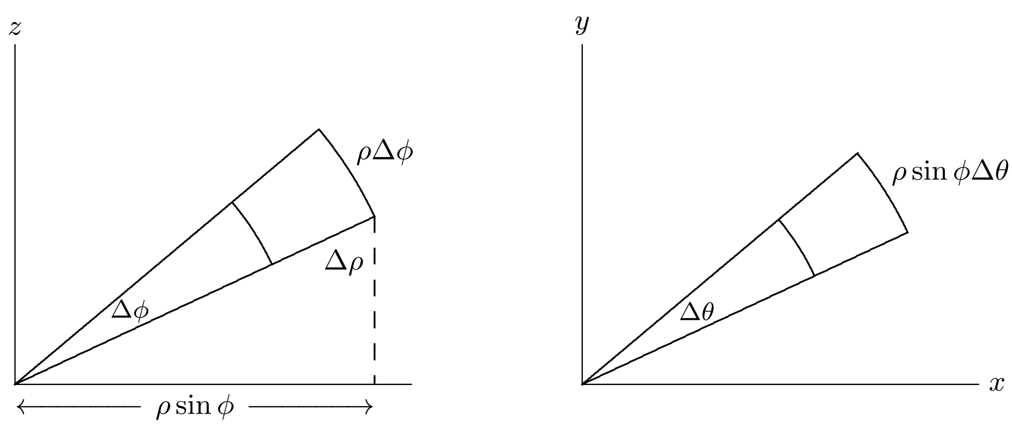


Figure 15.6.2 Setting up integration in spherical coordinates.

other arc is governed by θ, we need to imagine looking straight down the z axis, so that the apparent angle we see is $\Delta\theta$. In this view, the axes really are the x and y axes. In this graph, the apparent distance from the origin is not ρ but $\rho\sin\phi$, as indicated in the left graph.

The upshot is that the volume of the little box is approximately $\Delta\rho(\rho\Delta\phi)(\rho\sin\phi\Delta\theta) = \rho^2\sin\phi\Delta\rho\Delta\phi\Delta\theta$, or in the limit $\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

EXAMPLE 15.6.3 Suppose the temperature at (x, y, z) is $T = 1/(1 + x^2 + y^2 + z^2)$. Find the average temperature in the unit sphere centered at the origin.

In two dimensions we add up the temperature at "each" point and divide by the area; here we add up the temperatures and divide by the volume, $(4/3)\pi$:

$$\frac{3}{4\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}\frac{1}{1+x^2+y^2+z^2}\,dz\,dy\,dx$$

This looks quite messy; since everything in the problem is closely related to a sphere, we'll convert to spherical coordinates.

$$\frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^1\frac{1}{1+\rho^2}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{3}{4\pi}(4\pi-\pi^2)=3-\frac{3\pi}{4}.$$

□

Exercises 15.6.

1. Evaluate $\displaystyle\int_0^1\int_0^x\int_0^{\sqrt{x^2+y^2}}\frac{(x^2+y^2)^{3/2}}{x^2+y^2+z^2}\,dz\,dy\,dx$. ⇒

2. Evaluate $\displaystyle\int_{-1}^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$. ⇒

3. Evaluate $\displaystyle\iiint x^2\,dV$ over the interior of the cylinder $x^2+y^2=1$ between $z=0$ and $z=5$. ⇒

4. Evaluate $\displaystyle\iiint xy\,dV$ over the interior of the cylinder $x^2+y^2=1$ between $z=0$ and $z=5$. ⇒

5. Evaluate $\displaystyle\iiint z\,dV$ over the region above the x-y plane, inside $x^2+y^2-2x=0$ and under $x^2+y^2+z^2=4$. ⇒

6. Evaluate $\displaystyle\iiint yz\,dV$ over the region in the first octant, inside $x^2+y^2-2x=0$ and under $x^2+y^2+z^2=4$. ⇒

7. Evaluate $\displaystyle\iiint x^2+y^2\,dV$ over the interior of $x^2+y^2+z^2=4$. ⇒

8. Evaluate $\displaystyle\iiint \sqrt{x^2+y^2}\,dV$ over the interior of $x^2+y^2+z^2=4$. ⇒

9. Compute $\displaystyle\iiint x+y+z\,dV$ over the region inside $x^2+y^2+z^2=1$ in the first octant. ⇒

10. Find the mass of a right circular cone of height h and base radius a if the density is proportional to the distance from the base. ⇒

11. Find the mass of a right circular cone of height h and base radius a if the density is proportional to the distance from its axis of symmetry. ⇒

12. An object occupies the region inside the unit sphere at the origin, and has density equal to the distance from the x-axis. Find the mass. ⇒

13. An object occupies the region inside the unit sphere at the origin, and has density equal to the square of the distance from the origin. Find the mass. ⇒

14. An object occupies the region between the unit sphere at the origin and a sphere of radius 2 with center at the origin, and has density equal to the distance from the origin. Find the mass. ⇒

15. An object occupies the region in the first octant bounded by the cones $\phi = \pi/4$ and $\phi = \arctan 2$, and the sphere $\rho = \sqrt{6}$, and has density proportional to the distance from the origin. Find the mass. ⇒

15.7 Change of Variables

One of the most useful techniques for evaluating integrals is substitution, both "u-substitution" and trigonometric substitution, in which we change the variable to something more convenient. As we have seen, sometimes changing from rectangular coordinates to another coordinate system is helpful, and this too changes the variables. This is certainly a more complicated change, since instead of changing one variable for another we change an entire suite of variables, but as it turns out it is really very similar to the kinds of change of variables we already know as substitution.

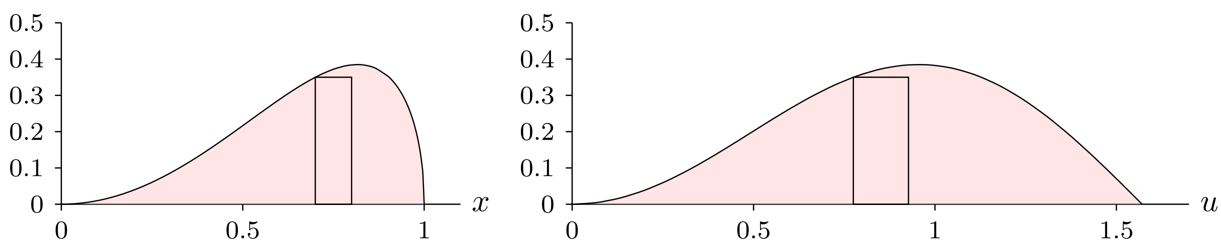


Figure 15.7.1 Single change of variable.

Let's examine the single variable case again, from a slightly different perspective than we have previously used. Suppose we start with the problem

$$\int_0^1 x^2\sqrt{1-x^2}\,dx;$$

this computes the area in the left graph of figure 15.7.1. We use the substitution $x = \sin u$ to transform the function from $x^2\sqrt{1-x^2}$ to $\sin^2 u\sqrt{1-\sin^2 u}$, and we also convert dx to $\cos u\,du$. Finally, we convert the limits 0 and 1 to 0 and $\pi/2$. This transforms the integral:

$$\int_0^1 x^2\sqrt{1-x^2}\,dx = \int_0^{\pi/2} \sin^2 u\sqrt{1-\sin^2 u}\cos u\,du.$$

We want to notice that there are three different conversions: the main function, the differential dx, and the interval of integration. The function is converted to $\sin^2 u\sqrt{1-\sin^2 u}$, shown in the right-hand graph of figure 15.7.1. It is evident that the two curves pictured there have the same y-values in the same order, but the horizontal scale has been changed. Even though the heights are the same, the two integrals

$$\int_0^1 x^2\sqrt{1-x^2}\,dx \qquad \text{and} \qquad \int_0^{\pi/2} \sin^2 u\sqrt{1-\sin^2 u}\,du$$

are not the same; clearly the right hand area is larger. One way to understand the problem is to note that if both areas are approximated using, say, ten subintervals, that the approximating rectangles on the right are wider than their counterparts on the left, as indicated. In the picture, the width of the rectangle on the left is $\Delta x = 0.1$, between 0.7 and 0.8. The rectangle on the right is situated between the corresponding values $\arcsin(0.7)$ and $\arcsin(0.8)$ so that $\Delta u = \arcsin(0.8) - \arcsin(0.7)$. To make the widths match, and the areas therefore the same, we can multiply Δu by a correction factor; in this case the correction factor is approximately $\cos u = \cos(\arcsin(0.7))$, which we compute when we convert dx to $\cos u\,du$.

Now let's move to functions of two variables. Suppose we want to convert an integral

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} f(x,y)\,dy\,dx$$

to use new variables u and v. In the single variable case, there's typically just one reason to want to change the variable: to make the function "nicer" so that we can find an antiderivative. In the two variable case, there is a second potential reason: the two-dimensional region over which we need to integrate is somehow unpleasant, and we want the region in terms of u and v to be nicer—to be a rectangle, for example. Ideally, of course, the new function and the new region will be no worse than the originals, and at least one of them will be better; this doesn't always pan out.

As before, there are three parts to the conversion: the function itself must be rewritten in terms of u and v, $dy\,dx$ must be converted to $du\,dv$, and the old region must be converted to the new region. We will develop the necessary techniques by considering a particular example, and we will use an example we already know how to do by other means.

Consider

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dy\,dx.$$

The limits correspond to integrating over the top half of a circular disk, and we recognize that the function will simplify in polar coordinates, so we would normally convert to polar coordinates:

$$\int_0^{\pi} \int_0^1 \sqrt{r^2}\, r\, dr\, d\theta = \frac{\pi}{3}.$$

But let's instead approach this as a substitution problem, starting with $x = r\cos\theta$, $y = r\sin\theta$. This pair of equations describes a function from "r-θ space" to "x-y space", and because it involves familiar concepts, it is not too hard to understand what it does. In figure 15.7.2 we have indicated geometrically a bit about how this function behaves. The four dots labeled a–d in the r-θ plane correspond to the three dots in the x-y plane; dots a and b both go to the origin because $r = 0$. The horizontal arrow in the r-θ plane has $r = 1$ everywhere and θ ranges from 0 to π, so the corresponding points $x = r\cos\theta$, $y = r\sin\theta$ start at $(1, 0)$ and follow the unit circle counter-clockwise. Finally, the vertical arrow has $\theta = \pi/4$ and r ranges from 0 to 1, so it maps to the straight arrow in the x-y plane. Extrapolating from these few examples, it's not hard to see that every vertical line in the r-θ plane is transformed to a line through the origin in the x-y plane, and every horizontal line in the r-θ plane is transformed to a circle with center at the origin in the x-y plane. Since we are interested in integrating over the half-disk in the x-y plane, we will integrate over the rectangle $[0, \pi] \times [0, 1]$ in the r-θ plane, because we now see that the points in this rectangle are sent precisely to the upper half disk by $x = r\cos\theta$ and $y = r\sin\theta$.

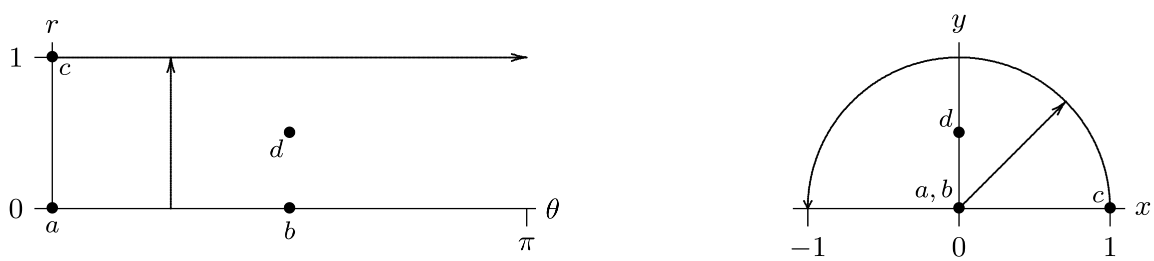


Figure 15.7.2 Double change of variable.

At this point we are two-thirds done with the task: we know the r-θ limits of integration, and we can easily convert the function to the new variables:

$$\sqrt{x^2 + y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r\sqrt{\cos^2\theta + \sin^2\theta} = r. \qquad (15.7.1)$$

The final, and most difficult, task is to figure out what replaces $dx\,dy$. (Of course, we actually know the answer, because we are in effect converting to polar coordinates. What we really want is a series of steps that gets to that right answer but that will also work for other substitutions that are not so familiar.)

Let's take a step back and remember how integration arises from approximation. When we approximate the integral in the x-y plane, we are computing the volumes of tall thin boxes, in this case boxes that are $\Delta x \times \Delta y \times \sqrt{x^2+y^2}$. We are aiming to come up with an integral in the r-θ plane that looks like this:

$$\int_0^\pi \int_0^1 r(?)\,dr\,d\theta. \qquad (15.7.2)$$

What we're missing is exactly the right quantity to replace the "?" so that we get the correct answer. Of course, this integral is also the result of an approximation, in which we add up volumes of boxes that are $\Delta r \times \Delta\theta \times$ height; the problem is that the height that will give us the correct answer is not simply r. Or put another way, we can think of the correct height as r, but the area of the base $\Delta r \Delta\theta$ as being wrong. The height r comes from equation 15.7.1, which is to say, it is precisely the same as the corresponding height in the x-y version of the integral. The problem is that the area of the base $\Delta x \times \Delta y$ is not the same as the area of the base $\Delta r \times \Delta\theta$. We can think of the "?" in the integral as a correction factor that is needed so that $?\,dr\,d\theta = dx\,dy$.

So let's think about what that little base $\Delta r \times \Delta\theta$ corresponds to. We know that each bit of horizontal line in the r-θ plane corresponds to a bit of circular arc in the x-y plane, and each bit of vertical line in the r-θ plane corresponds to a bit of "radial line" in the x-y plane. In figure 15.7.3 we show a typical rectangle in the r-θ plane and its corresponding area in the x-y plane.

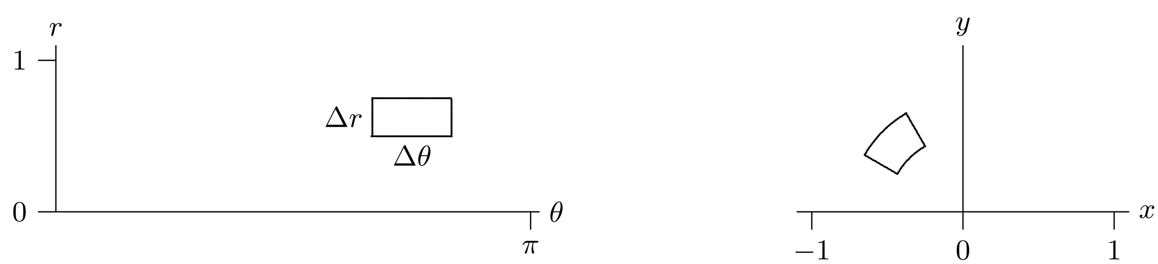


Figure 15.7.3 Corresponding areas.

In this case, the region in the x-y plane is approximately a rectangle with dimensions $\Delta r \times r\Delta\theta$, but in general the corner angles will not be right angles, so the region will typically be (almost) a parallelogram. We need to compute the area of this parallelogram. We know a neat way to do this: compute the length of a certain cross product (page 315). If we can determine an appropriate two vectors we'll be nearly done.

Fortunately, we've really done this before. The sides of the region in the x-y plane are formed by temporarily fixing either r or θ and letting the other variable range over a small interval. In figure 15.7.3, for example, the upper right edge of the region is formed by fixing $\theta = 2\pi/3$ and letting r run from 0.5 to 0.75. In other words, we have a vector function $\mathbf{v}(r) = \langle r\cos\theta_0, r\sin\theta_0, 0\rangle$, and we are interested in a restricted set of values for r. A vector tangent to this path is given by the derivative $\mathbf{v}'(r) = \langle \cos\theta_0, \sin\theta_0, 0\rangle$, and a small tangent vector, with length approximately equal to the side of the region, is $\langle \cos\theta_0, \sin\theta_0, 0\rangle\, dr$. Likewise, if we fix $r = r_0 = 0.5$, we get the vector function $\mathbf{w}(\theta) = \langle r_0\cos\theta, r_0\sin\theta, 0\rangle$ with derivative $\mathbf{w}'(\theta) = \langle -r_0\sin\theta, r_0\cos\theta, 0\rangle$ and a small tangent vector $\langle -r_0\sin\theta_0, r_0\cos\theta_0, 0\rangle\, d\theta$ when $\theta = \theta_0$ (at the corner we're focusing on). These vectors are shown in figure 15.7.4, with the actual region outlined by a dotted boundary. Of course, since both Δr and $\Delta\theta$ are quite large, the parallelogram is not a particularly good approximation to the true area.

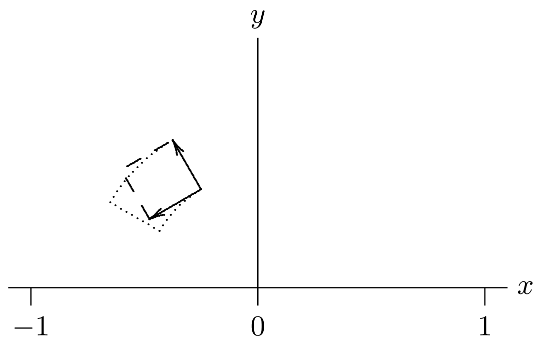


Figure 15.7.4 The approximating parallelogram.

The area of this parallelogram is the length of the cross product:

$$\begin{aligned}\langle -r_0\sin\theta_0, r_0\cos\theta_0, 0\rangle\, d\theta \times \langle \cos\theta_0, \sin\theta_0, 0\rangle\, dr &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r_0\sin\theta_0 & r_0\cos\theta_0 & 0 \\ \cos\theta_0 & \sin\theta_0 & 0 \end{vmatrix} d\theta\, dr \\ &= \langle 0, 0, -r_0\sin^2\theta_0 - r_0\cos^2\theta_0\rangle\, d\theta\, dr \\ &= \langle 0, 0, -r_0\rangle\, d\theta\, dr.\end{aligned}$$

The length of this vector is $r_0\, dr\, d\theta$. So in general, for any values of r and θ, the area in the x-y plane corresponding to a small rectangle anchored at (θ, r) in the r-θ plane is approximately $r\, dr\, d\theta$. In other words, "r" replaces the "?" in equation 15.7.2.

In general, a substitution will start with equations $x = f(u, v)$ and $y = g(u, v)$. Again, it will be straightforward to convert the function being integrated. Converting the limits will require, as above, an understanding of just how the functions f and g transform the u-v plane into the x-y plane. Finally, the small vectors we need to approximate an area will be $\langle f_u, g_u, 0\rangle\, du$ and $\langle f_v, g_v, 0\rangle\, dv$. The cross product of these is $\langle 0, 0, f_u g_v - g_u f_v\rangle\, du\, dv$ with length $|f_u g_v - g_u f_v|\, du\, dv$. The quantity $|f_u g_v - g_u f_v|$ is usually denoted

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = |f_u g_v - g_u f_v|$$

and called the **Jacobian**. Note that this is the absolute value of the two by two determinant

$$\begin{vmatrix} f_u & g_u \\ f_v & g_v \end{vmatrix},$$

which may be easier to remember. (Confusingly, the matrix, the determinant of the matrix, and the absolute value of the determinant are all called the Jacobian by various authors.)

Because there are two things to worry about, namely, the form of the function and the region of integration, transformations in two (or more) variables are quite tricky to discover.

EXAMPLE 15.7.1 Integrate $x^2 - xy + y^2$ over the region $x^2 - xy + y^2 \le 2$.

The equation $x^2 - xy + y^2 = 2$ describes an ellipse as in figure 15.7.5; the region of integration is the interior of the ellipse. We will use the transformation $x = \sqrt{2}u - \sqrt{2/3}v$, $y = \sqrt{2}u + \sqrt{2/3}v$. Substituting into the function itself we get

$$x^2 - xy + y^2 = 2u^2 + 2v^2.$$

The boundary of the ellipse is $x^2 - xy + y^2 = 2$, so the boundary of the corresponding region in the u-v plane is $2u^2 + 2v^2 = 2$ or $u^2 + v^2 = 1$, the unit circle, so this substitution makes the region of integration simpler.

Next, we compute the Jacobian, using $f = \sqrt{2}u - \sqrt{2/3}v$ and $g = \sqrt{2}u + \sqrt{2/3}v$:

$$f_u g_v - g_u f_v = \sqrt{2}\sqrt{2/3} + \sqrt{2}\sqrt{2/3} = \frac{4}{\sqrt{3}}.$$

Hence the new integral is

$$\iint\limits_R (2u^2 + 2v^2)\frac{4}{\sqrt{3}}\,du\,dv,$$

where R is the interior of the unit circle. This is still not an easy integral, but it is easily transformed to polar coordinates, and then easily integrated. □

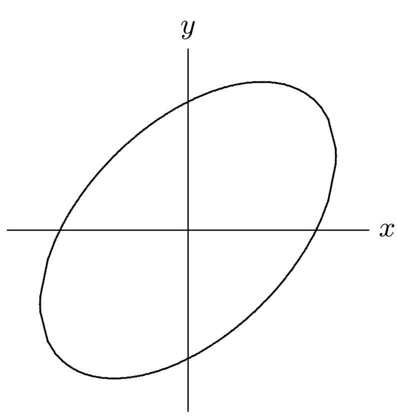


Figure 15.7.5 $x^2 - xy + y^2 = 2$

There is a similar change of variables formula for triple integrals, though it is a bit more difficult to derive. Suppose we use three substitution functions, $x = f(u, v, w)$, $y = g(u, v, w)$, and $z = h(u, v, w)$. The Jacobian determinant is now

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} f_u & g_u & h_u \\ f_v & g_v & h_v \\ f_w & g_w & h_w \end{vmatrix}.$$

Then the integral is transformed in a similar fashion:

$$\iiint_R F(x,y,z)\,dV = \iiint_S F(f(u,v,w), g(u,v,w), h(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw,$$

where of course the region S in uvw space corresponds to the region R in xyz space.

Exercises 15.7.

1. Complete example 15.7.1 by converting to polar coordinates and evaluating the integral. ⇒

2. Evaluate $\iint xy\,dx\,dy$ over the square with corners $(0,0)$, $(1,1)$, $(2,0)$, and $(1,-1)$ in two ways: directly, and using $x = (u+v)/2$, $y = (u-v)/2$. ⇒

3. Evaluate $\iint x^2 + y^2\,dx\,dy$ over the square with corners $(-1,0)$, $(0,1)$, $(1,0)$, and $(0,-1)$ in two ways: directly, and using $x = (u+v)/2$, $y = (u-v)/2$. ⇒

4. Evaluate $\iint (x+y)e^{x-y}\,dx\,dy$ over the triangle with corners $(0,0)$, $(-1,1)$, and $(1,1)$ in two ways: directly, and using $x = (u+v)/2$, $y = (u-v)/2$. ⇒

5. Evaluate $\iint y(x-y)\,dx\,dy$ over the parallelogram with corners $(0,0)$, $(3,3)$, $(7,3)$, and $(4,0)$ in two ways: directly, and using $x = u+v$, $y = u$. ⇒

6. Evaluate $\iint \sqrt{x^2+y^2}\,dx\,dy$ over the triangle with corners $(0,0)$, $(4,4)$, and $(4,0)$ using $x = u$, $y = uv$. ⇒

7. Evaluate $\iint y\sin(xy)\,dx\,dy$ over the region bounded by $xy = 1$, $xy = 4$, $y = 1$, and $y = 4$ using $x = u/v$, $y = v$. ⇒

8. Evaluate $\iint \sin(9x^2 + 4y^2)\,dA$, over the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$. ⇒

9. Compute the Jacobian for the substitutions $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.

10. Evaluate $\iiint_E dV$ where E is the solid enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

using the transformation $x = au$, $y = bv$, and $z = cw$. ⇒

16

Vector Calculus

16.1 Vector Fields

This chapter is concerned with applying calculus in the context of **vector fields**. A two-dimensional vector field is a function f that maps each point (x, y) in $\mathbb{R}^2$ to a two-dimensional vector $\langle u, v\rangle$, and similarly a three-dimensional vector field maps (x, y, z) to $\langle u, v, w\rangle$. Since a vector has no position, we typically indicate a vector field in graphical form by placing the vector $f(x, y)$ with its tail at (x, y). Figure 16.1.1 shows a representation of the vector field $f(x, y) = \langle -x/\sqrt{x^2 + y^2 + 4}, y/\sqrt{x^2 + y^2 + 4}\rangle$. For such a graph to be readable, the vectors must be fairly short, which is accomplished by using a different scale for the vectors than for the axes. Such graphs are thus useful for understanding the sizes of the vectors relative to each other but not their absolute size.

Vector fields have many important applications, as they can be used to represent many physical quantities: the vector at a point may represent the strength of some force (gravity, electricity, magnetism) or a velocity (wind speed or the velocity of some other fluid).

We have already seen a particularly important kind of vector field—the gradient. Given a function $f(x, y)$, recall that the gradient is $\langle f_x(x, y), f_y(x, y)\rangle$, a vector that depends on (is a function of) x and y. We usually picture the gradient vector with its tail at (x, y), pointing in the direction of maximum increase. Vector fields that are gradients have some particularly nice properties, as we will see. An important example is

$$\mathbf{F} = \left\langle \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-z}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle,$$

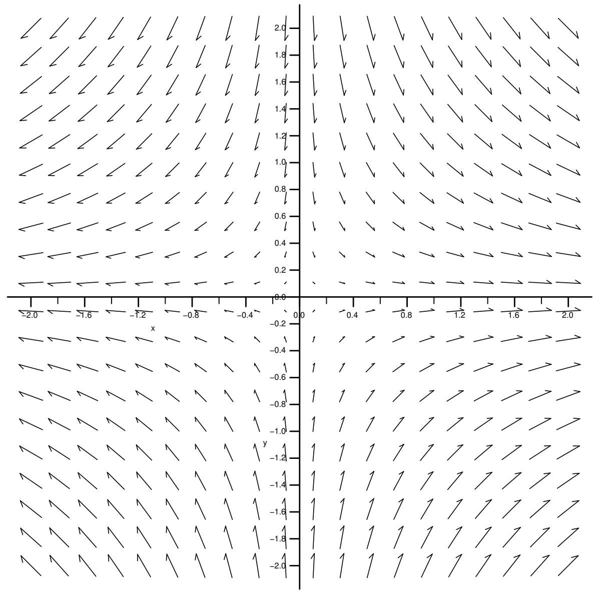


Figure 16.1.1 A vector field.

which points from the point (x, y, z) toward the origin and has length

$$\frac{\sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)^{3/2}} = \frac{1}{(\sqrt{x^2+y^2+z^2})^2},$$

which is the reciprocal of the square of the distance from (x, y, z) to the origin—in other words, $\mathbf{F}$ is an "inverse square law". The vector $\mathbf{F}$ is a gradient:

$$\mathbf{F} = \nabla \frac{1}{\sqrt{x^2+y^2+z^2}}, \tag{16.1.1}$$

which turns out to be extremely useful.

Exercises 16.1.

Sketch the vector fields; check your work with Sage's `plot_vector_field` function.

1. $\langle x, y\rangle$

2. $\langle -x, -y\rangle$

3. $\langle x, -y\rangle$

4. $\langle \sin x, \cos y\rangle$

5. $\langle y, 1/x\rangle$

6. $\langle x+1, x+3\rangle$

7. Verify equation 16.1.1.

16.2 Line Integrals

We have so far integrated "over" intervals, areas, and volumes with single, double, and triple integrals. We now investigate integration over or "along" a curve—"line integrals" are really "curve integrals".

As with other integrals, a geometric example may be easiest to understand. Consider the function $f = x + y$ and the parabola $y = x^2$ in the x-y plane, for $0 \le x \le 2$. Imagine that we extend the parabola up to the surface f, to form a curved wall or curtain, as in figure 16.2.1. What is the area of the surface thus formed? We already know one way to compute surface area, but here we take a different approach that is more useful for the problems to come.

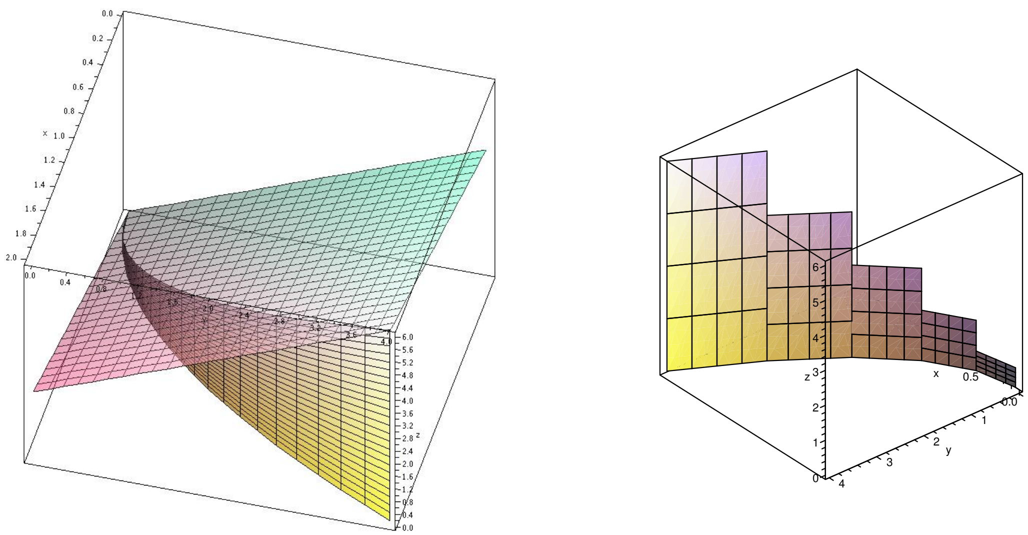


Figure 16.2.1 Approximating the area under a curve. (AP)

As usual, we start by thinking about how to approximate the area. We pick some points along the part of the parabola we're interested in, and connect adjacent points by straight lines; when the points are close together, the length of each line segment will be close to the length along the parabola. Using each line segment as the base of a rectangle, we choose the height to be the height of the surface f above the line segment. If we add up the areas of these rectangles, we get an approximation to the desired area, and in the limit this sum turns into an integral.

Typically the curve is in vector form, or can easily be put in vector form; in this example we have $\mathbf{v}(t) = \langle t, t^2 \rangle$. Then as we have seen in section 13.3 on arc length, the length of one of the straight line segments in the approximation is approximately

$ds = |\mathbf{v}'|\,dt = \sqrt{1+4t^2}\,dt$, so the integral is

$$\int_0^2 f(t,t^2)\sqrt{1+4t^2}\,dt = \int_0^2 (t+t^2)\sqrt{1+4t^2}\,dt = \frac{167}{48}\sqrt{17} - \frac{1}{12} - \frac{1}{64}\ln(4+\sqrt{17}).$$

This integral of a function along a curve C is often written in abbreviated form as

$$\int_C f(x,y)\,ds.$$

EXAMPLE 16.2.1 Compute $\int_C ye^x\,ds$ where C is the line segment from $(1,2)$ to $(4,7)$.

We write the line segment as a vector function: $\mathbf{v} = \langle 1,2\rangle + t\langle 3,5\rangle$, $0 \le t \le 1$, or in parametric form $x = 1+3t$, $y = 2+5t$. Then

$$\int_C ye^x\,ds = \int_0^1 (2+5t)e^{1+3t}\sqrt{3^2+5^2}\,dt = \frac{16}{9}\sqrt{34}e^4 - \frac{1}{9}\sqrt{34}\,e.$$

□

All of these ideas extend to three dimensions in the obvious way.

EXAMPLE 16.2.2 Compute $\int_C x^2z\,ds$ where C is the line segment from $(0,6,-1)$ to $(4,1,5)$.

We write the line segment as a vector function: $\mathbf{v} = \langle 0,6,-1\rangle + t\langle 4,-5,6\rangle$, $0 \le t \le 1$, or in parametric form $x = 4t$, $y = 6-5t$, $z = -1+6t$. Then

$$\int_C x^2z\,ds = \int_0^1 (4t)^2(-1+6t)\sqrt{16+25+36}\,dt = 16\sqrt{77}\int_0^1 -t^2+6t^3\,dt = \frac{56}{3}\sqrt{77}.$$

□

Now we turn to a perhaps more interesting example. Recall that in the simplest case, the work done by a force on an object is equal to the magnitude of the force times the distance the object moves; this assumes that the force is constant and in the direction of motion. We have already dealt with examples in which the force is not constant; now we are prepared to examine what happens when the force is not parallel to the direction of motion.

We have already examined the idea of components of force, in example 12.3.4: the component of a force $\mathbf{F}$ in the direction of a vector $\mathbf{v}$ is

$$\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v},$$

the projection of $\mathbf{F}$ onto $\mathbf{v}$. The length of this vector, that is, the magnitude of the force in the direction of $\mathbf{v}$, is

$$\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|},$$

the scalar projection of $\mathbf{F}$ onto $\mathbf{v}$. If an object moves subject to this (constant) force, in the direction of $\mathbf{v}$, over a distance equal to the length of $\mathbf{v}$, the work done is

$$\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|}|\mathbf{v}| = \mathbf{F}\cdot\mathbf{v}.$$

Thus, work in the vector setting is still "force times distance", except that "times" means "dot product".

If the force varies from point to point, it is represented by a vector field $\mathbf{F}$; the displacement vector $\mathbf{v}$ may also change, as an object may follow a curving path in two or three dimensions. Suppose that the path of an object is given by a vector function $\mathbf{r}(t)$; at any point along the path, the (small) tangent vector $\mathbf{r}'\,\Delta t$ gives an approximation to its motion over a short time Δt, so the work done during that time is approximately $\mathbf{F}\cdot\mathbf{r}'\,\Delta t$; the total work over some time period is then

$$\int_{t_0}^{t_1} \mathbf{F}\cdot\mathbf{r}'\,dt.$$

It is useful to rewrite this in various ways at different times. We start with

$$\int_{t_0}^{t_1} \mathbf{F}\cdot\mathbf{r}'\,dt = \int_C \mathbf{F}\cdot\,d\mathbf{r},$$

abbreviating $\mathbf{r}'\,dt$ by $d\mathbf{r}$. Or we can write

$$\int_{t_0}^{t_1} \mathbf{F}\cdot\mathbf{r}'\,dt = \int_{t_0}^{t_1} \mathbf{F}\cdot\frac{\mathbf{r}'}{|\mathbf{r}'|}|\mathbf{r}'|\,dt = \int_{t_0}^{t_1} \mathbf{F}\cdot\mathbf{T}\,|\mathbf{r}'|\,dt = \int_C \mathbf{F}\cdot\mathbf{T}\,ds,$$

using the unit tangent vector $\mathbf{T}$, abbreviating $|\mathbf{r}'|\,dt$ as ds, and indicating the path of the object by C. In other words, work is computed using a particular line integral of the form we have considered. Alternately, we sometimes write

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{r}'\,dt = \int_C \langle f,g,h\rangle\cdot\langle x',y',z'\rangle\,dt = \int_C \left(f\frac{dx}{dt}+g\frac{dy}{dt}+h\frac{dz}{dt}\right)\,dt\\ = \int_C f\,dx+g\,dy+h\,dz = \int_C f\,dx+\int_C g\,dy+\int_C h\,dz,\end{aligned}$$

and similarly for two dimensions, leaving out references to z.

EXAMPLE 16.2.3 Suppose an object moves from $(-1,1)$ to $(2,4)$ along the path $\mathbf{r}(t)=\langle t,t^2\rangle$, subject to the force $\mathbf{F}=\langle x\sin y, y\rangle$. Find the work done.

We can write the force in terms of t as $\langle t\sin(t^2),t^2\rangle$, and compute $\mathbf{r}'(t)=\langle 1,2t\rangle$, and then the work is

$$\int_{-1}^{2}\langle t\sin(t^2),t^2\rangle\cdot\langle 1,2t\rangle\,dt = \int_{-1}^{2} t\sin(t^2)+2t^3\,dt = \frac{15}{2}+\frac{\cos(1)-\cos(4)}{2}.$$

Alternately, we might write

$$\int_C x\sin y\,dx+\int_C y\,dy = \int_{-1}^{2} x\sin(x^2)\,dx+\int_1^4 y\,dy = -\frac{\cos(4)}{2}+\frac{\cos(1)}{2}+\frac{16}{2}-\frac{1}{2}$$

getting the same answer. □

Exercises 16.2.

1. Compute $\int_C xy^2\,ds$ along the line segment from $(1,2,0)$ to $(2,1,3)$. ⇒

2. Compute $\int_C \sin x\,ds$ along the line segment from $(-1,2,1)$ to $(1,2,5)$. ⇒

3. Compute $\int_C z\cos(xy)\,ds$ along the line segment from $(1,0,1)$ to $(2,2,3)$. ⇒

4. Compute $\int_C \sin x\,dx+\cos y\,dy$ along the top half of the unit circle, from $(1,0)$ to $(-1,0)$. ⇒

5. Compute $\int_C xe^y\,dx+x^2y\,dy$ along the line segment $y=3$, $0\le x\le 2$. ⇒

6. Compute $\int_C xe^y\,dx+x^2y\,dy$ along the line segment $x=4$, $0\le y\le 4$. ⇒

7. Compute $\int_C xe^y\,dx+x^2y\,dy$ along the curve $x=3t$, $y=t^2$, $0\le t\le 1$. ⇒

8. Compute $\int_C xe^y\,dx+x^2y\,dy$ along the curve $\langle e^t,e^t\rangle$, $-1\le t\le 1$. ⇒

9. Compute $\int_C \langle\cos x,\sin y\rangle\cdot d\mathbf{r}$ along the curve $\langle t,t\rangle$, $0\le t\le 1$. ⇒

10. Compute $\int_C \langle 1/xy, 1/(x+y)\rangle \cdot d\mathbf{r}$ along the path from $(1,1)$ to $(3,1)$ to $(3,6)$ using straight line segments. ⇒

11. Compute $\int_C \langle 1/xy, 1/(x+y)\rangle \cdot d\mathbf{r}$ along the curve $\langle 2t, 5t\rangle$, $1 \le t \le 4$. ⇒

12. Compute $\int_C \langle 1/xy, 1/(x+y)\rangle \cdot d\mathbf{r}$ along the curve $\langle t, t^2\rangle$, $1 \le t \le 4$. ⇒

13. Compute $\int_C yz\,dx + xz\,dy + xy\,dz$ along the curve $\langle t, t^2, t^3\rangle$, $0 \le t \le 1$. ⇒

14. Compute $\int_C yz\,dx + xz\,dy + xy\,dz$ along the curve $\langle \cos t, \sin t, \tan t\rangle$, $0 \le t \le \pi$. ⇒

15. An object moves from $(1,1)$ to $(4,8)$ along the path $\mathbf{r}(t) = \langle t^2, t^3\rangle$, subject to the force $\mathbf{F} = \langle x^2, \sin y\rangle$. Find the work done. ⇒

16. An object moves along the line segment from $(1,1)$ to $(2,5)$, subject to the force $\mathbf{F} = \langle x/(x^2+y^2), y/(x^2+y^2)\rangle$. Find the work done. ⇒

17. An object moves along the parabola $\mathbf{r}(t) = \langle t, t^2\rangle$, $0 \le t \le 1$, subject to the force $\mathbf{F} = \langle 1/(y+1), -1/(x+1)\rangle$. Find the work done. ⇒

18. An object moves along the line segment from $(0,0,0)$ to $(3,6,10)$, subject to the force $\mathbf{F} = \langle x^2, y^2, z^2\rangle$. Find the work done. ⇒

19. An object moves along the curve $\mathbf{r}(t) = \langle \sqrt{t}, 1/\sqrt{t}, t\rangle$ $1 \le t \le 4$, subject to the force $\mathbf{F} = \langle y, z, x\rangle$. Find the work done. ⇒

20. An object moves from $(1,1,1)$ to $(2,4,8)$ along the path $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$, subject to the force $\mathbf{F} = \langle \sin x, \sin y, \sin z\rangle$. Find the work done. ⇒

21. An object moves from $(1,0,0)$ to $(-1,0,\pi)$ along the path $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, subject to the force $\mathbf{F} = \langle y^2, y^2, xz\rangle$. Find the work done. ⇒

22. Give an example of a non-trivial force field $\mathbf{F}$ and non-trivial path $\mathbf{r}(t)$ for which the total work done moving along the path is zero.

16.3 The Fundamental Theorem of Line Integrals

One way to write the Fundamental Theorem of Calculus (7.2.1) is:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

That is, to compute the integral of a derivative f' we need only compute the values of f at the endpoints. Something similar is true for line integrals of a certain form.

THEOREM 16.3.1 Fundamental Theorem of Line Integrals Suppose a curve C is given by the vector function $\mathbf{r}(t)$, with $\mathbf{a} = \mathbf{r}(a)$ and $\mathbf{b} = \mathbf{r}(b)$. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}),$$

provided that $\mathbf{r}$ is sufficiently nice.

Proof. We write $\mathbf{r} = \langle x(t), y(t), z(t)\rangle$, so that $\mathbf{r}' = \langle x'(t), y'(t), z'(t)\rangle$. Also, we know that $\nabla f = \langle f_x, f_y, f_z\rangle$. Then

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \langle f_x, f_y, f_z\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\, dt = \int_a^b f_x x' + f_y y' + f_z z'\, dt.$$

By the chain rule (see section 14.4) $f_x x' + f_y y' + f_z z' = df/dt$, where f in this context means $f(x(t), y(t), z(t))$, a function of t. In other words, all we have is

$$\int_a^b f'(t)\, dt = f(b) - f(a).$$

In this context, $f(a) = f(x(a), y(a), z(a))$. Since $\mathbf{a} = \mathbf{r}(a) = \langle x(a), y(a), z(a)\rangle$, we can write $f(a) = f(\mathbf{a})$—this is a bit of a cheat, since we are simultaneously using f to mean $f(t)$ and $f(x, y, z)$, and since $f(x(a), y(a), z(a))$ is not technically the same as $f(\langle x(a), y(a), z(a)\rangle)$, but the concepts are clear and the different uses are compatible. Doing the same for b, we get

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b f'(t)\, dt = f(b) - f(a) = f(\mathbf{b}) - f(\mathbf{a}).$$ ∎

This theorem, like the Fundamental Theorem of Calculus, says roughly that if we integrate a "derivative-like function" (f' or ∇f) the result depends only on the values of the original function (f) at the endpoints.

If a vector field $\mathbf{F}$ is the gradient of a function, $\mathbf{F} = \nabla f$, we say that $\mathbf{F}$ is a **conservative vector field**. If $\mathbf{F}$ is a conservative force field, then the integral for work, $\int_C \mathbf{F} \cdot d\mathbf{r}$, is in the form required by the Fundamental Theorem of Line Integrals. This means that in a conservative force field, the amount of work required to move an object from point $\mathbf{a}$ to point $\mathbf{b}$ depends only on those points, not on the path taken between them.

EXAMPLE 16.3.2 An object moves in the force field

$$\mathbf{F} = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle,$$

along the curve $\mathbf{r} = \langle 1+t, t^3, t\cos(\pi t)\rangle$ as t ranges from 0 to 1. Find the work done by the force on the object.

The straightforward way to do this involves substituting the components of $\mathbf{r}$ into $\mathbf{F}$, forming the dot product $\mathbf{F} \cdot \mathbf{r}'$, and then trying to compute the integral, but this integral is extraordinarily messy, perhaps impossible to compute. But since $\mathbf{F} = \nabla(1/\sqrt{x^2+y^2+z^2})$ we need only substitute:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left. \frac{1}{\sqrt{x^2+y^2+z^2}} \right|_{(1,0,0)}^{(2,1,-1)} = \frac{1}{\sqrt{6}} - 1.$$

□

Another immediate consequence of the Fundamental Theorem involves **closed paths**. A path C is closed if it forms a loop, so that traveling over the C curve brings you back to the starting point. If C is a closed path, we can integrate around it starting at any point $\mathbf{a}$; since the starting and ending points are the same,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{a}) - f(\mathbf{a}) = 0.$$

For example, in a gravitational field (an inverse square law field) the amount of work required to move an object around a closed path is zero. Of course, it's only the *net* amount of work that is zero. It may well take a great deal of work to get from point $\mathbf{a}$ to point $\mathbf{b}$, but then the return trip will "produce" work. For example, it takes work to pump water from a lower to a higher elevation, but if you then let gravity pull the water back down, you can recover work by running a water wheel or generator. (In the real world you won't recover all the work because of various losses along the way.)

To make use of the Fundamental Theorem of Line Integrals, we need to be able to spot conservative vector fields $\mathbf{F}$ and to compute f so that $\mathbf{F} = \nabla f$. Suppose that $\mathbf{F} = \langle P, Q\rangle = \nabla f$. Then $P = f_x$ and $Q = f_y$, and provided that f is sufficiently nice, we know from Clairaut's Theorem (14.6.2) that $P_y = f_{xy} = f_{yx} = Q_x$. If we compute P_y and Q_x and find that they are not equal, then $\mathbf{F}$ is not conservative. If $P_y = Q_x$, then, again provided that $\mathbf{F}$ is sufficiently nice, we can be assured that $\mathbf{F}$ is conservative. Ultimately, what's important is that we be able to find f; as this amounts to finding anti-derivatives, we may not always succeed.

EXAMPLE 16.3.3 Find an f so that $\langle 3 + 2xy, x^2 - 3y^2\rangle = \nabla f$.

First, note that

$$\frac{\partial}{\partial y}(3 + 2xy) = 2x \qquad \text{and} \qquad \frac{\partial}{\partial x}(x^2 - 3y^2) = 2x,$$

so the desired f does exist. This means that $f_x = 3 + 2xy$, so that $f = 3x + x^2y + g(y)$; the first two terms are needed to get $3+2xy$, and the $g(y)$ could be any function of y, as it would disappear upon taking a derivative with respect to x. Likewise, since $f_y = x^2 - 3y^2$, $f = x^2y - y^3 + h(x)$. The question now becomes, is it possible to find $g(y)$ and $h(x)$ so that

$$3x + x^2y + g(y) = x^2y - y^3 + h(x),$$

and of course the answer is yes: $g(y) = -y^3$, $h(x) = 3x$. Thus, $f = 3x + x^2y - y^3$. □

We can test a vector field $\mathbf{F} = \langle P, Q, R\rangle$ in a similar way. Suppose that $\langle P, Q, R\rangle = \langle f_x, f_y, f_z\rangle$. If we temporarily hold z constant, then $f(x, y, z)$ is a function of x and y, and by Clairaut's Theorem $P_y = f_{xy} = f_{yx} = Q_x$. Likewise, holding y constant implies $P_z = f_{xz} = f_{zx} = R_x$, and with x constant we get $Q_z = f_{yz} = f_{zy} = R_y$. Conversely, if we find that $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ then $\mathbf{F}$ is conservative.

Exercises 16.3.

1. Find an f so that $\nabla f = \langle 2x + y^2, 2y + x^2\rangle$, or explain why there is no such f. ⇒
2. Find an f so that $\nabla f = \langle x^3, -y^4\rangle$, or explain why there is no such f. ⇒
3. Find an f so that $\nabla f = \langle xe^y, ye^x\rangle$, or explain why there is no such f. ⇒
4. Find an f so that $\nabla f = \langle y\cos x, y\sin x\rangle$, or explain why there is no such f. ⇒
5. Find an f so that $\nabla f = \langle y\cos x, \sin x\rangle$, or explain why there is no such f. ⇒
6. Find an f so that $\nabla f = \langle x^2y^3, xy^4\rangle$, or explain why there is no such f. ⇒
7. Find an f so that $\nabla f = \langle yz, xz, xy\rangle$, or explain why there is no such f. ⇒
8. Evaluate $\int_C (10x^4 - 2xy^3)\,dx - 3x^2y^2\,dy$ where C is the part of the curve $x^5 - 5x^2y^2 - 7x^2 = 0$ from $(0,0)$ to $(3,2)$. ⇒
9. Let $\mathbf{F} = \langle yz, xz, xy\rangle$. Find the work done by this force field on an object that moves from $(1,0,2)$ to $(1,2,3)$. ⇒
10. Let $\mathbf{F} = \langle e^y, xe^y + \sin z, y\cos z\rangle$. Find the work done by this force field on an object that moves from $(0,0,0)$ to $(1,-1,3)$. ⇒
11. Let
$$\mathbf{F} = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle.$$
Find the work done by this force field on an object that moves from $(1,1,1)$ to $(4,5,6)$. ⇒

16.4 Green's Theorem

We now come to the first of three important theorems that extend the Fundamental Theorem of Calculus to higher dimensions. (The Fundamental Theorem of Line Integrals has already done this in one way, but in that case we were still dealing with an essentially one-dimensional integral.) They all share with the Fundamental Theorem the following rather vague description: *To compute a certain sort of integral over a region, we may do a computation on the boundary of the region that involves one fewer integrations.*

Note that this does indeed describe the Fundamental Theorem of Calculus and the Fundamental Theorem of Line Integrals: to compute a single integral over an interval, we do a computation on the boundary (the endpoints) that involves one fewer integrations, namely, no integrations at all.

THEOREM 16.4.1 Green's Theorem If the vector field $\mathbf{F} = \langle P, Q\rangle$ and the region D are sufficiently nice, and if C is the boundary of D (C is a closed curve), then

$$\iint\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \int_C P\,dx + Q\,dy,$$

provided the integration on the right is done counter-clockwise around C. □

To indicate that an integral $\int_C$ is being done over a closed curve in the counter-clockwise direction, we usually write $\oint_C$. We also use the notation ∂D to mean the boundary of D **oriented** in the counterclockwise direction. With this notation, $\oint_C = \int_{\partial D}$.

We already know one case, not particularly interesting, in which this theorem is true: If $\mathbf{F}$ is conservative, we know that the integral $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$, because any integral of a conservative vector field around a closed curve is zero. We also know in this case that $\partial P/\partial y = \partial Q/\partial x$, so the double integral in the theorem is simply the integral of the zero function, namely, 0. So in the case that $\mathbf{F}$ is conservative, the theorem says simply that $0 = 0$.

EXAMPLE 16.4.2 We illustrate the theorem by computing both sides of

$$\int_{\partial D} x^4\,dx + xy\,dy = \iint\limits_D y - 0\,dA,$$

where D is the triangular region with corners $(0,0)$, $(1,0)$, $(0,1)$.

Starting with the double integral:

$$\iint\limits_D y - 0\,dA = \int_0^1\int_0^{1-x} y\,dy\,dx = \int_0^1 \frac{(1-x)^2}{2}\,dx = -\frac{(1-x)^3}{6}\bigg|_0^1 = \frac{1}{6}.$$

There is no single formula to describe the boundary of D, so to compute the left side directly we need to compute three separate integrals corresponding to the three sides of the triangle, and each of these integrals we break into two integrals, the "dx" part and the "dy" part. The three sides are described by $y = 0$, $y = 1 - x$, and $x = 0$. The integrals are then

$$\begin{aligned}\int_{\partial D} x^4\,dx + xy\,dy &= \int_0^1 x^4\,dx + \int_0^0 0\,dy + \int_1^0 x^4\,dx + \int_0^1 (1-y)y\,dy + \int_0^0 0\,dx + \int_1^0 0\,dy\\ &= \frac{1}{5} + 0 - \frac{1}{5} + \frac{1}{6} + 0 + 0 = \frac{1}{6}.\end{aligned}$$

Alternately, we could describe the three sides in vector form as $\langle t, 0\rangle$, $\langle 1-t, t\rangle$, and $\langle 0, 1-t\rangle$. Note that in each case, as t ranges from 0 to 1, we follow the corresponding side in the correct direction. Now

$$\begin{aligned}\int_{\partial D} x^4\,dx + xy\,dy &= \int_0^1 t^4 + t\cdot 0\,dt + \int_0^1 -(1-t)^4 + (1-t)t\,dt + \int_0^1 0 + 0\,dt \\ &= \int_0^1 t^4\,dt + \int_0^1 -(1-t)^4 + (1-t)t\,dt = \frac{1}{6}.\end{aligned}$$

□

In this case, none of the integrations are difficult, but the second approach is somewhat tedious because of the necessity to set up three different integrals. In different circumstances, either of the integrals, the single or the double, might be easier to compute. Sometimes it is worthwhile to turn a single integral into the corresponding double integral, sometimes exactly the opposite approach is best.

Here is a clever use of Green's Theorem: We know that areas can be computed using double integrals, namely,

$$\iint_D 1\,dA$$

computes the area of region D. If we can find P and Q so that $\partial Q/\partial x - \partial P/\partial y = 1$, then the area is also

$$\int_{\partial D} P\,dx + Q\,dy.$$

It is quite easy to do this: $P = 0, Q = x$ works, as do $P = -y, Q = 0$ and $P = -y/2, Q = x/2$.

EXAMPLE 16.4.3 An ellipse centered at the origin, with its two principal axes aligned with the x and y axes, is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We find the area of the interior of the ellipse via Green's theorem. To do this we need a vector equation for the boundary; one such equation is $\langle a\cos t, b\sin t\rangle$, as t ranges from 0 to 2π. We can easily verify this by substitution:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1.$$

Let's consider the three possibilities for P and Q above: Using 0 and x gives

$$\oint_C 0\,dx + x\,dy = \int_0^{2\pi} a\cos(t)b\cos(t)\,dt = \int_0^{2\pi} ab\cos^2(t)\,dt.$$

Using $-y$ and 0 gives

$$\oint_C -y\,dx + 0\,dy = \int_0^{2\pi} -b\sin(t)(-a\sin(t))\,dt = \int_0^{2\pi} ab\sin^2(t)\,dt.$$

Finally, using $-y/2$ and $x/2$ gives

$$\begin{aligned}\oint_C -\frac{y}{2}\,dx + \frac{x}{2}\,dy &= \int_0^{2\pi} -\frac{b\sin(t)}{2}(-a\sin(t))\,dt + \frac{a\cos(t)}{2}(b\cos(t))\,dt \\ &= \int_0^{2\pi} \frac{ab\sin^2 t}{2} + \frac{ab\cos^2 t}{2}\,dt = \int_0^{2\pi} \frac{ab}{2}\,dt = \pi ab.\end{aligned}$$

The first two integrals are not particularly difficult, but the third is very easy, though the choice of P and Q seems more complicated. □

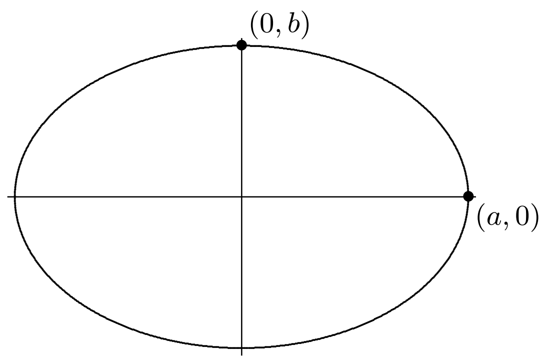


Figure 16.4.1 A "standard" ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Proof of Green's Theorem. We cannot here prove Green's Theorem in general, but we can do a special case. We seek to prove that

$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA.$$

It is sufficient to show that

$$\oint_C P\,dx = \iint_D -\frac{\partial P}{\partial y}\,dA \qquad \text{and} \qquad \oint_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA,$$

which we can do if we can compute the double integral in both possible ways, that is, using $dA = dy\,dx$ and $dA = dx\,dy$.

For the first equation, we start with

$$\iint\limits_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\,dx = \int_a^b P(x,g_2(x)) - P(x,g_1(x))\,dx.$$

Here we have simply used the ordinary Fundamental Theorem of Calculus, since for the inner integral we are integrating a derivative with respect to y: an antiderivative of $\partial P/\partial y$ with respect to y is simply $P(x,y)$, and then we substitute g_1 and g_2 for y and subtract.

Now we need to manipulate $\oint_C P\,dx$. The boundary of region D consists of 4 parts, given by the equations $y = g_1(x)$, $x = b$, $y = g_2(x)$, and $x = a$. On the portions $x = b$ and $x = a$, $dx = 0\,dt$, so the corresponding integrals are zero. For the other two portions, we use the parametric forms $x = t$, $y = g_1(t)$, $a \le t \le b$, and $x = t$, $y = g_2(t)$, letting t range from b to a, since we are integrating counter-clockwise around the boundary. The resulting integrals give us

$$\begin{aligned}\oint_C P\,dx = \int_a^b P(t,g_1(t))\,dt + \int_b^a P(t,g_2(t))\,dt &= \int_a^b P(t,g_1(t))\,dt - \int_a^b P(t,g_2(t))\,dt \\ &= \int_a^b P(t,g_1(t)) - P(t,g_2(t))\,dt\end{aligned}$$

which is the result of the double integral times -1, as desired.

The equation involving Q is essentially the same, and left as an exercise. ∎

Exercises 16.4.

1. Compute $\int_{\partial D} 2y\,dx + 3x\,dy$, where D is described by $0 \le x \le 1$, $0 \le y \le 1$. ⇒

2. Compute $\int_{\partial D} xy\,dx + xy\,dy$, where D is described by $0 \le x \le 1$, $0 \le y \le 1$. ⇒

3. Compute $\int_{\partial D} e^{2x+3y}\,dx + e^{xy}\,dy$, where D is described by $-2 \le x \le 2$, $-1 \le y \le 1$. ⇒

4. Compute $\int_{\partial D} y\cos x\,dx + y\sin x\,dy$, where D is described by $0 \le x \le \pi/2$, $1 \le y \le 2$. ⇒

5. Compute $\int_{\partial D} x^2y\,dx + xy^2\,dy$, where D is described by $0 \le x \le 1$, $0 \le y \le x$. ⇒

6. Compute $\int_{\partial D} x\sqrt{y}\,dx + \sqrt{x+y}\,dy$, where D is described by $1 \le x \le 2$, $2x \le y \le 4$. ⇒

7. Compute $\int_{\partial D} (x/y)\,dx + (2+3x)\,dy$, where D is described by $1 \le x \le 2$, $1 \le y \le x^2$. ⇒

8. Compute $\int_{\partial D} \sin y\,dx + \sin x\,dy$, where D is described by $0 \le x \le \pi/2$, $x \le y \le \pi/2$. ⇒

9. Compute $\int_{\partial D} x\ln y\,dx$, where D is described by $1 \le x \le 2$, $e^x \le y \le e^{x^2}$. ⇒

10. Compute $\int_{\partial D} \sqrt{1+x^2}\,dy$, where D is described by $-1 \le x \le 1$, $x^2 \le y \le 1$. ⇒

11. Compute $\int_{\partial D} x^2 y\,dx - xy^2\,dy$, where D is described by $x^2 + y^2 \le 1$. ⇒

12. Compute $\int_{\partial D} y^3\,dx + 2x^3\,dy$, where D is described by $x^2 + y^2 \le 4$. ⇒

13. Evaluate $\oint_C (y - \sin(x))\,dx + \cos(x)\,dy$, where C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ oriented counter-clockwise. ⇒

14. Finish our proof of Green's Theorem by showing that $\oint_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$.

16.5 Divergence and Curl

Divergence and curl are two measurements of vector fields that are very useful in a variety of applications. Both are most easily understood by thinking of the vector field as representing a flow of a liquid or gas; that is, each vector in the vector field should be interpreted as a velocity vector. Roughly speaking, divergence measures the tendency of the fluid to collect or disperse at a point, and curl measures the tendency of the fluid to swirl around the point. Divergence is a scalar, that is, a single number, while curl is itself a vector. The magnitude of the curl measures how much the fluid is swirling, the direction indicates the axis around which it tends to swirl. These ideas are somewhat subtle in practice, and are beyond the scope of this course. You can find additional information on the web, for example at http://mathinsight.org/curl_idea and http://mathinsight.org/divergence_idea and in many books including *Div, Grad, Curl, and All That: An Informal Text on Vector Calculus*, by H. M. Schey.

Recall that if f is a function, the gradient of f is given by

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

A useful mnemonic for this (and for the divergence and curl, as it turns out) is to let

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle,$$

that is, we pretend that ∇ is a vector with rather odd looking entries. Recalling that $\langle u, v, w \rangle a = \langle ua, va, wa \rangle$, we can then think of the gradient as

$$\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle,$$

that is, we simply multiply the f into the vector.

The divergence and curl can now be defined in terms of this same odd vector ∇ by using the cross product and dot product. The divergence of a vector field $\mathbf{F} = \langle f, g, h\rangle$ is

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h\rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle.$$

Here are two simple but useful facts about divergence and curl.

THEOREM 16.5.1 $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. ∎

In words, this says that the divergence of the curl is zero.

THEOREM 16.5.2 $\nabla \times (\nabla f) = \mathbf{0}$. ∎

That is, the curl of a gradient is the zero vector. Recalling that gradients are conservative vector fields, this says that the curl of a conservative vector field is the zero vector. Under suitable conditions, it is also true that if the curl of $\mathbf{F}$ is $\mathbf{0}$ then $\mathbf{F}$ is conservative. (Note that this is exactly the same test that we discussed on page 427.)

EXAMPLE 16.5.3 Let $\mathbf{F} = \langle e^z, 1, xe^z\rangle$. Then $\nabla \times \mathbf{F} = \langle 0, e^z - e^z, 0\rangle = \mathbf{0}$. Thus, $\mathbf{F}$ is conservative, and we can exhibit this directly by finding the corresponding f.

Since $f_x = e^z$, $f = xe^z + g(y, z)$. Since $f_y = 1$, it must be that $g_y = 1$, so $g(y, z) = y + h(z)$. Thus $f = xe^z + y + h(z)$ and

$$xe^z = f_z = xe^z + 0 + h'(z),$$

so $h'(z) = 0$, i.e., $h(z) = C$, and $f = xe^z + y + C$. □

We can rewrite Green's Theorem using these new ideas; these rewritten versions in turn are closer to some later theorems we will see.

Suppose we write a two dimensional vector field in the form $\mathbf{F} = \langle P, Q, 0\rangle$, where P and Q are functions of x and y. Then

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \langle 0, 0, Q_x - P_y\rangle,$$

and so $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \langle 0, 0, Q_x - P_y\rangle \cdot \langle 0, 0, 1\rangle = Q_x - P_y$. So Green's Theorem says

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D} P\,dx + Q\,dy = \iint_D Q_x - P_y\,dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA. \quad (16.5.1)$$

Roughly speaking, the right-most integral adds up the curl (tendency to swirl) at each point in the region; the left-most integral adds up the tangential components of the vector field around the entire boundary. Green's Theorem says these are equal, or roughly, that the sum of the "microscopic" swirls over the region is the same as the "macroscopic" swirl around the boundary.

Next, suppose that the boundary ∂D has a vector form $\mathbf{r}(t)$, so that $\mathbf{r}'(t)$ is tangent to the boundary, and $\mathbf{T} = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ is the usual unit tangent vector. Writing $\mathbf{r} = \langle x(t), y(t)\rangle$ we get

$$\mathbf{T} = \frac{\langle x', y'\rangle}{|\mathbf{r}'(t)|}$$

and then

$$\mathbf{N} = \frac{\langle y', -x'\rangle}{|\mathbf{r}'(t)|}$$

is a unit vector perpendicular to $\mathbf{T}$, that is, a unit normal to the boundary. Now

$$\begin{aligned}\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds &= \int_{\partial D} \langle P, Q\rangle \cdot \frac{\langle y', -x'\rangle}{|\mathbf{r}'(t)|}|\mathbf{r}'(t)|dt = \int_{\partial D} Py'\,dt - Qx'\,dt \\ &= \int_{\partial D} P\,dy - Q\,dx = \int_{\partial D} -Q\,dx + P\,dy.\end{aligned}$$

So far, we've just rewritten the original integral using alternate notation. The last integral looks just like the right side of Green's Theorem (16.4.1) except that P and Q have traded places and Q has acquired a negative sign. Then applying Green's Theorem we get

$$\int_{\partial D} -Q\,dx + P\,dy = \iint_D P_x + Q_y\,dA = \iint_D \nabla\cdot\mathbf{F}\,dA.$$

Summarizing the long string of equalities,

$$\int_{\partial D} \mathbf{F}\cdot\mathbf{N}\,ds = \iint_D \nabla\cdot\mathbf{F}\,dA. \qquad (16.5.2)$$

Roughly speaking, the first integral adds up the flow across the boundary of the region, from inside to out, and the second sums the divergence (tendency to spread) at each point in the interior. The theorem roughly says that the sum of the "microscopic" spreads is the same as the total spread across the boundary and out of the region.

Exercises 16.5.

1. Let $\mathbf{F} = \langle xy, -xy \rangle$ and let D be given by $0 \le x \le 1$, $0 \le y \le 1$. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F} \cdot \mathbf{N}\, ds$. ⇒

2. Let $\mathbf{F} = \langle ax^2, by^2 \rangle$ and let D be given by $0 \le x \le 1$, $0 \le y \le 1$. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F} \cdot \mathbf{N}\, ds$. ⇒

3. Let $\mathbf{F} = \langle ay^2, bx^2 \rangle$ and let D be given by $0 \le x \le 1$, $0 \le y \le x$. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F} \cdot \mathbf{N}\, ds$. ⇒

4. Let $\mathbf{F} = \langle \sin x \cos y, \cos x \sin y \rangle$ and let D be given by $0 \le x \le \pi/2$, $0 \le y \le x$. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F} \cdot \mathbf{N}\, ds$. ⇒

5. Let $\mathbf{F} = \langle y, -x \rangle$ and let D be given by $x^2 + y^2 \le 1$. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F} \cdot \mathbf{N}\, ds$. ⇒

6. Let $\mathbf{F} = \langle x, y \rangle$ and let D be given by $x^2 + y^2 \le 1$. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{\partial D} \mathbf{F} \cdot \mathbf{N}\, ds$. ⇒

7. Prove theorem 16.5.1.

8. Prove theorem 16.5.2.

9. If $\nabla \cdot \mathbf{F} = 0$, $\mathbf{F}$ is said to be **incompressible**. Show that any vector field of the form $\mathbf{F}(x, y, z) = \langle f(y, z), g(x, z), h(x, y) \rangle$ is incompressible. Give a non-trivial example.

16.6 Vector Functions for Surfaces

We have dealt extensively with vector equations for curves, $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. A similar technique can be used to represent surfaces in a way that is more general than the equations for surfaces we have used so far. Recall that when we use $\mathbf{r}(t)$ to represent a curve, we imagine the vector $\mathbf{r}(t)$ with its tail at the origin, and then we follow the head of the arrow as t changes. The vector "draws" the curve through space as t varies.

Suppose we instead have a vector function of two variables,

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle.$$

As both u and v vary, we again imagine the vector $\mathbf{r}(u, v)$ with its tail at the origin, and its head sweeps out a surface in space. A useful analogy is the technology of CRT video screens, in which an electron gun fires electrons in the direction of the screen. The gun's direction sweeps horizontally and vertically to "paint" the screen with the desired image. In practice, the gun moves horizontally through an entire line, then moves vertically to the next line and repeats the operation. In the same way, it can be useful to imagine fixing a value of v and letting $\mathbf{r}(u, v)$ sweep out a curve as u changes. Then v can change a bit, and $\mathbf{r}(u, v)$ sweeps out a new curve very close to the first. Put enough of these curves together and they form a surface.

EXAMPLE 16.6.1 Consider the function $\mathbf{r}(u, v) = \langle v \cos u, v \sin u, v \rangle$. For a fixed value of v, as u varies from 0 to 2π, this traces a circle of radius v at height v above the x-y plane. Put lots and lots of these together,and they form a cone, as in figure 16.6.1. □

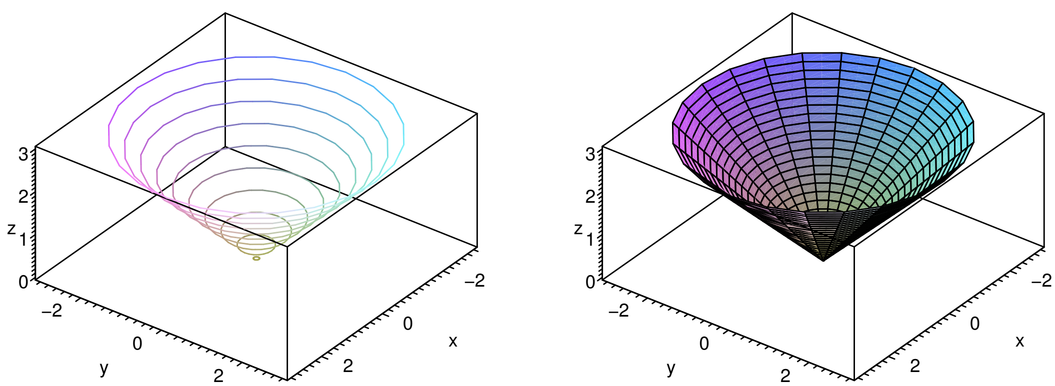


Figure 16.6.1 Tracing a surface.

EXAMPLE 16.6.2 Let $\mathbf{r} = \langle v \cos u, v \sin u, u \rangle$. If v is constant, the resulting curve is a helix (as in figure 13.1.1). If u is constant, the resulting curve is a straight line at height u in the direction u radians from the positive x axis. Note in figure 16.6.2 how the helixes and the lines both paint the same surface in a different way. □

This technique allows us to represent many more surfaces than previously.

EXAMPLE 16.6.3 The curve given by

$$\mathbf{r} = \langle (2 + \cos(3u/2)) \cos u, (2 + \cos(3u/2)) \sin u, \sin(3u/2) \rangle$$

is called a trefoil knot. Recall that from the vector equation of the curve we can compute the unit tangent $\mathbf{T}$, the unit normal $\mathbf{N}$, and the binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$; you may want to review section 13.3. The binormal is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$; one way to interpret this is that $\mathbf{N}$ and $\mathbf{B}$ define a plane perpendicular to $\mathbf{T}$, that is, perpendicular to the curve; since $\mathbf{N}$ and $\mathbf{B}$ are perpendicular to each other, they can function just as $\mathbf{i}$ and

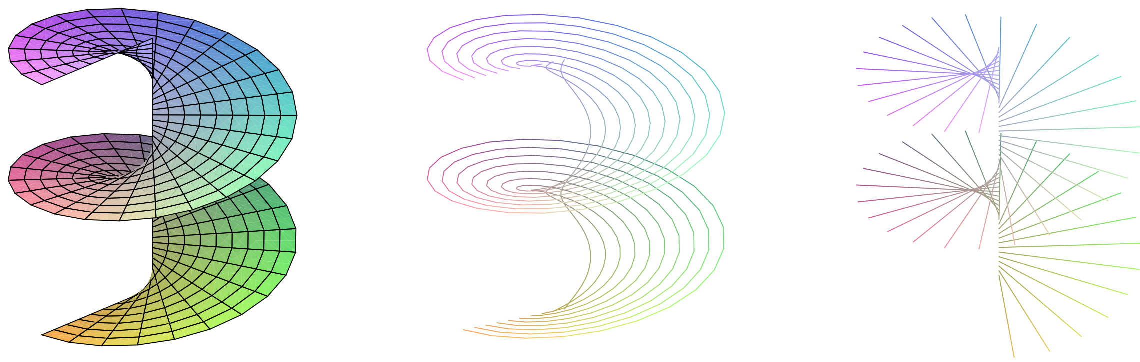


Figure 16.6.2 Tracing a surface. (AP)

$\mathbf{j}$ do for the x-y plane. So, for example, $\mathbf{c}(v) = \mathbf{N}\cos v + \mathbf{B}\sin v$ is a vector equation for a unit circle in a plane perpendicular to the curve described by $\mathbf{r}$, except that the usual interpretation of $\mathbf{c}$ would put its center at the origin. We can fix that simply by adding $\mathbf{c}$ to the original $\mathbf{r}$: let $\mathbf{f} = \mathbf{r}(u) + \mathbf{c}(v)$. For a fixed u this draws a circle around the point $\mathbf{r}(u)$; as u varies we get a sequence of such circles around the curve $\mathbf{r}$, that is, a tube of radius 1 with $\mathbf{r}$ at its center. We can easily change the radius; for example $\mathbf{r}(u) + a\mathbf{c}(v)$ gives the tube radius a; we can make the radius vary as we move along the curve with $\mathbf{r}(u) + g(u)\mathbf{c}(v)$, where $g(u)$ is a function of u. As shown in figure 16.6.3, it is hard to see that the plain knot is knotted; the tube makes the structure apparent. Of course, there is nothing special about the trefoil knot in this example; we can put a tube around (almost) any curve in the same way. □

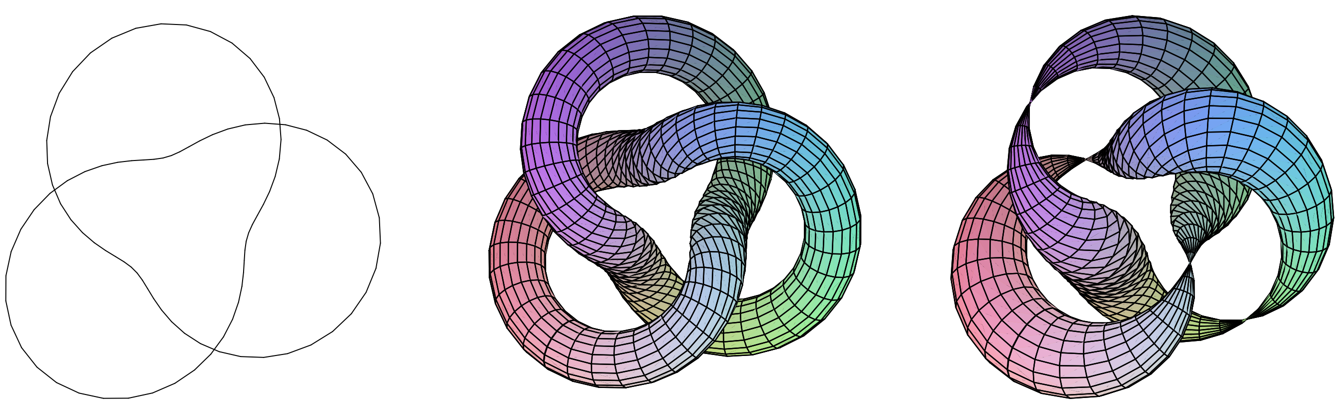


Figure 16.6.3 Tubes around a trefoil knot, with radius 1/2 and $3\cos(u)/4$. (AP)

We have previously examined surfaces given in the form $f(x, y)$. It is sometimes useful to represent such surfaces in the more general vector form, which is quite easy:

$\mathbf{r}(u, v) = \langle u, v, f(u, v) \rangle$. The names of the variables are not important of course; instead of disguising x and y, we could simply write $\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle$.

We have also previously dealt with surfaces that are not functions of x and y; many of these are easy to represent in vector form. One common type of surface that cannot be represented as $z = f(x, y)$ is a surface given by an equation involving only x and y. For example, $x + y = 1$ and $y = x^2$ are "vertical" surfaces. For every point (x, y) in the plane that satisfies the equation, the point (x, y, z) is on the surface, for every value of z. Thus, a corresponding vector form for the surface is something like $\langle f(u), g(u), v \rangle$; for example, $x + y = 1$ becomes $\langle u, 1 - u, v \rangle$ and $y = x^2$ becomes $\langle u, u^2, v \rangle$.

Yet another sort of example is the sphere, say $x^2 + y^2 + z^2 = 1$. This cannot be written in the form $z = f(x, y)$, but it is easy to write in vector form; indeed this particular surface is much like the cone, since it has circular cross-sections, or we can think of it as a tube around a portion of the z-axis, with a radius that varies depending on where along the axis we are. One vector expression for the sphere is $\langle \sqrt{1 - v^2} \cos u, \sqrt{1 - v^2} \sin u, v \rangle$— this emphasizes the tube structure, as it is naturally viewed as drawing a circle of radius $\sqrt{1 - v^2}$ around the z-axis at height v. We could also take a cue from spherical coordinates, and write $\langle \sin u \cos v, \sin u \sin v, \cos u \rangle$, where in effect u and v are ϕ and θ in disguise.

It is quite simple in Sage to plot any surface for which you have a vector representation. Using different vector functions sometimes gives different looking plots, because Sage in effect draws the surface by holding one variable constant and then the other. For example, you might have noticed in figure 16.6.2 that the curves in the two right-hand graphs are superimposed on the left-hand graph; the graph of the surface is just the combination of the two sets of curves, with the spaces filled in with color.

Here's a simple but striking example: the plane $x + y + z = 1$ can be represented quite naturally as $\langle u, v, 1 - u - v \rangle$. But we could also think of painting the same plane by choosing a particular point on the plane, say $(1, 0, 0)$, and then drawing circles or ellipses (or any of a number of other curves) as if that point were the origin in the plane. For example, $\langle 1 - v \cos u - v \sin u, v \sin u, v \cos u \rangle$ is one such vector function. Note that while it may not be obvious where this came from, it is quite easy to see that the sum of the x, y, and z components of the vector is always 1. Computer renderings of the plane using these two functions are shown in figure 16.6.4.

Suppose we know that a plane contains a particular point (x_0, y_0, z_0) and that two vectors $\mathbf{u} = \langle u_0, u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_0, v_1, v_2 \rangle$ are parallel to the plane but not to each other. We know how to get an equation for the plane in the form $ax + by + cz = d$, by first computing $\mathbf{u} \times \mathbf{v}$. It's even easier to get a vector equation:

$$\mathbf{r}(u, v) = \langle x_0, y_0, z_0 \rangle + u\mathbf{u} + v\mathbf{v}.$$

The first vector gets to the point (x_0, y_0, z_0) and then by varying u and v, $u\mathbf{u} + v\mathbf{v}$ gets to every point in the plane.

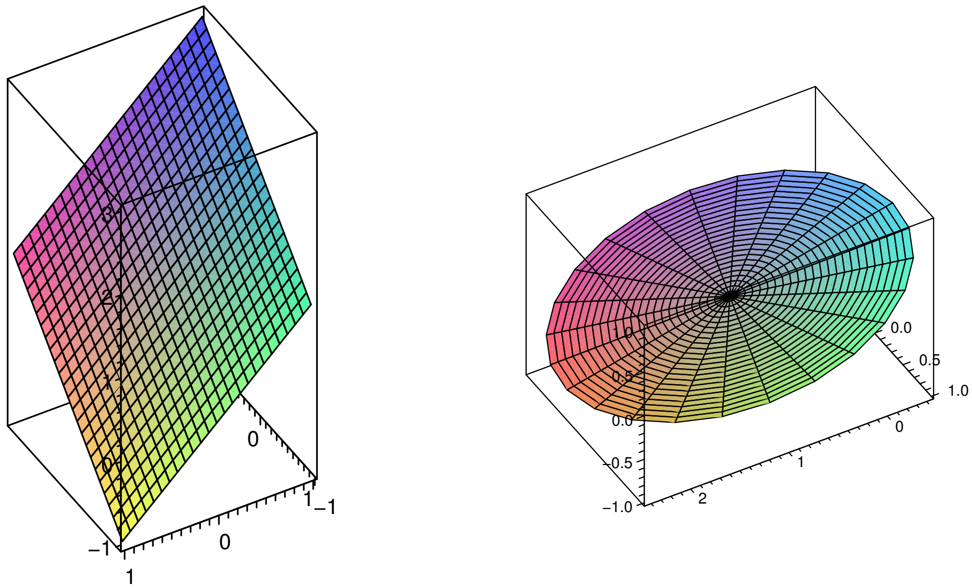


Figure 16.6.4 Two representations of the same plane. (AP)

Returning to $x + y + z = 1$, the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ are all on the plane. By subtracting coordinates we see that $\langle -1,0,1\rangle$ and $\langle -1,1,0\rangle$ are parallel to the plane, so a third vector form for this plane is

$$\langle 1,0,0\rangle + u\langle -1,0,1\rangle + v\langle -1,1,0\rangle = \langle 1-u-v, v, u\rangle.$$

This is clearly quite similar to the first form we found.

We have already seen (section 15.4) how to find the area of a surface when it is defined in the form $f(x,y)$. Finding the area when the surface is given as a vector function is very similar. Looking at the plots of surfaces we have just seen, it is evident that the two sets of curves that fill out the surface divide it into a grid, and that the spaces in the grid are approximately parallelograms. As before this is the key: we can write down the area of a typical little parallelogram and add them all up with an integral.

Suppose we want to approximate the area of the surface $\mathbf{r}(u,v)$ near $\mathbf{r}(u_0,v_0)$. The functions $\mathbf{r}(u,v_0)$ and $\mathbf{r}(u_0,v)$ define two curves that intersect at $\mathbf{r}(u_0,v_0)$. The derivatives of $\mathbf{r}$ give us vectors tangent to these two curves: $\mathbf{r}_u(u_0,v_0)$ and $\mathbf{r}_v(u_0,v_0)$, and then $\mathbf{r}_u(u_0,v_0)\,du$ and $\mathbf{r}_v(u_0,v_0)\,dv$ are two small tangent vectors, whose lengths can be used as the lengths of the sides of an approximating parallelogram. Finally, the area of this parallelogram is $|\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv$ and so the total surface area is

$$\int_a^b\int_c^d |\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv.$$

EXAMPLE 16.6.4 We find the area of the surface $\langle v\cos u, v\sin u, u\rangle$ for $0 \le u \le \pi$ and $0 \le v \le 1$; this is a portion of the helical surface in figure 16.6.2. We compute $\mathbf{r}_u = \langle -v\sin u, v\cos u, 1\rangle$ and $\mathbf{r}_v = \langle \cos u, \sin u, 0\rangle$. The cross product of these two vectors is $\langle \sin u, -\cos u, v\rangle$ with length $\sqrt{1+v^2}$, and the surface area is

$$\int_0^\pi \int_0^1 \sqrt{1+v^2}\,dv\,du = \frac{\pi\sqrt{2}}{2} + \frac{\pi\ln(\sqrt{2}+1)}{2}.$$

□

Exercises 16.6.

1. Describe or sketch the surface with the given vector function.

a. $\mathbf{r}(u,v) = \langle u+v, 3-v, 1+4u+5v\rangle$

b. $\mathbf{r}(u,v) = \langle 2\sin u, 3\cos u, v\rangle$

c. $\mathbf{r}(s,t) = \langle s, t, t^2 - s^2\rangle$

d. $\mathbf{r}(s,t) = \langle s\sin 2t, s^2, s\cos 2t\rangle$

2. Find a parametric representation, $\mathbf{r}(u,v)$, for the surface.

a. The plane that passes through the point $(1,2,-3)$ and is parallel to the vectors $\langle 1,1,-1\rangle$ and $\langle 1,-1,1\rangle$.

b. The lower half of the ellipsoid $2x^2 + 4y^2 + z^2 = 1$.

c. The part of the sphere of radius 4 centered at the origin that lies between the planes $z = -2$ and $z = 2$.

3. Find the area of the portion of $x + 2y + 4z = 10$ in the first octant. ⇒

4. Find the area of the portion of $2x + 4y + z = 0$ inside $x^2 + y^2 = 1$. ⇒

5. Find the area of $z = x^2 + y^2$ that lies below $z = 1$. ⇒

6. Find the area of $z = \sqrt{x^2 + y^2}$ that lies below $z = 2$. ⇒

7. Find the area of the portion of $x^2 + y^2 + z^2 = a^2$ that lies in the first octant. ⇒

8. Find the area of the portion of $x^2 + y^2 + z^2 = a^2$ that lies above $x^2 + y^2 \le b^2$, $b \le a$. ⇒

9. Find the area of $z = x^2 - y^2$ that lies inside $x^2 + y^2 = a^2$. ⇒

10. Find the area of $z = xy$ that lies inside $x^2 + y^2 = a^2$. ⇒

11. Find the area of $x^2 + y^2 + z^2 = a^2$ that lies above the interior of the circle given in polar coordinates by $r = a\cos\theta$. ⇒

12. Find the area of the cone $z = k\sqrt{x^2 + y^2}$ that lies above the interior of the circle given in polar coordinates by $r = a\cos\theta$. ⇒

13. Find the area of the plane $z = ax + by + c$ that lies over a region D with area A. ⇒

14. Find the area of the cone $z = k\sqrt{x^2 + y^2}$ that lies over a region D with area A. ⇒

15. Find the area of the cylinder $x^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = a^2$. ⇒

16. The surface $f(x,y)$ can be represented with the vector function $\langle x, y, f(x,y)\rangle$. Set up the surface area integral using this vector function and compare to the integral of section 15.4.

16.7 Surface Integrals

In the integral for surface area,

$$\int_a^b \int_c^d |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv,$$

the integrand $|\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$ is the area of a tiny parallelogram, that is, a very small surface area, so it is reasonable to abbreviate it dS; then a shortened version of the integral is

$$\iint_D 1 \cdot dS.$$

We have already seen that if D is a region in the plane, the area of D may be computed with

$$\iint_D 1 \cdot dA,$$

so this is really quite familiar, but the dS hides a little more detail than does dA.

Just as we can integrate functions $f(x, y)$ over regions in the plane, using

$$\iint_D f(x, y) \, dA,$$

so we can compute integrals over surfaces in space, using

$$\iint_D f(x, y, z) \, dS.$$

In practice this means that we have a vector function $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ for the surface, and the integral we compute is

$$\int_a^b \int_c^d f(x(u, v), y(u, v), z(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv.$$

That is, we express everything in terms of u and v, and then we can do an ordinary double integral.

EXAMPLE 16.7.1 Suppose a thin object occupies the upper hemisphere of $x^2 + y^2 + z^2 = 1$ and has density $\sigma(x, y, z) = z$. Find the mass and center of mass of the object. (Note that the object is just a thin shell; it does not occupy the interior of the hemisphere.)

We write the hemisphere as $\mathbf{r}(\phi,\theta) = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$, $0 \le \phi \le \pi/2$ and $0 \le \theta \le 2\pi$. So $\mathbf{r}_\theta = \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0\rangle$ and $\mathbf{r}_\phi = \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\rangle$. Then

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = \langle -\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\cos\phi\sin\phi\rangle$$

and

$$|\mathbf{r}_\theta \times \mathbf{r}_\phi| = |\sin\phi| = \sin\phi,$$

since we are interested only in $0 \le \phi \le \pi/2$. Finally, the density is $z = \cos\phi$ and the integral for mass is

$$\int_0^{2\pi}\int_0^{\pi/2} \cos\phi\sin\phi\,d\phi\,d\theta = \pi.$$

By symmetry, the center of mass is clearly on the z-axis, so we only need to find the z-coordinate of the center of mass. The moment around the x-y plane is

$$\int_0^{2\pi}\int_0^{\pi/2} z\cos\phi\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} \cos^2\phi\sin\phi\,d\phi\,d\theta = \frac{2\pi}{3},$$

so the center of mass is at $(0,0,2/3)$. □

Now suppose that $\mathbf{F}$ is a vector field; imagine that it represents the velocity of some fluid at each point in space. We would like to measure how much fluid is passing through a surface D, the **flux** across D. As usual, we imagine computing the flux across a very small section of the surface, with area dS, and then adding up all such small fluxes over D with an integral. Suppose that vector $\mathbf{N}$ is a unit normal to the surface at a point; $\mathbf{F}\cdot\mathbf{N}$ is the scalar projection of $\mathbf{F}$ onto the direction of $\mathbf{N}$, so it measures how fast the fluid is moving across the surface. In one unit of time the fluid moving across the surface will fill a volume of $\mathbf{F}\cdot\mathbf{N}\,dS$, which is therefore the rate at which the fluid is moving across a small patch of the surface. Thus, the total flux across D is

$$\iint_D \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D \mathbf{F}\cdot\,d\mathbf{S},$$

defining $d\mathbf{S} = \mathbf{N}\,dS$. As usual, certain conditions must be met for this to work out; chief among them is the nature of the surface. As we integrate over the surface, we must choose the normal vectors $\mathbf{N}$ in such a way that they point "the same way" through the surface. For example, if the surface is roughly horizontal in orientation, we might want to measure the flux in the "upwards" direction, or if the surface is closed, like a sphere, we might want to measure the flux "outwards" across the surface. In the first case we would choose $\mathbf{N}$ to have positive z component, in the second we would make sure that $\mathbf{N}$ points away from the origin. Unfortunately, there are surfaces that are not **orientable**: they have only one side, so that it is not possible to choose the normal vectors to point in the "same way" through the surface. The most famous such surface is the Möbius strip shown in figure 16.7.1. It is quite easy to make such a strip with a piece of paper and some tape. If you have never done this, it is quite instructive; in particular, you should draw a line down the center of the strip until you return to your starting point. No matter how unit normal vectors are assigned to the points of the Möbius strip, there will be normal vectors very close to each other pointing in opposite directions.

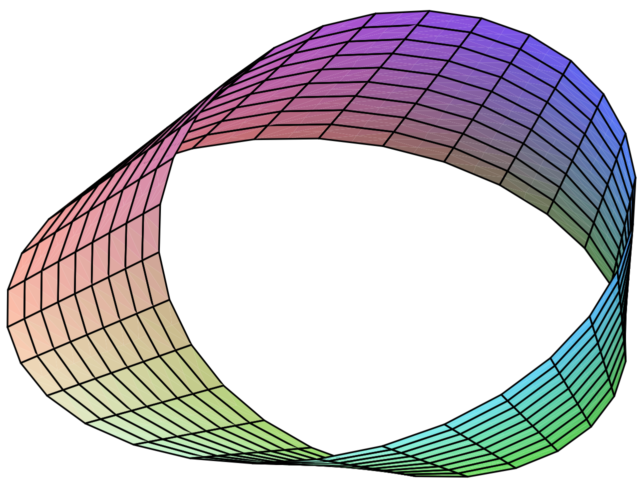


Figure 16.7.1 A Möbius strip. (AP)

Assuming that the quantities involved are well behaved, however, the flux of the vector field across the surface $\mathbf{r}(u,v)$ is

$$\iint_D \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D \mathbf{F}\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}|\mathbf{r}_u\times\mathbf{r}_v|\,dA = \iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA.$$

In practice, we may have to use $\mathbf{r}_v\times\mathbf{r}_u$ or even something a bit more complicated to make sure that the normal vector points in the desired direction.

EXAMPLE 16.7.2 Compute the flux of $\mathbf{F}=\langle x,y,z^4\rangle$ across the cone $z=\sqrt{x^2+y^2}$, $0\le z\le 1$, in the downward direction.

We write the cone as a vector function: $\mathbf{r}=\langle v\cos u, v\sin u, v\rangle$, $0\le u\le 2\pi$ and $0\le v\le 1$. Then $\mathbf{r}_u=\langle -v\sin u, v\cos u, 0\rangle$ and $\mathbf{r}_v=\langle\cos u,\sin u,1\rangle$ and $\mathbf{r}_u\times\mathbf{r}_v =$

$\langle v\cos u, v\sin u, -v\rangle$. The third coordinate $-v$ is negative, which is exactly what we desire, that is, the normal vector points down through the surface. Then

$$\begin{aligned}\int_0^{2\pi}\int_0^1 \langle x,y,z^4\rangle\cdot\langle v\cos u, v\sin u, -v\rangle\,dv\,du &= \int_0^{2\pi}\int_0^1 xv\cos u + yv\sin u - z^4 v\,dv\,du\\ &= \int_0^{2\pi}\int_0^1 v^2\cos^2 u + v^2\sin^2 u - v^5\,dv\,du\\ &= \int_0^{2\pi}\int_0^1 v^2 - v^5\,dv\,du = \frac{\pi}{3}.\end{aligned}$$

□

Exercises 16.7.

1. Find the center of mass of an object that occupies the upper hemisphere of $x^2+y^2+z^2=1$ and has density x^2+y^2. ⇒
2. Find the center of mass of an object that occupies the surface $z=xy$, $0\le x\le 1$, $0\le y\le 1$ and has density $\sqrt{1+x^2+y^2}$. ⇒
3. Find the center of mass of an object that occupies the surface $z=\sqrt{x^2+y^2}$, $1\le z\le 4$ and has density x^2z. ⇒
4. Find the centroid of the surface of a right circular cone of height h and base radius r, not including the base. ⇒
5. Evaluate $\iint_D \langle 2,-3,4\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=x^2+y^2$, $-1\le x\le 1$, $-1\le y\le 1$, oriented up. ⇒
6. Evaluate $\iint_D \langle x,y,3\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=3x-5y$, $1\le x\le 2$, $0\le y\le 2$, oriented up. ⇒
7. Evaluate $\iint_D \langle x,y,-2\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=1-x^2-y^2$, $x^2+y^2\le 1$, oriented up. ⇒
8. Evaluate $\iint_D \langle xy,yz,zx\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=x+y^2+2$, $0\le x\le 1$, $x\le y\le 1$, oriented up. ⇒
9. Evaluate $\iint_D \langle e^x,e^y,z\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=xy$, $0\le x\le 1$, $-x\le y\le x$, oriented up. ⇒
10. Evaluate $\iint_D \langle xz,yz,z\rangle\cdot\mathbf{N}\,dS$, where D is given by $z=a^2-x^2-y^2$, $x^2+y^2\le b^2$, oriented up. ⇒
11. A fluid has density 870 kg/m^3 and flows with velocity $\mathbf{v}=\langle z,y^2,x^2\rangle$, where distances are in meters and the components of $\mathbf{v}$ are in meters per second. Find the rate of flow outward through the portion of the cylinder $x^2+y^2=4$, $0\le z\le 1$ for which $y\ge 0$. ⇒

12. Gauss's Law says that the net charge, Q, enclosed by a closed surface, S, is

$$Q = \epsilon_0 \iint \mathbf{E} \cdot \mathbf{N} \, dS$$

where $\mathbf{E}$ is an electric field and ϵ_0 (the permittivity of free space) is a known constant; $\mathbf{N}$ is oriented outward. Use Gauss's Law to find the charge contained in the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is $\mathbf{E} = \langle x, y, z \rangle$. ⇒

16.8 Stokes's Theorem

Recall that one version of Green's Theorem (see equation 16.5.1) is

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA.$$

Here D is a region in the x-y plane and $\mathbf{k}$ is a unit normal to D at every point. If D is instead an orientable surface in space, there is an obvious way to alter this equation, and it turns out still to be true:

THEOREM 16.8.1 Stokes's Theorem Provided that the quantities involved are sufficiently nice, and in particular if D is orientable,

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N} \, dS,$$

if ∂D is oriented counter-clockwise relative to $\mathbf{N}$. □

Note how little has changed: $\mathbf{k}$ becomes $\mathbf{N}$, a unit normal to the surface, and dA becomes dS, since this is now a general surface integral. The phrase "counter-clockwise relative to $\mathbf{N}$" means roughly that if we take the direction of $\mathbf{N}$ to be "up", then we go around the boundary counter-clockwise when viewed from "above". In many cases, this description is inadequate. A slightly more complicated but general description is this: imagine standing on the side of the surface considered positive; walk to the boundary and turn left. You are now following the boundary in the correct direction.

EXAMPLE 16.8.2 Let $\mathbf{F} = \langle e^{xy} \cos z, x^2 z, xy \rangle$ and the surface D be $x = \sqrt{1 - y^2 - z^2}$, oriented in the positive x direction. It quickly becomes apparent that the surface integral in Stokes's Theorem is intractable, so we try the line integral. The boundary of D is the unit circle in the y-z plane, $\mathbf{r} = \langle 0, \cos u, \sin u \rangle$, $0 \le u \le 2\pi$. The integral is

$$\int_0^{2\pi} \langle e^{xy} \cos z, x^2 z, xy \rangle \cdot \langle 0, -\sin u, \cos u \rangle \, du = \int_0^{2\pi} 0 \, du = 0,$$

because $x = 0$. □

EXAMPLE 16.8.3 Consider the cylinder $\mathbf{r} = \langle \cos u, \sin u, v\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 2$, oriented outward, and $\mathbf{F} = \langle y, zx, xy\rangle$. We compute

$$\iint_D \nabla \times \mathbf{F} \cdot \mathbf{N}\, dS = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

in two ways.

First, the double integral is

$$\int_0^{2\pi}\int_0^2 \langle 0, -\sin u, v-1\rangle \cdot \langle \cos u, \sin u, 0\rangle\, dv\, du = \int_0^{2\pi}\int_0^2 -\sin^2 u\, dv\, du = -2\pi.$$

The boundary consists of two parts, the bottom circle $\langle \cos t, \sin t, 0\rangle$, with t ranging from 0 to 2π, and $\langle \cos t, \sin t, 2\rangle$, with t ranging from 2π to 0. We compute the corresponding integrals and add the results:

$$\int_0^{2\pi} -\sin^2 t\, dt + \int_{2\pi}^0 -\sin^2 t + 2\cos^2 t = -\pi - \pi = -2\pi,$$

as before. □

An interesting consequence of Stokes's Theorem is that if D and E are two orientable surfaces with the same boundary, then

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial E} \mathbf{F} \cdot d\mathbf{r} = \iint_E (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS.$$

Sometimes both of the integrals

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS \qquad \text{and} \qquad \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

are difficult, but you may be able to find a second surface E so that

$$\iint_E (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$

has the same value but is easier to compute.

EXAMPLE 16.8.4 In example 16.8.2 the line integral was easy to compute. But we might also notice that another surface E with the same boundary is the flat disk $y^2 + z^2 \le 1$.

The unit normal $\mathbf{N}$ for this surface is simply $\mathbf{i} = \langle 1, 0, 0\rangle$. We compute the curl:

$$\nabla \times \mathbf{F} = \langle x - x^2, -e^{xy}\sin z - y, 2xz - xe^{xy}\cos z\rangle.$$

Since $x = 0$ everywhere on the surface,

$$(\nabla \times \mathbf{F}) \cdot \mathbf{N} = \langle 0, -e^{xy}\sin z - y, 2xz - xe^{xy}\cos z\rangle \cdot \langle 1, 0, 0\rangle = 0,$$

so the surface integral is

$$\iint_E 0\,dS = 0,$$

as before. In this case, of course, it is still somewhat easier to compute the line integral, avoiding $\nabla \times \mathbf{F}$ entirely. □

EXAMPLE 16.8.5 Let $\mathbf{F} = \langle -y^2, x, z^2\rangle$, and let the curve C be the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$, oriented counter-clockwise when viewed from above. We compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways.

First we do it directly: a vector function for C is $\mathbf{r} = \langle \cos u, \sin u, 2 - \sin u\rangle$, so $\mathbf{r}' = \langle -\sin u, \cos u, -\cos u\rangle$, and the integral is then

$$\int_0^{2\pi} y^2 \sin u + x\cos u - z^2 \cos u\,du = \int_0^{2\pi} \sin^3 u + \cos^2 u - (2 - \sin u)^2 \cos u\,du = \pi.$$

To use Stokes's Theorem, we pick a surface with C as the boundary; the simplest such surface is that portion of the plane $y + z = 2$ inside the cylinder. This has vector equation $\mathbf{r} = \langle v\cos u, v\sin u, 2 - v\sin u\rangle$. We compute $\mathbf{r}_u = \langle -v\sin u, v\cos u, -v\cos u\rangle$, $\mathbf{r}_v = \langle \cos u, \sin u, -\sin u\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, -v, -v\rangle$. To match the orientation of C we need to use the normal $\langle 0, v, v\rangle$. The curl of $\mathbf{F}$ is $\langle 0, 0, 1 + 2y\rangle = \langle 0, 0, 1 + 2v\sin u\rangle$, and the surface integral from Stokes's Theorem is

$$\int_0^{2\pi} \int_0^1 (1 + 2v\sin u)v\,dv\,du = \pi.$$

In this case the surface integral was more work to set up, but the resulting integral is somewhat easier. □

Proof of Stokes's Theorem. We can prove here a special case of Stokes's Theorem, which perhaps not too surprisingly uses Green's Theorem.

Suppose the surface D of interest can be expressed in the form $z = g(x, y)$, and let $\mathbf{F} = \langle P, Q, R\rangle$. Using the vector function $\mathbf{r} = \langle x, y, g(x, y)\rangle$ for the surface we get the surface integral

$$\begin{aligned}\iint\limits_D \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint\limits_E \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle \cdot \langle -g_x, -g_y, 1\rangle\, dA \\ &= \iint\limits_E -R_y g_x + Q_z g_x - P_z g_y + R_x g_y + Q_x - P_y\, dA.\end{aligned}$$

Here E is the region in the x-y plane directly below the surface D.

For the line integral, we need a vector function for ∂D. If $\langle x(t), y(t)\rangle$ is a vector function for ∂E then we may use $\mathbf{r}(t) = \langle x(t), y(t), g(x(t), y(t))\rangle$ to represent ∂D. Then

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_a^b P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\, dt = \int_a^b P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right) dt.$$

using the chain rule for dz/dt. Now we continue to manipulate this:

$$\begin{aligned}&\int_a^b P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right) dt \\ &\qquad = \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\right] dt \\ &\qquad = \int_{\partial E} \left(P + R\frac{\partial z}{\partial x}\right) dx + \left(Q + R\frac{\partial z}{\partial y}\right) dy,\end{aligned}$$

which now looks just like the line integral of Green's Theorem, except that the functions P and Q of Green's Theorem have been replaced by the more complicated $P + R(\partial z/\partial x)$ and $Q + R(\partial z/\partial y)$. We can apply Green's Theorem to get

$$\int_{\partial E} \left(P + R\frac{\partial z}{\partial x}\right) dx + \left(Q + R\frac{\partial z}{\partial y}\right) dy = \iint\limits_E \frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right) dA.$$

Now we can use the chain rule again to evaluate the derivatives inside this integral, and it becomes

$$\begin{aligned}&\iint\limits_E Q_x + Q_z g_x + R_x g_y + R_z g_x g_y + R g_{yx} - (P_y + P_z g_y + R_y g_x + R_z g_y g_x + R g_{xy})\, dA \\ &\quad = \iint\limits_E Q_x + Q_z g_x + R_x g_y - P_y - P_z g_y - R_y g_x\, dA,\end{aligned}$$

which is the same as the expression we obtained for the surface integral. ∎

Exercises 16.8.

1. Let $\mathbf{F} = \langle z, x, y\rangle$. The plane $z = 2x + 2y - 1$ and the paraboloid $z = x^2 + y^2$ intersect in a closed curve. Stokes's Theorem implies that

$$\iint_{D_1} (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{D_2} (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS,$$

where the line integral is computed over the intersection C of the plane and the paraboloid, and the two surface integrals are computed over the portions of the two surfaces that have boundary C (provided, of course, that the orientations all match). Compute all three integrals. ⇒

2. Let D be the portion of $z = 1 - x^2 - y^2$ above the x-y plane, oriented up, and let $\mathbf{F} = \langle xy^2, -x^2y, xyz\rangle$. Compute $\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$. ⇒

3. Let D be the portion of $z = 2x + 5y$ inside $x^2 + y^2 = 1$, oriented up, and let $\mathbf{F} = \langle y, z, -x\rangle$. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$. ⇒

4. Compute $\oint_C x^2 z\, dx + 3x\, dy - y^3\, dz$, where C is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise. ⇒

5. Let D be the portion of $z = px + qy + r$ over a region in the x-y plane that has area A, oriented up, and let $\mathbf{F} = \langle ax + by + cz, ax + by + cz, ax + by + cz\rangle$. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$. ⇒

6. Let D be any surface and let $\mathbf{F} = \langle P(x), Q(y), R(z)\rangle$ (P depends only on x, Q only on y, and R only on z). Show that $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = 0$.

7. Show that $\int_C f\nabla g + g\nabla f \cdot d\mathbf{r} = 0$, where $\mathbf{r}$ describes a closed curve C to which Stokes's Theorem applies. (See theorems 12.4.1 and 16.5.2.)

16.9 The Divergence Theorem

The third version of Green's Theorem (equation 16.5.2) we saw was:

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D \nabla \cdot \mathbf{F}\, dA.$$

With minor changes this turns into another equation, the Divergence Theorem:

THEOREM 16.9.1 Divergence Theorem Under suitable conditions, if E is a region of three dimensional space and D is its boundary surface, oriented outward, then

$$\iint_D \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_E \nabla \cdot \mathbf{F}\, dV.$$

Proof. Again this theorem is too difficult to prove here, but a special case is easier. In the proof of a special case of Green's Theorem, we needed to know that we could describe the region of integration in both possible orders, so that we could set up one double integral using $dx\,dy$ and another using $dy\,dx$. Similarly here, we need to be able to describe the three-dimensional region E in different ways.

We start by rewriting the triple integral:

$$\iiint_E \nabla\cdot\mathbf{F}\,dV = \iiint_E (P_x+Q_y+R_z)\,dV = \iiint_E P_x\,dV + \iiint_E Q_y\,dV + \iiint_E R_z\,dV.$$

The double integral may be rewritten:

$$\iint_D \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D (P\mathbf{i}+Q\mathbf{j}+R\mathbf{k})\cdot\mathbf{N}\,dS = \iint_D P\mathbf{i}\cdot\mathbf{N}\,dS + \iint_D Q\mathbf{j}\cdot\mathbf{N}\,dS + \iint_D R\mathbf{k}\cdot\mathbf{N}\,dS.$$

To prove that these give the same value it is sufficient to prove that

$$\begin{aligned}\iint_D P\mathbf{i}\cdot\mathbf{N}\,dS &= \iiint_E P_x\,dV,\\ \iint_D Q\mathbf{j}\cdot\mathbf{N}\,dS &= \iiint_E Q_y\,dV, \text{ and}\\ \iint_D R\mathbf{k}\cdot\mathbf{N}\,dS &= \iiint_E R_z\,dV.\end{aligned}\qquad (16.9.1)$$

Not surprisingly, these are all pretty much the same; we'll do the first one.

We set the triple integral up with dx innermost:

$$\iiint_E P_x\,dV = \iint_B \int_{g_1(y,z)}^{g_2(y,z)} P_x\,dx\,dA = \iint_B P(g_2(y,z),y,z) - P(g_1(y,z),y,z)\,dA,$$

where B is the region in the y-z plane over which we integrate. The boundary surface of E consists of a "top" $x = g_2(y,z)$, a "bottom" $x = g_1(y,z)$, and a "wrap-around side" that is vertical to the y-z plane. To integrate over the entire boundary surface, we can integrate over each of these (top, bottom, side) and add the results. Over the side surface, the vector $\mathbf{N}$ is perpendicular to the vector $\mathbf{i}$, so

$$\iint_{\text{side}} P\mathbf{i}\cdot\mathbf{N}\,dS = \iint_{\text{side}} 0\,dS = 0.$$

Thus, we are left with just the surface integral over the top plus the surface integral over the bottom. For the top, we use the vector function $\mathbf{r} = \langle g_2(y,z), y, z\rangle$ which gives

$\mathbf{r}_y \times \mathbf{r}_z = \langle 1, -g_{2y}, -g_{2z}\rangle$; the dot product of this with $\mathbf{i} = \langle 1,0,0\rangle$ is 1. Then

$$\iint_{\text{top}} P\mathbf{i}\cdot\mathbf{N}\,dS = \iint_B P(g_2(y,z),y,z)\,dA.$$

In almost identical fashion we get

$$\iint_{\text{bottom}} P\mathbf{i}\cdot\mathbf{N}\,dS = -\iint_B P(g_1(y,z),y,z)\,dA,$$

where the negative sign is needed to make $\mathbf{N}$ point in the negative x direction. Now

$$\iint_D P\mathbf{i}\cdot\mathbf{N}\,dS = \iint_B P(g_2(y,z),y,z)\,dA - \iint_B P(g_1(y,z),y,z)\,dA,$$

which is the same as the value of the triple integral above. ■

EXAMPLE 16.9.2 Let $\mathbf{F} = \langle 2x, 3y, z^2\rangle$, and consider the three-dimensional volume inside the cube with faces parallel to the principal planes and opposite corners at $(0,0,0)$ and $(1,1,1)$. We compute the two integrals of the divergence theorem.

The triple integral is the easier of the two:

$$\int_0^1\int_0^1\int_0^1 2+3+2z\,dx\,dy\,dz = 6.$$

The surface integral must be separated into six parts, one for each face of the cube. One face is $z=0$ or $\mathbf{r} = \langle u,v,0\rangle$, $0 \le u,v \le 1$. Then $\mathbf{r}_u = \langle 1,0,0\rangle$, $\mathbf{r}_v = \langle 0,1,0\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle 0,0,1\rangle$. We need this to be oriented downward (out of the cube), so we use $\langle 0,0,-1\rangle$ and the corresponding integral is

$$\int_0^1\int_0^1 -z^2\,du\,dv = \int_0^1\int_0^1 0\,du\,dv = 0.$$

Another face is $y=1$ or $\mathbf{r} = \langle u,1,v\rangle$. Then $\mathbf{r}_u = \langle 1,0,0\rangle$, $\mathbf{r}_v = \langle 0,0,1\rangle$, and $\mathbf{r}_u\times\mathbf{r}_v = \langle 0,-1,0\rangle$. We need a normal in the positive y direction, so we convert this to $\langle 0,1,0\rangle$, and the corresponding integral is

$$\int_0^1\int_0^1 3y\,du\,dv = \int_0^1\int_0^1 3\,du\,dv = 3.$$

The remaining four integrals have values 0, 0, 2, and 1, and the sum of these is 6, in agreement with the triple integral. □

EXAMPLE 16.9.3 Let $\mathbf{F} = \langle x^3, y^3, z^2\rangle$, and consider the cylindrical volume $x^2+y^2 \le 9$, $0 \le z \le 2$. The triple integral (using cylindrical coordinates) is

$$\int_0^{2\pi}\int_0^3\int_0^2 (3r^2+2z)r\,dz\,dr\,d\theta = 279\pi.$$

For the surface we need three integrals. The top of the cylinder can be represented by $\mathbf{r} = \langle v\cos u, v\sin u, 2\rangle$; $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, -v\rangle$, which points down into the cylinder, so we convert it to $\langle 0, 0, v\rangle$. Then

$$\int_0^{2\pi}\int_0^3 \langle v^3\cos^3 u, v^3\sin^3 u, 4\rangle \cdot \langle 0,0,v\rangle\,dv\,du = \int_0^{2\pi}\int_0^3 4v\,dv\,du = 36\pi.$$

The bottom is $\mathbf{r} = \langle v\cos u, v\sin u, 0\rangle$; $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, -v\rangle$ and

$$\int_0^{2\pi}\int_0^3 \langle v^3\cos^3 u, v^3\sin^3 u, 0\rangle \cdot \langle 0,0,-v\rangle\,dv\,du = \int_0^{2\pi}\int_0^3 0\,dv\,du = 0.$$

The side of the cylinder is $\mathbf{r} = \langle 3\cos u, 3\sin u, v\rangle$; $\mathbf{r}_u \times \mathbf{r}_v = \langle 3\cos u, 3\sin u, 0\rangle$ which does point outward, so

$$\begin{aligned}\int_0^{2\pi}\int_0^2 &\langle 27\cos^3 u, 27\sin^3 u, v^2\rangle \cdot \langle 3\cos u, 3\sin u, 0\rangle\,dv\,du\\ &= \int_0^{2\pi}\int_0^2 81\cos^4 u + 81\sin^4 u\,dv\,du = 243\pi.\end{aligned}$$

The total surface integral is thus $36\pi + 0 + 243\pi = 279\pi$. □

Exercises 16.9.

1. Using $\mathbf{F} = \langle 3x, y^3, -2z^2\rangle$ and the region bounded by $x^2+y^2 = 9$, $z = 0$, and $z = 5$, compute both integrals from the Divergence Theorem. ⇒
2. Let E be the volume described by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, and $\mathbf{F} = \langle x^2, y^2, z^2\rangle$. Compute $\displaystyle\iint_{\partial E} \mathbf{F}\cdot\mathbf{N}\,dS$. ⇒
3. Let E be the volume described by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, and $\mathbf{F} = \langle 2xy, 3xy, ze^{x+y}\rangle$. Compute $\displaystyle\iint_{\partial E} \mathbf{F}\cdot\mathbf{N}\,dS$. ⇒
4. Let E be the volume described by $0 \le x \le 1$, $0 \le y \le x$, $0 \le z \le x+y$, and $\mathbf{F} = \langle x, 2y, 3z\rangle$. Compute $\displaystyle\iint_{\partial E} \mathbf{F}\cdot\mathbf{N}\,dS$. ⇒

5. Let E be the volume described by $x^2 + y^2 + z^2 \leq 4$, and $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS$. ⇒

6. Let E be the hemisphere described by $0 \leq z \leq \sqrt{1 - x^2 - y^2}$, and $\mathbf{F} = \langle \sqrt{x^2 + y^2 + z^2}, \sqrt{x^2 + y^2 + z^2}, \sqrt{x^2 + y^2 + z^2} \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS$. ⇒

7. Let E be the volume described by $x^2 + y^2 \leq 1$, $0 \leq z \leq 4$, and $\mathbf{F} = \langle xy^2, yz, x^2 z \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS$. ⇒

8. Let E be the solid cone above the x-y plane and inside $z = 1 - \sqrt{x^2 + y^2}$, and $\mathbf{F} = \langle x \cos^2 z, y \sin^2 z, \sqrt{x^2 + y^2} z \rangle$. Compute $\iint\limits_{\partial E} \mathbf{F} \cdot \mathbf{N}\, dS$. ⇒

9. Prove the other two equations in the display 16.9.1.

10. Suppose D is a closed surface, and that D and F are sufficiently nice. Show that

$$\iint\limits_{D} (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = 0$$

where $\mathbf{N}$ is the outward pointing unit normal.

11. Suppose D is a closed surface, D is sufficiently nice, and $F = \langle a, b, c \rangle$ is a constant vector field. Show that

$$\iint\limits_{D} \mathbf{F} \cdot \mathbf{N}\, dS = 0$$

where $\mathbf{N}$ is the outward pointing unit normal.

12. We know that the volume of a region E may often be computed as $\iiint\limits_{E} dx\, dy\, dz$. Show that this volume may also be computed as $\frac{1}{3} \iint\limits_{\partial E} \langle x, y, z \rangle \cdot \mathbf{N}\, dS$ where $\mathbf{N}$ is the outward pointing unit normal to ∂E.

17

Differential Equations

Many physical phenomena can be modeled using the language of calculus. For example, observational evidence suggests that the temperature of a cup of tea (or some other liquid) in a room of constant temperature will cool over time at a rate proportional to the difference between the room temperature and the temperature of the tea.

In symbols, if t is the time, M is the room temperature, and $f(t)$ is the temperature of the tea at time t then $f'(t) = k(M - f(t))$ where $k > 0$ is a constant which will depend on the kind of tea (or more generally the kind of liquid) but not on the room temperature or the temperature of the tea. This is **Newton's law of cooling** and the equation that we just wrote down is an example of a **differential equation**. Ideally we would like to solve this equation, namely, find the function $f(t)$ that describes the temperature over time, though this often turns out to be impossible, in which case various approximation techniques must be used. The use and solution of differential equations is an important field of mathematics; here we see how to solve some simple but useful types of differential equation.

Informally, a differential equation is an equation in which one or more of the derivatives of some function appear. Typically, a scientific theory will produce a differential equation (or a system of differential equations) that describes or governs some physical process, but the theory will not produce the desired function or functions directly.

Recall from section 6.2 that when the variable is time the derivative of a function $y(t)$ is sometimes written as $\dot{y}$ instead of y'; this is quite common in the study of differential equations.

17.1 First Order Differential Equations

We start by considering equations in which only the first derivative of the function appears.

DEFINITION 17.1.1 A **first order differential equation** is an equation of the form $F(t, y, \dot{y}) = 0$. A solution of a first order differential equation is a function $f(t)$ that makes $F(t, f(t), f'(t)) = 0$ for every value of t. □

Here, F is a function of three variables which we label t, y, and $\dot{y}$. It is understood that $\dot{y}$ will explicitly appear in the equation although t and y need not. The term "first order" means that the first derivative of y appears, but no higher order derivatives do.

EXAMPLE 17.1.2 The equation from Newton's law of cooling, $\dot{y} = k(M - y)$ is a first order differential equation; $F(t, y, \dot{y}) = k(M - y) - \dot{y}$. □

EXAMPLE 17.1.3 $\dot{y} = t^2 + 1$ is a first order differential equation; $F(t, y, \dot{y}) = \dot{y} - t^2 - 1$. All solutions to this equation are of the form $t^3/3 + t + C$. □

DEFINITION 17.1.4 A **first order initial value problem** is a system of equations of the form $F(t, y, \dot{y}) = 0$, $y(t_0) = y_0$. Here t_0 is a fixed time and y_0 is a number. A solution of an initial value problem is a solution $f(t)$ of the differential equation that also satisfies the **initial condition** $f(t_0) = y_0$. □

EXAMPLE 17.1.5 The initial value problem $\dot{y} = t^2 + 1$, $y(1) = 4$ has solution $f(t) = t^3/3 + t + 8/3$. □

The general first order equation is rather too general, that is, we can't describe methods that will work on them all, or even a large portion of them. We can make progress with specific kinds of first order differential equations. For example, much can be said about equations of the form $\dot{y} = \phi(t, y)$ where ϕ is a function of the two variables t and y. Under reasonable conditions on ϕ, such an equation has a solution and the corresponding initial value problem has a unique solution. However, in general, these equations can be very difficult or impossible to solve explicitly.

EXAMPLE 17.1.6 Consider this specific example of an initial value problem for Newton's law of cooling: $\dot{y} = 2(25 - y)$, $y(0) = 40$. We first note that if $y(t_0) = 25$, the right hand side of the differential equation is zero, and so the constant function $y(t) = 25$ is a solution to the differential equation. It is not a solution to the initial value problem, since $y(0) \neq 40$. (The physical interpretation of this constant solution is that if a liquid is at the same temperature as its surroundings, then the liquid will stay at that temperature.)

So long as y is not 25, we can rewrite the differential equation as

$$\begin{aligned}\frac{dy}{dt}\frac{1}{25-y} &= 2\\ \frac{1}{25-y}\,dy &= 2\,dt,\end{aligned}$$

so

$$\int \frac{1}{25-y}\,dy = \int 2\,dt,$$

that is, the two anti-derivatives must be the same except for a constant difference. We can calculate these anti-derivatives and rearrange the results:

$$\begin{aligned}\int \frac{1}{25-y}\,dy &= \int 2\,dt\\ (-1)\ln|25-y| &= 2t + C_0\\ \ln|25-y| &= -2t - C_0 = -2t + C\\ |25-y| &= e^{-2t+C} = e^{-2t}e^{C}\\ y - 25 &= \pm\, e^{C}e^{-2t}\\ y &= 25 \pm e^{C}e^{-2t} = 25 + Ae^{-2t}.\end{aligned}$$

Here $A = \pm\, e^{C} = \pm\, e^{-C_0}$ is some non-zero constant. Since we want $y(0) = 40$, we substitute and solve for A:

$$\begin{aligned}40 &= 25 + Ae^{0}\\ 15 &= A,\end{aligned}$$

and so $y = 25 + 15e^{-2t}$ is a solution to the initial value problem. Note that y is never 25, so this makes sense for all values of t. However, if we allow $A = 0$ we get the solution $y = 25$ to the differential equation, which would be the solution to the initial value problem if we were to require $y(0) = 25$. Thus, $y = 25 + Ae^{-2t}$ describes all solutions to the differential equation $\dot{y} = 2(25 - y)$, and all solutions to the associated initial value problems. □

Why could we solve this problem? Our solution depended on rewriting the equation so that all instances of y were on one side of the equation and all instances of t were on the other; of course, in this case the only t was originally hidden, since we didn't write dy/dt in the original equation. This is not required, however.

EXAMPLE 17.1.7 Solve the differential equation $\dot{y} = 2t(25 - y)$. This is almost identical to the previous example. As before, $y(t) = 25$ is a solution. If $y \neq 25$,

$$\begin{aligned}
\int \frac{1}{25-y}\,dy &= \int 2t\,dt \\
(-1)\ln|25-y| &= t^2 + C_0 \\
\ln|25-y| &= -t^2 - C_0 = -t^2 + C \\
|25-y| &= e^{-t^2+C} = e^{-t^2}e^{C} \\
y - 25 &= \pm\, e^{C}e^{-t^2} \\
y &= 25 \pm e^{C}e^{-t^2} = 25 + Ae^{-t^2}.
\end{aligned}$$

As before, all solutions are represented by $y = 25 + Ae^{-t^2}$, allowing A to be zero. □

DEFINITION 17.1.8 A first order differential equation is **separable** if it can be written in the form $\dot{y} = f(t)g(y)$. □

As in the examples, we can attempt to solve a separable equation by converting to the form

$$\int \frac{1}{g(y)}\,dy = \int f(t)\,dt.$$

This technique is called **separation of variables**. The simplest (in principle) sort of separable equation is one in which $g(y) = 1$, in which case we attempt to solve

$$\int 1\,dy = \int f(t)\,dt.$$

We can do this if we can find an anti-derivative of $f(t)$.

Also as we have seen so far, a differential equation typically has an infinite number of solutions. Ideally, but certainly not always, a corresponding initial value problem will have just one solution. A solution in which there are no unknown constants remaining is called a **particular solution**.

The general approach to separable equations is this: Suppose we wish to solve $\dot{y} = f(t)g(y)$ where f and g are continuous functions. If $g(a) = 0$ for some a then $y(t) = a$ is a constant solution of the equation, since in this case $\dot{y} = 0 = f(t)g(a)$. For example, $\dot{y} = y^2 - 1$ has constant solutions $y(t) = 1$ and $y(t) = -1$.

To find the nonconstant solutions, we note that the function $1/g(y)$ is continuous where $g \neq 0$, so $1/g$ has an antiderivative G. Let F be an antiderivative of f. Now we write

$$G(y) = \int \frac{1}{g(y)}\,dy = \int f(t)\,dt = F(t) + C,$$

so $G(y) = F(t) + C$. Now we solve this equation for y.

Of course, there are a few places this ideal description could go wrong: we need to be able to find the antiderivatives G and F, and we need to solve the final equation for y. The upshot is that the solutions to the original differential equation are the constant solutions, if any, and all functions y that satisfy $G(y) = F(t) + C$.

EXAMPLE 17.1.9 Consider the differential equation $\dot{y} = ky$. When $k > 0$, this describes certain simple cases of population growth: it says that the change in the population y is proportional to the population. The underlying assumption is that each organism in the current population reproduces at a fixed rate, so the larger the population the more new organisms are produced. While this is too simple to model most real populations, it is useful in some cases over a limited time. When $k < 0$, the differential equation describes a quantity that decreases in proportion to the current value; this can be used to model radioactive decay.

The constant solution is $y(t) = 0$; of course this will not be the solution to any interesting initial value problem. For the non-constant solutions, we proceed much as before:

$$\begin{aligned} \int \frac{1}{y}\,dy &= \int k\,dt \\ \ln|y| &= kt + C \\ |y| &= e^{kt}e^{C} \\ y &= \pm\, e^{C}e^{kt} \\ y &= Ae^{kt}. \end{aligned}$$

Again, if we allow $A = 0$ this includes the constant solution, and we can simply say that $y = Ae^{kt}$ is the general solution. With an initial value we can easily solve for A to get the solution of the initial value problem. In particular, if the initial value is given for time $t = 0$, $y(0) = y_0$, then $A = y_0$ and the solution is $y = y_0 e^{kt}$. □

Exercises 17.1.

1. Which of the following equations are separable?

- **a.** $\dot{y} = \sin(ty)$
- **b.** $\dot{y} = e^t e^y$
- **c.** $y\dot{y} = t$
- **d.** $\dot{y} = (t^3 - t)\arcsin(y)$
- **e.** $\dot{y} = t^2 \ln y + 4t^3 \ln y$

2. Solve $\dot{y} = 1/(1 + t^2)$. ⇒

3. Solve the initial value problem $\dot{y} = t^n$ with $y(0) = 1$ and $n \geq 0$. ⇒

4. Solve $\dot{y} = \ln t$. ⇒

5. Identify the constant solutions (if any) of $\dot{y} = t \sin y$. ⇒

6. Identify the constant solutions (if any) of $\dot{y} = te^y$. ⇒

7. Solve $\dot{y} = t/y$. ⇒

8. Solve $\dot{y} = y^2 - 1$. ⇒

9. Solve $\dot{y} = t/(y^3 - 5)$. You may leave your solution in implicit form: that is, you may stop once you have done the integration, without solving for y. ⇒

10. Find a non-constant solution of the initial value problem $\dot{y} = y^{1/3}$, $y(0) = 0$, using separation of variables. Note that the constant function $y(t) = 0$ also solves the initial value problem. This shows that an initial value problem can have more than one solution. ⇒

11. Solve the equation for Newton's law of cooling leaving M and k unknown. ⇒

12. After 10 minutes in Jean-Luc's room, his tea has cooled to 40° Celsius from 100° Celsius. The room temperature is 25° Celsius. How much longer will it take to cool to 35°? ⇒

13. Solve the **logistic equation** $\dot{y} = ky(M - y)$. (This is a somewhat more reasonable population model in most cases than the simpler $\dot{y} = ky$.) Sketch the graph of the solution to this equation when $M = 1000$, $k = 0.002$, $y(0) = 1$. ⇒

14. Suppose that $\dot{y} = ky$, $y(0) = 2$, and $\dot{y}(0) = 3$. What is y? ⇒

15. A radioactive substance obeys the equation $\dot{y} = ky$ where $k < 0$ and y is the mass of the substance at time t. Suppose that initially, the mass of the substance is $y(0) = M > 0$. At what time does half of the mass remain? (This is known as the half life. Note that the half life depends on k but not on M.) ⇒

16. Bismuth-210 has a half life of five days. If there is initially 600 milligrams, how much is left after 6 days? When will there be only 2 milligrams left? ⇒

17. The half life of carbon-14 is 5730 years. If one starts with 100 milligrams of carbon-14, how much is left after 6000 years? How long do we have to wait before there is less than 2 milligrams? ⇒

18. A certain species of bacteria doubles its population (or its mass) every hour in the lab. The differential equation that models this phenomenon is $\dot{y} = ky$, where $k > 0$ and y is the population of bacteria at time t. What is y? ⇒

19. If a certain microbe doubles its population every 4 hours and after 5 hours the total population has mass 500 grams, what was the initial mass? ⇒

17.2 First Order Homogeneous Linear Equations

A simple, but important and useful, type of separable equation is the **first order homogeneous linear equation**:

DEFINITION 17.2.1 A first order homogeneous linear differential equation is one of the form $\dot{y} + p(t)y = 0$ or equivalently $\dot{y} = -p(t)y$. □

"Linear" in this definition indicates that both $\dot{y}$ and y occur to the first power; "homogeneous" refers to the zero on the right hand side of the first form of the equation.

EXAMPLE 17.2.2 The equation $\dot{y} = 2t(25 - y)$ can be written $\dot{y} + 2ty = 50t$. This is linear, but not homogeneous. The equation $\dot{y} = ky$, or $\dot{y} - ky = 0$ is linear and homogeneous, with a particularly simple $p(t) = -k$. □

Because first order homogeneous linear equations are separable, we can solve them in the usual way:

$$\begin{aligned}
\dot{y} &= -p(t)y \\
\int \frac{1}{y}\,dy &= \int -p(t)\,dt \\
\ln|y| &= P(t) + C \\
y &= \pm\, e^{P(t)} \\
y &= Ae^{P(t)},
\end{aligned}$$

where $P(t)$ is an anti-derivative of $-p(t)$. As in previous examples, if we allow $A = 0$ we get the constant solution $y = 0$.

EXAMPLE 17.2.3 Solve the initial value problems $\dot{y} + y\cos t = 0$, $y(0) = 1/2$ and $y(2) = 1/2$. We start with

$$P(t) = \int -\cos t\,dt = -\sin t,$$

so the general solution to the differential equation is

$$y = Ae^{-\sin t}.$$

To compute A we substitute:

$$\frac{1}{2} = Ae^{-\sin 0} = A,$$

so the solutions is

$$y = \frac{1}{2}e^{-\sin t}.$$

For the second problem,

$$\begin{aligned}
\frac{1}{2} &= Ae^{-\sin 2} \\
A &= \frac{1}{2}e^{\sin 2}
\end{aligned}$$

so the solution is

$$y = \frac{1}{2}e^{\sin 2}e^{-\sin t}.$$

□

EXAMPLE 17.2.4 Solve the initial value problem $y\dot{y} + 3y = 0$, $y(1) = 2$, assuming $t > 0$. We write the equation in standard form: $\dot{y} + 3y/t = 0$. Then

$$P(t) = \int -\frac{3}{t}\,dt = -3\ln t$$

and

$$y = Ae^{-3\ln t} = At^{-3}.$$

Substituting to find A: $2 = A(1)^{-3} = A$, so the solution is $y = 2t^{-3}$. □

Exercises 17.2.

Find the general solution of each equation in 1–4.

1. $\dot{y} + 5y = 0$ ⇒

2. $\dot{y} - 2y = 0$ ⇒

3. $\dot{y} + \dfrac{y}{1+t^2} = 0$ ⇒

4. $\dot{y} + t^2y = 0$ ⇒

In 5–14, solve the initial value problem.

5. $\dot{y} + y = 0$, $y(0) = 4$ ⇒

6. $\dot{y} - 3y = 0$, $y(1) = -2$ ⇒

7. $\dot{y} + y\sin t = 0$, $y(\pi) = 1$ ⇒

8. $\dot{y} + ye^t = 0$, $y(0) = e$ ⇒

9. $\dot{y} + y\sqrt{1+t^4} = 0$, $y(0) = 0$ ⇒

10. $\dot{y} + y\cos(e^t) = 0$, $y(0) = 0$ ⇒

11. $t\dot{y} - 2y = 0$, $y(1) = 4$ ⇒

12. $t^2\dot{y} + y = 0$, $y(1) = -2$, $t > 0$ ⇒

13. $t^3\dot{y} = 2y$, $y(1) = 1$, $t > 0$ ⇒

14. $t^3\dot{y} = 2y$, $y(1) = 0$, $t > 0$ ⇒

15. A function $y(t)$ is a solution of $\dot{y} + ky = 0$. Suppose that $y(0) = 100$ and $y(2) = 4$. Find k and find $y(t)$. ⇒

16. A function $y(t)$ is a solution of $\dot{y} + t^ky = 0$. Suppose that $y(0) = 1$ and $y(1) = e^{-13}$. Find k and find $y(t)$. ⇒

17. A bacterial culture grows at a rate proportional to its population. If the population is one million at $t = 0$ and 1.5 million at $t = 1$ hour, find the population as a function of time. ⇒

18. A radioactive element decays with a half-life of 6 years. If a mass of the element weighs ten pounds at $t = 0$, find the amount of the element at time t. ⇒

17.3 First Order Linear Equations

As you might guess, a first order linear differential equation has the form $\dot{y}+p(t)y=f(t)$. Not only is this closely related in form to the first order homogeneous linear equation, we can use what we know about solving homogeneous equations to solve the general linear equation.

Suppose that $y_1(t)$ and $y_2(t)$ are solutions to $\dot{y}+p(t)y=f(t)$. Let $g(t)=y_1-y_2$. Then

$$\begin{aligned} g'(t)+p(t)g(t) &= y_1'-y_2'+p(t)(y_1-y_2) \\ &= (y_1'+p(t)y_1)-(y_2'+p(t)y_2) \\ &= f(t)-f(t)=0. \end{aligned}$$

In other words, $g(t)=y_1-y_2$ is a solution to the homogeneous equation $\dot{y}+p(t)y=0$. Turning this around, any solution to the linear equation $\dot{y}+p(t)y=f(t)$, call it y_1, can be written as $y_2+g(t)$, for some particular y_2 and some solution $g(t)$ of the homogeneous equation $\dot{y}+p(t)y=0$. Since we already know how to find all solutions of the homogeneous equation, finding just one solution to the equation $\dot{y}+p(t)y=f(t)$ will give us all of them.

How might we find that one particular solution to $\dot{y}+p(t)y=f(t)$? Again, it turns out that what we already know helps. We know that the general solution to the homogeneous equation $\dot{y}+p(t)y=0$ looks like $Ae^{P(t)}$. We now make an inspired guess: consider the function $v(t)e^{P(t)}$, in which we have replaced the constant parameter A with the function $v(t)$. This technique is called **variation of parameters**. For convenience write this as $s(t)=v(t)h(t)$ where $h(t)=e^{P(t)}$ is a solution to the homogeneous equation. Now let's compute a bit with $s(t)$:

$$\begin{aligned} s'(t)+p(t)s(t) &= v(t)h'(t)+v'(t)h(t)+p(t)v(t)h(t) \\ &= v(t)(h'(t)+p(t)h(t))+v'(t)h(t) \\ &= v'(t)h(t). \end{aligned}$$

The last equality is true because $h'(t)+p(t)h(t)=0$, since $h(t)$ is a solution to the homogeneous equation. We are hoping to find a function $s(t)$ so that $s'(t)+p(t)s(t)=f(t)$; we will have such a function if we can arrange to have $v'(t)h(t)=f(t)$, that is, $v'(t)=f(t)/h(t)$. But this is as easy (or hard) as finding an anti-derivative of $f(t)/h(t)$. Putting this all together, the general solution to $\dot{y}+p(t)y=f(t)$ is

$$v(t)h(t)+Ae^{P(t)}=v(t)e^{P(t)}+Ae^{P(t)}.$$

EXAMPLE 17.3.1 Find the solution of the initial value problem $\dot{y}+3y/t=t^2$, $y(1)=1/2$. First we find the general solution; since we are interested in a solution with a given condition at $t = 1$, we may assume $t > 0$. We start by solving the homogeneous equation as usual; call the solution g:

$$g = Ae^{-\int (3/t)\,dt} = Ae^{-3\ln t} = At^{-3}.$$

Then as in the discussion, $h(t) = t^{-3}$ and $v'(t) = t^2/t^{-3} = t^5$, so $v(t) = t^6/6$. We know that every solution to the equation looks like

$$v(t)t^{-3} + At^{-3} = \frac{t^6}{6}t^{-3} + At^{-3} = \frac{t^3}{6} + At^{-3}.$$

Finally we substitute to find A:

$$\begin{aligned}\frac{1}{2} &= \frac{(1)^3}{6} + A(1)^{-3} = \frac{1}{6} + A\\ A &= \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.\end{aligned}$$

The solution is then

$$y = \frac{t^3}{6} + \frac{1}{3}t^{-3}.$$

□

Here is an alternate method for finding a particular solution to the differential equation, using an **integrating factor**. In the differential equation $\dot{y} + p(t)y = f(t)$, we note that if we multiply through by a function $I(t)$ to get $I(t)\dot{y} + I(t)p(t)y = I(t)f(t)$, the left hand side looks like it could be a derivative computed by the product rule:

$$\frac{d}{dt}(I(t)y) = I(t)\dot{y} + I'(t)y.$$

Now if we could choose $I(t)$ so that $I'(t) = I(t)p(t)$, this would be exactly the left hand side of the differential equation. But this is just a first order homogeneous linear equation, and we know a solution is $I(t) = e^{Q(t)}$, where $Q(t) = \int p\,dt$; note that $Q(t) = -P(t)$, where $P(t)$ appears in the variation of parameters method and $P'(t) = -p$. Now the modified differential equation is

$$\begin{aligned}e^{-P(t)}\dot{y} + e^{-P(t)}p(t)y &= e^{-P(t)}f(t)\\ \frac{d}{dt}(e^{-P(t)}y) &= e^{-P(t)}f(t).\end{aligned}$$

Integrating both sides gives

$$\begin{aligned} e^{-P(t)}y &= \int e^{-P(t)} f(t)\,dt \\ y &= e^{P(t)} \int e^{-P(t)} f(t)\,dt. \end{aligned}$$

If you look carefully, you will see that this is exactly the same solution we found by variation of parameters, because $e^{-P(t)} f(t) = f(t)/h(t)$.

Some people find it easier to remember how to use the integrating factor method than variation of parameters. Since ultimately they require the same calculation, you should use whichever of the two you find easier to recall. Using this method, the solution of the previous example would look just a bit different: Starting with $\dot{y}+3y/t = t^2$, we recall that the integrating factor is $e^{\int 3/t} = e^{3\ln t} = t^3$. Then we multiply through by the integrating factor and solve:

$$\begin{aligned} t^3\dot{y} + t^3 3y/t &= t^3 t^2 \\ t^3\dot{y} + t^2 3y &= t^5 \\ \frac{d}{dt}(t^3 y) &= t^5 \\ t^3 y &= t^6/6 \\ y &= t^3/6. \end{aligned}$$

This is the same answer, of course, and the problem is then finished just as before.

Exercises 17.3.

In problems 1–10, find the general solution of the equation.

1. $\dot{y} + 4y = 8$ ⇒

2. $\dot{y} - 2y = 6$ ⇒

3. $\dot{y} + ty = 5t$ ⇒

4. $\dot{y} + e^t y = -2e^t$ ⇒

5. $\dot{y} - y = t^2$ ⇒

6. $2\dot{y} + y = t$ ⇒

7. $t\dot{y} - 2y = 1/t$, $t > 0$ ⇒

8. $t\dot{y} + y = \sqrt{t}$, $t > 0$ ⇒

9. $\dot{y}\cos t + y\sin t = 1$, $-\pi/2 < t < \pi/2$ ⇒

10. $\dot{y} + y\sec t = \tan t$, $-\pi/2 < t < \pi/2$ ⇒

17.4 Approximation

We have seen how to solve a restricted collection of differential equations, or more accurately, how to attempt to solve them—we may not be able to find the required antiderivatives. Not surprisingly, non-linear equations can be even more difficult to solve. Yet much is known about solutions to some more general equations.

Suppose $\phi(t,y)$ is a function of two variables. A more general class of first order differential equations has the form $\dot y=\phi(t,y)$. This is not necessarily a linear first order equation, since ϕ may depend on y in some complicated way; note however that $\dot y$ appears in a very simple form. Under suitable conditions on the function ϕ, it can be shown that every such differential equation has a solution, and moreover that for each initial condition the associated initial value problem has exactly one solution. In practical applications this is obviously a very desirable property.

EXAMPLE 17.4.1 The equation $\dot y=t-y^2$ is a first order non-linear equation, because y appears to the second power. We will not be able to solve this equation. □

EXAMPLE 17.4.2 The equation $\dot y=y^2$ is also non-linear, but it is separable and can be solved by separation of variables. □

Not all differential equations that are important in practice can be solved exactly, so techniques have been developed to approximate solutions. We describe one such technique, **Euler's Method**, which is simple though not particularly useful compared to some more sophisticated techniques.

Suppose we wish to approximate a solution to the initial value problem $\dot y=\phi(t,y)$, $y(t_0)=y_0$, for $t\ge t_0$. Under reasonable conditions on ϕ, we know the solution exists, represented by a curve in the t-y plane; call this solution $f(t)$. The point (t_0,y_0) is of course on this curve. We also know the slope of the curve at this point, namely $\phi(t_0,y_0)$. If we follow the tangent line for a brief distance, we arrive at a point that should be almost on the graph of $f(t)$, namely $(t_0+\Delta t, y_0+\phi(t_0,y_0)\Delta t)$; call this point (t_1,y_1). Now we pretend, in effect, that this point really is on the graph of $f(t)$, in which case we again know the slope of the curve through (t_1,y_1), namely $\phi(t_1,y_1)$. So we can compute a new point, $(t_2,y_2)=(t_1+\Delta t, y_1+\phi(t_1,y_1)\Delta t)$ that is a little farther along, still close to the graph of $f(t)$ but probably not quite so close as (t_1,y_1). We can continue in this way, doing a sequence of straightforward calculations, until we have an approximation (t_n,y_n) for whatever time t_n we need. At each step we do essentially the same calculation, namely

$$(t_{i+1},y_{i+1})=(t_i+\Delta t, y_i+\phi(t_i,y_i)\Delta t).$$

We expect that smaller time steps Δt will give better approximations, but of course it will require more work to compute to a specified time. It is possible to compute a guaranteed upper bound on how far off the approximation might be, that is, how far y_n is from $f(t_n)$. Suffice it to say that the bound is not particularly good and that there are other more complicated approximation techniques that do better.

EXAMPLE 17.4.3 Let us compute an approximation to the solution for $\dot{y} = t - y^2$, $y(0) = 0$, when $t = 1$. We will use $\Delta t = 0.2$, which is easy to do even by hand, though we should not expect the resulting approximation to be very good. We get

$$
\begin{aligned}
(t_1, y_1) &= (0 + 0.2, 0 + (0 - 0^2)0.2) = (0.2, 0)\\
(t_2, y_2) &= (0.2 + 0.2, 0 + (0.2 - 0^2)0.2) = (0.4, 0.04)\\
(t_3, y_3) &= (0.6, 0.04 + (0.4 - 0.04^2)0.2) = (0.6, 0.11968)\\
(t_4, y_4) &= (0.8, 0.11968 + (0.6 - 0.11968^2)0.2) = (0.8, 0.23681533952)\\
(t_5, y_5) &= (1.0, 0.23681533952 + (0.6 - 0.23681533952^2)0.2) = (1.0, 0.385599038513605)
\end{aligned}
$$

So $y(1) \approx 0.3856$. As it turns out, this is not accurate to even one decimal place. Figure 17.4.1 shows these points connected by line segments (the lower curve) compared to a solution obtained by a much better approximation technique. Note that the shape is approximately correct even though the end points are quite far apart.

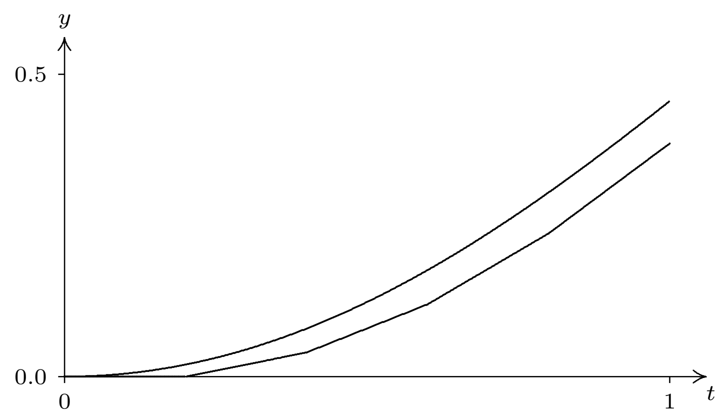


Figure 17.4.1 Approximating a solution to $\dot{y} = t - y^2$, $y(0) = 0$.

If you need to do Euler's method by hand, it is useful to construct a table to keep track of the work, as shown in figure 17.4.2. Each row holds the computation for a single step: the starting point (t_i, y_i); the stepsize Δt; the computed slope $\phi(t_i, y_i)$; the change in y, $\Delta y = \phi(t_i, y_i)\Delta t$; and the new point, $(t_{i+1}, y_{i+1}) = (t_i + \Delta t, y_i + \Delta y)$. The starting point in each row is the newly computed point from the end of the previous row.

It is easy to write a short function in Sage to do Euler's method; see this Sage worksheet. □

(t, y)	Δt	$\phi(t, y)$	$\Delta y = \phi(t, y)\Delta t$	$(t + \Delta t, y + \Delta y)$
$(0, 0)$	0.2	0	0	$(0.2, 0)$
$(0.2, 0)$	0.2	0.2	0.04	$(0.4, 0.04)$
$(0.4, 0.04)$	0.2	0.3984	0.07968	$(0.6, 0.11968)$
$(0.6, 0.11968)$	0.2	$0.58\ldots$	$0.117\ldots$	$(0.8, 0.236\ldots)$
$(0.8, 0.236\ldots)$	0.2	$0.743\ldots$	$0.148\ldots$	$(1.0, 0.385\ldots)$

Figure 17.4.2 Computing with Euler's Method.

Euler's method is related to another technique that can help in understanding a differential equation in a qualitative way. Euler's method is based on the ability to compute the slope of a solution curve at any point in the plane, simply by computing $\phi(t, y)$. If we compute $\phi(t, y)$ at many points, say in a grid, and plot a small line segment with that slope at the point, we can get an idea of how solution curves must look. Such a plot is called a **slope field**. A slope field for $\phi = t - y^2$ is shown in figure 17.4.3; compare this to figure 17.4.1. With a little practice, one can sketch reasonably accurate solution curves based on the slope field, in essence doing Euler's method visually.

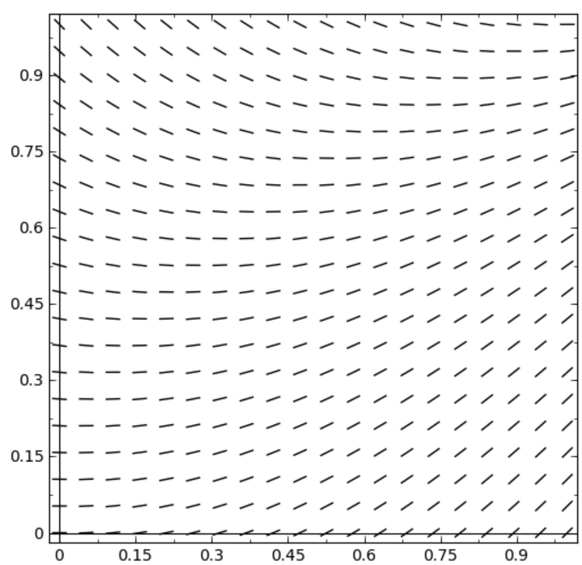


Figure 17.4.3 A slope field for $\dot{y} = t - y^2$.

Even when a differential equation can be solved explicitly, the slope field can help in understanding what the solutions look like with various initial conditions. Recall the logistic equation from exercise 13 in section 17.1, $\dot{y} = ky(M - y)$: y is a population at time t, M is a measure of how large a population the environment can support, and k measures the reproduction rate of the population. Figure 17.4.4 shows a slope field for this equation that is quite informative. It is apparent that if the initial population is smaller than M it rises to M over the long term, while if the initial population is greater than M it decreases to M. It is quite easy to generate slope fields with Sage; follow the AP link in the figure caption.

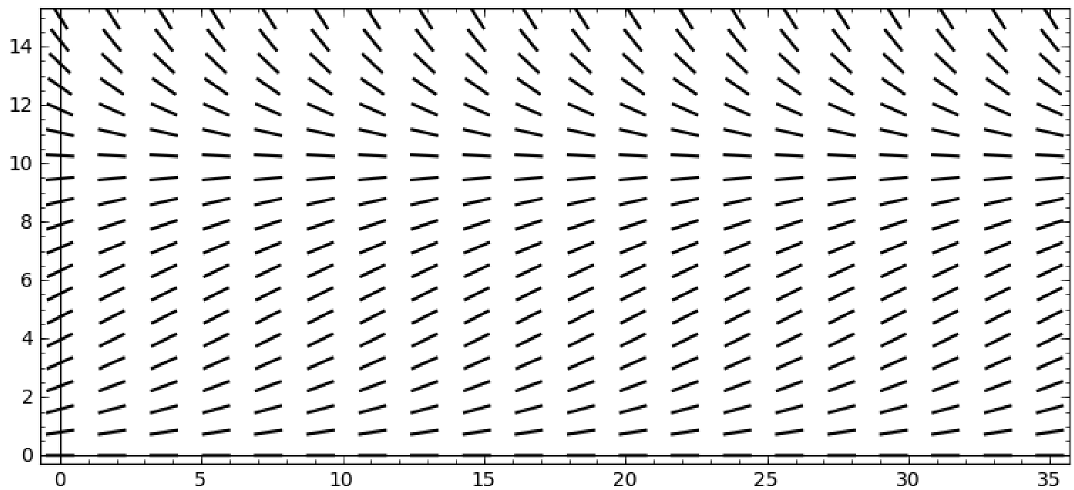


Figure 17.4.4 A slope field for $\dot{y} = 0.2y(10 - y)$.

Exercises 17.4.

In problems 1–4, compute the Euler approximations for the initial value problem for $0 \le t \le 1$ and $\Delta t = 0.2$. If you have access to Sage, generate the slope field first and attempt to sketch the solution curve. Then use Sage to compute better approximations with smaller values of Δt.

1. $\dot{y} = t/y$, $y(0) = 1$ ⇒

2. $\dot{y} = t + y^3$, $y(0) = 1$ ⇒

3. $\dot{y} = \cos(t + y)$, $y(0) = 1$ ⇒

4. $\dot{y} = t \ln y$, $y(0) = 2$ ⇒

17.5 Second Order Homogeneous Equations

A second order differential equation is one containing the second derivative. These are in general quite complicated, but one fairly simple type is useful: the second order linear equation with constant coefficients.

EXAMPLE 17.5.1 Consider the intial value problem $\ddot{y} - \dot{y} - 2y = 0$, $y(0) = 5$, $\dot{y}(0) = 0$. We make an inspired guess: might there be a solution of the form e^{rt}? This seems at least plausible, since in this case $\ddot{y}$, $\dot{y}$, and y all involve e^{rt}.

If such a function is a solution then

$$\begin{aligned}
r^2e^{rt} - re^{rt} - 2e^{rt} &= 0\\
e^{rt}(r^2 - r - 2) &= 0\\
(r^2 - r - 2) &= 0\\
(r-2)(r+1) &= 0,
\end{aligned}$$

so r is 2 or -1. Not only are $f = e^{2t}$ and $g = e^{-t}$ solutions, but notice that $y = Af + Bg$ is also, for any constants A and B:

$$\begin{aligned}
(Af+Bg)'' - (Af+Bg)' - 2(Af+Bg) &= Af'' + Bg'' - Af' - Bg' - 2Af - 2Bg\\
&= A(f'' - f' - 2f) + B(g'' - g' - 2g)\\
&= A(0) + B(0) = 0.
\end{aligned}$$

Can we find A and B so that this is a solution to the initial value problem? Let's substitute:

$$5 = y(0) = Af(0) + Bg(0) = Ae^0 + Be^0 = A + B$$

and

$$0 = \dot{y}(0) = Af'(0) + Bg'(0) = A2e^0 + B(-1)e^0 = 2A - B.$$

So we need to find A and B that make both $5 = A + B$ and $0 = 2A - B$ true. This is a simple set of simultaneous equations: solve $B = 2A$, substitute to get $5 = A + 2A = 3A$. Then $A = 5/3$ and $B = 10/3$, and the desired solution is $(5/3)e^{2t} + (10/3)e^{-t}$. You now see why the initial condition in this case included both $y(0)$ and $\dot{y}(0)$: we needed two equations in the two unknowns A and B □

You should of course wonder whether there might be other solutions; the answer is no. We will not prove this, but here is the theorem that tells us what we need to know:

THEOREM 17.5.2 Given the differential equation $a\ddot{y} + b\dot{y} + cy = 0$, $a \neq 0$, consider the quadratic polynomial $ax^2 + bx + c$, called the **characteristic polynomial**. Using the quadratic formula, this polynomial always has one or two roots, call them r and s. The general solution of the differential equation is:

(a) $y = Ae^{rt} + Be^{st}$, if the roots r and s are real numbers and $r \neq s$.

(b) $y = Ae^{rt} + Bte^{rt}$, if $r = s$ is real.

(c) $y = A\cos(\beta t)e^{\alpha t} + B\sin(\beta t)e^{\alpha t}$, if the roots r and s are complex numbers $\alpha + \beta i$ and $\alpha - \beta i$.

■

EXAMPLE 17.5.3 Suppose a mass m is hung on a spring with spring constant k. If the spring is compressed or stretched and then released, the mass will oscillate up and down. Because of friction, the oscillation will be damped: eventually the motion will cease. The damping will depend on the amount of friction; for example, if the system is suspended in oil the motion will cease sooner than if the system is in air. Using some simple physics, it is not hard to see that the position of the mass is described by this differential equation: $m\ddot{y} + b\dot{y} + ky = 0$. Using $m = 1$, $b = 4$, and $k = 5$ we find the motion of the mass. The characteristic polynomial is $x^2 + 4x + 5$ with roots $(-4 \pm \sqrt{16-20})/2 = -2 \pm i$. Thus the general solution is $y = A\cos(t)e^{-2t} + B\sin(t)e^{-2t}$. Suppose we know that $y(0) = 1$ and $\dot{y}(0) = 2$. Then as before we form two simultaneous equations: from $y(0) = 1$ we get $1 = A\cos(0)e^0 + B\sin(0)e^0 = A$. For the second we compute

$$\ddot{y} = -2Ae^{-2t}\cos(t) + Ae^{-2t}(-\sin(t)) - 2Be^{-2t}\sin(t) + Be^{-2t}\cos(t),$$

and then

$$2 = -2Ae^0\cos(0) - Ae^0\sin(0) - 2Be^0\sin(0) + Be^0\cos(0) = -2A + B.$$

So we get $A = 1$, $B = 4$, and $y = \cos(t)e^{-2t} + 4\sin(t)e^{-2t}$.

Here is a useful trick that makes this easier to understand: We have $y = (\cos t + 4\sin t)e^{-2t}$. The expression $\cos t + 4\sin t$ is a bit reminiscent of the trigonometric formula $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ with $\alpha = t$. Let's rewrite it a bit as

$$\sqrt{17}\left(\frac{1}{\sqrt{17}}\cos t + \frac{4}{\sqrt{17}}\sin t\right).$$

Note that $(1/\sqrt{17})^2 + (4/\sqrt{17})^2 = 1$, which means that there is an angle β with $\cos\beta = 1/\sqrt{17}$ and $\sin\beta = 4/\sqrt{17}$ (of course, β may not be a "nice" angle). Then

$$\cos t + 4\sin t = \sqrt{17}\,(\cos t\cos\beta + \sin\beta\sin t) = \sqrt{17}\cos(t - \beta).$$

Thus, the solution may also be written $y = \sqrt{17}e^{-2t}\cos(t - \beta)$. This is a cosine curve that has been shifted β to the right; the $\sqrt{17}e^{-2t}$ has the effect of diminishing the amplitude of the cosine as t increases; see figure 17.5.1. The oscillation is damped very quickly, so in the first graph it is not clear that this is an oscillation. The second graph shows a restricted range for t. □

Other physical systems that oscillate can also be described by such differential equations. Some electric circuits, for example, generate oscillating current.

EXAMPLE 17.5.4 Find the solution to the intial value problem $\ddot{y} - 4\dot{y} + 4y = 0$, $y(0) = -3$, $\dot{y}(0) = 1$. The characteristic polynomial is $x^2 - 4x + 4 = (x-2)^2$, so there

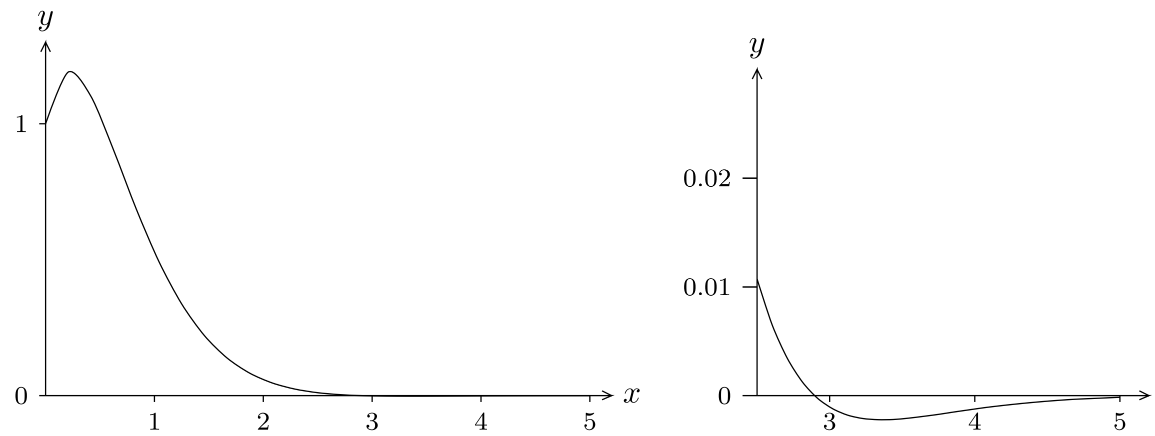


Figure 17.5.1 Graph of a damped oscillation.

is one root, $r = 2$, and the general solution is $Ae^{2t} + Bte^{2t}$. Substituting $t = 0$ we get $-3 = A + 0 = A$. The first derivative is $2Ae^{2t} + 2Bte^{2t} + Be^{2t}$; substituting $t = 0$ gives $1 = 2A + 0 + B = 2A + B = 2(-3) + B = -6 + B$, so $B = 7$. The solution is $-3e^{2t} + 7te^{2t}$. □

Exercises 17.5.

1. Verify that the function in part (a) of theorem 17.5.2 is a solution to the differential equation $a\ddot{y} + b\dot{y} + cy = 0$.

2. Verify that the function in part (b) of theorem 17.5.2 is a solution to the differential equation $a\ddot{y} + b\dot{y} + cy = 0$.

3. Verify that the function in part (c) of theorem 17.5.2 is a solution to the differential equation $a\ddot{y} + b\dot{y} + cy = 0$.

4. Solve the initial value problem $\ddot{y} - \omega^2 y = 0$, $y(0) = 1$, $\dot{y}(0) = 1$, assuming $\omega \neq 0$. ⇒

5. Solve the initial value problem $2\ddot{y} + 18y = 0$, $y(0) = 2$, $\dot{y}(0) = 15$. ⇒

6. Solve the initial value problem $\ddot{y} + 6\dot{y} + 5y = 0$, $y(0) = 1$, $\dot{y}(0) = 0$. ⇒

7. Solve the initial value problem $\ddot{y} - \dot{y} - 12y = 0$, $y(0) = 0$, $\dot{y}(0) = 14$. ⇒

8. Solve the initial value problem $\ddot{y} + 12\dot{y} + 36y = 0$, $y(0) = 5$, $\dot{y}(0) = -10$. ⇒

9. Solve the initial value problem $\ddot{y} - 8\dot{y} + 16y = 0$, $y(0) = -3$, $\dot{y}(0) = 4$. ⇒

10. Solve the initial value problem $\ddot{y} + 5y = 0$, $y(0) = -2$, $\dot{y}(0) = 5$. ⇒

11. Solve the initial value problem $\ddot{y} + y = 0$, $y(\pi/4) = 0$, $\dot{y}(\pi/4) = 2$. ⇒

12. Solve the initial value problem $\ddot{y} + 12\dot{y} + 37y = 0$, $y(0) = 4$, $\dot{y}(0) = 0$. ⇒

13. Solve the initial value problem $\ddot{y} + 6\dot{y} + 18y = 0$, $y(0) = 0$, $\dot{y}(0) = 6$. ⇒

14. Solve the initial value problem $\ddot{y} + 4y = 0$, $y(0) = \sqrt{3}$, $\dot{y}(0) = 2$. Put your answer in the form developed at the end of exercise 17.5.3. ⇒

15. Solve the initial value problem $\ddot{y} + 100y = 0$, $y(0) = 5$, $\dot{y}(0) = 50$. Put your answer in the form developed at the end of exercise 17.5.3. ⇒

16. Solve the initial value problem $\ddot{y} + 4\dot{y} + 13y = 0$, $y(0) = 1$, $\dot{y}(0) = 1$. Put your answer in the form developed at the end of exercise 17.5.3. ⇒

17. Solve the initial value problem $\ddot{y} - 8\dot{y} + 25y = 0$, $y(0) = 3$, $\dot{y}(0) = 0$. Put your answer in the form developed at the end of exercise 17.5.3. ⇒

18. A mass-spring system $m\ddot{y} + b\dot{y} + kx$ has $k = 29$, $b = 4$, and $m = 1$. At time $t = 0$ the position is $y(0) = 2$ and the velocity is $\dot{y}(0) = 1$. Find $y(t)$. ⇒

19. A mass-spring system $m\ddot{y} + b\dot{y} + kx$ has $k = 24$, $b = 12$, and $m = 3$. At time $t = 0$ the position is $y(0) = 0$ and the velocity is $\dot{y}(0) = -1$. Find $y(t)$. ⇒

20. Consider the differential equation $a\ddot{y} + b\dot{y} = 0$, with a and b both non-zero. Find the general solution by the method of this section. Now let $g = \dot{y}$; the equation may be written as $a\dot{g} + bg = 0$, a first order linear homogeneous equation. Solve this for g, then use the relationship $g = \dot{y}$ to find y.

21. Suppose that $y(t)$ is a solution to $a\ddot{y} + b\dot{y} + cy = 0$, $y(t_0) = 0$, $\dot{y}(t_0) = 0$. Show that $y(t) = 0$.

17.6 Second Order Linear Equations

Now we consider second order equations of the form $a\ddot{y} + b\dot{y} + cy = f(t)$, with a, b, and c constant. Of course, if $a = 0$ this is really a first order equation, so we assume $a \neq 0$. Also, much as in exercise 20 of section 17.5, if $c = 0$ we can solve the related first order equation $a\dot{h} + bh = f(t)$, and then solve $h = \dot{y}$ for y. So we will only examine examples in which $c \neq 0$.

Suppose that $y_1(t)$ and $y_2(t)$ are solutions to $a\ddot{y} + b\dot{y} + cy = f(t)$, and consider the function $h = y_1 - y_2$. We substitute this function into the left hand side of the differential equation and simplify:

$$a(y_1 - y_2)'' + b(y_1 - y_2)' + c(y_1 - y_2) = ay_1'' + by_1' + cy_1 - (ay_2'' + by_2' + cy_2) = f(t) - f(t) = 0.$$

So h is a solution to the homogeneous equation $a\ddot{y} + b\dot{y} + cy = 0$. Since we know how to find all such h, then with just one particular solution y_2 we can express all possible solutions y_1, namely, $y_1 = h + y_2$, where now h is the general solution to the homogeneous equation. Of course, this is exactly how we approached the first order linear equation.

To make use of this observation we need a method to find a single solution y_2. This turns out to be somewhat more difficult than the first order case, but if $f(t)$ is of a certain simple form, we can find a solution using the **method of undetermined coefficients**, sometimes more whimsically called the **method of judicious guessing**.

EXAMPLE 17.6.1 Solve the differential equation $\ddot{y} - \dot{y} - 6y = 18t^2 + 5$. The general solution of the homogeneous equation is $Ae^{3t} + Be^{-2t}$. We guess that a solution to the non-homogeneous equation might look like $f(t)$ itself, namely, a quadratic $y = at^2 + bt + c$.

Substituting this guess into the differential equation we get

$$\ddot{y} - \dot{y} - 6y = 2a - (2at + b) - 6(at^2 + bt + c) = -6at^2 + (-2a - 6b)t + (2a - b - 6c).$$

We want this to equal $18t^2 + 5$, so we need

$$\begin{aligned} -6a &= 18 \\ -2a - 6b &= 0 \\ 2a - b - 6c &= 5 \end{aligned}$$

This is a system of three equations in three unknowns and is not hard to solve: $a = -3$, $b = 1$, $c = -2$. Thus the general solution to the differential equation is $Ae^{3t} + Be^{-2t} - 3t^2 + t - 2$. □

So the "judicious guess" is a function with the same form as $f(t)$ but with undetermined (or better, yet to be determined) coefficients. This works whenever $f(t)$ is a polynomial.

EXAMPLE 17.6.2 Consider the initial value problem $m\ddot{y} + ky = -mg$, $y(0) = 2$, $\dot{y}(0) = 50$. The left hand side represents a mass-spring system with no damping, i.e., $b = 0$. Unlike the homogeneous case, we now consider the force due to gravity, $-mg$, assuming the spring is vertical at the surface of the earth, so that $g = 980$. To be specific, let us take $m = 1$ and $k = 100$. The general solution to the homogeneous equation is $A\cos(10t) + B\sin(10t)$. For the solution to the non-homogeneous equation we guess simply a constant $y = a$, since $-mg = -980$ is a constant. Then $\ddot{y} + 100y = 100a$ so $a = -980/100 = -9.8$. The desired general solution is then $A\cos(10t) + B\sin(10t) - 9.8$. Substituting the initial conditions we get

$$\begin{aligned} 2 &= A - 9.8 \\ 50 &= 10B \end{aligned}$$

so $A = 11.8$ and $B = 5$ and the solution is $11.8\cos(10t) + 5\sin(10t) - 9.8$. □

More generally, this method can be used when a function similar to $f(t)$ has derivatives that are also similar to $f(t)$; in the examples so far, since $f(t)$ was a polynomial, so were its derivatives. The method will work if $f(t)$ has the form $p(t)e^{\alpha t}\cos(\beta t) + q(t)e^{\alpha t}\sin(\beta t)$, where $p(t)$ and $q(t)$ are polynomials; when $\alpha = \beta = 0$ this is simply $p(t)$, a polynomial. In the most general form it is not simple to describe the appropriate judicious guess; we content ourselves with some examples to illustrate the process.

EXAMPLE 17.6.3 Find the general solution to $\ddot{y} + 7\dot{y} + 10y = e^{3t}$. The characteristic equation is $r^2 + 7r + 10 = (r + 5)(r + 2)$, so the solution to the homogeneous equation is

$Ae^{-5t} + Be^{-2t}$. For a particular solution to the inhomogeneous equation we guess Ce^{3t}. Substituting we get

$$9Ce^{3t} + 21Ce^{3t} + 10Ce^{3t} = e^{3t}40C.$$

When $C = 1/40$ this is equal to $f(t) = e^{3t}$, so the solution is $Ae^{-5t} + Be^{-2t} + (1/40)e^{3t}$. □

EXAMPLE 17.6.4 Find the general solution to $\ddot{y} + 7\dot{y} + 10y = e^{-2t}$. Following the last example we might guess Ce^{-2t}, but since this is a solution to the homogeneous equation it cannot work. Instead we guess Cte^{-2t}. Then

$$(-2Ce^{-2t} - 2Ce^{-2t} + 4Cte^{-2t}) + 7(Ce^{-2t} - 2Cte^{-2t}) + 10Cte^{-2t} = e^{-2t}(-3C).$$

Then $C = -1/3$ and the solution is $Ae^{-5t} + Be^{-2t} - (1/3)te^{-2t}$. □

In general, if $f(t) = e^{kt}$ and k is one of the roots of the characteristic equation, then we guess Cte^{kt} instead of Ce^{kt}. If k is the only root of the characteristic equation, then Cte^{kt} will not work, and we must guess Ct^2e^{kt}.

EXAMPLE 17.6.5 Find the general solution to $\ddot{y} - 6\dot{y} + 9y = e^{3t}$. The characteristic equation is $r^2 - 6r + 9 = (r-3)^2$, so the general solution to the homogeneous equation is $Ae^{3t} + Bte^{3t}$. Guessing Ct^2e^{3t} for the particular solution, we get

$$(9Ct^2e^{3t} + 6Cte^{3t} + 6Cte^{3t} + 2Ce^{3t}) - 6(3Ct^2e^{3t} + 2Cte^{3t}) + 9Ct^2e^{3t} = e^{3t}2C.$$

The solution is thus $Ae^{3t} + Bte^{3t} + (1/2)t^2e^{3t}$. □

It is common in various physical systems to encounter an $f(t)$ of the form $a\cos(\omega t) + b\sin(\omega t)$.

EXAMPLE 17.6.6 Find the general solution to $\ddot{y} + 6\dot{y} + 25y = \cos(4t)$. The roots of the characteristic equation are $-3 \pm 4i$, so the solution to the homogeneous equation is $e^{-3t}(A\cos(4t) + B\sin(4t))$. For a particular solution, we guess $C\cos(4t) + D\sin(4t)$. Substituting as usual:

$$\begin{aligned}(-16C\cos(4t) + -16D\sin(4t)) + 6(-4C\sin(4t) + 4D\cos(4t)) + 25(C\cos(4t) + D\sin(4t))\\ = (24D + 9C)\cos(4t) + (-24C + 9D)\sin(4t).\end{aligned}$$

To make this equal to $\cos(4t)$ we need

$$\begin{aligned}24D + 9C &= 1\\ 9D - 24C &= 0\end{aligned}$$

which gives $C = 1/73$ and $D = 8/219$. The full solution is then $e^{-3t}(A\cos(4t) + B\sin(4t)) + (1/73)\cos(4t) + (8/219)\sin(4t)$.

The function $e^{-3t}(A\cos(4t) + B\sin(4t))$ is a damped oscillation as in example 17.5.3, while $(1/73)\cos(4t) + (8/219)\sin(4t)$ is a simple undamped oscillation. As t increases, the sum $e^{-3t}(A\cos(4t) + B\sin(4t))$ approaches zero, so the solution

$$e^{-3t}(A\cos(4t) + B\sin(4t)) + (1/73)\cos(4t) + (8/219)\sin(4t)$$

becomes more and more like the simple oscillation $(1/73)\cos(4t) + (8/219)\sin(4t)$—notice that the initial conditions don't matter to this long term behavior. The damped portion is called the **transient** part of the solution, and the simple oscillation is called the **steady state** part of the solution. A physical example is a mass-spring system. If the only force on the mass is due to the spring, then the behavior of the system is a damped oscillation. If in addition an external force is applied to the mass, and if the force varies according to a function of the form $a\cos(\omega t) + b\sin(\omega t)$, then the long term behavior will be a simple oscillation determined by the steady state part of the general solution; the initial position of the mass will not matter. □

As with the exponential form, such a simple guess may not work.

EXAMPLE 17.6.7 Find the general solution to $\ddot{y} + 16y = -\sin(4t)$. The roots of the characteristic equation are $\pm 4i$, so the solution to the homogeneous equation is $A\cos(4t) + B\sin(4t)$. Since both $\cos(4t)$ and $\sin(4t)$ are solutions to the homogeneous equation, $C\cos(4t) + D\sin(4t)$ is also, so it cannot be a solution to the non-homogeneous equation. Instead, we guess $Ct\cos(4t) + Dt\sin(4t)$. Then substituting:

$$\begin{aligned}(-16Ct\cos(4t) - 16D\sin(4t) + 8D\cos(4t) - 8C\sin(4t))) + 16(Ct\cos(4t) + Dt\sin(4t))\\ = 8D\cos(4t) - 8C\sin(4t).\end{aligned}$$

Thus $C = 1/8$, $D = 0$, and the solution is $C\cos(4t) + D\sin(4t) + (1/8)t\cos(4t)$. □

In general, if $f(t) = a\cos(\omega t) + b\sin(\omega t)$, and $\pm\omega i$ are the roots of the characteristic equation, then instead of $C\cos(\omega t) + D\sin(\omega t)$ we guess $Ct\cos(\omega t) + Dt\sin(\omega t)$.

Exercises 17.6.

Find the general solution to the differential equation.

1. $\ddot{y} - 10\dot{y} + 25y = \cos t$ ⇒

2. $\ddot{y} + 2\sqrt{2}\dot{y} + 2y = 10$ ⇒

3. $\ddot{y} + 16y = 8t^2 + 3t - 4$ ⇒

4. $\ddot{y} + 2y = \cos(5t) + \sin(5t)$ ⇒

5. $\ddot{y} - 2\dot{y} + 2y = e^{2t}$ ⇒

6. $\ddot{y} - 6y + 13 = 1 + 2t + e^{-t}$ ⇒

7. $\ddot{y} + \dot{y} - 6y = e^{-3t}$ ⇒

8. $\ddot{y} - 4\dot{y} + 3y = e^{3t}$ ⇒

9. $\ddot{y} + 16y = \cos(4t)$ ⇒

10. $\ddot{y} + 9y = 3\sin(3t)$ ⇒

11. $\ddot{y} + 12\dot{y} + 36y = 6e^{-6t}$ ⇒

12. $\ddot{y} - 8\dot{y} + 16y = -2e^{4t}$ ⇒

13. $\ddot{y} + 6\dot{y} + 5y = 4$ ⇒

14. $\ddot{y} - \dot{y} - 12y = t$ ⇒

15. $\ddot{y} + 5y = 8\sin(2t)$ ⇒

16. $\ddot{y} - 4y = 4e^{2t}$ ⇒

Solve the initial value problem.

17. $\ddot{y} - y = 3t + 5$, $y(0) = 0$, $\dot{y}(0) = 0$ ⇒

18. $\ddot{y} + 9y = 4t$, $y(0) = 0$, $\dot{y}(0) = 0$ ⇒

19. $\ddot{y} + 12\dot{y} + 37y = 10e^{-4t}$, $y(0) = 4$, $\dot{y}(0) = 0$ ⇒

20. $\ddot{y} + 6\dot{y} + 18y = \cos t - \sin t$, $y(0) = 0$, $\dot{y}(0) = 2$ ⇒

21. Find the solution for the mass-spring equation $\ddot{y} + 4\dot{y} + 29y = 689\cos(2t)$. ⇒

22. Find the solution for the mass-spring equation $3\ddot{y} + 12\dot{y} + 24y = 2\sin t$. ⇒

23. Consider the differential equation $m\ddot{y} + b\dot{y} + ky = \cos(\omega t)$, with m, b, and k all positive and $b^2 < 2mk$; this equation is a model for a damped mass-spring system with external driving force $\cos(\omega t)$. Show that the steady state part of the solution has amplitude

$$\frac{1}{\sqrt{(k - m\omega^2)^2 + \omega^2 b^2}}.$$

Show that this amplitude is largest when $\omega = \dfrac{\sqrt{4mk - 2b^2}}{2m}$. This is the **resonant frequency** of the system.

17.7 Second Order Linear Equations, take two

The method of the last section works only when the function $f(t)$ in $a\ddot{y} + b\dot{y} + cy = f(t)$ has a particularly nice form, namely, when the derivatives of f look much like f itself. In other cases we can try variation of parameters as we did in the first order case.

Since as before $a \neq 0$, we can always divide by a to make the coefficient of $\ddot{y}$ equal to 1. Thus, to simplify the discussion, we assume $a = 1$. We know that the differential equation $\ddot{y} + b\dot{y} + cy = 0$ has a general solution $Ay_1 + By_2$. As before, we guess a particular solution to $\ddot{y} + b\dot{y} + cy = f(t)$; this time we use the guess $y = u(t)y_1 + v(t)y_2$. Compute the derivatives:

$$\dot{y} = \dot{u}y_1 + u\dot{y}_1 + \dot{v}y_2 + v\dot{y}_2$$

$$\ddot{y} = \ddot{u}y_1 + \dot{u}\dot{y}_1 + \dot{u}\dot{y}_1 + u\ddot{y}_1 + \ddot{v}y_2 + \dot{v}\dot{y}_2 + \dot{v}\dot{y}_2 + v\ddot{y}_2.$$

Now substituting:

$$\begin{aligned}\ddot{y} + b\dot{y} + cy &= \ddot{u}y_1 + \dot{u}\dot{y}_1 + \dot{u}\dot{y}_1 + u\ddot{y}_1 + \ddot{v}y_2 + \dot{v}\dot{y}_2 + \dot{v}\dot{y}_2 + v\ddot{y}_2\\ &\qquad + b\dot{u}y_1 + bu\dot{y}_1 + b\dot{v}y_2 + bv\dot{y}_2 + cuy_1 + cvy_2\\ &= (u\ddot{y}_1 + bu\dot{y}_1 + cuy_1) + (v\ddot{y}_2 + bv\dot{y}_2 + cvy_2)\\ &\qquad + b(\dot{u}y_1 + \dot{v}y_2) + (\ddot{u}y_1 + \dot{u}\dot{y}_1 + \ddot{v}y_2 + \dot{v}\dot{y}_2) + (\dot{u}\dot{y}_1 + \dot{v}\dot{y}_2)\\ &= 0 + 0 + b(\dot{u}y_1 + \dot{v}y_2) + (\ddot{u}y_1 + \dot{u}\dot{y}_1 + \ddot{v}y_2 + \dot{v}\dot{y}_2) + (\dot{u}\dot{y}_1 + \dot{v}\dot{y}_2).\end{aligned}$$

The first two terms in parentheses are zero because y_1 and y_2 are solutions to the associated homogeneous equation. Now we engage in some wishful thinking. If $\dot{u}y_1 + \dot{v}y_2 = 0$ then also $\ddot{u}y_1 + \dot{u}\dot{y}_1 + \ddot{v}y_2 + \dot{v}\dot{y}_2 = 0$, by taking derivatives of both sides. This reduces the entire expression to $\dot{u}\dot{y}_1 + \dot{v}\dot{y}_2$. We want this to be $f(t)$, that is, we need $\dot{u}\dot{y}_1 + \dot{v}\dot{y}_2 = f(t)$. So we would very much like these equations to be true:

$$\begin{aligned}\dot{u}y_1 + \dot{v}y_2 &= 0\\ \dot{u}\dot{y}_1 + \dot{v}\dot{y}_2 &= f(t).\end{aligned}$$

This is a system of two equations in the two unknowns $\dot{u}$ and $\dot{v}$, so we can solve as usual to get $\dot{u} = g(t)$ and $\dot{v} = h(t)$. Then we can find u and v by computing antiderivatives. This is of course the sticking point in the whole plan, since the antiderivatives may be impossible to find. Nevertheless, this sometimes works out and is worth a try.

EXAMPLE 17.7.1 Consider the equation $\ddot{y} - 5\dot{y} + 6y = \sin t$. We can solve this by the method of undetermined coefficients, but we will use variation of parameters. The solution to the homogeneous equation is $Ae^{2t} + Be^{3t}$, so the simultaneous equations to be solved are

$$\begin{aligned}\dot{u}e^{2t} + \dot{v}e^{3t} &= 0\\ 2\dot{u}e^{2t} + 3\dot{v}e^{3t} &= \sin t.\end{aligned}$$

If we multiply the first equation by 2 and subtract it from the second equation we get

$$\begin{aligned}\dot{v}e^{3t} &= \sin t\\ \dot{v} &= e^{-3t}\sin t\\ v &= -\frac{1}{10}(3\sin t + \cos t)e^{-3t},\end{aligned}$$

using integration by parts. Then from the first equation:

$$\begin{aligned}\dot{u} &= -e^{-2t}\dot{v}e^{3t} = -e^{-2t}e^{-3t}\sin(t)e^{3t} = -e^{-2t}\sin t\\ u &= \frac{1}{5}(2\sin t + \cos t)e^{-2t}.\end{aligned}$$

Now the particular solution we seek is

$$\begin{aligned} ue^{2t} + ve^{3t} &= \frac{1}{5}(2\sin t + \cos t)e^{-2t}e^{2t} - \frac{1}{10}(3\sin t + \cos t)e^{-3t}e^{3t} \\ &= \frac{1}{5}(2\sin t + \cos t) - \frac{1}{10}(3\sin t + \cos t) \\ &= \frac{1}{10}(\sin t + \cos t), \end{aligned}$$

and the solution to the differential equation is $Ae^{2t} + Be^{3t} + (\sin t + \cos t)/10$. For comparison (and practice) you might want to solve this using the method of undetermined coefficients. □

EXAMPLE 17.7.2 The differential equation $\ddot{y} - 5\dot{y} + 6y = e^t \sin t$ can be solved using the method of undetermined coefficients, though we have not seen any examples of such a solution. Again, we will solve it by variation of parameters. The equations to be solved are

$$\begin{aligned} \dot{u}e^{2t} + \dot{v}e^{3t} &= 0 \\ 2\dot{u}e^{2t} + 3\dot{v}e^{3t} &= e^t \sin t. \end{aligned}$$

If we multiply the first equation by 2 and subtract it from the second equation we get

$$\begin{aligned} \dot{v}e^{3t} &= e^t \sin t \\ \dot{v} &= e^{-3t}e^t \sin t = e^{-2t}\sin t \\ v &= -\frac{1}{5}(2\sin t + \cos t)e^{-2t}. \end{aligned}$$

Then substituting we get

$$\begin{aligned} \dot{u} &= -e^{-2t}\dot{v}e^{3t} = -e^{-2t}e^{-2t}\sin(t)e^{3t} = -e^{-t}\sin t \\ u &= \frac{1}{2}(\sin t + \cos t)e^{-t}. \end{aligned}$$

The particular solution is

$$\begin{aligned} ue^{2t} + ve^{3t} &= \frac{1}{2}(\sin t + \cos t)e^{-t}e^{2t} - \frac{1}{5}(2\sin t + \cos t)e^{-2t}e^{3t} \\ &= \frac{1}{2}(\sin t + \cos t)e^{t} - \frac{1}{5}(2\sin t + \cos t)e^{t} \\ &= \frac{1}{10}(\sin t + 3\cos t)e^{t}, \end{aligned}$$

and the solution to the differential equation is $Ae^{2t} + Be^{3t} + e^t(\sin t + 3\cos t)/10$. □

EXAMPLE 17.7.3 The differential equation $\ddot{y} - 2\dot{y} + y = e^t/t^2$ is not of the form amenable to the method of undetermined coefficients. The solution to the homogeneous equation is $Ae^t + Bte^t$ and so the simultaneous equations are

$$\begin{aligned} \dot{u}e^t + \dot{v}te^t &= 0 \\ \dot{u}e^t + \dot{v}te^t + \dot{v}e^t &= \frac{e^t}{t^2}. \end{aligned}$$

Subtracting the equations gives

$$\begin{aligned} \dot{v}e^t &= \frac{e^t}{t^2} \\ \dot{v} &= \frac{1}{t^2} \\ v &= -\frac{1}{t}. \end{aligned}$$

Then substituting we get

$$\begin{aligned} \dot{u}e^t &= -\dot{v}te^t = -\frac{1}{t^2}te^t \\ \dot{u} &= -\frac{1}{t} \\ u &= -\ln t. \end{aligned}$$

The solution is $Ae^t + Bte^t - e^t \ln t - e^t$. □

Exercises 17.7.

Find the general solution to the differential equation using variation of parameters.

1. $\ddot{y} + y = \tan x$ ⇒
2. $\ddot{y} + y = e^{2t}$ ⇒
3. $\ddot{y} + 4y = \sec x$ ⇒
4. $\ddot{y} + 4y = \tan x$ ⇒
5. $\ddot{y} + \dot{y} - 6y = t^2 e^{2t}$ ⇒
6. $\ddot{y} - 2\dot{y} + 2y = e^t \tan(t)$ ⇒
7. $\ddot{y} - 2\dot{y} + 2y = \sin(t)\cos(t)$ (This is rather messy when done by variation of parameters; compare to undetermined coefficients.) ⇒

A

Selected Answers

1.1.1. $(2/3)x + (1/3)$

1.1.2. $y = -2x$

1.1.3. $(-2/3)x + (1/3)$

1.1.4. $y = 2x + 2$, 2, -1

1.1.5. $y = -x + 6$, 6, 6

1.1.6. $y = x/2 + 1/2$, $1/2$, -1

1.1.7. $y = 3/2$, y-intercept: $3/2$, no x-intercept

1.1.8. $y = (-2/3)x - 2$, -2, -3

1.1.9. yes

1.1.10. $y = 0$, $y = -2x + 2$, $y = 2x + 2$

1.1.11. $y = 75t$ (t in hours); 164 minutes

1.1.12. $y = (9/5)x + 32$, $(-40, -40)$

1.1.13. $y = 0.15x + 10$

1.1.14. $0.03x + 1.2$

1.1.15. (a) $y = \begin{cases} 0 & 0 \le x < 100 \\ (x/10) - 10 & 100 \le x \le 1000 \\ x - 910 & 1000 < x \end{cases}$

1.1.16. $y = \begin{cases} 0.15x & 0 \le x \le 19450 \\ 0.28x - 2528.50 & 19450 < x \le 47050 \\ 0.33x - 4881 & 47050 < x \le 97620 \end{cases}$

1.1.17. (a) $P = -0.0001x + 2$
(b) $x = -10000P + 20000$

1.1.18. $(2/25)x - (16/5)$

1.2.1. (a) $x^2 + y^2 = 9$
(b) $(x - 5)^2 + (y - 6)^2 = 9$
(c) $(x + 5)^2 + (y + 6)^2 = 9$

1.2.2. (a) $\Delta x = 2$, $\Delta y = 3$, $m = 3/2$, $y = (3/2)x - 3$, $\sqrt{13}$
(b) $\Delta x = -1$, $\Delta y = 3$, $m = -3$, $y = -3x + 2$, $\sqrt{10}$
(c) $\Delta x = -2$, $\Delta y = -2$, $m = 1$, $y = x$, $\sqrt{8}$

1.2.6. $(x + 2/7)^2 + (y - 41/7)^2 = 1300/49$

1.3.1. $\{x \mid x \ge 3/2\}$

1.3.2. $\{x \mid x \ne -1\}$

1.3.3. $\{x \mid x \ne 1 \text{ and } x \ne -1\}$

1.3.4. $\{x \mid x < 0\}$

1.3.5. $\{x \mid x \in \mathbb{R}\}$, i.e., all x

1.3.6. $\{x \mid x \geq 0\}$

1.3.7. $\{x \mid h - r \leq x \leq h + r\}$

1.3.8. $\{x \mid x \geq 1 \text{ or } x < 0\}$

1.3.9. $\{x \mid -1/3 < x < 1/3\}$

1.3.10. $\{x \mid x \geq 0 \text{ and } x \neq 1\}$

1.3.11. $\{x \mid x \geq 0 \text{ and } x \neq 1\}$

1.3.12. $\mathbb{R}$

1.3.13. $\{x \mid x \geq 3\}$, $\{x \mid x \geq 0\}$

1.3.14. $A = x(500 - 2x)$, $\{x \mid 0 \leq x \leq 250\}$

1.3.15. $V = r(50 - \pi r^2)$, $\{r \mid 0 < r \leq \sqrt{50/\pi}\}$

1.3.16. $A = 2\pi r^2 + 2000/r$, $\{r \mid 0 < r < \infty\}$

2.1.1. -5, -2.47106145, -2.4067927, -2.400676, -2.4

2.1.2. $-4/3$, $-24/7$, $7/24$, $3/4$

2.1.3. -0.107526881, -0.11074197, -0.1110741, $\dfrac{-1}{3(3+\Delta x)} \to \dfrac{-1}{9}$

2.1.4. $\dfrac{3 + 3\Delta x + \Delta x^2}{1 + \Delta x} \to 3$

2.1.5. 3.31, 3.003001, 3.0000, $3 + 3\Delta x + \Delta x^2 \to 3$

2.1.6. m

2.2.1. 10, 25/2, 20, 15, 25, 35.

2.2.2. 5, 4.1, 4.01, 4.001, $4 + \Delta t \to 4$

2.2.3. -10.29, -9.849, -9.8049, $-9.8 - 4.9\Delta t \to -9.8$

2.3.1. 7

2.3.2. 5

2.3.3. 0

2.3.4. undefined

2.3.5. 1/6

2.3.6. 0

2.3.7. 3

2.3.8. 172

2.3.9. 0

2.3.10. 2

2.3.11. does not exist

2.3.12. $\sqrt{2}$

2.3.13. $3a^2$

2.3.14. 512

2.3.15. -4

2.3.16. 0

2.3.18. (a) 8, (b) 6, (c) dne, (d) -2, (e) -1, (f) 8, (g) 7, (h) 6, (i) 3, (j) $-3/2$, (k) 6, (l) 2

2.4.1. $-x/\sqrt{169 - x^2}$

2.4.2. $-9.8t$

2.4.3. $2x + 1/x^2$

2.4.4. $2ax + b$

2.4.5. $3x^2$

2.4.8. $-2/(2x+1)^{3/2}$

2.4.9. $5/(t+2)^2$

2.4.10. $y = -13x + 17$

2.4.11. -8

3.1.1. $100x^{99}$

3.1.2. $-100x^{-101}$

3.1.3. $-5x^{-6}$

3.1.4. $\pi x^{\pi - 1}$

3.1.5. $(3/4)x^{-1/4}$

3.1.6. $-(9/7)x^{-16/7}$

3.2.1. $15x^2 + 24x$

3.2.2. $-20x^4 + 6x + 10/x^3$

3.2.3. $-30x + 25$

3.2.4. $6x^2 + 2x - 8$

3.2.5. $3x^2 + 6x - 1$

3.2.6. $9x^2 - x/\sqrt{625 - x^2}$

3.2.7. $y = 13x/4 + 5$

3.2.8. $y = 24x - 48 - \pi^3$

3.2.9. $-49t/5 + 5$, $-49/5$

3.2.11. $\sum_{k=1}^{n} k a_k x^{k-1}$

3.2.12. $x^3/16 - 3x/4 + 4$

3.3.1. $3x^2(x^3 - 5x + 10) + x^3(3x^2 - 5)$

3.3.2. $(x^2 + 5x - 3)(5x^4 - 18x^2 + 6x - 7) + (2x + 5)(x^5 - 6x^3 + 3x^2 - 7x + 1)$

3.3.3. $\dfrac{\sqrt{625 - x^2}}{2\sqrt{x}} - \dfrac{x\sqrt{x}}{\sqrt{625 - x^2}}$

3.3.4. $\dfrac{-1}{x^{19}\sqrt{625 - x^2}} - \dfrac{20\sqrt{625 - x^2}}{x^{21}}$

3.3.5. $f' = 4(2x - 3)$, $y = 4x - 7$

3.4.1. $\dfrac{3x^2}{x^3 - 5x + 10} - \dfrac{x^3(3x^2 - 5)}{(x^3 - 5x + 10)^2}$

3.4.2. $\dfrac{2x + 5}{x^5 - 6x^3 + 3x^2 - 7x + 1} - \dfrac{(x^2 + 5x - 3)(5x^4 - 18x^2 + 6x - 7)}{(x^5 - 6x^3 + 3x^2 - 7x + 1)^2}$

3.4.3. $\dfrac{1}{2\sqrt{x}\sqrt{625 - x^2}} + \dfrac{x^{3/2}}{(625 - x^2)^{3/2}}$

3.4.4. $\dfrac{-1}{x^{19}\sqrt{625 - x^2}} - \dfrac{20\sqrt{625 - x^2}}{x^{21}}$

3.4.5. $y = 17x/4 - 41/4$

3.4.6. $y = 11x/16 - 15/16$

3.4.8. $y = 19/169 - 5x/338$

3.4.9. $13/18$

3.5.1. $4x^3 - 9x^2 + x + 7$

3.5.2. $3x^2 - 4x + 2/\sqrt{x}$

3.5.3. $6(x^2 + 1)^2 x$

3.5.4. $\sqrt{169 - x^2} - x^2/\sqrt{169 - x^2}$

3.5.5. $(2x - 4)\sqrt{25 - x^2} - (x^2 - 4x + 5)x/\sqrt{25 - x^2}$

3.5.6. $-x/\sqrt{r^2 - x^2}$

3.5.7. $2x^3/\sqrt{1 + x^4}$

3.5.8. $\dfrac{1}{4\sqrt{x}(5 - \sqrt{x})^{3/2}}$

3.5.9. $6 + 18x$

3.5.10. $\dfrac{2x + 1}{1 - x} + \dfrac{x^2 + x + 1}{(1 - x)^2}$

3.5.11. $-1/\sqrt{25 - x^2} - \sqrt{25 - x^2}/x^2$

3.5.12. $\dfrac{1}{2}\left(\dfrac{-169}{x^2} - 1\right) \Big/ \sqrt{\dfrac{169}{x} - x}$

3.5.13. $\dfrac{3x^2 - 2x + 1/x^2}{2\sqrt{x^3 - x^2 - (1/x)}}$

3.5.14. $\dfrac{300x}{(100 - x^2)^{5/2}}$

3.5.15. $\dfrac{1 + 3x^2}{3(x + x^3)^{2/3}}$

3.5.16. $\left(4x(x^2 + 1) + \dfrac{4x^3 + 4x}{2\sqrt{1 + (x^2 + 1)^2}}\right) \Big/ 2\sqrt{(x^2 + 1)^2 + \sqrt{1 + (x^2 + 1)^2}}$

3.5.17. $5(x + 8)^4$

3.5.18. $-3(4 - x)^2$

3.5.19. $6x(x^2 + 5)^2$

3.5.20. $-12x(6 - 2x^2)^2$

3.5.21. $24x^2(1 - 4x^3)^{-3}$

3.5.22. $5 + 5/x^2$

3.5.23. $-8(4x - 1)(2x^2 - x + 3)^{-3}$

3.5.24. $1/(x + 1)^2$

3.5.25. $3(8x - 2)/(4x^2 - 2x + 1)^2$

3.5.26. $-3x^2 + 5x - 1$

3.5.27. $6x(2x - 4)^3 + 6(3x^2 + 1)(2x - 4)^2$

3.5.28. $-2/(x-1)^2$

3.5.29. $4x/(x^2+1)^2$

3.5.30. $(x^2-6x+7)/(x-3)^2$

3.5.31. $-5/(3x-4)^2$

3.5.32. $60x^4+72x^3+18x^2+18x-6$

3.5.33. $(5-4x)/((2x+1)^2(x-3)^2)$

3.5.34. $1/(2(2+3x)^2)$

3.5.35. $56x^6+72x^5+110x^4+100x^3+60x^2+28x+6$

3.5.36. $y=23x/96-29/96$

3.5.37. $y=3-2x/3$

3.5.38. $y=13x/2-23/2$

3.5.39. $y=2x-11$

3.5.40. $y=\dfrac{20+2\sqrt{5}}{5\sqrt{4+\sqrt{5}}}x+\dfrac{3\sqrt{5}}{5\sqrt{4+\sqrt{5}}}$

4.1.1. $2n\pi-\pi/2$, any integer n

4.1.2. $n\pi\pm\pi/6$, any integer n

4.1.3. $(\sqrt{2}+\sqrt{6})/4$

4.1.4. $-(1+\sqrt{3})/(1-\sqrt{3})=2+\sqrt{3}$

4.1.11. $t=\pi/2$

4.3.1. 5

4.3.2. $7/2$

4.3.3. $3/4$

4.3.4. 1

4.3.5. $-\sqrt{2}/2$

4.3.6. 7

4.3.7. 2

4.4.1. $\sin(\sqrt{x})\cos(\sqrt{x})/\sqrt{x}$

4.4.2. $\dfrac{\sin x}{2\sqrt{x}}+\sqrt{x}\cos x$

4.4.3. $-\dfrac{\cos x}{\sin^2 x}$

4.4.4. $\dfrac{(2x+1)\sin x-(x^2+x)\cos x}{\sin^2 x}$

4.4.5. $\dfrac{-\sin x\cos x}{\sqrt{1-\sin^2 x}}$

4.5.1. $\cos^2 x-\sin^2 x$

4.5.2. $-\sin x\cos(\cos x)$

4.5.3. $\dfrac{\tan x+x\sec^2 x}{2\sqrt{x\tan x}}$

4.5.4. $\dfrac{\sec^2 x(1+\sin x)-\tan x\cos x}{(1+\sin x)^2}$

4.5.5. $-\csc^2 x$

4.5.6. $-\csc x\cot x$

4.5.7. $3x^2\sin(23x^2)+46x^4\cos(23x^2)$

4.5.8. 0

4.5.9. $-6\cos(\cos(6x))\sin(6x)$

4.5.10. $\sin\theta/(\cos\theta+1)^2$

4.5.11. $5t^4\cos(6t)-6t^5\sin(6t)$

4.5.12. $3t^2(\sin(3t)+t\cos(3t))/\cos(2t)+2t^3\sin(3t)\sin(2t)/\cos^2(2t)$

4.5.13. $n\pi/2$, any integer n

4.5.14. $\pi/2+n\pi$, any integer n

4.5.15. $\sqrt{3}x/2+3/4-\sqrt{3}\pi/6$

4.5.16. $8\sqrt{3}x+4-8\sqrt{3}\pi/3$

4.5.17. $3\sqrt{3}x/2-\sqrt{3}\pi/4$

4.5.18. $\pi/6+2n\pi$, $5\pi/6+2n\pi$, any integer n

4.7.1. $2\ln(3)x3^{x^2}$

4.7.2. $\dfrac{\cos x-\sin x}{e^x}$

4.7.3. $2e^{2x}$

4.7.4. $e^x\cos(e^x)$

4.7.5. $\cos(x)e^{\sin x}$

4.7.6. $x^{\sin x}\left(\cos x\ln x+\dfrac{\sin x}{x}\right)$

4.7.7. $3x^2e^x + x^3e^x$

4.7.8. $1 + 2^x\ln(2)$

4.7.9. $-2x\ln(3)(1/3)^{x^2}$

4.7.10. $e^{4x}(4x-1)/x^2$

4.7.11. $(3x^2+3)/(x^3+3x)$

4.7.12. $-\tan(x)$

4.7.13. $(1-\ln(x^2))/(x^2\sqrt{\ln(x^2)})$

4.7.14. $\sec(x)$

4.7.15. $x^{\cos(x)}(\cos(x)/x - \sin(x)\ln(x))$

4.7.20. e

4.8.1. x/y

4.8.2. $-(2x+y)/(x+2y)$

4.8.3. $(2xy-3x^2-y^2)/(2xy-3y^2-x^2)$

4.8.4. $\sin(x)\sin(y)/(\cos(x)\cos(y))$

4.8.5. $-\sqrt{y}/\sqrt{x}$

4.8.6. $(y\sec^2(x/y)-y^2)/(x\sec^2(x/y)+y^2)$

4.8.7. $(y-\cos(x+y))/(\cos(x+y)-x)$

4.8.8. $-y^2/x^2$

4.8.9. 1

4.8.12. $y = 2x \pm 6$

4.8.13. $y = x/2 \pm 3$

4.8.14. $(\sqrt{3}, 2\sqrt{3})$, $(-\sqrt{3}, -2\sqrt{3})$, $(2\sqrt{3}, \sqrt{3})$, $(-2\sqrt{3}, -\sqrt{3})$

4.8.15. $y = 7x/\sqrt{3} - 8/\sqrt{3}$

4.8.16. $y = (-y_1^{1/3}x + y_1^{1/3}x_1 + x_1^{1/3}y_1)/x_1^{1/3}$

4.8.17. $(y-y_1)/(x-x_1) = (2x_1^3 + 2x_1y_1^2 - x_1)/(2y_1^3 + 2y_1x_1^2 + y_1)$

4.9.3. $\dfrac{-1}{1+x^2}$

4.9.5. $\dfrac{2x}{\sqrt{1-x^4}}$

4.9.6. $\dfrac{e^x}{1+e^{2x}}$

4.9.7. $-3x^2\cos(x^3)/\sqrt{1-\sin^2(x^3)}$

4.9.8. $\dfrac{2}{(\arcsin x)\sqrt{1-x^2}}$

4.9.9. $-e^x/\sqrt{1-e^{2x}}$

4.9.10. 0

4.9.11. $\dfrac{(1+\ln x)x^x}{\ln 5(1+x^{2x})\arctan(x^x)}$

4.10.1. 0

4.10.2. ∞

4.10.3. 1

4.10.4. 0

4.10.5. 0

4.10.6. 1

4.10.7. $1/6$

4.10.8. $-\infty$

4.10.9. $1/16$

4.10.10. $1/3$

4.10.11. 0

4.10.12. $3/2$

4.10.13. $-1/4$

4.10.14. -3

4.10.15. $1/2$

4.10.16. 0

4.10.17. -1

4.10.18. $-1/2$

4.10.19. 5

4.10.20. ∞

4.10.21. ∞

4.10.22. $2/7$

4.10.23. 2

4.10.24. $-\infty$

4.10.25. 1

4.10.26. 1

4.10.27. 2

4.10.28. 1

4.10.29. 0

4.10.30. 1/2

4.10.31. 2

4.10.32. 0

4.10.33. ∞

4.10.34. 1/2

4.10.35. 0

4.10.36. 1/2

4.10.37. 5

4.10.38. $2\sqrt{2}$

4.10.39. $-1/2$

4.10.40. 2

4.10.41. 0

4.10.42. ∞

4.10.43. 0

4.10.44. 3/2

4.10.45. ∞

4.10.46. 5

4.10.47. $-1/2$

4.10.48. does not exist

4.10.49. ∞

4.10.50. $y = 1$ and $y = -1$

5.1.1. min at $x = 1/2$

5.1.2. min at $x = -1$, max at $x = 1$

5.1.3. max at $x = 2$, min at $x = 4$

5.1.4. min at $x = \pm 1$, max at $x = 0$.

5.1.5. min at $x = 1$

5.1.6. none

5.1.7. none

5.1.8. min at $x = 7\pi/12 + k\pi$, max at $x = -\pi/12 + k\pi$, for integer k.

5.1.9. none

5.1.10. local max at $x = 5$

5.1.11. local min at $x = 49$

5.1.12. local min at $x = 0$

5.1.15. one

5.2.1. min at $x = 1/2$

5.2.2. min at $x = -1$, max at $x = 1$

5.2.3. max at $x = 2$, min at $x = 4$

5.2.4. min at $x = \pm 1$, max at $x = 0$.

5.2.5. min at $x = 1$

5.2.6. none

5.2.7. none

5.2.8. min at $x = 7\pi/12 + k\pi$, max at $x = -\pi/12 + k\pi$, for integer k.

5.2.9. none

5.2.10. max at $x = 0$, min at $x = \pm 11$

5.2.11. min at $x = -3/2$, neither at $x = 0$

5.2.13. min at $n\pi$, max at $\pi/2 + n\pi$

5.2.14. min at $2n\pi$, max at $(2n + 1)\pi$

5.2.15. min at $\pi/2+2n\pi$, max at $3\pi/2+2n\pi$

5.3.1. min at $x = 1/2$

5.3.2. min at $x = -1$, max at $x = 1$

5.3.3. max at $x = 2$, min at $x = 4$

5.3.4. min at $x = \pm 1$, max at $x = 0$.

5.3.5. min at $x = 1$

5.3.6. none

5.3.7. none

5.3.8. min at $x = 7\pi/12 + n\pi$, max at $x = -\pi/12 + n\pi$, for integer n.

5.3.9. max at $x = 63/64$

5.3.10. max at $x = 7$

5.3.11. max at $-5^{-1/4}$, min at $5^{-1/4}$

5.3.12. none

5.3.13. max at -1, min at 1

5.3.14. min at $2^{-1/3}$

5.3.15. none

5.3.16. min at $n\pi$

5.3.17. max at $n\pi$, min at $\pi/2 + n\pi$

5.3.18. max at $\pi/2+2n\pi$, min at $3\pi/2+2n\pi$

5.4.1. concave up everywhere

5.4.2. concave up when $x < 0$, concave down when $x > 0$

5.4.3. concave down when $x < 3$, concave up when $x > 3$

5.4.4. concave up when $x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$, concave down when $-1/\sqrt{3} < x < 1/\sqrt{3}$

5.4.5. concave up when $x < 0$ or $x > 2/3$, concave down when $0 < x < 2/3$

5.4.6. concave up when $x < 0$, concave down when $x > 0$

5.4.7. concave up when $x < -1$ or $x > 1$, concave down when $-1 < x < 0$ or $0 < x < 1$

5.4.8. concave down on $((8n-1)\pi/4, (8n+3)\pi/4)$, concave up on $((8n+3)\pi/4, (8n+7)\pi/4)$, for integer n

5.4.9. concave down everywhere

5.4.10. concave up on $(-\infty, (21-\sqrt{497})/4)$ and $(21+\sqrt{497})/4, \infty)$

5.4.11. concave up on $(0, \infty)$

5.4.12. concave down on $(2n\pi/3, (2n+1)\pi/3)$

5.4.13. concave up on $(0, \infty)$

5.4.14. concave up on $(-\infty, -1)$ and $(0, \infty)$

5.4.15. concave down everywhere

5.4.16. concave up everywhere

5.4.17. concave up on $(\pi/4+n\pi, 3\pi/4+n\pi)$

5.4.18. inflection points at $n\pi$, $\pm\arcsin(\sqrt{2/3}) + n\pi$

5.4.19. up/incr: $(3, \infty)$, up/decr: $(-\infty, 0)$, $(2, 3)$, down/decr: $(0, 2)$

6.1.1. max at $(2, 5)$, min at $(0, 1)$

6.1.2. 25×25

6.1.3. $P/4 \times P/4$

6.1.4. $w = l = 2 \cdot 5^{2/3}$, $h = 5^{2/3}$, $h/w = 1/2$

6.1.5. $\sqrt[3]{100} \times \sqrt[3]{100} \times 2\sqrt[3]{100}$, $h/s = 2$

6.1.6. $w = l = 2^{1/3}V^{1/3}$, $h = V^{1/3}/2^{2/3}$, $h/w = 1/2$

6.1.7. 1250 square feet

6.1.8. $l^2/8$ square feet

6.1.9. \$5000

6.1.10. 100

6.1.11. r^2

6.1.12. $h/r = 2$

6.1.13. $h/r = 2$

6.1.14. $r = 5$ cm, $h = 40/\pi$ cm, $h/r = 8/\pi$

6.1.15. $8/\pi$

6.1.16. $4/27$

6.1.17. Go direct from A to D.

6.1.18. (a) 2, (b) 7/2

6.1.19. $\dfrac{\sqrt{3}}{6} \times \dfrac{\sqrt{3}}{6} + \dfrac{1}{2} \times \dfrac{1}{4} - \dfrac{\sqrt{3}}{12}$

6.1.20. (a) $a/6$, (b) $(a + b - \sqrt{a^2 - ab + b^2})/6$

6.1.21. 1.5 meters wide by 1.25 meters tall

6.1.22. If $k \le 2/\pi$ the ratio is $(2 - k\pi)/4$; if $k \ge 2/\pi$, the ratio is zero: the window should be semicircular with no rectangular part.

6.1.23. a/b

6.1.24. $w = 2r/\sqrt{3}$, $h = 2\sqrt{2}r/\sqrt{3}$

6.1.25. $1/\sqrt{3} \approx 58\%$

6.1.26. $18 \times 18 \times 36$

6.1.27. $r = 5/(2\pi)^{1/3} \approx 2.7$ cm, $h = 5 \cdot 2^{5/3}/\pi^{1/3} = 4r \approx 10.8$ cm

6.1.28. $h = \dfrac{750}{\pi}\left(\dfrac{2\pi^2}{750^2}\right)^{1/3}$, $r = \left(\dfrac{750^2}{2\pi^2}\right)^{1/6}$

6.1.29. $h/r = \sqrt{2}$

6.1.30. The ratio of the volume of the sphere to the volume of the cone is $1033/4096 + 33/4096\sqrt{17} \approx 0.2854$, so the cone occupies approximately 28.54% of the sphere.

6.1.31. P should be at distance $c\sqrt[3]{a}/(\sqrt[3]{a} + \sqrt[3]{b})$ from charge A.

6.1.32. $1/2$

6.1.33. \$7000

6.1.34. There is a critical point when $\sin\theta_1/v_1 = \sin\theta_2/v_2$, and the second derivative is positive, so there is a minimum at the critical point.

6.2.1. $1/(16\pi)$ cm/s

6.2.2. $3/(1000\pi)$ meters/second

6.2.3. $1/4$ m/s

6.2.4. $-6/25$ m/s

6.2.5. 80π mi/min

6.2.6. $3\sqrt{5}$ ft/s

6.2.7. $20/(3\pi)$ cm/s

6.2.8. $13/20$ ft/s

6.2.9. $5\sqrt{10}/2$ m/s

6.2.10. $75/64$ m/min

6.2.11. $145\pi/72$ m/s

6.2.12. $25\pi/144$ m/min

6.2.13. $\pi\sqrt{2}/36$ ft^3/s

6.2.14. tip: 6 ft/s, length: 5/2 ft/s

6.2.15. tip: 20/11 m/s, length: 9/11 m/s

6.2.16. $380/\sqrt{3} - 150 \approx 69.4$ mph

6.2.17. $500/\sqrt{3} - 200 \approx 88.7$ km/hr

6.2.18. 18 m/s

6.2.19. $136\sqrt{475}/19 \approx 156$ km/hr

6.2.20. -50 m/s

6.2.21. 68 m/s

6.2.22. $3800/\sqrt{329} \approx 210$ km/hr

6.2.23. $820/\sqrt{329} + 150\sqrt{57}/\sqrt{47} \approx 210$ km/hr

6.2.24. $4000/49$ m/s

6.2.25. (a) $x = a\cos\theta - a\sin\theta\cot(\theta+\beta) = a\sin\beta/\sin(\theta+\beta)$, (c) $\dot{x} \approx 3.79$ cm/s

6.3.1. $x_3 = 1.475773162$

6.3.2. 2.15

6.3.3. 3.36

6.3.4. 2.19 or 1.26

6.4.1. $\Delta y = 65/16$, $dy = 2$

6.4.2. $\Delta y = \sqrt{11/10} - 1$, $dy = 0.05$

6.4.3. $\Delta y = \sin(\pi/50)$, $dy = \pi/50$

6.4.4. $dV = 8\pi/25$

6.5.1. $c = 1/2$

6.5.2. $c = \sqrt{18} - 2$

6.5.6. $x^3/3 + 47x^2/2 - 5x + k$

6.5.7. $\arctan x + k$

6.5.8. $x^4/4 - \ln x + k$

6.5.9. $-\cos(2x)/2 + k$

7.1.1. 10

7.1.2. 35/3

7.1.3. x^2

7.1.4. $2x^2$

7.1.5. $2x^2 - 8$

7.1.6. $2b^2 - 2a^2$

7.1.7. 4 rectangles: $41/4 = 10.25$, 8 rectangles: $183/16 = 11.4375$

7.1.8. 23/4

7.2.1. $(16/3)x^{3/2} + C$

7.2.2. $t^3 + t + C$

7.2.3. $8\sqrt{x} + C$

7.2.4. $-2/z + C$

7.2.5. $7 \ln s + C$

7.2.6. $(5x+1)^3/15 + C$

7.2.7. $(x-6)^3/3 + C$

7.2.8. $2x^{5/2}/5 + C$

7.2.9. $-4/\sqrt{x} + C$

7.2.10. $4t - t^2 + C$, $t < 2$; $t^2 - 4t + 8 + C$, $t \geq 2$

7.2.11. 87/2

7.2.12. 2

7.2.13. $\ln(10)$

7.2.14. $e^5 - 1$

7.2.15. $3^4/4$

7.2.16. $2^6/6 - 1/6$

7.2.17. $x^2 - 3x$

7.2.18. $2x(x^4 - 3x^2)$

7.2.19. e^{x^2}

7.2.20. $2xe^{x^4}$

7.2.21. $\tan(x^2)$

7.2.22. $2x \tan(x^4)$

7.3.1. It rises until $t = 100/49$, then falls. The position of the object at time t is $s(t) = -4.9t^2 + 20t + k$. The net distance traveled is $-45/2$, that is, it ends up $45/2$ meters below where it started. The total distance traveled is $6205/98$ meters.

7.3.2. $\int_0^{2\pi} \sin t \, dt = 0$

7.3.3. net: 2π, total: $2\pi/3 + 4\sqrt{3}$

7.3.4. 8

7.3.5. 17/3

7.3.6. $A = 18$, $B = 44/3$, $C = 10/3$

8.1.1. $-(1-t)^{10}/10 + C$

8.1.2. $x^5/5 + 2x^3/3 + x + C$

8.1.3. $(x^2+1)^{101}/202 + C$

8.1.4. $-3(1-5t)^{2/3}/10 + C$

8.1.5. $(\sin^4 x)/4 + C$

8.1.6. $-(100 - x^2)^{3/2}/3 + C$

8.1.7. $-2\sqrt{1-x^3}/3 + C$

8.1.8. $\sin(\sin \pi t)/\pi + C$

8.1.9. $1/(2\cos^2 x) = (1/2)\sec^2 x + C$

8.1.10. $-\ln|\cos x| + C$

8.1.11. 0

8.1.12. $\tan^2(x)/2 + C$

8.1.13. 1/4

8.1.14. $-\cos(\tan x) + C$

8.1.15. 1/10

8.1.16. $\sqrt{3}/4$

8.1.17. $(27/8)(x^2-7)^{8/9}+C$

8.1.18. $-(3^7+1)/14$

8.1.19. 0

8.1.20. $f(x)^2/2$

8.2.1. $x/2-\sin(2x)/4+C$

8.2.2. $-\cos x+(\cos^3 x)/3+C$

8.2.3. $3x/8-(\sin 2x)/4+(\sin 4x)/32+C$

8.2.4. $(\cos^5 x)/5-(\cos^3 x)/3+C$

8.2.5. $\sin x-(\sin^3 x)/3+C$

8.2.6. $x/8-(\sin 4x)/32+C$

8.2.7. $(\sin^3 x)/3-(\sin^5 x)/5+C$

8.2.8. $-2(\cos x)^{5/2}/5+C$

8.2.9. $\tan x-\cot x+C$

8.2.10. $(\sec^3 x)/3-\sec x+C$

8.3.1. $-\ln|\csc x+\cot x|+C$

8.3.2. $-\csc x\cot x/2-(1/2)\ln|\csc x+\cot x|+C$

8.3.3. $x\sqrt{x^2-1}/2-\ln|x+\sqrt{x^2-1}|/2+C$

8.3.4. $x\sqrt{9+4x^2}/2+(9/4)\ln|2x+\sqrt{9+4x^2}|+C$

8.3.5. $-(1-x^2)^{3/2}/3+C$

8.3.6. $\arcsin(x)/8-\sin(4\arcsin x)/32+C$

8.3.7. $\ln|x+\sqrt{1+x^2}|+C$

8.3.8. $(x+1)\sqrt{x^2+2x}/2-\ln|x+1+\sqrt{x^2+2x}|/2+C$

8.3.9. $-\arctan x-1/x+C$

8.3.10. $2\arcsin(x/2)-x\sqrt{4-x^2}/2+C$

8.3.11. $\arcsin(\sqrt{x})-\sqrt{x}\sqrt{1-x}+C$

8.3.12. $(2x^2+1)\sqrt{4x^2-1}/24+C$

8.4.1. $\cos x+x\sin x+C$

8.4.2. $x^2\sin x-2\sin x+2x\cos x+C$

8.4.3. $(x-1)e^x+C$

8.4.4. $(1/2)e^{x^2}+C$

8.4.5. $(x/2)-\sin(2x)/4+C=(x/2)-(\sin x\cos x)/2+C$

8.4.6. $x\ln x-x+C$

8.4.7. $(x^2\arctan x+\arctan x-x)/2+C$

8.4.8. $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x+C$

8.4.9. $x^3\sin x+3x^2\cos x-6x\sin x-6\cos x+C$

8.4.10. $x^2/4-(\cos^2 x)/4-(x\sin x\cos x)/2+C$

8.4.11. $x/4-(x\cos^2 x)/2+(\cos x\sin x)/4+C$

8.4.12. $x\arctan(\sqrt{x})+\arctan(\sqrt{x})-\sqrt{x}+C$

8.4.13. $2\sin(\sqrt{x})-2\sqrt{x}\cos(\sqrt{x})+C$

8.4.14. $\sec x\csc x-2\cot x+C$

8.5.1. $-\ln|x-2|/4+\ln|x+2|/4+C$

8.5.2. $-x^3/3-4x-4\ln|x-2|+4\ln|x+2|+C$

8.5.3. $-1/(x+5)+C$

8.5.4. $-x-\ln|x-2|+\ln|x+2|+C$

8.5.5. $-4x+x^3/3+8\arctan(x/2)+C$

8.5.6. $(1/2)\arctan(x/2+5/2)+C$

8.5.7. $x^2/2-2\ln(4+x^2)+C$

8.5.8. $(1/4)\ln|x+3|-(1/4)\ln|x+7|+C$

8.5.9. $(1/5)\ln|2x-3|-(1/5)\ln|1+x|+C$

8.5.10. $(1/3)\ln|x|-(1/3)\ln|x+3|+C$

8.6.1. T,S: 4 ± 0

8.6.2. T: 9.28125 ± 0.281125; S: 9 ± 0

8.6.3. T: 60.75 ± 1; S: 60 ± 0

8.6.4. T: 1.1167 ± 0.0833; S: 1.1000 ± 0.0167

8.6.5. T: 0.3235 ± 0.0026; S: 0.3217 ± 0.000065

8.6.6. T: 0.6478 ± 0.0052; S: 0.6438 ± 0.000033

8.6.7. T: 2.8833 ± 0.0834; S: 2.9000 ± 0.0167

8.6.8. T: 1.1170 ± 0.0077; S: 1.1114 ± 0.0002

8.6.9. T: 1.097 ± 0.0147; S: 1.089 ± 0.0003

8.6.10. T: 3.63 ± 0.087; S: 3.62 ± 0.032

8.7.1. $\frac{(t+4)^4}{4} + C$

8.7.2. $\frac{(t^2-9)^{5/2}}{5} + C$

8.7.3. $\frac{(e^{t^2}+16)^2}{4} + C$

8.7.4. $\cos t - \frac{2}{3}\cos^3 t + C$

8.7.5. $\frac{\tan^2 t}{2} + C$

8.7.6. $\ln|t^2+t+3| + C$

8.7.7. $\frac{1}{8}\ln|1-4/t^2| + C$

8.7.8. $\frac{1}{25}\tan(\arcsin(t/5)) + C = \frac{t}{25\sqrt{25-t^2}} + C$

8.7.9. $\frac{2}{3}\sqrt{\sin 3t} + C$

8.7.10. $t\tan t + \ln|\cos t| + C$

8.7.11. $2\sqrt{e^t+1} + C$

8.7.12. $\frac{3t}{8} + \frac{\sin 2t}{4} + \frac{\sin 4t}{32} + C$

8.7.13. $\frac{\ln|t|}{3} - \frac{\ln|t+3|}{3} + C$

8.7.14. $\frac{-1}{\sin\arctan t} + C = -\sqrt{1+t^2}/t + C$

8.7.15. $\frac{-1}{2(1+\tan t)^2} + C$

8.7.16. $\frac{(t^2+1)^{5/2}}{5} - \frac{(t^2+1)^{3/2}}{3} + C$

8.7.17. $\frac{e^t\sin t - e^t\cos t}{2} + C$

8.7.18. $\frac{(t^{3/2}+47)^4}{6} + C$

8.7.19. $\frac{2}{3(2-t^2)^{3/2}} - \frac{1}{(2-t^2)^{1/2}} + C$

8.7.20. $\frac{\ln|\sin(\arctan(2t/3))|}{9} + C = (\ln(4t^2) - \ln(9+4t^2))/18 + C$

8.7.21. $\frac{(\arctan(2t))^2}{4} + C$

8.7.22. $\frac{3\ln|t+3|}{4} + \frac{\ln|t-1|}{4} + C$

8.7.23. $\frac{\cos^7 t}{7} - \frac{\cos^5 t}{5} + C$

8.7.24. $\frac{-1}{t-3} + C$

8.7.25. $\frac{-1}{\ln t} + C$

8.7.26. $\frac{t^2(\ln t)^2}{2} - \frac{t^2\ln t}{2} + \frac{t^2}{4} + C$

8.7.27. $(t^3 - 3t^2 + 6t - 6)e^t + C$

8.7.28. $\frac{5+\sqrt{5}}{10}\ln(2t+1-\sqrt{5}) + \frac{5-\sqrt{5}}{10}\ln(2t+1+\sqrt{5}) + C$

9.1.1. $8\sqrt{2}/15$

9.1.2. $1/12$

9.1.3. $9/2$

9.1.4. $4/3$

9.1.5. $2/3 - 2/\pi$

9.1.6. $3/\pi - 3\sqrt{3}/(2\pi) - 1/8$

9.1.7. $1/3$

9.1.8. $10\sqrt{5}/3 - 6$

9.1.9. $500/3$

9.1.10. 2

9.1.11. 1/5

9.1.12. 1/6

9.2.1. $1/\pi$, $5/\pi$

9.2.2. 0, 245

9.2.3. 20, 28

9.2.4. $(3-\pi)/(2\pi)$, $(18-12\sqrt{3}+\pi)/(4\pi)$

9.2.5. 10/49 meters, 20/49 seconds

9.2.6. 45/98 meters, 30/49 seconds

9.2.7. 25000/49 meters, 1000/49 seconds

9.2.8. $s(t)=\cos t$, $v(t)=-\sin t$, maximum distance is 1, maximum speed is 1

9.2.9. $s(t)=-\sin(\pi t)/\pi^2+t/\pi$, $v(t)=-\cos(\pi t)/\pi+1/\pi$, maximum speed is $2/\pi$

9.2.10. $s(t)=t^2/2-\sin(\pi t)/\pi^2+t/\pi$, $v(t)=t-\cos(\pi t)/\pi+1/\pi$

9.2.11. $s(t)=t^2/2+\sin(\pi t)/\pi^2-t/\pi$, $v(t)=t+\cos(\pi t)/\pi-1/\pi$

9.3.5. $8\pi/3$

9.3.6. $\pi/30$

9.3.7. $\pi(\pi/2-1)$

9.3.8. (a) $114\pi/5$ (b) $74\pi/5$ (c) 20π (d) 4π

9.3.9. 16π, 24π

9.3.11. $\pi h^2(3r-h)/3$

9.3.13. 2π

9.4.1. $2/\pi$; $2/\pi$; 0

9.4.2. 4/3

9.4.3. $1/A$

9.4.4. $\pi/4$

9.4.5. $-1/3$, 1

9.4.6. $-4\sqrt{1224}$ ft/s; $-8\sqrt{1224}$ ft/s

9.5.1. $\approx 5,305,028,516$ N-m

9.5.2. $\approx 4,457,854,041$ N-m

9.5.3. $367,500\pi$ N-m

9.5.4. $49000\pi+196000/3$ N-m

9.5.5. 2450π N-m

9.5.6. 0.05 N-m

9.5.7. 6/5 N-m

9.5.8. 3920 N-m

9.5.9. 23520 N-m

9.5.10. 12740 N-m

9.6.1. 15/2

9.6.2. 5

9.6.3. 16/5

9.6.5. $\bar{x}=45/28$, $\bar{y}=93/70$

9.6.6. $\bar{x}=0$, $\bar{y}=4/(3\pi)$

9.6.7. $\bar{x}=1/2$, $\bar{y}=2/5$

9.6.8. $\bar{x}=0$, $\bar{y}=8/5$

9.6.9. $\bar{x}=4/7$, $\bar{y}=2/5$

9.6.10. $\bar{x}=\bar{y}=1/5$

9.6.11. $\bar{x}=0$, $\bar{y}=28/(9\pi)$

9.6.12. $\bar{x}=\bar{y}=28/(9\pi)$

9.6.13. $\bar{x}=0$, $\bar{y}=244/(27\pi)\approx 2.88$

9.7.1. ∞

9.7.2. 1/2

9.7.3. diverges

9.7.4. diverges

9.7.5. 1

9.7.6. diverges

9.7.7. 2

9.7.8. diverges

9.7.9. $\pi/6$

9.7.10. diverges, 0

9.7.11. diverges, 0

9.7.12. diverges, no CPV

9.7.13. π

9.7.14. 80 mph: 90.8 to 95.3 N
90 mph: 114.9 to 120.6 N
100.9 mph: 144.5 to 151.6 N

9.8.2. $\mu = 1/c$, $\sigma = 1/c$

9.8.3. $\mu = (a+b)/2$, $\sigma = \dfrac{(b-a)}{2\sqrt{3}}$

9.8.4. $7/2$

9.8.5. $21/2$

9.8.9. $r = 6$

9.9.1. $(22\sqrt{22}-8)/27$

9.9.2. $\ln(2) + 3/8$

9.9.3. $a + a^3/3$

9.9.4. $\ln((\sqrt{2}+1)/\sqrt{3})$

9.9.6. $3/4$

9.9.7. ≈ 3.82

9.9.8. ≈ 1.01

9.9.9. $\sqrt{1+e^2} - \sqrt{2} + \dfrac{1}{2}\ln\left(\dfrac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right) + \dfrac{1}{2}\ln(3+2\sqrt{2})$

9.10.1. $8\pi\sqrt{3} - \dfrac{16\pi\sqrt{2}}{3}$

9.10.3. $\dfrac{730\pi\sqrt{730}}{27} - \dfrac{10\pi\sqrt{10}}{27}$

9.10.4. $\pi + 2\pi e + \dfrac{1}{4}\pi e^2 - \dfrac{\pi}{4e^2} - \dfrac{2\pi}{e}$

9.10.6. $8\pi^2$

9.10.7. $2\pi + \dfrac{8\pi^2}{3\sqrt{3}}$

9.10.8. $a > b$: $2\pi b^2 + \dfrac{2\pi a^2 b}{\sqrt{a^2-b^2}}\arcsin(\sqrt{a^2-b^2}/a)$,
$a < b$: $2\pi b^2 + \dfrac{2\pi a^2 b}{\sqrt{b^2-a^2}}\ln\left(\dfrac{b}{a} + \dfrac{\sqrt{b^2-a^2}}{a}\right)$

10.1.2. $\theta = \arctan(3)$

10.1.3. $r = -4\csc\theta$

10.1.4. $r^3\cos\theta\sin^2\theta = 1$

10.1.5. $r = \sqrt{5}$

10.1.6. $r^2 = \sin\theta\sec^3\theta$

10.1.7. $r\sin\theta = \sin(r\cos\theta)$

10.1.8. $r = 2/(\sin\theta - 5\cos\theta)$

10.1.9. $r = 2\sec\theta$

10.1.10. $0 = r^2\cos^2\theta - r\sin\theta + 1$

10.1.11. $0 = 3r^2\cos^2\theta - 2r\cos\theta - r\sin\theta$

10.1.12. $r = \sin\theta$

10.1.21. $(x^2+y^2)^2 = 4x^2y - (x^2+y^2)y$

10.1.22. $(x^2+y^2)^{3/2} = y^2$

10.1.23. $x^2 + y^2 = x^2y^2$

10.1.24. $x^4 + x^2y^2 = y^2$

10.2.1. $(\theta\cos\theta + \sin\theta)/(-\theta\sin\theta + \cos\theta)$,
$(\theta^2+2)/(-\theta\sin\theta+\cos\theta)^3$

10.2.2. $\dfrac{\cos\theta + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta - \sin\theta}$,
$\dfrac{3(1+\sin\theta)}{(\cos^2\theta - \sin^2\theta - \sin\theta)^3}$

10.2.3. $(\sin^2\theta - \cos^2\theta)/(2\sin\theta\cos\theta)$,
$-1/(4\sin^3\theta\cos^3\theta)$

10.2.4. $\dfrac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$, $\dfrac{2}{(\cos^2\theta - \sin^2\theta)^3}$

10.2.5. undefined

10.2.6. $\dfrac{2\sin\theta - 3\sin^3\theta}{3\cos^3\theta - 2\cos\theta}$,
$\dfrac{3\cos^4\theta - 3\cos^2\theta + 2}{2\cos^3\theta(3\cos^2\theta - 2)^3}$

10.3.1. 1

10.3.2. $9\pi/2$

10.3.3. $\sqrt{3}/3$

10.3.4. $\pi/12 + \sqrt{3}/16$

10.3.5. πa^2

10.3.6. $41\pi/2$

10.3.7. $2 - \pi/2$

10.3.8. $\pi/12$

10.3.9. $3\pi/16$

10.3.10. $\pi/4 - 3\sqrt{3}/8$

10.3.11. $\pi/2 + 3\sqrt{3}/8$

10.3.12. $1/2$

10.3.13. $3/2 - \pi/4$

10.3.14. $\pi/3 + \sqrt{3}/2$

10.3.15. $\pi/3 - \sqrt{3}/4$

10.3.16. $4\pi^3/3$

10.3.17. π^2

10.3.18. $5\pi/24 - \sqrt{3}/4$

10.3.19. $7\pi/12 - \sqrt{3}$

10.3.20. $4\pi - \sqrt{15}/2 - 7\arccos(1/4)$

10.3.21. $3\pi^3$

10.4.6. $x = t - \dfrac{\sin(t)}{2}$, $y = 1 - \dfrac{\cos(t)}{2}$

10.4.7. $x = 4\cos t - \cos(4t)$, $y = 4\sin t - \sin(4t)$

10.4.8. $x = 2\cos t + \cos(2t)$, $y = 2\sin t - \sin(2t)$

10.4.9. $x = \cos t + t\sin t$, $y = \sin t - t\cos t$

10.5.1. There is a horizontal tangent at all multiples of π.

10.5.2. $9\pi/4$

10.5.3. $\displaystyle\int_0^{2\pi} \frac{1}{2}\sqrt{5 - 4\cos t}\,dt$

10.5.4. Four points: $\left(\dfrac{-3 - 3\sqrt{5}}{4}, \pm\sqrt{\dfrac{5 - \sqrt{5}}{8}}\right)$, $\left(\dfrac{-3 + 3\sqrt{5}}{4}, \pm\sqrt{\dfrac{5 + \sqrt{5}}{8}}\right)$

10.5.5. $11\pi/3$

10.5.6. $32/3$

10.5.7. 2π

10.5.8. $16/3$

10.5.9. $(\pi/2, 1)$

10.5.10. $5\pi^3/6$

10.5.11. $2\pi^2$

10.5.12. $(2\pi\sqrt{4\pi^2 + 1} + \ln(2\pi + \sqrt{4\pi^2 + 1}))/2$

11.1.1. 1

11.1.3. 0

11.1.4. 1

11.1.5. 1

11.1.6. 0

11.2.1. $\lim_{n\to\infty} n^2/(2n^2 + 1) = 1/2$

11.2.2. $\lim_{n\to\infty} 5/(2^{1/n} + 14) = 1/3$

11.2.3. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\displaystyle\sum_{n=1}^{\infty} 3\frac{1}{n}$ diverges

11.2.4. $-3/2$

11.2.5. 11

11.2.6. 20

11.2.7. $3/4$

11.2.8. $3/2$

11.2.9. $3/10$

11.3.1. diverges

11.3.2. diverges

11.3.3. converges

11.3.4. converges

11.3.5. converges

11.3.6. converges

11.3.7. diverges

11.3.8. converges

11.3.9. $N = 5$

11.3.10. $N = 10$

11.3.11. $N = 1687$

11.3.12. any integer greater than e^{200}

11.4.1. converges

11.4.2. converges

11.4.3. diverges

11.4.4. converges

11.4.5. 0.90

11.4.6. 0.95

11.5.1. converges

11.5.2. converges

11.5.3. converges

11.5.4. diverges

11.5.5. diverges

11.5.6. diverges

11.5.7. converges

11.5.8. diverges

11.5.9. converges

11.5.10. diverges

11.6.1. converges absolutely

11.6.2. diverges

11.6.3. converges conditionally

11.6.4. converges absolutely

11.6.5. converges conditionally

11.6.6. converges absolutely

11.6.7. diverges

11.6.8. converges conditionally

11.7.5. converges

11.7.6. converges

11.7.7. converges

11.7.8. diverges

11.8.1. $R = 1$, $I = (-1, 1)$

11.8.2. $R = \infty$, $I = (-\infty, \infty)$

11.8.3. $R = e$, $I = (-e, e)$

11.8.4. $R = e$, $I = (2 - e, 2 + e)$

11.8.5. $R = 0$, converges only when $x = 2$

11.8.6. $R = 1$, $I = [-6, -4]$

11.9.1. the alternating harmonic series

11.9.2. $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$

11.9.3. $\displaystyle\sum_{n=0}^{\infty}(n+1)(n+2)x^n$

11.9.4. $\displaystyle\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n$, $R = 1$

11.9.5. $\displaystyle C + \sum_{n=0}^{\infty}\frac{-1}{(n+1)(n+2)}x^{n+2}$

11.10.1. $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$, $R = \infty$

11.10.2. $\displaystyle\sum_{n=0}^{\infty} x^n/n!$, $R = \infty$

11.10.3. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x-5)^n}{5^{n+1}}$, $R = 5$

11.10.4. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$, $R = 1$

11.10.5. $\ln(2)+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-2)^n}{n2^n}$, $R=2$

11.10.6. $\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$, $R=1$

11.10.7. $1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!2^n}x^n =$
$1+\sum_{n=1}^{\infty}\frac{(2n-1)!}{2^{2n-1}(n-1)!\,n!}x^n$, $R=1$

11.10.8. $x+x^3/3$

11.10.9. $\sum_{n=0}^{\infty}(-1)^n x^{4n+1}/(2n)!$

11.10.10. $\sum_{n=0}^{\infty}(-1)^n x^{n+1}/n!$

11.11.1. $1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots+\frac{x^{12}}{12!}$

11.11.2. 1000; 8

11.11.3. $x+\frac{x^3}{3}+\frac{2x^5}{15}$, error ± 1.27.

11.12.1. diverges

11.12.2. converges

11.12.3. converges

11.12.4. diverges

11.12.5. diverges

11.12.6. diverges

11.12.7. converges

11.12.8. converges

11.12.9. converges

11.12.10. converges

11.12.11. converges

11.12.12. converges

11.12.13. converges

11.12.14. converges

11.12.15. converges

11.12.16. converges

11.12.17. diverges

11.12.18. $(-\infty,\infty)$

11.12.19. $(-3,3)$

11.12.20. $(-3,3)$

11.12.21. $(-1,1)$

11.12.22. radius is 0—it converges only when $x=0$

11.12.23. $(-\sqrt{3},\sqrt{3})$

11.12.24. $(-\infty,\infty)$

11.12.25. $\sum_{n=0}^{\infty}\frac{(\ln(2))^n}{n!}x^n$

11.12.26. $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}$

11.12.27. $\sum_{n=0}^{\infty}\frac{2}{2n+1}x^{2n+1}$

11.12.28. $1+x/2+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^n$

11.12.29. $\sum_{n=0}^{\infty}(-1)^n x^{2n}$

11.12.30. $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$

11.12.31. $\pi=\sum_{n=0}^{\infty}(-1)^n\frac{4}{2n+1}$

12.1.6. 3, $\sqrt{26}$, $\sqrt{29}$

12.1.7. $\sqrt{14}$, $2\sqrt{14}$, $3\sqrt{14}$.

12.1.8. $(x-1)^2+(y-1)^2+(z-1)^2=4$.

12.1.9. $(x-2)^2+(y+1)^2+(z-3)^2=25$.

12.1.10. $(x-3)^2+(y+2)^2+(z-1)^2=33$.

12.1.11. $(x-2)^2+(y-1)^2+(z+1)^2=16$, $(y-1)^2+(z+1)^2=12$

12.2.6. $\sqrt{10}$, $\langle 0,-2\rangle$, $\langle 2,8\rangle$, 2, $2\sqrt{17}$, $\langle -2,-6\rangle$

12.2.7. $\sqrt{14}$, $\langle 0,4,0\rangle$, $\langle 2,0,6\rangle$, 4, $2\sqrt{10}$, $\langle -2,-4,-6\rangle$

12.2.8. $\sqrt{2}$, $\langle 0,-2,3\rangle$, $\langle 2,2,-1\rangle$, $\sqrt{13}$, 3, $\langle -2,0,-2\rangle$

12.2.9. $\sqrt{3}$, $\langle 1,-1,4\rangle$, $\langle 1,-1,-2\rangle$, $3\sqrt{2}$, $\sqrt{6}$, $\langle -2,2,-2\rangle$

12.2.10. $\sqrt{14}$, $\langle 2,1,0\rangle$, $\langle 4,3,2\rangle$, $\sqrt{5}$, $\sqrt{29}$, $\langle -6,-4,-2\rangle$

12.2.11. $\langle -3,-3,-11\rangle$, $\langle -3/\sqrt{139},-3/\sqrt{139},-11/\sqrt{139}\rangle$ $\langle -12/\sqrt{139},-12/\sqrt{139},-44/\sqrt{139}\rangle$

12.2.12. $\langle 0,0,0\rangle$

12.2.13. $\mathbf{0}$; $\langle -r\sqrt{3}/2,r/2\rangle$; $\langle 0,-12r\rangle$; where r is the radius of the clock

12.3.1. 3

12.3.2. 0

12.3.3. 2

12.3.4. -6

12.3.5. 42

12.3.6. $\sqrt{6}/\sqrt{7}$, ≈ 0.39

12.3.7. $-11\sqrt{14}\sqrt{29}/406$, ≈ 2.15

12.3.8. 0, $\pi/2$

12.3.9. $1/2$, $\pi/3$

12.3.10. $-1/\sqrt{3}$, ≈ 2.19

12.3.11. $\arccos(1/\sqrt{3})\approx 0.96$

12.3.12. $\sqrt{5}$, $\langle 1,2,0\rangle$.

12.3.13. $3\sqrt{14}/7$, $\langle 9/7,6/7,3/7\rangle$.

12.3.14. $\langle 0,5\rangle$, $\langle 5\sqrt{3},0\rangle$

12.3.15. $\langle 0,15\sqrt{2}/2\rangle$,$\langle 15\sqrt{2}/2,0\rangle$

12.3.16. Any vector of the form $\langle a,-7a/2,-2a\rangle$

12.3.17. $\langle 1/\sqrt{3},-1/\sqrt{3},1/\sqrt{3}\rangle$

12.3.18. No.

12.3.19. Yes.

12.4.1. $\langle 1,-2,1\rangle$

12.4.2. $\langle 4,-6,-2\rangle$

12.4.3. $\langle -7,13,-9\rangle$

12.4.4. $\langle 0,-1,0\rangle$

12.4.5. 3

12.4.6. $21\sqrt{2}/2$

12.4.7. 1

12.5.1. $(x-6)+(y-2)+(z-1)=0$

12.5.2. $4(x+1)+5(y-2)-(z+3)=0$

12.5.3. $(x-1)-(y-2)=0$

12.5.4. $-2(x-1)+3y-2z=0$

12.5.5. $4(x-1)-6y=0$

12.5.6. $x+3y=0$

12.5.7. $\langle 1,0,3\rangle+t\langle 0,2,1\rangle$

12.5.8. $\langle 1,0,3\rangle+t\langle 1,2,-1\rangle$

12.5.9. $t\langle 1,1,-1\rangle$

12.5.10. $-2/5$, $13/5$

12.5.12. neither

12.5.13. parallel

12.5.14. intersect at $(3,6,5)$

12.5.15. same line

12.5.19. $7/\sqrt{3}$

12.5.20. $4/\sqrt{14}$

12.5.21. $\sqrt{131}/\sqrt{14}$

12.5.22. $\sqrt{68}/3$

12.5.23. $\sqrt{42}/7$

12.5.24. $\sqrt{21}/6$

12.6.1. $(\sqrt{2}, \pi/4, 1)$, $(\sqrt{3}, \pi/4, \arccos(1/\sqrt{3}))$; $(7\sqrt{2}, 7\pi/4, 5)$, $(\sqrt{123}, 7\pi/4, \arccos(5/\sqrt{123})$; $(1, 1, 1)$, $(\sqrt{2}, 1, \pi/4)$; $(0, 0, -\pi)$, $(\pi, 0, \pi)$

12.6.2. $r^2 + z^2 = 4$

12.6.3. $r\cos\theta = 0$

12.6.4. $r^2 + 2z^2 + 2z - 5 = 0$

12.6.5. $z = e^{-r^2}$

12.6.6. $z = r$

12.6.7. $\sin\theta = 0$

12.6.8. $1 = \rho\cos\phi$

12.6.9. $\rho = 2\sin\theta\sin\phi$.

12.6.10. $\rho\sin\phi = 3$

12.6.11. $\phi = \pi/4$

12.6.13. $z = mr$; $\cot\phi = m$ if $m \neq 0$, $\phi = 0$ if $m = 0$

12.6.14. A sphere with radius $1/2$, center at $(0, 1/2, 0)$

12.6.15. $0 < \theta < \pi/2$, $0 < \phi < \pi/2$, $\rho > 0$; $0 < \theta < \pi/2$, $r > 0$, $z > 0$

13.1.5. $\langle 3\cos t, 3\sin t, 2 - 3\sin t\rangle$

13.1.6. $\langle 0, t\cos t, t\sin t\rangle$

13.2.1. $\langle 2t, 0, 1\rangle$, $\mathbf{r}'/\sqrt{1+4t^2}$

13.2.2. $\langle -\sin t, 2\cos 2t, 2t\rangle$, $\mathbf{r}'/\sqrt{\sin^2 t + 4\cos^2(2t) + 4t^2}$

13.2.3. $\langle -e^t\sin(e^t), e^t\cos(e^t), \cos t\rangle$, $\mathbf{r}'/\sqrt{e^{2t} + \cos^2 t}$

13.2.4. $\langle \sqrt{2}/2, \sqrt{2}/2, \pi/4\rangle + t\langle -\sqrt{2}/2, \sqrt{2}/2, 1\rangle$

13.2.5. $\langle 1/2, \sqrt{3}/2, -1/2\rangle + t\langle -\sqrt{3}/2, 1/2, 2\sqrt{3}\rangle$

13.2.6. $2/\sqrt{5}/\sqrt{4+\pi^2}$

13.2.7. $7\sqrt{5}\sqrt{17}/85$, $-9\sqrt{5}\sqrt{17}/85$

13.2.9. $\langle 0, t\cos t, t\sin t\rangle$, $\langle 0, \cos t - t\sin t, \sin t + t\cos t\rangle$, $\mathbf{r}'/\sqrt{1+t^2}$, $\sqrt{1+t^2}$

13.2.10. $\langle \sin t, 1 - \cos t, t^2/2\rangle$

13.2.11. $t = 4$

13.2.12. $\sqrt{37}$, 1

13.2.13. $\langle t^2/2, t^3/3, \sin t\rangle$

13.2.16. $(1, 1, 1)$ when $t = 1$ and $s = 0$; $\theta = \arccos(3/\sqrt{14})$; no

13.2.17. $-6x + (y - \pi) = 0$

13.2.18. $-x/\sqrt{2} + y/\sqrt{2} + 6z = 0$

13.2.19. $(-1, -3, 1)$

13.2.20. $\langle 1/\sqrt{2}, 1/\sqrt{2}, 0\rangle + t\langle -1, 1, 6\sqrt{2}\rangle$

13.3.1. $2\pi\sqrt{13}$

13.3.2. $(-8 + 13\sqrt{13})/27$

13.3.3. $\sqrt{5}/2 + \ln(\sqrt{5} + 2)/4$

13.3.4. $(85\sqrt{85} - 13\sqrt{13})/27$

13.3.5. $\int_0^5 \sqrt{1+e^{2t}}\,dt$

13.3.6. $2\sqrt{2}/(2+4t^2)^{3/2}$

13.3.7. $2\sqrt{2}/(1+8t^2)^{3/2}$

13.3.8. $2\sqrt{1+9t^2+9t^4}/(1+4t^2+9t^4)^{3/2}$

13.3.9. $12\sqrt{17}/289$

13.4.1. $\langle -\sin t, \cos t, 1\rangle$, $\langle -\cos t, -\sin t, 0\rangle$, 0, 1

13.4.2. $\langle -\sin t, \cos t, 2t\rangle$, $\langle -\cos t, -\sin t, 2\rangle$, $4t/\sqrt{4t^2+1}$, $\sqrt{4t^2+5}/\sqrt{4t^2+1}$

13.4.3. $\langle -\sin t, \cos t, e^t\rangle$, $\langle -\cos t, -\sin t, e^t\rangle$, $e^{2t}/\sqrt{e^{2t}+1}$, $\sqrt{2e^{2t}+1}/\sqrt{e^{2t}+1}$

13.4.4. $\langle e^t, \cos t, e^t\rangle$, $\langle e^t, -\sin t, e^t\rangle$, $(2e^{2t} - \cos t \sin t)/\sqrt{2e^{2t} + \cos^2 t}$, $\sqrt{2}e^t|\cos t + \sin t|/\sqrt{2e^{2t} + \cos^2 t}$

13.4.5. $\langle -3\sin t, 2\cos t, 0\rangle$, $\langle 3\cos t, 2\sin t, 0\rangle$

13.4.6. $\langle -3\sin t, 2\cos t + 0.1, 0\rangle$, $\langle 3\cos t, 2\sin t + t/10, 0\rangle$

13.4.7. $\langle -3\sin t, 2\cos t, 1\rangle$, $\langle 3\cos t, 2\sin t, t\rangle$

13.4.8. $\langle -3\sin t, 2\cos t + 1/10, 1\rangle$, $\langle 3\cos t, 2\sin t + t/10, t\rangle$

14.1.1. $z = y^2$, $z = x^2$, $z = 0$, lines of slope 1

14.1.2. $z = |y|$, $z = |x|$, $z = 2|x|$, diamonds

14.1.3. $z = e^{-y^2}\sin(y^2)$, $z = e^{-x^2}\sin(x^2)$, $z = e^{-2x^2}\sin(2x^2)$, circles

14.1.4. $z = -\sin(y)$, $z = \sin(x)$, $z = 0$, lines of slope 1

14.1.5. $z = y^4$, $z = x^4$, $z = 0$, hyperbolas

14.1.6. (a) $\{(x,y) \mid |x| \le 3 \text{ and } |y| \ge 2\}$
(b) $\{(x,y) \mid 1 \le x^2 + y^2 \le 3\}$
(c) $\{(x,y) \mid x^2 + 4y^2 \le 16\}$

14.2.1. No limit; use $x = 0$ and $y = 0$.

14.2.2. No limit; use $x = 0$ and $x = y$.

14.2.3. No limit; use $x = 0$ and $x = y$.

14.2.4. Limit is zero.

14.2.5. Limit is 1.

14.2.6. Limit is zero.

14.2.7. Limit is -1.

14.2.8. Limit is zero.

14.2.9. No limit; use $x = 0$ and $y = 0$.

14.2.10. Limit is zero.

14.2.11. Limit is -1.

14.2.12. Limit is zero.

14.3.1. $-2xy\sin(x^2y)$, $-x^2\sin(x^2y) + 3y^2$

14.3.2. $(y^2 - x^2y)/(x^2+y)^2$, $x^3/(x^2+y)^2$

14.3.3. $2xe^{x^2+y^2}$, $2ye^{x^2+y^2}$

14.3.4. $y\ln(xy) + y$, $x\ln(xy) + x$

14.3.5. $-x/\sqrt{1-x^2-y^2}$, $-y/\sqrt{1-x^2-y^2}$

14.3.6. $\tan y$, $x\sec^2 y$

14.3.7. $-1/(x^2y)$, $-1/(xy^2)$

14.3.8. $z = -2(x-1) - 3(y-1) - 1$

14.3.9. $z = 1$

14.3.10. $z = 6(x-3) + 3(y-1) + 10$

14.3.11. $z = (x-2) + 4(y-1/2)$

14.3.12. $\mathbf{r}(t) = \langle 2,1,4\rangle + t\langle 2,4,-1\rangle$

14.4.1. $4xt\cos(x^2+y^2) + 6yt^2\cos(x^2+y^2)$

14.4.2. $2xy\cos t + 2x^2t$

14.4.3. $2xyt\cos(st) + 2x^2s$, $2xys\cos(st) + 2x^2t$

14.4.4. $2xy^2t - 4yx^2s$, $2xy^2s + 4yx^2t$

14.4.5. x/z, $3y/(2z)$

14.4.6. $-2x/z$, $-y/z$

14.4.7. (a) $V' = (nR - 0.2V)/P$
(b) $P' = (nR + 0.6P)/2V$
(c) $T' = (3P - 0.4V)/(nR)$

14.5.1. $9\sqrt{5}/5$

14.5.2. $\sqrt{2}\cos 3$

14.5.3. $e\sqrt{2}(\sqrt{3}-1)/4$

14.5.4. $\sqrt{3} + 5$

14.5.5. $-\sqrt{6}(2+\sqrt{3})/72$

14.5.6. $-1/5$, 0

14.5.7. $4(x-2) + 8(y-1) = 0$

14.5.8. $2(x-3) + 3(y-2) = 0$

14.5.9. $\langle -1, -1 - \cos 1, -\cos 1\rangle$, $-\sqrt{2 + 2\cos 1 + 2\cos^2 1}$

14.5.10. Any direction perpendicular to $\nabla T = \langle 1,1,1\rangle$, for example, $\langle -1,1,0\rangle$

14.5.11. $2(x-1)-6(y-1)+6(z-3)=0$

14.5.12. $6(x-1)+3(y-2)+2(z-3)=0$

14.5.13. $\langle 2+4t,-3-12t,-1-8t\rangle$

14.5.14. $\langle 4+8t,2+4t,-2-36t\rangle$

14.5.15. $\langle 4+8t,2+20t,6-12t\rangle$

14.5.16. $\langle 0,1\rangle$, $\langle 4/5,-3/5\rangle$

14.5.18. (a) $\langle 4,9\rangle$ (b) $\langle -81,2\rangle$ or $\langle 81,-2\rangle$

14.5.19. in the direction of $\langle 8,1\rangle$

14.5.20. $\nabla g(-1,3)=\langle 2,1\rangle$

14.6.1. $f_{xx}=(2x^3y-6xy^3)/(x^2+y^2)^3$, $f_{yy}=(2xy^3-6x^3y)/(x^2+y^2)^3$

14.6.2. $f_x=3x^2y^2$, $f_y=2x^3y+5y^4$, $f_{xx}=6xy^2$, $f_{yy}=2x^3+20y^3$, $f_{xy}=6x^2y$

14.6.3. $f_x=12x^2+y^2$, $f_y=2xy$, $f_{xx}=24x$, $f_{yy}=2x$, $f_{xy}=2y$

14.6.4. $f_x=\sin y$, $f_y=x\cos y$, $f_{xx}=0$, $f_{yy}=-x\sin y$, $f_{xy}=\cos y$

14.6.5. $f_x=3\cos(3x)\cos(2y)$, $f_y=-2\sin(3x)\sin(2y)$, $f_{xy}=-6\cos(3x)\sin(2y)$, $f_{yy}=-4\sin(3x)\cos(2y)$, $f_{xx}=-9\sin(3x)\cos(2y)$

14.6.6. $f_x=e^{x+y^2}$, $f_y=2ye^{x+y^2}$, $f_{xx}=e^{x+y^2}$, $f_{yy}=4y^2e^{x+y^2}+2e^{x+y^2}$, $f_{xy}=2ye^{x+y^2}$

14.6.7. $f_x=\dfrac{3x^2}{2(x^3+y^4)}$, $f_y=\dfrac{2y^3}{x^3+y^4}$, $f_{xx}=\dfrac{3x}{x^3+y^4}-\dfrac{9x^4}{2(x^3+y^4)^2}$, $f_{yy}=\dfrac{6y^2}{x^3+y^4}-\dfrac{8y^6}{(x^3+y^4)^2}$, $f_{xy}=\dfrac{-6x^2y^3}{(x^3+y^4)^2}$

14.6.8. $z_x=\dfrac{-x}{16z}$, $z_y=\dfrac{-y}{4z}$, $z_{xx}=-\dfrac{16z^2+x^2}{16^2z^3}$, $z_{yy}=-\dfrac{4z^2+y^2}{16z^3}$, $z_{xy}=\dfrac{-xy}{64z^3}$

14.6.9. $z_x=-\dfrac{y+z}{x+y}$, $z_y=-\dfrac{x+z}{x+y}$, $z_{xx}=2\dfrac{y+z}{(x+y)^2}$, $z_{yy}=2\dfrac{x+z}{(x+y)^2}$, $z_{xy}=\dfrac{2z}{(x+y)^2}$

14.7.1. minimum at $(1,-1)$

14.7.2. none

14.7.3. none

14.7.4. maximum at $(1,-1/6)$

14.7.5. none

14.7.6. minimum at $(2,-1)$

14.7.7. $f(2,2)=-2$, $f(2,0)=4$

14.7.8. a cube $1/\sqrt[3]{2}$ on a side

14.7.9. $65/3\cdot 65/3\cdot 130/3=2\cdot 65^3/27\approx$ 20343 cubic inches.

14.7.10. It has a square base, and is one and one half times as tall as wide. If the volume is V the dimensions are $\sqrt[3]{2V/3}\times\sqrt[3]{2V/3}\times\sqrt[3]{9V/4}$.

14.7.11. $\sqrt{100/3}$

14.7.12. $|ax_0+by_0+cz_0-d|/\sqrt{a^2+b^2+c^2}$

14.7.13. The sides and bottom should all be 2/3 meter, and the sides should be bent up at angle $\pi/3$.

14.7.14. $(3, 4/3)$

14.7.16. $|b|$ if $b \le 1/2$, otherwise $\sqrt{b - 1/4}$

14.7.17. $|b|$ if $b \le 1/2$, otherwise $\sqrt{b - 1/4}$

14.7.19. $256/\sqrt{3}$

14.8.1. a cube, $\sqrt[3]{1/2} \times \sqrt[3]{1/2} \times \sqrt[3]{1/2}$

14.8.2. $65/3 \cdot 65/3 \cdot 130/3 = 2 \cdot 65^3/27$

14.8.3. It has a square base, and is one and one half times as tall as wide. If the volume is V the dimensions are $\sqrt[3]{2V/3} \times \sqrt[3]{2V/3} \times \sqrt[3]{9V/4}$.

14.8.4. $|ax_0 + by_0 + cz_0 - d|/\sqrt{a^2 + b^2 + c^2}$

14.8.5. $(0, 0, 1)$, $(0, 0, -1)$

14.8.6. $\sqrt[3]{4V} \times \sqrt[3]{4V} \times \sqrt[3]{V/16}$

14.8.7. Farthest: $(-\sqrt{2}, \sqrt{2}, 2 + 2\sqrt{2})$; closest: $(2, 0, 0)$, $(0, -2, 0)$

14.8.8. $x = y = z = 16$

14.8.9. $(1, 2, 2)$

14.8.10. $(\sqrt{5}, 0, 0)$, $(-\sqrt{5}, 0, 0)$

14.8.11. standard \$65, deluxe \$75

14.8.12. $x = 9$, $\phi = \pi/3$

14.8.13. 35, -35

14.8.14. maximum e^4, no minimum

14.8.15. 5, $-9/2$

14.8.16. 3, 3, 3

14.8.17. a cube of side length $2/\sqrt{3}$

15.1.1. 16

15.1.2. 4

15.1.3. $15/8$

15.1.4. $1/2$

15.1.5. $5/6$

15.1.6. $12 - 65/(2e)$.

15.1.7. $1/2$

15.1.8. $\pi/64$

15.1.9. $(2/9)2^{3/2} - (2/9)$

15.1.10. $(1 - \cos(1))/4$

15.1.11. $(2\sqrt{2} - 1)/6$

15.1.12. $\pi - 2$

15.1.13. $(e^9 - 1)/6$

15.1.14. $\dfrac{4}{15} - \dfrac{\pi}{4}$

15.1.15. $1/3$

15.1.16. 448

15.1.17. $4/5$

15.1.18. 8π

15.1.19. 2

15.1.20. $5/3$

15.1.21. $81/2$

15.1.22. $2a^3/3$

15.1.23. 4π

15.1.24. $\pi/32$

15.1.25. $31/8$

15.1.26. $128/15$

15.1.27. 1800π m^3

15.1.28. $\dfrac{(e^2 + 8e + 16)}{15}\sqrt{e + 4} - \dfrac{5\sqrt{5}}{3} - \dfrac{e^{5/2}}{15} + \dfrac{1}{15}$

15.1.30. $16 - 8\sqrt{2}$

15.2.1. 4π

15.2.2. $32\pi/3 - 4\sqrt{3}\pi$

15.2.3. $(2 - \sqrt{2})\pi/3$

15.2.4. $4/9$

15.2.5. $5\pi/3$

15.2.6. $\pi/6$

15.2.7. $\pi/2$

15.2.8. $\pi/2-1$

15.2.9. $\sqrt{3}/4+\pi/6$

15.2.10. $8+\pi$

15.2.11. $\pi/12$

15.2.12. $(1-\cos(9))\pi/2$

15.2.13. $-a^5/15$

15.2.14. 12π

15.2.15. π

15.2.16. $16/3$

15.2.17. 21π

15.2.19. 2π

15.3.1. $\bar{x}=\bar{y}=2/3$

15.3.2. $\bar{x}=4/5$, $\bar{y}=8/15$

15.3.3. $\bar{x}=0$, $\bar{y}=3\pi/16$

15.3.4. $\bar{x}=0$, $\bar{y}=16/(15\pi)$

15.3.5. $\bar{x}=3/2$, $\bar{y}=9/4$

15.3.6. $\bar{x}=6/5$, $\bar{y}=12/5$

15.3.7. $\bar{x}=14/27$, $\bar{y}=28/55$

15.3.8. $(3/4,2/5)$

15.3.9. $\left(\dfrac{81\sqrt{3}}{80\pi},0\right)$

15.3.10. $\bar{x}=\pi/2$, $\bar{y}=\pi/8$

15.3.11. $M = \int_0^{2\pi}\int_0^{1+\cos\theta}(2+\cos\theta)r\,dr\,d\theta$,
$M_x = \int_0^{2\pi}\int_0^{1+\cos\theta}\sin\theta(2+\cos\theta)r^2\,dr\,d\theta$,
$M_y = \int_0^{2\pi}\int_0^{1+\cos\theta}\cos\theta(2+\cos\theta)r^2\,dr\,d\theta$.

15.3.12. $M=\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}(r+1)r\,dr\,d\theta$,
$M_x = \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\sin\theta(r+1)r^2\,dr\,d\theta$,
$M_y = \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\cos\theta(r+1)r^2\,dr\,d\theta$.

15.3.13. $M = \int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{1+\cos\theta} r\,dr\,d\theta + \int_{\pi/2}^{3\pi/2}\int_0^{1+\cos\theta} r\,dr\,d\theta$,
$M_x = \int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{1+\cos\theta} r^2\sin\theta\,dr\,d\theta + \int_{\pi/2}^{3\pi/2}\int_0^{1+\cos\theta} r^2\sin\theta\,dr\,d\theta$,
$M_y = \int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{1+\cos\theta} r^2\cos\theta\,dr\,d\theta + \int_{\pi/2}^{3\pi/2}\int_0^{1+\cos\theta} r^2\cos\theta\,dr\,d\theta$.

15.4.1. $\pi a\sqrt{h^2+a^2}$

15.4.2. $\pi a^2\sqrt{m^2+1}$

15.4.3. $\sqrt{3}/2$

15.4.4. $\pi\sqrt{2}$

15.4.5. $\pi\sqrt{2}/8$

15.4.6. $\pi/2-1$

15.4.7. $\dfrac{d^2\sqrt{a^2+b^2+c^2}}{2abc}$

15.4.8. $8\sqrt{3}\pi/3$

15.5.1. $11/24$

15.5.2. $623/60$

15.5.3. $-3e^2/4+2e-3/4$

15.5.4. $1/20$

15.5.5. $\pi/48$

15.5.6. $11/84$

15.5.7. $151/60$

15.5.8. π

15.5.10. $\dfrac{3\pi}{16}$

15.5.11. 32

15.5.12. $64/3$

15.5.13. $\bar{x} = \bar{y} = 0,\ \bar{z} = 16/15$

15.5.14. $\bar{x} = \bar{y} = 0,\ \bar{z} = 1/3$

15.6.1. $\pi/12$

15.6.2. $\pi(1 - \sqrt{2}/2)$

15.6.3. $5\pi/4$

15.6.4. 0

15.6.5. $5\pi/4$

15.6.6. $4/5$

15.6.7. $256\pi/15$

15.6.8. $4\pi^2$

15.6.9. $\dfrac{3\pi}{16}$

15.6.10. $\pi k h^2 a^2/12$

15.6.11. $\pi k h a^3/6$

15.6.12. $\pi^2/4$

15.6.13. $4\pi/5$

15.6.14. 15π

15.6.15. $9k\pi(5\sqrt{2} - 2\sqrt{5})/20$

15.7.1. $4\pi\sqrt{3}/3$

15.7.2. 0

15.7.3. $8/3$

15.7.4. $\dfrac{e^2 - 1}{2e^2}$

15.7.5. 36

15.7.6. $32(\sqrt{2} + \ln(1 + \sqrt{2}))/3$

15.7.7. $3\cos(1) - 3\cos(4)$

15.7.8. $\pi(1 - \cos(1))/24$

15.7.10. $(4/3)\pi abc$

16.2.1. $13\sqrt{11}/4$

16.2.2. 0

16.2.3. $3\sin(4)/2$

16.2.4. 0

16.2.5. $2e^3$

16.2.6. 128

16.2.7. $(9e - 3)/2$

16.2.8. $e^{e+1} - e^e - e^{1/e-1} + e^{1/e} + e^4/4 - e^{-4}/4$

16.2.9. $1 + \sin(1) - \cos(1)$

16.2.10. $3\ln 3 - 2\ln 2$

16.2.11. $3/20 + 10\ln(2)/7$

16.2.12. $2\ln 5 - 2\ln 2 + 15/32$

16.2.13. 1

16.2.14. 0

16.2.15. $21 + \cos(1) - \cos(8)$

16.2.16. $(\ln 29 - \ln 2)/2$

16.2.17. $2\ln 2 + \pi/4 - 2$

16.2.18. $1243/3$

16.2.19. $\ln 2 + 11/3$

16.2.20. $3\cos(1) - \cos(2) - \cos(4) - \cos(8)$

16.2.21. $-10/3$

16.3.1. no f

16.3.2. $x^4/4 - y^5/5$

16.3.3. no f

16.3.4. no f

16.3.5. $y\sin x$

16.3.6. no f

16.3.7. xyz

16.3.8. 414

16.3.9. 6

16.3.10. $1/e - \sin 3$

16.3.11. $1/\sqrt{77} - 1/\sqrt{3}$

16.4.1. 1

16.4.2. 0

16.4.3. $1/(2e) - 1/(2e^7) + e/2 - e^7/2$

16.4.4. $1/2$

16.4.5. $-1/6$

16.4.6. $(2\sqrt{3}-10\sqrt{5}+8\sqrt{6})/3-2\sqrt{2}/5+1/5$

16.4.7. $11/2 - \ln(2)$

16.4.8. $2 - \pi/2$

16.4.9. $-17/12$

16.4.10. 0

16.4.11. $-\pi/2$

16.4.12. 12π

16.4.13. $2\cos(1) - 2\sin(1) - 1$

16.5.1. -1, 0

16.5.2. 0, $a+b$

16.5.3. $(2b-a)/3$, 0

16.5.4. 0, 1

16.5.5. -2π, 0

16.5.6. 0, 2π

16.6.3. $25\sqrt{21}/4$

16.6.4. $\pi\sqrt{21}$

16.6.5. $\pi(5\sqrt{5}-1)/6$

16.6.6. $4\pi\sqrt{2}$

16.6.7. $\pi a^2/2$

16.6.8. $2\pi a(a - \sqrt{a^2-b^2})$

16.6.9. $\pi((1+4a^2)^{3/2}-1)/6$

16.6.10. $2\pi((1+a^2)^{3/2}-1)/3$

16.6.11. $\pi a^2 - 2a^2$

16.6.12. $\pi a^2\sqrt{1+k^2}/4$

16.6.13. $A\sqrt{1+a^2+b^2}$

16.6.14. $A\sqrt{k^2+1}$

16.6.15. $8a^2$

16.7.1. $(0,0,3/8)$

16.7.2. $(11/20, 11/20, 3/10)$

16.7.3. $(0,0,2275/682)$

16.7.4. on center axis, $h/3$ above the base

16.7.5. 16

16.7.6. 7

16.7.7. $-\pi$

16.7.8. $-137/120$

16.7.9. $-2/e$

16.7.10. $\pi b^2(-4b^4 - 3b^2 + 6a^2b^2 + 6a^2)/6$

16.7.11. 9280 kg/s

16.7.12. $24\epsilon_0$

16.8.1. -3π

16.8.2. 0

16.8.3. -4π

16.8.4. 3π

16.8.5. $A(p(c-b) + q(a-c) + a - b)$

16.9.1. both are $-45\pi/4$

16.9.2. $a^2bc + ab^2c + abc^2$

16.9.3. $e^2 - 2e + 7/2$

16.9.4. 3

16.9.5. $384\pi/5$

16.9.6. $\pi/3$

16.9.7. 10π

16.9.8. $\pi/2$

17.1.2. $y = \arctan t + C$

17.1.3. $y = \dfrac{t^{n+1}}{n+1} + 1$

17.1.4. $y = t\ln t - t + C$

17.1.5. $y = n\pi$, for any integer n.

17.1.6. none

17.1.7. $y = \pm\sqrt{t^2 + C}$

17.1.8. $y = \pm 1$, $y = (1 + Ae^{2t})/(1 - Ae^{2t})$

17.1.9. $y^4/4 - 5y = t^2/2 + C$

17.1.10. $y = (2t/3)^{3/2}$

17.1.11. $y = M + Ae^{-kt}$

17.1.12. $\dfrac{10\ln(15/2)}{\ln 5} \approx 2.52$ minutes

17.1.13. $y = \dfrac{M}{1 + Ae^{-Mkt}}$

17.1.14. $y = 2e^{3t/2}$

17.1.15. $t = -\dfrac{\ln 2}{k}$

17.1.16. $600e^{-6\ln 2/5} \approx 261$ mg; $\dfrac{5\ln 300}{\ln 2} \approx$ 41 days

17.1.17. $100e^{-200\ln 2/191} \approx 48$ mg; $\dfrac{5730\ln 50}{\ln 2} \approx 32339$ years

17.1.18. $y = y_0 e^{t\ln 2}$

17.1.19. $500e^{-5\ln 2/4} \approx 210$ g

17.2.1. $y = Ae^{-5t}$

17.2.2. $y = Ae^{2t}$

17.2.3. $y = Ae^{-\arctan t}$

17.2.4. $y = Ae^{-t^3/3}$

17.2.5. $y = 4e^{-t}$

17.2.6. $y = -2e^{3t-3}$

17.2.7. $y = e^{1+\cos t}$

17.2.8. $y = e^2 e^{-e^t}$

17.2.9. $y = 0$

17.2.10. $y = 0$

17.2.11. $y = 4t^2$

17.2.12. $y = -2e^{(1/t)-1}$

17.2.13. $y = e^{1-t^{-2}}$

17.2.14. $y = 0$

17.2.15. $k = \ln 5$, $y = 100e^{-t\ln 5}$

17.2.16. $k = -12/13$, $y = \exp(-13t^{1/13})$

17.2.17. $y = 10^6 e^{t\ln(3/2)}$

17.2.18. $y = 10e^{-t\ln(2)/6}$

17.3.1. $y = Ae^{-4t} + 2$

17.3.2. $y = Ae^{2t} - 3$

17.3.3. $y = Ae^{-(1/2)t^2} + 5$

17.3.4. $y = Ae^{-e^t} - 2$

17.3.5. $y = Ae^t - t^2 - 2t - 2$

17.3.6. $y = Ae^{-t/2} + t - 2$

17.3.7. $y = At^2 - \dfrac{1}{3t}$

17.3.8. $y = \dfrac{c}{t} + \dfrac{2}{3}\sqrt{t}$

17.3.9. $y = A\cos t + \sin t$

17.3.10. $y = \dfrac{A}{\sec t + \tan t} + 1 - \dfrac{t}{\sec t + \tan t}$

17.4.1. $y(1) \approx 1.355$

17.4.2. $y(1) \approx 40.31$

17.4.3. $y(1) \approx 1.05$

17.4.4. $y(1) \approx 2.30$

17.5.4. $\dfrac{\omega+1}{2\omega}e^{\omega t} + \dfrac{\omega-1}{2\omega}e^{-\omega t}$

17.5.5. $2\cos(3t) + 5\sin(3t)$

17.5.6. $-(1/4)e^{-5t} + (5/4)e^{-t}$

17.5.7. $-2e^{-3t} + 2e^{4t}$

17.5.8. $5e^{-6t} + 20te^{-6t}$

17.5.9. $(16t - 3)e^{4t}$

17.5.10. $-2\cos(\sqrt{5}t) + \sqrt{5}\sin(\sqrt{5}t)$

17.5.11. $-\sqrt{2}\cos t + \sqrt{2}\sin t$

17.5.12. $e^{-6t}(4\cos t + 24\sin t)$

17.5.13. $2e^{-3t}\sin(3t)$

17.5.14. $2\cos(2t - \pi/6)$

17.5.15. $5\sqrt{2}\cos(10t - \pi/4)$

17.5.16. $\sqrt{2}e^{-2t}\cos(3t - \pi/4)$

17.5.17. $5e^{4t}\cos(3t + \arcsin(4/5))$

17.5.18. $(2\cos(5t) + \sin(5t))e^{-2t}$

17.5.19. $-(1/2)e^{-2t}\sin(2t)$

17.6.1. $Ae^{5t} + Bte^{5t} + (6/169)\cos t - (5/338)\sin t$

17.6.2. $Ae^{-\sqrt{2}t} + Bte^{-\sqrt{2}t} + 5$

17.6.3. $A\cos(4t) + B\sin(4t) + (1/2)t^2 + (3/16)t - 5/16$

17.6.4. $A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t) - (\cos(5t) + \sin(5t))/23$

17.6.5. $e^t(A\cos t + B\sin t) + e^{2t}/2$

17.6.6. $Ae^{\sqrt{6}t} + Be^{-\sqrt{6}t} + 2 - t/3 - e^{-t}/5$

17.6.7. $Ae^{-3t} + Be^{2t} - (1/5)te^{-3t}$

17.6.8. $Ae^t + Be^{3t} + (1/2)te^{3t}$

17.6.9. $A\cos(4t) + B\sin(4t) + (1/8)t\sin(4t)$

17.6.10. $A\cos(3t) + B\sin(3t) - (1/2)t\cos(3t)$

17.6.11. $Ae^{-6t} + Bte^{-6t} + 3t^2e^{-6t}$

17.6.12. $Ae^{4t} + Bte^{4t} - t^2e^{4t}$

17.6.13. $Ae^{-t} + Be^{-5t} + (4/5)$

17.6.14. $Ae^{4t} + Be^{-3t} + (1/144) - (t/12)$

17.6.15. $A\cos(\sqrt{5}t) + B\sin(\sqrt{5}t) + 8\sin(2t)$

17.6.16. $Ae^{2t} + Be^{-2t} + te^{2t}$

17.6.17. $4e^t + e^{-t} - 3t - 5$

17.6.18. $-(4/27)\sin(3t) + (4/9)t$

17.6.19. $e^{-6t}(2\cos t + 20\sin t) + 2e^{-4t}$

17.6.20. $\left(-\frac{23}{325}\cos(3t) + \frac{592}{975}\sin(3t)\right)e^{-3t} + \frac{23}{325}\cos t - \frac{11}{325}\sin t$

17.6.21. $e^{-2t}(A\sin(5t) + B\cos(5t)) + 8\sin(2t) + 25\cos(2t)$

17.6.22. $e^{-2t}(A\sin(2t) + B\cos(2t)) + (14/195)\sin t - (8/195)\cos t$

17.7.1. $A\sin(t) + B\cos(t) - \cos t\ln|\sec t + \tan t|$

17.7.2. $A\sin(t) + B\cos(t) + \frac{1}{5}e^{2t}$

17.7.3. $A\sin(2t) + B\cos(2t) + \cos t - \sin t\cos t\ln|\sec t + \tan t|$

17.7.4. $A\sin(2t) + B\cos(2t) + \frac{1}{2}\sin(2t)\sin^2(t) + \frac{1}{2}\sin(2t)\ln|\cos t| - \frac{t}{2}\cos(2t) + \frac{1}{4}\sin(2t)\cos(2t)$

17.7.5. $Ae^{2t} + Be^{-3t} + \frac{t^3}{15}e^{2t} - \left(\frac{t^2}{5} - \frac{2t}{25} + \frac{2}{125}\right)\frac{e^{2t}}{5}$

17.7.6. $Ae^t\sin t + Be^t\cos t - e^t\cos t\ln|\sec t + \tan t|$

17.7.7. $Ae^t\sin t + Be^t\cos t - \frac{1}{10}\cos t(\cos^3 t + 3\sin^3 t - 2\cos t - \sin t) + \frac{1}{10}\sin t(\sin^3 t - 3\cos^3 t - 2\sin t + \cos t) = \frac{1}{10}\cos(2t) - \frac{1}{20}\sin(2t)$

B

Useful Formulas

Algebra

Remember that the common algebraic operations have **precedences** relative to each other: for example, multiplication and division take precedence over addition and subtraction, but are "tied" with each other. In the case of ties, work left to right. This means, for example, that $1/2x$ means $(1/2)x$: do the division, then the multiplication in left to right order. It sometimes is a good idea to use more parentheses than strictly necessary, for clarity, but it is also a bad idea to use too many parentheses.

Completing the square: $x^2 + bx + c = (x + \frac{b}{2})^2 - \frac{b^2}{4} + c$.

Quadratic formula: the roots of $ax^2 + bx + c$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Exponent rules:

$$a^b \cdot a^c = a^{b+c}$$
$$\frac{a^b}{a^c} = a^{b-c}$$
$$(a^b)^c = a^{bc}$$
$$a^{1/b} = \sqrt[b]{a}$$

Geometry

Circle: circumference $= 2\pi r$, area $= \pi r^2$.

Ellipse: area $= \pi ab$, where $2a$ and $2b$ are the lengths of the axes of the ellipse.

Sphere: vol $= 4\pi r^3/3$, surface area $= 4\pi r^2$.

Cylinder: vol $= \pi r^2 h$, lateral area $= 2\pi rh$, total surface area $= 2\pi rh + 2\pi r^2$.

Cone: vol $= \pi r^2 h/3$, lateral area $= \pi r\sqrt{r^2+h^2}$, total surface area $= \pi r\sqrt{r^2+h^2} + \pi r^2$.

Analytic geometry

Point-slope formula for straight line through the point (x_0, y_0) with slope m: $y = y_0 + m(x - x_0)$.

Circle with radius r centered at (h, k): $(x-h)^2 + (y-k)^2 = r^2$.

Ellipse with axes on the x-axis and y-axis: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Trigonometry

$\sin(\theta) = \text{opposite}/\text{hypotenuse}$

$\cos(\theta) = \text{adjacent}/\text{hypotenuse}$

$\tan(\theta) = \text{opposite}/\text{adjacent}$

$\sec(\theta) = 1/\cos(\theta)$

$\csc(\theta) = 1/\sin(\theta)$

$\cot(\theta) = 1/\tan(\theta)$

$\tan(\theta) = \sin(\theta)/\cos(\theta)$

$\cot(\theta) = \cos(\theta)/\sin(\theta)$

$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$

$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$

$\sin(\theta + \pi) = -\sin(\theta)$

$\cos(\theta + \pi) = -\cos(\theta)$

Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos A$

Law of sines: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Sine of sum of angles: $\sin(x+y) = \sin x \cos y + \cos x \sin y$

Sine of double angle: $\sin(2x) = 2\sin x \cos x$

Sine of difference of angles: $\sin(x-y) = \sin x \cos y - \cos x \sin y$

Cosine of sum of angles: $\cos(x+y) = \cos x \cos y - \sin x \sin y$

Cosine of double angle: $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Cosine of difference of angles: $\cos(x-y) = \cos x \cos y + \sin x \sin y$

Tangent of sum of angles: $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$\sin^2(\theta)$ and $\cos^2(\theta)$ formulas:

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 \\
\tan^2(\theta) + 1 &= \sec^2(\theta) \\
\sin^2(\theta) &= \frac{1 - \cos(2\theta)}{2} \\
\cos^2(\theta) &= \frac{1 + \cos(2\theta)}{2}
\end{aligned}$$

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