12 Multiple Integrals

In this chapter we extend the idea of a definite integral to double and triple integrals of functions of two or three variables. These ideas are then used to compute volumes, surface areas, masses, and centroids of more general regions than we were able to consider in Chapter 6. We also use double integrals to calculate probabilities when two random variables are involved.

12.1 Double Integrals over Rectangles

In much the same way that our attempt to solve the area problem led to the definition of a definite integral, we now seek to find the volume of a solid and in the process we arrive at the definition of a double integral.

Review of the Definite Integral

First let's recall the basic facts concerning definite integrals of functions of a single variable. If $f(x)$ is defined for $a \leqslant x \leqslant b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

1
$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

and take the limit of such sums as $n \to \infty$ to obtain the definite integral of f from a to b:

2
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

In the special case where $f(x) \geqslant 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles in Figure 1, and $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b.

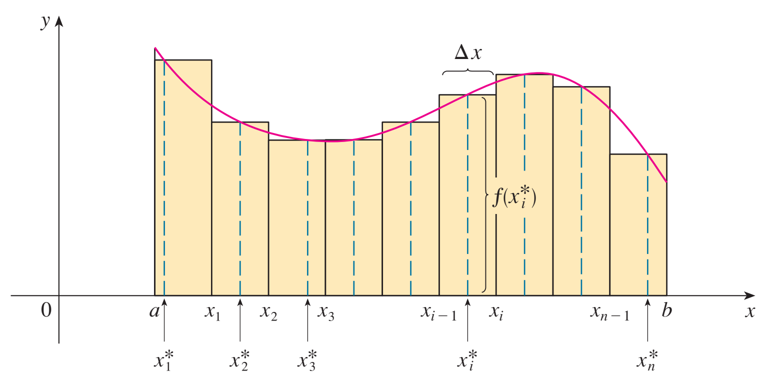


FIGURE 1

Volumes and Double Integrals

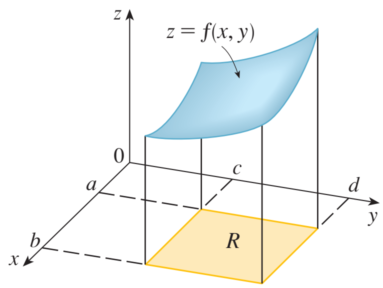


FIGURE 2

In a similar manner we consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$$

and we first suppose that $f(x, y) \geqslant 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant z \leqslant f(x, y),\ (x, y) \in R\}$$

(See Figure 2.) Our goal is to find the volume of S.

The first step is to divide the rectangle R into subrectangles. We do this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$. By drawing lines parallel to the coordinate axes through the endpoints of these subintervals as in Figure 3, we form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}$$

each with area $\Delta A = \Delta x\, \Delta y$.

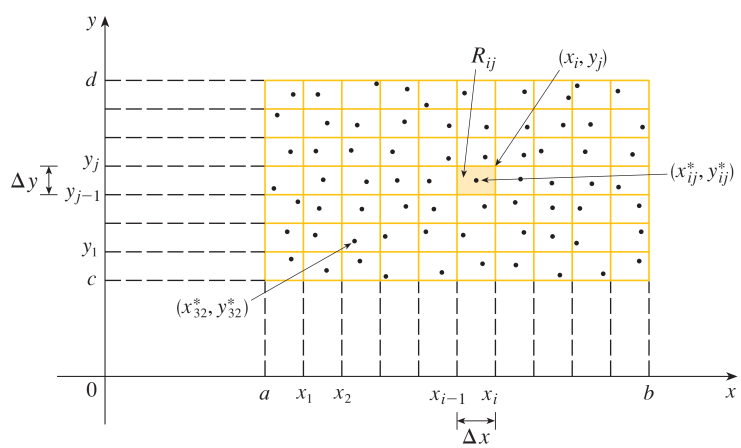


FIGURE 3
Dividing R into subrectangles

If we choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in Figure 4. (Compare with Figure 1.) The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\ \Delta A$$

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S:

3
$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\ \Delta A$$

(See Figure 5.) This double sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle, and then we add the results.

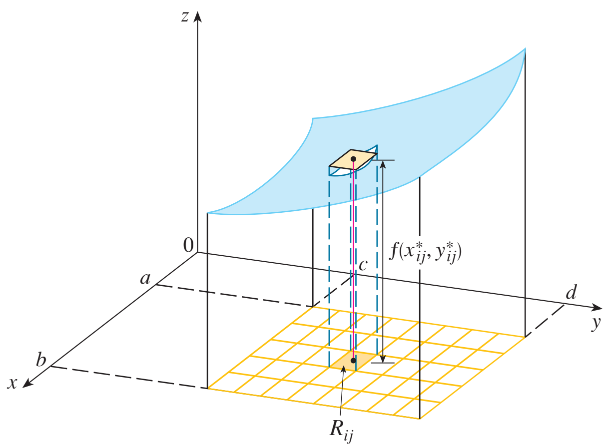


FIGURE 4

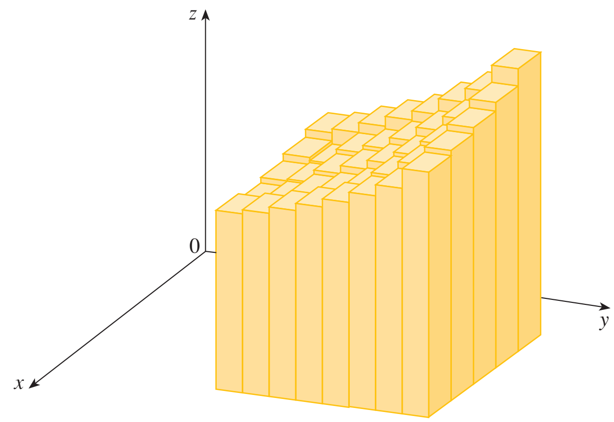


FIGURE 5

Our intuition tells us that the approximation given in (3) becomes better as m and n become larger and so we would expect that

▲ The meaning of the double limit in Equation 4 is that we can make the double sum as close as we like to the number V [for any choice of (x_{ij}^*, y_{ij}^*)] by taking m and n sufficiently large.

4
$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

We use the expression in Equation 4 to define the **volume** of the solid S that lies under the graph of f and above the rectangle R. (It can be shown that this definition is consistent with our formula for volume in Section 6.2.)

Limits of the type that appear in Equation 4 occur frequently, not just in finding volumes but in a variety of other situations as well—as we will see in Section 12.5—even when f is not a positive function. So we make the following definition.

▲ Notice the similarity between Definition 5 and the definition of a single integral in Equation 2.

5 Definition The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

if this limit exists.

It can be proved that the limit in Definition 5 exists if f is a continuous function. (It also exists for some discontinuous functions as long as they are reasonably "well behaved.")

The sample point (x_{ij}^*, y_{ij}^*) can be chosen to be any point in the subrectangle R_{ij}, but if we choose it to be the upper right-hand corner of R_{ij} [namely (x_i, y_j), see Figure 3], then the expression for the double integral looks simpler:

6
$$\iint_R f(x, y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\,\Delta A$$

By comparing Definitions 4 and 5, we see that a volume can be written as a double integral:

If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$

The sum in Definition 5,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral. [Notice how similar it is to the Riemann sum in (1) for a function of a single variable.] If f happens to be a *positive* function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 5, and is an approximation to the volume under the graph of f.

EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

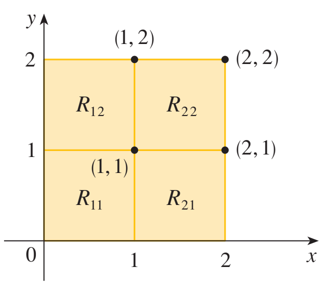


FIGURE 6

SOLUTION The squares are shown in Figure 6. The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is 1. Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$\begin{aligned} V &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A \\ &= f(1, 1)\, \Delta A + f(1, 2)\, \Delta A + f(2, 1)\, \Delta A + f(2, 2)\, \Delta A \\ &= 13(1) + 7(1) + 10(1) + 4(1) = 34 \end{aligned}$$

This is the volume of the approximating rectangular boxes shown in Figure 7.

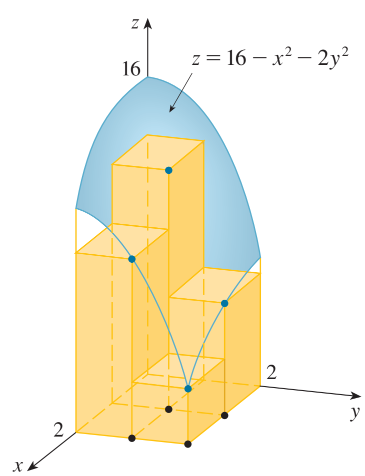
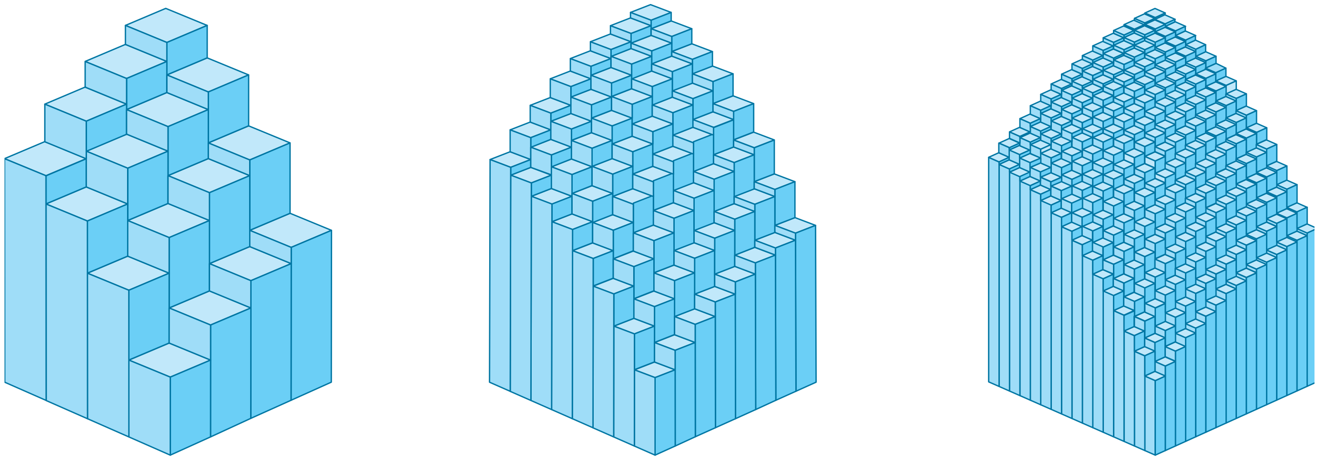


FIGURE 7

■

We get better approximations to the volume in Example 1 if we increase the number of squares. Figure 8 shows how the columns start to look more like the actual solid and the corresponding approximations become more accurate when we use 16, 64, and 256 squares. In the next section we will be able to show that the exact volume is 48.



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

FIGURE 8 The Riemann sum approximations to the volume under $z = 16 - x^2 - 2y^2$ become more accurate as m and n increase.

EXAMPLE 2 If $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$

SOLUTION It would be very difficult to evaluate this integral directly from Definition 5 but, because $\sqrt{1 - x^2} \geqslant 0$, we can compute the integral by interpreting it as a volume. If $z = \sqrt{1 - x^2}$, then $x^2 + z^2 = 1$ and $z \geqslant 0$, so the given double integral represents the volume of the solid S that lies below the circular cylinder $x^2 + z^2 = 1$ and above the rectangle R. (See Figure 9.) The volume of S is the area of a semicircle with radius 1 times the length of the cylinder. Thus

$$\iint_R \sqrt{1 - x^2}\, dA = \tfrac{1}{2}\pi(1)^2 \times 4 = 2\pi$$

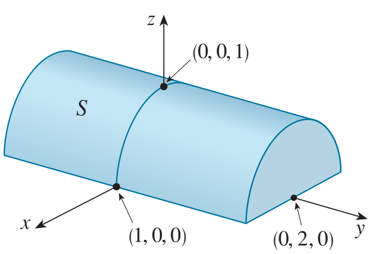


FIGURE 9

The Midpoint Rule

The methods that we used for approximating single integrals (the Midpoint Rule, the Trapezoidal Rule, Simpson's Rule) all have counterparts for double integrals. Here we consider only the Midpoint Rule for double integrals. This means that we use a double Riemann sum to approximate the double integral, where the sample point (x_{ij}^*, y_{ij}^*) in R_{ij} is chosen to be the center $(\bar{x}_i, \bar{y}_j)$ of R_{ij}. In other words, $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Midpoint Rule for Double Integrals

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

EXAMPLE 3 Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$.

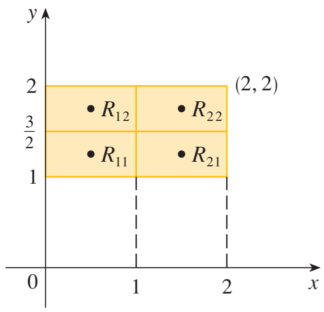


FIGURE 10

SOLUTION In using the Midpoint Rule with $m = n = 2$, we evaluate $f(x, y) = x - 3y^2$ at the centers of the four subrectangles shown in Figure 10. So $\bar{x}_1 = \frac{1}{2}$, $\bar{x}_2 = \frac{3}{2}$, $\bar{y}_1 = \frac{5}{4}$, and $\bar{y}_2 = \frac{7}{4}$. The area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus

$$\begin{aligned}
\iint_R (x - 3y^2)\, dA &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\, \Delta A \\
&= f(\bar{x}_1, \bar{y}_1)\, \Delta A + f(\bar{x}_1, \bar{y}_2)\, \Delta A + f(\bar{x}_2, \bar{y}_1)\, \Delta A + f(\bar{x}_2, \bar{y}_2)\, \Delta A \\
&= f\left(\tfrac{1}{2}, \tfrac{5}{4}\right) \Delta A + f\left(\tfrac{1}{2}, \tfrac{7}{4}\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{5}{4}\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{7}{4}\right) \Delta A \\
&= \left(-\tfrac{67}{16}\right)\tfrac{1}{2} + \left(-\tfrac{139}{16}\right)\tfrac{1}{2} + \left(-\tfrac{51}{16}\right)\tfrac{1}{2} + \left(-\tfrac{123}{16}\right)\tfrac{1}{2} \\
&= -\tfrac{95}{8} = -11.875
\end{aligned}$$

Thus, we have
$$\iint_R (x - 3y^2)\, dA \approx -11.875$$

Number of subrectangles	Midpoint Rule approximations
1	−11.5000
4	−11.8750
16	−11.9687
64	−11.9922
256	−11.9980
1024	−11.9995

NOTE • In the next section we will develop an efficient method for computing double integrals and then we will see that the exact value of the double integral in Example 3 is -12. (Remember that the interpretation of a double integral as a volume is valid only when the integrand f is a *positive* function. The integrand in Example 3 is not a positive function, so its integral is not a volume. In Examples 2 and 3 in Section 12.2 we will discuss how to interpret integrals of functions that are not always positive in terms of volumes.) If we keep dividing each subrectangle in Figure 10 into four smaller ones with similar shape, we get the Midpoint Rule approximations displayed in the chart in the margin. Notice how these approximations approach the exact value of the double integral, -12.

Average Value

Recall from Section 6.4 that the average value of a function f of one variable defined on an interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x)\, dx$$

In a similar fashion we define the **average value** of a function f of two variables defined on a rectangle R to be

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

where $A(R)$ is the area of R.

If $f(x, y) \geqslant 0$, the equation

$$A(R) \times f_{\text{ave}} = \iint_R f(x, y)\, dA$$

says that the box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f. [If $z = f(x, y)$ describes a mountainous region and you chop off the tops of the mountains at height f_{ave}, then you can use them to fill in the valleys so that the region becomes completely flat. See Figure 11.]

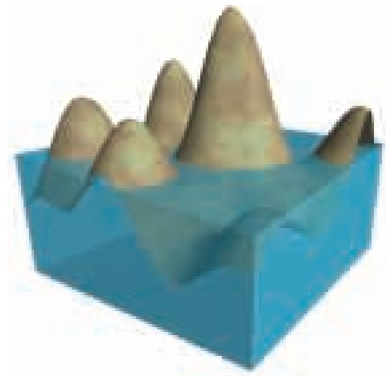


FIGURE 11

EXAMPLE 4 The contour map in Figure 12 shows the snowfall, in inches, that fell on the state of Colorado on December 24, 1982. (The state is in the shape of a rectangle that measures 388 mi west to east and 276 mi south to north.) Use the contour map to estimate the average snowfall for Colorado as a whole on December 24.

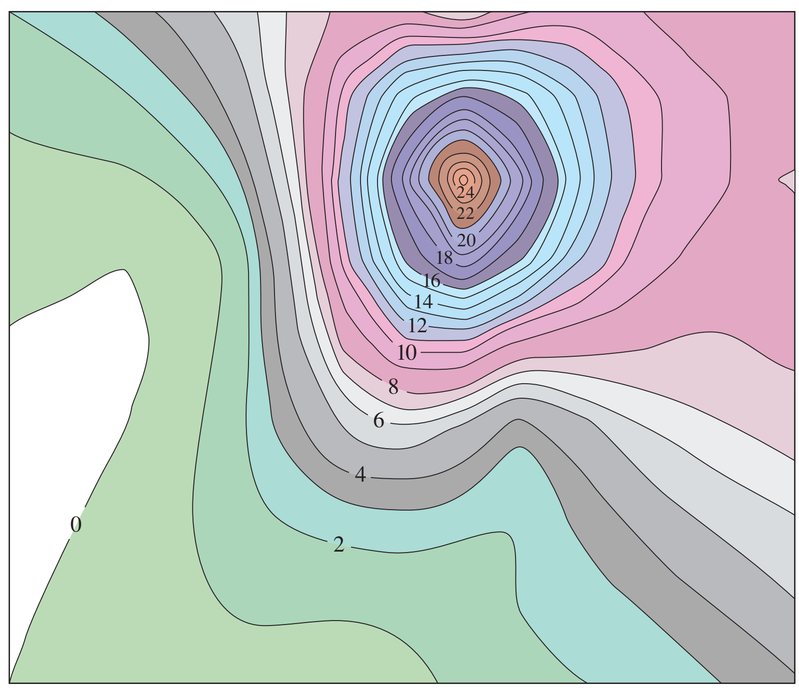


FIGURE 12

SOLUTION Let's place the origin at the southwest corner of the state. Then $0 \leqslant x \leqslant 388$, $0 \leqslant y \leqslant 276$, and $f(x, y)$ is the snowfall, in inches, at a location x miles to the east and y miles to the north of the origin. If R is the rectangle that represents Colorado, then the average snowfall for Colorado on December 24 was

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

where $A(R) = 388 \cdot 276$. To estimate the value of this double integral let's use the Midpoint Rule with $m = n = 4$. In other words, we divide R into 16 subrectangles of equal size, as in Figure 13. The area of each subrectangle is

$$\Delta A = \tfrac{1}{16}(388)(276) = 6693 \text{ mi}^2$$

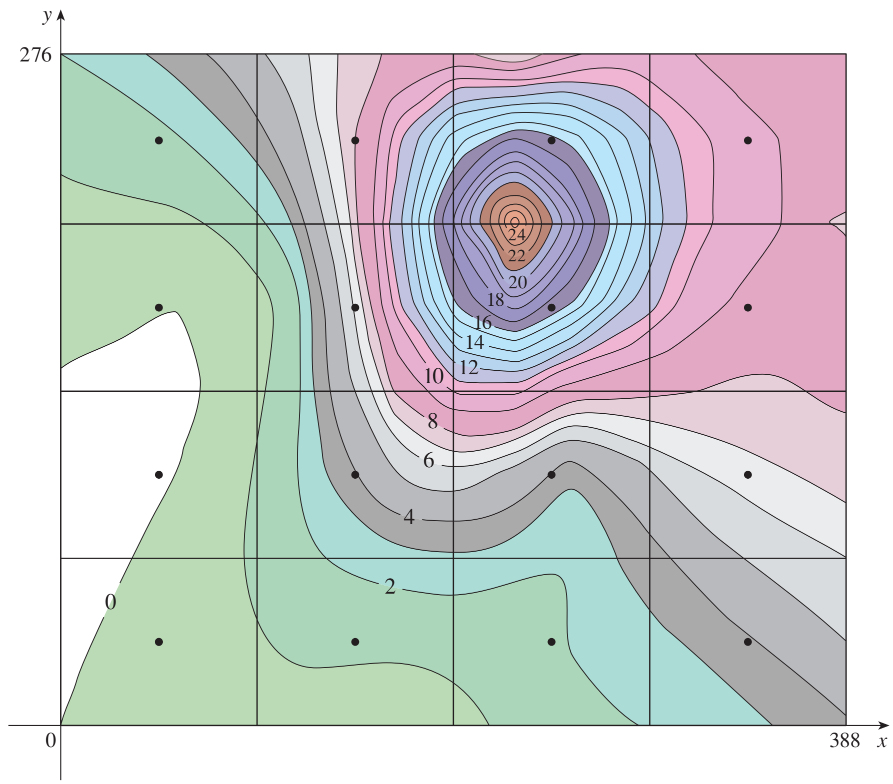


FIGURE 13

Using the contour map to estimate the value of f at the center of each subrectangle, we get

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{4} \sum_{j=1}^{4} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

$$\approx \Delta A[0.4 + 1.2 + 1.8 + 3.9 + 0 + 3.9 + 4.0 + 6.5$$
$$+ 0.1 + 6.1 + 16.5 + 8.8 + 1.8 + 8.0 + 16.2 + 9.4]$$
$$= (6693)(88.6)$$

Therefore $$f_{\text{ave}} \approx \frac{(6693)(88.6)}{(388)(276)} \approx 5.5$$

On December 24, 1982, Colorado received an average of approximately $5\frac{1}{2}$ inches of snow. ■

Properties of Double Integrals

We list here three properties of double integrals that can be proved in the same manner as in Section 5.2. We assume that all of the integrals exist. Properties 7 and 8 are referred to as the *linearity* of the integral.

7 $$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

▲ Double integrals behave this way because the double sums that define them behave this way.

8 $$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA \qquad \text{where } c \text{ is a constant}$$

If $f(x, y) \geqslant g(x, y)$ for all (x, y) in R, then

9 $$\iint_R f(x, y)\, dA \geqslant \iint_R g(x, y)\, dA$$

12.1 Exercises

1. Find approximations to $\iint_R (x - 3y^2)\, dA$ using the same subrectangles as in Example 3 but choosing the sample point to be the (a) upper left corner, (b) upper right corner, (c) lower left corner, (d) lower right corner of each subrectangle.

2. Find the approximation to the volume in Example 1 if the Midpoint Rule is used.

3. (a) Estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle $R = \{(x, y) \mid 0 \leqslant x \leqslant 6, 0 \leqslant y \leqslant 4\}$. Use a Riemann sum with $m = 3$, $n = 2$, and take the sample point to be the upper right corner of each subrectangle.
(b) Use the Midpoint Rule to estimate the volume of the solid in part (a).

4. If $R = [-1, 3] \times [0, 2]$, use a Riemann sum with $m = 4$, $n = 2$ to estimate the value of $\iint_R (y^2 - 2x^2)\, dA$. Take the sample points to be the upper left corners of the subrectangles.

5. A table of values is given for a function $f(x, y)$ defined on $R = [1, 3] \times [0, 4]$.
(a) Estimate $\iint_R f(x, y)\, dA$ using the Midpoint Rule with $m = n = 2$.
(b) Estimate the double integral with $m = n = 4$ by choosing the sample points to be the points farthest from the origin.

x \ y	0	1	2	3	4
1.0	2	0	−3	−6	−5
1.5	3	1	−4	−8	−6
2.0	4	3	0	−5	−8
2.5	5	5	3	−1	−4
3.0	7	8	6	3	0

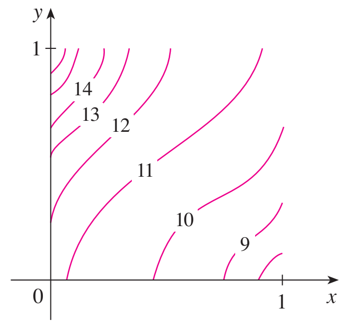
6. A 20-ft-by-30-ft swimming pool is filled with water. The depth is measured at 5-ft intervals, starting at one corner of the pool, and the values are recorded in the table. Estimate the volume of water in the pool.

	0	5	10	15	20	25	30
0	2	3	4	6	7	8	8
5	2	3	4	7	8	10	8
10	2	4	6	8	10	12	10
15	2	3	4	5	6	8	7
20	2	2	2	2	3	4	4

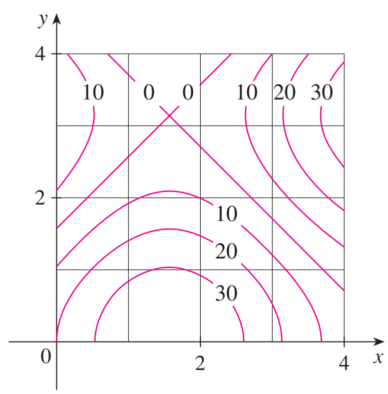
7. Let V be the volume of the solid that lies under the graph of $f(x, y) = \sqrt{52 - x^2 - y^2}$ and above the rectangle given by $2 \leqslant x \leqslant 4$, $2 \leqslant y \leqslant 6$. We use the lines $x = 3$ and $y = 4$

to divide R into subrectangles. Let L and U be the Riemann sums computed using lower left corners and upper right corners, respectively. Without calculating the numbers V, L, and U, arrange them in increasing order and explain your reasoning.

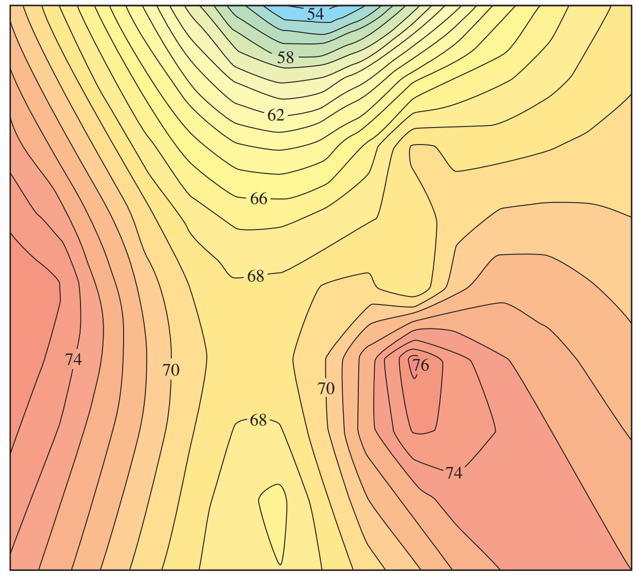
8. The figure shows level curves of a function f in the square $R = [0, 1] \times [0, 1]$. Use them to estimate $\iint_R f(x, y)\, dA$ to the nearest integer.



9. A contour map is shown for a function f on the square $R = [0, 4] \times [0, 4]$.
(a) Use the Midpoint Rule with $m = n = 2$ to estimate the value of $\iint_R f(x, y)\, dA$.
(b) Estimate the average value of f.



10. The contour map shows the temperature, in degrees Fahrenheit, at 3:00 P.M. on May 1, 1996, in Colorado. (The state measures 388 mi east to west and 276 mi north to south.) Use the Midpoint Rule to estimate the average temperature in Colorado at that time.



11–13 ■ Evaluate the double integral by first identifying it as the volume of a solid.

11. $\iint_R 3\, dA, \quad R = \{(x, y) \mid -2 \leqslant x \leqslant 2, 1 \leqslant y \leqslant 6\}$

12. $\iint_R (5 - x)\, dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 5, 0 \leqslant y \leqslant 3\}$

13. $\iint_R (4 - 2y)\, dA, \quad R = [0, 1] \times [0, 1]$

14. The integral $\iint_R \sqrt{9 - y^2}\, dA$, where $R = [0, 4] \times [0, 2]$, represents the volume of a solid. Sketch the solid.

15. Use a programmable calculator or computer (or the sum command on a CAS) to estimate

$$\iint_R e^{-x^2-y^2}\, dA$$

where $R = [0, 1] \times [0, 1]$. Use the Midpoint Rule with the following numbers of squares of equal size: 1, 4, 16, 64, 256, and 1024.

16. Repeat Exercise 15 for the integral $\iint_R \cos(x^4 + y^4)\, dA$.

17. If f is a constant function, $f(x, y) = k$, and $R = [a, b] \times [c, d]$, show that $\iint_R k\, dA = k(b - a)(d - c)$.

18. If $R = [0, 1] \times [0, 1]$, show that $0 \leqslant \iint_R \sin(x + y)\, dA \leqslant 1$.

12.2 Iterated Integrals

Recall that it is usually difficult to evaluate single integrals directly from the definition of an integral, but the Evaluation Theorem (Part 2 of the Fundamental Theorem of Calculus) provides a much easier method. The evaluation of double integrals from first principles is even more difficult, but in this section we see how to express a double integral as an iterated integral, which can then be evaluated by calculating two single integrals.

Suppose that f is a function of two variables that is continuous on the rectangle $R = [a, b] \times [c, d]$. We use the notation $\int_c^d f(x, y)\,dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called *partial integration with respect to y*. (Notice its similarity to partial differentiation.) Now $\int_c^d f(x, y)\,dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x, y)\,dy$$

If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get

1 $$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy\right] dx$$

The integral on the right side of Equation 1 is called an **iterated integral**. Usually the brackets are omitted. Thus

2 $$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy\right] dx$$

means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly, the iterated integral

3 $$\int_c^d \int_a^b f(x, y)\,dx\,dy = \int_c^d \left[\int_a^b f(x, y)\,dx\right] dy$$

means that we first integrate with respect to x (holding y fixed) from $x = a$ to $x = b$ and then we integrate the resulting function of y with respect to y from $y = c$ to $y = d$. Notice that in both Equations 2 and 3 we work *from the inside out.*

EXAMPLE 1 Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y\,dy\,dx$ (b) $\int_1^2 \int_0^3 x^2 y\,dx\,dy$

SOLUTION

(a) Regarding x as a constant, we obtain

$$\int_1^2 x^2 y\,dy = \left[x^2 \frac{y^2}{2}\right]_{y=1}^{y=2}$$

$$= x^2\left(\frac{2^2}{2}\right) - x^2\left(\frac{1^2}{2}\right) = \tfrac{3}{2}x^2$$

Thus, the function A in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example. We now integrate this function of x from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx$$
$$= \int_0^3 \tfrac{3}{2} x^2 \, dx = \frac{x^3}{2} \bigg]_0^3 = \frac{27}{2}$$

(b) Here we first integrate with respect to x:

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy = \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy$$
$$= \int_1^2 9y \, dy = 9 \frac{y^2}{2} \bigg]_1^2 = \frac{27}{2}$$

■

Notice that in Example 1 we obtained the same answer whether we integrated with respect to y or x first. In general, it turns out (see Theorem 4) that the two iterated integrals in Equations 2 and 3 are always equal; that is, the order of integration does not matter. (This is similar to Clairaut's Theorem on the equality of the mixed partial derivatives.)

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

▲ Theorem 4 is named after the Italian mathematician Guido Fubini (1879–1943), who proved a very general version of this theorem in 1907. But the version for continuous functions was known to the French mathematician Augustin-Louis Cauchy almost a century earlier.

4 Fubini's Theorem If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b, \, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

The proof of Fubini's Theorem is too difficult to include in this book, but we can at least give an intuitive indication of why it is true for the case where $f(x, y) \geqslant 0$. Recall that if f is positive, then we can interpret the double integral $\iint_R f(x, y) \, dA$ as the volume V of the solid S that lies above R and under the surface $z = f(x, y)$. But we have another formula that we used for volume in Chapter 6, namely,

$$V = \int_a^b A(x) \, dx$$

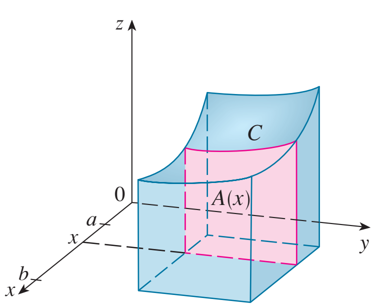


FIGURE 1

where $A(x)$ is the area of a cross-section of S in the plane through x perpendicular to the x-axis. From Figure 1 you can see that $A(x)$ is the area under the curve C whose equation is $z = f(x, y)$, where x is held constant and $c \leqslant y \leqslant d$. Therefore

$$A(x) = \int_c^d f(x, y) \, dy$$

and we have

$$\iint_R f(x, y) \, dA = V = \int_a^b A(x) \, dx = \int_a^b \int_c^d f(x, y) \, dy \, dx$$

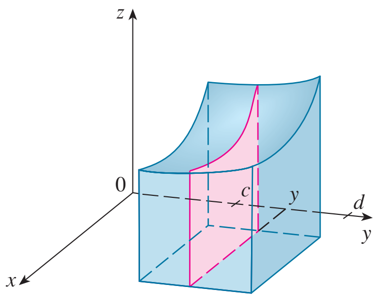


FIGURE 2

▲ Notice the negative answer in Example 2; nothing is wrong with that. The function f in that example is not a positive function, so its integral doesn't represent a volume. From Figure 3 we see that f is always negative on R, so the value of the integral is the *negative* of the volume that lies *above* the graph of f and *below* R.

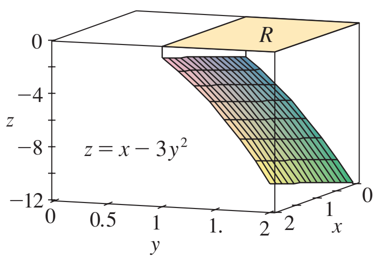


FIGURE 3

▲ For a function f that takes on both positive and negative values, $\iint_R f(x, y)\,dA$ is a difference of volumes: $V_1 - V_2$, where V_1 is the volume above R and below the graph of f and V_2 is the volume below R and above the graph. The fact that the integral in Example 3 is 0 means that these two volumes V_1 and V_2 are equal. (See Figure 4.)

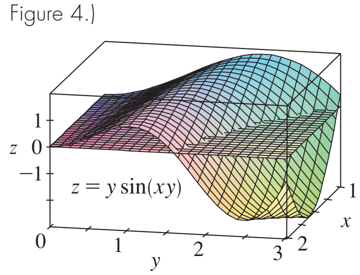


FIGURE 4

A similar argument, using cross-sections perpendicular to the y-axis as in Figure 2, shows that

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

EXAMPLE 2 Evaluate the double integral $\iint_R (x - 3y^2)\,dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$. (Compare with Example 3 in Section 12.1.)

SOLUTION 1 Fubini's Theorem gives

$$\iint_R (x - 3y^2)\,dA = \int_0^2 \int_1^2 (x - 3y^2)\,dy\,dx$$

$$= \int_0^2 \Big[xy - y^3\Big]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\,dx = \frac{x^2}{2} - 7x\Big]_0^2 = -12$$

SOLUTION 2 Again applying Fubini's Theorem, but this time integrating with respect to x first, we have

$$\iint_R (x - 3y^2)\,dA = \int_1^2 \int_0^2 (x - 3y^2)\,dx\,dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy$$

$$= \int_1^2 (2 - 6y^2)\,dy = 2y - 2y^3\Big]_1^2 = -12$$

■

EXAMPLE 3 Evaluate $\iint_R y\sin(xy)\,dA$, where $R = [1, 2] \times [0, \pi]$.

SOLUTION 1 If we first integrate with respect to x, we get

$$\iint_R y\sin(xy)\,dA = \int_0^\pi \int_1^2 y\sin(xy)\,dx\,dy$$

$$= \int_0^\pi \Big[-\cos(xy)\Big]_{x=1}^{x=2} dy$$

$$= \int_0^\pi (-\cos 2y + \cos y)\,dy$$

$$= -\tfrac{1}{2}\sin 2y + \sin y\Big]_0^\pi = 0$$

SOLUTION 2 If we reverse the order of integration, we get

$$\iint_R y\sin(xy)\,dA = \int_1^2 \int_0^\pi y\sin(xy)\,dy\,dx$$

To evaluate the inner integral we use integration by parts with

$$u = y \qquad dv = \sin(xy)\,dy$$

$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

and so
$$\int_0^\pi y\sin(xy)\,dy = -\frac{y\cos(xy)}{x}\Bigg]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^\pi \cos(xy)\,dy$$
$$= -\frac{\pi\cos\pi x}{x} + \frac{1}{x^2}\big[\sin(xy)\big]_{y=0}^{y=\pi}$$
$$= -\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}$$

If we now integrate the first term by parts with $u = -1/x$ and $dv = \pi\cos\pi x\,dx$, we get $du = dx/x^2$, $v = \sin\pi x$, and
$$\int\left(-\frac{\pi\cos\pi x}{x}\right)dx = -\frac{\sin\pi x}{x} - \int\frac{\sin\pi x}{x^2}\,dx$$

Therefore
$$\int\left(-\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}\right)dx = -\frac{\sin\pi x}{x}$$

and so
$$\int_1^2\int_0^\pi y\sin(xy)\,dy\,dx = \left[-\frac{\sin\pi x}{x}\right]_1^2$$
$$= -\frac{\sin 2\pi}{2} + \sin\pi = 0$$ ■

▲ In Example 2, Solutions 1 and 2 are equally straightforward, but in Example 3 the first solution is much easier than the second one. Therefore, when we evaluate double integrals it is wise to choose the order of integration that gives simpler integrals.

EXAMPLE 4 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

SOLUTION We first observe that S is the solid that lies under the surface $z = 16 - x^2 - 2y^2$ and above the square $R = [0, 2] \times [0, 2]$. (See Figure 5.) This solid was considered in Example 1 in Section 12.1, but we are now in a position to evaluate the double integral using Fubini's Theorem. Therefore

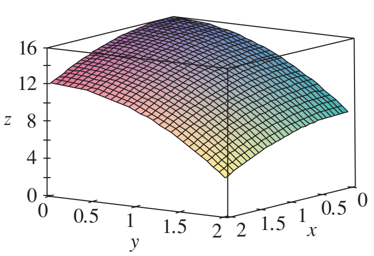


FIGURE 5

$$V = \iint_R (16 - x^2 - 2y^2)\,dA = \int_0^2\int_0^2 (16 - x^2 - 2y^2)\,dx\,dy$$
$$= \int_0^2 \left[16x - \tfrac{1}{3}x^3 - 2y^2x\right]_{x=0}^{x=2} dy$$
$$= \int_0^2 \left(\tfrac{88}{3} - 4y^2\right)dy = \left[\tfrac{88}{3}y - \tfrac{4}{3}y^3\right]_0^2 = 48$$ ■

In the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form. To be specific, suppose that $f(x, y) = g(x)h(y)$ and $R = [a, b] \times [c, d]$. Then Fubini's Theorem gives
$$\iint_R f(x, y)\,dA = \int_c^d\int_a^b g(x)h(y)\,dx\,dy = \int_c^d\left[\int_a^b g(x)h(y)\,dx\right]dy$$

In the inner integral y is a constant, so $h(y)$ is a constant and we can write
$$\int_c^d\left[\int_a^b g(x)h(y)\,dx\right]dy = \int_c^d\left[h(y)\left(\int_a^b g(x)\,dx\right)\right]dy$$
$$= \int_a^b g(x)\,dx\int_c^d h(y)\,dy$$

since $\int_a^b g(x)\,dx$ is a constant. Therefore, in this case, the double integral of f can be written as the product of two single integrals:

$$\iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy \qquad \text{where } R = [a, b] \times [c, d]$$

EXAMPLE 5 If $R = [0, \pi/2] \times [0, \pi/2]$, then

$$\iint_R \sin x \cos y\,dA = \int_0^{\pi/2} \sin x\,dx \int_0^{\pi/2} \cos y\,dy$$

$$= \big[-\cos x\big]_0^{\pi/2} \big[\sin y\big]_0^{\pi/2} = 1 \cdot 1 = 1$$

▲ The function $f(x, y) = \sin x \cos y$ in Example 5 is positive on R, so the integral represents the volume of the solid that lies above R and below the graph of f shown in Figure 6.

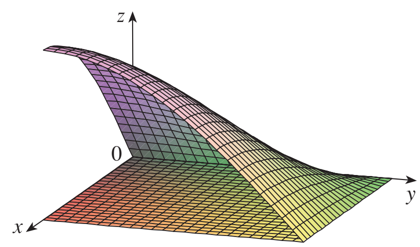


FIGURE 6

12.2 Exercises

1–2 ■ Find $\int_0^3 f(x, y)\,dx$ and $\int_0^4 f(x, y)\,dy$.

1. $f(x, y) = 2x + 3x^2y$

2. $f(x, y) = \dfrac{y}{x + 2}$

3–10 ■ Calculate the iterated integral.

3. $\int_1^3 \int_0^1 (1 + 4xy)\,dx\,dy$

4. $\int_2^4 \int_{-1}^1 (x^2 + y^2)\,dy\,dx$

5. $\int_0^3 \int_0^1 \sqrt{x + y}\,dx\,dy$

6. $\int_1^4 \int_0^2 (x + \sqrt{y})\,dx\,dy$

7. $\int_1^4 \int_1^2 \left(\dfrac{x}{y} + \dfrac{y}{x}\right) dy\,dx$

8. $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x + y)\,dy\,dx$

9. $\int_0^{\ln 2} \int_0^{\ln 5} e^{2x-y}\,dx\,dy$

10. $\int_0^1 \int_0^1 \dfrac{xy}{\sqrt{x^2 + y^2 + 1}}\,dy\,dx$

11–16 ■ Calculate the double integral.

11. $\iint_R (6x^2y^3 - 5y^4)\,dA,\quad R = \{(x, y) \mid 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant 1\}$

12. $\iint_R xye^y\,dA,\quad R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 1\}$

13. $\iint_R \dfrac{xy^2}{x^2 + 1}\,dA,\quad R = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ -3 \leqslant y \leqslant 3\}$

14. $\iint_R \dfrac{1 + x^2}{1 + y^2}\,dA,\quad R = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$

15. $\iint_R x\sin(x + y)\,dA,\quad R = [0, \pi/6] \times [0, \pi/3]$

16. $\iint_R xe^{xy}\,dA,\quad R = [0, 1] \times [0, 1]$

17–18 ■ Sketch the solid whose volume is given by the iterated integral.

17. $\int_0^1 \int_0^1 (4 - x - 2y)\,dx\,dy$

18. $\int_0^1 \int_0^1 (2 - x^2 - y^2)\,dy\,dx$

19. Find the volume of the solid lying under the plane $z = 2x + 5y + 1$ and above the rectangle $R = \{(x, y) \mid -1 \leqslant x \leqslant 0,\ 1 \leqslant y \leqslant 4\}$.

20. Find the volume of the solid lying under the circular paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2, 2] \times [-3, 3]$.

21. Find the volume of the solid lying under the elliptic paraboloid $x^2/4 + y^2/9 + z = 1$ and above the square $R = [-1, 1] \times [-2, 2]$.

22. Find the volume of the solid lying under the hyperbolic paraboloid $z = y^2 - x^2$ and above the square $R = [-1, 1] \times [1, 3]$.

23. Find the volume of the solid bounded by the surface $z = x\sqrt{x^2 + y}$ and the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, and $z = 0$.

24. Find the volume of the solid bounded by the elliptic paraboloid $z = 1 + (x - 1)^2 + 4y^2$, the planes $x = 3$ and $y = 2$, and the coordinate planes.

25. Find the volume of the solid in the first octant bounded by the cylinder $z = 9 - y^2$ and the plane $x = 2$.

26. (a) Find the volume of the solid bounded by the surface $z = 6 - xy$ and the planes $x = 2$, $x = -2$, $y = 0$, $y = 3$, and $z = 0$.
(b) Use a computer to draw the solid.

CAS **27.** Use a computer algebra system to find the exact value of the integral $\iint_R x^5 y^3 e^{xy}\, dA$, where $R = [0, 1] \times [0, 1]$. Then use the CAS to draw the solid whose volume is given by the integral.

CAS **28.** Graph the solid that lies between the surfaces $z = e^{-x^2}\cos(x^2 + y^2)$ and $z = 2 - x^2 - y^2$ for $|x| \leqslant 1$, $|y| \leqslant 1$. Use a computer algebra system to approximate the volume of this solid correct to four decimal places.

29–30 ■ Find the average value of f over the given rectangle.

29. $f(x, y) = x^2 y$,
R has vertices $(-1, 0)$, $(-1, 5)$, $(1, 5)$, $(1, 0)$

30. $f(x, y) = x \sin xy$, $\quad R = [0, \pi/2] \times [0, 1]$

CAS **31.** Use your CAS to compute the iterated integrals

$$\int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dy\, dx \qquad \text{and} \qquad \int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dx\, dy$$

Do the answers contradict Fubini's Theorem? Explain what is happening.

32. (a) In what way are the theorems of Fubini and Clairaut similar?
(b) If $f(x, y)$ is continuous on $[a, b] \times [c, d]$ and

$$g(x, y) = \int_a^x \int_c^y f(s, t)\, dt\, ds$$

for $a < x < b$, $c < y < d$, show that $g_{xy} = g_{yx} = f(x, y)$.

12.3 Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function f not just over rectangles but also over regions D of more general shape, such as the one illustrated in Figure 1. We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2. Then we define a new function F with domain R by

1
$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

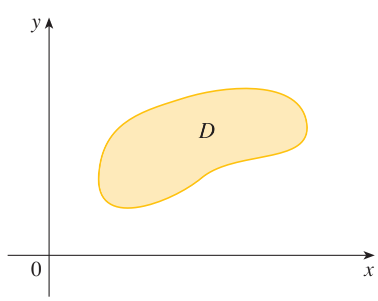


FIGURE 1

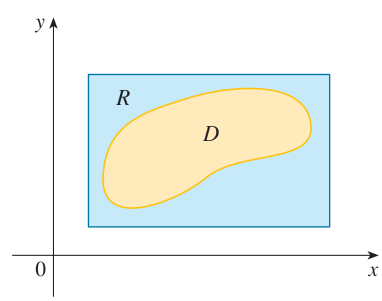


FIGURE 2

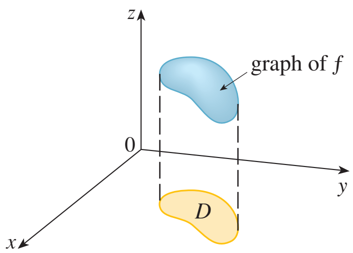


FIGURE 3

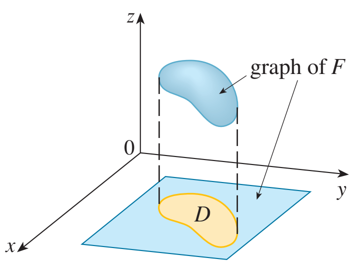


FIGURE 4

If the double integral of F exists over R, then we define the **double integral of f over D** by

$$\boxed{2} \qquad \iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}$$

Definition 2 makes sense because R is a rectangle and so $\iint_R F(x, y)\, dA$ has been previously defined in Section 12.1. The procedure that we have used is reasonable because the values of $F(x, y)$ are 0 when (x, y) lies outside D and so they contribute nothing to the integral. This means that it doesn't matter what rectangle R we use as long as it contains D.

In the case where $f(x, y) \geqslant 0$ we can still interpret $\iint_D f(x, y)\, dA$ as the volume of the solid that lies above D and under the surface $z = f(x, y)$ (the graph of f). You can see that this is reasonable by comparing the graphs of f and F in Figures 3 and 4 and remembering that $\iint_R F(x, y)\, dA$ is the volume under the graph of F.

Figure 4 also shows that F is likely to have discontinuities at the boundary points of D. Nonetheless, if f is continuous on D and the boundary curve of D is "well behaved" (in a sense outside the scope of this book), then it can be shown that $\iint_R F(x, y)\, dA$ exists and therefore $\iint_D f(x, y)\, dA$ exists. In particular, this is the case for the following types of regions.

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

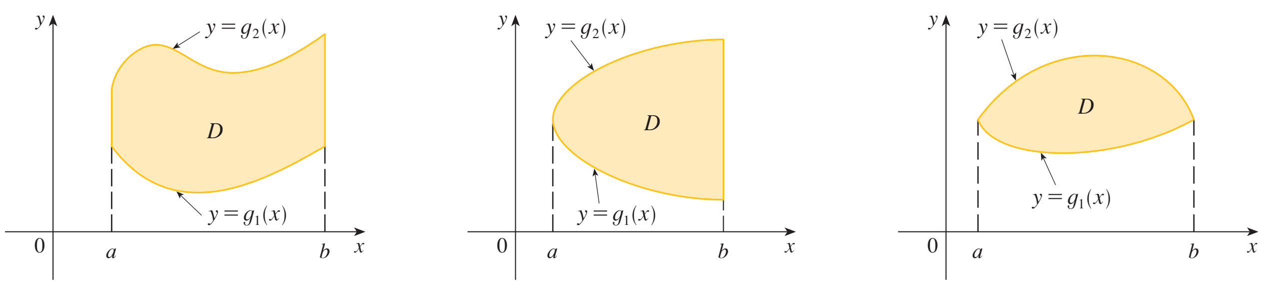


FIGURE 5 Some type I regions

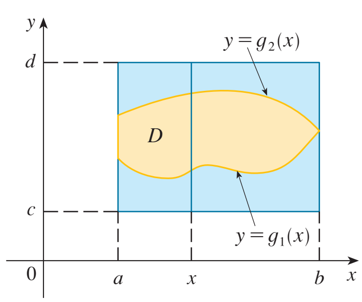


FIGURE 6

In order to evaluate $\iint_D f(x, y)\, dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D, as in Figure 6, and we let F be the function given by Equation 1; that is, F agrees with f on D and F is 0 outside D. Then, by Fubini's Theorem,

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Observe that $F(x, y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ because (x, y) then lies outside D. Therefore

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \leqslant y \leqslant g_2(x)$. Thus, we have the following formula that enables us to evaluate the double integral as an iterated integral.

3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

The integral on the right side of (3) is an iterated integral that is similar to the ones we considered in the preceding section, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

We also consider plane regions of **type II**, which can be expressed as

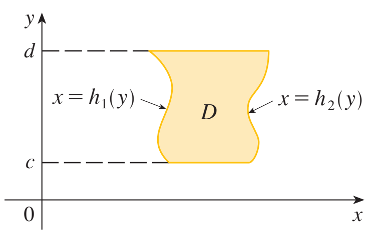
4
$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous. Two such regions are illustrated in Figure 7.

Using the same methods that were used in establishing (3), we can show that

5
$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region given by Equation 4.



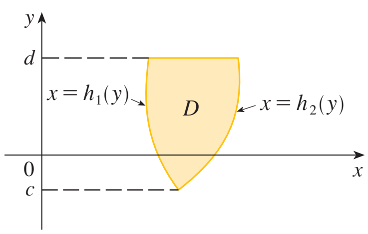


FIGURE 7
Some type II regions

EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

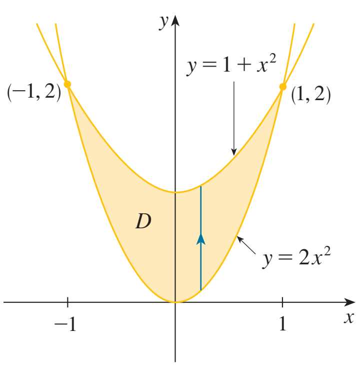


FIGURE 8

SOLUTION The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. We note that the region D, sketched in Figure 8, is a type I region but not a type II region and we can write

$$D = \{(x, y) \mid -1 \leqslant x \leqslant 1,\ 2x^2 \leqslant y \leqslant 1 + x^2\}$$

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 3 gives

$$\begin{aligned}\iint_D (x + 2y)\, dA &= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx \\ &= \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} dx \\ &= \int_{-1}^{1} \left[x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2\right] dx \\ &= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx \\ &= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Big]_{-1}^{1} = \frac{32}{15}\end{aligned}$$

■

NOTE • When we set up a double integral as in Example 1, it is essential to draw a diagram. Often it is helpful to draw a vertical arrow as in Figure 8. Then the limits of integration for the *inner* integral can be read from the diagram as follows: The arrow starts at the lower boundary $y = g_1(x)$, which gives the lower limit in the integral, and the arrow ends at the upper boundary $y = g_2(x)$, which gives the upper limit of integration. For a type II region the arrow is drawn horizontally from the left boundary to the right boundary.

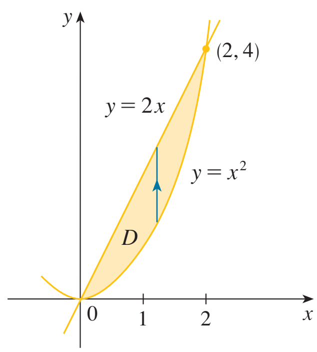


FIGURE 9
D as a type I region

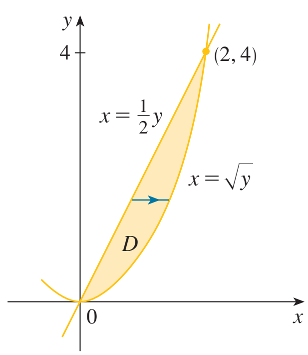


FIGURE 10
D as a type II region

EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

SOLUTION 1 From Figure 9 we see that D is a type I region and

$$D = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 2x\}$$

Therefore, the volume under $z = x^2 + y^2$ and above D is

$$V = \iint_D (x^2 + y^2)\,dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\,dy\,dx$$

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{y=x^2}^{y=2x} dx = \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2x^2 - \frac{(x^2)^3}{3}\right] dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3}\right) dx = -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6}\Bigg]_0^2 = \frac{216}{35}$$

SOLUTION 2 From Figure 10 we see that D can also be written as a type II region:

$$D = \left\{(x, y) \mid 0 \leqslant y \leqslant 4,\ \tfrac{1}{2}y \leqslant x \leqslant \sqrt{y}\right\}$$

Therefore, another expression for V is

$$V = \iint_D (x^2 + y^2)\,dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2)\,dx\,dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2x\right]_{x=\frac{1}{2}y}^{x=\sqrt{y}} dy = \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2}\right) dy$$

$$= \tfrac{2}{15}y^{5/2} + \tfrac{2}{7}y^{7/2} - \tfrac{13}{96}y^4\Big]_0^4 = \tfrac{216}{35}$$

▲ Figure 11 shows the solid whose volume is calculated in Example 2. It lies above the xy-plane, below the paraboloid $z = x^2 + y^2$, and between the plane $y = 2x$ and the parabolic cylinder $y = x^2$.

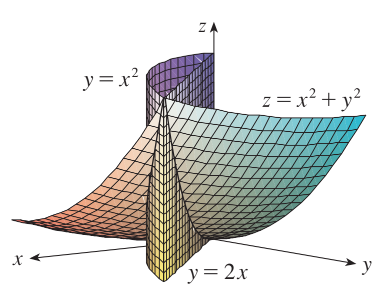


FIGURE 11

EXAMPLE 3 Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

SOLUTION The region D is shown in Figure 12. Again D is both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore, we prefer to express D as a type II region:

$$D = \left\{(x, y) \mid -2 \leqslant y \leqslant 4,\ \tfrac{1}{2}y^2 - 3 \leqslant x \leqslant y + 1\right\}$$

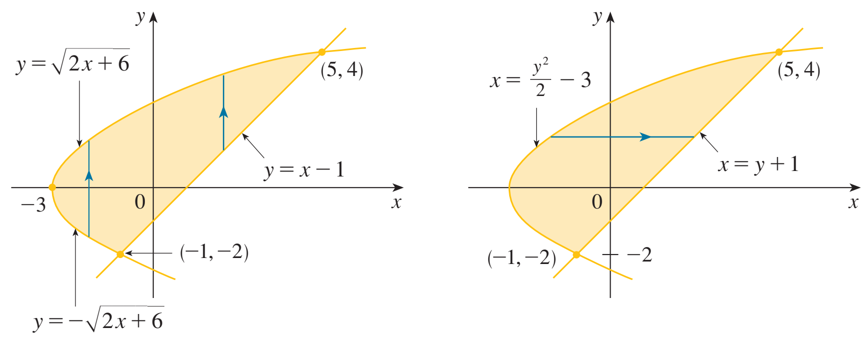


FIGURE 12 (a) D as a type I region (b) D as a type II region

Then (5) gives

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \frac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - \left(\tfrac{1}{2}y^2 - 3\right)^2\right] dy$$

$$= \frac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy$$

$$= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36$$

If we had expressed D as a type I region using Figure 12(a), then we would have obtained

$$\iint_D xy\,dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

but this would have involved more work than the other method. ■

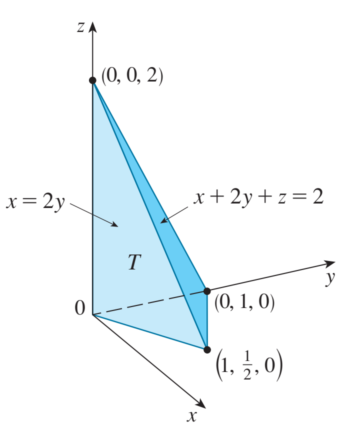


FIGURE 13

EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

SOLUTION In a question such as this, it's wise to draw two diagrams: one of the three-dimensional solid and another of the plane region D over which it lies. Figure 13 shows the tetrahedron T bounded by the coordinate planes $x = 0$, $z = 0$, the vertical plane $x = 2y$, and the plane $x + 2y + z = 2$. Since the plane $x + 2y + z = 2$ intersects the xy-plane (whose equation is $z = 0$) in the line $x + 2y = 2$, we see that T

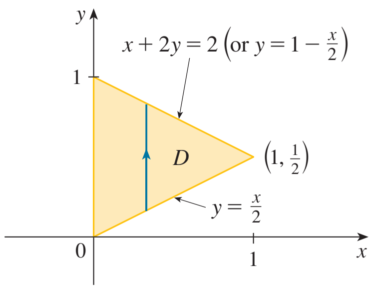


FIGURE 14

lies above the triangular region D in the xy-plane bounded by the lines $x = 2y$, $x + 2y = 2$, and $x = 0$. (See Figure 14.)

The plane $x + 2y + z = 2$ can be written as $z = 2 - x - 2y$, so the required volume lies under the graph of the function $z = 2 - x - 2y$ and above

$$D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ x/2 \leqslant y \leqslant 1 - x/2\}$$

Therefore

$$\begin{aligned} V &= \iint_D (2 - x - 2y)\, dA = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx \\ &= \int_0^1 \Big[2y - xy - y^2\Big]_{y=x/2}^{y=1-x/2}\, dx \\ &= \int_0^1 \left[2 - x - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4}\right] dx \\ &= \int_0^1 (x^2 - 2x + 1)\, dx = \frac{x^3}{3} - x^2 + x\Big]_0^1 = \frac{1}{3} \end{aligned}$$

EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

SOLUTION If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\, dy$. But it's impossible to do so in finite terms since $\int \sin(y^2)\, dy$ is not an elementary function. (See the end of Section 5.8.) So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. Using (3) backward, we have

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \iint_D \sin(y^2)\, dA$$

where $$D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ x \leqslant y \leqslant 1\}$$

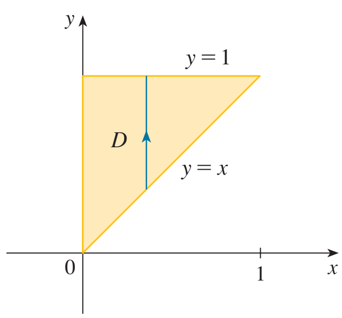


FIGURE 15
D as a type I region

We sketch this region D in Figure 15. Then from Figure 16 we see that an alternative description of D is

$$D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$$

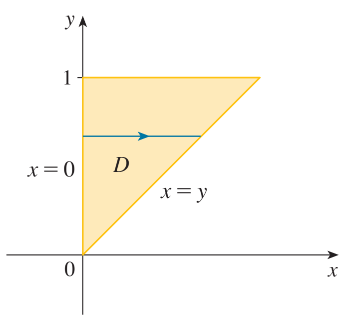


FIGURE 16
D as a type II region

This enables us to use (5) to express the double integral as an iterated integral in the reverse order:

$$\begin{aligned} \int_0^1 \int_x^1 \sin(y^2)\, dy\, dx &= \iint_D \sin(y^2)\, dA \\ &= \int_0^1 \int_0^y \sin(y^2)\, dx\, dy = \int_0^1 \Big[x \sin(y^2)\Big]_{x=0}^{x=y}\, dy \\ &= \int_0^1 y \sin(y^2)\, dy = -\tfrac{1}{2} \cos(y^2)\Big]_0^1 \\ &= \tfrac{1}{2}(1 - \cos 1) \end{aligned}$$

Properties of Double Integrals

We assume that all of the following integrals exist. The first three properties of double integrals over a region D follow immediately from Definition 2 and Properties 7, 8, and 9 in Section 12.1.

6 $$\iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$$

7 $$\iint_D cf(x, y)\, dA = c \iint_D f(x, y)\, dA$$

If $f(x, y) \geqslant g(x, y)$ for all (x, y) in D, then

8 $$\iint_D f(x, y)\, dA \geqslant \iint_D g(x, y)\, dA$$

The next property of double integrals is similar to the property of single integrals given by the equation $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$.

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

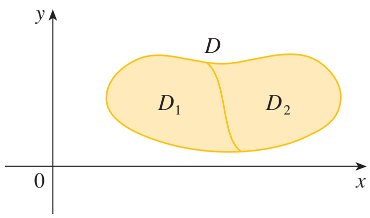


FIGURE 17

9 $$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA$$

Property 9 can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure. (See Exercises 41 and 42.)

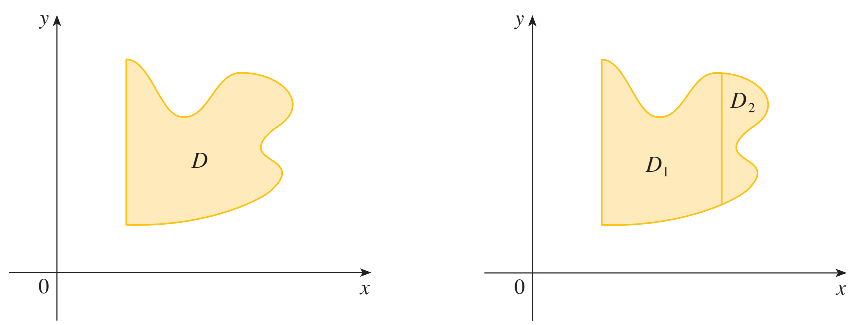


FIGURE 18 (a) D is neither type I nor type II. (b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

The next property of integrals says that if we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

10 $$\iint_D 1\, dA = A(D)$$

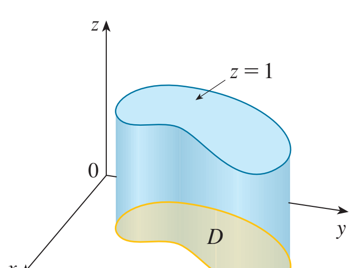


FIGURE 19
Cylinder with base D and height 1

Figure 19 illustrates why Equation 10 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\, dA$.

Finally, we can combine Properties 7, 8, and 10 to prove the following property. (See Exercise 45.)

11 If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$mA(D) \leqslant \iint_D f(x, y)\, dA \leqslant MA(D)$$

EXAMPLE 6 Use Property 11 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

SOLUTION Since $-1 \leqslant \sin x \leqslant 1$ and $-1 \leqslant \cos y \leqslant 1$, we have $-1 \leqslant \sin x \cos y \leqslant 1$ and therefore

$$e^{-1} \leqslant e^{\sin x \cos y} \leqslant e^1 = e$$

Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D) = \pi(2)^2$ in Property 11, we obtain

$$\frac{4\pi}{e} \leqslant \iint_D e^{\sin x \cos y}\, dA \leqslant 4\pi e$$

12.3 Exercises

1–6 ■ Evaluate the iterated integral.

1. $\int_0^1 \int_0^{x^2} (x + 2y)\, dy\, dx$

2. $\int_1^2 \int_y^2 xy\, dx\, dy$

3. $\int_0^1 \int_y^{e^y} \sqrt{x}\, dx\, dy$

4. $\int_0^1 \int_x^{2-x} (x^2 - y)\, dy\, dx$

5. $\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\, dr\, d\theta$

6. $\int_0^1 \int_0^v \sqrt{1 - v^2}\, du\, dv$

7–16 ■ Evaluate the double integral.

7. $\iint_D x^3 y^2\, dA$, $D = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ -x \leqslant y \leqslant x\}$

8. $\iint_D \frac{4y}{x^3 + 2}\, dA$, $D = \{(x, y) \mid 1 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 2x\}$

9. $\iint_D \frac{2y}{x^2 + 1}\, dA$, $D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant \sqrt{x}\}$

10. $\iint_D e^{y^2}\, dA$, $D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$

11. $\iint_D x \cos y\, dA$, D is bounded by $y = 0$, $y = x^2$, $x = 1$

12. $\iint_D x\sqrt{y^2 - x^2}\, dA$, $D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$

13. $\iint_D y^3\, dA$,
D is the triangular region with vertices (0, 2), (1, 1), and (3, 2)

14. $\iint_D (x + y)\, dA$, D is bounded by $y = \sqrt{x}$, $y = x^2$

15. $\iint_D (2x - y)\, dA$,
D is bounded by the circle with center the origin and radius 2

16. $\iint_D ye^x\, dA$, D is the triangular region with vertices (0, 0), (2, 4), and (6, 0)

17–24 ■ Find the volume of the given solid.

17. Under the paraboloid $z = x^2 + y^2$ and above the region bounded by $y = x^2$ and $x = y^2$

18. Under the paraboloid $z = 3x^2 + y^2$ and above the region bounded by $y = x$ and $x = y^2 - y$

19. Under the surface $z = xy$ and above the triangle with vertices (1, 1), (4, 1), and (1, 2)

20. Bounded by the paraboloid $z = x^2 + y^2 + 4$ and the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$

21. Bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$

22. Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant

23. Bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, $z = 0$ in the first octant

24. Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$

25. Use a graphing calculator or computer to estimate the x-coordinates of the points of intersection of the curves $y = x^4$ and $y = 3x - x^2$. If D is the region bounded by these curves, estimate $\iint_D x\, dA$.

26. Find the approximate volume of the solid in the first octant that is bounded by the planes $y = x$, $z = 0$, and $z = x$ and the cylinder $y = \cos x$. (Use a graphing device to estimate the points of intersection.)

CAS **27–28** ■ Use a computer algebra system to find the exact volume of the solid.

27. Under the surface $z = x^3y^4 + xy^2$ and above the region bounded by the curves $y = x^3 - x$ and $y = x^2 + x$ for $x \geqslant 0$

28. Between the paraboloids $z = 2x^2 + y^2$ and $z = 8 - x^2 - 2y^2$ and inside the cylinder $x^2 + y^2 = 1$

29–34 ■ Sketch the region of integration and change the order of integration.

29. $\int_0^1 \int_0^x f(x, y)\, dy\, dx$

30. $\int_0^{\pi/2} \int_0^{\sin x} f(x, y)\, dy\, dx$

31. $\int_1^2 \int_0^{\ln x} f(x, y)\, dy\, dx$

32. $\int_0^1 \int_{y^2}^{2-y} f(x, y)\, dx\, dy$

33. $\int_0^4 \int_{y/2}^2 f(x, y)\, dx\, dy$

34. $\int_0^1 \int_{\arctan x}^{\pi/4} f(x, y)\, dy\, dx$

35–40 ■ Evaluate the integral by reversing the order of integration.

35. $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$

36. $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1}\, dx\, dy$

37. $\int_0^3 \int_{y^2}^9 y \cos(x^2)\, dx\, dy$

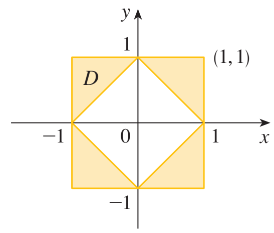
38. $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx$

39. $\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x}\, dx\, dy$

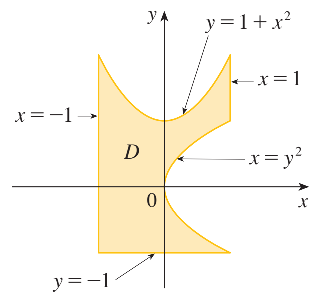
40. $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$

41–42 ■ Express D as a union of regions of type I or type II and evaluate the integral.

41. $\iint_D x^2\, dA$



42. $\iint_D xy\, dA$



43–44 ■ Use Property 11 to estimate the value of the integral.

43. $\iint_D \sqrt{x^3 + y^3}\, dA$, $\quad D = [0, 1] \times [0, 1]$

44. $\iint_D e^{x^2+y^2}\, dA$,

D is the disk with center the origin and radius $\frac{1}{2}$

45. Prove Property 11.

46. In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

$$\iint_D f(x, y)\, dA = \int_0^1 \int_0^{2y} f(x, y)\, dx\, dy + \int_1^3 \int_0^{3-y} f(x, y)\, dx\, dy$$

Sketch the region D and express the double integral as an iterated integral with reversed order of integration.

47. Evaluate $\iint_D (x^2 \tan x + y^3 + 4)\, dA$, where $D = \{(x, y) \mid x^2 + y^2 \leqslant 2\}$.

[*Hint:* Exploit the fact that D is symmetric with respect to both axes.]

48. Use symmetry to evaluate $\iint_D (2 - 3x + 4y)\, dA$, where D is the region bounded by the square with vertices $(\pm 5, 0)$ and $(0, \pm 5)$.

49. Compute $\iint_D \sqrt{1 - x^2 - y^2}\, dA$, where D is the disk $x^2 + y^2 \leqslant 1$, by first identifying the integral as the volume of a solid.

CAS **50.** Graph the solid bounded by the plane $x + y + z = 1$ and the paraboloid $z = 4 - x^2 - y^2$ and find its exact volume. (Use your CAS to do the graphing, to find the equations of the boundary curves of the region of integration, and to evaluate the double integral.)

12.4 Double Integrals in Polar Coordinates

▲ See Appendix H for information about polar coordinates.

Suppose that we want to evaluate a double integral $\iint_R f(x, y)\, dA$, where R is one of the regions shown in Figure 1. In either case the description of R in terms of rectangular coordinates is rather complicated but R is easily described using polar coordinates.

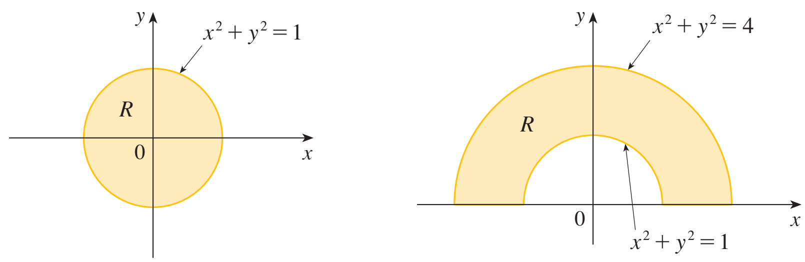


FIGURE 1

(a) $R = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

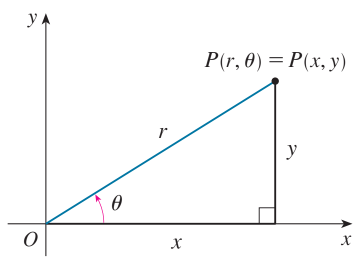


FIGURE 2

Recall from Figure 2 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

The regions in Figure 1 are special cases of a **polar rectangle**

$$R = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

which is shown in Figure 3. In order to compute the double integral $\iint_R f(x, y)\, dA$, where R is a polar rectangle, we divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b - a)/m$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = (\beta - \alpha)/n$. Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles shown in Figure 4.

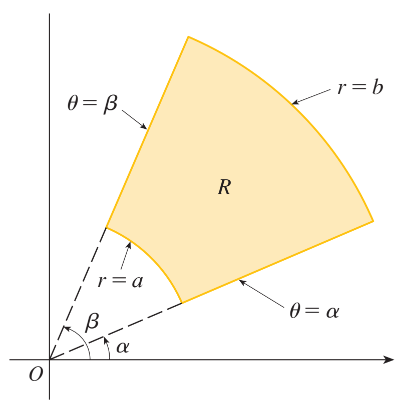


FIGURE 3
Polar rectangle

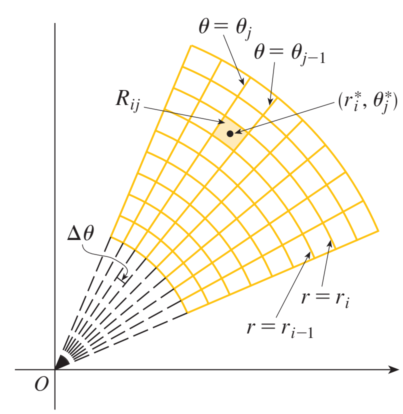


FIGURE 4
Dividing R into polar subrectangles

The "center" of the polar subrectangle

$$R_{ij} = \{(r, \theta) \mid r_{i-1} \leqslant r \leqslant r_i,\ \theta_{j-1} \leqslant \theta \leqslant \theta_j\}$$

has polar coordinates

$$r_i^* = \tfrac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \tfrac{1}{2}(\theta_{j-1} + \theta_j)$$

We compute the area of R_{ij} using the fact that the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$. Subtracting the areas of two such sectors, each of which has central angle $\Delta\theta = \theta_j - \theta_{j-1}$, we find that the area of R_{ij} is

$$\begin{aligned}\Delta A_i &= \tfrac{1}{2}r_i^2\,\Delta\theta - \tfrac{1}{2}r_{i-1}^2\,\Delta\theta \\ &= \tfrac{1}{2}(r_i^2 - r_{i-1}^2)\,\Delta\theta \\ &= \tfrac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\,\Delta\theta \\ &= r_i^*\,\Delta r\,\Delta\theta\end{aligned}$$

Although we have defined the double integral $\iint_R f(x, y)\,dA$ in terms of ordinary rectangles, it can be shown that, for continuous functions f, we always obtain the same answer using polar rectangles. The rectangular coordinates of the center of R_{ij} are $(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)$, so a typical Riemann sum is

$$\boxed{1} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\,\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\,r_i^*\,\Delta r\,\Delta\theta$$

If we write $g(r, \theta) = rf(r\cos\theta, r\sin\theta)$, then the Riemann sum in Equation 1 can be written as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r\,\Delta\theta$$

which is a Riemann sum for the double integral

$$\int_\alpha^\beta \int_a^b g(r, \theta)\,dr\,d\theta$$

Therefore, we have

$$\begin{aligned}\iint_R f(x, y)\,dA &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\,\Delta A_i \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r\,\Delta\theta = \int_\alpha^\beta \int_a^b g(r, \theta)\,dr\,d\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta\end{aligned}$$

2 Change to Polar Coordinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x, y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

The formula in (2) says that we convert from rectangular to polar coordinates in a double integral by writing $x = r\cos\theta$ and $y = r\sin\theta$, using the appropriate limits of integration for r and θ, and replacing dA by $r\,dr\,d\theta$. Be careful not to forget the additional factor r on the right side of Formula 2. A classical method for remembering this is shown in Figure 5, where the "infinitesimal" polar rectangle can be thought of as an ordinary rectangle with dimensions $r\,d\theta$ and dr and therefore has "area" $dA = r\,dr\,d\theta$.

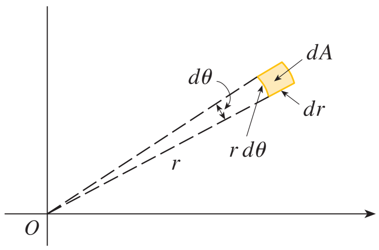


FIGURE 5

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

SOLUTION The region R can be described as

$$R = \{(x, y) \mid y \geqslant 0,\ 1 \leqslant x^2 + y^2 \leqslant 4\}$$

It is the half-ring shown in Figure 1(b), and in polar coordinates it is given by $1 \leqslant r \leqslant 2$, $0 \leqslant \theta \leqslant \pi$. Therefore, by Formula 2,

$$\iint_R (3x + 4y^2)\,dA = \int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta$$

$$= \int_0^{\pi} \Big[r^3\cos\theta + r^4\sin^2\theta\Big]_{r=1}^{r=2} d\theta = \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\,d\theta$$

$$= \int_0^{\pi} \big[7\cos\theta + \tfrac{15}{2}(1 - \cos 2\theta)\big]\,d\theta$$

$$= 7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta\Big]_0^{\pi} = \frac{15\pi}{2}$$

▲ Here we use trigonometric identity

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

as discussed in Section 5.7. Alternatively, we could have used Formula 63 in the Table of Integrals:

$$\int \sin^2 u\,du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C$$

EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

SOLUTION If we put $z = 0$ in the equation of the paraboloid, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$, so the solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \leqslant 1$ [see Figures 6 and 1(a)]. In polar coordinates D is given by $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$. Since $1 - x^2 - y^2 = 1 - r^2$, the volume is

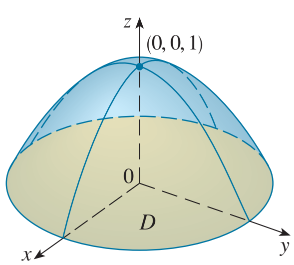


FIGURE 6

$$V = \iint_D (1 - x^2 - y^2)\,dA = \int_0^{2\pi}\int_0^1 (1 - r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\,dr = 2\pi\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{\pi}{2}$$

If we had used rectangular coordinates instead of polar coordinates, then we would have obtained

$$V = \iint_D (1 - x^2 - y^2)\,dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\,dy\,dx$$

which is not easy to evaluate because it involves finding the following integrals:

$$\int \sqrt{1 - x^2}\,dx \qquad \int x^2\sqrt{1 - x^2}\,dx \qquad \int (1 - x^2)^{3/2}\,dx$$

What we have done so far can be extended to the more complicated type of region shown in Figure 7. It's similar to the type II rectangular regions considered in Section 12.3. In fact, by combining Formula 2 in this section with Formula 12.3.5, we obtain the following formula.

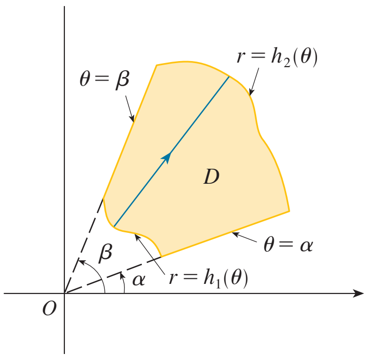


FIGURE 7
$D=\{(r,\theta)\mid \alpha\leqslant\theta\leqslant\beta,\ h_1(\theta)\leqslant r\leqslant h_2(\theta)\}$

3 If f is continuous on a polar region of the form

$$D=\{(r,\theta)\mid \alpha\leqslant\theta\leqslant\beta,\ h_1(\theta)\leqslant r\leqslant h_2(\theta)\}$$

then
$$\iint_D f(x,y)\,dA=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\,r\sin\theta)\,r\,dr\,d\theta$$

In particular, taking $f(x,y)=1$, $h_1(\theta)=0$, and $h_2(\theta)=h(\theta)$ in this formula, we see that the area of the region D bounded by $\theta=\alpha$, $\theta=\beta$, and $r=h(\theta)$ is

$$A(D)=\iint_D 1\,dA=\int_\alpha^\beta\int_0^{h(\theta)} r\,dr\,d\theta$$
$$=\int_\alpha^\beta\left[\frac{r^2}{2}\right]_0^{h(\theta)}d\theta=\int_\alpha^\beta \tfrac{1}{2}[h(\theta)]^2\,d\theta$$

and this agrees with Formula 3 in Appendix H.2.

EXAMPLE 3 Find the volume of the solid that lies under the paraboloid $z=x^2+y^2$, above the xy-plane, and inside the cylinder $x^2+y^2=2x$.

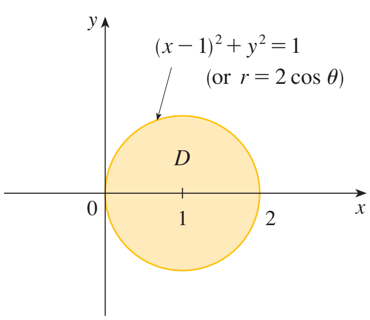


FIGURE 8

SOLUTION The solid lies above the disk D whose boundary circle has equation $x^2+y^2=2x$ or, after completing the square,

$$(x-1)^2+y^2=1$$

(see Figures 8 and 9). In polar coordinates we have $x^2+y^2=r^2$ and $x=r\cos\theta$, so the boundary circle becomes $r^2=2r\cos\theta$, or $r=2\cos\theta$. Thus, the disk D is given by

$$D=\{(r,\theta)\mid -\pi/2\leqslant\theta\leqslant\pi/2,\ 0\leqslant r\leqslant 2\cos\theta\}$$

and, by Formula 3, we have

$$V=\iint_D (x^2+y^2)\,dA=\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,r\,dr\,d\theta$$
$$=\int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\cos\theta}d\theta=4\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$$
$$=8\int_0^{\pi/2}\cos^4\theta\,d\theta$$

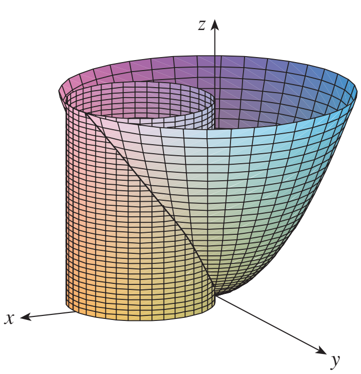


FIGURE 9

Using Formula 74 in the Table of Integrals with $n=4$, we get

$$V=8\int_0^{\pi/2}\cos^4\theta\,d\theta=8\left(\tfrac{1}{4}\cos^3\theta\,\sin\theta\Big]_0^{\pi/2}+\tfrac{3}{4}\int_0^{\pi/2}\cos^2\theta\,d\theta\right)$$
$$=6\int_0^{\pi/2}\cos^2\theta\,d\theta$$

▲ Instead of using tables, we could have used the identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ twice.

Now we use Formula 64 in the Table of Integrals:

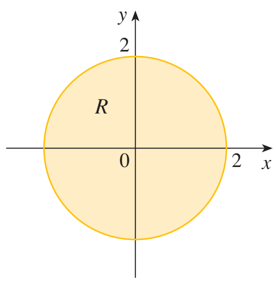
$$V = 6\int_0^{\pi/2} \cos^2\theta\, d\theta = 6\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$

$$= 6 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{2}$$

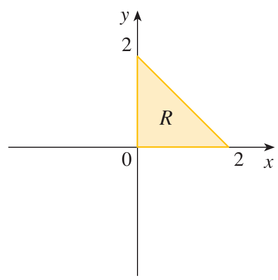
12.4 Exercises

1–6 ■ A region R is shown. Decide whether to use polar coordinates or rectangular coordinates and write $\iint_R f(x, y)\, dA$ as an iterated integral, where f is an arbitrary continuous function on R.

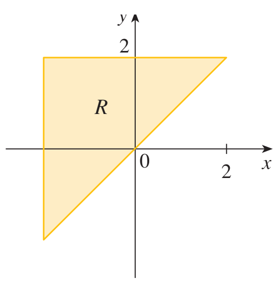
1.



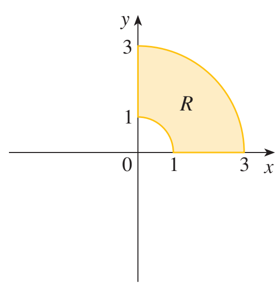
2.



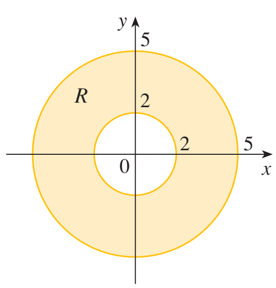
3.



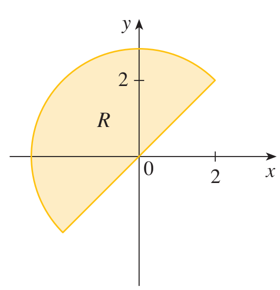
4.



5.



6.



7–8 ■ Sketch the region whose area is given by the integral and evaluate the integral.

7. $\displaystyle\int_{\pi}^{2\pi}\int_4^7 r\, dr\, d\theta$

8. $\displaystyle\int_0^{\pi/2}\int_0^{4\cos\theta} r\, dr\, d\theta$

9–14 ■ Evaluate the given integral by changing to polar coordinates.

9. $\iint_D xy\, dA$,
where D is the disk with center the origin and radius 3

10. $\iint_R \sqrt{x^2 + y^2}\, dA$,
where $R = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 9,\ y \geq 0\}$

11. $\iint_D e^{-x^2-y^2}\, dA$, where D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis

12. $\iint_R ye^x\, dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 25$

13. $\iint_R \arctan(y/x)\, dA$,
where $R = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4,\ -x \leq y \leq x\}$

14. $\iint_D x\, dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$

15–21 ■ Use polar coordinates to find the volume of the given solid.

15. Under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 9$

16. Inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$

17. A sphere of radius a

18. Bounded by the paraboloid $z = 10 - 3x^2 - 3y^2$ and the plane $z = 4$

19. Above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$

20. Bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$

21. Inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$

22. (a) A cylindrical drill with radius r_1 is used to bore a hole through the center of a sphere of radius r_2. Find the volume of the ring-shaped solid that remains.
(b) Express the volume in part (a) in terms of the height h of the ring. Notice that the volume depends only on h, not on r_1 or r_2.

23–24 ■ Use a double integral to find the area of the region.

23. One loop of the rose $r = \cos 3\theta$

24. The region enclosed by the cardioid $r = 1 - \sin\theta$

25–28 ■ Evaluate the iterated integral by converting to polar coordinates.

25. $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$

26. $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)^{3/2}\,dx\,dy$

27. $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2y^2\,dx\,dy$

28. $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

29. A swimming pool is circular with a 40-ft diameter. The depth is constant along east-west lines and increases linearly from 2 ft at the south end to 7 ft at the north end. Find the volume of water in the pool.

30. An agricultural sprinkler distributes water in a circular pattern of radius 100 ft. It supplies water to a depth of e^{-r} feet per hour at a distance of r feet from the sprinkler.
(a) What is the total amount of water supplied per hour to the region inside the circle of radius R centered at the sprinkler?
(b) Determine an expression for the average amount of water per hour per square foot supplied to the region inside the circle of radius R.

31. Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_1^{\sqrt{2}} \int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-x^2}} xy\,dy\,dx$$

into one double integral. Then evaluate the double integral.

32. (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx$$
$$= \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\,dA$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA = \pi$$

(b) An equivalent definition of the improper integral in part (a) is

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\,dA$$

where S_a is the square with vertices $(\pm a, \pm a)$. Use this to show that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \pi$$

(c) Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

(d) By making the change of variable $t = \sqrt{2}\,x$, show that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

(This is a fundamental result for probability and statistics.)

33. Use the result of Exercise 32 part (c) to evaluate the following integrals.

(a) $\int_0^{\infty} x^2 e^{-x^2}\,dx$ (b) $\int_0^{\infty} \sqrt{x}\,e^{-x}\,dx$

12.5 Applications of Double Integrals

We have already seen one application of double integrals: computing volumes. Another geometric application is finding areas of surfaces and this will be done in the next section. In this section we explore physical applications such as computing mass, electric charge, center of mass, and moment of inertia. We will see that these physical ideas are also important when applied to probability density functions of two random variables.

Density and Mass

In Chapter 6 we were able to use single integrals to compute moments and the center of mass of a thin plate or lamina with constant density. But now, equipped with the double integral, we can consider a lamina with variable density. Suppose the lamina occupies a region D of the xy-plane and its **density** (in units of mass per unit area) at

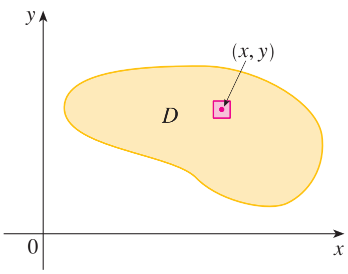


FIGURE 1

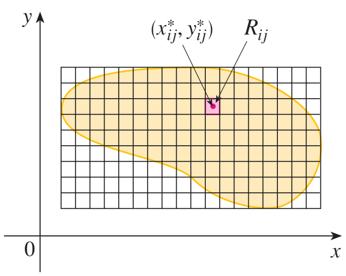


FIGURE 2

a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D. This means that

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$

where Δm and ΔA are the mass and area of a small rectangle that contains (x, y) and the limit is taken as the dimensions of the rectangle approach 0. (See Figure 1.)

To find the total mass m of the lamina we divide a rectangle R containing D into subrectangles R_{ij} of the same size (as in Figure 2) and consider $\rho(x, y)$ to be 0 outside D. If we choose a point (x_{ij}^*, y_{ij}^*) in R_{ij}, then the mass of the part of the lamina that occupies R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\,\Delta A$, where ΔA is the area of R_{ij}. If we add all such masses, we get an approximation to the total mass:

$$m \approx \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A$$

If we now increase the number of subrectangles, we obtain the total mass m of the lamina as the limiting value of the approximations:

1
$$m = \lim_{k, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_D \rho(x, y)\,dA$$

Physicists also consider other types of density that can be treated in the same manner. For example, if an electric charge is distributed over a region D and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$ at a point (x, y) in D, then the total charge Q is given by

2
$$Q = \iint_D \sigma(x, y)\,dA$$

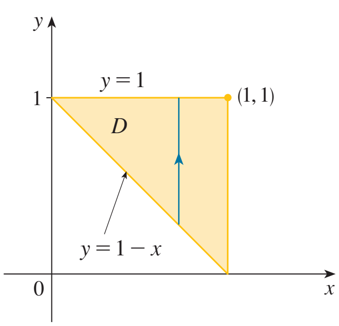


FIGURE 3

EXAMPLE 1 Charge is distributed over the triangular region D in Figure 3 so that the charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.

SOLUTION From Equation 2 and Figure 3 we have

$$Q = \iint_D \sigma(x, y)\,dA = \int_0^1 \int_{1-x}^1 xy\,dy\,dx$$

$$= \int_0^1 \left[x\frac{y^2}{2} \right]_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2}\left[1^2 - (1-x)^2\right] dx$$

$$= \tfrac{1}{2}\int_0^1 (2x^2 - x^3)\,dx = \frac{1}{2}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{5}{24}$$

Thus, the total charge is $\frac{5}{24}$ C. ■

Moments and Centers of Mass

In Section 6.5 we found the center of mass of a lamina with constant density; here we consider a lamina with variable density. Suppose the lamina occupies a region D and has density function $\rho(x, y)$. Recall from Chapter 6 that we defined the moment of a particle about an axis as the product of its mass and its directed distance from the axis. We divide D into small rectangles as in Figure 2. Then the mass of R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\,\Delta A$, so we can approximate the moment of R_{ij} with respect to the x-axis by

$$[\rho(x_{ij}^*, y_{ij}^*)\,\Delta A]\,y_{ij}^*$$

If we now add these quantities and take the limit as the number of subrectangles becomes large, we obtain the **moment** of the entire lamina **about the *x*-axis**:

3 $$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^*\,\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_D y\rho(x, y)\,dA$$

Similarly, the **moment about the *y*-axis** is

4 $$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^*\,\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_D x\rho(x, y)\,dA$$

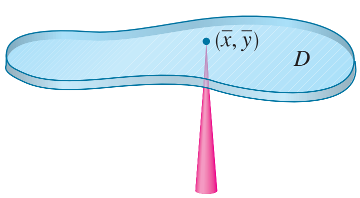


FIGURE 4

As before, we define the center of mass $(\bar{x}, \bar{y})$ so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$. The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass. Thus, the lamina balances horizontally when supported at its center of mass (see Figure 4).

5 The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)\,dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)\,dA$$

where the mass m is given by

$$m = \iint_D \rho(x, y)\,dA$$

EXAMPLE 2 Find the mass and center of mass of a triangular lamina with vertices (0, 0), (1, 0), and (0, 2) if the density function is $\rho(x, y) = 1 + 3x + y$.

SOLUTION The triangle is shown in Figure 5. (Note that the equation of the upper boundary is $y = 2 - 2x$.) The mass of the lamina is

$$m = \iint_D \rho(x, y)\,dA = \int_0^1 \int_0^{2-2x} (1 + 3x + y)\,dy\,dx$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= 4\int_0^1 (1 - x^2)\,dx = 4\left[x - \frac{x^3}{3} \right]_0^1 = \frac{8}{3}$$

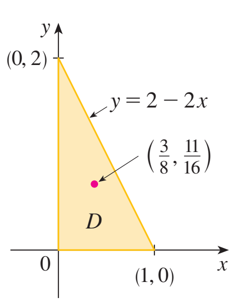


FIGURE 5

Then the formulas in (5) give

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x, y)\,dA = \tfrac{3}{8}\int_0^1\int_0^{2-2x}(x + 3x^2 + xy)\,dy\,dx$$
$$= \frac{3}{8}\int_0^1\left[xy + 3x^2y + x\frac{y^2}{2}\right]_{y=0}^{y=2-2x}dx$$
$$= \frac{3}{2}\int_0^1(x - x^3)\,dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{8}$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)\,dA = \tfrac{3}{8}\int_0^1\int_0^{2-2x}(y + 3xy + y^2)\,dy\,dx$$
$$= \frac{3}{8}\int_0^1\left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3}\right]_{y=0}^{y=2-2x}dx = \tfrac{1}{4}\int_0^1(7 - 9x - 3x^2 + 5x^3)\,dx$$
$$= \frac{1}{4}\left[7x - 9\frac{x^2}{2} - x^3 + 5\frac{x^4}{4}\right]_0^1 = \frac{11}{16}$$

The center of mass is at the point $\left(\frac{3}{8}, \frac{11}{16}\right)$. ■

EXAMPLE 3 The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

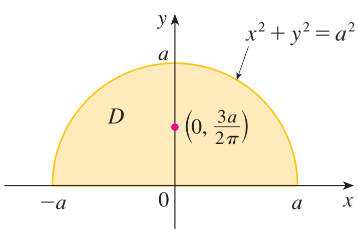


FIGURE 6

SOLUTION Let's place the lamina as the upper half of the circle $x^2 + y^2 = a^2$ (see Figure 6). Then the distance from a point (x, y) to the center of the circle (the origin) is $\sqrt{x^2 + y^2}$. Therefore, the density function is

$$\rho(x, y) = K\sqrt{x^2 + y^2}$$

where K is some constant. Both the density function and the shape of the lamina suggest that we convert to polar coordinates. Then $\sqrt{x^2 + y^2} = r$ and the region D is given by $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant \pi$. Thus, the mass of the lamina is

$$m = \iint_D \rho(x, y)\,dA = \iint_D K\sqrt{x^2 + y^2}\,dA$$
$$= \int_0^\pi\int_0^a (Kr)\,r\,dr\,d\theta = K\int_0^\pi d\theta\int_0^a r^2\,dr$$
$$= K\pi\frac{r^3}{3}\bigg]_0^a = \frac{K\pi a^3}{3}$$

Both the lamina and the density function are symmetric with respect to the y-axis, so the center of mass must lie on the y-axis, that is, $\bar{x} = 0$. The y-coordinate is given by

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)\,dA = \frac{3}{K\pi a^3}\int_0^\pi\int_0^a r\sin\theta\,(Kr)\,r\,dr\,d\theta$$
$$= \frac{3}{\pi a^3}\int_0^\pi \sin\theta\,d\theta\int_0^a r^3\,dr = \frac{3}{\pi a^3}\left[-\cos\theta\right]_0^\pi\left[\frac{r^4}{4}\right]_0^a$$
$$= \frac{3}{\pi a^3}\,\frac{2a^4}{4} = \frac{3a}{2\pi}$$

▲ Compare the location of the center of mass in Example 3 with Example 6 in Section 6.5 where we found that the center of mass of a lamina with the same shape but uniform density is located at the point $(0, 4a/(3\pi))$.

Therefore, the center of mass is located at the point $(0, 3a/(2\pi))$. ■

Moment of Inertia

The **moment of inertia** (also called the **second moment**) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. We extend this concept to a lamina with density function $\rho(x, y)$ and occupying a region D by proceeding as we did for ordinary moments. We divide D into small rectangles, approximate the moment of inertia of each subrectangle about the x-axis, and take the limit of the sum as the number of subrectangles becomes large. The result is the **moment of inertia** of the lamina **about the x-axis**:

$$I_x = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D y^2 \rho(x, y)\, dA \tag{6}$$

Similarly, the **moment of inertia about the y-axis** is

$$I_y = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x^2 \rho(x, y)\, dA \tag{7}$$

It is also of interest to consider the **moment of inertia about the origin**, also called the **polar moment of inertia**:

$$I_0 = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D (x^2 + y^2) \rho(x, y)\, dA \tag{8}$$

Note that $I_0 = I_x + I_y$.

EXAMPLE 4 Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x, y) = \rho$, center the origin, and radius a.

SOLUTION The boundary of D is the circle $x^2 + y^2 = a^2$ and in polar coordinates D is described by $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant r \leqslant a$. Let's compute I_0 first:

$$I_0 = \iint_D (x^2 + y^2)\rho\, dA = \rho \int_0^{2\pi} \int_0^a r^2\, r\, dr\, d\theta$$

$$= \rho \int_0^{2\pi} d\theta \int_0^a r^3\, dr = 2\pi\rho \left[\frac{r^4}{4}\right]_0^a = \frac{\pi\rho a^4}{2}$$

Instead of computing I_x and I_y directly, we use the facts that $I_x + I_y = I_0$ and $I_x = I_y$ (from the symmetry of the problem). Thus

$$I_x = I_y = \frac{I_0}{2} = \frac{\pi\rho a^4}{4}$$ ■

In Example 4 notice that the mass of the disk is

$$m = \text{density} \times \text{area} = \rho(\pi a^2)$$

so the moment of inertia of the disk about the origin (like a wheel about its axle) can be written as

$$I_0 = \tfrac{1}{2} m a^2$$

Thus, if we increase the mass or the radius of the disk, we thereby increase the moment of inertia. In general, the moment of inertia plays much the same role in rotational motion that mass plays in linear motion. The moment of inertia of a wheel is what makes it difficult to start or stop the rotation of the wheel, just as the mass of a car is what makes it difficult to start or stop the motion of the car.

Probability

In Section 6.7 we considered the *probability density function* f of a continuous random variable X. This means that $f(x) \geqslant 0$ for all x, $\int_{-\infty}^{\infty} f(x)\,dx = 1$, and the probability that X lies between a and b is found by integrating f from a to b:

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx$$

Now we consider a pair of continuous random variables X and Y, such as the lifetimes of two components of a machine or the height and weight of an adult female chosen at random. The **joint density function** of X and Y is a function f of two variables such that the probability that (X, Y) lies in a region D is

$$P((X, Y) \in D) = \iint_D f(x, y)\,dA$$

In particular, if the region is a rectangle, the probability that X lies between a and b and Y lies between c and d is

$$P(a \leqslant X \leqslant b,\ c \leqslant Y \leqslant d) = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

(See Figure 7.)

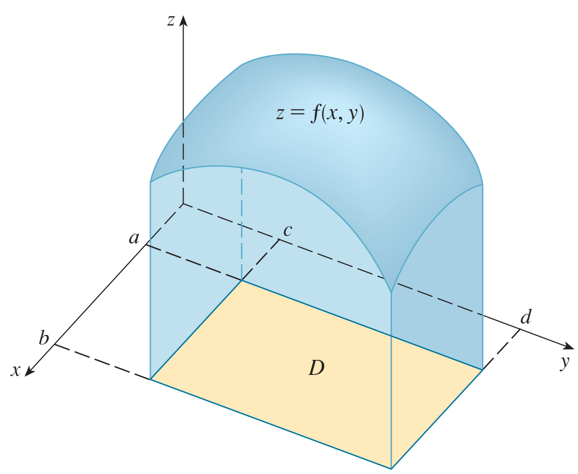


FIGURE 7 The probability that X lies between a and b and Y lies between c and d is the volume that lies above the rectangle $D = [a, b] \times [c, d]$ and below the graph of the joint density function.

Because probabilities aren't negative and are measured on a scale from 0 to 1, the joint density function has the following properties:

$$f(x, y) \geqslant 0 \qquad \iint_{\mathbb{R}^2} f(x, y)\, dA = 1$$

As in Exercise 32 in Section 12.4, the double integral over $\mathbb{R}^2$ is an improper integral defined as the limit of double integrals over expanding circles or squares and we can write

$$\iint_{\mathbb{R}^2} f(x, y)\, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

EXAMPLE 5 If the joint density function for X and Y is given by

$$f(x, y) = \begin{cases} C(x + 2y) & \text{if } 0 \leqslant x \leqslant 10,\ 0 \leqslant y \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$$

find the value of the constant C. Then find $P(X \leqslant 7, Y \geqslant 2)$.

SOLUTION We find the value of C by ensuring that the double integral of f is equal to 1. Because $f(x, y) = 0$ outside the rectangle $[0, 10] \times [0, 10]$, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dy\, dx = \int_0^{10} \int_0^{10} C(x + 2y)\, dy\, dx = C \int_0^{10} \Big[xy + y^2\Big]_{y=0}^{y=10} dx$$

$$= C \int_0^{10} (10x + 100)\, dx = 1500C$$

Therefore, $1500C = 1$ and so $C = \frac{1}{1500}$.

Now we can compute the probability that X is at most 7 and Y is at least 2:

$$P(X \leqslant 7, Y \geqslant 2) = \int_{-\infty}^{7} \int_{2}^{\infty} f(x, y)\, dy\, dx = \int_0^7 \int_2^{10} \tfrac{1}{1500}(x + 2y)\, dy\, dx$$

$$= \frac{1}{1500} \int_0^7 \Big[xy + y^2\Big]_{y=2}^{y=10} dx = \frac{1}{1500} \int_0^7 (8x + 96)\, dx$$

$$= \frac{868}{1500} \approx 0.5787$$

■

Suppose X is a random variable with probability density function $f_1(x)$ and Y is a random variable with density function $f_2(y)$. Then X and Y are called **independent random variables** if their joint density function is the product of their individual density functions:

$$f(x, y) = f_1(x) f_2(y)$$

In Section 6.7 we modeled waiting times by using exponential density functions

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mu^{-1} e^{-t/\mu} & \text{if } t \geqslant 0 \end{cases}$$

where μ is the mean waiting time. In the next example we consider a situation with two independent waiting times.

EXAMPLE 6 The manager of a movie theater determines that the average time moviegoers wait in line to buy a ticket for this week's film is 10 minutes and the average time they wait to buy popcorn is 5 minutes. Assuming that the waiting times are independent, find the probability that a moviegoer waits a total of less than 20 minutes before taking his or her seat.

SOLUTION Assuming that both the waiting time X for the ticket purchase and the waiting time Y in the refreshment line are modeled by exponential probability density functions, we can write the individual density functions as

$$f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{10}e^{-x/10} & \text{if } x \geqslant 0 \end{cases} \qquad f_2(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{5}e^{-y/5} & \text{if } y \geqslant 0 \end{cases}$$

Since X and Y are independent, the joint density function is the product:

$$f(x, y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{50}e^{-x/10}e^{-y/5} & \text{if } x \geqslant 0,\ y \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

We are asked for the probability that $X + Y < 20$:

$$P(X + Y < 20) = P((X, Y) \in D)$$

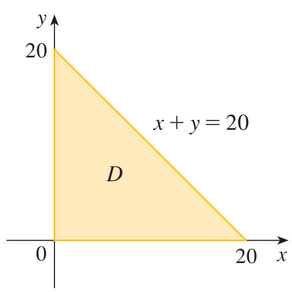


FIGURE 8

where D is the triangular region shown in Figure 8. Thus

$$\begin{aligned} P(X + Y < 20) &= \iint_D f(x, y)\, dA = \int_0^{20}\int_0^{20-x} \tfrac{1}{50}e^{-x/10}e^{-y/5}\, dy\, dx \\ &= \tfrac{1}{50}\int_0^{20} \Big[e^{-x/10}(-5)e^{-y/5}\Big]_{y=0}^{y=20-x} dx \\ &= \tfrac{1}{10}\int_0^{20} e^{-x/10}\big(1 - e^{(x-20)/5}\big)\, dx \\ &= \tfrac{1}{10}\int_0^{20} \big(e^{-x/10} - e^{-4}e^{x/10}\big)\, dx = 1 + e^{-4} - 2e^{-2} \approx 0.7476 \end{aligned}$$

This means that about 75% of the moviegoers wait less than 20 minutes before taking their seats. ■

Expected Values

Recall from Section 6.7 that if X is a random variable with probability density function f, then its *mean* is

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

Now if X and Y are random variables with joint density function f, we define the ***X*-mean** and ***Y*-mean**, also called the **expected values** of X and Y, to be

9 $$\mu_1 = \iint_{\mathbb{R}^2} x f(x, y)\, dA \qquad \mu_2 = \iint_{\mathbb{R}^2} y f(x, y)\, dA$$

Notice how closely the expressions for μ_1 and μ_2 in (9) resemble the moments M_x and M_y of a lamina with density function ρ in Equations 3 and 4. In fact, we can think of probability as being like continuously distributed mass. We calculate probability the way we calculate mass—by integrating a density function. And because the total "probability mass" is 1, the expressions for $\bar{x}$ and $\bar{y}$ in (5) show that we can think of the expected values of X and Y, μ_1 and μ_2, as the coordinates of the "center of mass" of the probability distribution.

In the next example we deal with normal distributions. As in Section 6.7, a single random variable is *normally distributed* if its probability density function is of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}$$

where μ is the mean and σ is the standard deviation.

EXAMPLE 7 A factory produces (cylindrically shaped) roller bearings that are sold as having diameter 4.0 cm and length 6.0 cm. In fact, the diameters X are normally distributed with mean 4.0 cm and standard deviation 0.01 cm while the lengths Y are normally distributed with mean 6.0 cm and standard deviation 0.01 cm. Assuming that X and Y are independent, write the joint density function and graph it. Find the probability that a bearing randomly chosen from the production line has either length or diameter that differs from the mean by more than 0.02 cm.

SOLUTION We are given that X and Y are normally distributed with $\mu_1 = 4.0$, $\mu_2 = 6.0$, and $\sigma_1 = \sigma_2 = 0.01$. So the individual density functions for X and Y are

$$f_1(x) = \frac{1}{0.01\sqrt{2\pi}}\, e^{-(x-4)^2/0.0002} \qquad f_2(y) = \frac{1}{0.01\sqrt{2\pi}}\, e^{-(y-6)^2/0.0002}$$

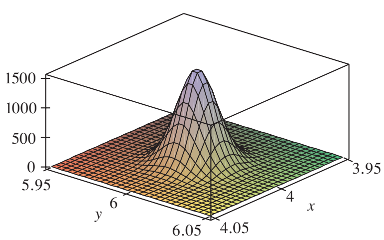


FIGURE 9
Graph of the bivariate normal joint density function in Example 7

Since X and Y are independent, the joint density function is the product:

$$f(x, y) = f_1(x)f_2(y) = \frac{1}{0.0002\pi}\, e^{-(x-4)^2/0.0002}e^{-(y-6)^2/0.0002}$$

$$= \frac{5000}{\pi}\, e^{-5000[(x-4)^2+(y-6)^2]}$$

A graph of this function is shown in Figure 9.

Let's first calculate the probability that both X and Y differ from their means by less than 0.02 cm. Using a calculator or computer to estimate the integral, we have

$$P(3.98 < X < 4.02,\ 5.98 < Y < 6.02) = \int_{3.98}^{4.02}\int_{5.98}^{6.02} f(x, y)\, dy\, dx$$

$$= \frac{5000}{\pi}\int_{3.98}^{4.02}\int_{5.98}^{6.02} e^{-5000[(x-4)^2+(y-6)^2]}\, dy\, dx$$

$$\approx 0.91$$

Then the probability that either X or Y differs from its mean by more than 0.02 cm is approximately

$$1 - 0.91 = 0.09$$ ■

12.5 Exercises

1. Electric charge is distributed over the rectangle $1 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant 2$ so that the charge density at (x, y) is $\sigma(x, y) = 2xy + y^2$ (measured in coulombs per square meter). Find the total charge on the rectangle.

2. Electric charge is distributed over the disk $x^2 + y^2 \leqslant 4$ so that the charge density at (x, y) is $\sigma(x, y) = x + y + x^2 + y^2$ (measured in coulombs per square meter). Find the total charge on the disk.

3–8 ■ Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ.

3. $D = \{(x, y) \mid 0 \leqslant x \leqslant 2, -1 \leqslant y \leqslant 1\}$; $\rho(x, y) = xy^2$

4. $D = \{(x, y) \mid 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b\}$; $\rho(x, y) = cxy$

5. D is the triangular region with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

6. D is bounded by the parabola $y = 9 - x^2$ and the x-axis; $\rho(x, y) = y$

7. D is bounded by the parabola $x = y^2$ and the line $y = x - 2$; $\rho(x, y) = 3$

8. $D = \{(x, y) \mid 0 \leqslant y \leqslant \cos x,\ 0 \leqslant x \leqslant \pi/2\}$; $\rho(x, y) = x$

9. A lamina occupies the part of the disk $x^2 + y^2 \leqslant 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the x-axis.

10. Find the center of mass of the lamina in Exercise 9 if the density at any point is proportional to the square of its distance from the origin.

11. Find the center of mass of a lamina in the shape of an isosceles right triangle with equal sides of length a if the density at any point is proportional to the square of the distance from the vertex opposite the hypotenuse.

12. A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ but outside the circle $x^2 + y^2 = 1$. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

13. Find the moments of inertia I_x, I_y, I_0 for the lamina of Exercise 3.

14. Find the moments of inertia I_x, I_y, I_0 for the lamina of Exercise 10.

15. Find the moments of inertia I_x, I_y, I_0 for the lamina of Exercise 7.

16. Consider a square fan blade with sides of length 2 and the lower left corner placed at the origin. If the density of the blade is $\rho(x, y) = 1 + 0.1x$, is it more difficult to rotate the blade about the x-axis or the y-axis?

CAS **17–18** ■ Use a computer algebra system to find the mass, center of mass, and moments of inertia of the lamina that occupies the region D and has the given density function.

17. $D = \{(x, y) \mid 0 \leqslant y \leqslant \sin x,\ 0 \leqslant x \leqslant \pi\}$; $\rho(x, y) = xy$

18. D is enclosed by the cardioid $r = 1 + \cos\theta$; $\rho(x, y) = \sqrt{x^2 + y^2}$

19. The joint density function for a pair of random variables X and Y is

$$f(x, y) = \begin{cases} Cx(1 + y) & \text{if } 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant C.
(b) Find $P(X \leqslant 1, Y \leqslant 1)$.
(c) Find $P(X + Y \leqslant 1)$.

20. (a) Verify that

$$f(x, y) = \begin{cases} 4xy & \text{if } 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

is a joint density function.
(b) If X and Y are random variables whose joint density function is the function f in part (a), find
(i) $P(X \geqslant \frac{1}{2})$ (ii) $P(X \geqslant \frac{1}{2}, Y \leqslant \frac{1}{2})$
(c) Find the expected values of X and Y.

21. Suppose X and Y are random variables with joint density function

$$f(x, y) = \begin{cases} 0.1e^{-(0.5x + 0.2y)} & \text{if } x \geqslant 0,\ y \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that f is indeed a joint density function.
(b) Find the following probabilities.
(i) $P(Y \geqslant 1)$ (ii) $P(X \leqslant 2, Y \leqslant 4)$
(c) Find the expected values of X and Y.

22. (a) A lamp has two bulbs of a type with an average lifetime of 1000 hours. Assuming that we can model the probability of failure of these bulbs by an exponential density function with mean $\mu = 1000$, find the probability that both of the lamp's bulbs fail within 1000 hours.
(b) Another lamp has just one bulb of the same type as in part (a). If one bulb burns out and is replaced by a bulb of the same type, find the probability that the two bulbs fail within a total of 1000 hours.

CAS **23.** Suppose that X and Y are independent random variables, where X is normally distributed with mean 45 and standard deviation 0.5 and Y is normally distributed with mean 20 and standard deviation 0.1.
(a) Find $P(40 \leqslant X \leqslant 50, 20 \leqslant Y \leqslant 25)$.
(b) Find $P(4(X - 45)^2 + 100(Y - 20)^2 \leqslant 2)$.

24. Xavier and Yolanda both have classes that end at noon and they agree to meet every day after class. They arrive at the coffee shop independently. Xavier's arrival time is X and Yolanda's arrival time is Y, where X and Y are measured in minutes after noon. The individual density functions are

$$f_1(x) = \begin{cases} e^{-x} & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f_2(y) = \begin{cases} \frac{1}{50}y & \text{if } 0 \leqslant y \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$$

(Xavier arrives sometime after noon and is more likely to arrive promptly than late. Yolanda always arrives by 12:10 P.M. and is more likely to arrive late than promptly.) After Yolanda arrives, she'll wait for up to half an hour for Xavier, but he won't wait for her. Find the probability that they meet.

25. When studying the spread of an epidemic, we assume that the probability that an infected individual will spread the disease to an uninfected individual is a function of the distance between them. Consider a circular city of radius 10 mi in which the population is uniformly distributed. For an uninfected individual at a fixed point $A(x_0, y_0)$, assume that the probability function is given by

$$f(P) = \tfrac{1}{20}[20 - d(P, A)]$$

where $d(P, A)$ denotes the distance between P and A.

(a) Suppose the exposure of a person to the disease is the sum of the probabilities of catching the disease from all members of the population. Assume that the infected people are uniformly distributed throughout the city, with k infected individuals per square mile. Find a double integral that represents the exposure of a person residing at A.

(b) Evaluate the integral for the case in which A is the center of the city and for the case in which A is located on the edge of the city. Where would you prefer to live?

12.6 Surface Area

In this section we apply double integrals to the problem of computing the area of a surface. We start by finding a formula for the area of a parametric surface and then, as a special case, we deduce a formula for the surface area of the graph of a function of two variables.

We recall from Section 10.5 that a parametric surface S is defined by a vector-valued function of two parameters

$$\boxed{1} \qquad \mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

or, equivalently, by parametric equations

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

where (u, v) varies throughout a region D in the uv-plane.

We will find the area of S by dividing S into patches and approximating the area of each patch by the area of a piece of a tangent plane. So first let's recall from Section 11.4 how to find tangent planes to parametric surfaces.

Let P_0 be a point on S with position vector $\mathbf{r}(u_0, v_0)$. If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_1 lying on S. (See Figure 1.) The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v:

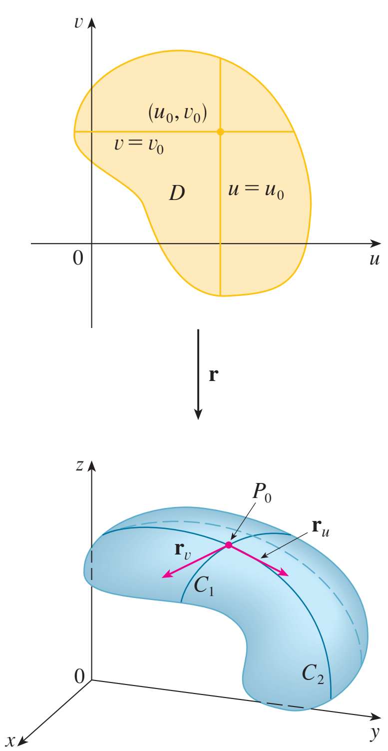


FIGURE 1

$$\boxed{2} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\,\mathbf{k}$$

Similarly, if we keep v constant by putting $v = v_0$, we get a grid curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S, and its tangent vector at P_0 is

$$\boxed{3} \qquad \mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\mathbf{k}$$

If the **normal vector** $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$, then the surface S is called **smooth**. (It has no "corners"). In this case the tangent plane to S at P_0 exists and can be found using the normal vector.

Now we define the surface area of a general parametric surface given by Equation 1. For simplicity we start by considering a surface whose parameter domain D is a rectangle, and we divide it into subrectangles R_{ij}. Let's choose (u_i^*, v_j^*) to be the lower left corner of R_{ij}. (See Figure 2.) The part S_{ij} of the surface S that corresponds to R_{ij} is called a *patch* and has the point P_{ij} with position vector $\mathbf{r}(u_i^*, v_j^*)$ as one of its corners. Let

$$\mathbf{r}_u^* = \mathbf{r}_u(u_i^*, v_j^*) \qquad \text{and} \qquad \mathbf{r}_v^* = \mathbf{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at P_{ij} as given by Equations 3 and 2.

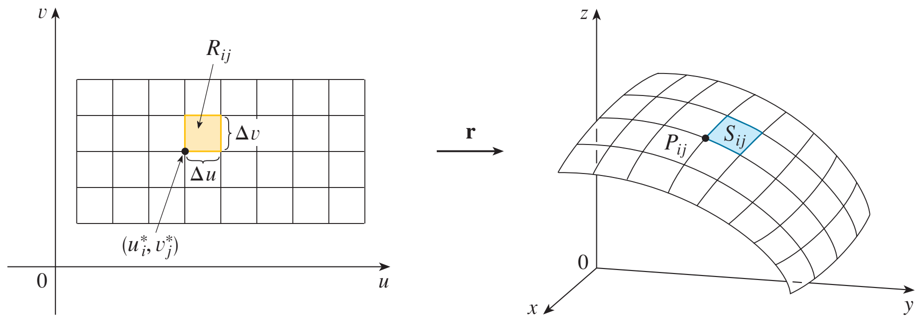


FIGURE 2
The image of the subrectangle R_{ij} is the patch S_{ij}.

Figure 3(a) shows how the two edges of the patch that meet at P_{ij} can be approximated by vectors. These vectors, in turn, can be approximated by the vectors $\Delta u\, \mathbf{r}_u^*$ and $\Delta v\, \mathbf{r}_v^*$ because partial derivatives can be approximated by difference quotients. So we approximate S_{ij} by the parallelogram determined by the vectors $\Delta u\, \mathbf{r}_u^*$ and $\Delta v\, \mathbf{r}_v^*$. This parallelogram is shown in Figure 3(b) and lies in the tangent plane to S at P_{ij}. The area of this parallelogram is

$$|(\Delta u\, \mathbf{r}_u^*) \times (\Delta v\, \mathbf{r}_v^*)| = |\mathbf{r}_u^* \times \mathbf{r}_v^*|\, \Delta u\, \Delta v$$

and so an approximation to the area of S is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_u^* \times \mathbf{r}_v^*|\, \Delta u\, \Delta v$$

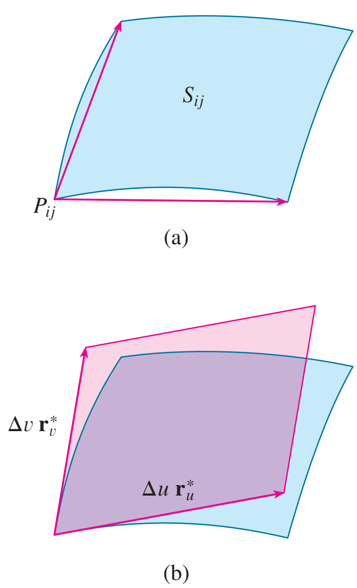


FIGURE 3
Approximating a patch by a parallelogram

Our intuition tells us that this approximation gets better as we increase the number of subrectangles, and we recognize the double sum as a Riemann sum for the double integral $\iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$. This motivates the following definition.

4 Definition If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where $$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

EXAMPLE 1 Find the surface area of a sphere of radius a.

SOLUTION In Example 4 in Section 10.5 we found the parametric representation

$$x = a \sin\phi \cos\theta \qquad y = a \sin\phi \sin\theta \qquad z = a\cos\phi$$

where the parameter domain is

$$D = \{(\phi, \theta) \mid 0 \leqslant \phi \leqslant \pi,\ 0 \leqslant \theta \leqslant 2\pi\}$$

We first compute the cross product of the tangent vectors:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial z}{\partial \phi} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi \cos\theta & a\cos\phi \sin\theta & -a\sin\phi \\ -a\sin\phi \sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\phi \cos\theta\, \mathbf{i} + a^2\sin^2\phi \sin\theta\, \mathbf{j} + a^2 \sin\phi \cos\phi\, \mathbf{k}$$

Thus

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\sin^4\phi \cos^2\theta + a^4\sin^4\phi \sin^2\theta + a^4\sin^2\phi \cos^2\phi}$$

$$= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi \cos^2\phi} = a^2\sqrt{\sin^2\phi} = a^2\sin\phi$$

since $\sin\phi \geqslant 0$ for $0 \leqslant \phi \leqslant \pi$. Therefore, by Definition 4, the area of the sphere is

$$A = \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, dA = \int_0^{2\pi}\int_0^{\pi} a^2 \sin\phi\, d\phi\, d\theta$$

$$= a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi = a^2(2\pi)2 = 4\pi a^2$$

■

Surface Area of a Graph

For the special case of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives, we take x and y as parameters. The parametric equations are

$$x = x \qquad y = y \qquad z = f(x, y)$$

so

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

and

5
$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{\partial f}{\partial x} \\ 0 & 1 & \dfrac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

▲ Notice the similarity between the surface area formula in Equation 6 and the arc length formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

from Section 6.3.

Thus, the surface area formula in Definition 4 becomes

6
$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

EXAMPLE 2 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

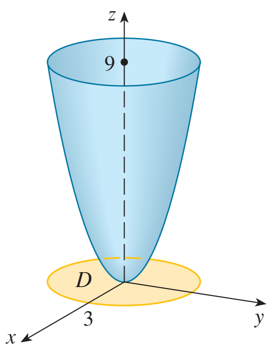


FIGURE 4

SOLUTION The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Therefore, the given surface lies above the disk D with center the origin and radius 3. (See Figure 4.) Using Formula 6, we have

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

$$= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA$$

Converting to polar coordinates, we obtain

$$A = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1 + 4r^2}\, dr$$

$$= 2\pi\left(\tfrac{1}{8}\right)\tfrac{2}{3}(1 + 4r^2)^{3/2}\Big]_0^3 = \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$

A common type of surface is a **surface of revolution** S obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis, where $f(x) \geqslant 0$ and f' is continuous. In Exercise 23 you are asked to use a parametric representation of S and Definition 4 to prove the following formula for the area of a surface of revolution:

7
$$A = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

12.6 Exercises

1–12 ■ Find the area of the surface.

1. The part of the plane $z = 2 + 3x + 4y$ that lies above the rectangle $[0, 5] \times [1, 4]$
2. The part of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$
3. The part of the plane $3x + 2y + z = 6$ that lies in the first octant
4. The part of the plane with vector equation $\mathbf{r}(u, v) = \langle 1 + v, u - 2v, 3 - 5u + v\rangle$ that is given by $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 1$
5. The part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$
6. The part of the surface $z = x + y^2$ that lies above the triangle with vertices (0, 0), (1, 1), and (0, 1)
7. The surface with parametric equations $x = uv$, $y = u + v$, $z = u - v$, $u^2 + v^2 \leqslant 1$
8. The helicoid (or spiral ramp) with vector equation $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$
9. The part of the surface $y = 4x + z^2$ that lies between the planes $x = 0$, $x = 1$, $z = 0$, and $z = 1$
10. The part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$
11. The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$

12. The surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

13. (a) Use the Midpoint Rule for double integrals (see Section 12.1) with four squares to estimate the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies above the square $[0, 1] \times [0, 1]$.

CAS (b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).

14. (a) Use the Midpoint Rule for double integrals with $m = n = 2$ to estimate the area of the surface $z = xy + x^2 + y^2$, $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 2$.

CAS (b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).

CAS **15.** Find the area of the surface with vector equation $\mathbf{r}(u, v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$, $0 \leqslant u \leqslant \pi$, $0 \leqslant v \leqslant 2\pi$. State your answer correct to four decimal places.

CAS **16.** Find, to four decimal places, the area of the part of the surface $z = (1 + x^2)/(1 + y^2)$ that lies above the square $|x| + |y| \leqslant 1$. Illustrate by graphing this part of the surface.

CAS **17.** Find the exact area of the surface $z = 1 + 2x + 3y + 4y^2$, $1 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 1$.

18. (a) Set up, but do not evaluate, a double integral for the area of the surface with parametric equations $x = au \cos v$, $y = bu \sin v$, $z = u^2$, $0 \leqslant u \leqslant 2$, $0 \leqslant v \leqslant 2\pi$.

(b) Eliminate the parameters to show that the surface is an elliptic paraboloid and set up another double integral for the surface area.

(c) Use the parametric equations in part (a) with $a = 2$ and $b = 3$ to graph the surface.

CAS (d) For the case $a = 2$, $b = 3$, use a computer algebra system to find the surface area correct to four decimal places.

19. (a) Show that the parametric equations $x = a \sin u \cos v$, $y = b \sin u \sin v$, $z = c \cos u$, $0 \leqslant u \leqslant \pi$, $0 \leqslant v \leqslant 2\pi$, represent an ellipsoid.

(b) Use the parametric equations in part (a) to graph the ellipsoid for the case $a = 1$, $b = 2$, $c = 3$.

(c) Set up, but do not evaluate, a double integral for the surface area of the ellipsoid in part (b).

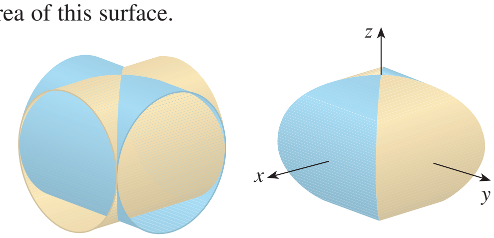
20. (a) Show that the parametric equations $x = a \cosh u \cos v$, $y = b \cosh u \sin v$, $z = c \sinh u$, represent a hyperboloid of one sheet.

(b) Use the parametric equations in part (a) to graph the hyperboloid for the case $a = 1$, $b = 2$, $c = 3$.

(c) Set up, but do not evaluate, a double integral for the surface area of the part of the hyperboloid in part (b) that lies between the planes $z = -3$ and $z = 3$.

21. Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.

22. The figure shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Find the area of this surface.



23. Use Definition 4 and the parametric equations for a surface of revolution (see Equations 10.5.3) to derive Formula 7.

24–25 ■ Use Formula 7 to find the area of the surface obtained by rotating the given curve about the x-axis.

24. $y = x^3$, $0 \leqslant x \leqslant 2$

25. $y = \sqrt{x}$, $4 \leqslant x \leqslant 9$

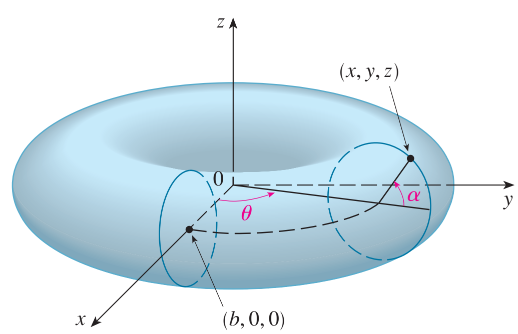
26. The figure shows the torus obtained by rotating about the z-axis the circle in the xz-plane with center $(b, 0, 0)$ and radius $a < b$. Parametric equations for the torus are

$$x = b \cos\theta + a \cos\alpha \cos\theta$$

$$y = b \sin\theta + a \cos\alpha \sin\theta$$

$$z = a \sin\alpha$$

where θ and α are the angles shown in the figure. Find the surface area of the torus.



12.7 Triple Integrals

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables. Let's first deal with the simplest case where f is defined on a rectangular box:

1 $$B = \{(x, y, z) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d,\ r \leqslant z \leqslant s\}$$

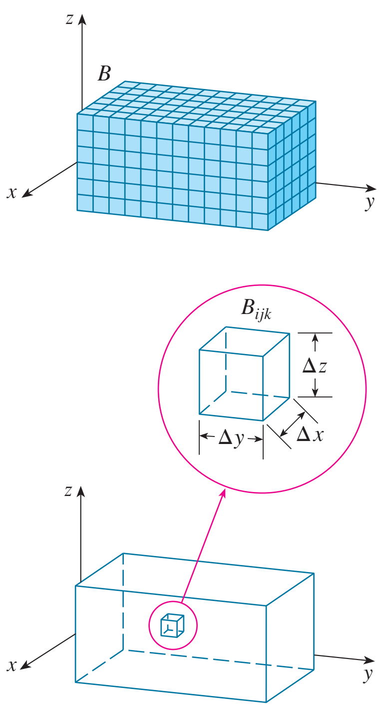


FIGURE 1

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz. The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in Figure 1. Each sub-box has volume $\Delta V = \Delta x\, \Delta y\, \Delta z$.

Then we form the **triple Riemann sum**

2 $$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in B_{ijk}. By analogy with the definition of a double integral (12.1.5), we define the triple integral as the limit of the triple Riemann sums in (2).

3 Definition The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists.

Again, the triple integral always exists if f is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point (x_i, y_j, z_k) we get a simpler-looking expression for the triple integral:

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\, \Delta V$$

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

4 Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z. There are five other possible orders in which we can integrate, all of which give the same value. For instance, if we integrate with respect to y, then z, and then x, we have

$$\iiint_B f(x, y, z)\, dV = \int_a^b \int_r^s \int_c^d f(x, y, z)\, dy\, dz\, dx$$

EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

SOLUTION We could use any of the six possible orders of integration. If we choose to integrate with respect to x, then y, and then z, we obtain

$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz = \int_0^3 \left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3z^2}{4}\, dz = \frac{z^3}{4}\bigg]_0^3 = \frac{27}{4}$$

Now we define the **triple integral over a general bounded region E** in three-dimensional space (a solid) by much the same procedure that we used for double integrals (12.3.2). We enclose E in a box B of the type given by Equation 1. Then we define a function F so that it agrees with f on E but is 0 for points in B that are outside E. By definition,

$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$$

This integral exists if f is continuous and the boundary of E is "reasonably smooth." The triple integral has essentially the same properties as the double integral (Properties 6–9 in Section 12.3).

We restrict our attention to continuous functions f and to certain simple types of regions. A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

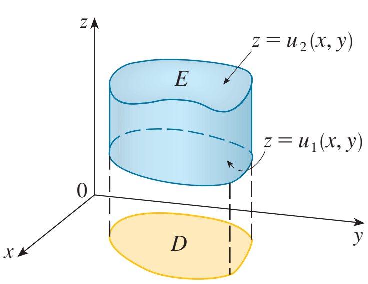


FIGURE 2
A type 1 solid region

5 $$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is the projection of E onto the xy-plane as shown in Figure 2. Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

By the same sort of argument that led to (12.3.3), it can be shown that if E is a type 1 region given by Equation 5, then

6 $$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\right] dA$$

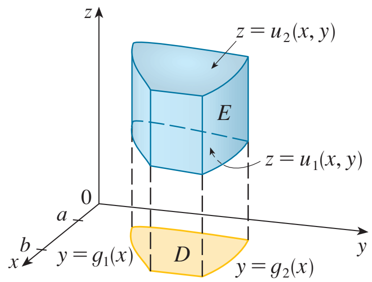


FIGURE 3
A type 1 solid region

The meaning of the inner integral on the right side of Equation 6 is that x and y are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to z.

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 3), then

$$E = \{(x, y, z) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

and Equation 6 becomes

7
$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx$$

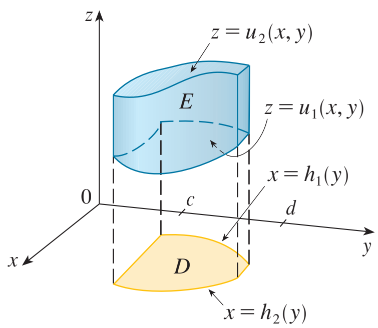


FIGURE 4
Another type 1 solid region

If, on the other hand, D is a type II plane region (as in Figure 4), then

$$E = \{(x, y, z) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

and Equation 6 becomes

8
$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$

EXAMPLE 2 Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

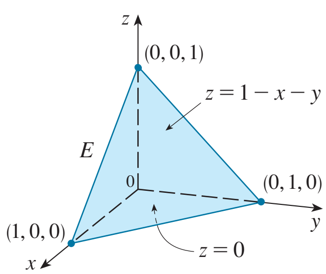


FIGURE 5

SOLUTION When we set up a triple integral it's wise to draw *two* diagrams: one of the solid region E (see Figure 5) and one of its projection D on the xy-plane (see Figure 6). The lower boundary of the tetrahedron is the plane $z = 0$ and the upper boundary is the plane $x + y + z = 1$ (or $z = 1 - x - y$), so we use $u_1(x, y) = 0$ and $u_2(x, y) = 1 - x - y$ in Formula 7. Notice that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ (or $y = 1 - x$) in the xy-plane. So the projection of E is the triangular region shown in Figure 6, and we have

9
$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y\}$$

This description of E as a type 1 region enables us to evaluate the integral as follows:

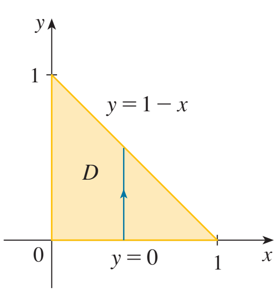


FIGURE 6

$$\iiint_E z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\, dx$$

$$= \tfrac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y)^2\, dy\, dx$$

$$= \tfrac{1}{2} \int_0^1 \left[-\frac{(1 - x - y)^3}{3}\right]_{y=0}^{y=1-x} dx$$

$$= \tfrac{1}{6} \int_0^1 (1 - x)^3\, dx = \frac{1}{6}\left[-\frac{(1 - x)^4}{4}\right]_0^1 = \frac{1}{24}$$

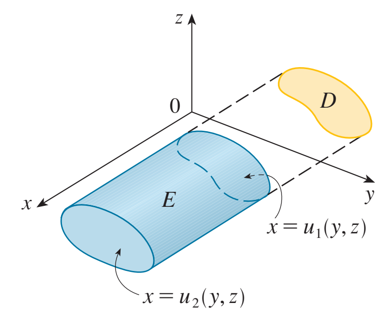


FIGURE 7
A type 2 region

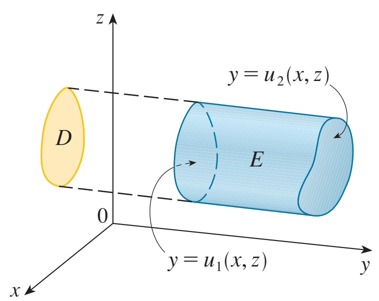


FIGURE 8
A type 3 region

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \leqslant x \leqslant u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz-plane (see Figure 7). The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

10 $$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\, dx \right] dA$$

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \leqslant y \leqslant u_2(x, z)\}$$

where D is the projection of E onto the xz-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 8). For this type of region we have

11 $$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\, dy \right] dA$$

In each of Equations 10 and 11 there may be two possible expressions for the integral depending on whether D is a type I or type II plane region (and corresponding to Equations 7 and 8).

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

SOLUTION The solid E is shown in Figure 9. If we regard it as a type 1 region, then we need to consider its projection D_1 onto the xy-plane, which is the parabolic region in Figure 10. (The trace of $y = x^2 + z^2$ in the plane $z = 0$ is the parabola $y = x^2$.)

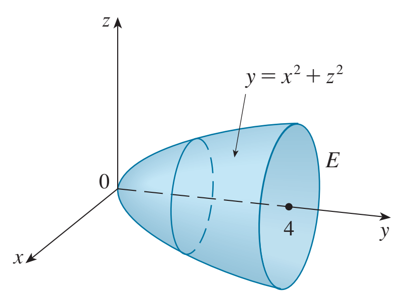


FIGURE 9
Region of integration

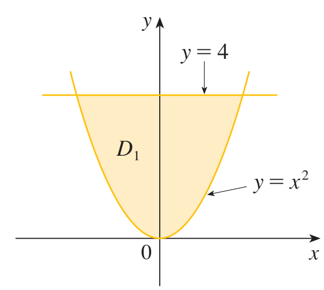


FIGURE 10
Projection on xy-plane

From $y = x^2 + z^2$ we obtain $z = \pm\sqrt{y - x^2}$, so the lower boundary surface of E is $z = -\sqrt{y - x^2}$ and the upper surface is $z = \sqrt{y - x^2}$. Therefore, the description of E as a type 1 region is

$$E = \left\{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 4,\ -\sqrt{y - x^2} \leqslant z \leqslant \sqrt{y - x^2}\right\}$$

and so we obtain

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx$$

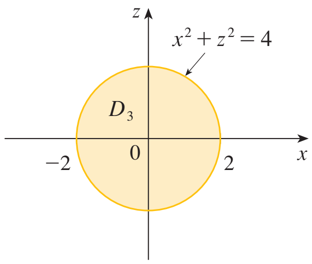


FIGURE 11
Projection on *xz*-plane

Although this expression is correct, it is extremely difficult to evaluate. So let's instead consider E as a type 3 region. As such, its projection D_3 onto the xz-plane is the disk $x^2 + z^2 \leqslant 4$ shown in Figure 11.

Then the left boundary of E is the paraboloid $y = x^2 + z^2$ and the right boundary is the plane $y = 4$, so taking $u_1(x, z) = x^2 + z^2$ and $u_2(x, z) = 4$ in Equation 11, we have

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_{D_3} \left[\int_{x^2+z^2}^{4} \sqrt{x^2 + z^2}\, dy \right] dA$$

$$= \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dA$$

Although this integral could be written as

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx$$

it's easier to convert to polar coordinates in the xz-plane: $x = r\cos\theta$, $z = r\sin\theta$. This gives

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dA$$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2) r\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4)\, dr$$

$$= 2\pi \left[\frac{4r^3}{3} - \frac{r^5}{5} \right]_0^2 = \frac{128\pi}{15}$$

⊘ The most difficult step in evaluating a triple integral is setting up an expression for the region of integration (such as Equation 9 in Example 2). Remember that the limits of integration in the inner integral contain at most two variables, the limits of integration in the middle integral contain at most one variable, and the limits of integration in the outer integral must be constants.

Applications of Triple Integrals

Recall that if $f(x) \geqslant 0$, then the single integral $\int_a^b f(x)\, dx$ represents the area under the curve $y = f(x)$ from a to b, and if $f(x, y) \geqslant 0$, then the double integral $\iint_D f(x, y)\, dA$ represents the volume under the surface $z = f(x, y)$ and above D. The corresponding interpretation of a triple integral $\iiint_E f(x, y, z)\, dV$, where $f(x, y, z) \geqslant 0$, is not very useful because it would be the "hypervolume" of a four-dimensional object and, of course, that is very difficult to visualize. (Remember that E is just the *domain* of the function f; the graph of f lies in four-dimensional space.) Nonetheless, the triple integral $\iiint_E f(x, y, z)\, dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of x, y, z and $f(x, y, z)$.

Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does represent the volume of E:

12
$$V(E) = \iiint_E dV$$

For example, you can see this in the case of a type 1 region by putting $f(x, y, z) = 1$ in Formula 6:

$$\iiint_E 1\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} dz \right] dA = \iint_D [u_2(x, y) - u_1(x, y)]\, dA$$

and from Section 12.3 we know this represents the volume that lies between the surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$.

EXAMPLE 4 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

SOLUTION The tetrahedron T and its projection D on the xy-plane are shown in Figures 12 and 13. The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$. Therefore, we have

$$V(T) = \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\, dy\, dx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx = \tfrac{1}{3}$$

by the same calculation as in Example 4 in Section 12.3.

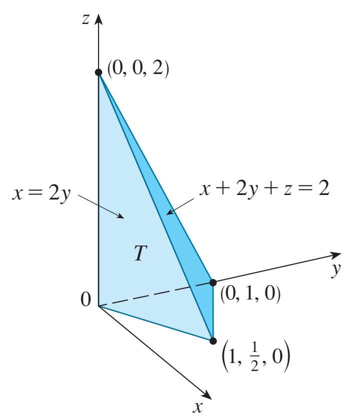


FIGURE 12

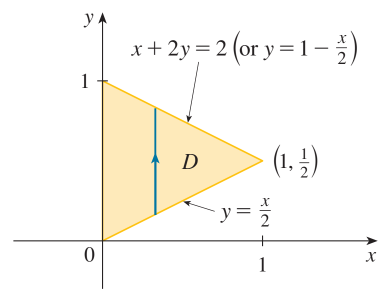


FIGURE 13

(Notice that it is not necessary to use triple integrals to compute volumes. They simply give an alternative method for setting up the calculation.) ■

All the applications of double integrals in Section 12.5 can be immediately extended to triple integrals. For example, if the density function of a solid object that occupies the region E is $\rho(x, y, z)$, in units of mass per unit volume, at any given point (x, y, z), then its **mass** is

13
$$m = \iiint_E \rho(x, y, z)\, dV$$

and its **moments** about the three coordinate planes are

14
$$M_{yz} = \iiint_E x\rho(x, y, z)\, dV \qquad M_{xz} = \iiint_E y\rho(x, y, z)\, dV$$

$$M_{xy} = \iiint_E z\rho(x, y, z)\, dV$$

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

15 $$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

If the density is constant, the center of mass of the solid is called the **centroid** of E. The **moments of inertia** about the three coordinate axes are

16 $$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\, dV \qquad I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\, dV$$

$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\, dV$$

As in Section 12.5, the total **electric charge** on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)\, dV$$

If we have three continuous random variables X, Y, and Z, their **joint density function** is a function of three variables such that the probability that (X, Y, Z) lies in E is

$$P((X, Y, Z) \in E) = \iiint_E f(x, y, z)\, dV$$

In particular,

$$P(a \leqslant X \leqslant b,\ c \leqslant Y \leqslant d,\ r \leqslant Z \leqslant s) = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx$$

The joint density function satisfies

$$f(x, y, z) \geqslant 0 \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dz\, dy\, dx = 1$$

EXAMPLE 5 Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

SOLUTION The solid E and its projection onto the xy-plane are shown in Figure 14. The lower and upper surfaces of E are the planes $z = 0$ and $z = x$, so we describe E as a type 1 region:

$$E = \{(x, y, z) \mid -1 \leqslant y \leqslant 1,\ y^2 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant x\}$$

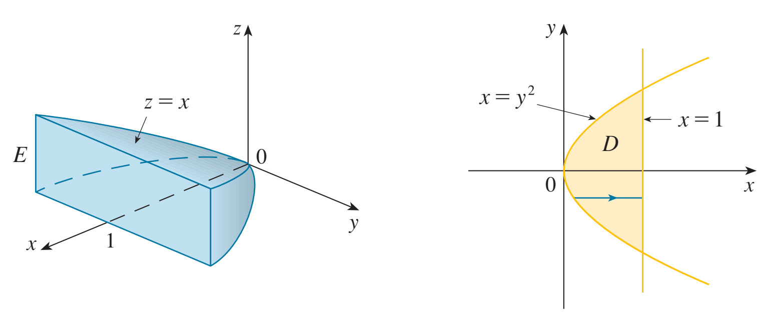


FIGURE 14

Then, if the density is $\rho(x, y, z) = \rho$, the mass is

$$m = \iiint_E \rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x \rho \, dz \, dx \, dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x \, dx \, dy = \rho \int_{-1}^{1} \left[\frac{x^2}{2} \right]_{x=y^2}^{x=1} dy$$

$$= \frac{\rho}{2} \int_{-1}^{1} (1 - y^4) \, dy = \rho \int_0^1 (1 - y^4) \, dy$$

$$= \rho \left[y - \frac{y^5}{5} \right]_0^1 = \frac{4\rho}{5}$$

Because of the symmetry of E and ρ about the xz-plane, we can immediately say that $M_{xz} = 0$ and, therefore, $\bar{y} = 0$. The other moments are

$$M_{yz} = \iiint_E x\rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x x\rho \, dz \, dx \, dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x^2 \, dx \, dy = \rho \int_{-1}^{1} \left[\frac{x^3}{3} \right]_{x=y^2}^{x=1} dy$$

$$= \frac{2\rho}{3} \int_0^1 (1 - y^6) \, dy = \frac{2\rho}{3} \left[y - \frac{y^7}{7} \right]_0^1 = \frac{4\rho}{7}$$

$$M_{xy} = \iiint_E z\rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x z\rho \, dz \, dx \, dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} \left[\frac{z^2}{2} \right]_{z=0}^{z=x} dx \, dy = \frac{\rho}{2} \int_{-1}^{1} \int_{y^2}^{1} x^2 \, dx \, dy$$

$$= \frac{\rho}{3} \int_0^1 (1 - y^6) \, dy = \frac{2\rho}{7}$$

Therefore, the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right) = \left(\tfrac{5}{7}, 0, \tfrac{5}{14} \right)$$ ■

12.7 Exercises

1. Evaluate the integral in Example 1, integrating first with respect to z, then x, and then y.

2. Evaluate the integral $\iiint_E (xz - y^3) \, dV$, where

$$E = \{(x, y, z) \mid -1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 2, 0 \leqslant z \leqslant 1\}$$

using three different orders of integration.

3–6 ■ Evaluate the iterated integral.

3. $\int_0^1 \int_0^z \int_0^{x+z} 6xz \, dy \, dx \, dz$

4. $\int_0^1 \int_x^{2x} \int_0^y 2xyz \, dz \, dy \, dx$

5. $\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y \, dx \, dz \, dy$

6. $\int_0^1 \int_0^z \int_0^y ze^{-y^2} \, dx \, dy \, dz$

7–14 ■ Evaluate the triple integral.

7. $\iiint_E 2x\,dV$, where
$E = \{(x, y, z) \mid 0 \leqslant y \leqslant 2,\ 0 \leqslant x \leqslant \sqrt{4 - y^2},\ 0 \leqslant z \leqslant y\}$

8. $\iiint_E yz\cos(x^5)\,dV$, where
$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x,\ x \leqslant z \leqslant 2x\}$

9. $\iiint_E 6xy\,dV$, where E lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$

10. $\iiint_E xz\,dV$, where E is the solid tetrahedron with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, and $(0, 1, 1)$

11. $\iiint_E z\,dV$, where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $y + z = 1$, and $x + z = 1$

12. $\iiint_E (x + 2y)\,dV$, where E is bounded by the parabolic cylinder $y = x^2$ and the planes $x = z$, $x = y$, and $z = 0$

13. $\iiint_E x\,dV$, where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$

14. $\iiint_E z\,dV$, where E is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant

15–18 ■ Use a triple integral to find the volume of the given solid.

15. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

16. The solid bounded by the elliptic cylinder $4x^2 + z^2 = 4$ and the planes $y = 0$ and $y = z + 2$

17. The solid bounded by the cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$

18. The solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 18 - x^2 - y^2$

19. (a) Express the volume of the wedge in the first octant that is cut from the cylinder $y^2 + z^2 = 1$ by the planes $y = x$ and $x = 1$ as a triple integral.

CAS (b) Use either the Table of Integrals (on the back Reference Pages) or a computer algebra system to find the exact value of the triple integral in part (a).

20. (a) In the **Midpoint Rule for triple integrals** we use a triple Riemann sum to approximate a triple integral over a box B, where $f(x, y, z)$ is evaluated at the center $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ of the box B_{ijk}. Use the Midpoint Rule to estimate $\iiint_B e^{-x^2-y^2-z^2}\,dV$, where B is the cube defined by $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$. Divide B into eight cubes of equal size.

CAS (b) Use a computer algebra system to approximate the integral in part (a) correct to two decimal places. Compare with the answer to part (a).

21–22 ■ Use the Midpoint Rule for triple integrals (Exercise 20) to estimate the value of the integral. Divide B into eight sub-boxes of equal size.

21. $\iiint_B \dfrac{1}{\ln(1 + x + y + z)}\,dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant 8,\ 0 \leqslant z \leqslant 4\}$

22. $\iiint_B \sin(xy^2z^3)\,dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 1\}$

23–24 ■ Sketch the solid whose volume is given by the iterated integral.

23. $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\,dz\,dx$

24. $\displaystyle\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\,dz\,dy$

25–28 ■ Express the integral $\iiint_E f(x, y, z)\,dV$ as an iterated integral in six different ways, where E is the solid bounded by the given surfaces.

25. $x^2 + z^2 = 4$, $y = 0$, $y = 6$

26. $z = 0$, $x = 0$, $y = 2$, $z = y - 2x$

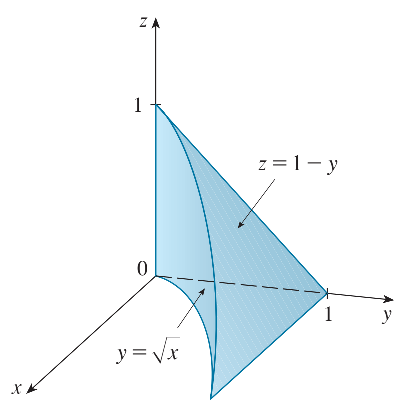
27. $z = 0$, $z = y$, $x^2 = 1 - y$

28. $9x^2 + 4y^2 + z^2 = 1$

29. The figure shows the region of integration for the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\,dz\,dy\,dx$$

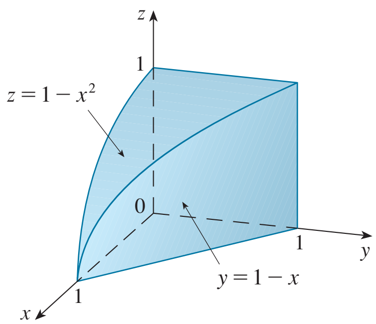
Rewrite this integral as an equivalent iterated integral in the five other orders.



30. The figure shows the region of integration for the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.



31–32 ■ Write five other iterated integrals that are equal to the given iterated integral.

31. $\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dz\, dx\, dy$

32. $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$

33–36 ■ Find the mass and center of mass of the given solid E with the given density function ρ.

33. E is the solid of Exercise 9; $\rho(x, y, z) = 2$

34. E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $x + z = 1$, $x = 0$, and $z = 0$; $\rho(x, y, z) = 4$

35. E is the cube given by $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant a$, $0 \leqslant z \leqslant a$; $\rho(x, y, z) = x^2 + y^2 + z^2$

36. E is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$; $\rho(x, y, z) = y$

37–38 ■ Set up, but do not evaluate, integral expressions for (a) the mass, (b) the center of mass, and (c) the moment of inertia about the z-axis.

37. The solid of Exercise 13; $\rho(x, y, z) = x^2 + y^2 + z^2$

38. The hemisphere $x^2 + y^2 + z^2 \leqslant 1$, $z \geqslant 0$; $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

CAS **39.** Let E be the solid in the first octant bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, and $z = 0$ with the density function $\rho(x, y, z) = 1 + x + y + z$. Use a computer algebra system to find the exact values of the following quantities for E.
(a) The mass
(b) The center of mass
(c) The moment of inertia about the z-axis

CAS **40.** If E is the solid of Exercise 14 with density function $\rho(x, y, z) = x^2 + y^2$, find the following quantities, correct to three decimal places.
(a) The mass
(b) The center of mass
(c) The moment of inertia about the z-axis

41. Find the moments of inertia for a cube of constant density k and side length L if one vertex is located at the origin and three edges lie along the coordinate axes.

42. Find the moments of inertia for a rectangular brick with dimensions a, b, and c, mass M, and constant density if the center of the brick is situated at the origin and the edges are parallel to the coordinate axes.

43. The joint density function for random variables X, Y, and Z is $f(x, y, z) = Cxyz$ if $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 2$, $0 \leqslant z \leqslant 2$, and $f(x, y, z) = 0$ otherwise.
(a) Find the value of the constant C.
(b) Find $P(X \leqslant 1, Y \leqslant 1, Z \leqslant 1)$.
(c) Find $P(X + Y + Z \leqslant 1)$.

44. Suppose X, Y, and Z are random variables with joint density function $f(x, y, z) = Ce^{-(0.5x+0.2y+0.1z)}$ if $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, and $f(x, y, z) = 0$ otherwise.
(a) Find the value of the constant C.
(b) Find $P(X \leqslant 1, Y \leqslant 1)$.
(c) Find $P(X \leqslant 1, Y \leqslant 1, Z \leqslant 1)$.

45–46 ■ The **average value** of a function $f(x, y, z)$ over a solid region E is defined to be

$$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV$$

where $V(E)$ is the volume of E. For instance, if ρ is a density function, then ρ_{ave} is the average density of E.

45. Find the average value of the function $f(x, y, z) = xyz$ over the cube with side length L that lies in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

46. Find the average value of the function $f(x, y, z) = x + y + z$ over the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

47. Find the region E for which the triple integral

$$\iiint_E (1 - x^2 - 2y^2 - 3z^2)\, dV$$

is a maximum.

Discovery Project

Volumes of Hyperspheres

In this project we find formulas for the volume enclosed by a hypersphere in n-dimensional space.

1. Use a double integral and the trigonometric substitution $y = r \sin \theta$, together with Formula 64 in the Table of Integrals, to find the area of a circle with radius r.
2. Use a triple integral and trigonometric substitution to find the volume of a sphere with radius r.
3. Use a quadruple integral to find the hypervolume enclosed by the hypersphere $x^2 + y^2 + z^2 + w^2 = r^2$ in $\mathbb{R}^4$. (Use only trigonometric substitution and the reduction formulas for $\int \sin^n x\, dx$ or $\int \cos^n x\, dx$.)
4. Use an n-tuple integral to find the volume enclosed by a hypersphere of radius r in n-dimensional space $\mathbb{R}^n$. [*Hint:* The formulas are different for n even and n odd.]

12.8 Triple Integrals in Cylindrical and Spherical Coordinates

We saw in Section 12.4 that some double integrals are easier to evaluate using polar coordinates. In this section we see that some triple integrals are easier to evaluate using cylindrical or spherical coordinates.

Cylindrical Coordinates

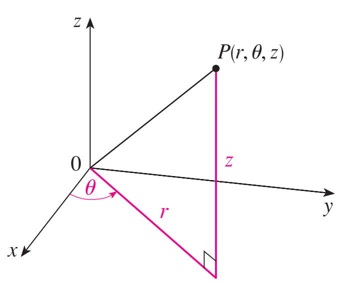


FIGURE 1

Recall from Section 9.7 that the cylindrical coordinates of a point P are (r, θ, z), where r, θ, and z are shown in Figure 1. Suppose that E is a type 1 region whose projection D on the xy-plane is conveniently described in polar coordinates (see Figure 2). In particular, suppose that f is continuous and

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

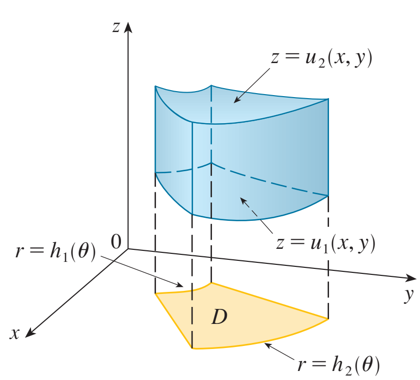


FIGURE 2

We know from Equation 12.7.6 that

1 $$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

But we also know how to evaluate double integrals in polar coordinates. In fact, combining Equation 1 with Equation 12.4.3, we obtain

2 $$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

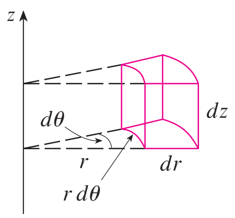


FIGURE 3
Volume element in cylindrical coordinates: $dV = r\,dz\,dr\,d\theta$

Formula 2 is the **formula for triple integration in cylindrical coordinates**. It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$, leaving z as it is, using the appropriate limits of integration for z, r, and θ, and replacing dV by $r\,dz\,dr\,d\theta$. (Figure 3 shows how to remember this.) It is worthwhile to use this formula when E is a solid region easily described in cylindrical coordinates, and especially when the function $f(x, y, z)$ involves the expression $x^2 + y^2$.

EXAMPLE 1 A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 4.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

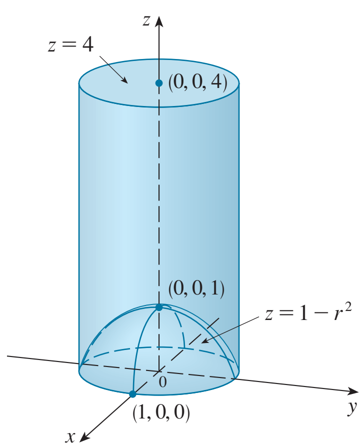


FIGURE 4

SOLUTION In cylindrical coordinates the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$, so we can write

$$E = \{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 1,\ 1 - r^2 \leqslant z \leqslant 4\}$$

Since the density at (x, y, z) is proportional to the distance from the z-axis, the density function is

$$f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

where K is the proportionality constant. Therefore, from Formula 12.7.13, the mass of E is

$$m = \iiint_E K\sqrt{x^2 + y^2}\, dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 Kr^2[4 - (1 - r^2)]\, dr\, d\theta = K \int_0^{2\pi} d\theta \int_0^1 (3r^2 + r^4)\, dr$$

$$= 2\pi K\left[r^3 + \frac{r^5}{5} \right]_0^1 = \frac{12\pi K}{5}$$ ■

EXAMPLE 2 Evaluate $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx$.

SOLUTION This iterated integral is a triple integral over the solid region

$$E = \left\{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ -\sqrt{4 - x^2} \leqslant y \leqslant \sqrt{4 - x^2},\ \sqrt{x^2 + y^2} \leqslant z \leqslant 2\right\}$$

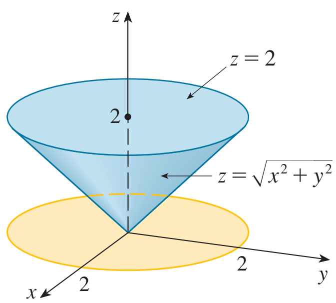


FIGURE 5

and the projection of E onto the xy-plane is the disk $x^2 + y^2 \leqslant 4$. The lower surface of E is the cone $z = \sqrt{x^2 + y^2}$ and its upper surface is the plane $z = 2$. (See Figure 5.) This region has a much simpler description in cylindrical coordinates:

$$E = \{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 2,\ r \leqslant z \leqslant 2\}$$

Therefore, we have

$$\begin{aligned}\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx &= \iiint_E (x^2 + y^2)\, dV \\ &= \int_0^{2\pi} \int_0^2 \int_r^2 r^2\, r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^2 r^3(2 - r)\, dr \\ &= 2\pi\left[\tfrac{1}{2}r^4 - \tfrac{1}{5}r^5\right]_0^2 = \frac{16\pi}{5}\end{aligned}$$

Spherical Coordinates

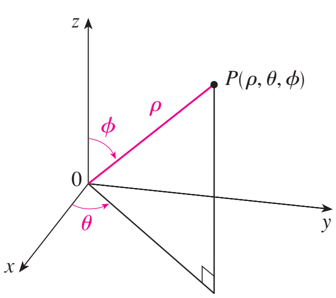


FIGURE 6
Spherical coordinates of P

In Section 9.7 we defined the spherical coordinates (ρ, θ, ϕ) of a point (see Figure 6) and we demonstrated the following relationships between rectangular coordinates and spherical coordinates:

3 $\qquad x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$

In this coordinate system the counterpart of a rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

where $a \geqslant 0$, $\beta - \alpha \leqslant 2\pi$, and $d - c \leqslant \pi$. Although we defined triple integrals by dividing solids into small boxes, it can be shown that dividing a solid into small spherical wedges always gives the same result. So we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$. Figure 7 shows that E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i\,\Delta\phi$ (arc of a circle with radius ρ_i, angle $\Delta\phi$), and $\rho_i \sin\phi_k\,\Delta\theta$ (arc of a circle with radius $\rho_i \sin\phi_k$, angle $\Delta\theta$). So an approximation to the volume of E_{ijk} is given by

$$(\Delta\rho) \times (\rho_i\,\Delta\phi) \times (\rho_i \sin\phi_k\,\Delta\theta) = \rho_i^2 \sin\phi_k\,\Delta\rho\,\Delta\theta\,\Delta\phi$$

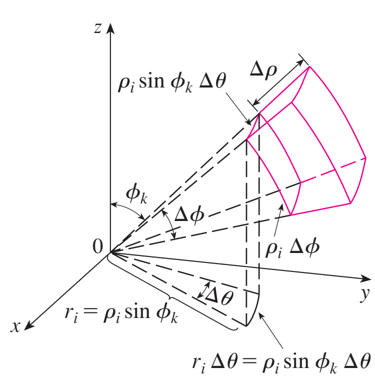


FIGURE 7

Thus, an approximation to a typical triple Riemann sum is

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\rho_i \sin\phi_k \cos\theta_j,\ \rho_i \sin\phi_k \sin\theta_j,\ \rho_i \cos\phi_k)\,\rho_i^2 \sin\phi_k\,\Delta\rho\,\Delta\theta\,\Delta\phi$$

But this sum is a Riemann sum for the function

$$F(\rho, \theta, \phi) = \rho^2 \sin\phi\, f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$$

Consequently, the following **formula for triple integration in spherical coordinates** is plausible.

4 $$\iiint_E f(x, y, z)\,dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

Formula 4 says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

using the appropriate limits of integration, and replacing dV by $\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$. This is illustrated in Figure 8.

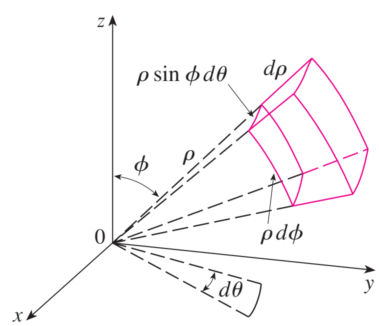


FIGURE 8
Volume element in spherical coordinates: $dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

This formula can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta, \phi) \mid \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d,\ g_1(\theta, \phi) \leqslant \rho \leqslant g_2(\theta, \phi)\}$$

In this case the formula is the same as in (4) except that the limits of integration for ρ are $g_1(\theta, \phi)$ and $g_2(\theta, \phi)$.

Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

EXAMPLE 3 Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$$

SOLUTION Since the boundary of B is a sphere, we use spherical coordinates:

$$B = \{(\rho, \theta, \phi) \mid 0 \leqslant \rho \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi\}$$

In addition, spherical coordinates are appropriate because

$$x^2 + y^2 + z^2 = \rho^2$$

Thus, (4) gives

$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3}\, d\rho$$

$$= \Big[-\cos\phi\Big]_0^{\pi} (2\pi) \Big[\tfrac{1}{3} e^{\rho^3}\Big]_0^1 = \frac{4\pi}{3}(e - 1)$$

NOTE • It would have been extremely awkward to evaluate the integral in Example 3 without spherical coordinates. In rectangular coordinates the iterated integral would have been

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx$$

EXAMPLE 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See Figure 9.)

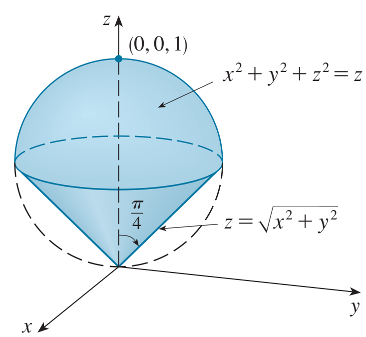


FIGURE 9

▲ Figure 10 gives another look (this time drawn by Maple) at the solid of Example 4.

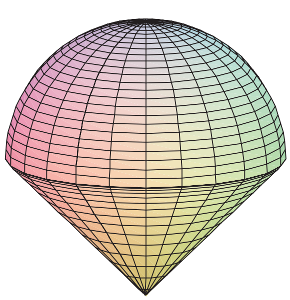


FIGURE 10

SOLUTION Notice that the sphere passes through the origin and has center $\left(0, 0, \tfrac{1}{2}\right)$. We write the equation of the sphere in spherical coordinates as

$$\rho^2 = \rho\cos\phi \qquad \text{or} \qquad \rho = \cos\phi$$

The equation of the cone can be written as

$$\rho\cos\phi = \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta} = \rho\sin\phi$$

This gives $\sin\phi = \cos\phi$, or $\phi = \pi/4$. Therefore, the description of the solid E in spherical coordinates is

$$E = \{(\rho, \theta, \phi) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi/4,\ 0 \leqslant \rho \leqslant \cos\phi\}$$

Figure 11 shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ. The volume of E is

$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=\cos\phi} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \cos^3\phi \, d\phi = \frac{2\pi}{3} \left[-\frac{\cos^4\phi}{4} \right]_0^{\pi/4} = \frac{\pi}{8}$$

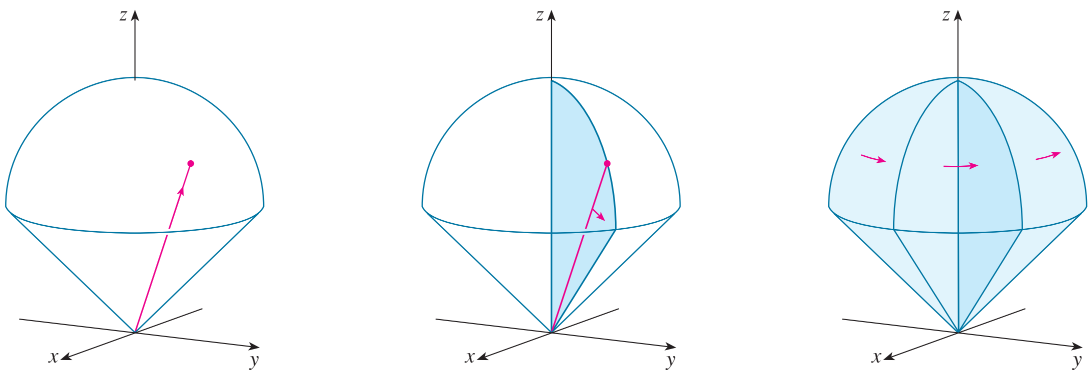


FIGURE 11

ρ varies from 0 to $\cos\phi$ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π. ■

12.8 Exercises

1–4 ■ Sketch the solid whose volume is given by the integral and evaluate the integral.

1. $\int_0^4 \int_0^{2\pi} \int_r^4 r \, dz \, d\theta \, dr$

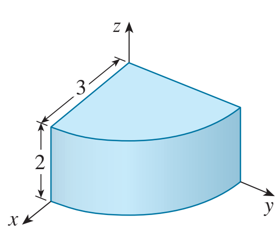
2. $\int_0^{\pi/2} \int_0^2 \int_0^{9-r^2} r \, dz \, dr \, d\theta$

3. $\int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

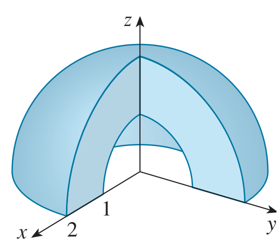
4. $\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

5–6 ■ Set up the triple integral of an arbitrary continuous function $f(x, y, z)$ in cylindrical or spherical coordinates over the solid shown.

5.



6.



7–14 ■ Use cylindrical coordinates.

7. Evaluate $\iiint_E \sqrt{x^2 + y^2} \, dV$, where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

8. Evaluate $\iiint_E (x^3 + xy^2) \, dV$, where E is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$.

9. Evaluate $\iiint_E y \, dV$, where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, above the xy-plane, and below the plane $z = x + 2$.

10. Evaluate $\iiint_E xz\,dV$, where E is bounded by the planes $z = 0$, $z = y$, and the cylinder $x^2 + y^2 = 1$ in the half-space $y \geqslant 0$.

11. Evaluate $\iiint_E x^2\,dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

12. (a) Find the volume of the solid that the cylinder $r = a\cos\theta$ cuts out of the sphere of radius a centered at the origin.
(b) Illustrate the solid of part (a) by graphing the sphere and the cylinder on the same screen.

13. Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$ $(a > 0)$ if S has constant density K.

14. (a) Find the volume of the region E bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.
(b) Find the centroid of the region E in part (a).

15–24 ■ Use spherical coordinates.

15. Evaluate $\iiint_B (x^2 + y^2 + z^2)\,dV$, where B is the unit ball $x^2 + y^2 + z^2 \leqslant 1$.

16. Evaluate $\iiint_H (x^2 + y^2)\,dV$, where H is the hemispherical region that lies above the xy-plane and below the sphere $x^2 + y^2 + z^2 = 1$.

17. Evaluate $\iiint_E z\,dV$, where E lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.

18. Evaluate $\iiint_E xe^{(x^2+y^2+z^2)^2}\,dV$, where E is the solid that lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.

19. Evaluate $\iiint_E \sqrt{x^2 + y^2 + z^2}\,dV$, where E is bounded below by the cone $\phi = \pi/6$ and above by the sphere $\rho = 2$.

20. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

21. (a) Find the volume of the solid that lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 4\cos\phi$.
(b) Find the centroid of the solid in part (a).

22. Let H be a solid hemisphere of radius a whose density at any point is proportional to its distance from the center of the base.
(a) Find the mass of H.
(b) Find the center of mass of H.
(c) Find the moment of inertia of H about its axis.

23. (a) Find the centroid of a solid homogeneous hemisphere of radius a.
(b) Find the moment of inertia of the solid in part (a) about a diameter of its base.

24. Find the mass and center of mass of a solid hemisphere of radius a if the density at any point is proportional to its distance from the base.

25–28 ■ Use cylindrical or spherical coordinates, whichever seems more appropriate.

25. Find the volume and centroid of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

26. Find the volume of the smaller wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\pi/6$.

CAS **27.** Evaluate $\iiint_E z\,dV$, where E lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 2y$. Use either the Table of Integrals (on the back Reference Pages) or a computer algebra system to evaluate the integral.

28. (a) Find the volume enclosed by the torus $\rho = \sin\phi$.
(b) Use a computer to draw the torus.

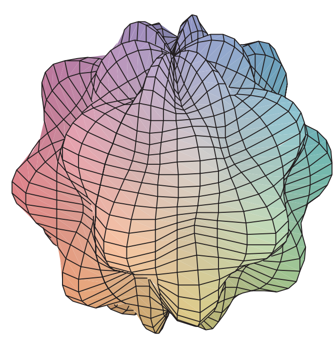
29. Evaluate the integral by changing to cylindrical coordinates:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2} (x^2 + y^2)^{3/2}\,dz\,dy\,dx$$

30. Evaluate the integral by changing to spherical coordinates:

$$\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2)\,dz\,dx\,dy$$

CAS **31.** In the Laboratory Project on page 699 we investigated the family of surfaces $\rho = 1 + \frac{1}{5}\sin m\theta\sin n\phi$ that have been used as models for tumors. The "bumpy sphere" with $m = 6$ and $n = 5$ is shown. Use a computer algebra system to find its volume.



32. Show that

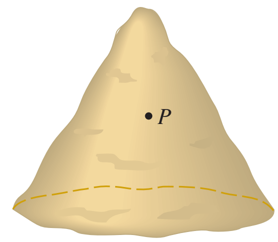
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2 + y^2 + z^2}\,e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz = 2\pi$$

(The improper triple integral is defined as the limit of a triple integral over a solid sphere as the radius of the sphere increases indefinitely.)

33. When studying the formation of mountain ranges, geologists estimate the amount of work required to lift a mountain from sea level. Consider a mountain that is essentially in the shape of a right circular cone. Suppose that the weight density of the material in the vicinity of a point P is $g(P)$ and the height is $h(P)$.

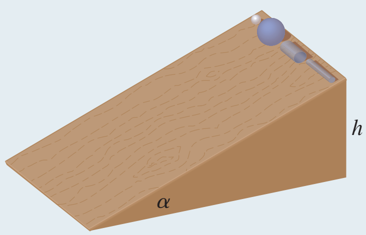
(a) Find a definite integral that represents the total work done in forming the mountain.

(b) Assume that Mount Fuji in Japan is in the shape of a right circular cone with radius 62,000 ft, height 12,400 ft, and density a constant 200 lb/ft^3. How much work was done in forming Mount Fuji if the land was initially at sea level?



Applied Project

Roller Derby



Suppose that a solid ball (a marble), a hollow ball (a squash ball), a solid cylinder (a steel bar), and a hollow cylinder (a lead pipe) roll down a slope. Which of these objects reaches the bottom first? (Make a guess before proceeding.)

To answer this question we consider a ball or cylinder with mass m, radius r, and moment of inertia I (about the axis of rotation). If the vertical drop is h, then the potential energy at the top is mgh. Suppose the object reaches the bottom with velocity v and angular velocity ω, so $v = \omega r$. The kinetic energy at the bottom consists of two parts: $\frac{1}{2}mv^2$ from translation (moving down the slope) and $\frac{1}{2}I\omega^2$ from rotation. If we assume that energy loss from rolling friction is negligible, then conservation of energy gives

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

1. Show that

$$v^2 = \frac{2gh}{1 + I^*} \qquad \text{where } I^* = \frac{I}{mr^2}$$

2. If $y(t)$ is the vertical distance traveled at time t, then the same reasoning as used in Problem 1 shows that $v^2 = 2gy/(1 + I^*)$ at any time t. Use this result to show that y satisfies the differential equation

$$\frac{dy}{dt} = \sqrt{\frac{2g}{1 + I^*}}\,(\sin\alpha)\sqrt{y}$$

where α is the angle of inclination of the plane.

3. By solving the differential equation in Problem 2, show that the total travel time is

$$T = \sqrt{\frac{2h(1 + I^*)}{g\sin^2\alpha}}$$

This shows that the object with the smallest value of I^* wins the race.

4. Show that $I^* = \frac{1}{2}$ for a solid cylinder and $I^* = 1$ for a hollow cylinder.

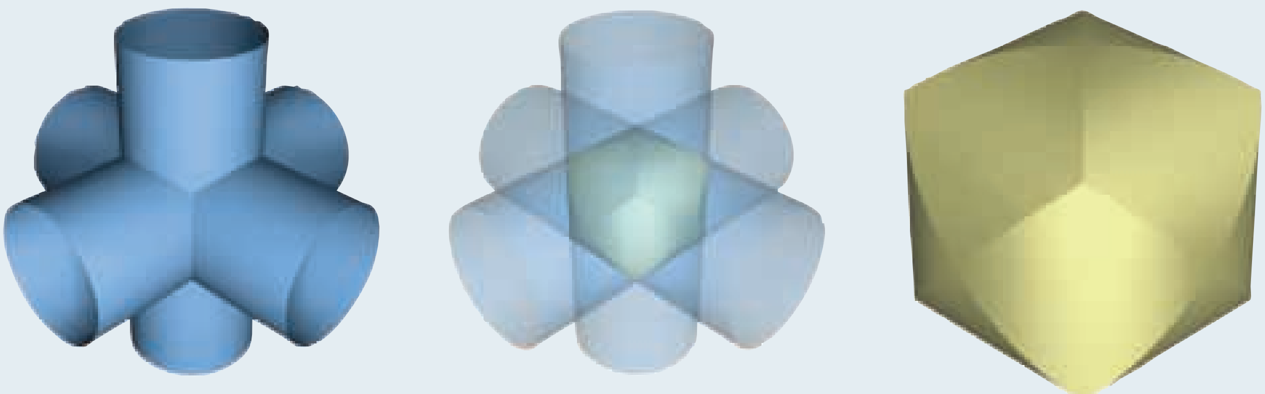
5. Calculate I^* for a partly hollow ball with inner radius a and outer radius r. Express your answer in terms of $b = a/r$. What happens as $a \to 0$ and as $a \to r$?

6. Show that $I^* = \frac{2}{5}$ for a solid ball and $I^* = \frac{2}{3}$ for a hollow ball. Thus, the objects finish in the following order: solid ball, solid cylinder, hollow ball, hollow cylinder.

Discovery Project

The Intersection of Three Cylinders

The figure shows the solid enclosed by three circular cylinders with the same diameter that intersect at right angles. In this project we compute its volume and determine how its shape changes if the cylinders have different diameters.



1. Sketch carefully the solid enclosed by the three cylinders $x^2 + y^2 = 1$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$. Indicate the positions of the coordinate axes and label the faces with the equations of the corresponding cylinders.
2. Find the volume of the solid in Problem 1.
3. CAS Use a computer algebra system to draw the edges of the solid.
4. What happens to the solid in Problem 1 if the radius of the first cylinder is different from 1? Illustrate with a hand-drawn sketch or a computer graph.
5. If the first cylinder is $x^2 + y^2 = a^2$, where $a < 1$, set up, but do not evaluate, a double integral for the volume of the solid. What if $a > 1$?

12.9 Change of Variables in Multiple Integrals

In one-dimensional calculus we often use a change of variable (a substitution) to simplify an integral. By reversing the roles of x and u, we can write the Substitution Rule (5.5.5) as

$$\boxed{1} \qquad \int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$. Another way of writing Formula 1 is as follows:

$$\boxed{2} \qquad \int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

A change of variables can also be useful in double integrals. We have already seen one example of this: conversion to polar coordinates. The new variables r and θ are related to the old variables x and y by the equations

$$x = r\cos\theta \qquad y = r\sin\theta$$

and the change of variables formula (12.4.2) can be written as

$$\iint_R f(x, y)\, dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

More generally, we consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

3 $$x = g(u, v) \qquad y = h(u, v)$$

or, as we sometimes write,

$$x = x(u, v) \qquad y = y(u, v)$$

We usually assume that T is a **C^1 transformation**, which means that g and h have continuous first-order partial derivatives.

A transformation T is really just a function whose domain and range are both subsets of $\mathbb{R}^2$. If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1). If no two points have the same image, T is called **one-to-one**. Figure 1 shows the effect of a transformation T on a region S in the uv-plane. T transforms S into a region R in the xy-plane called the **image of S**, consisting of the images of all points in S.

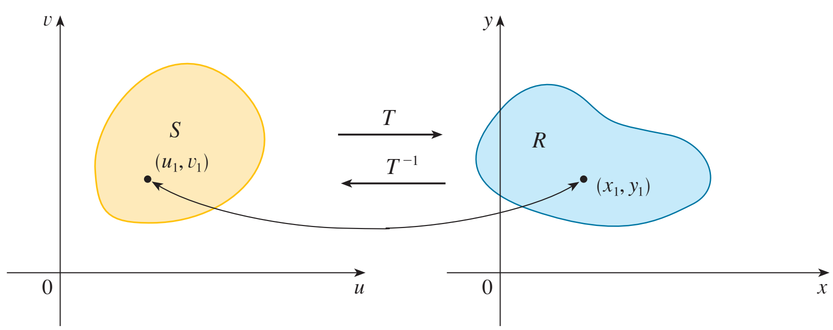


FIGURE 1

If T is a one-to-one transformation, then it has an **inverse transformation** T^{-1} from the xy-plane to the uv-plane and it may be possible to solve Equations 3 for u and v in terms of x and y:

$$u = G(x, y) \qquad v = H(x, y)$$

EXAMPLE 1 A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1\}$.

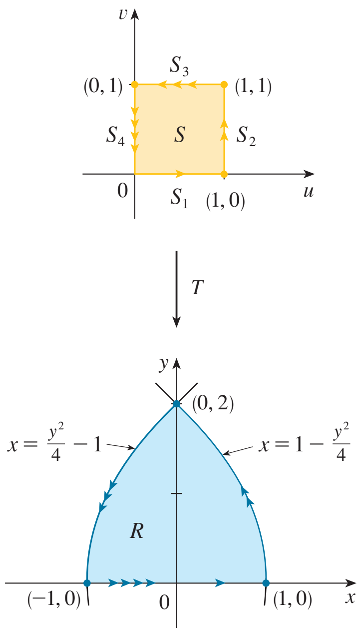


FIGURE 2

SOLUTION The transformation maps the boundary of S into the boundary of the image. So we begin by finding the images of the sides of S. The first side, S_1, is given by $v = 0$ $(0 \leqslant u \leqslant 1)$. (See Figure 2.) From the given equations we have $x = u^2$, $y = 0$, and so $0 \leqslant x \leqslant 1$. Thus, S_1 is mapped into the line segment from $(0, 0)$ to $(1, 0)$ in the xy-plane. The second side, S_2, is $u = 1$ $(0 \leqslant v \leqslant 1)$ and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2 \qquad y = 2v$$

Eliminating v, we obtain

4 $$x = 1 - \frac{y^2}{4} \qquad 0 \leqslant x \leqslant 1$$

which is part of a parabola. Similarly, S_3 is given by $v = 1$ $(0 \leqslant u \leqslant 1)$, whose image is the parabolic arc

5 $$x = \frac{y^2}{4} - 1 \qquad -1 \leqslant x \leqslant 0$$

Finally, S_4 is given by $u = 0$ $(0 \leqslant v \leqslant 1)$ whose image is $x = -v^2$, $y = 0$, that is, $-1 \leqslant x \leqslant 0$. (Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.) The image of S is the region R (shown in Figure 2) bounded by the x-axis and the parabolas given by Equations 4 and 5. ■

Now let's see how a change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv. (See Figure 3.)

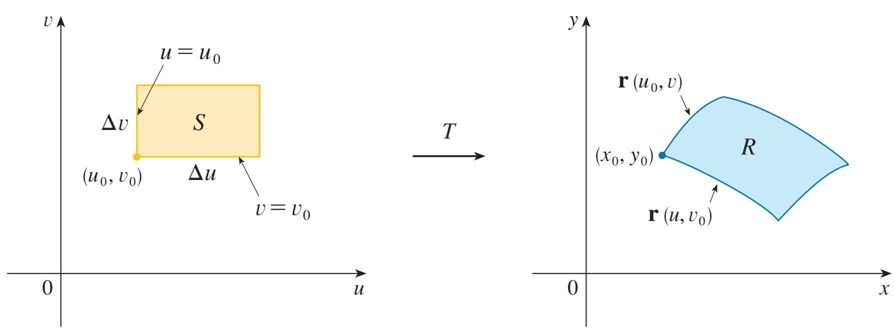


FIGURE 3

The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$. The vector

$$\mathbf{r}(u, v) = g(u, v)\,\mathbf{i} + h(u, v)\,\mathbf{j}$$

is the position vector of the image of the point (u, v). The equation of the lower side of S is $v = v_0$, whose image curve is given by the vector function $\mathbf{r}(u, v_0)$. The tangent vector at (x_0, y_0) to this image curve is

$$\mathbf{r}_u = g_u(u_0, v_0)\,\mathbf{i} + h_u(u_0, v_0)\,\mathbf{j} = \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j}$$

Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S (namely, $u = u_0$) is

$$\mathbf{r}_v = g_v(u_0, v_0)\mathbf{i} + h_v(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \qquad \mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$$

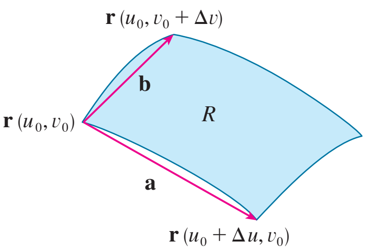


FIGURE 4

shown in Figure 4. But

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

and so $\qquad \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u\, \mathbf{r}_u$

Similarly $\qquad \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v\, \mathbf{r}_v$

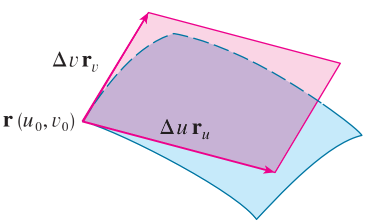


FIGURE 5

This means that we can approximate R by a parallelogram determined by the vectors $\Delta u\, \mathbf{r}_u$ and $\Delta v\, \mathbf{r}_v$. (See Figure 5.) Therefore, we can approximate the area of R by the area of this parallelogram, which, from Section 9.4, is

6 $$|(\Delta u\, \mathbf{r}_u) \times (\Delta v\, \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v$$

Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

The determinant that arises in this calculation is called the *Jacobian* of the transformation and is given a special notation.

▲ The Jacobian is named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851). Although the French mathematician Cauchy first used these special determinants involving partial derivatives, Jacobi developed them into a method for evaluating multiple integrals.

7 Definition The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

With this notation we can use Equation 6 to give an approximation to the area ΔA of R:

8 $$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u\, \Delta v$$

where the Jacobian is evaluated at (u_0, v_0).

Next we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}. (See Figure 6.) Applying the approximation (8) to each R_{ij}, we approximate the double integral of f over R as follows:

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u\, \Delta v$$

where the Jacobian is evaluated at (u_i, v_j). Notice that this double sum is a Riemann sum for the integral

$$\iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

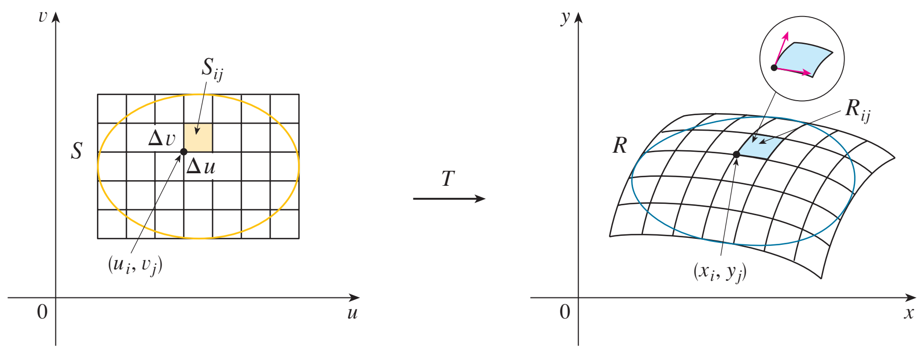


FIGURE 6

The foregoing argument suggests that the following theorem is true. (A full proof is given in books on advanced calculus.)

9 Change of Variables in a Double Integral Suppose that T is a one-to-one C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Theorem 9 says that we change from an integral in x and y to an integral in u and v by expressing x and y in terms of u and v and writing

$$dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Notice the similarity between Theorem 9 and the one-dimensional formula in Equation 2. Instead of the derivative dx/du, we have the absolute value of the Jacobian, that is, $|\partial(x, y)/\partial(u, v)|$.

As a first illustration of Theorem 9, we show that the formula for integration in polar coordinates is just a special case. Here the transformation T from the $r\theta$-plane to the xy-plane is given by

$$x = g(r, \theta) = r \cos \theta \qquad y = h(r, \theta) = r \sin \theta$$

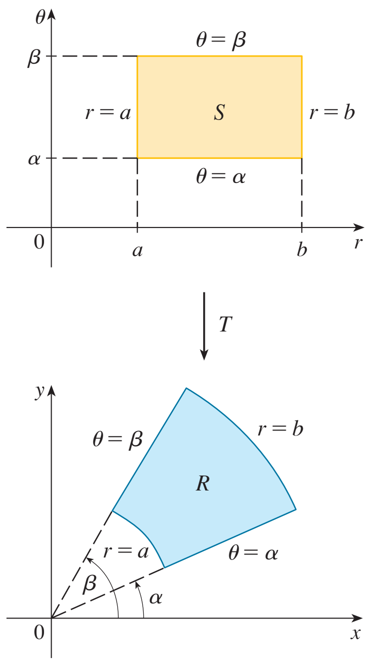


FIGURE 7
The polar coordinate transformation

and the geometry of the transformation is shown in Figure 7. T maps an ordinary rectangle in the $r\theta$-plane to a polar rectangle in the xy-plane. The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0$$

Thus, Theorem 9 gives

$$\iint_R f(x, y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| dr\,d\theta$$
$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

which is the same as Formula 12.4.2.

EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$.

SOLUTION The region R is pictured in Figure 2. In Example 1 we discovered that $T(S) = R$, where S is the square $[0, 1] \times [0, 1]$. Indeed, the reason for making the change of variables to evaluate the integral is that S is a much simpler region than R. First we need to compute the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

Therefore, by Theorem 9,

$$\iint_R y\,dA = \iint_S 2uv\left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA = \int_0^1\int_0^1 (2uv)4(u^2 + v^2)\,du\,dv$$
$$= 8\int_0^1\int_0^1 (u^3v + uv^3)\,du\,dv = 8\int_0^1 \left[\tfrac{1}{4}u^4v + \tfrac{1}{2}u^2v^3\right]_{u=0}^{u=1} dv$$
$$= \int_0^1 (2v + 4v^3)\,dv = \left[v^2 + v^4\right]_0^1 = 2$$

■

NOTE • Example 2 was not a very difficult problem to solve because we were given a suitable change of variables. If we are not supplied with a transformation, then the first step is to think of an appropriate change of variables. If $f(x, y)$ is difficult to integrate, then the form of $f(x, y)$ may suggest a transformation. If the region of integration R is awkward, then the transformation should be chosen so that the corresponding region S in the uv-plane has a convenient description.

EXAMPLE 3 Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.

SOLUTION Since it isn't easy to integrate $e^{(x+y)/(x-y)}$, we make a change of variables suggested by the form of this function:

10 $$u = x + y \qquad v = x - y$$

These equations define a transformation T^{-1} from the xy-plane to the uv-plane. Theorem 9 talks about a transformation T from the uv-plane to the xy-plane. It is obtained by solving Equations 10 for x and y:

11 $$x = \tfrac{1}{2}(u + v) \qquad y = \tfrac{1}{2}(u - v)$$

The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

To find the region S in the uv-plane corresponding to R, we note that the sides of R lie on the lines

$$y = 0 \qquad x - y = 2 \qquad x = 0 \qquad x - y = 1$$

and, from either Equations 10 or Equations 11, the image lines in the uv-plane are

$$u = v \qquad v = 2 \qquad u = -v \qquad v = 1$$

Thus, the region S is the trapezoidal region with vertices $(1, 1)$, $(2, 2)$, $(-2, 2)$, and $(-1, 1)$ shown in Figure 8. Since

$$S = \{(u, v) \mid 1 \leqslant v \leqslant 2,\ -v \leqslant u \leqslant v\}$$

Theorem 9 gives

$$\iint_R e^{(x+y)/(x-y)}\, dA = \iint_S e^{u/v} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

$$= \int_1^2 \int_{-v}^{v} e^{u/v} \left(\tfrac{1}{2}\right) du\, dv = \tfrac{1}{2} \int_1^2 \Big[v e^{u/v} \Big]_{u=-v}^{u=v} dv$$

$$= \tfrac{1}{2} \int_1^2 (e - e^{-1}) v\, dv = \tfrac{3}{4}(e - e^{-1})$$

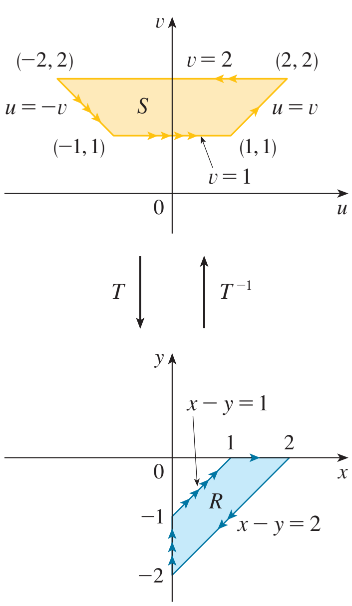


FIGURE 8

Triple Integrals

There is a similar change of variables formula for triple integrals. Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

$$x = g(u, v, w) \qquad y = h(u, v, w) \qquad z = k(u, v, w)$$

The **Jacobian** of T is the following 3×3 determinant:

12
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Under hypotheses similar to those in Theorem 9, we have the following formula for triple integrals:

13
$$\iiint_R f(x, y, z)\, dV$$
$$= \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

EXAMPLE 4 Use Formula 13 to derive the formula for triple integration in spherical coordinates.

SOLUTION Here the change of variables is given by

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

We compute the Jacobian as follows:

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \end{vmatrix} - \rho \sin\phi \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta \end{vmatrix}$$

$$= \cos\phi\, (-\rho^2 \sin\phi \cos\phi \sin^2\theta - \rho^2 \sin\phi \cos\phi \cos^2\theta)$$
$$- \rho \sin\phi\, (\rho \sin^2\phi \cos^2\theta + \rho \sin^2\phi \sin^2\theta)$$
$$= -\rho^2 \sin\phi \cos^2\phi - \rho^2 \sin\phi \sin^2\phi = -\rho^2 \sin\phi$$

Since $0 \leqslant \phi \leqslant \pi$, we have $\sin\phi \geqslant 0$. Therefore

$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| = \left| -\rho^2 \sin\phi \right| = \rho^2 \sin\phi$$

and Formula 13 gives

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

which is equivalent to Formula 12.8.4. ■

12.9 Exercises

1–6 ■ Find the Jacobian of the transformation.

1. $x = u + 4v, \quad y = 3u - 2v$

2. $x = u^2 - v^2, \quad y = u^2 + v^2$

3. $x = \dfrac{u}{u+v}, \quad y = \dfrac{v}{u-v}$

4. $x = \alpha \sin \beta, \quad y = \alpha \cos \beta$

5. $x = uv, \quad y = vw, \quad z = uw$

6. $x = e^{u-v}, \quad y = e^{u+v}, \quad z = e^{u+v+w}$

7–10 ■ Find the image of the set S under the given transformation.

7. $S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\}$; $x = 2u + 3v,\ y = u - v$

8. S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $\quad x = v,\ y = u(1 + v^2)$

9. S is the triangular region with vertices (0, 0), (1, 1), (0, 1); $x = u^2,\ y = v$

10. S is the disk given by $u^2 + v^2 \leqslant 1$; $\quad x = au,\ y = bv$

11–16 ■ Use the given transformation to evaluate the integral.

11. $\iint_R (3x + 4y)\, dA$, where R is the region bounded by the lines $y = x$, $y = x - 2$, $y = -2x$, and $y = 3 - 2x$; $x = \frac{1}{3}(u + v),\ y = \frac{1}{3}(v - 2u)$

12. $\iint_R (x + y)\, dA$, where R is the square with vertices (0, 0), (2, 3), (5, 1), and (3, −2); $\quad x = 2u + 3v,\ y = 3u - 2v$

13. $\iint_R x^2\, dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$; $\quad x = 2u,\ y = 3v$

14. $\iint_R (x^2 - xy + y^2)\, dA$, where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$; $x = \sqrt{2}u - \sqrt{2/3}v,\ y = \sqrt{2}u + \sqrt{2/3}v$

15. $\iint_R xy\, dA$, where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$, $xy = 3$; $\quad x = u/v,\ y = v$

16. $\iint_R y^2\, dA$, where R is the region bounded by the curves $xy = 1$, $xy = 2$, $xy^2 = 1$, $xy^2 = 2$; $\quad u = xy,\ v = xy^2$. Illustrate by using a graphing calculator or computer to draw R.

17. (a) Evaluate $\iiint_E dV$, where E is the solid enclosed by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Use the transformation $x = au,\ y = bv,\ z = cw$.

(b) Earth is not a perfect sphere; rotation has resulted in flattening at the poles. So the shape can be approximated by an ellipsoid with $a = b = 6378$ km and $c = 6356$ km. Use part (a) to estimate the volume of Earth.

18. Evaluate $\iiint_E x^2 y\, dV$, where E is the solid of Exercise 17(a).

19–23 ■ Evaluate the integral by making an appropriate change of variables.

19. $\iint_R xy\, dA$, where R is the region bounded by the lines $2x - y = 1$, $2x - y = -3$, $3x + y = 1$, and $3x + y = -2$

20. $\displaystyle\iint_R \frac{x + 2y}{\cos(x - y)}\, dA$, where R is the parallelogram bounded by the lines $y = x$, $y = x - 1$, $x + 2y = 0$, and $x + 2y = 2$

21. $\displaystyle\iint_R \cos\left(\frac{y - x}{y + x}\right) dA$, where R is the trapezoidal region with vertices (1, 0), (2, 0), (0, 2), and (0, 1)

22. $\iint_R \sin(9x^2 + 4y^2)\, dA$, where R is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$

23. $\iint_R e^{x+y}\, dA$, where R is given by the inequality $|x| + |y| \leqslant 1$

24. Let f be continuous on [0, 1] and let R be the triangular region with vertices (0, 0), (1, 0), and (0, 1). Show that

$$\iint_R f(x + y)\, dA = \int_0^1 u f(u)\, du$$

12 Review

CONCEPT CHECK

1. Suppose f is a continuous function defined on a rectangle $R = [a, b] \times [c, d]$.
(a) Write an expression for a double Riemann sum of f. If $f(x, y) \geq 0$, what does the sum represent?
(b) Write the definition of $\iint_R f(x, y)\, dA$ as a limit.
(c) What is the geometric interpretation of $\iint_R f(x, y)\, dA$ if $f(x, y) \geq 0$? What if f takes on both positive and negative values?
(d) How do you evaluate $\iint_R f(x, y)\, dA$?
(e) What does the Midpoint Rule for double integrals say?
(f) Write an expression for the average value of f.

2. (a) How do you define $\iint_D f(x, y)\, dA$ if D is a bounded region that is not a rectangle?
(b) What is a type I region? How do you evaluate $\iint_D f(x, y)\, dA$ if D is a type I region?
(c) What is a type II region? How do you evaluate $\iint_D f(x, y)\, dA$ if D is a type II region?
(d) What properties do double integrals have?

3. How do you change from rectangular coordinates to polar coordinates in a double integral? Why would you want to do it?

4. If a lamina occupies a plane region D and has density function $\rho(x, y)$, write expressions for each of the following in terms of double integrals.
(a) The mass
(b) The moments about the axes
(c) The center of mass
(d) The moments of inertia about the axes and the origin

5. Let f be a joint density function of a pair of continuous random variables X and Y.
(a) Write a double integral for the probability that X lies between a and b and Y lies between c and d.
(b) What properties does f possess?
(c) What are the expected values of X and Y?

6. Write an expression for the area of a surface S for each of the following cases.
(a) S is a parametric surface given by a vector function $\mathbf{r}(u, v)$, $(u, v) \in D$.
(b) S has the equation $z = f(x, y)$, $(x, y) \in D$.
(c) S is the surface of revolution obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis.

7. (a) Write the definition of the triple integral of f over a rectangular box B.
(b) How do you evaluate $\iiint_B f(x, y, z)\, dV$?
(c) How do you define $\iiint_E f(x, y, z)\, dV$ if E is a bounded solid region that is not a box?
(d) What is a type 1 solid region? How do you evaluate $\iiint_E f(x, y, z)\, dV$ if E is such a region?
(e) What is a type 2 solid region? How do you evaluate $\iiint_E f(x, y, z)\, dV$ if E is such a region?
(f) What is a type 3 solid region? How do you evaluate $\iiint_E f(x, y, z)\, dV$ if E is such a region?

8. Suppose a solid object occupies the region E and has density function $\rho(x, y, z)$. Write expressions for each of the following.
(a) The mass
(b) The moments about the coordinate planes
(c) The coordinates of the center of mass
(d) The moments of inertia about the axes

9. (a) How do you change from rectangular coordinates to cylindrical coordinates in a triple integral?
(b) How do you change from rectangular coordinates to spherical coordinates in a triple integral?
(c) In what situations would you change to cylindrical or spherical coordinates?

10. (a) If a transformation T is given by $x = g(u, v)$, $y = h(u, v)$, what is the Jacobian of T?
(b) How do you change variables in a double integral?
(c) How do you change variables in a triple integral?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\int_{-1}^{2} \int_{0}^{6} x^2 \sin(x - y)\, dx\, dy = \int_{0}^{6} \int_{-1}^{2} x^2 \sin(x - y)\, dy\, dx$

2. $\int_{-1}^{1} \int_{0}^{1} e^{x^2+y^2} \sin y\, dx\, dy = 0$

3. If D is the disk given by $x^2 + y^2 \leqslant 4$, then

$$\iint_D \sqrt{4 - x^2 - y^2}\, dA = \frac{16\pi}{3}$$

4. $\int_{1}^{4} \int_{0}^{1} (x^2 + \sqrt{y}) \sin(x^2 y^2)\, dx\, dy \leqslant 9$

5. The integral

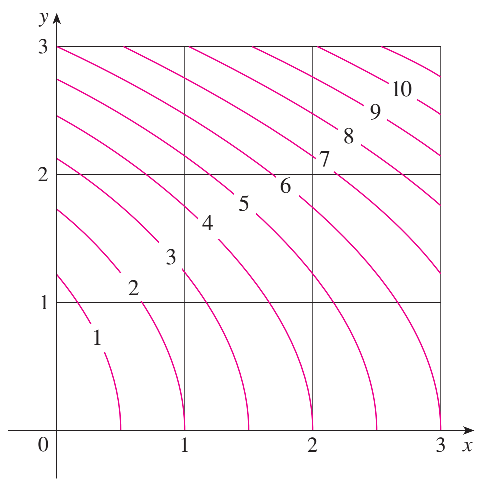
$$\int_{0}^{2\pi} \int_{0}^{2} \int_{r}^{2} dz\, dr\, d\theta$$

represents the volume enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$.

6. The integral $\iiint_E kr^3\, dz\, dr\, d\theta$ represents the moment of inertia about the z-axis of a solid E with constant density k.

EXERCISES

1. A contour map is shown for a function f on the square $R = [0, 3] \times [0, 3]$. Use a Riemann sum with nine terms to estimate the value of $\iint_R f(x, y)\, dA$. Take the sample points to be the upper right corners of the squares.



2. Use the Midpoint Rule to estimate the integral in Exercise 1.

3–8 ■ Calculate the iterated integral.

3. $\int_{1}^{2} \int_{0}^{2} (y + 2xe^y)\, dx\, dy$

4. $\int_{0}^{1} \int_{0}^{1} ye^{xy}\, dx\, dy$

5. $\int_{0}^{1} \int_{0}^{x} \cos(x^2)\, dy\, dx$

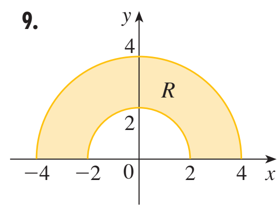
6. $\int_{0}^{1} \int_{x}^{e^x} 3xy^2\, dy\, dx$

7. $\int_{0}^{\pi} \int_{0}^{1} \int_{0}^{\sqrt{1-y^2}} y \sin x\, dz\, dy\, dx$

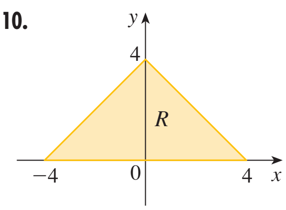
8. $\int_{0}^{1} \int_{\sqrt{y}}^{1} \int_{0}^{y} xy\, dz\, dx\, dy$

9–10 ■ Write $\iint_R f(x, y)\, dA$ as an iterated integral, where R is the region shown and f is an arbitrary continuous function on R.

9.



10.



11. Describe the region whose area is given by the integral

$$\int_{0}^{\pi} \int_{1}^{1+\sin\theta} r\, dr\, d\theta$$

12. Describe the solid whose volume is given by the integral

$$\int_{0}^{2\pi} \int_{0}^{\pi/6} \int_{1}^{3} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

and evaluate the integral.

13–14 ■ Calculate the iterated integral by first reversing the order of integration.

13. $\int_{0}^{1} \int_{x}^{1} e^{x/y}\, dy\, dx$

14. $\int_{0}^{1} \int_{y^2}^{1} y \sin(x^2)\, dx\, dy$

15–28 ■ Calculate the value of the multiple integral.

15. $\iint_R \frac{1}{(x - y)^2}\, dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ 2 \leqslant y \leqslant 4\}$

16. $\iint_D x^3\, dA$, where $D = \{(x, y) \mid -1 \leqslant x \leqslant 1,\ x^2 - 1 \leqslant y \leqslant x + 1\}$

17. $\iint_D xy\,dA$, where D is bounded by $y^2 = x^3$ and $y = x$

18. $\iint_D xe^y\,dA$, where D is bounded by $y = 0$, $y = x^2$, $x = 1$

19. $\iint_D (xy + 2x + 3y)\,dA$, where D is the region in the first quadrant bounded by $x = 1 - y^2$, $y = 0$, $x = 0$

20. $\iint_D y\,dA$, where D is the region in the first quadrant that lies above the hyperbola $xy = 1$ and the line $y = x$ and below the line $y = 2$

21. $\iint_D (x^2 + y^2)^{3/2}\,dA$, where D is the region in the first quadrant bounded by the lines $y = 0$ and $y = \sqrt{3}x$ and the circle $x^2 + y^2 = 9$

22. $\iint_D \sqrt{x^2 + y^2}\,dA$, where D is the closed disk with radius 1 and center $(0, 1)$

23. $\iiint_E x^2 z\,dV$, where $E = \{(x, y, z) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 2x,\ 0 \leqslant z \leqslant x\}$

24. $\iiint_T y\,dV$, where T is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + z = 2$

25. $\iiint_E y^2 z^2\,dV$, where E is bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$

26. $\iiint_E z\,dV$, where E is bounded by the planes $y = 0$, $z = 0$, $x + y = 2$ and the cylinder $y^2 + z^2 = 1$ in the first octant

27. $\iiint_E yz\,dV$, where E lies above the plane $z = 0$, below the plane $z = y$, and inside the cylinder $x^2 + y^2 = 4$

28. $\iiint_H z^3\sqrt{x^2 + y^2 + z^2}\,dV$, where H is the solid hemisphere with center the origin, radius 1, that lies above the xy-plane

29–34 ■ Find the volume of the given solid.

29. Under the paraboloid $z = x^2 + 4y^2$ and above the rectangle $R = [0, 2] \times [1, 4]$

30. Under the surface $z = x^2 y$ and above the triangle in the xy-plane with vertices $(1, 0)$, $(2, 1)$, and $(4, 0)$

31. The solid tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 1)$, $(0, 2, 0)$, and $(2, 2, 0)$

32. Bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 3$

33. One of the wedges cut from the cylinder $x^2 + 9y^2 = a^2$ by the planes $z = 0$ and $z = mx$

34. Above the paraboloid $z = x^2 + y^2$ and below the half-cone $z = \sqrt{x^2 + y^2}$

35. Consider a lamina that occupies the region D bounded by the parabola $x = 1 - y^2$ and the coordinate axes in the first quadrant with density function $\rho(x, y) = y$.
(a) Find the mass of the lamina.
(b) Find the center of mass.
(c) Find the moments of inertia about the x- and y-axes.

36. A lamina occupies the part of the disk $x^2 + y^2 \leqslant a^2$ that lies in the first quadrant.
(a) Find the centroid of the lamina.
(b) Find the center of mass of the lamina if the density function is $\rho(x, y) = xy^2$.

37. (a) Find the centroid of a right circular cone with height h and base radius a. (Place the cone so that its base is in the xy-plane with center the origin and its axis along the positive z-axis.)
(b) Find the moment of inertia of the cone about its axis (the z-axis).

38. (a) Set up, but don't evaluate, an integral for the surface area of the parametric surface given by the vector function $\mathbf{r}(u, v) = v^2\,\mathbf{i} - uv\,\mathbf{j} + u^2\,\mathbf{k}$, $0 \leqslant u \leqslant 3$, $-3 \leqslant v \leqslant 3$.
CAS (b) Use a computer algebra system to approximate the surface area correct to four significant digits.

39. Find the area of the part of the surface $z = x^2 + y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$.

CAS **40.** Graph the surface $z = x \sin y$, $-3 \leqslant x \leqslant 3$, $-\pi \leqslant y \leqslant \pi$, and find its surface area correct to four decimal places.

41. Use polar coordinates to evaluate

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{1 + x^2 + y^2}\,dx\,dy$$

42. Use spherical coordinates to evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2 + y^2 + z^2)^2\,dz\,dy\,dx$$

43. If D is the region bounded by the curves $y = 1 - x^2$ and $y = e^x$, find the approximate value of the integral $\iint_D y^2\,dA$. (Use a graphing device to estimate the points of intersection of the curves.)

CAS **44.** Find the center of mass of the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$ and density function $\rho(x, y, z) = x^2 + y^2 + z^2$.

45. The joint density function for random variables X and Y is

$$f(x, y) = \begin{cases} C(x + y) & \text{if } 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant C.
(b) Find $P(X \leqslant 2, Y \geqslant 1)$.
(c) Find $P(X + Y \leqslant 1)$.

46. A lamp has three bulbs, each of a type with average lifetime 800 hours. If we model the probability of failure of the bulbs by an exponential density function with mean 800, find the probability that all three bulbs fail within a total of 1000 hours.

47. Rewrite the integral

$$\int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} f(x, y, z)\, dz\, dy\, dx$$

as an iterated integral in the order $dx\, dy\, dz$.

48. Give five other iterated integrals that are equal to

$$\int_{0}^{2} \int_{0}^{y^3} \int_{0}^{y^2} f(x, y, z)\, dz\, dx\, dy$$

49. Use the transformation $u = x - y$, $v = x + y$ to evaluate $\iint_R (x - y)/(x + y)\, dA$, where R is the square with vertices $(0, 2)$, $(1, 1)$, $(2, 2)$, and $(1, 3)$.

50. Use the transformation $x = u^2$, $y = v^2$, $z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.

51. Use the change of variables formula and an appropriate transformation to evaluate $\iint_R xy\, dA$, where R is the square with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$, and $(1, -1)$.

52. (a) Evaluate $\displaystyle\iint_D \frac{1}{(x^2 + y^2)^{n/2}}\, dA$, where n is an integer and D is the region bounded by the circles with center the origin and radii r and R, $0 < r < R$.

(b) For what values of n does the integral in part (a) have a limit as $r \to 0^+$?

(c) Find $\displaystyle\iiint_E \frac{1}{(x^2 + y^2 + z^2)^{n/2}}\, dV$, where E is the region bounded by the spheres with center the origin and radii r and R, $0 < r < R$.

(d) For what values of n does the integral in part (c) have a limit as $r \to 0^+$?

Focus on Problem Solving

1. If $[\![x]\!]$ denotes the greatest integer in x, evaluate the integral

$$\iint_R [\![x + y]\!]\, dA$$

where $R = \{(x, y) \mid 1 \leqslant x \leqslant 3,\ 2 \leqslant y \leqslant 5\}$.

2. Evaluate the integral

$$\int_0^1 \int_0^1 e^{\max\{x^2, y^2\}}\, dy\, dx$$

where $\max\{x^2, y^2\}$ means the larger of the numbers x^2 and y^2.

3. Find the average value of the function $f(x) = \int_x^1 \cos(t^2)\, dt$ on the interval $[0, 1]$.

4. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are constant vectors, $\mathbf{r}$ is the position vector $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and E is given by the inequalities $0 \leqslant \mathbf{a} \cdot \mathbf{r} \leqslant \alpha,\ 0 \leqslant \mathbf{b} \cdot \mathbf{r} \leqslant \beta,\ 0 \leqslant \mathbf{c} \cdot \mathbf{r} \leqslant \gamma$, show that

$$\iiint_E (\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})(\mathbf{c} \cdot \mathbf{r})\, dV = \frac{(\alpha\beta\gamma)^2}{8|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}$$

5. The double integral $\int_0^1 \int_0^1 \frac{1}{1 - xy}\, dx\, dy$ is an improper integral and could be defined as the limit of double integrals over the rectangle $[0, t] \times [0, t]$ as $t \to 1^-$. But if we expand the integrand as a geometric series, we can express the integral as the sum of an infinite series. Show that

$$\int_0^1 \int_0^1 \frac{1}{1 - xy}\, dx\, dy = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

6. Leonhard Euler was able to find the exact sum of the series in Problem 5. In 1736 he proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

In this problem we ask you to prove this fact by evaluating the double integral in Problem 5. Start by making the change of variables

$$x = \frac{u - v}{\sqrt{2}} \qquad y = \frac{u + v}{\sqrt{2}}$$

This gives a rotation about the origin through the angle $\pi/4$. You will need to sketch the corresponding region in the uv-plane.

[*Hint:* If, in evaluating the integral, you encounter either of the expressions $(1 - \sin\theta)/\cos\theta$ or $(\cos\theta)/(1 + \sin\theta)$, you might like to use the identity $\cos\theta = \sin((\pi/2) - \theta)$ and the corresponding identity for $\sin\theta$.]

7. (a) Show that

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1 - xyz}\, dx\, dy\, dz = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

(Nobody has ever been able to find the exact value of the sum of this series.)

(b) Show that

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1 + xyz}\, dx\, dy\, dz = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$$

Use this equation to evaluate the triple integral correct to two decimal places.

8. Show that

$$\int_0^{\infty} \frac{\arctan \pi x - \arctan x}{x}\, dx = \frac{\pi}{2} \ln \pi$$

by first expressing the integral as an iterated integral.

9. If f is continuous, show that

$$\int_0^x \int_0^y \int_0^z f(t)\, dt\, dz\, dy = \tfrac{1}{2} \int_0^x (x - t)^2 f(t)\, dt$$

10. (a) A lamina has constant density ρ and takes the shape of a disk with center the origin and radius R. Use Newton's Law of Gravitation (see page 728) to show that the magnitude of the force of attraction that the lamina exerts on a body with mass m located at the point $(0, 0, d)$ on the positive z-axis is

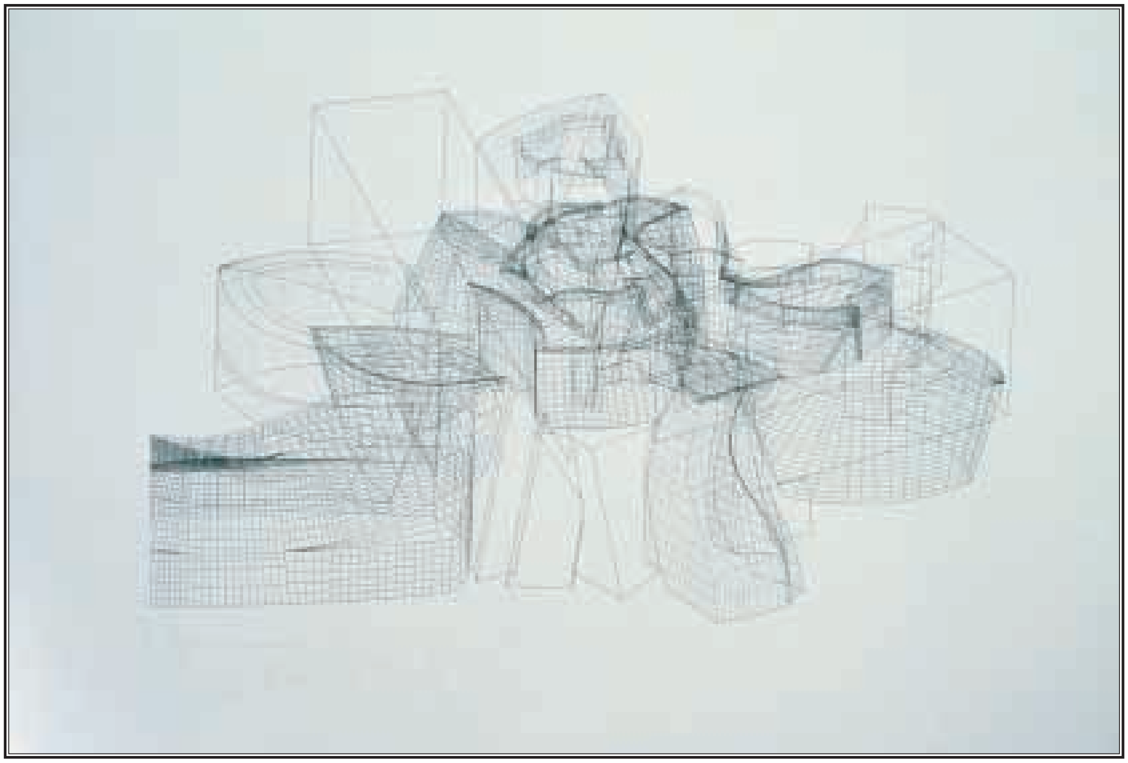
$$F = 2\pi G m \rho d \left(\frac{1}{d} - \frac{1}{\sqrt{R^2 + d^2}} \right)$$

[*Hint:* Divide the disk as in Figure 4 in Section 12.4 and first compute the vertical component of the force exerted by the polar subrectangle R_{ij}.]

(b) Show that the magnitude of the force of attraction of a lamina with density ρ that occupies an entire plane on an object with mass m located at a distance d from the plane is

$$F = 2\pi G m \rho$$

Notice that this expression does not depend on d.

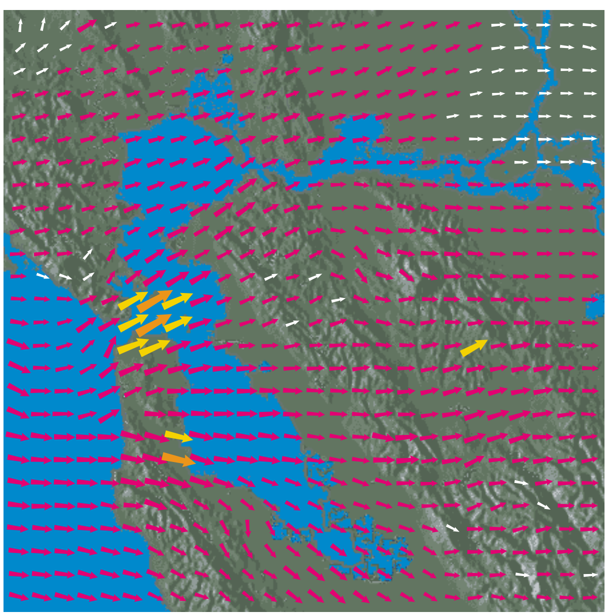


13 Vector Calculus

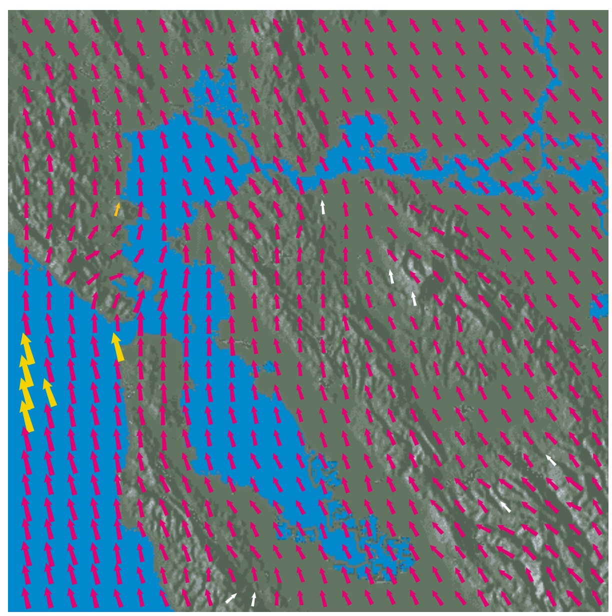
In this chapter we study the calculus of vector fields. (These are functions that assign vectors to points in space.) In particular we define line integrals (which can be used to find the work done by a force field in moving an object along a curve). Then we define surface integrals (which can be used to find the rate of fluid flow across a surface). The connections between these new types of integrals and the single, double, and triple integrals that we have already met are given by the higher-dimensional versions of the Fundamental Theorem of Calculus: Green's Theorem, Stokes' Theorem, and the Divergence Theorem.

13.1 Vector Fields

The vectors in Figure 1 are air velocity vectors that indicate the wind speed and direction at points 10 m above the surface elevation in the San Francisco Bay area. We see at a glance from the largest arrows in part (a) that the greatest wind speeds at that time occurred as the winds entered the bay across the Golden Gate Bridge. Part (b) shows the very different wind pattern at a later date. Associated with every point in the air we can imagine a wind velocity vector. This is an example of a *velocity vector field.*



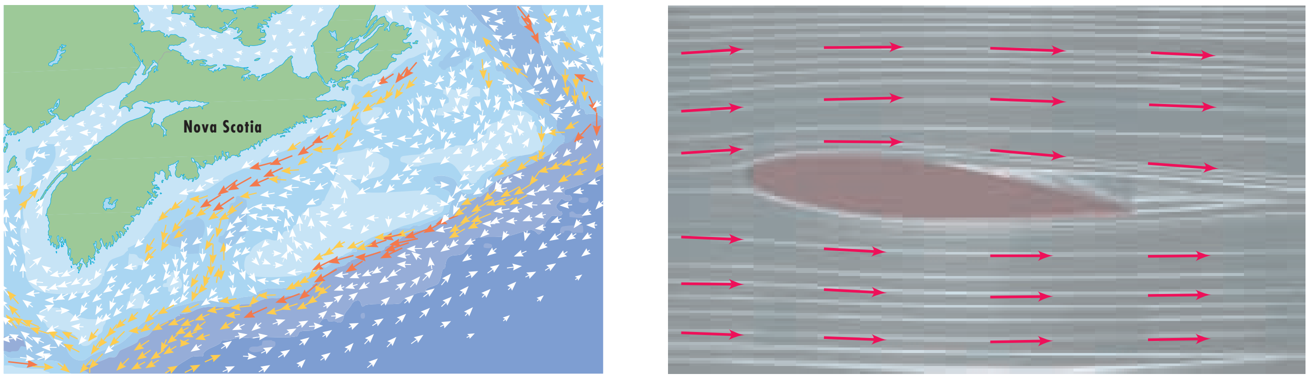
(a) 5:00 P.M., August 6, 1997



(b) 2:00 P.M., March 7, 2000

FIGURE 1 Velocity vector fields showing San Francisco Bay wind patterns

Other examples of velocity vector fields are illustrated in Figure 2: ocean currents and flow past an airfoil.



(a) Ocean currents off the coast of Nova Scotia

(b) Airflow past an inclined airfoil

FIGURE 2 Velocity vector fields

Another type of vector field, called a *force field,* associates a force vector with each point in a region. An example is the gravitational force field that we will look at in Example 4.

In general, a vector field is a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in V_2 (or V_3).

> **1 Definition** Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

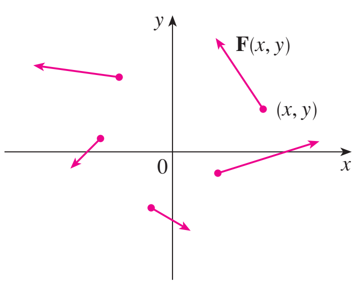


FIGURE 3
Vector field on $\mathbb{R}^2$

The best way to picture a vector field is to draw the arrow representing the vector $\mathbf{F}(x, y)$ starting at the point (x, y). Of course, it's impossible to do this for all points (x, y), but we can gain a reasonable impression of $\mathbf{F}$ by doing it for a few representative points in D as in Figure 3. Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

or, for short, $$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$$

Notice that P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

> **2 Definition** Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

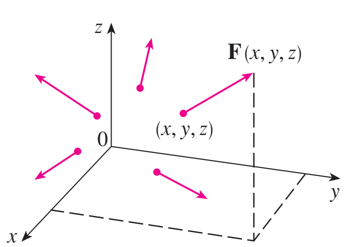


FIGURE 4
Vector field on $\mathbb{R}^3$

A vector field $\mathbf{F}$ on $\mathbb{R}^3$ is pictured in Figure 4. We can express it in terms of its component functions P, Q, and R as

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

As with the vector functions in Section 10.1, we can define continuity of vector fields and show that **F** is continuous if and only if its component functions P, Q, and R are continuous.

We sometimes identify a point (x, y, z) with its position vector $\mathbf{x} = \langle x, y, z \rangle$ and write $\mathbf{F}(\mathbf{x})$ instead of $\mathbf{F}(x, y, z)$. Then **F** becomes a function that assigns a vector $\mathbf{F}(\mathbf{x})$ to a vector **x**.

EXAMPLE 1 A vector field on $\mathbb{R}^2$ is defined by

$$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$$

Describe **F** by sketching some of the vectors $\mathbf{F}(x, y)$ as in Figure 3.

SOLUTION Since $\mathbf{F}(1, 0) = \mathbf{j}$, we draw the vector $\mathbf{j} = \langle 0, 1 \rangle$ starting at the point $(1, 0)$ in Figure 5. Since $\mathbf{F}(0, 1) = -\mathbf{i}$, we draw the vector $\langle -1, 0 \rangle$ with starting point $(0, 1)$. Continuing in this way, we draw a number of representative vectors to represent the vector field in Figure 5.

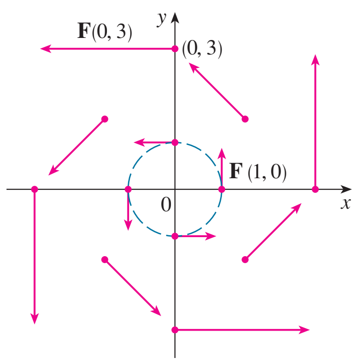


FIGURE 5
$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

It appears that each arrow is tangent to a circle with center the origin. To confirm this, we take the dot product of the position vector $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j}$ with the vector $\mathbf{F}(\mathbf{x}) = \mathbf{F}(x, y)$:

$$\begin{aligned}\mathbf{x} \cdot \mathbf{F}(\mathbf{x}) &= (x\,\mathbf{i} + y\,\mathbf{j}) \cdot (-y\,\mathbf{i} + x\,\mathbf{j}) \\ &= -xy + yx = 0\end{aligned}$$

This shows that $\mathbf{F}(x, y)$ is perpendicular to the position vector $\langle x, y \rangle$ and is therefore tangent to a circle with center the origin and radius $|\mathbf{x}| = \sqrt{x^2 + y^2}$. Notice also that

$$|\mathbf{F}(x, y)| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} = |\mathbf{x}|$$

so the magnitude of the vector $\mathbf{F}(x, y)$ is equal to the radius of the circle. ■

Some computer algebra systems are capable of plotting vector fields in two or three dimensions. They give a better impression of the vector field than is possible by hand because the computer can plot a large number of representative vectors. Figure 6 shows a computer plot of the vector field in Example 1; Figures 7 and 8 show two other vector fields. Notice that the computer scales the lengths of the vectors so they are not too long and yet are proportional to their true lengths.

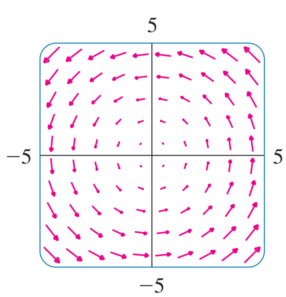


FIGURE 6
$\mathbf{F}(x, y) = \langle -y, x \rangle$

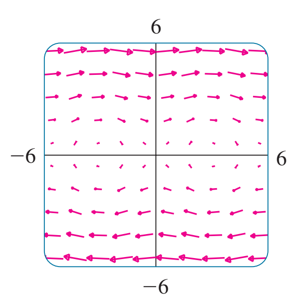


FIGURE 7
$\mathbf{F}(x, y) = \langle y, \sin x \rangle$

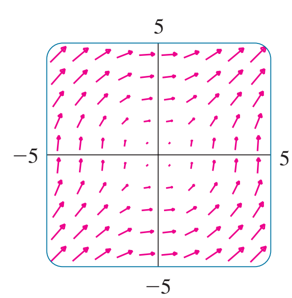


FIGURE 8
$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$

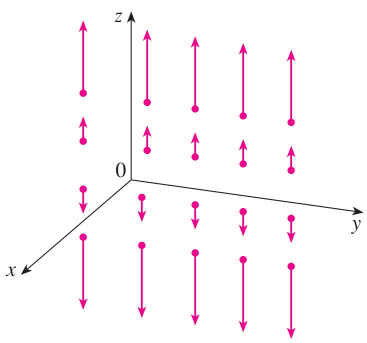


FIGURE 9
$\mathbf{F}(x, y, z) = z\,\mathbf{k}$

EXAMPLE 2 Sketch the vector field on $\mathbb{R}^3$ given by $\mathbf{F}(x, y, z) = z\,\mathbf{k}$.

SOLUTION The sketch is shown in Figure 9. Notice that all vectors are vertical and point upward above the xy-plane or downward below it. The magnitude increases with the distance from the xy-plane. ■

We were able to draw the vector field in Example 2 by hand because of its particularly simple formula. Most three-dimensional vector fields, however, are virtually impossible to sketch by hand and so we need to resort to a computer algebra system. Examples are shown in Figures 10, 11, and 12. Notice that the vector fields in Figures 10 and 11 have similar formulas, but all the vectors in Figure 11 point in the general direction of the negative y-axis because their y-components are all -2. If the vector field in Figure 12 represents a velocity field, then a particle would be swept upward and would spiral around the z-axis in the clockwise direction as viewed from above.

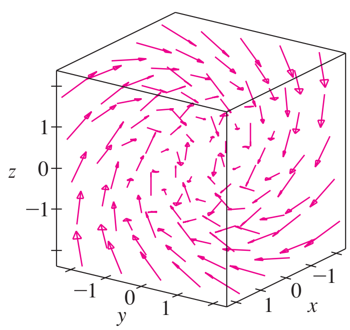


FIGURE 10
$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$

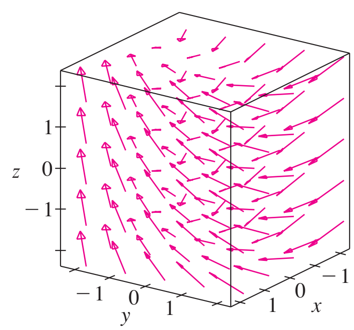


FIGURE 11
$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2\,\mathbf{j} + x\,\mathbf{k}$

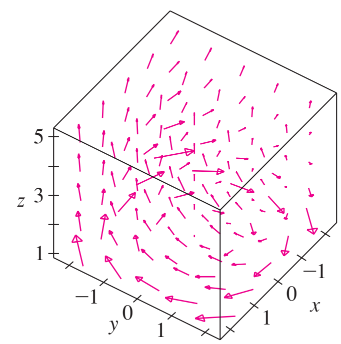


FIGURE 12
$\mathbf{F}(x, y, z) = \dfrac{y}{z}\,\mathbf{i} - \dfrac{x}{z}\,\mathbf{j} + \dfrac{z}{4}\,\mathbf{k}$

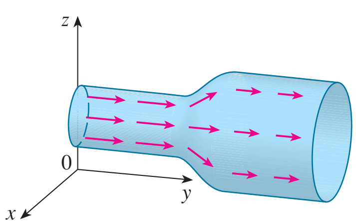


FIGURE 13
Velocity field in fluid flow

EXAMPLE 3 Imagine a fluid flowing steadily along a pipe and let $\mathbf{V}(x, y, z)$ be the velocity vector at a point (x, y, z). Then $\mathbf{V}$ assigns a vector to each point (x, y, z) in a certain domain E (the interior of the pipe) and so $\mathbf{V}$ is a vector field on $\mathbb{R}^3$ called a **velocity field**. A possible velocity field is illustrated in Figure 13. The speed at any given point is indicated by the length of the arrow.

Velocity fields also occur in other areas of physics. For instance, the vector field in Example 1 could be used as the velocity field describing the counterclockwise rotation of a wheel. We have seen other examples of velocity fields in Figures 1 and 2. ■

EXAMPLE 4 Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

where r is the distance between the objects and G is the gravitational constant. (This is an example of an inverse square law.) Let's assume that the object with mass M is located at the origin in $\mathbb{R}^3$. (For instance, M could be the mass of Earth and the origin would be at its center.) Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z \rangle$. Then $r = |\mathbf{x}|$, so $r^2 = |\mathbf{x}|^2$. The gravitational force exerted on this second object acts toward the origin, and the unit vector in this direction is

$$-\frac{\mathbf{x}}{|\mathbf{x}|}$$

Therefore, the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z \rangle$ is

$$\boxed{3} \qquad \mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

[Physicists often use the notation **r** instead of **x** for the position vector, so you may see Formula 3 written in the form $\mathbf{F} = -(mMG/r^3)\mathbf{r}$.] The function given by Equation 3 is an example of a vector field, called the **gravitational field**, because it associates a vector [the force $\mathbf{F}(\mathbf{x})$] with every point **x** in space.

Formula 3 is a compact way of writing the gravitational field, but we can also write it in terms of its component functions by using the facts that $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$:

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

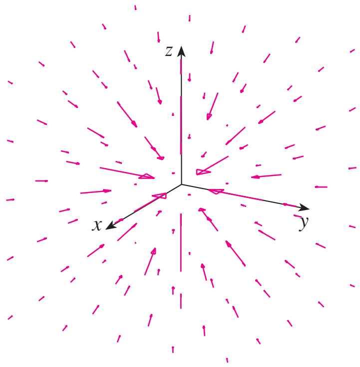


FIGURE 14
Gravitational force field

The gravitational field **F** is pictured in Figure 14. ■

EXAMPLE 5 Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}(\mathbf{x})$ exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = \langle x, y, z \rangle$ is

$$\boxed{4} \qquad \mathbf{F}(\mathbf{x}) = \frac{\varepsilon qQ}{|\mathbf{x}|^3}\mathbf{x}$$

where ε is a constant (that depends on the units used). For like charges, we have $qQ > 0$ and the force is repulsive; for unlike charges, we have $qQ < 0$ and the force is attractive. Notice the similarity between Formulas 3 and 4. Both vector fields are examples of **force fields**.

Instead of considering the electric force **F**, physicists often consider the force per unit charge:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q}\mathbf{F}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

Then **E** is a vector field on $\mathbb{R}^3$ called the **electric field** of Q. ■

Gradient Fields

If f is a scalar function of two variables, recall from Section 11.6 that its gradient ∇f (or grad f) is defined by

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

Therefore, ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$

EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

SOLUTION The gradient vector field is given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = 2xy\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

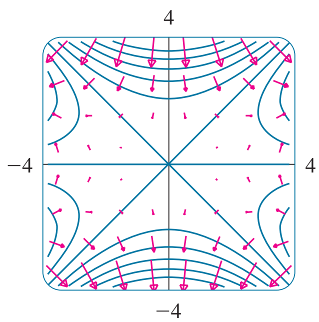


FIGURE 15

Figure 15 shows a contour map of f with the gradient vector field. Notice that the gradient vectors are perpendicular to the level curves, as we would expect from Section 11.6. Notice also that the gradient vectors are long where the level curves are close to each other and short where they are farther apart. That's because the length of the gradient vector is the value of the directional derivative of f and close level curves indicate a steep graph. ■

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for $\mathbf{F}$.

Not all vector fields are conservative, but such fields do arise frequently in physics. For example, the gravitational field $\mathbf{F}$ in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k} \\ &= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k} \\ &= \mathbf{F}(x, y, z)\end{aligned}$$

In Sections 13.3 and 13.5 we will learn how to tell whether or not a given vector field is conservative.

13.1 Exercises

1–10 ■ Sketch the vector field $\mathbf{F}$ by drawing a diagram like Figure 5 or Figure 9.

1. $\mathbf{F}(x, y) = \frac{1}{2}(\mathbf{i} + \mathbf{j})$

2. $\mathbf{F}(x, y) = \mathbf{i} + x\,\mathbf{j}$

3. $\mathbf{F}(x, y) = x\,\mathbf{i} + y\,\mathbf{j}$

4. $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$

5. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

6. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

7. $\mathbf{F}(x, y, z) = \mathbf{j}$

8. $\mathbf{F}(x, y, z) = z\,\mathbf{j}$

9. $\mathbf{F}(x, y, z) = y\,\mathbf{j}$

10. $\mathbf{F}(x, y, z) = \mathbf{j} - \mathbf{i}$

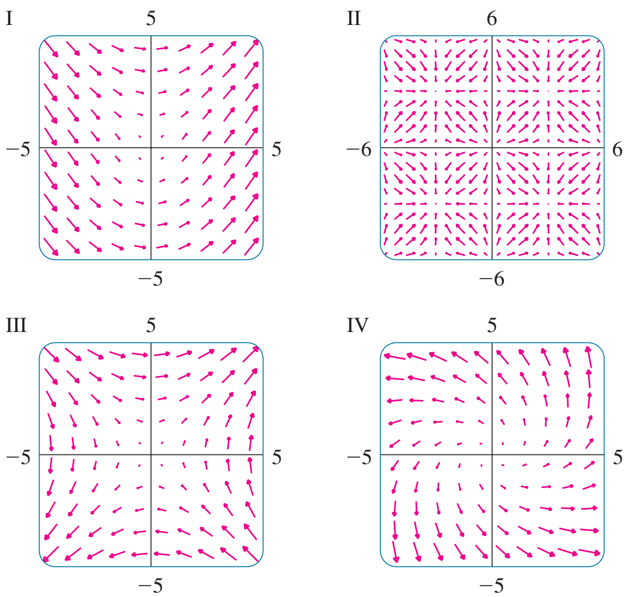
11–14 ■ Match the vector fields $\mathbf{F}$ with the plots labeled I–IV. Give reasons for your choices.

11. $\mathbf{F}(x, y) = \langle y, x \rangle$

12. $\mathbf{F}(x, y) = \langle 2x - 3y, 2x + 3y \rangle$

13. $\mathbf{F}(x, y) = \langle \sin x, \sin y \rangle$

14. $\mathbf{F}(x, y) = \langle \ln(1 + x^2 + y^2), x \rangle$



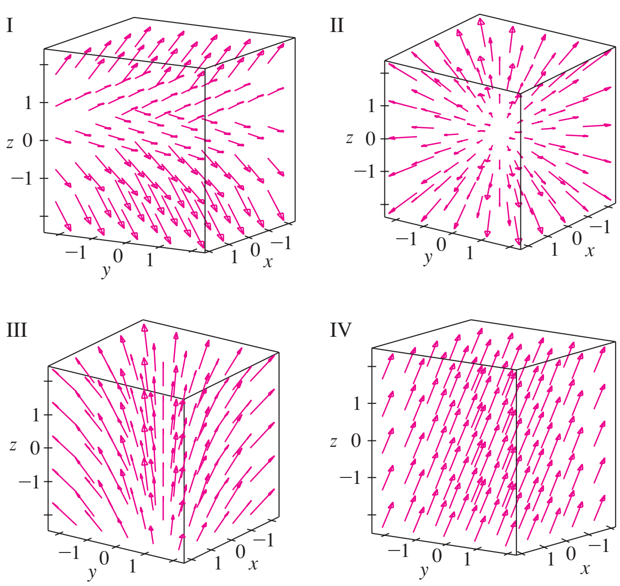
15–18 ■ Match the vector fields $\mathbf{F}$ on $\mathbb{R}^3$ with the plots labeled I–IV. Give reasons for your choices.

15. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$

16. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$

17. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$

18. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$



CAS **19.** If you have a CAS that plots vector fields (the command is fieldplot in Maple and PlotVectorField in Mathematica), use it to plot

$$\mathbf{F}(x, y) = (y^2 - 2xy)\,\mathbf{i} + (3xy - 6x^2)\,\mathbf{j}$$

Explain the appearance by finding the set of points (x, y) such that $\mathbf{F}(x, y) = \mathbf{0}$.

CAS **20.** Let $\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\mathbf{x}$, where $\mathbf{x} = \langle x, y \rangle$ and $r = |\mathbf{x}|$. Use a CAS to plot this vector field in various domains until you can see what is happening. Describe the appearance of the plot and explain it by finding the points where $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

21–24 ■ Find the gradient vector field of f.

21. $f(x, y) = \ln(x + 2y)$

22. $f(x, y) = x^{\alpha} e^{-\beta x}$

23. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

24. $f(x, y, z) = x \cos(y/z)$

25–26 ■ Find the gradient vector field ∇f of f and sketch it.

25. $f(x, y) = xy - 2x$

26. $f(x, y) = \frac{1}{4}(x + y)^2$

CAS **27–28** ■ Plot the gradient vector field of f together with a contour map of f. Explain how they are related to each other.

27. $f(x, y) = \sin x + \sin y$

28. $f(x, y) = \sin(x + y)$

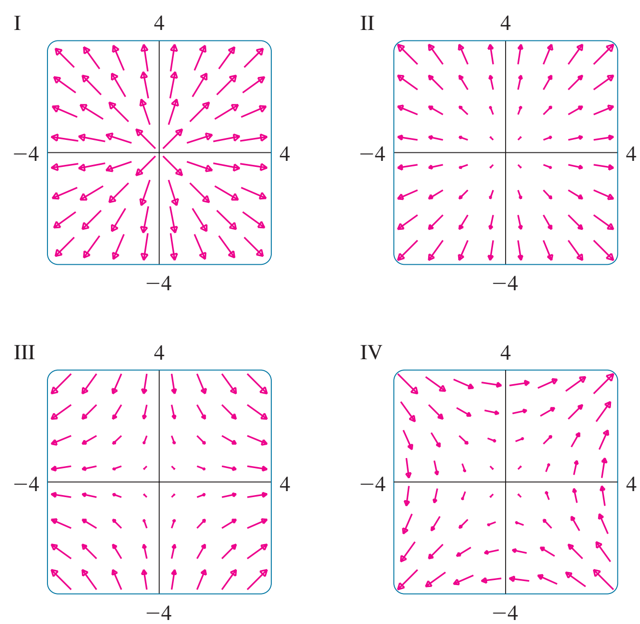
29–32 ■ Match the functions f with the plots of their gradient vector fields (labeled I–IV). Give reasons for your choices.

29. $f(x, y) = xy$

30. $f(x, y) = x^2 - y^2$

31. $f(x, y) = x^2 + y^2$

32. $f(x, y) = \sqrt{x^2 + y^2}$



33. The **flow lines** (or **streamlines**) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Thus, the vectors in a vector field are tangent to the flow lines.

(a) Use a sketch of the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ to draw some flow lines. From your sketches, can you guess the equations of the flow lines?

(b) If parametric equations of a flow line are $x = x(t)$, $y = y(t)$, explain why these functions satisfy the differential equations $dx/dt = x$ and $dy/dt = -y$. Then solve the differential equations to find an equation of the flow line that passes through the point (1, 1).

34. (a) Sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\,\mathbf{j}$ and then sketch some flow lines. What shape do these flow lines appear to have?

(b) If parametric equations of the flow lines are $x = x(t)$, $y = y(t)$, what differential equations do these functions satisfy? Deduce that $dy/dx = x$.

(c) If a particle starts at the origin in the velocity field given by $\mathbf{F}$, find an equation of the path it follows.

13.2 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve C. Such integrals are called *line integrals,* although "curve integrals" would be better terminology. They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

We start with a plane curve C given by the parametric equations

$$\boxed{1} \qquad x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, and we assume that C is a smooth curve. [This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$. See Section 10.2.] If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. (See Figure 1.) We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc. (This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.) Now if f is any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum

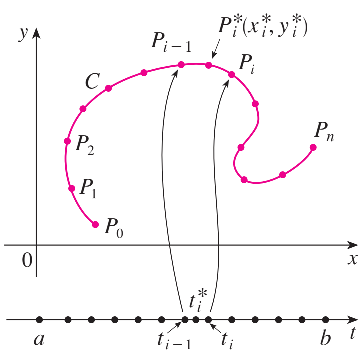


FIGURE 1

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

which is similar to a Riemann sum. Then we take the limit of these sums and make the following definition by analogy with a single integral.

> $\boxed{2}$ **Definition** If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is
>
> $$\int_C f(x, y)\,ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$
>
> if this limit exists.

In Section 6.3 we found that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

A similar type of argument can be used to show that if f is a continuous function, then the limit in Definition 2 always exists and the following formula can be used to evaluate the line integral:

3
$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

If $s(t)$ is the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, then

▲ The arc length function s is discussed in Section 10.3.

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

So the way to remember Formula 3 is to express everything in terms of the parameter t. Use the parametric equations to express x and y in terms of t and write ds as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

In the special case where C is the line segment that joins $(a, 0)$ to $(b, 0)$, using x as the parameter, we can write the parametric equations of C as follows: $x = x$, $y = 0$, $a \leqslant x \leqslant b$. Formula 3 then becomes

$$\int_C f(x, y)\, ds = \int_a^b f(x, 0)\, dx$$

and so the line integral reduces to an ordinary single integral in this case.

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area. In fact, if $f(x, y) \geqslant 0$, $\int_C f(x, y)\, ds$ represents the area of one side of the "fence" or "curtain" in Figure 2, whose base is C and whose height above the point (x, y) is $f(x, y)$.

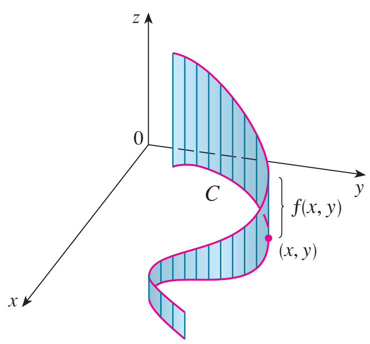


FIGURE 2

EXAMPLE 1 Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

SOLUTION In order to use Formula 3 we first need parametric equations to represent C. Recall that the unit circle can be parametrized by means of the equations

$$x = \cos t \qquad y = \sin t$$

and the upper half of the circle is described by the parameter interval $0 \leqslant t \leqslant \pi$. (See Figure 3.) Therefore, Formula 3 gives

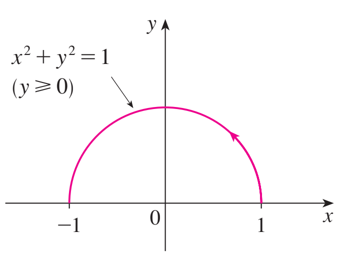


FIGURE 3

$$\begin{aligned}\int_C (2 + x^2 y)\, ds &= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t}\, dt \\ &= \int_0^\pi (2 + \cos^2 t \sin t)\, dt = \left[2t - \frac{\cos^3 t}{3}\right]_0^\pi \\ &= 2\pi + \tfrac{2}{3}\end{aligned}$$

■

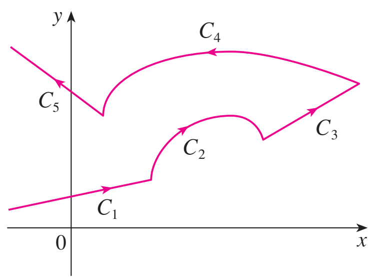


FIGURE 4
A piecewise-smooth curve

Suppose now that C is a **piecewise-smooth curve**; that is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where, as illustrated in Figure 4, the initial point of C_{i+1} is the terminal point of C_i. Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds$$

EXAMPLE 2 Evaluate $\int_C 2x\, ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the vertical line segment C_2 from (1, 1) to (1, 2).

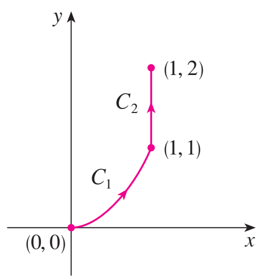


FIGURE 5
$C = C_1 \cup C_2$

SOLUTION The curve C is shown in Figure 5. C_1 is the graph of a function of x, so we can choose x as the parameter and the equations for C_1 become

$$x = x \qquad y = x^2 \qquad 0 \leqslant x \leqslant 1$$

Therefore

$$\int_{C_1} 2x\, ds = \int_0^1 2x \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 2x\sqrt{1 + 4x^2}\, dx = \tfrac{1}{4} \cdot \tfrac{2}{3}(1 + 4x^2)^{3/2}\Big]_0^1 = \frac{5\sqrt{5} - 1}{6}$$

On C_2 we choose y as the parameter, so the equations of C_2 are

$$x = 1 \qquad y = y \qquad 1 \leqslant y \leqslant 2$$

and
$$\int_{C_2} 2x\, ds = \int_1^2 2(1)\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\, dy = \int_1^2 2\, dy = 2$$

Thus
$$\int_C 2x\, ds = \int_{C_1} 2x\, ds + \int_{C_2} 2x\, ds = \frac{5\sqrt{5} - 1}{6} + 2$$

Any physical interpretation of a line integral $\int_C f(x, y)\, ds$ depends on the physical interpretation of the function f. Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C. Then the mass of the part of the wire from P_{i-1} to P_i in Figure 1 is approximately $\rho(x_i^*, y_i^*)\, \Delta s_i$ and so the total mass of the wire is approximately $\Sigma\, \rho(x_i^*, y_i^*)\, \Delta s_i$. By taking more and more points on the curve, we obtain the **mass** m of the wire as the limiting value of these approximations:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\, \Delta s_i = \int_C \rho(x, y)\, ds$$

[For example, if $f(x, y) = 2 + x^2 y$ represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.] The **center of mass** of the wire with density function ρ is located at the point $(\bar{x}, \bar{y})$, where

4
$$\bar{x} = \frac{1}{m}\int_C x\rho(x, y)\, ds \qquad \bar{y} = \frac{1}{m}\int_C y\rho(x, y)\, ds$$

Other physical interpretations of line integrals will be discussed later in this chapter.

EXAMPLE 3 A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geqslant 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

SOLUTION As in Example 1 we use the parametrization $x = \cos t$, $y = \sin t$, $0 \leqslant t \leqslant \pi$, and find that $ds = dt$. The linear density is

$$\rho(x, y) = k(1 - y)$$

where k is a constant, and so the mass of the wire is

$$m = \int_C k(1 - y)\, ds = \int_0^\pi k(1 - \sin t)\, dt$$
$$= k\big[t + \cos t\big]_0^\pi = k(\pi - 2)$$

From Equations 4 we have

$$\bar{y} = \frac{1}{m}\int_C y\rho(x, y)\, ds = \frac{1}{k(\pi - 2)}\int_C yk(1 - y)\, ds$$
$$= \frac{1}{\pi - 2}\int_0^\pi (\sin t - \sin^2 t)\, dt = \frac{1}{\pi - 2}\left[-\cos t - \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^\pi$$
$$= \frac{4 - \pi}{2(\pi - 2)}$$

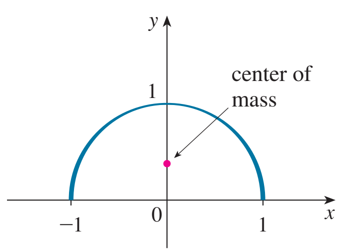


FIGURE 6

By symmetry we see that $\bar{x} = 0$, so the center of mass is

$$\left(0, \frac{4 - \pi}{2(\pi - 2)}\right) \approx (0, 0.38)$$

See Figure 6. ■

Two other line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition 2. They are called the **line integrals of f along C with respect to x and y**:

5 $$\int_C f(x, y)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta x_i$$

6 $$\int_C f(x, y)\, dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta y_i$$

When we want to distinguish the original line integral $\int_C f(x, y)\, ds$ from those in Equations 5 and 6, we call it the **line integral with respect to arc length**.

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t: $x = x(t)$, $y = y(t)$, $dx = x'(t)\, dt$, $dy = y'(t)\, dt$.

7 $$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$
$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

It frequently happens that line integrals with respect to x and y occur together. When this happens, it's customary to abbreviate by writing

$$\int_C P(x, y)\,dx + \int_C Q(x, y)\,dy = \int_C P(x, y)\,dx + Q(x, y)\,dy$$

When we are setting up a line integral, sometimes the most difficult thing is to think of a parametric representation for a curve whose geometric description is given. In particular, we often need to parametrize a line segment, so it's useful to remember that a vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

8 $$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\,\mathbf{r}_1 \qquad 0 \leqslant t \leqslant 1$$

(See Equation 9.5.1 with $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$.)

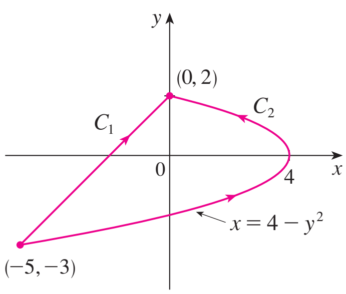


FIGURE 7

EXAMPLE 4 Evaluate $\int_C y^2\,dx + x\,dy$, where (a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. (See Figure 7.)

SOLUTION

(a) A parametric representation for the line segment is

$$x = 5t - 5 \qquad y = 5t - 3 \qquad 0 \leqslant t \leqslant 1$$

(Use Equation 8 with $\mathbf{r}_0 = \langle -5, -3 \rangle$ and $\mathbf{r}_1 = \langle 0, 2 \rangle$.) Then $dx = 5\,dt$, $dy = 5\,dt$, and Formula 7 gives

$$\int_{C_1} y^2\,dx + x\,dy = \int_0^1 (5t - 3)^2(5\,dt) + (5t - 5)(5\,dt)$$

$$= 5\int_0^1 (25t^2 - 25t + 4)\,dt$$

$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_0^1 = -\frac{5}{6}$$

(b) Since the parabola is given as a function of y, let's take y as the parameter and write C_2 as

$$x = 4 - y^2 \qquad y = y \qquad -3 \leqslant y \leqslant 2$$

Then $dx = -2y\,dy$ and by Formula 7 we have

$$\int_{C_2} y^2\,dx + x\,dy = \int_{-3}^{2} y^2(-2y)\,dy + (4 - y^2)\,dy$$

$$= \int_{-3}^{2} (-2y^3 - y^2 + 4)\,dy$$

$$= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right]_{-3}^{2} = 40\tfrac{5}{6}$$ ■

Notice that we got different answers in parts (a) and (b) of Example 4 even though the two curves had the same endpoints. Thus, in general, the value of a line integral depends not just on the endpoints of the curve but also on the path. (But see Section 13.3 for conditions under which the integral is independent of the path.)

Notice also that the answers in Example 4 depend on the direction, or orientation, of the curve. If $-C_1$ denotes the line segment from $(0, 2)$ to $(-5, -3)$, you can verify, using the parametrization

$$x = -5t \qquad y = 2 - 5t \qquad 0 \leqslant t \leqslant 1$$

that

$$\int_{-C_1} y^2\,dx + x\,dy = \frac{5}{6}$$

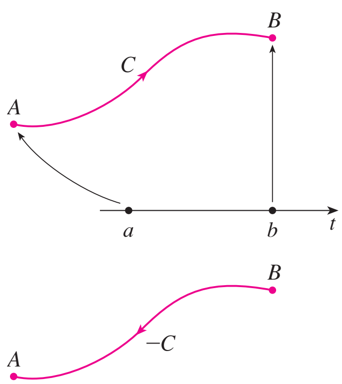


FIGURE 8

In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \leqslant t \leqslant b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing values of the parameter t. (See Figure 8, where the initial point A corresponds to the parameter value a and the terminal point B corresponds to $t = b$.)

If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point B to terminal point A in Figure 8), then we have

$$\int_{-C} f(x, y)\,dx = -\int_C f(x, y)\,dx \qquad \int_{-C} f(x, y)\,dy = -\int_C f(x, y)\,dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y)\,ds = \int_C f(x, y)\,ds$$

This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C.

Line Integrals in Space

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \leqslant t \leqslant b$$

or by a vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. If f is a function of three variables that is continuous on some region containing C, then we define the **line integral of f along C** (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x, y, z)\,ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i$$

We evaluate it using a formula similar to Formula 3:

9 $$\int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Observe that the integrals in both Formulas 3 and 9 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t))\,|\,\mathbf{r}'(t)\,|\,dt$$

For the special case $f(x, y, z) = 1$, we get

$$\int_C ds = \int_a^b |\mathbf{r}'(t)|\, dt = L$$

where L is the length of the curve C (see Formula 10.3.3).

Line integrals along C with respect to x, y, and z can also be defined. For example,

$$\int_C f(x, y, z)\, dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta z_i$$

$$= \int_a^b f(x(t), y(t), z(t))\, z'(t)\, dt$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

10 $$\int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

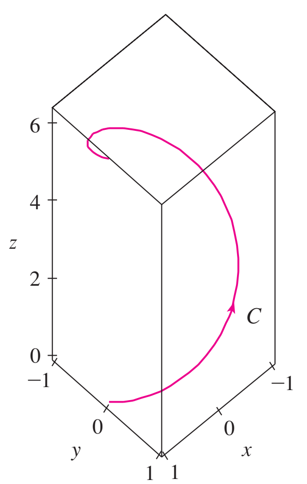


FIGURE 9

EXAMPLE 5 Evaluate $\int_C y \sin z\, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)

SOLUTION Formula 9 gives

$$\int_C y \sin z\, ds = \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1}\, dt$$

$$= \sqrt{2} \int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2t)\, dt = \frac{\sqrt{2}}{2} \left[t - \tfrac{1}{2} \sin 2t\right]_0^{2\pi} = \sqrt{2}\pi$$ ■

EXAMPLE 6 Evaluate $\int_C y\, dx + z\, dy + x\, dz$, where C consists of the line segment C_1 from $(2, 0, 0)$ to $(3, 4, 5)$ followed by the vertical line segment C_2 from $(3, 4, 5)$ to $(3, 4, 0)$.

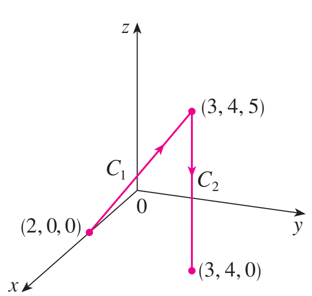


FIGURE 10

SOLUTION The curve C is shown in Figure 10. Using Equation 8, we write C_1 as

$$\mathbf{r}(t) = (1 - t)\langle 2, 0, 0\rangle + t\langle 3, 4, 5\rangle = \langle 2 + t, 4t, 5t\rangle$$

or, in parametric form, as

$$x = 2 + t \qquad y = 4t \qquad z = 5t \qquad 0 \leqslant t \leqslant 1$$

Thus

$$\int_{C_1} y\, dx + z\, dy + x\, dz = \int_0^1 (4t)\, dt + (5t)4\, dt + (2 + t)5\, dt$$

$$= \int_0^1 (10 + 29t)\, dt = 10t + 29\frac{t^2}{2}\Big]_0^1 = 24.5$$

Likewise, C_2 can be written in the form

$$\mathbf{r}(t) = (1 - t)\langle 3, 4, 5 \rangle + t\langle 3, 4, 0 \rangle = \langle 3, 4, 5 - 5t \rangle$$

or $$x = 3 \qquad y = 4 \qquad z = 5 - 5t \qquad 0 \leqslant t \leqslant 1$$

Then $dx = 0 = dy$, so

$$\int_{C_2} y\,dx + z\,dy + x\,dz = \int_0^1 3(-5)\,dt = -15$$

Adding the values of these integrals, we obtain

$$\int_C y\,dx + z\,dy + x\,dz = 24.5 - 15 = 9.5$$

Line Integrals of Vector Fields

Recall from Section 6.5 that the work done by a variable force $f(x)$ in moving a particle from a to b along the x-axis is $W = \int_a^b f(x)\,dx$. Then in Section 9.3 we found that the work done by a constant force $\mathbf{F}$ in moving an object from a point P to another point Q in space is $W = \mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D} = \overrightarrow{PQ}$ is the displacement vector.

Now suppose that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$, such as the gravitational field of Example 4 in Section 13.1 or the electric force field of Example 5 in Section 13.1. (A force field on $\mathbb{R}^2$ could be regarded as a special case where $R = 0$ and P and Q depend only on x and y.) We wish to compute the work done by this force in moving a particle along a smooth curve C.

We divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width. (See Figure 1 for the two-dimensional case or Figure 11 for the three-dimensional case.) Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the ith subarc corresponding to the parameter value t_i^*. If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction of $\mathbf{T}(t_i^*)$, the unit tangent vector at P_i^*. Thus, the work done by the force $\mathbf{F}$ in moving the particle from P_{i-1} to P_i is approximately

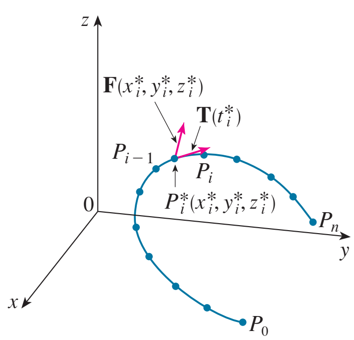


FIGURE 11

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i\,\mathbf{T}(t_i^*)] = [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)]\,\Delta s_i$$

and the total work done in moving the particle along C is approximately

11 $$\sum_{i=1}^{n} [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)]\,\Delta s_i$$

where $\mathbf{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on C. Intuitively, we see that these approximations ought to become better as n becomes larger. Therefore, we define the **work** W done by the force field $\mathbf{F}$ as the limit of the Riemann sums in (11), namely,

12 $$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\,ds = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$$

Equation 12 says that *work is the line integral with respect to arc length of the tangential component of the force.*

If the curve C is given by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, then $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$, so using Equation 9 we can rewrite Equation 12 in the form

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right] |\mathbf{r}'(t)|\, dt$$
$$= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

This integral is often abbreviated as $\int_C \mathbf{F} \cdot d\mathbf{r}$ and occurs in other areas of physics as well. Therefore, we make the following definition for the line integral of *any* continuous vector field.

13 Definition Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along *C*** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

When using Definition 13, remember that $\mathbf{F}(\mathbf{r}(t))$ is just an abbreviation for $\mathbf{F}(x(t), y(t), z(t))$, so we evaluate $\mathbf{F}(\mathbf{r}(t))$ simply by putting $x = x(t)$, $y = y(t)$, and $z = z(t)$ in the expression for $\mathbf{F}(x, y, z)$. Notice also that we can formally write $d\mathbf{r} = \mathbf{r}'(t)\, dt$.

▲ Figure 12 shows the force field and the curve in Example 7. The work done is negative because the field impedes movement along the curve.

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$.

SOLUTION Since $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\,\mathbf{i} - \cos t \sin t\,\mathbf{j}$$

and
$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Therefore, the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^{\pi/2} (-2\cos^2 t \sin t)\, dt$$
$$= 2\,\frac{\cos^3 t}{3}\bigg]_0^{\pi/2} = -\frac{2}{3}$$

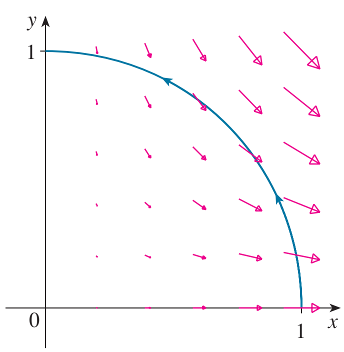


FIGURE 12

NOTE • Even though $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

since the unit tangent vector $\mathbf{T}$ is replaced by its negative when C is replaced by $-C$.

▲ Figure 13 shows the twisted cubic C in Example 8 and some typical vectors acting at three points on C.

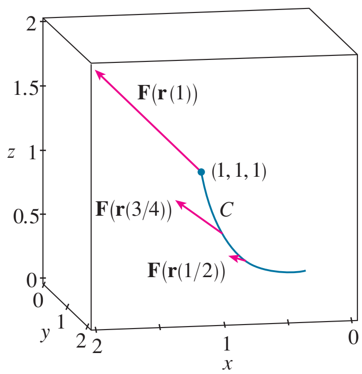


FIGURE 13

EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leqslant t \leqslant 1$$

SOLUTION We have

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = t^3\mathbf{i} + t^5\mathbf{j} + t^4\mathbf{k}$$

Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^1 (t^3 + 5t^6)\,dt = \frac{t^4}{4} + \frac{5t^7}{7}\bigg]_0^1 = \frac{27}{28}$$

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given in component form by the equation $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. We use Definition 13 to compute its line integral along C:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_a^b (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot (x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k})\,dt$$

$$= \int_a^b [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)]\,dt$$

But this last integral is precisely the line integral in (10). Therefore, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz \qquad \text{where } \mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

For example, the integral $\int_C y\,dx + z\,dy + x\,dz$ in Example 6 could be expressed as $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$$

13.2 Exercises

1–12 ■ Evaluate the line integral, where C is the given curve.

1. $\int_C y\,ds$, $C\colon x = t^2,\ y = t,\ 0 \leqslant t \leqslant 2$

2. $\int_C (y/x)\,ds$, $C\colon x = t^4,\ y = t^3,\ \frac{1}{2} \leqslant t \leqslant 1$

3. $\int_C xy^4\,ds$, C is the right half of the circle $x^2 + y^2 = 16$

4. $\int_C \sin x\,dx$,
C is the arc of the curve $x = y^4$ from $(1, -1)$ to $(1, 1)$

5. $\int_C xy\,dx + (x - y)\,dy$, C consists of line segments from $(0, 0)$ to $(2, 0)$ and from $(2, 0)$ to $(3, 2)$

6. $\int_C x\sqrt{y}\,dx + 2y\sqrt{x}\,dy$,
C consists of the shortest arc of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$ and the line segment from $(0, 1)$ to $(4, 3)$

7. $\int_C xy^3\,ds$,
C: $x = 4\sin t,\ y = 4\cos t,\ z = 3t,\ 0 \leqslant t \leqslant \pi/2$

8. $\int_C x^2 z\,ds$, C is the line segment from $(0, 6, -1)$ to $(4, 1, 5)$

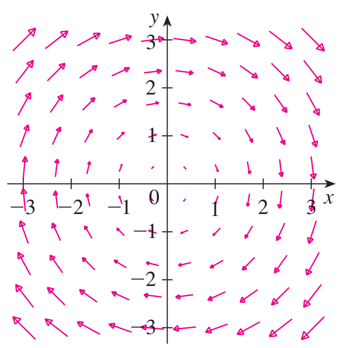
9. $\int_C xe^{yz}\,ds$, C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$

10. $\int_C yz\,dy + xy\,dz$, C: $x = \sqrt{t},\ y = t,\ z = t^2,\ 0 \leqslant t \leqslant 1$

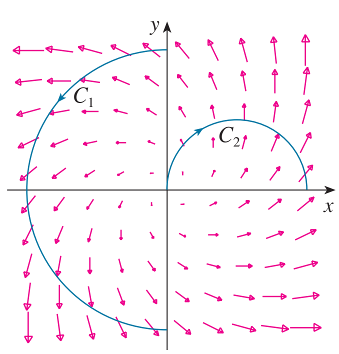
11. $\int_C z^2\,dx - z\,dy + 2y\,dz$,
C consists of line segments from $(0, 0, 0)$ to $(0, 1, 1)$, from $(0, 1, 1)$ to $(1, 2, 3)$, and from $(1, 2, 3)$ to $(1, 2, 4)$

12. $\int_C yz\,dx + xz\,dy + xy\,dz$,
C consists of line segments from $(0, 0, 0)$ to $(2, 0, 0)$, from $(2, 0, 0)$ to $(1, 3, -1)$, and from $(1, 3, -1)$ to $(1, 3, 0)$

13. Let **F** be the vector field shown in the figure.
(a) If C_1 is the vertical line segment from $(-3, -3)$ to $(-3, 3)$, determine whether $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.
(b) If C_2 is the counterclockwise-oriented circle with radius 3 and center the origin, determine whether $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.



14. The figure shows a vector field **F** and two curves C_1 and C_2. Are the line integrals of **F** over C_1 and C_2 positive, negative, or zero? Explain.



15–18 ■ Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

15. $\mathbf{F}(x, y) = x^2y^3\,\mathbf{i} - y\sqrt{x}\,\mathbf{j}$,
$\mathbf{r}(t) = t^2\,\mathbf{i} - t^3\,\mathbf{j},\quad 0 \leqslant t \leqslant 1$

16. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$,
$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k},\quad 0 \leqslant t \leqslant 2$

17. $\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + xz\,\mathbf{k}$,
$\mathbf{r}(t) = t^3\,\mathbf{i} - t^2\,\mathbf{j} + t\,\mathbf{k},\quad 0 \leqslant t \leqslant 1$

18. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z^2\,\mathbf{k}$,
$\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t^2\,\mathbf{k},\quad 0 \leqslant t \leqslant \pi/2$

CAS **19–20** ■ Use a graph of the vector field **F** and the curve C to guess whether the line integral of **F** over C is positive, negative, or zero. Then evaluate the line integral.

19. $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + xy\,\mathbf{j}$,
C is the arc of the circle $x^2 + y^2 = 4$ traversed counterclockwise from $(2, 0)$ to $(0, -2)$

20. $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\,\mathbf{j}$,
C is the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$

21. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = e^{x-1}\,\mathbf{i} + xy\,\mathbf{j}$ and C is given by $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j}$, $0 \leqslant t \leqslant 1$.
(b) Illustrate part (a) by using a graphing calculator or computer to graph C and the vectors from the vector field corresponding to $t = 0$, $1/\sqrt{2}$, and 1 (as in Figure 13).

22. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$ and C is given by $\mathbf{r}(t) = 2t\,\mathbf{i} + 3t\,\mathbf{j} - t^2\,\mathbf{k}$, $-1 \leqslant t \leqslant 1$.
(b) Illustrate part (a) by using a computer to graph C and the vectors from the vector field corresponding to $t = \pm 1$ and $\pm\frac{1}{2}$ (as in Figure 13).

CAS **23.** Find the exact value of $\int_C x^3y^5\,ds$, where C is the part of the astroid $x = \cos^3 t$, $y = \sin^3 t$ in the first quadrant.

24. (a) Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + xy\,\mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction.
CAS (b) Use a computer algebra system to graph the force field and circle on the same screen. Use the graph to explain your answer to part (a).

25. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4$, $x \geqslant 0$. If the linear density is a constant k, find the mass and center of mass of the wire.

26. Find the mass and center of mass of a thin wire in the shape of a quarter-circle $x^2 + y^2 = r^2$, $x \geqslant 0$, $y \geqslant 0$, if the density function is $\rho(x, y) = x + y$.

27. (a) Write the formulas similar to Equations 4 for the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of a thin wire with density function $\rho(x, y, z)$ in the shape of a space curve C.
(b) Find the center of mass of a wire in the shape of the helix $x = 2\sin t$, $y = 2\cos t$, $z = 3t$, $0 \leqslant t \leqslant 2\pi$, if the density is a constant k.

28. Find the mass and center of mass of a wire in the shape of the helix $x = t$, $y = \cos t$, $z = \sin t$, $0 \leqslant t \leqslant 2\pi$, if the density at any point is equal to the square of the distance from the origin.

29. If a wire with linear density $\rho(x, y)$ lies along a plane curve C, its **moments of inertia** about the x- and y-axes are defined as

$$I_x = \int_C y^2 \rho(x, y)\, ds \qquad I_y = \int_C x^2 \rho(x, y)\, ds$$

Find the moments of inertia for the wire in Example 3.

30. If a wire with linear density $\rho(x, y, z)$ lies along a space curve C, its **moments of inertia** about the x-, y-, and z-axes are defined as

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\, ds$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\, ds$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\, ds$$

Find the moments of inertia for the wire in Exercise 27.

31. Find the work done by the force field $\mathbf{F}(x, y) = x\,\mathbf{i} + (y + 2)\,\mathbf{j}$ in moving an object along an arch of the cycloid $\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$.

32. Find the work done by the force field $\mathbf{F}(x, y) = x\sin y\,\mathbf{i} + y\,\mathbf{j}$ on a particle that moves along the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

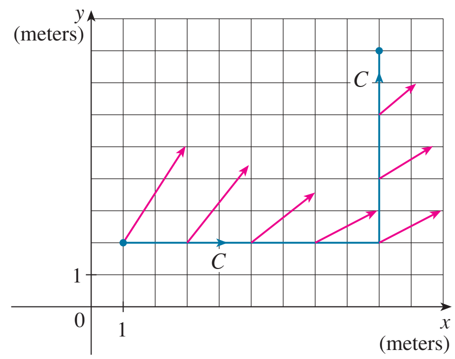
33. Find the work done by the force field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yx\,\mathbf{j} + zy\,\mathbf{k}$ on a particle that moves along the curve $\mathbf{r}(t) = t^2\,\mathbf{i} - t^3\,\mathbf{j} + t^4\,\mathbf{k}$, $0 \leqslant t \leqslant 1$.

34. The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\mathbf{r} = \langle x, y, z\rangle$ is $\mathbf{F}(\mathbf{r}) = K\mathbf{r}/|\mathbf{r}|^3$ where K is a constant. (See Example 5 in Section 13.1.) Find the work done as the particle moves along a straight line from $(2, 0, 0)$ to $(2, 1, 5)$.

35. A 160-lb man carries a 25-lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft high and the man makes exactly three complete revolutions, how much work is done by the man against gravity in climbing to the top?

36. Suppose there is a hole in the can of paint in Exercise 35 and 9 lb of paint leak steadily out of the can during the man's ascent. How much work is done?

37. An object moves along the curve C shown in the figure from $(1, 2)$ to $(9, 8)$. The lengths of the vectors in the force field $\mathbf{F}$ are measured in newtons by the scales on the axes. Estimate the work done by $\mathbf{F}$ on the object.

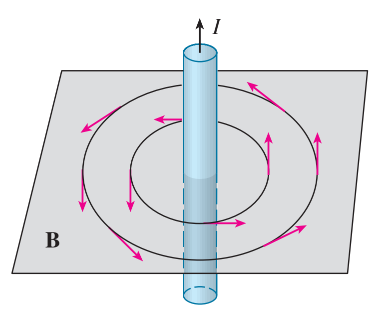


38. Experiments show that a steady current I in a long wire produces a magnetic field $\mathbf{B}$ that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). *Ampère's Law* relates the electric current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where I is the net current that passes through any surface bounded by a closed curve C and μ_0 is a constant called the permeability of free space. By taking C to be a circle with radius r, show that the magnitude $B = |\mathbf{B}|$ of the magnetic field at a distance r from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$



13.3 The Fundamental Theorem for Line Integrals

Recall from Section 5.4 that Part 2 of the Fundamental Theorem of Calculus can be written as

$$\boxed{1} \qquad \int_a^b F'(x)\,dx = F(b) - F(a)$$

where F' is continuous on $[a, b]$. We also called Equation 1 the Total Change Theorem: The integral of a rate of change is the total change.

If we think of the gradient vector ∇f of a function f of two or three variables as a sort of derivative of f, then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

2 Theorem Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

NOTE • Theorem 2 says that we can evaluate the line integral of a conservative vector field (the gradient vector field of the potential function f) simply by knowing the value of f at the endpoints of C. In fact, Theorem 2 says that the line integral of ∇f is the total change in f. If f is a function of two variables and C is a plane curve with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, as in Figure 1, then Theorem 2 becomes

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$$

If f is a function of three variables and C is a space curve joining the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$, then we have

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Let's prove Theorem 2 for this case.

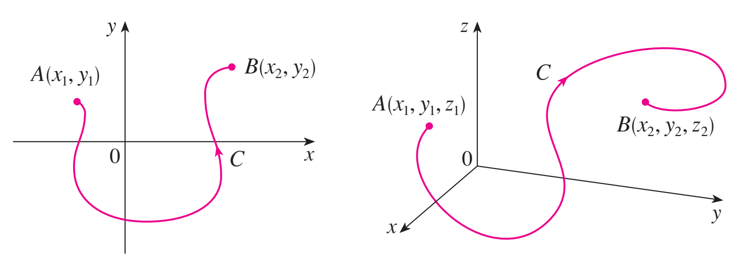


FIGURE 1

Proof of Theorem 2 Using Definition 13.2.13, we have

$$\begin{aligned}\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt \\ &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt \qquad \text{(by the Chain Rule)} \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a))\end{aligned}$$

The last step follows from the Fundamental Theorem of Calculus (Equation 1). ■

Although we have proved Theorem 2 for smooth curves, it is also true for piecewise-smooth curves. This can be seen by subdividing C into a finite number of smooth curves and adding the resulting integrals.

EXAMPLE 1 Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

in moving a particle with mass m from the point (3, 4, 12) to the point (2, 2, 0) along a piecewise-smooth curve C. (See Example 4 in Section 13.1.)

SOLUTION From Section 13.1 we know that $\mathbf{F}$ is a conservative vector field and, in fact, $\mathbf{F} = \nabla f$, where

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

Therefore, by Theorem 2, the work done is

$$\begin{aligned}W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} \\ &= f(2, 2, 0) - f(3, 4, 12) \\ &= \frac{mMG}{\sqrt{2^2 + 2^2}} - \frac{mMG}{\sqrt{3^2 + 4^2 + 12^2}} = mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)\end{aligned}$$ ■

Independence of Path

Suppose C_1 and C_2 are two piecewise-smooth curves (which are called **paths**) that have the same initial point A and terminal point B. We know from Example 4 in Section 13.2 that, in general, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. But one implication of Theorem 2 is that

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

whenever ∇f is continuous. In other words, the line integral of a conservative vector field depends only on the initial point and terminal point of a curve.

In general, if $\mathbf{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths

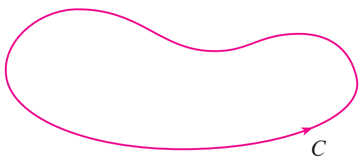


FIGURE 2
A closed curve

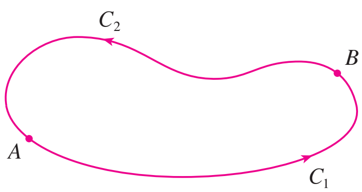


FIGURE 3

C_1 and C_2 in D that have the same initial and terminal points. With this terminology we can say that *line integrals of conservative vector fields are independent of path.*

A curve is called **closed** if its terminal point coincides with its initial point, that is, $\mathbf{r}(b) = \mathbf{r}(a)$. (See Figure 2.) If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D and C is any closed path in D, we can choose any two points A and B on C and regard C as being composed of the path C_1 from A to B followed by the path C_2 from B to A. (See Figure 3.) Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

since C_1 and $-C_2$ have the same initial and terminal points.

Conversely, if it is true that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ whenever C is a closed path in D, then we demonstrate independence of path as follows. Take any two paths C_1 and C_2 from A to B in D and define C to be the curve consisting of C_1 followed by $-C_2$. Then

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

and so $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Thus, we have proved the following theorem.

> **3 Theorem** $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Since we know that the line integral of any conservative vector field $\mathbf{F}$ is independent of path, it follows that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path. The physical interpretation is that the work done by a conservative force field (such as the gravitational or electric field in Section 13.1) as it moves an object around a closed path is 0.

The following theorem says that the *only* vector fields that are independent of path are conservative. It is stated and proved for plane curves, but there is a similar version for space curves. We assume that D is **open**, which means that for every point P in D there is a disk with center P that lies entirely in D. (So D doesn't contain any of its boundary points.) In addition, we assume that D is **connected**. This means that any two points in D can be joined by a path that lies in D.

> **4 Theorem** Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Proof Let $A(a, b)$ be a fixed point in D. We construct the desired potential function f by defining

$$f(x, y) = \int_{(a, b)}^{(x, y)} \mathbf{F} \cdot d\mathbf{r}$$

for any point (x, y) in D. Since $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, it does not matter which path C from (a, b) to (x, y) is used to evaluate $f(x, y)$. Since D is open, there exists a disk contained in D with center (x, y). Choose any point (x_1, y) in the disk with $x_1 < x$ and let C consist of any path C_1 from (a, b) to (x_1, y) followed by the horizontal line segment C_2 from (x_1, y) to (x, y). (See Figure 4.) Then

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{(a, b)}^{(x_1, y)} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

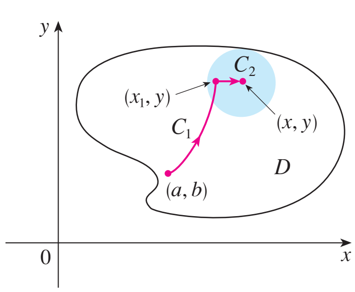


FIGURE 4

Notice that the first of these integrals does not depend on x, so

$$\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

If we write $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, then

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} P\,dx + Q\,dy$$

On C_2, y is constant, so $dy = 0$. Using t as the parameter, where $x_1 \leqslant t \leqslant x$, we have

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial x} \int_{x_1}^{x} P(t, y)\,dt = P(x, y)$$

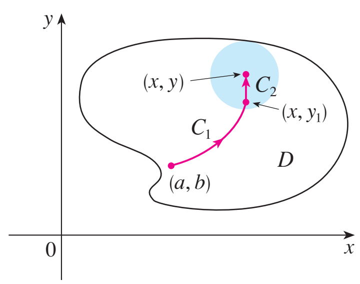


FIGURE 5

by Part 1 of the Fundamental Theorem of Calculus (see Section 5.4). A similar argument, using a vertical line segment (see Figure 5), shows that

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial y} \int_{y_1}^{y} Q(x, t)\,dt = Q(x, y)$$

Thus
$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = \nabla f$$

which says that $\mathbf{F}$ is conservative. ■

The question remains: How is it possible to determine whether or not a vector field $\mathbf{F}$ is conservative? Suppose it is known that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is conservative, where P and Q have continuous first-order partial derivatives. Then there is a function f such that $\mathbf{F} = \nabla f$, that is,

$$P = \frac{\partial f}{\partial x} \qquad \text{and} \qquad Q = \frac{\partial f}{\partial y}$$

Therefore, by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial Q}{\partial x}$$

5 Theorem If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

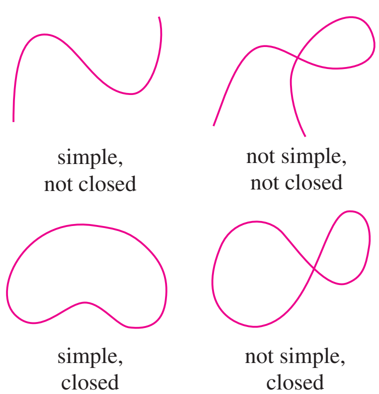


FIGURE 6
Types of curves

The converse of Theorem 5 is true only for a special type of region. To explain this, we first need the concept of a **simple curve**, which is a curve that doesn't intersect itself anywhere between its endpoints. [See Figure 6; $\mathbf{r}(a) = \mathbf{r}(b)$ for a simple closed curve, but $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$.]

In Theorem 4 we needed an open connected region. For the next theorem we need a stronger condition. A **simply-connected region** in the plane is a connected region

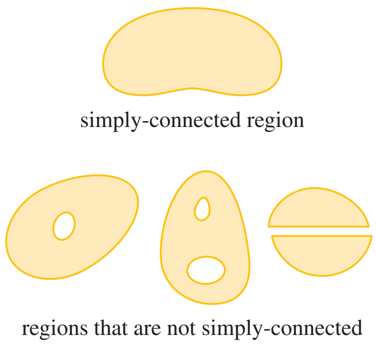


FIGURE 7

D such that every simple closed curve in D encloses only points that are in D. Notice from Figure 7 that, intuitively speaking, a simply-connected region contains no hole and can't consist of two separate pieces.

In terms of simply-connected regions we can now state a partial converse to Theorem 5 that gives a convenient method for verifying that a vector field on $\mathbb{R}^2$ is conservative. The proof will be sketched in the next section as a consequence of Green's Theorem.

> **6 Theorem** Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and
>
> $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$
>
> Then $\mathbf{F}$ is conservative.

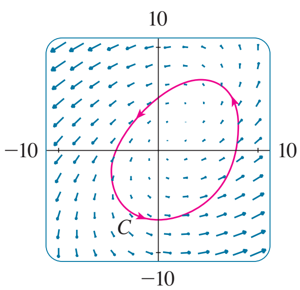


FIGURE 8

▲ Figures 8 and 9 show the vector fields in Examples 2 and 3, respectively. The vectors in Figure 8 that start on the closed curve C all appear to point in roughly the same direction as C. So it looks as if $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$ and therefore $\mathbf{F}$ is not conservative. The calculation in Example 2 confirms this impression. Some of the vectors near the curves C_1 and C_2 in Figure 9 point in approximately the same direction as the curves, whereas others point in the opposite direction. So it appears plausible that line integrals around all closed paths are 0. Example 3 shows that $\mathbf{F}$ is indeed conservative.

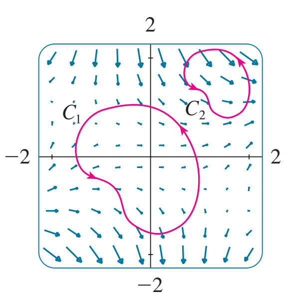


FIGURE 9

EXAMPLE 2 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$$

is conservative.

SOLUTION Let $P(x, y) = x - y$ and $Q(x, y) = x - 2$. Then

$$\frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} = 1$$

Since $\partial P/\partial y \neq \partial Q/\partial x$, $\mathbf{F}$ is not conservative by Theorem 5. ■

EXAMPLE 3 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

is conservative.

SOLUTION Let $P(x, y) = 3 + 2xy$ and $Q(x, y) = x^2 - 3y^2$. Then

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$$

Also, the domain of $\mathbf{F}$ is the entire plane ($D = \mathbb{R}^2$), which is open and simply-connected. Therefore, we can apply Theorem 6 and conclude that $\mathbf{F}$ is conservative. ■

In Example 3, Theorem 6 told us that $\mathbf{F}$ is conservative, but it did not tell us how to find the (potential) function f such that $\mathbf{F} = \nabla f$. The proof of Theorem 4 gives us a clue as to how to find f. We use "partial integration" as in the following example.

EXAMPLE 4

(a) If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.
(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j}$, $0 \leqslant t \leqslant \pi$.

SOLUTION

(a) From Example 3 we know that $\mathbf{F}$ is conservative and so there exists a function f with $\nabla f = \mathbf{F}$, that is,

$$f_x(x, y) = 3 + 2xy \tag{7}$$

$$f_y(x, y) = x^2 - 3y^2 \tag{8}$$

Integrating (7) with respect to x, we obtain

$$f(x, y) = 3x + x^2y + g(y) \tag{9}$$

Notice that the constant of integration is a constant with respect to x, that is, a function of y, which we have called $g(y)$. Next we differentiate both sides of (9) with respect to y:

$$f_y(x, y) = x^2 + g'(y) \tag{10}$$

Comparing (8) and (10), we see that

$$g'(y) = -3y^2$$

Integrating with respect to y, we have

$$g(y) = -y^3 + K$$

where K is a constant. Putting this in (9), we have

$$f(x, y) = 3x + x^2y - y^3 + K$$

as the desired potential function.

(b) To use Theorem 2 all we have to know are the initial and terminal points of C, namely, $\mathbf{r}(0) = (0, 1)$ and $\mathbf{r}(\pi) = (0, -e^\pi)$. In the expression for $f(x, y)$ in part (a), any value of the constant K will do, so let's choose $K = 0$. Then we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0, -e^\pi) - f(0, 1)$$

$$= e^{3\pi} - (-1) = e^{3\pi} + 1$$

This method is much shorter than the straightforward method for evaluating line integrals that we learned in Section 13.2. ■

A criterion for determining whether or not a vector field $\mathbf{F}$ on $\mathbb{R}^3$ is conservative is given in Section 13.5. Meanwhile, the next example shows that the technique for finding the potential function is much the same as for vector fields on $\mathbb{R}^2$.

EXAMPLE 5 If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

SOLUTION If there is such a function f, then

$$f_x(x, y, z) = y^2 \tag{11}$$

$$f_y(x, y, z) = 2xy + e^{3z} \tag{12}$$

$$f_z(x, y, z) = 3ye^{3z} \tag{13}$$

Integrating (11) with respect to x, we get

$$f(x, y, z) = xy^2 + g(y, z) \tag{14}$$

where $g(y, z)$ is a constant with respect to x. Then differentiating (14) with respect to y, we have

$$f_y(x, y, z) = 2xy + g_y(y, z)$$

and comparison with (12) gives

$$g_y(y, z) = e^{3z}$$

Thus, $g(y, z) = ye^{3z} + h(z)$ and we rewrite (14) as

$$f(x, y, z) = xy^2 + ye^{3z} + h(z)$$

Finally, differentiating with respect to z and comparing with (13), we obtain $h'(z) = 0$ and, therefore, $h(z) = K$, a constant. The desired function is

$$f(x, y, z) = xy^2 + ye^{3z} + K$$

It is easily verified that $\nabla f = \mathbf{F}$. ■

Conservation of Energy

Let's apply the ideas of this chapter to a continuous force field $\mathbf{F}$ that moves an object along a path C given by $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, where $\mathbf{r}(a) = A$ is the initial point and $\mathbf{r}(b) = B$ is the terminal point of C. According to Newton's Second Law of Motion (see Section 10.4), the force $\mathbf{F}(\mathbf{r}(t))$ at a point on C is related to the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ by the equation

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So the work done by the force on the object is

$$\begin{aligned}
W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\, dt \\
&= \frac{m}{2} \int_a^b \frac{d}{dt} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)]\, dt && \text{(Theorem 10.2.3, Formula 4)} \\
&= \frac{m}{2} \int_a^b \frac{d}{dt} |\mathbf{r}'(t)|^2\, dt \\
&= \frac{m}{2} \left[|\mathbf{r}'(t)|^2 \right]_a^b && \text{(Fundamental Theorem of Calculus)} \\
&= \frac{m}{2} \left(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2 \right)
\end{aligned}$$

Therefore

15 $$W = \tfrac{1}{2}m\,|\mathbf{v}(b)|^2 - \tfrac{1}{2}m\,|\mathbf{v}(a)|^2$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity.

The quantity $\tfrac{1}{2}m\,|\mathbf{v}(t)|^2$, that is, half the mass times the square of the speed, is called the **kinetic energy** of the object. Therefore, we can rewrite Equation 15 as

16 $$W = K(B) - K(A)$$

which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Now let's further assume that $\mathbf{F}$ is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$. In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = -\nabla P$. Then by Theorem 2 we have

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} \\ &= -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] \\ &= P(A) - P(B) \end{aligned}$$

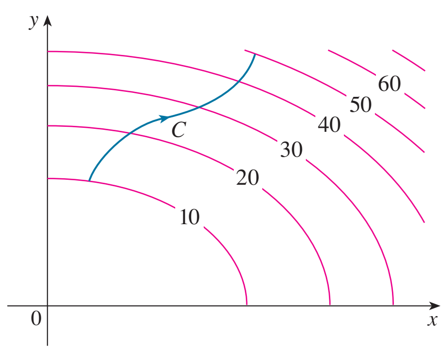
Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative.*

13.3 Exercises

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



2. A table of values of a function f with continuous gradient is given. Find $\int_C \nabla f \cdot d\mathbf{r}$, where C has parametric equations $x = t^2 + 1$, $y = t^3 + t$, $0 \leqslant t \leqslant 1$.

x \ y	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9

3–10 ■ Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

3. $\mathbf{F}(x, y) = (6x + 5y)\,\mathbf{i} + (5x + 4y)\,\mathbf{j}$

4. $\mathbf{F}(x, y) = (x^3 + 4xy)\,\mathbf{i} + (4xy - y^3)\,\mathbf{j}$

5. $\mathbf{F}(x, y) = xe^y\,\mathbf{i} + ye^x\,\mathbf{j}$

6. $\mathbf{F}(x, y) = e^y\,\mathbf{i} + xe^y\,\mathbf{j}$

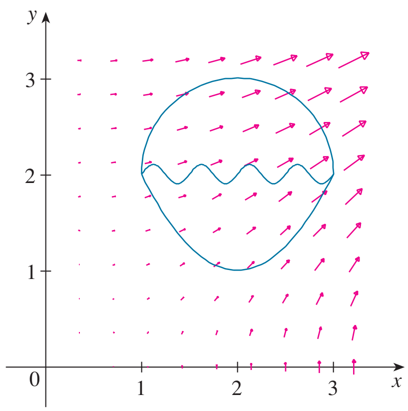
7. $\mathbf{F}(x, y) = (2x\cos y - y\cos x)\,\mathbf{i} + (-x^2\sin y - \sin x)\,\mathbf{j}$

8. $\mathbf{F}(x, y) = (1 + 2xy + \ln x)\,\mathbf{i} + x^2\,\mathbf{j}$

9. $\mathbf{F}(x, y) = (ye^x + \sin y)\,\mathbf{i} + (e^x + x\cos y)\,\mathbf{j}$

10. $\mathbf{F}(x, y) = (ye^{xy} + 4x^3 y)\,\mathbf{i} + (xe^{xy} + x^4)\,\mathbf{j}$

11. The figure shows the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$ and three curves that start at (1, 2) and end at (3, 2).
(a) Explain why $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all three curves.
(b) What is this common value?



12–18 ■ (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

12. $\mathbf{F}(x, y) = y\,\mathbf{i} + (x + 2y)\,\mathbf{j}$,
C is the upper semicircle that starts at (0, 1) and ends at (2, 1)

13. $\mathbf{F}(x, y) = x^3 y^4\,\mathbf{i} + x^4 y^3\,\mathbf{j}$,
C: $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (1 + t^3)\,\mathbf{j}$, $\quad 0 \leqslant t \leqslant 1$

14. $\mathbf{F}(x, y) = e^{2y}\,\mathbf{i} + (1 + 2xe^{2y})\,\mathbf{j}$,
C: $\mathbf{r}(t) = te^t\,\mathbf{i} + (1 + t)\,\mathbf{j}$, $\quad 0 \leqslant t \leqslant 1$

15. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (xy + 2z)\,\mathbf{k}$,
C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$

16. $\mathbf{F}(x, y, z) = (2xz + y^2)\,\mathbf{i} + 2xy\,\mathbf{j} + (x^2 + 3z^2)\,\mathbf{k}$,
C: $x = t^2$, $y = t + 1$, $z = 2t - 1$, $\quad 0 \leqslant t \leqslant 1$

17. $\mathbf{F}(x, y, z) = y^2 \cos z\,\mathbf{i} + 2xy \cos z\,\mathbf{j} - xy^2 \sin z\,\mathbf{k}$,
C: $\mathbf{r}(t) = t^2\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$, $\quad 0 \leqslant t \leqslant \pi$

18. $\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + xe^y\,\mathbf{j} + (z + 1)e^z\,\mathbf{k}$,
C: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $\quad 0 \leqslant t \leqslant 1$

19–20 ■ Show that the line integral is independent of path and evaluate the integral.

19. $\int_C 2x \sin y\,dx + (x^2 \cos y - 3y^2)\,dy$,
C is any path from $(-1, 0)$ to $(5, 1)$

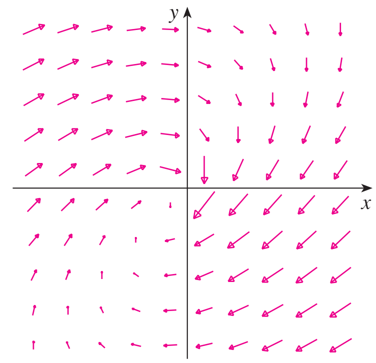
20. $\int_C (2y^2 - 12x^3 y^3)\,dx + (4xy - 9x^4 y^2)\,dy$,
C is any path from (1, 1) to (3, 2)

21–22 ■ Find the work done by the force field $\mathbf{F}$ in moving an object from P to Q.

21. $\mathbf{F}(x, y) = x^2 y^3\,\mathbf{i} + x^3 y^2\,\mathbf{j}$; $\quad P(0, 0)$, $Q(2, 1)$

22. $\mathbf{F}(x, y) = (y^2/x^2)\,\mathbf{i} - (2y/x)\,\mathbf{j}$; $\quad P(1, 1)$, $Q(4, -2)$

23. Is the vector field shown in the figure conservative? Explain.



CAS **24–25** ■ From a plot of $\mathbf{F}$ guess whether it is conservative. Then determine whether your guess is correct.

24. $\mathbf{F}(x, y) = (2xy + \sin y)\,\mathbf{i} + (x^2 + x \cos y)\,\mathbf{j}$

25. $\mathbf{F}(x, y) = \dfrac{(x - 2y)\,\mathbf{i} + (x - 2)\,\mathbf{j}}{\sqrt{1 + x^2 + y^2}}$

26. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equation.

(a) $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$ $\qquad$ (b) $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$

27. Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

28. Use Exercise 27 to show that the line integral $\int_C y\,dx + x\,dy + xyz\,dz$ is not independent of path.

29–32 ■ Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.

29. $\{(x, y) \mid x > 0, y > 0\}$ $\qquad$ **30.** $\{(x, y) \mid x \neq 0\}$

31. $\{(x, y) \mid 1 < x^2 + y^2 < 4\}$

32. $\{(x, y) \mid x^2 + y^2 \leqslant 1 \text{ or } 4 \leqslant x^2 + y^2 \leqslant 9\}$

33. Let $\mathbf{F}(x, y) = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$.
(a) Show that $\partial P/\partial y = \partial Q/\partial x$.
(b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path. [*Hint:* Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$.] Does this contradict Theorem 6?

34. (a) Suppose that **F** is an inverse square force field, that is,

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Find the work done by **F** in moving an object from a point P_1 along a path to a point P_2 in terms of the distances d_1 and d_2 from these points to the origin.

(b) An example of an inverse square field is the gravitational field $\mathbf{F} = -(mMG)\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 4 in Section 13.1. Use part (a) to find the work done by the gravitational field when Earth moves from aphelion (at a maximum distance of 1.52×10^8 km from the Sun) to perihelion (at a minimum distance of 1.47×10^8 km). (Use the values $m = 5.97 \times 10^{24}$ kg, $M = 1.99 \times 10^{30}$ kg, and $G = 6.67 \times 10^{-11}$ N·m^2/kg^2.)

(c) Another example of an inverse square field is the electric field $\mathbf{E} = \varepsilon q Q\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 5 in Section 13.1. Suppose that an electron with a charge of -1.6×10^{-19} C is located at the origin. A positive unit charge is positioned a distance 10^{-12} m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric field. (Use the value $\varepsilon = 8.985 \times 10^{10}$.)

13.4 Green's Theorem

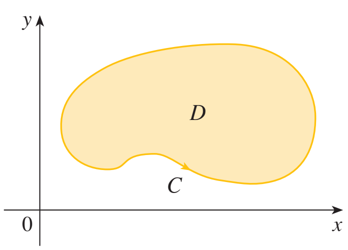


FIGURE 1

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. (See Figure 1. We assume that D consists of all points inside C as well as all points on C.) In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. Thus, if C is given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C. (See Figure 2.)

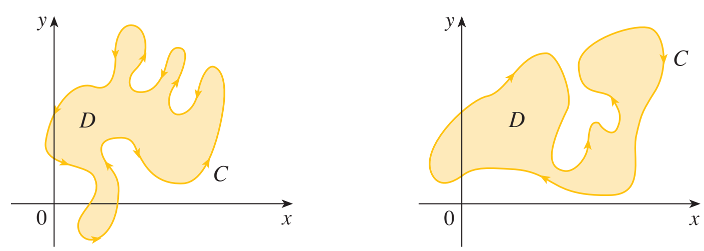


FIGURE 2

(a) Positive orientation

(b) Negative orientation

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

NOTE • The notation

$$\oint_C P\,dx + Q\,dy \qquad \text{or} \qquad \circlearrowleft\!\!\!\!\int_C P\,dx + Q\,dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C. Another notation for the positively oriented boundary curve of D is ∂D, so the equation in Green's Theorem can be written as

$$\boxed{1} \qquad \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P\,dx + Q\,dy$$

Green's Theorem should be regarded as the counterpart of the Fundamental Theorem of Calculus for double integrals. Compare Equation 1 with the statement of the Fundamental Theorem of Calculus, Part 2, in the following equation:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

In both cases there is an integral involving derivatives (F', $\partial Q/\partial x$, and $\partial P/\partial y$) on the left side of the equation. And in both cases the right side involves the values of the original functions (F, Q, and P) only on the *boundary* of the domain. (In the one-dimensional case, the domain is an interval $[a, b]$ whose boundary consists of just two points, a and b.)

▲ Green's Theorem is named after the self-taught English scientist George Green (1793–1841). He worked full-time in his father's bakery from the age of nine and taught himself mathematics from library books. In 1828 he published privately *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism,* but only 100 copies were printed and most of those went to his friends. This pamphlet contained a theorem that is equivalent to what we know as Green's Theorem, but it didn't become widely known at that time. Finally, at age 40, Green entered Cambridge University as an undergraduate but died four years after graduation. In 1846 William Thomson (Lord Kelvin) located a copy of Green's essay, realized its significance, and had it reprinted. Green was the first person to try to formulate a mathematical theory of electricity and magnetism. His work was the basis for the subsequent electromagnetic theories of Thomson, Stokes, Rayleigh, and Maxwell.

Green's Theorem is not easy to prove in the generality stated in Theorem 1, but we can give a proof for the special case where the region is both of type I and of type II (see Section 12.3). Let's call such regions **simple regions**.

Proof of Green's Theorem for the Case in Which D Is a Simple Region Notice that Green's Theorem will be proved if we can show that

$$\boxed{2} \qquad \int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$$

and

$$\boxed{3} \qquad \int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$$

We prove Equation 2 by expressing D as a type I region:

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous functions. This enables us to compute the double integral on the right side of Equation 2 as follows:

$$\boxed{4} \qquad \iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y)\,dy\,dx = \int_a^b [P(x, g_2(x)) - P(x, g_1(x))]\,dx$$

where the last step follows from the Fundamental Theorem of Calculus.

Now we compute the left side of Equation 2 by breaking up C as the union of the four curves C_1, C_2, C_3, and C_4 shown in Figure 3. On C_1 we take x as the parameter and write the parametric equations as $x = x$, $y = g_1(x)$, $a \leqslant x \leqslant b$. Thus

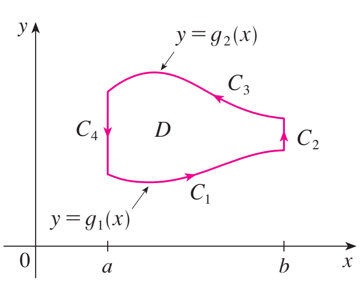


FIGURE 3

$$\int_{C_1} P(x, y)\,dx = \int_a^b P(x, g_1(x))\,dx$$

Observe that C_3 goes from right to left but $-C_3$ goes from left to right, so we can write the parametric equations of $-C_3$ as $x = x$, $y = g_2(x)$, $a \leqslant x \leqslant b$. Therefore

$$\int_{C_3} P(x, y)\,dx = -\int_{-C_3} P(x, y)\,dx = -\int_a^b P(x, g_2(x))\,dx$$

On C_2 or C_4 (either of which might reduce to just a single point), x is constant, so $dx = 0$ and

$$\int_{C_2} P(x, y)\, dx = 0 = \int_{C_4} P(x, y)\, dx$$

Hence

$$\int_C P(x, y)\, dx = \int_{C_1} P(x, y)\, dx + \int_{C_2} P(x, y)\, dx + \int_{C_3} P(x, y)\, dx + \int_{C_4} P(x, y)\, dx$$

$$= \int_a^b P(x, g_1(x))\, dx - \int_a^b P(x, g_2(x))\, dx$$

Comparing this expression with the one in Equation 4, we see that

$$\int_C P(x, y)\, dx = -\iint_D \frac{\partial P}{\partial y}\, dA$$

Equation 3 can be proved in much the same way by expressing D as a type II region (see Exercise 28). Then, by adding Equations 2 and 3, we obtain Green's Theorem. ■

EXAMPLE 1 Evaluate $\int_C x^4\, dx + xy\, dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

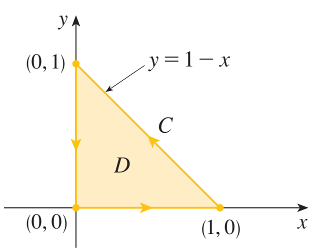


FIGURE 4

SOLUTION Although the given line integral could be evaluated as usual by the methods of Section 13.2, that would involve setting up three separate integrals along the three sides of the triangle, so let's use Green's Theorem instead. Notice that the region D enclosed by C is simple and C has positive orientation (see Figure 4). If we let $P(x, y) = x^4$ and $Q(x, y) = xy$, then we have

$$\int_C x^4\, dx + xy\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{1-x} (y - 0)\, dy\, dx$$

$$= \int_0^1 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \tfrac{1}{2}\int_0^1 (1 - x)^2\, dx$$

$$= -\tfrac{1}{6}(1 - x)^3\Big]_0^1 = \tfrac{1}{6}$$ ■

EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\, dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where C is the circle $x^2 + y^2 = 9$ oriented counterclockwise.

SOLUTION The region D bounded by C is the disk $x^2 + y^2 \leqslant 9$, so let's change to polar coordinates after applying Green's Theorem:

$$\oint_C (3y - e^{\sin x})\, dx + \left(7x + \sqrt{y^4 + 1}\right) dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(7x + \sqrt{y^4 + 1}\right) - \frac{\partial}{\partial y}(3y - e^{\sin x})\right] dA$$

$$= \int_0^{2\pi} \int_0^3 (7 - 3)\, r\, dr\, d\theta$$

$$= 4\int_0^{2\pi} d\theta \int_0^3 r\, dr = 36\pi$$ ■

▲ Instead of using polar coordinates, we could simply use the fact that D is a disk of radius 3 and write

$$\iint_D 4\, dA = 4 \cdot \pi(3)^2 = 36\pi$$

In Examples 1 and 2 we found that the double integral was easier to evaluate than the line integral. (Try setting up the line integral in Example 2 and you'll soon be convinced!) But sometimes it's easier to evaluate the line integral, and Green's Theorem is used in the reverse direction. For instance, if it is known that $P(x, y) = Q(x, y) = 0$ on the curve C, then Green's Theorem gives

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy = 0$$

no matter what values P and Q assume in the region D.

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of D is $\iint_D 1\,dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities:

$$P(x, y) = 0 \qquad P(x, y) = -y \qquad P(x, y) = -\tfrac{1}{2}y$$

$$Q(x, y) = x \qquad Q(x, y) = 0 \qquad Q(x, y) = \tfrac{1}{2}x$$

Then Green's Theorem gives the following formulas for the area of D:

5
$$A = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C x\,dy - y\,dx$$

EXAMPLE 3 Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

SOLUTION The ellipse has parametric equations $x = a\cos t$ and $y = b\sin t$, where $0 \leqslant t \leqslant 2\pi$. Using the third formula in Equation 5, we have

$$A = \tfrac{1}{2}\int_C x\,dy - y\,dx$$

$$= \tfrac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt$$

$$= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab$$

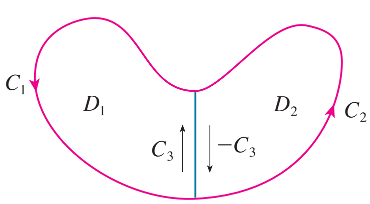


FIGURE 5

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a finite union of simple regions. For example, if D is the region shown in Figure 5, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple. The boundary of D_1 is $C_1 \cup C_3$ and the boundary of D_2 is $C_2 \cup (-C_3)$ so, applying Green's Theorem to D_1 and D_2 separately, we get

$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

If we add these two equations, the line integrals along C_3 and $-C_3$ cancel, so we get

$$\int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

which is Green's Theorem for $D = D_1 \cup D_2$, since its boundary is $C = C_1 \cup C_2$.

The same sort of argument allows us to establish Green's Theorem for any finite union of simple regions (see Figure 6).

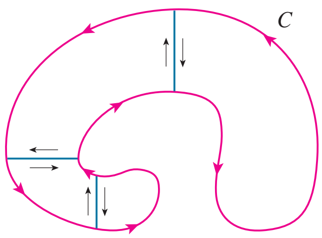


FIGURE 6

EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

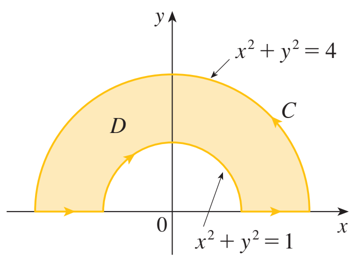


FIGURE 7

SOLUTION Notice that although D is not simple, the y-axis divides it into two simple regions (see Figure 7). In polar coordinates we can write

$$D = \{(r, \theta) \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \pi\}$$

Therefore, Green's Theorem gives

$$\begin{aligned}
\int_C y^2\,dx + 3xy\,dy &= \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2) \right] dA \\
&= \iint_D y\,dA = \int_0^{\pi} \int_1^2 (r \sin\theta)\, r\,dr\,d\theta \\
&= \int_0^{\pi} \sin\theta\,d\theta \int_1^2 r^2\,dr = \Big[-\cos\theta\Big]_0^{\pi} \Big[\tfrac{1}{3}r^3\Big]_1^2 = \frac{14}{3}
\end{aligned}$$

Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary C of the region D in Figure 8 consists of two simple closed curves C_1 and C_2. We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. Thus, the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2. If we divide D into two regions D' and D'' by means of the lines shown in Figure 9 and then apply Green's Theorem to each of D' and D'', we get

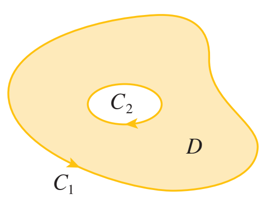


FIGURE 8

$$\begin{aligned}
\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA &= \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
&= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy
\end{aligned}$$

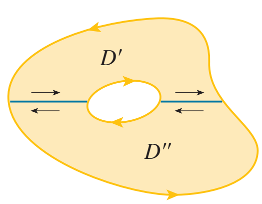


FIGURE 9

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy$$

which is Green's Theorem for the region D.

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every simple closed path that encloses the origin.

SOLUTION Since C is an *arbitrary* closed path that encloses the origin, it's difficult to compute the given integral directly. So let's consider a counterclockwise-oriented circle

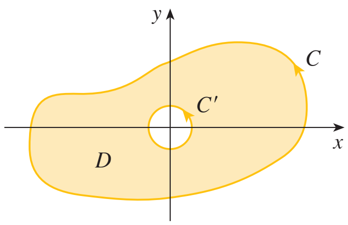


FIGURE 10

C' with center the origin and radius a, where a is chosen to be small enough that C' lies inside C. (See Figure 10.) Let D be the region bounded by C and C'. Then its positively oriented boundary is $C \cup (-C')$ and so the general version of Green's Theorem gives

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2}\right] dA$$

$$= 0$$

Therefore $$\int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$$

that is, $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$

We now easily compute this last integral using the parametrization given by $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$. Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\,dt$$

$$= \int_0^{2\pi} dt = 2\pi$$ ■

We end this section by using Green's Theorem to discuss a result that was stated in the preceding section.

Sketch of Proof of Theorem 13.3.6 We're assuming that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is a vector field on an open simply-connected region D, that P and Q have continuous first-order partial derivatives, and that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

If C is any simple closed path in D and R is the region that C encloses, then Green's Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R 0\,dA = 0$$

A curve that is not simple crosses itself at one or more points and can be broken up into a number of simple curves. We have shown that the line integrals of $\mathbf{F}$ around these simple curves are all 0 and, adding these integrals, we see that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C. Therefore, $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D by Theorem 13.3.3. It follows that $\mathbf{F}$ is a conservative vector field. ■

13.4 Exercises

1–4 ■ Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

1. $\oint_C xy^2\,dx + x^3\,dy$,
C is the rectangle with vertices (0, 0), (2, 0), (2, 3), and (0, 3)

2. $\oint_C y\,dx - x\,dy$,
C is the circle with center the origin and radius 1

3. $\oint_C xy\,dx + x^2y^3\,dy$,
C is the triangle with vertices (0, 0), (1, 0), and (1, 2)

4. $\oint_C (x^2 + y^2)\,dx + 2xy\,dy$, C consists of the arc of the parabola $y = x^2$ from (0, 0) to (2, 4) and the line segments from (2, 4) to (0, 4) and from (0, 4) to (0, 0)

CAS **5–6** ■ Verify Green's Theorem by using a computer algebra system to evaluate both the line integral and the double integral.

5. $P(x, y) = x^4y^5$, $Q(x, y) = -x^7y^6$,
C is the circle $x^2 + y^2 = 1$

6. $P(x, y) = y^2 \sin x$, $Q(x, y) = x^2 \sin y$,
C consists of the arc of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the line segment from (1, 1) to (0, 0)

7–16 ■ Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

7. $\int_C e^y\,dx + 2xe^y\,dy$,
C is the square with sides $x = 0$, $x = 1$, $y = 0$, and $y = 1$

8. $\int_C x^2y^2\,dx + 4xy^3\,dy$,
C is the triangle with vertices (0, 0), (1, 3), and (0, 3)

9. $\int_C \left(y + e^{\sqrt{x}}\right)dx + (2x + \cos y^2)\,dy$,
C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$

10. $\int_C (y^2 - \tan^{-1}x)\,dx + (3x + \sin y)\,dy$,
C is the boundary of the region enclosed by the parabola $y = x^2$ and the line $y = 4$

11. $\int_C y^3\,dx - x^3\,dy$, C is the circle $x^2 + y^2 = 4$

12. $\int_C \sin y\,dx + x\cos y\,dy$, C is the ellipse $x^2 + xy + y^2 = 1$

13. $\int_C xy\,dx + 2x^2\,dy$,
C consists of the line segment from $(-2, 0)$ to (2, 0) and the top half of the circle $x^2 + y^2 = 4$

14. $\int_C (x^3 - y^3)\,dx + (x^3 + y^3)\,dy$,
C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$

15. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (y^2 - x^2y)\,\mathbf{i} + xy^2\,\mathbf{j}$,
C consists of the circle $x^2 + y^2 = 4$ from (2, 0) to $\left(\sqrt{2}, \sqrt{2}\right)$ and the line segments from $\left(\sqrt{2}, \sqrt{2}\right)$ to (0, 0) and from (0, 0) to (2, 0)

16. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = y^6\,\mathbf{i} + xy^5\,\mathbf{j}$,
C is the ellipse $4x^2 + y^2 = 1$

17. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y)\,\mathbf{i} + xy^2\,\mathbf{j}$ in moving a particle from the origin along the x-axis to (1, 0), then along the line segment to (0, 1), and then back to the origin along the y-axis.

18. A particle starts at the point $(-2, 0)$, moves along the x-axis to (2, 0), and then along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \langle x, x^3 + 3xy^2 \rangle$.

19–20 ■ Find the area of the given region using one of the formulas in Equations 5.

19. The region bounded by the hypocycloid with vector equation $\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$

20. The region bounded by the curve with vector equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$

21. (a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), show that

$$\int_C x\,dy - y\,dx = x_1y_2 - x_2y_1$$

(b) If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, show that the area of the polygon is

$$A = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)]$$

(c) Find the area of the pentagon with vertices (0, 0), (2, 1), (1, 3), (0, 2), and $(-1, 1)$.

22. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are

$$\bar{x} = \frac{1}{2A}\oint_C x^2\,dy \qquad \bar{y} = -\frac{1}{2A}\oint_C y^2\,dx$$

where A is the area of D.

23. Use Exercise 22 to find the centroid of the triangle with vertices (0, 0), (1, 0), and (0, 1).

24. Use Exercise 22 to find the centroid of a semicircular region of radius a.

25. A plane lamina with constant density $\rho(x, y) = \rho$ occupies a region in the xy-plane bounded by a simple closed path C. Show that its moments of inertia about the axes are

$$I_x = -\frac{\rho}{3}\oint_C y^3\,dx \qquad I_y = \frac{\rho}{3}\oint_C x^3\,dy$$

26. Use Exercise 25 to find the moment of inertia of a circular disk of radius a with constant density ρ about a diameter. (Compare with Example 4 in Section 12.5.)

27. If $\mathbf{F}$ is the vector field of Example 5, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed path that does not pass through or enclose the origin.

28. Complete the proof of the special case of Green's Theorem by proving Equation 3.

29. Use Green's Theorem to prove the change of variables formula for a double integral (Formula 12.9.9) for the case where $f(x, y) = 1$:

$$\iint_R dx\, dy = \iint_S \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Here R is the region in the xy-plane that corresponds to the region S in the uv-plane under the transformation given by $x = g(u, v)$, $y = h(u, v)$.

[*Hint:* Note that the left side is $A(R)$ and apply the first part of Equation 5. Convert the line integral over ∂R to a line integral over ∂S and apply Green's Theorem in the uv-plane.]

13.5 Curl and Divergence

In this section we define two operations that can be performed on vector fields and that play a basic role in the applications of vector calculus to fluid flow and electricity and magnetism. Each operation resembles differentiation, but one produces a vector field whereas the other produces a scalar field.

Curl

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \tag{1}$$

As an aid to our memory, let's rewrite Equation 1 using operator notation. We introduce the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

It has meaning when it operates on a scalar function to produce the gradient of f:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field $\mathbf{F}$ as follows:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\ &= \text{curl}\,\mathbf{F}\end{aligned}$$

Thus, the easiest way to remember Definition 1 is by means of the symbolic expression

2
$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$

EXAMPLE 1 If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find curl $\mathbf{F}$.

SOLUTION Using Equation 2, we have

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\mathbf{k}$$

$$= (-2y - xy)\,\mathbf{i} - (0 - x)\,\mathbf{j} + (yz - 0)\,\mathbf{k}$$

$$= -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$

▲ Most computer algebra systems have commands that compute the curl and divergence of vector fields. If you have access to a CAS, use these commands to check the answers to the examples and exercises in this section.

Recall that the gradient of a function f of three variables is a vector field on $\mathbb{R}^3$ and so we can compute its curl. The following theorem says that the curl of a gradient vector field is $\mathbf{0}$.

3 Theorem If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

Proof We have

$$\text{curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z\,\partial x} - \frac{\partial^2 f}{\partial x\,\partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x\,\partial y} - \frac{\partial^2 f}{\partial y\,\partial x}\right)\mathbf{k}$$

$$= 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0}$$

by Clairaut's Theorem.

▲ Notice the similarity to what we know from Section 9.4: $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for every three-dimensional vector $\mathbf{a}$.

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, Theorem 3 can be rephrased as follows:

If $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \mathbf{0}$.

▲ Compare this with Exercise 27 in Section 13.3.

This gives us a way of verifying that a vector field is not conservative.

EXAMPLE 2 Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ is not conservative.

SOLUTION In Example 1 we showed that

$$\text{curl}\,\mathbf{F} = -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$

This shows that curl $\mathbf{F} \neq \mathbf{0}$ and so, by Theorem 3, $\mathbf{F}$ is not conservative. ■

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if $\mathbf{F}$ is defined everywhere. (More generally it is true if the domain is simply-connected, that is, "has no hole.") Theorem 4 is the three-dimensional version of Theorem 13.3.6. Its proof requires Stokes' Theorem and is sketched at the end of Section 13.7.

4 Theorem If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

EXAMPLE 3

(a) Show that $\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$ is a conservative vector field.
(b) Find a function f such that $\mathbf{F} = \nabla f$.

SOLUTION

(a) We compute the curl of $\mathbf{F}$:

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)\,\mathbf{i} - (3y^2z^2 - 3y^2z^2)\,\mathbf{j} + (2yz^3 - 2yz^3)\,\mathbf{k}$$

$$= \mathbf{0}$$

Since curl $\mathbf{F} = \mathbf{0}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^3$, $\mathbf{F}$ is a conservative vector field by Theorem 4.

(b) The technique for finding f was given in Section 13.3. We have

5 $$f_x(x, y, z) = y^2z^3$$

6 $$f_y(x, y, z) = 2xyz^3$$

7 $$f_z(x, y, z) = 3xy^2z^2$$

Integrating (5) with respect to x, we obtain

8 $$f(x, y, z) = xy^2z^3 + g(y, z)$$

Differentiating (8) with respect to y, we get $f_y(x, y, z) = 2xyz^3 + g_y(y, z)$, so comparison with (6) gives $g_y(y, z) = 0$. Thus, $g(y, z) = h(z)$ and

$$f_z(x, y, z) = 3xy^2z^2 + h'(z)$$

Then (7) gives $h'(z) = 0$. Therefore

$$f(x, y, z) = xy^2z^3 + K$$

The reason for the name *curl* is that the curl vector is associated with rotations. One connection is explained in Exercise 33. Another occurs when **F** represents the velocity field in fluid flow (see Example 3 in Section 13.1). Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of curl $\mathbf{F}(x, y, z)$ and the length of this curl vector is a measure of how quickly the particles move around the axis (see Figure 1). If curl $\mathbf{F} = \mathbf{0}$ at a point P, then the fluid is free from rotations at P and **F** is called **irrotational** at P. In other words, there is no whirlpool or eddy at P. If curl $\mathbf{F} = \mathbf{0}$, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis. If curl $\mathbf{F} \neq \mathbf{0}$, the paddle wheel rotates about its axis. We give a more detailed explanation in Section 13.7 as a consequence of Stokes' Theorem.

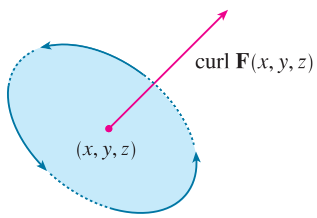


FIGURE 1

Divergence

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

9
$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Observe that curl **F** is a vector field but div **F** is a scalar field. In terms of the gradient operator $\nabla = (\partial/\partial x)\,\mathbf{i} + (\partial/\partial y)\,\mathbf{j} + (\partial/\partial z)\,\mathbf{k}$, the divergence of **F** can be written symbolically as the dot product of ∇ and **F**:

10
$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$$

EXAMPLE 4 If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find div **F**.

SOLUTION By the definition of divergence (Equation 9 or 10) we have

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2)$$
$$= z + xz$$

If **F** is a vector field on $\mathbb{R}^3$, then curl **F** is also a vector field on $\mathbb{R}^3$. As such, we can compute its divergence. The next theorem shows that the result is 0.

11 Theorem If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0$$

▲ Note the analogy with the scalar triple product: $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

Proof Using the definitions of divergence and curl, we have

$$\begin{aligned}
\operatorname{div} \operatorname{curl} \mathbf{F} &= \nabla \cdot (\nabla \times \mathbf{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \frac{\partial^2 R}{\partial x\,\partial y} - \frac{\partial^2 Q}{\partial x\,\partial z} + \frac{\partial^2 P}{\partial y\,\partial z} - \frac{\partial^2 R}{\partial y\,\partial x} + \frac{\partial^2 Q}{\partial z\,\partial x} - \frac{\partial^2 P}{\partial z\,\partial y} \\
&= 0
\end{aligned}$$

because the terms cancel in pairs by Clairaut's Theorem. ■

EXAMPLE 5 Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ can't be written as the curl of another vector field, that is, $\mathbf{F} \neq \operatorname{curl} \mathbf{G}$.

SOLUTION In Example 4 we showed that

$$\operatorname{div} \mathbf{F} = z + xz$$

and therefore $\operatorname{div} \mathbf{F} \neq 0$. If it were true that $\mathbf{F} = \operatorname{curl} \mathbf{G}$, then Theorem 11 would give

$$\operatorname{div} \mathbf{F} = \operatorname{div} \operatorname{curl} \mathbf{G} = 0$$

which contradicts $\operatorname{div} \mathbf{F} \neq 0$. Therefore, $\mathbf{F}$ is not the curl of another vector field. ■

▲ The reason for this interpretation of $\operatorname{div} \mathbf{F}$ will be explained at the end of Section 13.8 as a consequence of the Divergence Theorem.

Again, the reason for the name *divergence* can be understood in the context of fluid flow. If $\mathbf{F}(x, y, z)$ is the velocity of a fluid (or gas), then $\operatorname{div} \mathbf{F}(x, y, z)$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume. In other words, $\operatorname{div} \mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z). If $\operatorname{div} \mathbf{F} = 0$, then $\mathbf{F}$ is said to be **incompressible**.

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f. If f is a function of three variables, we have

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and this expression occurs so often that we abbreviate it as $\nabla^2 f$. The operator

$$\nabla^2 = \nabla \cdot \nabla$$

is called the **Laplace operator** because of its relation to **Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

We can also apply the Laplace operator ∇^2 to a vector field

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

in terms of its components:

$$\nabla^2 \mathbf{F} = \nabla^2 P\,\mathbf{i} + \nabla^2 Q\,\mathbf{j} + \nabla^2 R\,\mathbf{k}$$

Vector Forms of Green's Theorem

The curl and divergence operators allow us to rewrite Green's Theorem in versions that will be useful in our later work. We suppose that the plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem. Then we consider the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$. Its line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy$$

and, regarding $\mathbf{F}$ as a vector field on $\mathbb{R}^3$ with third component 0, we have

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

Therefore $$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can now rewrite the equation in Green's Theorem in the vector form

12 $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dA$$

Equation 12 expresses the line integral of the tangential component of $\mathbf{F}$ along C as the double integral of the vertical component of curl $\mathbf{F}$ over the region D enclosed by C. We now derive a similar formula involving the *normal* component of $\mathbf{F}$.

If C is given by the vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad a \leqslant t \leqslant b$$

then the unit tangent vector (see Section 10.2) is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

You can verify that the outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

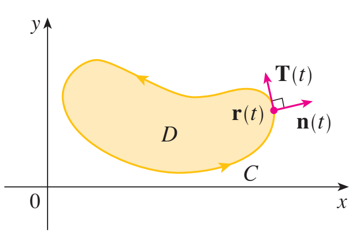


FIGURE 2

(See Figure 2.) Then, from Equation 13.2.3, we have

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (\mathbf{F} \cdot \mathbf{n})(t)\, |\mathbf{r}'(t)|\, dt$$

$$= \int_a^b \left[\frac{P(x(t), y(t))\, y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t), y(t))\, x'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)|\, dt$$

$$= \int_a^b P(x(t), y(t))\, y'(t)\, dt - Q(x(t), y(t))\, x'(t)\, dt$$

$$= \int_C P\, dy - Q\, dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

by Green's Theorem. But the integrand in this double integral is just the divergence of **F**. So we have a second vector form of Green's Theorem.

13
$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

This version says that the line integral of the normal component of **F** along C is equal to the double integral of the divergence of **F** over the region D enclosed by C.

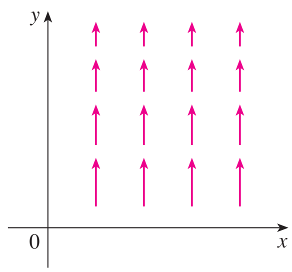
13.5 Exercises

1–6 ■ Find (a) the curl and (b) the divergence of the vector field.

1. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$
2. $\mathbf{F}(x, y, z) = (x - 2z)\,\mathbf{i} + (x + y + z)\,\mathbf{j} + (x - 2y)\,\mathbf{k}$
3. $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} - x^2y\,\mathbf{k}$
4. $\mathbf{F}(x, y, z) = xe^y\,\mathbf{j} + ye^z\,\mathbf{k}$
5. $\mathbf{F}(x, y, z) = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j} + z\,\mathbf{k}$
6. $\mathbf{F}(x, y, z) = \dfrac{x}{x^2 + y^2 + z^2}\,\mathbf{i} + \dfrac{y}{x^2 + y^2 + z^2}\,\mathbf{j} + \dfrac{z}{x^2 + y^2 + z^2}\,\mathbf{k}$

7–9 ■ The vector field **F** is shown in the xy-plane and looks the same in all other horizontal planes. (In other words, **F** is independent of z and its z-component is 0.)
(a) Is div **F** positive, negative, or zero? Explain.
(b) Determine whether curl **F** = **0**. If not, in which direction does curl **F** point?

7.



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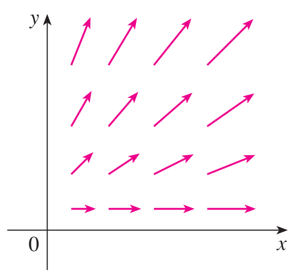

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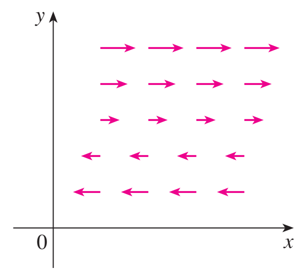


10. Let f be a scalar field and **F** a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.
(a) curl f (b) grad f
(c) div **F** (d) curl(grad f)
(e) grad **F** (f) grad(div **F**)
(g) div(grad f) (h) grad(div f)
(i) curl(curl **F**) (j) div(div **F**)
(k) (grad f) × (div **F**) (l) div(curl(grad f))

11–16 ■ Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

11. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$
12. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$
13. $\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + 2yz)\,\mathbf{j} + y^2\,\mathbf{k}$
14. $\mathbf{F}(x, y, z) = xy^2z^3\,\mathbf{i} + 2x^2yz^3\,\mathbf{j} + 3x^2y^2z^2\,\mathbf{k}$
15. $\mathbf{F}(x, y, z) = e^x\,\mathbf{i} + e^z\,\mathbf{j} + e^y\,\mathbf{k}$
16. $\mathbf{F}(x, y, z) = yze^{xz}\,\mathbf{i} + e^{xz}\,\mathbf{j} + xye^{xz}\,\mathbf{k}$

17. Is there a vector field **G** on $\mathbb{R}^3$ such that curl $\mathbf{G} = xy^2\,\mathbf{i} + yz^2\,\mathbf{j} + zx^2\,\mathbf{k}$? Explain.

18. Is there a vector field **G** on $\mathbb{R}^3$ such that curl $\mathbf{G} = yz\,\mathbf{i} + xyz\,\mathbf{j} + xy\,\mathbf{k}$? Explain.

19. Show that any vector field of the form
$$\mathbf{F}(x, y, z) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$$
where f, g, h are differentiable functions, is irrotational.

20. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(y, z)\,\mathbf{i} + g(x, z)\,\mathbf{j} + h(x, y)\,\mathbf{k}$$

is incompressible.

21–27 ■ Prove the identity, assuming that the appropriate partial derivatives exist and are continuous. If f is a scalar field and $\mathbf{F}$, $\mathbf{G}$ are vector fields, then $f\mathbf{F}$, $\mathbf{F}\cdot\mathbf{G}$, and $\mathbf{F}\times\mathbf{G}$ are defined by

$$(f\mathbf{F})(x, y, z) = f(x, y, z)\mathbf{F}(x, y, z)$$
$$(\mathbf{F}\cdot\mathbf{G})(x, y, z) = \mathbf{F}(x, y, z)\cdot\mathbf{G}(x, y, z)$$
$$(\mathbf{F}\times\mathbf{G})(x, y, z) = \mathbf{F}(x, y, z)\times\mathbf{G}(x, y, z)$$

21. $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$

22. $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G}$

23. $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$

24. $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + (\nabla f)\times\mathbf{F}$

25. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$

26. $\operatorname{div}(\nabla f\times\nabla g) = 0$

27. $\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F}$

28–30 ■ Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $r = |\mathbf{r}|$.

28. Verify each identity.
(a) $\nabla\cdot\mathbf{r} = 3$ (b) $\nabla\cdot(r\mathbf{r}) = 4r$
(c) $\nabla^2 r^3 = 12r$

29. Verify each identity.
(a) $\nabla r = \mathbf{r}/r$ (b) $\nabla\times\mathbf{r} = \mathbf{0}$
(c) $\nabla(1/r) = -\mathbf{r}/r^3$ (d) $\nabla\ln r = \mathbf{r}/r^2$

30. If $\mathbf{F} = \mathbf{r}/r^p$, find $\operatorname{div}\mathbf{F}$. Is there a value of p for which $\operatorname{div}\mathbf{F} = 0$?

31. Use Green's Theorem in the form of Equation 13 to prove **Green's first identity**:

$$\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA$$

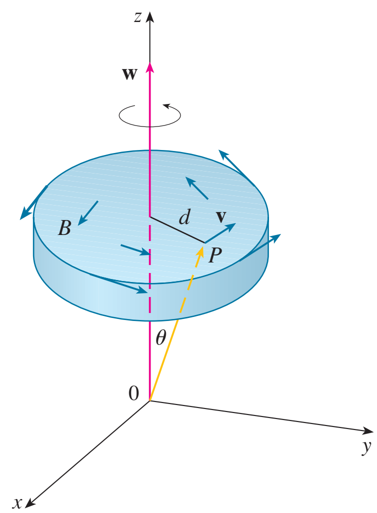
where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous. (The quantity $\nabla g\cdot\mathbf{n} = D_{\mathbf{n}}\,g$ occurs in the line integral. This is the directional derivative in the direction of the normal vector $\mathbf{n}$ and is called the **normal derivative** of g.)

32. Use Green's first identity (Exercise 31) to prove **Green's second identity**:

$$\iint_D (f\nabla^2 g - g\nabla^2 f)\,dA = \oint_C (f\nabla g - g\nabla f)\cdot\mathbf{n}\,ds$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous.

33. This exercise demonstrates a connection between the curl vector and rotations. Let B be a rigid body rotating about the z-axis. The rotation can be described by the vector $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed of B, that is, the tangential speed of any point P in B divided by the distance d from the axis of rotation. Let $\mathbf{r} = \langle x, y, z\rangle$ be the position vector of P.
(a) By considering the angle θ in the figure, show that the velocity field of B is given by $\mathbf{v} = \mathbf{w}\times\mathbf{r}$.
(b) Show that $\mathbf{v} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$.
(c) Show that $\operatorname{curl}\mathbf{v} = 2\mathbf{w}$.



34. Maxwell's equations relating the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\operatorname{div}\mathbf{E} = 0 \qquad \operatorname{div}\mathbf{H} = 0$$
$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \qquad \operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations to prove the following:

(a) $\nabla\times(\nabla\times\mathbf{E}) = -\dfrac{1}{c^2}\dfrac{\partial^2\mathbf{E}}{\partial t^2}$

(b) $\nabla\times(\nabla\times\mathbf{H}) = -\dfrac{1}{c^2}\dfrac{\partial^2\mathbf{H}}{\partial t^2}$

(c) $\nabla^2\mathbf{E} = \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{E}}{\partial t^2}$ [*Hint:* Use Exercise 27.]

(d) $\nabla^2\mathbf{H} = \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{H}}{\partial t^2}$

13.6 Surface Integrals

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length. Suppose f is a function of three variables whose domain includes a surface S. We will define the surface integral of f over S in such a way that, in the case where $f(x, y, z) = 1$, the value of the surface integral is equal to the surface area of S. We start with parametric surfaces and then deal with the special case where S is the graph of a function of two variables.

Parametric Surfaces

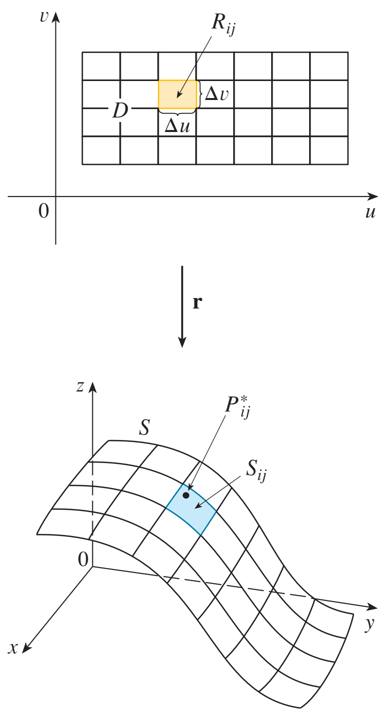


FIGURE 1

Suppose that a surface S has a vector equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

We first assume that the parameter domain D is a rectangle and we divide it into subrectangles R_{ij} with dimensions Δu and Δv. Then the surface S is divided into corresponding patches S_{ij} as in Figure 1. We evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij} of the patch, and form the Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

Then we take the limit as the number of patches increases and define the **surface integral of *f* over the surface *S*** as

1
$$\iint_S f(x, y, z)\,dS = \lim_{m,\,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

Notice the analogy with the definition of a line integral (13.2.2) and also the analogy with the definition of a double integral (12.1.5).

To evaluate the surface integral in Equation 1 we approximate the patch area ΔS_{ij} by the area of an approximating parallelogram in the tangent plane. In our discussion of surface area in Section 12.6 we made the approximation

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u\,\Delta v$$

where
$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

are the tangent vectors at a corner of S_{ij}. If the components are continuous and $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel in the interior of D, it can be shown from Definition 1, even when D is not a rectangle, that

2
$$\iint_S f(x, y, z)\,dS = \iint_D f(\mathbf{r}(u, v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

▲ We assume that the surface is covered only once as (u, v) ranges throughout D. The value of the surface integral does not depend on the parametrization that is used.

This should be compared with the formula for a line integral:

$$\int_C f(x, y, z)\, ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\, dt$$

Observe also that

$$\iint_S 1\, dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = A(S)$$

Formula 2 allows us to compute a surface integral by converting it into a double integral over the parameter domain D. When using this formula, remember that $f(\mathbf{r}(u, v))$ is evaluated by writing $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ in the formula for $f(x, y, z)$.

EXAMPLE 1 Compute the surface integral $\iint_S x^2\, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION As in Example 4 in Section 10.5, we use the parametric representation

$$x = \sin\phi\cos\theta \qquad y = \sin\phi\sin\theta \qquad z = \cos\phi \qquad 0 \leqslant \phi \leqslant \pi \qquad 0 \leqslant \theta \leqslant 2\pi$$

that is, $\qquad \mathbf{r}(\phi, \theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$

As in Example 1 in Section 12.6, we can compute that

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$$

Therefore, by Formula 2,

$$\begin{aligned}\iint_S x^2\, dS &= \iint_D (\sin\phi\cos\theta)^2\,|\mathbf{r}_\phi \times \mathbf{r}_\theta|\, dA \\ &= \int_0^{2\pi}\int_0^{\pi} \sin^2\phi\cos^2\theta\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\cos^2\theta\, d\theta\int_0^{\pi}\sin^3\phi\, d\phi \\ &= \int_0^{2\pi}\tfrac{1}{2}(1 + \cos 2\theta)\, d\theta\int_0^{\pi}(\sin\phi - \sin\phi\cos^2\phi)\, d\phi \\ &= \tfrac{1}{2}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{2\pi}\Big[-\cos\phi + \tfrac{1}{3}\cos^3\phi\Big]_0^{\pi} = \frac{4\pi}{3}\end{aligned}$$

▲ Here we use the identities

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2\phi = 1 - \cos^2\phi$$

Instead, we could use Formulas 64 and 67 in the Table of Integrals.

Surface integrals have applications similar to those for the integrals we have previously considered. For example, if a thin sheet (say, of aluminum foil) has the shape of a surface S and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$, then the total **mass** of the sheet is

$$m = \iint_S \rho(x, y, z)\, dS$$

and the **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m}\iint_S x\rho(x, y, z)\, dS \qquad \bar{y} = \frac{1}{m}\iint_S y\rho(x, y, z)\, dS \qquad \bar{z} = \frac{1}{m}\iint_S z\rho(x, y, z)\, dS$$

Moments of inertia can also be defined as before (see Exercise 35).

Graphs

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x, y)$$

and so we have $\quad \mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$

Thus

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \tag{3}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Therefore, in this case, Formula 2 becomes

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \tag{4}$$

Similar formulas apply when it is more convenient to project S onto the yz-plane or xz-plane. For instance, if S is a surface with equation $y = h(x, z)$ and D is its projection on the xz-plane, then

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, h(x, z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA$$

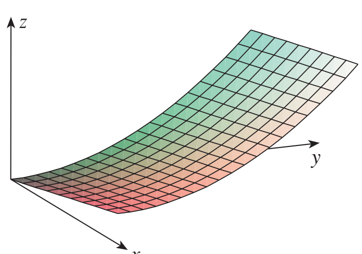


FIGURE 2

EXAMPLE 2 Evaluate $\iint_S y\, dS$, where S is the surface $z = x + y^2$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$. (See Figure 2.)

SOLUTION Since

$$\frac{\partial z}{\partial x} = 1 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 2y$$

Formula 4 gives

$$\begin{aligned}\iint_S y\, dS &= \iint_D y \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \int_0^1 \int_0^2 y\sqrt{1 + 1 + 4y^2}\, dy\, dx \\ &= \int_0^1 dx\, \sqrt{2} \int_0^2 y\sqrt{1 + 2y^2}\, dy \\ &= \sqrt{2}\left(\tfrac{1}{4}\right)\tfrac{2}{3}(1 + 2y^2)^{3/2}\Big]_0^2 = \frac{13\sqrt{2}}{3}\end{aligned}$$

■

If S is a piecewise-smooth surface, that is, a finite union of smooth surfaces S_1, $S_2, \ldots, S_n$ that intersect only along their boundaries, then the surface integral of f over S is defined by

$$\iint_S f(x, y, z)\, dS = \iint_{S_1} f(x, y, z)\, dS + \cdots + \iint_{S_n} f(x, y, z)\, dS$$

EXAMPLE 3 Evaluate $\iint_S z\, dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \leqslant 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

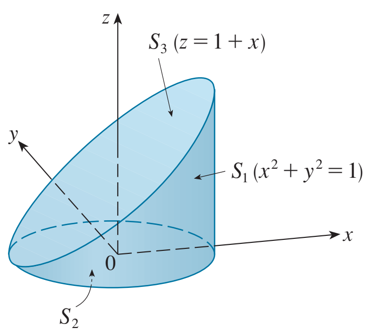


FIGURE 3

SOLUTION The surface S is shown in Figure 3. (We have changed the usual position of the axes to get a better look at S.) For S_1 we use θ and z as parameters (see Example 5 in Section 10.5) and write its parametric equations as

$$x = \cos\theta \qquad y = \sin\theta \qquad z = z$$

where

$$0 \leqslant \theta \leqslant 2\pi \qquad \text{and} \qquad 0 \leqslant z \leqslant 1 + x = 1 + \cos\theta$$

Therefore

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

and

$$|\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Thus, the surface integral over S_1 is

$$\begin{aligned} \iint_{S_1} z\, dS &= \iint_D z\,|\mathbf{r}_\theta \times \mathbf{r}_z|\, dA \\ &= \int_0^{2\pi} \int_0^{1+\cos\theta} z\, dz\, d\theta = \int_0^{2\pi} \tfrac{1}{2}(1 + \cos\theta)^2\, d\theta \\ &= \tfrac{1}{2} \int_0^{2\pi} \left[1 + 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta \\ &= \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3\pi}{2} \end{aligned}$$

Since S_2 lies in the plane $z = 0$, we have

$$\iint_{S_2} z\, dS = \iint_{S_2} 0\, dS = 0$$

The top surface S_3 lies above the unit disk D and is part of the plane $z = 1 + x$. So, taking $g(x, y) = 1 + x$ in Formula 4 and converting to polar coordinates, we have

$$\iint_{S_3} z\, dS = \iint_D (1 + x)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + r\cos\theta)\sqrt{1 + 1 + 0}\, r\, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \int_0^1 (r + r^2\cos\theta)\, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{1}{3}\cos\theta\right) d\theta = \sqrt{2}\left[\frac{\theta}{2} + \frac{\sin\theta}{3}\right]_0^{2\pi} = \sqrt{2}\pi$$

Therefore

$$\iint_S z\, dS = \iint_{S_1} z\, dS + \iint_{S_2} z\, dS + \iint_{S_3} z\, dS$$

$$= \frac{3\pi}{2} + 0 + \sqrt{2}\pi = \left(\tfrac{3}{2} + \sqrt{2}\right)\pi$$

Oriented Surfaces

In order to define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip shown in Figure 4. [It is named after the German geometer August Möbius (1790–1868).] You can construct one for yourself by taking a long rectangular strip of paper, giving it a half-twist, and taping the short edges together as in Figure 5. If an ant were to crawl along the Möbius strip starting at a point P, it would end up on the "other side" of the strip (that is, with its upper side pointing in the opposite direction). Then, if the ant continued to crawl in the same direction, it would end up back at the same point P without ever having crossed an edge. (If you have constructed a Möbius strip, try drawing a pencil line down the middle.) Therefore, a Möbius strip really has only one side.

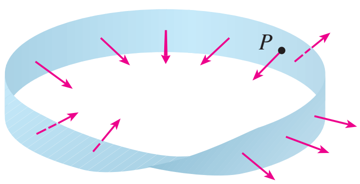


FIGURE 4
A Möbius strip

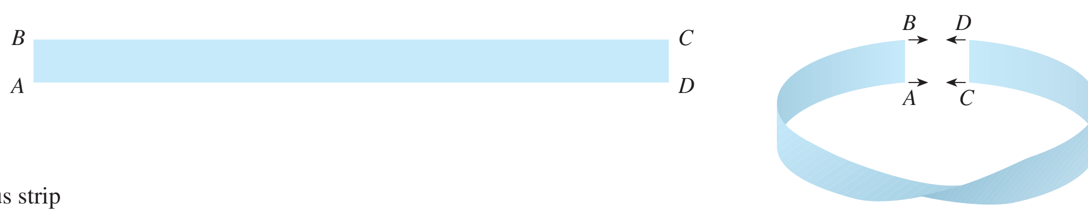


FIGURE 5
Constructing a Möbius strip

From now on we consider only orientable (two-sided) surfaces. We start with a surface S that has a tangent plane at every point (x, y, z) on S (except at any boundary point). There are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at (x, y, z). (See Figure 6.)

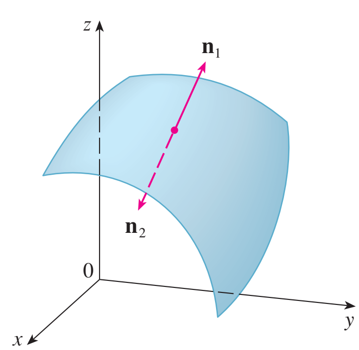


FIGURE 6

If it is possible to choose a unit normal vector **n** at every such point (x, y, z) so that **n** varies continuously over S, then S is called an **oriented surface** and the given choice of **n** provides S with an **orientation**. There are two possible orientations for any orientable surface (see Figure 7).

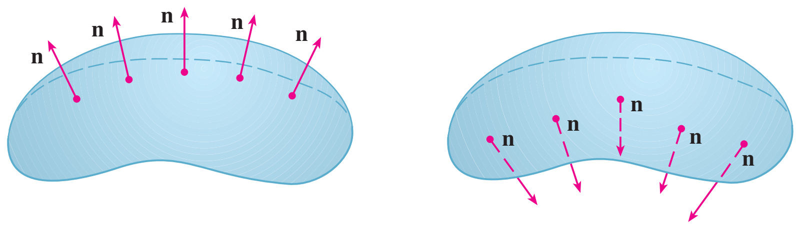


FIGURE 7
The two orientations of an orientable surface

For a surface $z = g(x, y)$ given as the graph of g, we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

5
$$\mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}}$$

Since the **k**-component is positive, this gives the *upward* orientation of the surface.

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

6
$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

and the opposite orientation is given by $-\mathbf{n}$. For instance, in Example 4 in Section 10.5 we found the parametric representation

$$\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta\,\mathbf{i} + a \sin\phi \sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}$$

for the sphere $x^2 + y^2 + z^2 = a^2$. Then in Example 1 in Section 12.6 we found that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2\sin^2\phi\cos\theta\,\mathbf{i} + a^2\sin^2\phi\sin\theta\,\mathbf{j} + a^2\sin\phi\cos\phi\,\mathbf{k}$$

and
$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2\sin\phi$$

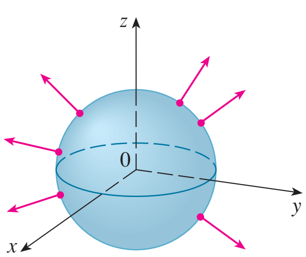


FIGURE 8
Positive orientation

So the orientation induced by $\mathbf{r}(\phi, \theta)$ is defined by the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} = \frac{1}{a}\mathbf{r}(\phi, \theta)$$

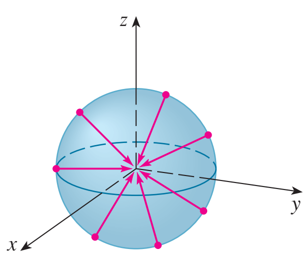


FIGURE 9
Negative orientation

Observe that **n** points in the same direction as the position vector, that is, outward from the sphere (see Figure 8). The opposite (inward) orientation would have been obtained (see Figure 9) if we had reversed the order of the parameters because $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$.

For a **closed surface**, that is, a surface that is the boundary of a solid region E, the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from E, and inward-pointing normals give the negative orientation (see Figures 8 and 9).

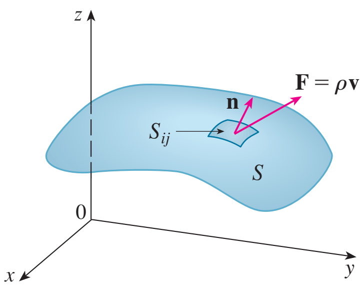


FIGURE 10

Surface Integrals of Vector Fields

Suppose that S is an oriented surface with unit normal vector $\mathbf{n}$, and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing through S. (Think of S as an imaginary surface that doesn't impede the fluid flow, like a fishing net across a stream.) Then the rate of flow (mass per unit time) per unit area is $\rho\mathbf{v}$. If we divide S into small patches S_{ij}, as in Figure 10 (compare with Figure 1), then S_{ij} is nearly planar and so we can approximate the mass of fluid crossing S_{ij} in the direction of the normal $\mathbf{n}$ per unit time by the quantity

$$(\rho\mathbf{v} \cdot \mathbf{n})A(S_{ij})$$

where ρ, $\mathbf{v}$, and $\mathbf{n}$ are evaluated at some point on S_{ij}. (Recall that the component of the vector $\rho\mathbf{v}$ in the direction of the unit vector $\mathbf{n}$ is $\rho\mathbf{v} \cdot \mathbf{n}$.) By summing these quantities and taking the limit we get, according to Definition 1, the surface integral of the function $\rho\mathbf{v} \cdot \mathbf{n}$ over S:

7 $$\iint_S \rho\mathbf{v} \cdot \mathbf{n}\, dS = \iint_S \rho(x, y, z)\mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z)\, dS$$

and this is interpreted physically as the rate of flow through S.

If we write $\mathbf{F} = \rho\mathbf{v}$, then $\mathbf{F}$ is also a vector field on $\mathbb{R}^3$ and the integral in Equation 7 becomes

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

A surface integral of this form occurs frequently in physics, even when $\mathbf{F}$ is not $\rho\mathbf{v}$, and is called the *surface integral* (or *flux integral*) of $\mathbf{F}$ over S.

8 Definition If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the **surface integral of F over *S*** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

In words, Definition 8 says that the surface integral of a vector field over S is equal to the surface integral of its normal component over S (as previously defined).

If S is given by a vector function $\mathbf{r}(u, v)$, then $\mathbf{n}$ is given by Equation 6, and from Definition 8 and Equation 2 we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS$$

$$= \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where D is the parameter domain. Thus, we have

▲ Compare Equation 9 to the similar expression for evaluating line integrals of vector fields in Definition 13.2.13:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

▲ Figure 11 shows the vector field $\mathbf{F}$ in Example 4 at points on the unit sphere.

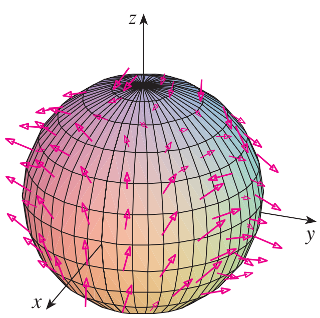


FIGURE 11

9
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

EXAMPLE 4 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION Using the parametric representation

$$\mathbf{r}(\phi, \theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} \qquad 0 \leqslant \phi \leqslant \pi \qquad 0 \leqslant \theta \leqslant 2\pi$$

we have

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) = \cos\phi\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\theta\,\mathbf{k}$$

and, from Example 1 in Section 12.6,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k}$$

Therefore

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \cos\phi\sin^2\phi\cos\theta + \sin^3\phi\sin^2\theta + \sin^2\phi\cos\phi\cos\theta$$

and, by Formula 9, the flux is

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, dA \\
&= \int_0^{2\pi} \int_0^{\pi} (2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\, d\phi\, d\theta \\
&= 2\int_0^{\pi} \sin^2\phi\cos\phi\, d\phi \int_0^{2\pi} \cos\theta\, d\theta + \int_0^{\pi} \sin^3\phi\, d\phi \int_0^{2\pi} \sin^2\theta\, d\theta \\
&= 0 + \int_0^{\pi} \sin^3\phi\, d\phi \int_0^{2\pi} \sin^2\theta\, d\theta \qquad \left(\text{since } \int_0^{2\pi} \cos\theta\, d\theta = 0\right) \\
&= \frac{4\pi}{3}
\end{aligned}$$

by the same calculation as in Example 1. ■

If, for instance, the vector field in Example 4 is a velocity field describing the flow of a fluid with density 1, then the answer, $4\pi/3$, represents the rate of flow through the unit sphere in units of mass per unit time.

In the case of a surface S given by a graph $z = g(x, y)$, we can think of x and y as parameters and use Equation 3 to write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\,\mathbf{i} - \frac{\partial g}{\partial y}\,\mathbf{j} + \mathbf{k}\right)$$

Thus, Formula 9 becomes

10
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

This formula assumes the upward orientation of S; for a downward orientation we multiply by -1. Similar formulas can be worked out if S is given by $y = h(x, z)$ or $x = k(y, z)$. (See Exercises 31 and 32.)

EXAMPLE 5 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

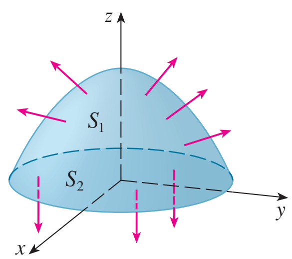


FIGURE 12

SOLUTION S consists of a parabolic top surface S_1 and a circular bottom surface S_2. (See Figure 12.) Since S is a closed surface, we use the convention of positive (outward) orientation. This means that S_1 is oriented upward and we can use Equation 10 with D being the projection of S_1 on the xy-plane, namely, the disk $x^2 + y^2 \leqslant 1$. Since

$$P(x, y, z) = y \qquad Q(x, y, z) = x \qquad R(x, y, z) = z = 1 - x^2 - y^2$$

on S_1 and

$$\frac{\partial g}{\partial x} = -2x \qquad \frac{\partial g}{\partial y} = -2y$$

we have

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA \\
&= \iint_D \left[-y(-2x) - x(-2y) + 1 - x^2 - y^2 \right] dA \\
&= \iint_D (1 + 4xy - x^2 - y^2)\, dA \\
&= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos\theta \sin\theta - r^2)\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 (r - r^3 + 4r^3 \cos\theta \sin\theta)\, dr\, d\theta \\
&= \int_0^{2\pi} \left(\tfrac{1}{4} + \cos\theta \sin\theta \right) d\theta = \tfrac{1}{4}(2\pi) + 0 = \frac{\pi}{2}
\end{aligned}$$

The disk S_2 is oriented downward, so its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and we have

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot (-\mathbf{k})\, dS = \iint_D (-z)\, dA = \iint_D 0\, dA = 0$$

since $z = 0$ on S_2. Finally, we compute, by definition, $\iint_S \mathbf{F} \cdot d\mathbf{S}$ as the sum of the surface integrals of $\mathbf{F}$ over the pieces S_1 and S_2:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

■

Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations. For instance, if $\mathbf{E}$ is an electric field (see Example 5 in Section 13.1), then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of **E** through the surface S. One of the important laws of electrostatics is **Gauss's Law**, which says that the net charge enclosed by a closed surface S is

11
$$Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where ε_0 is a constant (called the permittivity of free space) that depends on the units used. (In the SI system, $\varepsilon_0 \approx 8.8542 \times 10^{-12}\ \mathrm{C^2/N{\cdot}m^2}$.) Therefore, if the vector field **F** in Example 4 represents an electric field, we can conclude that the charge enclosed by S is $Q = 4\pi\varepsilon_0/3$.

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K\,\nabla u$$

where K is an experimentally determined constant called the **conductivity** of the substance. The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$

EXAMPLE 6 The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

SOLUTION Taking the center of the ball to be at the origin, we have

$$u(x, y, z) = C(x^2 + y^2 + z^2)$$

where C is the proportionality constant. Then the heat flow is

$$\mathbf{F}(x, y, z) = -K\,\nabla u = -KC(2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k})$$

where K is the conductivity of the metal. Instead of using the usual parametrization of the sphere as in Example 4, we observe that the outward unit normal to the sphere $x^2 + y^2 + z^2 = a^2$ at the point (x, y, z) is

$$\mathbf{n} = \frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

and so
$$\mathbf{F} \cdot \mathbf{n} = -\frac{2KC}{a}(x^2 + y^2 + z^2)$$

But on S we have $x^2 + y^2 + z^2 = a^2$, so $\mathbf{F} \cdot \mathbf{n} = -2aKC$. Therefore, the rate of heat flow across S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -2aKC \iint_S dS$$
$$= -2aKCA(S) = -2aKC(4\pi a^2) = -8KC\pi a^3$$ ■

13.6 Exercises

1. Let S be the cube with vertices $(\pm 1, \pm 1, \pm 1)$. Approximate $\iint_S \sqrt{x^2 + 2y^2 + 3z^2}\, dS$ by using a Riemann sum as in Definition 1, taking the patches S_{ij} to be the squares that are the faces of the cube and the points P_{ij}^* to be the centers of the squares.

2. A surface S consists of the cylinder $x^2 + y^2 = 1$, $-1 \leqslant z \leqslant 1$, together with its top and bottom disks. Suppose you know that f is a continuous function with $f(\pm 1, 0, 0) = 2$, $f(0, \pm 1, 0) = 3$, and $f(0, 0, \pm 1) = 4$. Estimate the value of $\iint_S f(x, y, z)\, dS$ by using a Riemann sum, taking the patches S_{ij} to be four quarter-cylinders and the top and bottom disks.

3. Let H be the hemisphere $x^2 + y^2 + z^2 = 50$, $z \geqslant 0$, and suppose f is a continuous function with $f(3, 4, 5) = 7$, $f(3, -4, 5) = 8$, $f(-3, 4, 5) = 9$, and $f(-3, -4, 5) = 12$. By dividing H into four patches, estimate the value of $\iint_H f(x, y, z)\, dS$.

4. Suppose that $f(x, y, z) = g\left(\sqrt{x^2 + y^2 + z^2}\right)$, where g is a function of one variable such that $g(2) = -5$. Evaluate $\iint_S f(x, y, z)\, dS$, where S is the sphere $x^2 + y^2 + z^2 = 4$.

5–18 ■ Evaluate the surface integral.

5. $\iint_S yz\, dS$,
S is the surface with parametric equations $x = uv$, $y = u + v$, $z = u - v$, $u^2 + v^2 \leqslant 1$

6. $\iint_S \sqrt{1 + x^2 + y^2}\, dS$,
S is the helicoid with vector equation $\mathbf{r}(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + v\, \mathbf{k}$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$

7. $\iint_S x^2 yz\, dS$,
S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$

8. $\iint_S xy\, dS$,
S is the triangular region with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$

9. $\iint_S yz\, dS$,
S is the part of the plane $x + y + z = 1$ that lies in the first octant

10. $\iint_S y\, dS$,
S is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

11. $\iint_S x\, dS$,
S is the surface $y = x^2 + 4z$, $0 \leqslant x \leqslant 2$, $0 \leqslant z \leqslant 2$

12. $\iint_S (y^2 + z^2)\, dS$,
S is the part of the paraboloid $x = 4 - y^2 - z^2$ that lies in front of the plane $x = 0$

13. $\iint_S yz\, dS$,
S is the part of the plane $z = y + 3$ that lies inside the cylinder $x^2 + y^2 = 1$

14. $\iint_S xy\, dS$,
S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$

15. $\iint_S (x^2 z + y^2 z)\, dS$,
S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$

16. $\iint_S xyz\, dS$,
S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$

17. $\iint_S (x^2 y + z^2)\, dS$,
S is the part of the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$

18. $\iint_S (x^2 + y^2 + z^2)\, dS$,
S consists of the cylinder in Exercise 17 together with its top and bottom disks

19–27 ■ Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the flux of $\mathbf{F}$ across S. For closed surfaces, use the positive (outward) orientation.

19. $\mathbf{F}(x, y, z) = xy\, \mathbf{i} + yz\, \mathbf{j} + zx\, \mathbf{k}$, S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and has upward orientation

20. $\mathbf{F}(x, y, z) = y\, \mathbf{i} + x\, \mathbf{j} + z^2\, \mathbf{k}$,
S is the helicoid of Exercise 6 with upward orientation

21. $\mathbf{F}(x, y, z) = xze^y\, \mathbf{i} - xze^y\, \mathbf{j} + z\, \mathbf{k}$,
S is the part of the plane $x + y + z = 1$ in the first octant and has downward orientation

22. $\mathbf{F}(x, y, z) = x\, \mathbf{i} + y\, \mathbf{j} + z^4\, \mathbf{k}$,
S is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation

23. $\mathbf{F}(x, y, z) = x\, \mathbf{i} + y\, \mathbf{j} + z\, \mathbf{k}$,
S is the sphere $x^2 + y^2 + z^2 = 9$

24. $\mathbf{F}(x, y, z) = -y\, \mathbf{i} + x\, \mathbf{j} + 3z\, \mathbf{k}$, S is the hemisphere $z = \sqrt{16 - x^2 - y^2}$ with upward orientation

25. $\mathbf{F}(x, y, z) = y\, \mathbf{j} - z\, \mathbf{k}$,
S consists of the paraboloid $y = x^2 + z^2$, $0 \leqslant y \leqslant 1$, and the disk $x^2 + z^2 \leqslant 1$, $y = 1$

26. $\mathbf{F}(x, y, z) = x\, \mathbf{i} + y\, \mathbf{j} + 5\, \mathbf{k}$,
S is the surface of Exercise 14

27. $\mathbf{F}(x, y, z) = x\, \mathbf{i} + 2y\, \mathbf{j} + 3z\, \mathbf{k}$,
S is the cube with vertices $(\pm 1, \pm 1, \pm 1)$

CAS **28.** Let S be the surface $z = xy$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.
(a) Evaluate $\iint_S xyz\, dS$ correct to four decimal places.
(b) Find the exact value of $\iint_S x^2yz\, dS$.

CAS **29.** Find the value of $\iint_S x^2y^2z^2\, dS$ correct to four decimal places, where S is the part of the paraboloid $z = 3 - 2x^2 - y^2$ that lies above the xy-plane.

CAS **30.** Find the flux of $\mathbf{F}(x, y, z) = \sin(xyz)\,\mathbf{i} + x^2y\,\mathbf{j} + z^2e^{x/5}\,\mathbf{k}$ across the part of the cylinder $4y^2 + z^2 = 4$ that lies above the xy-plane and between the planes $x = -2$ and $x = 2$ with upward orientation. Illustrate by using a computer algebra system to draw the cylinder and the vector field on the same screen.

31. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $y = h(x, z)$ and $\mathbf{n}$ is the unit normal that points toward the left.

32. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $x = k(y, z)$ and $\mathbf{n}$ is the unit normal that points forward (that is, toward the viewer when the axes are drawn in the usual way).

33. Find the center of mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geqslant 0$, if it has constant density.

34. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leqslant z \leqslant 4$, if its density function is $\rho(x, y, z) = 10 - z$.

35. (a) Give an integral expression for the moment of inertia I_z about the z-axis of a thin sheet in the shape of a surface S if the density function is ρ.
(b) Find the moment of inertia about the z-axis of the funnel in Exercise 34.

36. The conical surface $z^2 = x^2 + y^2$, $0 \leqslant z \leqslant a$, has constant density k. Find (a) the center of mass and (b) the moment of inertia about the z-axis.

37. A fluid with density 1200 flows with velocity $\mathbf{v} = y\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k}$. Find the rate of flow upward through the paraboloid $z = 9 - \frac{1}{4}(x^2 + y^2)$, $x^2 + y^2 \leqslant 36$.

38. A fluid has density 1500 and velocity field $\mathbf{v} = -y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k}$. Find the rate of flow outward through the sphere $x^2 + y^2 + z^2 = 25$.

39. Use Gauss's Law to find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \leqslant a^2$, $z \geqslant 0$, if the electric field is $\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$.

40. Use Gauss's Law to find the charge enclosed by the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is $\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

41. The temperature at the point (x, y, z) in a substance with conductivity $K = 6.5$ is $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the cylindrical surface $y^2 + z^2 = 6$, $0 \leqslant x \leqslant 4$.

42. The temperature at a point in a ball with conductivity K is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

13.7 Stokes' Theorem

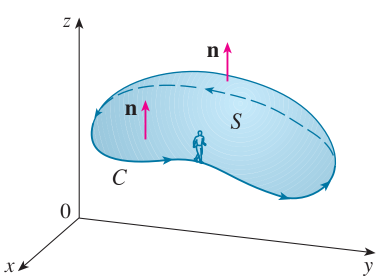


FIGURE 1

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve). Figure 1 shows an oriented surface with unit normal vector $\mathbf{n}$. The orientation of S induces the **positive orientation of the boundary curve C** shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of $\mathbf{n}$, then the surface will always be on your left.

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

▲ Stokes' Theorem is named after the Irish mathematical physicist Sir George Stokes (1819–1903). Stokes was a professor at Cambridge University (in fact he held the same position as Newton, Lucasian Professor of Mathematics) and was especially noted for his studies of fluid flow and light. What we call Stokes' Theorem was actually discovered by the Scottish physicist Sir William Thomson (1824–1907, known as Lord Kelvin). Stokes learned of this theorem in a letter from Thomson in 1850 and asked students to prove it on an examination at Cambridge University in 1854. We don't know if any of those students was able to do so.

Since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \qquad \text{and} \qquad \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl}\, \mathbf{F} \cdot \mathbf{n}\, dS$$

Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of $\mathbf{F}$ is equal to the surface integral of the normal component of the curl of $\mathbf{F}$.

The positively oriented boundary curve of the oriented surface S is often written as ∂S, so Stokes' Theorem can be expressed as

$$\boxed{1} \qquad \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

There is an analogy among Stokes' Theorem, Green's Theorem, and the Fundamental Theorem of Calculus. As before, there is an integral involving derivatives on the left side of Equation 1 (recall that curl $\mathbf{F}$ is a sort of derivative of $\mathbf{F}$) and the right side involves the values of $\mathbf{F}$ only on the *boundary* of S.

In fact, in the special case where the surface S is flat and lies in the xy-plane with upward orientation, the unit normal is $\mathbf{k}$, the surface integral becomes a double integral, and Stokes' Theorem becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \iint_S (\text{curl}\, \mathbf{F}) \cdot \mathbf{k}\, dA$$

This is precisely the vector form of Green's Theorem given in Equation 13.5.12. Thus, we see that Green's Theorem is really a special case of Stokes' Theorem.

Although Stokes' Theorem is too difficult for us to prove in its full generality, we can give a proof when S is a graph and $\mathbf{F}$, S, and C are well behaved.

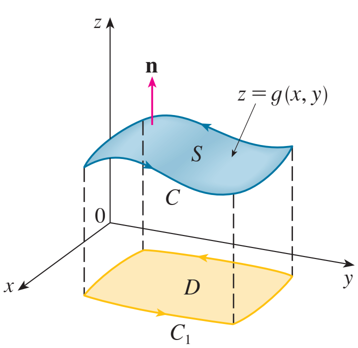


FIGURE 2

Proof of a Special Case of Stokes' Theorem We assume that the equation of S is $z = g(x, y)$, $(x, y) \in D$, where g has continuous second-order partial derivatives and D is a simple plane region whose boundary curve C_1 corresponds to C. If the orientation of S is upward, then the positive orientation of C corresponds to the positive orientation of C_1. (See Figure 2.) We are given that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, where the partial derivatives of P, Q, and R are continuous.

Since S is a graph of a function, we can apply Formula 13.6.10 with $\mathbf{F}$ replaced by curl $\mathbf{F}$. The result is

$$\boxed{2} \qquad \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$$

$$= \iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right] dA$$

where the partial derivatives of P, Q, and R are evaluated at $(x, y, g(x, y))$. If

$$x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

is a parametric representation of C_1, then a parametric representation of C is

$$x = x(t) \qquad y = y(t) \qquad z = g(x(t), y(t)) \qquad a \leqslant t \leqslant b$$

This allows us, with the aid of the Chain Rule, to evaluate the line integral as follows:

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt \\
&= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \right) \right] dt \\
&= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x} \right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y} \right)\frac{dy}{dt} \right] dt \\
&= \int_{C_1} \left(P + R\frac{\partial z}{\partial x} \right) dx + \left(Q + R\frac{\partial z}{\partial y} \right) dy \\
&= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x} \right) \right] dA
\end{aligned}$$

where we have used Green's Theorem in the last step. Then, using the Chain Rule again and remembering that P, Q, and R are functions of x, y, and z and that z is itself a function of x and y, we get

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \Bigg[&\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x\,\partial y} \right) \\
&- \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y\,\partial x} \right) \Bigg] dA
\end{aligned}$$

Four of the terms in this double integral cancel and the remaining six terms can be arranged to coincide with the right side of Equation 2. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

■

EXAMPLE 1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)

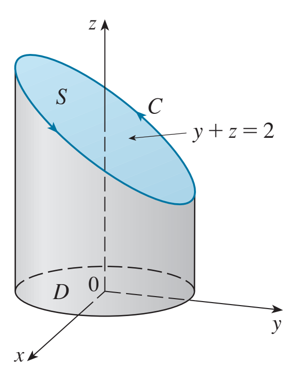


FIGURE 3

SOLUTION The curve C (an ellipse) is shown in Figure 3. Although $\int_C \mathbf{F} \cdot d\mathbf{r}$ could be evaluated directly, it's easier to use Stokes' Theorem. We first compute

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y)\,\mathbf{k}$$

Although there are many surfaces with boundary C, the most convenient choice is the elliptical region S in the plane $y + z = 2$ that is bounded by C. If we orient S upward, then C has the induced positive orientation. The projection D of S on the

xy-plane is the disk $x^2 + y^2 \leqslant 1$ and so using Equation 13.6.10 with $z = g(x, y) = 2 - y$, we have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y)\, dA \\ &= \int_0^{2\pi} \int_0^1 (1 + 2r \sin \theta)\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3} \sin \theta \right]_0^1 d\theta = \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{2}{3} \sin \theta \right) d\theta \\ &= \tfrac{1}{2}(2\pi) + 0 = \pi\end{aligned}$$

■

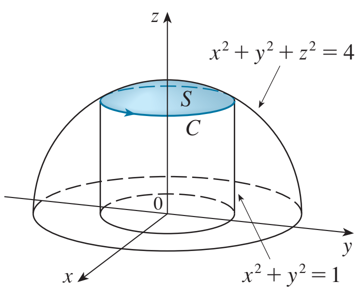


FIGURE 4

EXAMPLE 2 Use Stokes' Theorem to compute the integral $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. (See Figure 4.)

SOLUTION To find the boundary curve C we solve the equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$. Subtracting, we get $z^2 = 3$ and so $z = \sqrt{3}$ (since $z > 0$). Thus, C is the circle given by the equations $x^2 + y^2 = 1$, $z = \sqrt{3}$. A vector equation of C is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sqrt{3}\,\mathbf{k} \qquad 0 \leqslant t \leqslant 2\pi$$

so

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Also, we have

$$\mathbf{F}(\mathbf{r}(t)) = \sqrt{3} \sin t\,\mathbf{i} + \sqrt{3} \cos t\,\mathbf{j} + \cos t \sin t\,\mathbf{k}$$

Therefore, by Stokes' Theorem,

$$\begin{aligned}\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^{2\pi} \left(-\sqrt{3} \sin^2 t + \sqrt{3} \cos^2 t \right) dt \\ &= \sqrt{3} \int_0^{2\pi} \cos 2t\, dt = 0\end{aligned}$$

■

Note that in Example 2 we computed a surface integral simply by knowing the values of $\mathbf{F}$ on the boundary curve C. This means that if we have another oriented surface with the same boundary curve C, then we get exactly the same value for the surface integral!

In general, if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then

3 $$\iint_{S_1} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.

We now use Stokes' Theorem to throw some light on the meaning of the curl vector. Suppose that C is an oriented closed curve and $\mathbf{v}$ represents the velocity field in fluid flow. Consider the line integral

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_C \mathbf{v} \cdot \mathbf{T}\, ds$$

and recall that $\mathbf{v} \cdot \mathbf{T}$ is the component of $\mathbf{v}$ in the direction of the unit tangent vector $\mathbf{T}$. This means that the closer the direction of $\mathbf{v}$ is to the direction of $\mathbf{T}$, the larger the value of $\mathbf{v} \cdot \mathbf{T}$. Thus, $\int_C \mathbf{v} \cdot d\mathbf{r}$ is a measure of the tendency of the fluid to move around C and is called the **circulation** of $\mathbf{v}$ around C. (See Figure 5.)

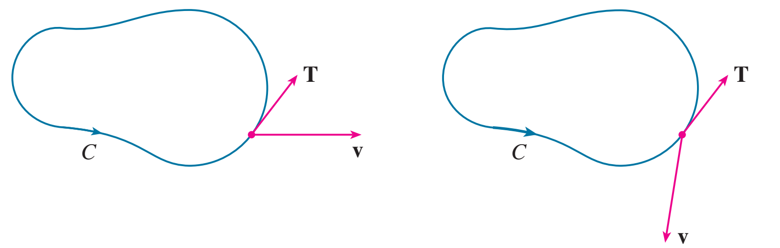


FIGURE 5 (a) $\int_C \mathbf{v} \cdot d\mathbf{r} > 0$, positive circulation (b) $\int_C \mathbf{v} \cdot d\mathbf{r} < 0$, negative circulation

Now let $P_0(x_0, y_0, z_0)$ be a point in the fluid and let S_a be a small disk with radius a and center P_0. Then $(\text{curl }\mathbf{F})(P) \approx (\text{curl }\mathbf{F})(P_0)$ for all points P on S_a because curl $\mathbf{F}$ is continuous. Thus, by Stokes' Theorem, we get the following approximation to the circulation around the boundary circle C_a:

$$\int_{C_a} \mathbf{v} \cdot d\mathbf{r} = \iint_{S_a} \text{curl }\mathbf{v} \cdot d\mathbf{S} = \iint_{S_a} \text{curl }\mathbf{v} \cdot \mathbf{n}\, dS$$

$$\approx \iint_{S_a} \text{curl }\mathbf{v}(P_0) \cdot \mathbf{n}(P_0)\, dS = \text{curl }\mathbf{v}(P_0) \cdot \mathbf{n}(P_0)\pi a^2$$

▲ Imagine a tiny paddle wheel placed in the fluid at a point P, as in Figure 6; the paddle wheel rotates fastest when its axis is parallel to curl $\mathbf{v}$.

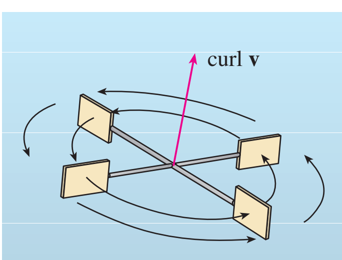


FIGURE 6

This approximation becomes better as $a \to 0$ and we have

4 $$\text{curl }\mathbf{v}(P_0) \cdot \mathbf{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{v} \cdot d\mathbf{r}$$

Equation 4 gives the relationship between the curl and the circulation. It shows that curl $\mathbf{v} \cdot \mathbf{n}$ is a measure of the rotating effect of the fluid about the axis $\mathbf{n}$. The curling effect is greatest about the axis parallel to curl $\mathbf{v}$.

Finally, we mention that Stokes' Theorem can be used to prove Theorem 13.5.4 (which states that if curl $\mathbf{F} = \mathbf{0}$ on all of $\mathbb{R}^3$, then $\mathbf{F}$ is conservative). From our previous work (Theorems 13.3.3 and 13.3.4), we know that $\mathbf{F}$ is conservative if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C. Given C, suppose we can find an orientable surface S whose boundary is C. (This can be done, but the proof requires advanced techniques.) Then Stokes' Theorem gives

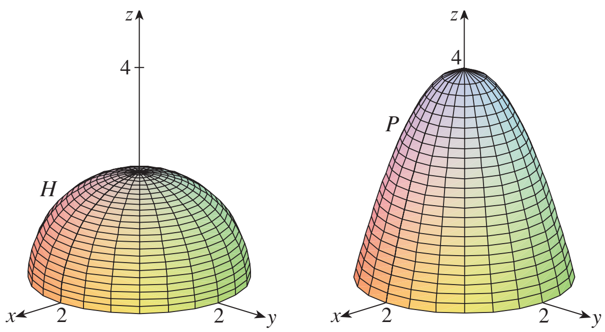
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl }\mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$$

A curve that is not simple can be broken into a number of simple curves, and the integrals around these simple curves are all 0. Adding these integrals, we obtain $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C.

13.7 Exercises

1. A hemisphere H and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



2–6 ■ Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

2. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$,
S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented upward

3. $\mathbf{F}(x, y, z) = x^2 e^{yz}\,\mathbf{i} + y^2 e^{xz}\,\mathbf{j} + z^2 e^{xy}\,\mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$, oriented upward

4. $\mathbf{F}(x, y, z) = (x + \tan^{-1}yz)\,\mathbf{i} + y^2 z\,\mathbf{j} + z\,\mathbf{k}$,
S is the part of the hemisphere $x = \sqrt{9 - y^2 - z^2}$ that lies inside the cylinder $y^2 + z^2 = 4$, oriented in the direction of the positive x-axis

5. $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + xy\,\mathbf{j} + x^2yz\,\mathbf{k}$,
S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm1, \pm1, \pm1)$, oriented outward [*Hint:* Use Equation 3.]

6. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + e^z\,\mathbf{j} + xy^2\,\mathbf{k}$,
S consists of the four sides of the pyramid with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, and $(0, 1, 0)$ that lie to the right of the xz-plane, oriented in the direction of the positive y-axis [*Hint:* Use Equation 3.]

7–10 ■ Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

7. $\mathbf{F}(x, y, z) = (x + y^2)\,\mathbf{i} + (y + z^2)\,\mathbf{j} + (z + x^2)\,\mathbf{k}$,
C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

8. $\mathbf{F}(x, y, z) = e^{-x}\,\mathbf{i} + e^x\,\mathbf{j} + e^z\,\mathbf{k}$,
C is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant

9. $\mathbf{F}(x, y, z) = 2z\,\mathbf{i} + 4x\,\mathbf{j} + 5y\,\mathbf{k}$,
C is the curve of intersection of the plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$

10. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2 + y^2)\,\mathbf{k}$,
C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant

11. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x^2 z\,\mathbf{i} + xy^2\,\mathbf{j} + z^2\,\mathbf{k}$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise as viewed from above.

(b) Graph both the plane and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

12. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2 y\,\mathbf{i} + \frac{1}{3}x^3\,\mathbf{j} + xy\,\mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$ oriented counterclockwise as viewed from above.

(b) Graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

13–15 ■ Verify that Stokes' Theorem is true for the given vector field $\mathbf{F}$ and surface S.

13. $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$,
S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, oriented upward

14. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + xyz\,\mathbf{k}$,
S is the part of the plane $2x + y + z = 2$ that lies in the first octant, oriented upward

15. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geqslant 0$, oriented in the direction of the positive y-axis

16. Let

$$\mathbf{F}(x, y, z) = \langle ax^3 - 3xz^2, x^2y + by^3, cz^3 \rangle$$

Let C be the curve in Exercise 12 and consider all possible smooth surfaces S whose boundary curve is C. Find the values of a, b, and c for which $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is independent of the choice of S.

17. Calculate the work done by the force field

$$\mathbf{F}(x, y, z) = (x^x + z^2)\,\mathbf{i} + (y^y + x^2)\,\mathbf{j} + (z^z + y^2)\,\mathbf{k}$$

when a particle moves under its influence around the edge of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant, in a counterclockwise direction as viewed from above.

18. Evaluate $\int_C (y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz$, where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$, $0 \leqslant t \leqslant 2\pi$. [*Hint:* Observe that C lies on the surface $z = 2xy$.]

19. If S is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

20. Suppose S and C satisfy the hypotheses of Stokes' Theorem and f, g have continuous second-order partial derivatives. Use Exercises 22 and 24 in Section 13.5 to show the following.

(a) $\int_C (f \nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$

(b) $\int_C (f \nabla f) \cdot d\mathbf{r} = 0$

(c) $\int_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} = 0$

Writing Project

Three Men and Two Theorems

Although two of the most important theorems in vector calculus are named after George Green and George Stokes, a third man, William Thomson (also known as Lord Kelvin), played a large role in the formulation, dissemination, and application of both of these results. All three men were interested in how the two theorems could help to explain and predict physical phenomena in electricity and magnetism and fluid flow. The basic facts of the story are given in the margin notes on pages 946 and 972.

Write a report on the historical origins of Green's Theorem and Stokes' Theorem. Explain the similarities and relationship between the theorems. Discuss the roles that Green, Thomson, and Stokes played in discovering these theorems and making them widely known. Show how both theorems arose from the investigation of electricity and magnetism and were later used to study a variety of physical problems.

The dictionary edited by Gillispie [2] is a good source for both biographical and scientific information. The book by Hutchinson [5] gives an account of Stokes' life and the book by Thompson [8] is a biography of Lord Kelvin. The articles by Grattan-Guinness [3] and Gray [4] and the book by Cannell [1] give background on the extraordinary life and works of Green. Additional historical and mathematical information is found in the books by Katz [6] and Kline [7].

▲ The photograph shows a stained-glass window at Cambridge University in honor of George Green.

1. D. M. Cannell, *George Green, Mathematician and Physicist 1793–1841: The Background to his Life and Work* (London: Athlone Press, 1993).

2. C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). See the article on Green by P. J. Wallis in Volume XV and the articles on Thomson by Jed Buchwald and on Stokes by E. M. Parkinson in Volume XIII.

3. I. Grattan-Guinness, "Why did George Green write his essay of 1828 on electricity and magnetism?" *Amer. Math. Monthly*, Vol. 102 (1995), pp. 387–396.

4. J. Gray, "There was a jolly miller." *The New Scientist*, Vol. 139 (1993), pp. 24–27.

5. G. E. Hutchinson, *The Enchanted Voyage* (New Haven: Yale University Press, 1962).

6. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), pp. 678–680.

7. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 683–685.

8. Sylvanus P. Thompson, *The Life of Lord Kelvin* (New York: Chelsea, 1976).

13.8 The Divergence Theorem

In Section 13.5 we rewrote Green's Theorem in a vector version as

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

where C is the positively oriented boundary curve of the plane region D. If we were seeking to extend this theorem to vector fields on $\mathbb{R}^3$, we might make the guess that

$$\boxed{1} \qquad \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}(x, y, z)\, dV$$

where S is the boundary surface of the solid region E. It turns out that Equation 1 is true, under appropriate hypotheses, and is called the Divergence Theorem. Notice its similarity to Green's Theorem and Stokes' Theorem in that it relates the integral of a derivative of a function (div **F** in this case) over a region to the integral of the original function **F** over the boundary of the region.

At this stage you may wish to review the various types of regions over which we were able to evaluate triple integrals in Section 12.7. We state and prove the Divergence Theorem for regions E that are simultaneously of types 1, 2, and 3 and we call such regions **simple solid regions**. (For instance, regions bounded by ellipsoids or rectangular boxes are simple solid regions.) The boundary of E is a closed surface, and we use the convention, introduced in Section 13.6, that the positive orientation is outward; that is, the unit normal vector **n** is directed outward from E.

▲ The Divergence Theorem is sometimes called Gauss's Theorem after the great German mathematician Karl Friedrich Gauss (1777–1855), who discovered this theorem during his investigation of electrostatics. In Eastern Europe the Divergence Theorem is known as Ostrogradsky's Theorem after the Russian mathematician Mikhail Ostrogradsky (1801–1862), who published this result in 1826.

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let **F** be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

Thus, the Divergence Theorem states that, under the given conditions, the flux of **F** across the boundary surface of E is equal to the triple integral of the divergence of **F** over E.

Proof Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. Then

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

so $$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E \frac{\partial P}{\partial x}\, dV + \iiint_E \frac{\partial Q}{\partial y}\, dV + \iiint_E \frac{\partial R}{\partial z}\, dV$$

If $\mathbf{n}$ is the unit outward normal of S, then the surface integral on the left side of the Divergence Theorem is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \mathbf{n}\, dS$$

$$= \iint_S P\,\mathbf{i} \cdot \mathbf{n}\, dS + \iint_S Q\,\mathbf{j} \cdot \mathbf{n}\, dS + \iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS$$

Therefore, to prove the Divergence Theorem, it suffices to prove the following three equations:

2 $$\iint_S P\,\mathbf{i} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial P}{\partial x}\, dV$$

3 $$\iint_S Q\,\mathbf{j} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial Q}{\partial y}\, dV$$

4 $$\iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial R}{\partial z}\, dV$$

To prove Equation 4 we use the fact that E is a type 1 region:

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is the projection of E onto the xy-plane. By Equation 12.7.6, we have

$$\iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z}(x, y, z)\, dz \right] dA$$

and, therefore, by the Fundamental Theorem of Calculus,

5 $$\iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D [R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))]\, dA$$

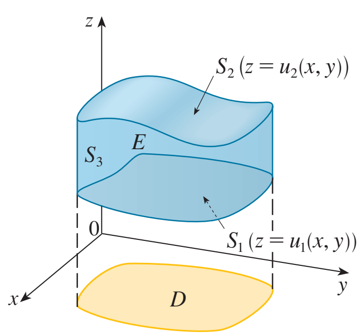


FIGURE 1

The boundary surface S consists of three pieces: the bottom surface S_1, the top surface S_2, and possibly a vertical surface S_3, which lies above the boundary curve of D. (See Figure 1. It might happen that S_3 doesn't appear, as in the case of a sphere.) Notice that on S_3 we have $\mathbf{k} \cdot \mathbf{n} = 0$, because $\mathbf{k}$ is vertical and $\mathbf{n}$ is horizontal, and so

$$\iint_{S_3} R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iint_{S_3} 0\, dS = 0$$

Thus, regardless of whether there is a vertical surface, we can write

6 $$\iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iint_{S_1} R\,\mathbf{k} \cdot \mathbf{n}\, dS + \iint_{S_2} R\,\mathbf{k} \cdot \mathbf{n}\, dS$$

The equation of S_2 is $z = u_2(x, y)$, $(x, y) \in D$, and the outward normal $\mathbf{n}$ points upward, so from Equation 13.6.10 (with $\mathbf{F}$ replaced by $R\,\mathbf{k}$) we have

$$\iint_{S_2} R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iint_D R(x, y, u_2(x, y))\, dA$$

On S_1 we have $z = u_1(x, y)$, but here the outward normal $\mathbf{n}$ points downward, so we multiply by -1:

$$\iint_{S_1} R\,\mathbf{k}\cdot\mathbf{n}\,dS = -\iint_D R(x, y, u_1(x, y))\,dA$$

Therefore, Equation 6 gives

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_D [R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))]\,dA$$

Comparison with Equation 5 shows that

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial R}{\partial z}\,dV$$

▲ Notice that the method of proof of the Divergence Theorem is very similar to that of Green's Theorem.

Equations 2 and 3 are proved in a similar manner using the expressions for E as a type 2 or type 3 region, respectively. ■

EXAMPLE 1 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION First we compute the divergence of $\mathbf{F}$:

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

The unit sphere S is the boundary of the unit ball B given by $x^2 + y^2 + z^2 \leqslant 1$. Thus, the Divergence Theorem gives the flux as

▲ The solution in Example 1 should be compared with the solution in Example 4 in Section 13.6.

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_B \text{div}\,\mathbf{F}\,dV = \iiint_B 1\,dV$$

$$= V(B) = \tfrac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$$ ■

EXAMPLE 2 Evaluate $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\,\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$. (See Figure 2.)

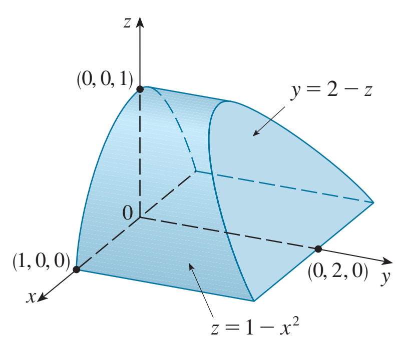


FIGURE 2

SOLUTION It would be extremely difficult to evaluate the given surface integral directly. (We would have to evaluate four surface integrals corresponding to the four pieces of S.) Furthermore, the divergence of $\mathbf{F}$ is much less complicated than $\mathbf{F}$ itself:

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \frac{\partial}{\partial z}(\sin xy) \\ &= y + 2y = 3y \end{aligned}$$

Therefore, we use the Divergence Theorem to transform the given surface integral into a triple integral. The easiest way to evaluate the triple integral is to express E as a type 3 region:

$$E = \{(x, y, z) \mid -1 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant 1 - x^2,\ 0 \leqslant y \leqslant 2 - z\}$$

Then we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E 3y\, dV \\ &= 3\int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} y\, dy\, dz\, dx \\ &= 3\int_{-1}^{1}\int_0^{1-x^2} \frac{(2-z)^2}{2}\, dz\, dx \\ &= \frac{3}{2}\int_{-1}^{1}\left[-\frac{(2-z)^3}{3}\right]_0^{1-x^2} dx \\ &= -\tfrac{1}{2}\int_{-1}^{1}\left[(x^2+1)^3 - 8\right] dx \\ &= -\int_0^1 (x^6 + 3x^4 + 3x^2 - 7)\, dx = \frac{184}{35} \end{aligned}$$

Although we have proved the Divergence Theorem only for simple solid regions, it can be proved for regions that are finite unions of simple solid regions. (The procedure is similar to the one we used in Section 13.4 to extend Green's Theorem.)

For example, let's consider the region E that lies between the closed surfaces S_1 and S_2, where S_1 lies inside S_2. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be outward normals of S_1 and S_2. Then the boundary surface of E is $S = S_1 \cup S_2$ and its normal $\mathbf{n}$ is given by $\mathbf{n} = -\mathbf{n}_1$ on S_1 and $\mathbf{n} = \mathbf{n}_2$ on S_2. (See Figure 3.) Applying the Divergence Theorem to S, we get

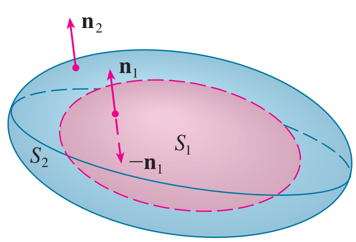


FIGURE 3

7
$$\begin{aligned} \iiint_E \operatorname{div} \mathbf{F}\, dV &= \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS \\ &= \iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS \\ &= -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} \end{aligned}$$

Let's apply this to the electric field (see Example 5 in Section 13.1):

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

where S_1 is a small sphere with radius a and center the origin. You can verify that div $\mathbf{E} = 0$ (see Exercise 19). Therefore, Equation 7 gives

$$\begin{aligned}\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} &= \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} + \iiint_E \operatorname{div} \mathbf{E}\, dV \\ &= \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n}\, dS\end{aligned}$$

The point of this calculation is that we can compute the surface integral over S_1 because S_1 is a sphere. The normal vector at $\mathbf{x}$ is $\mathbf{x}/|\mathbf{x}|$. Therefore

$$\begin{aligned}\mathbf{E} \cdot \mathbf{n} &= \frac{\varepsilon Q}{|\mathbf{x}|^3} \mathbf{x} \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) = \frac{\varepsilon Q}{|\mathbf{x}|^4} \mathbf{x} \cdot \mathbf{x} \\ &= \frac{\varepsilon Q}{|\mathbf{x}|^2} = \frac{\varepsilon Q}{a^2}\end{aligned}$$

since the equation of S_1 is $|\mathbf{x}| = a$. Thus, we have

$$\begin{aligned}\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} &= \iint_{S_1} \mathbf{E} \cdot \mathbf{n}\, dS \\ &= \frac{\varepsilon Q}{a^2} \iint_{S_1} dS = \frac{\varepsilon Q}{a^2} A(S_1) \\ &= \frac{\varepsilon Q}{a^2} 4\pi a^2 = 4\pi\varepsilon Q\end{aligned}$$

This shows that the electric flux of $\mathbf{E}$ is $4\pi\varepsilon Q$ through *any* closed surface S_2 that contains the origin. [This is a special case of Gauss's Law (Equation 13.6.11) for a single charge. The relationship between ε and ε_0 is $\varepsilon = 1/(4\pi\varepsilon_0)$.]

Another application of the Divergence Theorem occurs in fluid flow. Let $\mathbf{v}(x, y, z)$ be the velocity field of a fluid with constant density ρ. Then $\mathbf{F} = \rho\mathbf{v}$ is the rate of flow per unit area. If $P_0(x_0, y_0, z_0)$ is a point in the fluid and B_a is a ball with center P_0 and very small radius a, then div $\mathbf{F}(P) \approx$ div $\mathbf{F}(P_0)$ for all points in B_a since div $\mathbf{F}$ is continuous. We approximate the flux over the boundary sphere S_a as follows:

$$\begin{aligned}\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} &= \iiint_{B_a} \operatorname{div} \mathbf{F}\, dV \\ &\approx \iiint_{B_a} \operatorname{div} \mathbf{F}(P_0)\, dV \\ &= \operatorname{div} \mathbf{F}(P_0) V(B_a)\end{aligned}$$

This approximation becomes better as $a \to 0$ and suggests that

8
$$\operatorname{div} \mathbf{F}(P_0) = \lim_{a \to 0} \frac{1}{V(B_a)} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S}$$

Equation 8 says that div $\mathbf{F}(P_0)$ is the net rate of outward flux per unit volume at P_0. (This is the reason for the name *divergence.*) If div $\mathbf{F}(P) > 0$, the net flow is outward near P and P is called a **source**. If div $\mathbf{F}(P) < 0$, the net flow is inward near P and P is called a **sink**.

For the vector field in Figure 4, it appears that the vectors that end near P_1 are shorter than the vectors that start near P_1. Thus, the net flow is outward near P_1, so div $\mathbf{F}(P_1) > 0$ and P_1 is a source. Near P_2, on the other hand, the incoming arrows are longer than the outgoing arrows. Here the net flow is inward, so div $\mathbf{F}(P_2) < 0$ and P_2 is a sink. We can use the formula for $\mathbf{F}$ to confirm this impression. Since $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$, we have div $\mathbf{F} = 2x + 2y$, which is positive when $y > -x$. So the points above the line $y = -x$ are sources and those below are sinks.

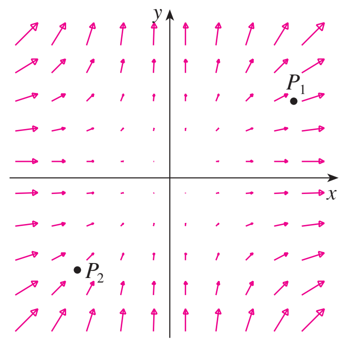
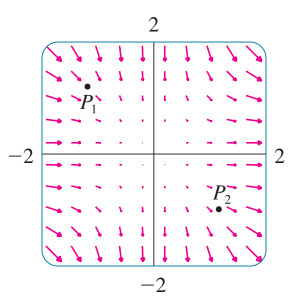


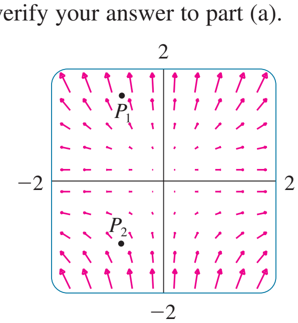
FIGURE 4
The vector field $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

13.8 Exercises

1. A vector field $\mathbf{F}$ is shown. Use the interpretation of divergence derived in this section to determine whether div $\mathbf{F}$ is positive or negative at P_1 and at P_2.



2. (a) Are the points P_1 and P_2 sources or sinks for the vector field $\mathbf{F}$ shown in the figure? Give an explanation based solely on the picture.
(b) Given that $\mathbf{F}(x, y) = \langle x, y^2 \rangle$, use the definition of divergence to verify your answer to part (a).



3–6 ■ Verify that the Divergence Theorem is true for the vector field $\mathbf{F}$ on the region E.

3. $\mathbf{F}(x, y, z) = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2xz\,\mathbf{k}$,
E is the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$

4. $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + 3z^2\,\mathbf{k}$,
E is the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$

5. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$,
E is the solid cylinder $x^2 + y^2 \leqslant 1$, $0 \leqslant z \leqslant 1$

6. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
E is the unit ball $x^2 + y^2 + z^2 \leqslant 1$

7–15 ■ Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S.

7. $\mathbf{F}(x, y, z) = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j} + yz^2\,\mathbf{k}$,
S is the surface of the box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 2$

8. $\mathbf{F}(x, y, z) = x^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + xz^4\,\mathbf{k}$,
S is the surface of the box with vertices $(\pm1, \pm2, \pm3)$

9. $\mathbf{F}(x, y, z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$

10. $\mathbf{F}(x, y, z) = x^3y\,\mathbf{i} - x^2y^2\,\mathbf{j} - x^2yz\,\mathbf{k}$,
S is the surface of the solid bounded by the hyperboloid $x^2 + y^2 - z^2 = 1$ and the planes $z = -2$ and $z = 2$

11. $\mathbf{F}(x, y, z) = xy\sin z\,\mathbf{i} + \cos(xz)\,\mathbf{j} + y\cos z\,\mathbf{k}$,
S is the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

12. $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + 2xz^2\,\mathbf{j} + 3y^2z\,\mathbf{k}$,
S is the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane

13. $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$,
S is the sphere $x^2 + y^2 + z^2 = 1$

14. $\mathbf{F}(x, y, z) = (x^3 + y\sin z)\,\mathbf{i} + (y^3 + z\sin x)\,\mathbf{j} + 3z\,\mathbf{k}$,
S is the surface of the solid bounded by the hemispheres $z = \sqrt{4 - x^2 - y^2}$, $z = \sqrt{1 - x^2 - y^2}$ and the plane $z = 0$

CAS **15.** $\mathbf{F}(x, y, z) = e^y\tan z\,\mathbf{i} + y\sqrt{3 - x^2}\,\mathbf{j} + x\sin y\,\mathbf{k}$,
S is the surface of the solid that lies above the xy-plane and below the surface $z = 2 - x^4 - y^4$, $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$

CAS **16.** Use a computer algebra system to plot the vector field $\mathbf{F}(x, y, z) = \sin x\cos^2 y\,\mathbf{i} + \sin^3 y\cos^4 z\,\mathbf{j} + \sin^5 z\cos^6 x\,\mathbf{k}$ in the cube cut from the first octant by the planes $x = \pi/2$, $y = \pi/2$, and $z = \pi/2$. Then compute the flux across the surface of the cube.

17. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = z^2x\,\mathbf{i} + \left(\tfrac{1}{3}y^3 + \tan z\right)\mathbf{j} + (x^2z + y^2)\,\mathbf{k}$$

and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$. [*Hint:* Note that S is not a closed surface. First compute integrals over S_1 and S_2, where S_1 is the disk $x^2 + y^2 \leqslant 1$, oriented downward, and $S_2 = S \cup S_1$.]

18. Let $\mathbf{F}(x, y, z) = z\tan^{-1}(y^2)\,\mathbf{i} + z^3\ln(x^2 + 1)\,\mathbf{j} + z\,\mathbf{k}$. Find the flux of $\mathbf{F}$ across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

19. Verify that $\operatorname{div}\mathbf{E} = 0$ for the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\,\mathbf{x}$$

20. Use the Divergence Theorem to evaluate

$$\iint_S (2x + 2y + z^2)\,dS$$

where S is the sphere $x^2 + y^2 + z^2 = 1$.

21–26 ■ Prove each identity, assuming that S and E satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

21. $\iint_S \mathbf{a}\cdot\mathbf{n}\,dS = 0$, where $\mathbf{a}$ is a constant vector

22. $V(E) = \frac{1}{3}\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

23. $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0$

24. $\iint_S D_{\mathbf{n}} f\,dS = \iiint_E \nabla^2 f\,dV$

25. $\iint_S (f\nabla g)\cdot\mathbf{n}\,dS = \iiint_E (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV$

26. $\iint_S (f\nabla g - g\nabla f)\cdot\mathbf{n}\,dS = \iiint_E (f\nabla^2 g - g\nabla^2 f)\,dV$

27. Suppose S and E satisfy the conditions of the Divergence Theorem and f is a scalar function with continuous partial derivatives. Prove that

$$\iint_S f\mathbf{n}\,dS = \iiint_E \nabla f\,dV$$

These surface and triple integrals of vector functions are vectors defined by integrating each component function. [*Hint:* Start by applying the Divergence Theorem to $\mathbf{F} = f\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector.]

28. A solid occupies a region E with surface S and is immersed in a liquid with constant density ρ. We set up a coordinate system so that the xy-plane coincides with the surface of the liquid and positive values of z are measured downward into the liquid. Then the pressure at depth z is $p = \rho g z$, where g is the acceleration due to gravity (see Section 6.5). The total buoyant force on the solid due to the pressure distribution is given by the surface integral

$$\mathbf{F} = -\iint_S p\mathbf{n}\,dS$$

where $\mathbf{n}$ is the outer unit normal. Use the result of Exercise 27 to show that $\mathbf{F} = -W\mathbf{k}$, where W is the weight of the liquid displaced by the solid. (Note that $\mathbf{F}$ is directed upward because z is directed downward.) The result is *Archimedes' principle:* The buoyant force on an object equals the weight of the displaced liquid.

13.9 Summary

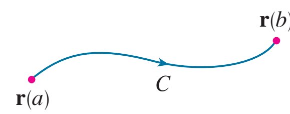
The main results of this chapter are all higher-dimensional versions of the Fundamental Theorem of Calculus. To help you remember them, we collect them together here (without hypotheses) so that you can see more easily their essential similarity. Notice that in each case we have an integral of a "derivative" over a region on the left side, and the right side involves the values of the original function only on the *boundary* of the region.

Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

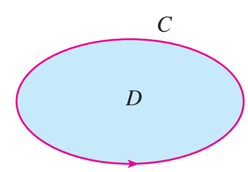
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



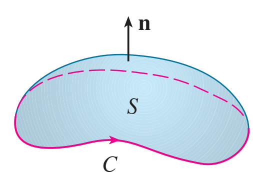
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$$



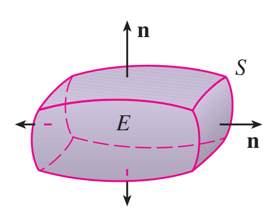
Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$



13 Review

CONCEPT CHECK

1. What is a vector field? Give three examples that have physical meaning.
2. (a) What is a conservative vector field?
   (b) What is a potential function?
3. (a) Write the definition of the line integral of a scalar function f along a smooth curve C with respect to arc length.
   (b) How do you evaluate such a line integral?
   (c) Write expressions for the mass and center of mass of a thin wire shaped like a curve C if the wire has linear density function $\rho(x, y)$.
   (d) Write the definitions of the line integrals along C of a scalar function f with respect to x, y, and z.
   (e) How do you evaluate these line integrals?
4. (a) Define the line integral of a vector field $\mathbf{F}$ along a smooth curve C given by a vector function $\mathbf{r}(t)$.
   (b) If $\mathbf{F}$ is a force field, what does this line integral represent?
   (c) If $\mathbf{F} = \langle P, Q, R \rangle$, what is the connection between the line integral of $\mathbf{F}$ and the line integrals of the component functions P, Q, and R?
5. State the Fundamental Theorem for Line Integrals.
6. (a) What does it mean to say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path?
   (b) If you know that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, what can you say about $\mathbf{F}$?
7. State Green's Theorem.
8. Write expressions for the area enclosed by a curve C in terms of line integrals around C.
9. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$.
   (a) Define curl $\mathbf{F}$.
   (b) Define div $\mathbf{F}$.
   (c) If $\mathbf{F}$ is a velocity field in fluid flow, what are the physical interpretations of curl $\mathbf{F}$ and div $\mathbf{F}$?
10. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, how do you test to determine whether $\mathbf{F}$ is conservative? What if $\mathbf{F}$ is a vector field on $\mathbb{R}^3$?
11. (a) Write the definition of the surface integral of a scalar function f over a surface S.
    (b) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?
    (c) What if S is given by an equation $z = g(x, y)$?
    (d) If a thin sheet has the shape of a surface S, and the density at (x, y, z) is $\rho(x, y, z)$, write expressions for the mass and center of mass of the sheet.
12. (a) What is an oriented surface? Give an example of a nonorientable surface.
    (b) Define the surface integral (or flux) of a vector field $\mathbf{F}$ over an oriented surface S with unit normal vector $\mathbf{n}$.
    (c) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?
    (d) What if S is given by an equation $z = g(x, y)$?
13. State Stokes' Theorem.
14. State the Divergence Theorem.
15. In what ways are the Fundamental Theorem for Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem similar to each other?

TRUE–FALSE QUIZ

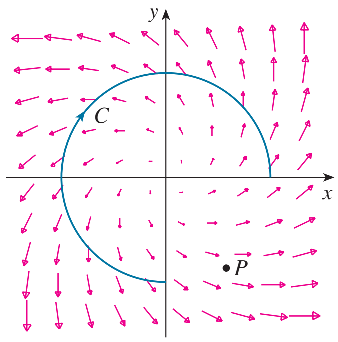
Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $\mathbf{F}$ is a vector field, then div $\mathbf{F}$ is a vector field.
2. If $\mathbf{F}$ is a vector field, then curl $\mathbf{F}$ is a vector field.
3. If f has continuous partial derivatives of all orders on $\mathbb{R}^3$, then $\operatorname{div}(\operatorname{curl} \nabla f) = 0$.
4. If f has continuous partial derivatives on $\mathbb{R}^3$ and C is any circle, then $\int_C \nabla f \cdot d\mathbf{r} = 0$.
5. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ and $P_y = Q_x$ in an open region D, then $\mathbf{F}$ is conservative.
6. $\int_{-C} f(x, y)\, ds = -\int_C f(x, y)\, ds$
7. If S is a sphere and $\mathbf{F}$ is a constant vector field, then $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$.
8. There is a vector field $\mathbf{F}$ such that

$$\operatorname{curl} \mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

EXERCISES

1. A vector field $\mathbf{F}$, a curve C, and a point P are shown.
(a) Is $\int_C \mathbf{F} \cdot d\mathbf{r}$ positive, negative, or zero? Explain.
(b) Is $\operatorname{div} \mathbf{F}(P)$ positive, negative, or zero? Explain.



2–9 ■ Evaluate the line integral.

2. $\int_C x\,ds$,
C is the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$

3. $\int_C x^3 z\,ds$,
C: $x = 2\sin t$, $y = t$, $z = 2\cos t$, $0 \leqslant t \leqslant \pi/2$

4. $\int_C xy\,dx + y\,dy$,
C is the sine curve $y = \sin x$, $0 \leqslant x \leqslant \pi/2$

5. $\int_C x^3 y\,dx - x\,dy$,
C is the circle $x^2 + y^2 = 1$ with counterclockwise orientation

6. $\int_C x \sin y\,dx + xyz\,dz$,
C is given by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $0 \leqslant t \leqslant 1$

7. $\int_C y\,dx + z\,dy + x\,dz$,
C consists of the line segments from $(0, 0, 0)$ to $(1, 1, 2)$ and from $(1, 1, 2)$ to $(3, 1, 4)$

8. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = x^2 y\,\mathbf{i} + e^y\,\mathbf{j}$ and C is given by $\mathbf{r}(t) = t^2\,\mathbf{i} - t^3\,\mathbf{j}$, $0 \leqslant t \leqslant 1$

9. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = (x + y)\,\mathbf{i} + z\,\mathbf{j} + x^2 y\,\mathbf{k}$ and C is given by $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + t^4\,\mathbf{k}$, $0 \leqslant t \leqslant 1$

10. Find the work done by the force field

$$\mathbf{F}(x, y, z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$$

in moving a particle from the point $(3, 0, 0)$ to the point $(0, \pi/2, 3)$ along
(a) A straight line
(b) The helix $x = 3\cos t$, $y = t$, $z = 3\sin t$

11–12 ■ Show that $\mathbf{F}$ is a conservative vector field. Then find a function f such that $\mathbf{F} = \nabla f$.

11. $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + (e^y + x^2 e^{xy})\,\mathbf{j}$

12. $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + x\cos y\,\mathbf{j} - \sin z\,\mathbf{k}$

13–14 ■ Show that $\mathbf{F}$ is conservative and use this fact to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve.

13. $\mathbf{F}(x, y) = (4x^3y^2 - 2xy^3)\,\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\,\mathbf{j}$,
C: $\mathbf{r}(t) = (t + \sin \pi t)\,\mathbf{i} + (2t + \cos \pi t)\,\mathbf{j}$, $0 \leqslant t \leqslant 1$

14. $\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + (xe^y + e^z)\,\mathbf{j} + ye^z\,\mathbf{k}$,
C is the line segment from $(0, 2, 0)$ to $(4, 0, 3)$

15. Verify that Green's Theorem is true for the line integral $\int_C xy^2\,dx - x^2 y\,dy$, where C consists of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$ and the line segment from $(1, 1)$ to $(-1, 1)$.

16. Use Green's Theorem to evaluate

$$\int_C \sqrt{1 + x^3}\,dx + 2xy\,dy$$

where C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$.

17. Use Green's Theorem to evaluate $\int_C x^2 y\,dx - xy^2\,dy$ where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

18. Find curl $\mathbf{F}$ and div $\mathbf{F}$ if

$$\mathbf{F}(x, y, z) = e^{-x}\sin y\,\mathbf{i} + e^{-y}\sin z\,\mathbf{j} + e^{-z}\sin x\,\mathbf{k}$$

19. Show that there is no vector field $\mathbf{G}$ such that

$$\operatorname{curl} \mathbf{G} = 2x\,\mathbf{i} + 3yz\,\mathbf{j} - xz^2\,\mathbf{k}$$

20. Show that, under conditions to be stated on the vector fields $\mathbf{F}$ and $\mathbf{G}$,

$$\operatorname{curl}(\mathbf{F} \times \mathbf{G}) = \mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

21. If C is any piecewise-smooth simple closed plane curve and f and g are differentiable functions, show that

$$\int_C f(x)\,dx + g(y)\,dy = 0$$

22. If f and g are twice differentiable functions, show that

$$\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$$

23. If f is a harmonic function, that is, $\nabla^2 f = 0$, show that the line integral $\int f_y\,dx - f_x\,dy$ is independent of path in any simple region D.

24. (a) Sketch the curve C with parametric equations

$$x = \cos t \qquad y = \sin t \qquad z = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

(b) Find $\int_C 2xe^{2y}\,dx + (2x^2e^{2y} + 2y\cot z)\,dy - y^2\csc^2 z\,dz$.

25–28 ■ Evaluate the surface integral.

25. $\iint_S z\,dS$, where S is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$

26. $\iint_S (x^2z + y^2z)\,dS$, where S is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $x^2 + y^2 = 4$

27. $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} - 2y\,\mathbf{j} + 3x\,\mathbf{k}$ and S is the sphere $x^2 + y^2 + z^2 = 4$ with outward orientation

28. $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ with upward orientation

29. Verify that Stokes' Theorem is true for the vector field

$$\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$$

where S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, and S has upward orientation.

30. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + yz^2\,\mathbf{j} + z^3e^{xy}\,\mathbf{k}$, S is the part of the sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$, and S is oriented upward.

31. Use Stokes' Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented counterclockwise as viewed from above.

32. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.

33. Verify that the Divergence Theorem is true for the vector field

$$\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

where E is the unit ball $x^2 + y^2 + z^2 \leqslant 1$.

34. Compute the outward flux of

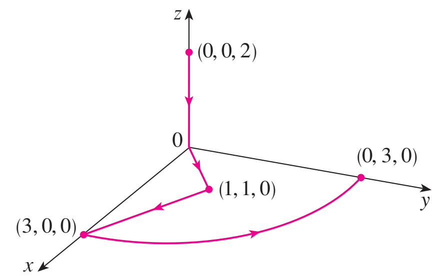
$$\mathbf{F}(x, y, z) = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

through the ellipsoid $4x^2 + 9y^2 + 6z^2 = 36$.

35. Let

$$\mathbf{F}(x, y, z) = (3x^2yz - 3y)\,\mathbf{i} + (x^3z - 3x)\,\mathbf{j} + (x^3y + 2z)\,\mathbf{k}$$

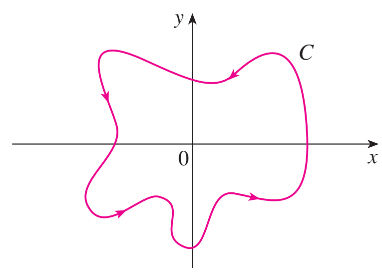
Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve with initial point $(0, 0, 2)$ and terminal point $(0, 3, 0)$ shown in the figure.



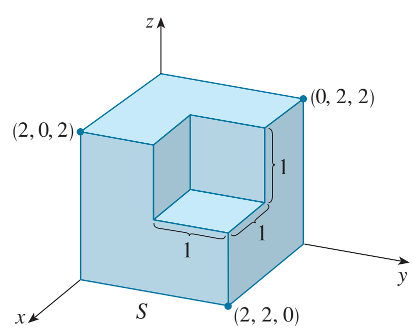
36. Let

$$\mathbf{F}(x, y) = \frac{(2x^3 + 2xy^2 - 2y)\,\mathbf{i} + (2y^3 + 2x^2y + 2x)\,\mathbf{j}}{x^2 + y^2}$$

Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where C is shown in the figure.



37. Find $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S is the outwardly oriented surface shown in the figure (the boundary surface of a cube with a unit corner cube removed).



38. If the components of $\mathbf{F}$ have continuous second partial derivatives and S is the boundary surface of a simple solid region, show that $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0$.

Focus on Problem Solving

1. Let S be a smooth parametric surface and let P be a point such that each line that starts at P intersects S at most once. The **solid angle** $\Omega(S)$ subtended by S at P is the set of lines starting at P and passing through S. Let $S(a)$ be the intersection of $\Omega(S)$ with the surface of the sphere with center P and radius a. Then the measure of the solid angle (in *steradians*) is defined to be

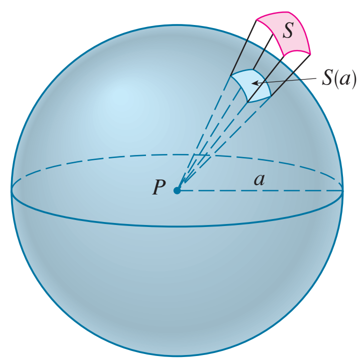
$$|\Omega(S)| = \frac{\text{area of } S(a)}{a^2}$$

Apply the Divergence Theorem to the part of $\Omega(S)$ between $S(a)$ and S to show that

$$|\Omega(S)| = \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3}\, dS$$

where $\mathbf{r}$ is the radius vector from P to any point on S, $r = |\mathbf{r}|$, and the unit normal vector $\mathbf{n}$ is directed away from P.

This shows that the definition of the measure of a solid angle is independent of the radius a of the sphere. Thus, the measure of the solid angle is equal to the area subtended on a *unit* sphere. (Note the analogy with the definition of radian measure.) The total solid angle subtended by a sphere at its center is thus 4π steradians.



2. Find the simple closed curve C for which the value of the line integral

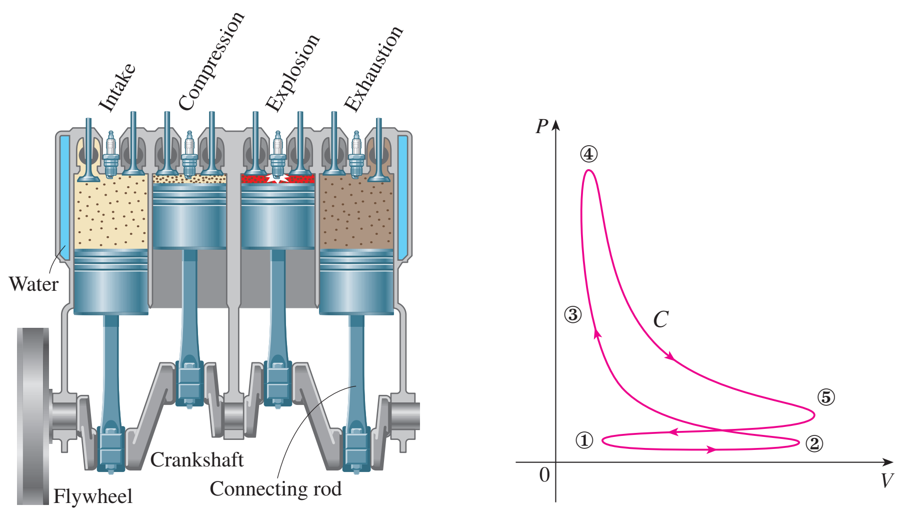
$$\int_C (y^3 - y)\, dx - 2x^3\, dy$$

is a maximum.

3. Let C be a simple closed piecewise-smooth space curve that lies in a plane with unit normal vector $\mathbf{n} = \langle a, b, c \rangle$ and has positive orientation with respect to $\mathbf{n}$. Show that the plane area enclosed by C is

$$\tfrac{1}{2} \int_C (bz - cy)\, dx + (cx - az)\, dy + (ay - bx)\, dz$$

4. The figure depicts the sequence of events in each cylinder of a four-cylinder internal combustion engine. Each piston moves up and down and is connected by a pivoted arm to a rotating crankshaft. Let $P(t)$ and $V(t)$ be the pressure and volume within a cylinder at time t, where $a \leqslant t \leqslant b$ gives the time required for a complete cycle. The graph shows how P and V vary through one cycle of a four-stroke engine.



During the intake stroke (from ① to ②) a mixture of air and gasoline at atmospheric pressure is drawn into a cylinder through the intake valve as the piston moves downward. Then the piston rapidly compresses the mix with the valves closed in the compression stroke (from ② to ③) during which the pressure rises and the volume decreases. At ③ the sparkplug ignites the fuel, raising the temperature and pressure at almost constant volume to ④. Then, with valves closed, the rapid expansion forces the piston downward during the power stroke (from ④ to ⑤). The exhaust valve opens, temperature and pressure drop, and mechanical energy stored in a rotating flywheel pushes the piston upward, forcing the waste products out of the exhaust valve in the exhaust stroke. The exhaust valve closes and the intake valve opens. We're now back at ① and the cycle starts again.

(a) Show that the work done on the piston during one cycle of a four-stroke engine is $W = \int_C P\, dV$, where C is the curve in the PV-plane shown in the figure.

[*Hint:* Let $x(t)$ be the distance from the piston to the top of the cylinder and note that the force on the piston is $\mathbf{F} = AP(t)\,\mathbf{i}$, where A is the area of the top of the piston. Then $W = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$, where C_1 is given by $\mathbf{r}(t) = x(t)\,\mathbf{i}$, $a \leqslant t \leqslant b$. An alternative approach is to work directly with Riemann sums.]

(b) Use Formula 13.4.5 to show that the work is the difference of the areas enclosed by the two loops of C.

Appendixes

A Intervals, Inequalities, and Absolute Values

Certain sets of real numbers, called **intervals,** occur frequently in calculus and correspond geometrically to line segments. For example, if $a < b$, the **open interval** from a to b consists of all numbers between a and b and is denoted by the symbol (a, b). Using set-builder notation, we can write

$$(a, b) = \{x \mid a < x < b\}$$

FIGURE 1
Open interval (a, b)

Notice that the endpoints of the interval—namely, a and b—are excluded. This is indicated by the round brackets ( ) and by the open dots in Figure 1. The **closed interval** from a to b is the set

$$[a, b] = \{x \mid a \leqslant x \leqslant b\}$$

FIGURE 2
Closed interval $[a, b]$

Here the endpoints of the interval are included. This is indicated by the square brackets [ ] and by the solid dots in Figure 2. It is also possible to include only one endpoint in an interval, as shown in Table 1.

We also need to consider infinite intervals such as

$$(a, \infty) = \{x \mid x > a\}$$

▲ Table 1 lists the nine possible types of intervals. When these intervals are discussed, it is always assumed that $a < b$.

This does not mean that ∞ ("infinity") is a number. The notation (a, ∞) stands for the set of all numbers that are greater than a, so the symbol ∞ simply indicates that the interval extends indefinitely far in the positive direction.

1 Table of Intervals

Notation	Set description	Picture	Notation	Set description	Picture
(a, b)	$\{x \mid a < x < b\}$		(a, ∞)	$\{x \mid x > a\}$	
$[a, b]$	$\{x \mid a \leqslant x \leqslant b\}$		$[a, \infty)$	$\{x \mid x \geqslant a\}$	
$[a, b)$	$\{x \mid a \leqslant x < b\}$		$(-\infty, b)$	$\{x \mid x < b\}$	
$(a, b]$	$\{x \mid a < x \leqslant b\}$		$(-\infty, b]$	$\{x \mid x \leqslant b\}$	
			$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

Inequalities

When working with inequalities, note the following rules.

Rules for Inequalities

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c < d$, then $a + c < b + d$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. If $a < b$ and $c < 0$, then $ac > bc$.
5. If $0 < a < b$, then $1/a > 1/b$.

Rule 1 says that we can add any number to both sides of an inequality, and Rule 2 says that two inequalities can be added. However, we have to be careful with multiplication. Rule 3 says that we can multiply both sides of an inequality by a *positive* number, but Rule 4 says that *if we multiply both sides of an inequality by a negative number, then we reverse the direction of the inequality.* For example, if we take the inequality $3 < 5$ and multiply by 2, we get $6 < 10$, but if we multiply by -2, we get $-6 > -10$. Finally, Rule 5 says that if we take reciprocals, then we reverse the direction of an inequality (provided the numbers are positive).

EXAMPLE 1 Solve the inequality $1 + x < 7x + 5$.

SOLUTION The given inequality is satisfied by some values of x but not by others. To *solve* an inequality means to determine the set of numbers x for which the inequality is true. This is called the *solution set.*

First we subtract 1 from each side of the inequality (using Rule 1 with $c = -1$):

$$x < 7x + 4$$

Then we subtract $7x$ from both sides (Rule 1 with $c = -7x$):

$$-6x < 4$$

Now we divide both sides by -6 $\left(\text{Rule 4 with } c = -\tfrac{1}{6}\right)$:

$$x > -\tfrac{4}{6} = -\tfrac{2}{3}$$

These steps can all be reversed, so the solution set consists of all numbers greater than $-\frac{2}{3}$. In other words, the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$. ■

EXAMPLE 2 Solve the inequality $x^2 - 5x + 6 \leqslant 0$.

SOLUTION First we factor the left side:

$$(x - 2)(x - 3) \leqslant 0$$

We know that the corresponding equation $(x - 2)(x - 3) = 0$ has the solutions 2 and 3. The numbers 2 and 3 divide the real line into three intervals:

$$(-\infty, 2) \qquad (2, 3) \qquad (3, \infty)$$

▲ A visual method for solving Example 2 is to use a graphing device to graph the parabola $y = x^2 - 5x + 6$ (as in Figure 3) and observe that the curve lies on or below the x-axis when $2 \leqslant x \leqslant 3$.

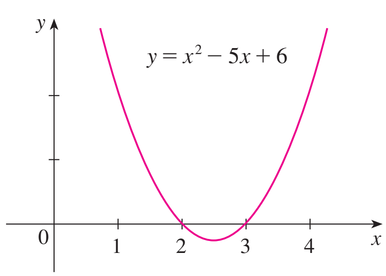


FIGURE 3

On each of these intervals we determine the signs of the factors. For instance,

$$x \in (-\infty, 2) \quad \Rightarrow \quad x < 2 \quad \Rightarrow \quad x - 2 < 0$$

Then we record these signs in the following chart:

Interval	$x - 2$	$x - 3$	$(x - 2)(x - 3)$
$x < 2$	$-$	$-$	$+$
$2 < x < 3$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$

Another method for obtaining the information in the chart is to use *test values.* For instance, if we use the test value $x = 1$ for the interval $(-\infty, 2)$, then substitution in $x^2 - 5x + 6$ gives

$$1^2 - 5(1) + 6 = 2$$

The polynomial $x^2 - 5x + 6$ doesn't change sign inside any of the three intervals, so we conclude that it is positive on $(-\infty, 2)$.

Then we read from the chart that $(x - 2)(x - 3)$ is negative when $2 < x < 3$. Thus, the solution of the inequality $(x - 2)(x - 3) \leqslant 0$ is

$$\{x \mid 2 \leqslant x \leqslant 3\} = [2, 3]$$

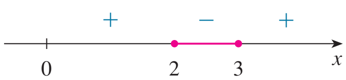


FIGURE 4

Notice that we have included the endpoints 2 and 3 because we are looking for values of x such that the product is either negative or zero. The solution is illustrated in Figure 4. ■

EXAMPLE 3 Solve $x^3 + 3x^2 > 4x$.

SOLUTION First we take all nonzero terms to one side of the inequality sign and factor the resulting expression:

$$x^3 + 3x^2 - 4x > 0 \qquad \text{or} \qquad x(x - 1)(x + 4) > 0$$

As in Example 2 we solve the corresponding equation $x(x - 1)(x + 4) = 0$ and use the solutions $x = -4$, $x = 0$, and $x = 1$ to divide the real line into four intervals $(-\infty, -4)$, $(-4, 0)$, $(0, 1)$, and $(1, \infty)$. On each interval the product keeps a constant sign as shown in the following chart.

Interval	x	$x - 1$	$x + 4$	$x(x - 1)(x + 4)$
$x < -4$	−	−	−	−
$-4 < x < 0$	−	−	+	+
$0 < x < 1$	+	−	+	−
$x > 1$	+	+	+	+

Then we read from the chart that the solution set is

$$\{x \mid -4 < x < 0 \text{ or } x > 1\} = (-4, 0) \cup (1, \infty)$$

FIGURE 5

The solution is illustrated in Figure 5. ■

Absolute Value

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geqslant 0 \qquad \text{for every number } a$$

For example,

$$|3| = 3 \qquad |-3| = 3 \qquad |0| = 0$$

$$|\sqrt{2} - 1| = \sqrt{2} - 1 \qquad |3 - \pi| = \pi - 3$$

In general, we have

▲ Remember that if a is negative, then $-a$ is positive.

2
$$|a| = a \quad \text{if } a \geqslant 0$$
$$|a| = -a \quad \text{if } a < 0$$

EXAMPLE 4 Express $|3x - 2|$ without using the absolute value symbol.

SOLUTION

$$|3x - 2| = \begin{cases} 3x - 2 & \text{if } 3x - 2 \geqslant 0 \\ -(3x - 2) & \text{if } 3x - 2 < 0 \end{cases}$$
$$= \begin{cases} 3x - 2 & \text{if } x \geqslant \frac{2}{3} \\ 2 - 3x & \text{if } x < \frac{2}{3} \end{cases}$$

Recall that the symbol $\sqrt{\ }$ means "the positive square root of." Thus, $\sqrt{r} = s$ means $s^2 = r$ and $s \geqslant 0$. Therefore, the equation $\sqrt{a^2} = a$ is not always true. It is true only when $a \geqslant 0$. If $a < 0$, then $-a > 0$, so we have $\sqrt{a^2} = -a$. In view of (2), we then have the equation

3
$$\sqrt{a^2} = |a|$$

which is true for all values of a.

Hints for the proofs of the following properties are given in the exercises.

Properties of Absolute Values Suppose a and b are any real numbers and n is an integer. Then

1. $|ab| = |a||b|$ **2.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|} \quad (b \neq 0)$ **3.** $|a^n| = |a|^n$

For solving equations or inequalities involving absolute values, it's often very helpful to use the following statements.

Suppose $a > 0$. Then

4. $|x| = a$ if and only if $x = \pm a$

5. $|x| < a$ if and only if $-a < x < a$

6. $|x| > a$ if and only if $x > a$ or $x < -a$

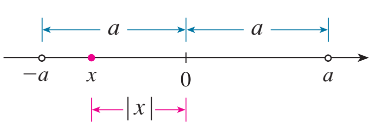


FIGURE 6

For instance, the inequality $|x| < a$ says that the distance from x to the origin is less than a, and you can see from Figure 6 that this is true if and only if x lies between $-a$ and a.

If a and b are any real numbers, then the distance between a and b is the absolute value of the difference, namely, $|a - b|$, which is also equal to $|b - a|$. (See Figure 7.)

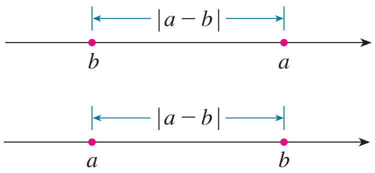


FIGURE 7
Length of a line segment = $|a - b|$

EXAMPLE 5 Solve $|2x - 5| = 3$.

SOLUTION By Property 4 of absolute values, $|2x - 5| = 3$ is equivalent to

$$2x - 5 = 3 \quad \text{or} \quad 2x - 5 = -3$$

So $2x = 8$ or $2x = 2$. Thus, $x = 4$ or $x = 1$. ■

EXAMPLE 6 Solve $|x - 5| < 2$.

SOLUTION 1 By Property 5 of absolute values, $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2$$

Therefore, adding 5 to each side, we have

$$3 < x < 7$$

and the solution set is the open interval $(3, 7)$.

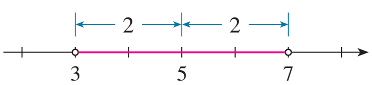


FIGURE 8

SOLUTION 2 Geometrically, the solution set consists of all numbers x whose distance from 5 is less than 2. From Figure 8 we see that this is the interval $(3, 7)$. ■

EXAMPLE 7 Solve $|3x + 2| \geqslant 4$.

SOLUTION By Properties 4 and 6 of absolute values, $|3x + 2| \geqslant 4$ is equivalent to

$$3x + 2 \geqslant 4 \quad \text{or} \quad 3x + 2 \leqslant -4$$

In the first case $3x \geqslant 2$, which gives $x \geqslant \frac{2}{3}$. In the second case $3x \leqslant -6$, which gives $x \leqslant -2$. So the solution set is

$$\left\{x \mid x \leqslant -2 \text{ or } x \geqslant \tfrac{2}{3}\right\} = (-\infty, -2] \cup \left[\tfrac{2}{3}, \infty\right)$$ ■

A Exercises

1–10 ■ Rewrite the expression without using the absolute value symbol.

1. $|5 - 23|$

2. $|\pi - 2|$

3. $|\sqrt{5} - 5|$

4. $||-2| - |-3||$

5. $|x - 2|$ if $x < 2$

6. $|x - 2|$ if $x > 2$

7. $|x + 1|$

8. $|2x - 1|$

9. $|x^2 + 1|$

10. $|1 - 2x^2|$

11–26 ■ Solve the inequality in terms of intervals and illustrate the solution set on the real number line.

11. $2x + 7 > 3$

12. $4 - 3x \geqslant 6$

13. $1 - x \leqslant 2$

14. $1 + 5x > 5 - 3x$

15. $0 \leqslant 1 - x < 1$

16. $1 < 3x + 4 \leqslant 16$

17. $(x - 1)(x - 2) > 0$

18. $x^2 < 2x + 8$

19. $x^2 < 3$

20. $x^2 \geqslant 5$

21. $x^3 - x^2 \leqslant 0$

22. $(x + 1)(x - 2)(x + 3) \geqslant 0$

23. $x^3 > x$

24. $x^3 + 3x < 4x^2$

25. $\dfrac{1}{x} < 4$

26. $-3 < \dfrac{1}{x} \leqslant 1$

27. The relationship between the Celsius and Fahrenheit temperature scales is given by $C = \frac{5}{9}(F - 32)$, where C is the temperature in degrees Celsius and F is the temperature in degrees Fahrenheit. What interval on the Celsius scale corresponds to the temperature range $50 \leqslant F \leqslant 95$?

28. Use the relationship between C and F given in Exercise 27 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \leqslant C \leqslant 30$.

29. As dry air moves upward, it expands and in so doing cools at a rate of about 1 °C for each 100-m rise, up to about 12 km.
(a) If the ground temperature is 20 °C, write a formula for the temperature at height h.
(b) What range of temperature can be expected if a plane takes off and reaches a maximum height of 5 km?

30. If a ball is thrown upward from the top of a building 128 ft high with an initial velocity of 16 ft/s, then the height h above the ground t seconds later will be

$$h = 128 + 16t - 16t^2$$

During what time interval will the ball be at least 32 ft above the ground?

31–32 ■ Solve the equation for x.

31. $|x + 3| = |2x + 1|$　　**32.** $|3x + 5| = 1$

33–40 ■ Solve the inequality.

33. $|x| < 3$　　**34.** $|x| \geqslant 3$

35. $|x - 4| < 1$　　**36.** $|x - 6| < 0.1$

37. $|x + 5| \geqslant 2$　　**38.** $|x + 1| \geqslant 3$

39. $|2x - 3| \leqslant 0.4$　　**40.** $|5x - 2| < 6$

41. Solve the inequality $a(bx - c) \geqslant bc$ for x, assuming that a, b, and c are positive constants.

42. Solve the inequality $ax + b < c$ for x, assuming that a, b, and c are negative constants.

43. Prove that $|ab| = |a||b|$. [*Hint:* Use Equation 3.]

44. Show that if $0 < a < b$, then $a^2 < b^2$.

B Coordinate Geometry

The points in a plane can be identified with ordered pairs of real numbers. We start by drawing two perpendicular coordinate lines that intersect at the origin O on each line. Usually one line is horizontal with positive direction to the right and is called the x-axis; the other line is vertical with positive direction upward and is called the y-axis.

Any point P in the plane can be located by a unique ordered pair of numbers as follows. Draw lines through P perpendicular to the x- and y-axes. These lines intersect the axes in points with coordinates a and b as shown in Figure 1. Then the point P is assigned the ordered pair (a, b). The first number a is called the ***x*-coordinate** of P; the second number b is called the ***y*-coordinate** of P. We say that P is the point with coordinates (a, b), and we denote the point by the symbol $P(a, b)$. Several points are labeled with their coordinates in Figure 2.

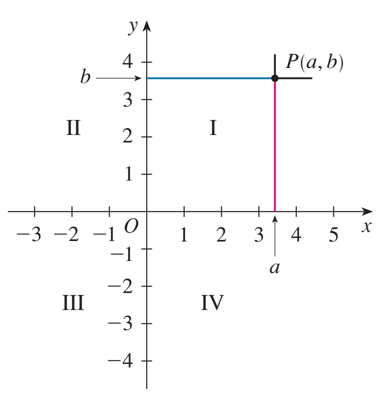


FIGURE 1

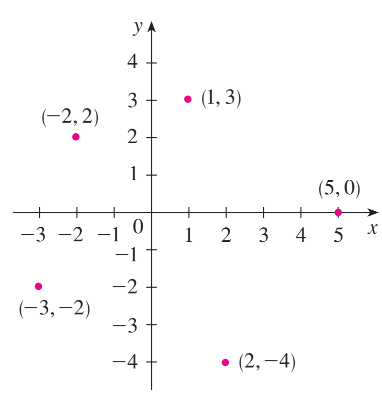


FIGURE 2

By reversing the preceding process we can start with an ordered pair (a, b) and arrive at the corresponding point P. Often we identify the point P with the ordered pair (a, b) and refer to "the point (a, b)." [Although the notation used for an open interval (a, b) is the same as the notation used for a point (a, b), you will be able to tell from the context which meaning is intended.]

This coordinate system is called the **rectangular coordinate system** or the **Cartesian coordinate system** in honor of the French mathematician René Descartes (1596–1650), even though another Frenchman, Pierre Fermat (1601–1665), invented the principles of analytic geometry at about the same time as Descartes. The plane supplied with this coordinate system is called the **coordinate plane** or the **Cartesian plane** and is denoted by $\mathbb{R}^2$.

The x- and y-axes are called the **coordinate axes** and divide the Cartesian plane into four quadrants, which are labeled I, II, III, and IV in Figure 1. Notice that the first quadrant consists of those points whose x- and y-coordinates are both positive.

EXAMPLE 1 Describe and sketch the regions given by the following sets.

(a) $\{(x, y) \mid x \geqslant 0\}$ (b) $\{(x, y) \mid y = 1\}$ (c) $\{(x, y) \,\big|\, |y| < 1\}$

SOLUTION

(a) The points whose x-coordinates are 0 or positive lie on the y-axis or to the right of it as indicated by the shaded region in Figure 3(a).

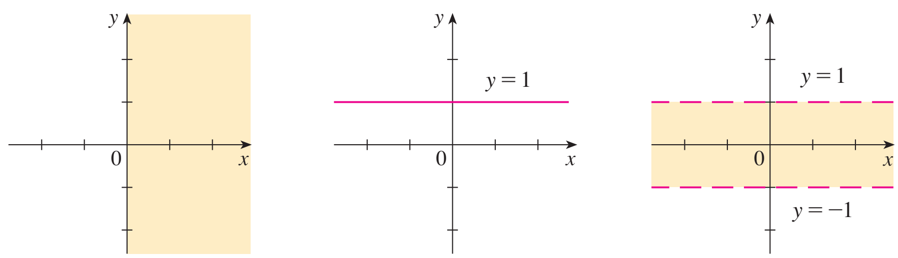


FIGURE 3 (a) $x \geqslant 0$ (b) $y = 1$ (c) $|y| < 1$

(b) The set of all points with y-coordinate 1 is a horizontal line one unit above the x-axis [see Figure 3(b)].

(c) Recall from Appendix A that

$$|y| < 1 \qquad \text{if and only if} \qquad -1 < y < 1$$

The given region consists of those points in the plane whose y-coordinates lie between -1 and 1. Thus, the region consists of all points that lie between (but not on) the horizontal lines $y = 1$ and $y = -1$. [These lines are shown as dashed lines in Figure 3(c) to indicate that the points on these lines don't lie in the set.] ■

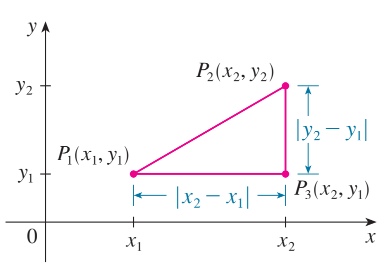


FIGURE 4

Recall from Appendix A that the distance between points a and b on a number line is $|a - b| = |b - a|$. Thus, the distance between points $P_1(x_1, y_1)$ and $P_3(x_2, y_1)$ on a horizontal line must be $|x_2 - x_1|$ and the distance between $P_2(x_2, y_2)$ and $P_3(x_2, y_1)$ on a vertical line must be $|y_2 - y_1|$. (See Figure 4.)

To find the distance $|P_1P_2|$ between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, we note that triangle $P_1P_2P_3$ in Figure 4 is a right triangle, and so by the Pythagorean

Theorem we have

$$|P_1P_2| = \sqrt{|P_1P_3|^2 + |P_2P_3|^2} = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance Formula The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For instance, the distance between $(1, -2)$ and $(5, 3)$ is

$$\sqrt{(5 - 1)^2 + [3 - (-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$

Circles

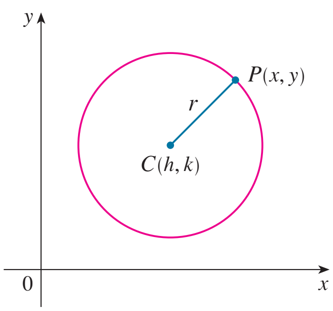


FIGURE 5

An **equation of a curve** is an equation satisfied by the coordinates of the points on the curve and by no other points. Let's use the distance formula to find the equation of a circle with radius r and center (h, k). By definition, the circle is the set of all points $P(x, y)$ whose distance from the center $C(h, k)$ is r. (See Figure 5.) Thus, P is on the circle if and only if $|PC| = r$. From the distance formula, we have

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

or equivalently, squaring both sides, we get

$$(x - h)^2 + (y - k)^2 = r^2$$

This is the desired equation.

Equation of a Circle An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

In particular, if the center is the origin $(0, 0)$, the equation is

$$x^2 + y^2 = r^2$$

For instance, an equation of the circle with radius 3 and center $(2, -5)$ is

$$(x - 2)^2 + (y + 5)^2 = 9$$

EXAMPLE 2 Sketch the graph of the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ by first showing that it represents a circle and then finding its center and radius.

SOLUTION We first group the x-terms and y-terms as follows:

$$(x^2 + 2x) + (y^2 - 6y) = -7$$

Then we complete the square within each grouping, adding the appropriate

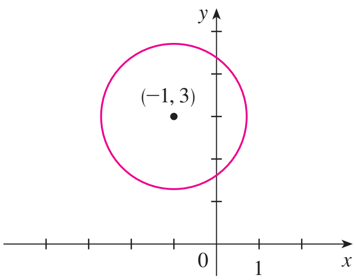


FIGURE 6
$x^2 + y^2 + 2x - 6y + 7 = 0$

constants (the squares of half the coefficients of x and y) to both sides of the equation:

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9$$

or

$$(x + 1)^2 + (y - 3)^2 = 3$$

Comparing this equation with the standard equation of a circle, we see that $h = -1$, $k = 3$, and $r = \sqrt{3}$, so the given equation represents a circle with center $(-1, 3)$ and radius $\sqrt{3}$. It is sketched in Figure 6.

Lines

To find the equation of a line L we use its *slope,* which is a measure of the steepness of the line.

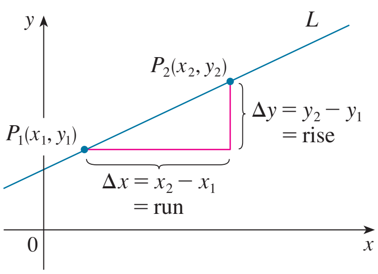


FIGURE 7

Definition The **slope** of a nonvertical line that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

Thus, the slope of a line is the ratio of the change in y, Δy, to the change in x, Δx (see Figure 7). The slope is therefore the rate of change of y with respect to x. The fact that the line is straight means that the rate of change is constant.

Figure 8 shows several lines labeled with their slopes. Notice that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. Notice also that the steepest lines are the ones for which the absolute value of the slope is largest, and a horizontal line has slope 0.

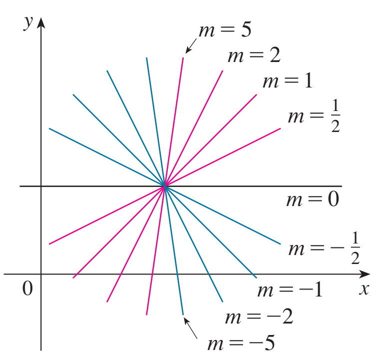


FIGURE 8

Now let's find an equation of the line that passes through a given point $P_1(x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line through P_1 and P is equal to m; that is,

$$\frac{y - y_1}{x - x_1} = m$$

This equation can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

and we observe that this equation is also satisfied when $x = x_1$ and $y = y_1$. Therefore, it is an equation of the given line.

Point-Slope Form of the Equation of a Line An equation of the line passing through the point $P_1(x_1, y_1)$ and having slope m is

$$y - y_1 = m(x - x_1)$$

EXAMPLE 3 Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

SOLUTION The slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{3}{2}$$

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\tfrac{3}{2}(x + 1)$$

which simplifies to $$3x + 2y = 1$$

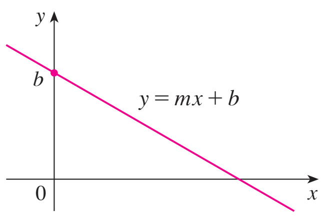


FIGURE 9

Suppose a nonvertical line has slope m and y-intercept b. (See Figure 9.) This means it intersects the y-axis at the point $(0, b)$, so the point-slope form of the equation of the line, with $x_1 = 0$ and $y_1 = b$, becomes

$$y - b = m(x - 0)$$

This simplifies as follows.

Slope-Intercept Form of the Equation of a Line An equation of the line with slope m and y-intercept b is

$$y = mx + b$$

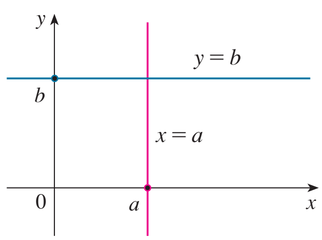


FIGURE 10

In particular, if a line is horizontal, its slope is $m = 0$, so its equation is $y = b$, where b is the y-intercept (see Figure 10). A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x-intercept, because the x-coordinate of every point on the line is a.

EXAMPLE 4 Graph the inequality $x + 2y > 5$.

SOLUTION We are asked to sketch the graph of the set $\{(x, y) \mid x + 2y > 5\}$ and we begin by solving the inequality for y:

$$x + 2y > 5$$

$$2y > -x + 5$$

$$y > -\tfrac{1}{2}x + \tfrac{5}{2}$$

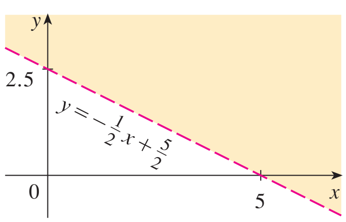


FIGURE 11

Compare this inequality with the equation $y = -\frac{1}{2}x + \frac{5}{2}$, which represents a line with slope $-\frac{1}{2}$ and y-intercept $\frac{5}{2}$. We see that the given graph consists of points whose y-coordinates are *larger* than those on the line $y = -\frac{1}{2}x + \frac{5}{2}$. Thus, the graph is the region that lies *above* the line, as illustrated in Figure 11.

Parallel and Perpendicular Lines

Slopes can be used to show that lines are parallel or perpendicular. The following facts are proved, for instance, in *Precalculus: Mathematics for Calculus, Third Edition* by Stewart, Redlin, and Watson (Brooks/Cole Publishing Co., Pacific Grove, CA, 1998).

Parallel and Perpendicular Lines

1. Two nonvertical lines are parallel if and only if they have the same slope.
2. Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$; that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

EXAMPLE 5 Find an equation of the line through the point $(5, 2)$ that is parallel to the line $4x + 6y + 5 = 0$.

SOLUTION The given line can be written in the form

$$y = -\tfrac{2}{3}x - \tfrac{5}{6}$$

which is in slope-intercept form with $m = -\frac{2}{3}$. Parallel lines have the same slope, so the required line has slope $-\frac{2}{3}$ and its equation in point-slope form is

$$y - 2 = -\tfrac{2}{3}(x - 5)$$

We can write this equation as $2x + 3y = 16$. ■

EXAMPLE 6 Show that the lines $2x + 3y = 1$ and $6x - 4y - 1 = 0$ are perpendicular.

SOLUTION The equations can be written as

$$y = -\tfrac{2}{3}x + \tfrac{1}{3} \qquad \text{and} \qquad y = \tfrac{3}{2}x - \tfrac{1}{4}$$

from which we see that the slopes are

$$m_1 = -\tfrac{2}{3} \qquad \text{and} \qquad m_2 = \tfrac{3}{2}$$

Since $m_1 m_2 = -1$, the lines are perpendicular. ■

Conic Sections

Here we review the geometric definitions of parabolas, ellipses, and hyperbolas and their standard equations. They are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane as shown in Figure 12.

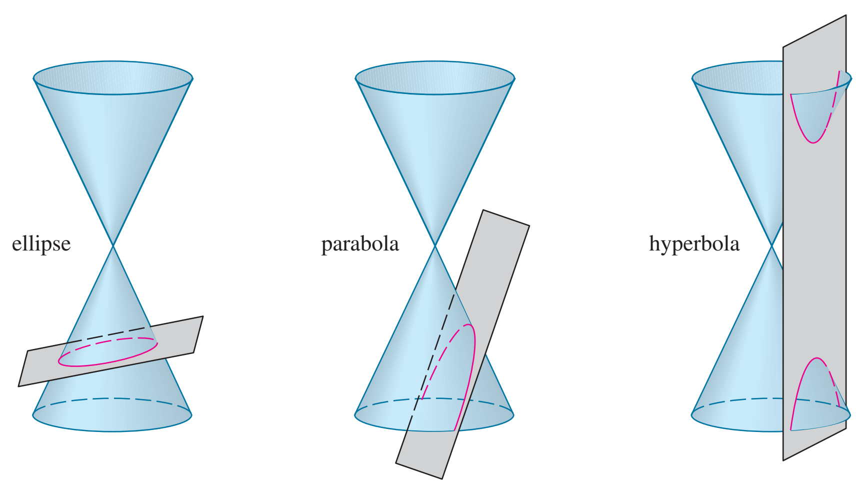


FIGURE 12
Conics

Parabolas

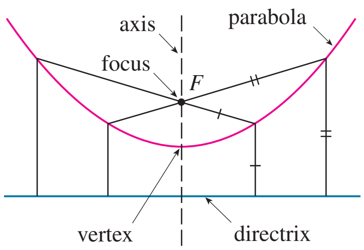


FIGURE 13

A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). This definition is illustrated by Figure 13. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**. The line through the focus perpendicular to the directrix is called the **axis** of the parabola.

In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges. (See Problem 16 on page 263 for the reflection property of parabolas that makes them so useful.)

We obtain a particularly simple equation for a parabola if we place its vertex at the origin O and its directrix parallel to the x-axis as in Figure 14. If the focus is the point $(0, p)$, then the directrix has the equation $y = -p$ and the parabola has the equation

$$x^2 = 4py$$

(See Exercise 47.)

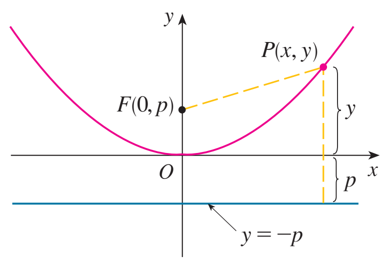


FIGURE 14

If we write $a = 1/(4p)$, then the equation of the parabola becomes

$$y = ax^2$$

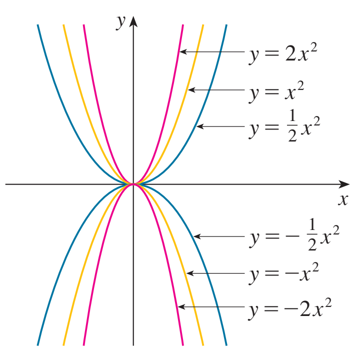


FIGURE 15

Figure 15 shows the graphs of several parabolas with equations of the form $y = ax^2$ for various values of the number a. We see that the parabola $y = ax^2$ opens upward if $a > 0$ and downward if $a < 0$ (as in Figure 16). The graph is symmetric with respect to the y-axis because its equation is unchanged when x is replaced by $-x$. This corresponds to the fact that the function $f(x) = ax^2$ is an even function.

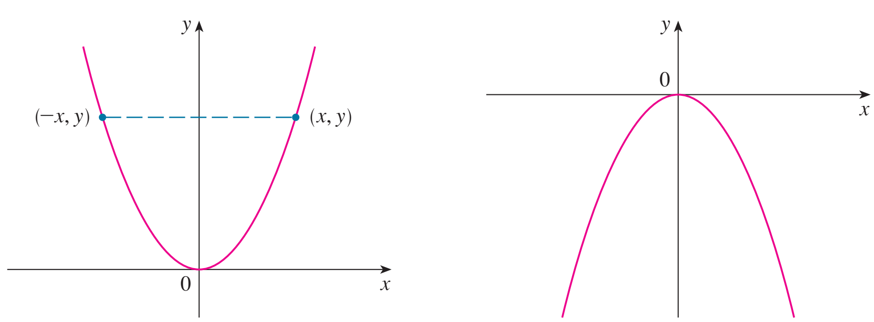


FIGURE 16 (a) $y = ax^2,\ a > 0$ (b) $y = ax^2,\ a < 0$

If we interchange x and y in the equation $y = ax^2$, the result is $x = ay^2$, which also represents a parabola. (Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola $x = ay^2$ opens to the right if $a > 0$ and to the left if $a < 0$. (See Figure 17.) This time the parabola is symmetric with respect to the x-axis because the equation is unchanged when y is replaced by $-y$.

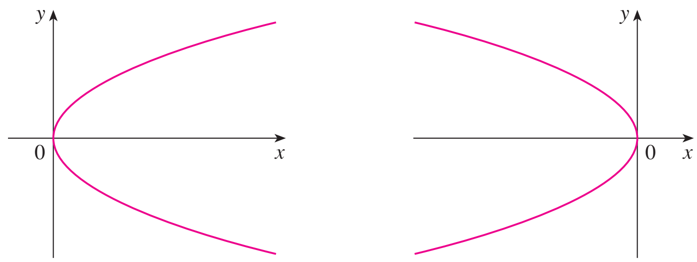


FIGURE 17

(a) $x = ay^2,\ a > 0$ (b) $x = ay^2,\ a < 0$

EXAMPLE 7 Sketch the region bounded by the parabola $x = 1 - y^2$ and the line $x + y + 1 = 0$.

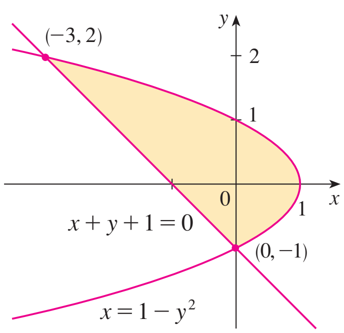


FIGURE 18

SOLUTION First we find the points of intersection by solving the two equations. Substituting $x = -y - 1$ into the equation $x = 1 - y^2$, we get $-y - 1 = 1 - y^2$, which gives

$$0 = y^2 - y - 2 = (y - 2)(y + 1)$$

so $y = 2$ or -1. Thus, the points of intersection are $(-3, 2)$ and $(0, -1)$, and we draw the line $x + y + 1 = 0$ passing through these points.

To sketch the parabola $x = 1 - y^2$ we start with the parabola $x = -y^2$ in Figure 17(b) and shift one unit to the right. We also make sure it passes through the points $(-3, 2)$ and $(0, -1)$. The region bounded by $x = 1 - y^2$ and $x + y + 1 = 0$ means the finite region whose boundaries are these curves. It is sketched in Figure 18. ■

Ellipses

An **ellipse** is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant (see Figure 19). These two fixed points are called the **foci** (plural of **focus**). One of Kepler's laws is that the orbits of the planets in the solar system are ellipses with the Sun at one focus.

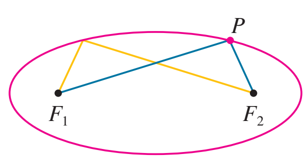


FIGURE 19

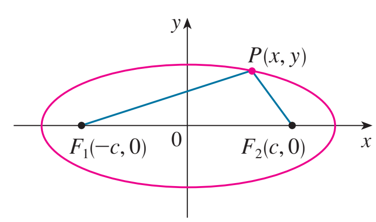


FIGURE 20

In order to obtain the simplest equation for an ellipse, we place the foci on the x-axis at the points $(-c, 0)$ and $(c, 0)$ as in Figure 20, so that the origin is halfway between the foci. If we let the sum of the distances from a point on the ellipse to the foci be $2a$, then we can write an equation of the ellipse as

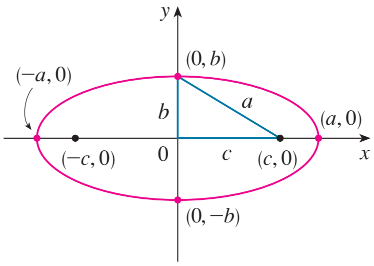


FIGURE 21
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

1
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 - b^2$. (See Exercise 49 and Figure 21.) Notice that the x-intercepts are $\pm a$, the y-intercepts are $\pm b$, the foci are $(\pm c, 0)$, and the ellipse is symmetric with respect to both axes. If the foci of an ellipse are located on the y-axis at $(0, \pm c)$, then we can find its equation by interchanging x and y in (1).

EXAMPLE 8 Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

SOLUTION Divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse, so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$. The x-intercepts are ± 4 and the y-intercepts are ± 3. Also, $c^2 = a^2 - b^2 = 7$, so $c = \sqrt{7}$ and the foci are $(\pm\sqrt{7}, 0)$. The graph is sketched in Figure 22. ■

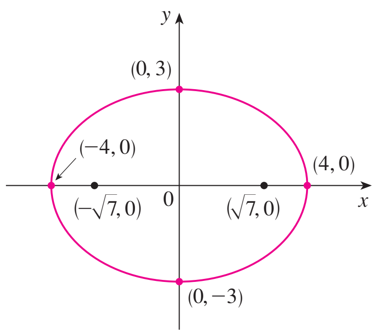


FIGURE 22
$9x^2 + 16y^2 = 144$

Like parabolas, ellipses have an interesting reflection property that has practical consequences. If a source of light or sound is placed at one focus of a surface with elliptical cross-sections, then all the light or sound is reflected off the surface to the other focus (see Exercise 55). This principle is used in *lithotripsy*, a treatment for kidney stones. A reflector with elliptical cross-section is placed in such a way that the kidney stone is at one focus. High-intensity sound waves generated at the other focus are reflected to the stone and destroy it without damaging surrounding tissue. The patient is spared the trauma of surgery and recovers within a few days.

Hyperbolas

A **hyperbola** is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the foci) is a constant. This definition is illustrated in Figure 23.

Notice that the definition of a hyperbola is similar to that of an ellipse; the only change is that the sum of distances has become a difference of distances. It is left as Exercise 51 to show that when the foci are on the x-axis at $(\pm c, 0)$ and the difference of distances is $|PF_1| - |PF_2| = \pm 2a$, then the equation of the hyperbola is

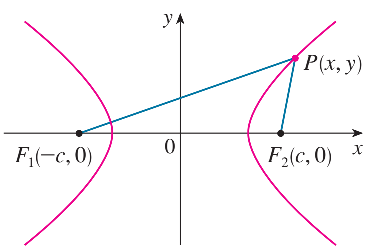


FIGURE 23
P is on the hyperbola when
$|PF_1| - |PF_2| = \pm 2a$

2
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$. Notice that the x-intercepts are again $\pm a$, But if we put $x = 0$ in Equation 2 we get $y = -b^2$, which is impossible, so there is no y-intercept. The hyperbola is symmetric with respect to both axes.

To analyze the hyperbola further, we look at Equation 2 and obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geqslant 1$$

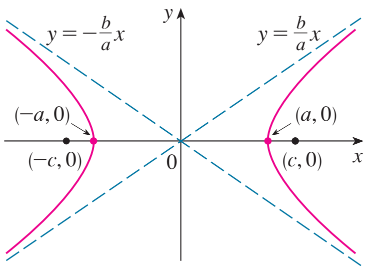


FIGURE 24
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

This shows that $x^2 \geqslant a^2$, so $|x| = \sqrt{x^2} \geqslant a$. Therefore, we have $x \geqslant a$ or $x \leqslant -a$. This means that the hyperbola consists of two parts, called its *branches*.

When we draw a hyperbola it is useful to first draw its *asymptotes*, which are the lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 24. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. If the foci of a hyperbola are on the y-axis, we find its equation by reversing the roles of x and y.

EXAMPLE 9 Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

SOLUTION If we divide both sides of the equation by 144, it becomes

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is of the form given in (2) with $a = 4$ and $b = 3$. Since $c^2 = 16 + 9 = 25$, the foci are $(\pm 5, 0)$. The asymptotes are the lines $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$. The graph is shown in Figure 25.

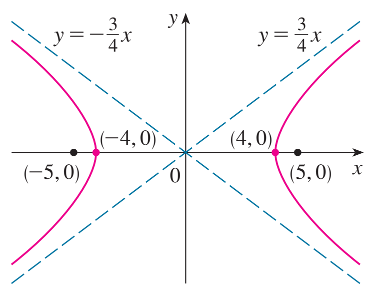


FIGURE 25
$9x^2-16y^2=144$

■

B Exercises

1–2 ■ Find the distance between the points.

1. (1, 1), (4, 5)

2. (1, −3), (5, 7)

3–4 ■ Find the slope of the line through P and Q.

3. $P(-3, 3)$, $Q(-1, -6)$

4. $P(-1, -4)$, $Q(6, 0)$

5. Show that the points (−2, 9), (4, 6), (1, 0), and (−5, 3) are the vertices of a square.

6. (a) Show that the points $A(-1, 3)$, $B(3, 11)$, and $C(5, 15)$ are collinear (lie on the same line) by showing that $|AB| + |BC| = |AC|$.
(b) Use slopes to show that A, B, and C are collinear.

7–10 ■ Sketch the graph of the equation.

7. $x = 3$

8. $y = -2$

9. $xy = 0$

10. $|y| = 1$

11–24 ■ Find an equation of the line that satisfies the given conditions.

11. Through (2, −3), slope 6

12. Through (−3, −5), slope $-\frac{7}{2}$

13. Through (2, 1) and (1, 6)

14. Through (−1, −2) and (4, 3)

15. Slope 3, y-intercept −2

16. Slope $\frac{2}{5}$, y-intercept 4

17. x-intercept 1, y-intercept −3

18. x-intercept −8, y-intercept 6

19. Through (4, 5), parallel to the x-axis

20. Through (4, 5), parallel to the y-axis

21. Through (1, −6), parallel to the line $x + 2y = 6$

22. y-intercept 6, parallel to the line $2x + 3y + 4 = 0$

23. Through $(-1, -2)$, perpendicular to the line $2x + 5y + 8 = 0$

24. Through $\left(\frac{1}{2}, -\frac{2}{3}\right)$, perpendicular to the line $4x - 8y = 1$

25–28 ■ Find the slope and y-intercept of the line and draw its graph.

25. $x + 3y = 0$

26. $2x - 3y + 6 = 0$

27. $3x - 4y = 12$

28. $4x + 5y = 10$

29–36 ■ Sketch the region in the xy-plane.

29. $\{(x, y) \mid x < 0\}$

30. $\{(x, y) \mid x \geqslant 1 \text{ and } y < 3\}$

31. $\{(x, y) \mid |x| \leqslant 2\}$

32. $\{(x, y) \mid |x| < 3 \text{ and } |y| < 2\}$

33. $\{(x, y) \mid 0 \leqslant y \leqslant 4 \text{ and } x \leqslant 2\}$

34. $\{(x, y) \mid y > 2x - 1\}$

35. $\{(x, y) \mid 1 + x \leqslant y \leqslant 1 - 2x\}$

36. $\{(x, y) \mid -x \leqslant y < \frac{1}{2}(x + 3)\}$

37–38 ■ Find an equation of a circle that satisfies the given conditions.

37. Center $(3, -1)$, radius 5

38. Center $(-1, 5)$, passes through $(-4, -6)$

39–40 ■ Show that the equation represents a circle and find the center and radius.

39. $x^2 + y^2 - 4x + 10y + 13 = 0$

40. $x^2 + y^2 + 6y + 2 = 0$

41. Show that the lines $2x - y = 4$ and $6x - 2y = 10$ are not parallel and find their point of intersection.

42. Show that the lines $3x - 5y + 19 = 0$ and $10x + 6y - 50 = 0$ are perpendicular and find their point of intersection.

43. Show that the midpoint of the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

44. Find the midpoint of the line segment joining the points $(1, 3)$ and $(7, 15)$.

45. Find an equation of the perpendicular bisector of the line segment joining the points $A(1, 4)$ and $B(7, -2)$.

46. (a) Show that if the x- and y-intercepts of a line are nonzero numbers a and b, then the equation of the line can be put in the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

This equation is called the **two-intercept form** of an equation of a line.

(b) Use part (a) to find an equation of the line whose x-intercept is 6 and whose y-intercept is -8.

47. Suppose that $P(x, y)$ is any point on the parabola with focus $(0, p)$ and directrix $y = -p$. (See Figure 14.) Use the definition of a parabola to show that $x^2 = 4py$.

48. Find the focus and directrix of the parabola $y = x^2$. Illustrate with a diagram.

49. Suppose an ellipse has foci $(\pm c, 0)$ and the sum of the distances from any point $P(x, y)$ on the ellipse to the foci is $2a$. Show that the coordinates of P satisfy Equation 1.

50. Find the foci of the ellipse $x^2 + 4y^2 = 4$ and sketch its graph.

51. Use the definition of a hyperbola to derive Equation 2 for a hyperbola with foci $(\pm c, 0)$.

52. (a) Find the foci and asymptotes of the hyperbola $x^2 - y^2 = 1$ and sketch its graph.

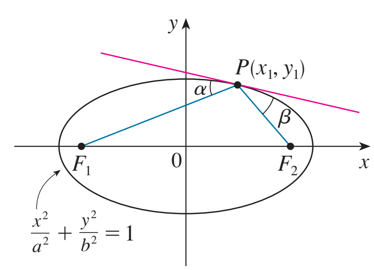
(b) Sketch the graph of $y^2 - x^2 = 1$.

53–54 ■ Sketch the region bounded by the curves.

53. $x + 4y = 8$ and $x = 2y^2 - 8$

54. $y = 4 - x^2$ and $x - 2y = 2$

55. Let $P_1(x_1, y_1)$ be a point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ with foci F_1 and F_2 and let α and β be the angles between the lines PF_1, PF_2 and the ellipse as in the figure. Prove that $\alpha = \beta$. This explains how whispering galleries and lithotripsy work. Sound coming from one focus is reflected and passes through the other focus. [*Hint:* Use the formula in Problem 15 on page 263 to show that $\tan \alpha = \tan \beta$.]



C Trigonometry

Here we review the aspects of trigonometry that are used in calculus: radian measure, trigonometric functions, trigonometric identities, and inverse trigonometric functions.

Angles

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains 360°, which is the same as 2π rad. Therefore

1 $$\pi \text{ rad} = 180°$$

and

2 $$1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ} \approx 57.3° \qquad 1° = \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad}$$

EXAMPLE 1
(a) Find the radian measure of 60°. (b) Express $5\pi/4$ rad in degrees.

SOLUTION
(a) From Equation 1 or 2 we see that to convert from degrees to radians we multiply by $\pi/180$. Therefore

$$60° = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{ rad}$$

(b) To convert from radians to degrees we multiply by $180/\pi$. Thus

$$\frac{5\pi}{4} \text{ rad} = \frac{5\pi}{4}\left(\frac{180}{\pi}\right) = 225°$$ ■

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

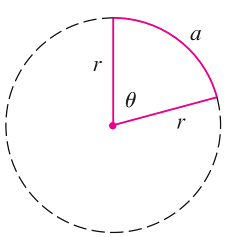


FIGURE 1

Figure 1 shows a sector of a circle with central angle θ and radius r subtending an arc with length a. Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference $2\pi r$ and central angle 2π, we have

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

Solving this equation for θ and for a, we obtain

3 $$\theta = \frac{a}{r} \qquad a = r\theta$$

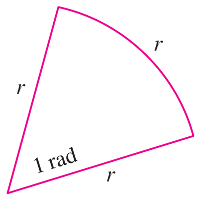


FIGURE 2

Remember that Equations 3 are valid only when θ is measured in radians.

In particular, putting $a = r$ in Equation 3, we see that an angle of 1 rad is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle (see Figure 2).

EXAMPLE 2

(a) If the radius of a circle is 5 cm, what angle is subtended by an arc of 6 cm?
(b) If a circle has radius 3 cm, what is the length of an arc subtended by a central angle of $3\pi/8$ rad?

SOLUTION

(a) Using Equation 3 with $a = 6$ and $r = 5$, we see that the angle is

$$\theta = \tfrac{6}{5} = 1.2 \text{ rad}$$

(b) With $r = 3$ cm and $\theta = 3\pi/8$ rad, the arc length is

$$a = r\theta = 3\left(\frac{3\pi}{8}\right) = \frac{9\pi}{8} \text{ cm}$$

The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x-axis as in Figure 3.

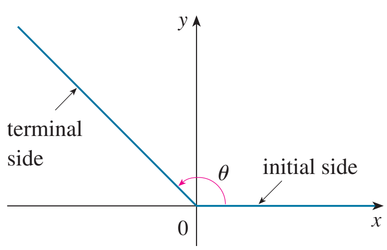


FIGURE 3
$\theta \geqslant 0$

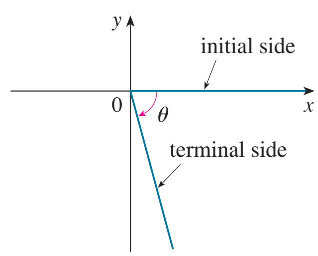


FIGURE 4
$\theta < 0$

A **positive** angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, **negative** angles are obtained by clockwise rotation as in Figure 4. Figure 5 shows several examples of angles in standard position. Notice that different angles can have the same terminal side. For instance, the angles $3\pi/4$, $-5\pi/4$, and $11\pi/4$ have the same initial and terminal sides because

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \qquad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and 2π rad represents a complete revolution.

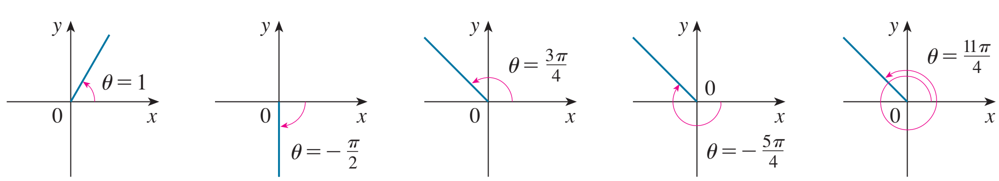


FIGURE 5
Angles in standard position

The Trigonometric Functions

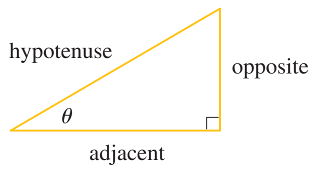


FIGURE 6

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 6).

4

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

This definition does not apply to obtuse or negative angles, so for a general angle θ in standard position we let $P(x, y)$ be any point on the terminal side of θ and we let r be the distance $|OP|$ as in Figure 7. Then we define

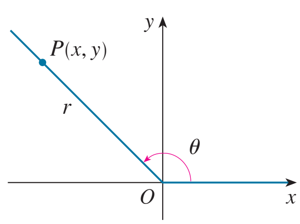


FIGURE 7

5

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Since division by 0 is not defined, $\tan\theta$ and $\sec\theta$ are undefined when $x = 0$ and $\csc\theta$ and $\cot\theta$ are undefined when $y = 0$. Notice that the definitions in (4) and (5) are consistent when θ is an acute angle.

If θ is a number, the convention is that $\sin\theta$ means the sine of the angle whose *radian* measure is θ. For example, the expression $\sin 3$ implies that we are dealing with an angle of 3 rad. When finding a calculator approximation to this number we must remember to set our calculator in radian mode, and then we obtain

$$\sin 3 \approx 0.14112$$

If we want to know the sine of the angle 3° we would write $\sin 3°$ and, with our calculator in degree mode, we find that

$$\sin 3° \approx 0.05234$$

The exact trigonometric ratios for certain angles can be read from the triangles in Figure 8. For instance,

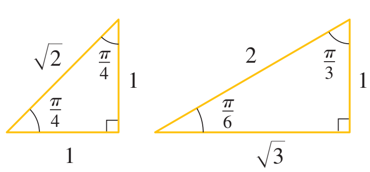


FIGURE 8

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

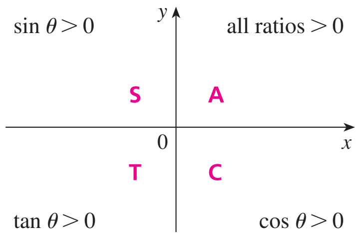


FIGURE 9

The signs of the trigonometric functions for angles in each of the four quadrants can be remembered by means of the rule "All Students Take Calculus" shown in Figure 9.

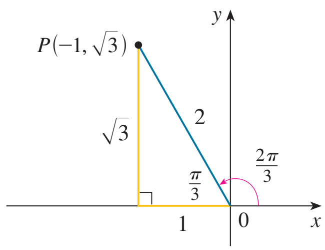


FIGURE 10

EXAMPLE 3 Find the exact trigonometric ratios for $\theta = 2\pi/3$.

SOLUTION From Figure 10 we see that a point on the terminal line for $\theta = 2\pi/3$ is $P(-1, \sqrt{3})$. Therefore, taking

$$x = -1 \qquad y = \sqrt{3} \qquad r = 2$$

in the definitions of the trigonometric ratios, we have

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec\frac{2\pi}{3} = -2 \qquad \cot\frac{2\pi}{3} = -\frac{1}{\sqrt{3}}$$

The following table gives some values of sin θ and cos θ found by the method of Example 3.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

EXAMPLE 4 If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.

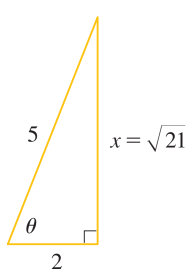


FIGURE 11

SOLUTION Since $\cos\theta = \frac{2}{5}$, we can label the hypotenuse as having length 5 and the adjacent side as having length 2 in Figure 11. If the opposite side has length x, then the Pythagorean Theorem gives $x^2 + 4 = 25$ and so $x^2 = 21$, or $x = \sqrt{21}$. We can now use the diagram to write the other five trigonometric functions:

$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta = \frac{5}{2} \qquad \cot\theta = \frac{2}{\sqrt{21}}$$

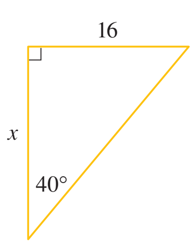


FIGURE 12

EXAMPLE 5 Use a calculator to approximate the value of x in Figure 12.

SOLUTION From the diagram we see that

$$\tan 40° = \frac{16}{x}$$

Therefore

$$x = \frac{16}{\tan 40°} \approx 19.07$$

Trigonometric Identities

A trigonometric identity is a relationship among the trigonometric functions. The most elementary are the following, which are immediate consequences of the definitions of the trigonometric functions.

6
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

For the next identity we refer back to Figure 7. The distance formula (or, equivalently, the Pythagorean Theorem) tells us that $x^2 + y^2 = r^2$. Therefore

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

We have therefore proved one of the most useful of all trigonometric identities:

7
$$\sin^2\theta + \cos^2\theta = 1$$

If we now divide both sides of Equation 7 by $\cos^2\theta$ and use Equations 6, we get

8
$$\tan^2\theta + 1 = \sec^2\theta$$

Similarly, if we divide both sides of Equation 7 by $\sin^2\theta$, we get

9
$$1 + \cot^2\theta = \csc^2\theta$$

The identities

10a
$$\sin(-\theta) = -\sin\theta$$

10b
$$\cos(-\theta) = \cos\theta$$

▲ Odd functions and even functions are discussed in Section 1.1.

show that sin is an odd function and cos is an even function. They are easily proved by drawing a diagram showing θ and $-\theta$ in standard position (see Exercise 19).

Since the angles θ and $\theta + 2\pi$ have the same terminal side, we have

11
$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$$

These identities show that the sine and cosine functions are periodic with period 2π.

The remaining trigonometric identities are all consequences of two basic identities called the **addition formulas**:

12a
$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

12b
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

The proofs of these addition formulas are outlined in Exercises 51, 52, and 53.

By substituting $-y$ for y in Equations 12a and 12b and using Equations 10a and 10b, we obtain the following **subtraction formulas**:

13a $$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

13b $$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

Then, by dividing the formulas in Equations 12 or Equations 13, we obtain the corresponding formulas for $\tan(x \pm y)$:

14a $$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

14b $$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

If we put $y = x$ in the addition formulas (12), we get the **double-angle formulas**:

15a $$\sin 2x = 2 \sin x \cos x$$

15b $$\cos 2x = \cos^2 x - \sin^2 x$$

Then, by using the identity $\sin^2 x + \cos^2 x = 1$, we obtain the following alternate forms of the double-angle formulas for $\cos 2x$:

16a $$\cos 2x = 2 \cos^2 x - 1$$

16b $$\cos 2x = 1 - 2 \sin^2 x$$

If we now solve these equations for $\cos^2 x$ and $\sin^2 x$, we get the following **half-angle formulas**, which are useful in integral calculus:

17a $$\cos^2 x = \frac{1 + \cos 2x}{2}$$

17b $$\sin^2 x = \frac{1 - \cos 2x}{2}$$

There are many other trigonometric identities, but those we have stated are the ones used most often in calculus. If you forget any of them, remember that they can all be deduced from Equations 12a and 12b.

EXAMPLE 6 Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

SOLUTION Using the double-angle formula (15a), we rewrite the given equation as

$$\sin x = 2 \sin x \cos x \qquad \text{or} \qquad \sin x\,(1 - 2 \cos x) = 0$$

Therefore, there are two possibilities:

$$\sin x = 0 \qquad \text{or} \qquad 1 - 2\cos x = 0$$

$$x = 0, \pi, 2\pi \qquad\qquad \cos x = \tfrac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

The given equation has five solutions: 0, $\pi/3$, π, $5\pi/3$, and 2π. ■

Graphs of the Trigonometric Functions

The graph of the function $f(x) = \sin x$, shown in Figure 13(a), is obtained by plotting points for $0 \leqslant x \leqslant 2\pi$ and then using the periodic nature of the function (from Equation 11) to complete the graph. Notice that the zeros of the sine function occur at the integer multiples of π, that is,

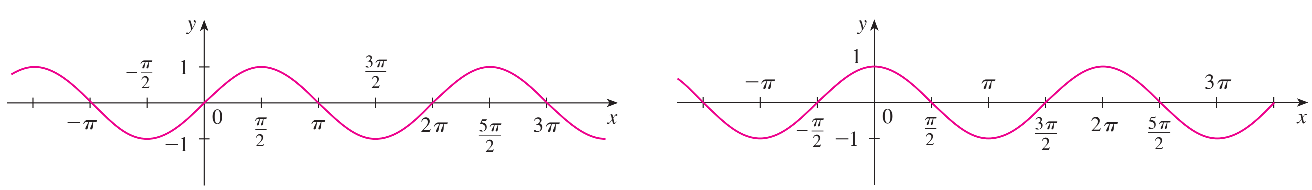
$$\sin x = 0 \qquad \text{whenever } x = n\pi, \quad n \text{ an integer}$$

Because of the identity

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

(which can be verified using Equation 12a), the graph of cosine is obtained by shifting the graph of sine by an amount $\pi/2$ to the left [see Figure 13(b)]. Note that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x, we have

$$-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1$$

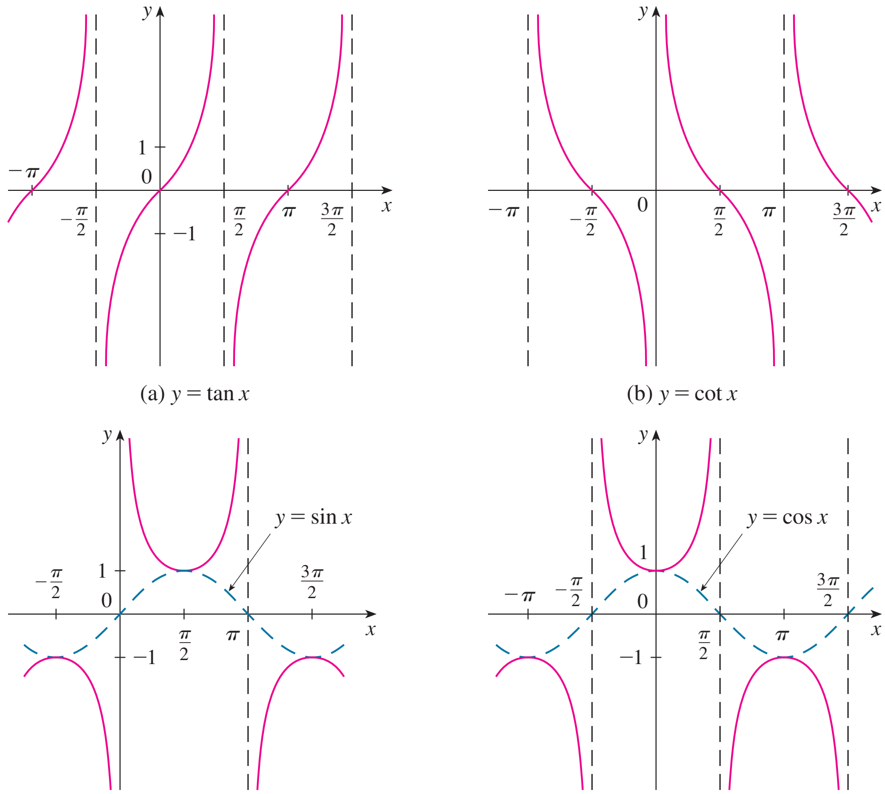


(a) $f(x) = \sin x$

(b) $g(x) = \cos x$

FIGURE 13

The graphs of the remaining four trigonometric functions are shown in Figure 14 and their domains are indicated there. Notice that tangent and cotangent have range $(-\infty, \infty)$, whereas cosecant and secant have range $(-\infty, -1] \cup [1, \infty)$. All four functions are periodic: tangent and cotangent have period π, whereas cosecant and secant have period 2π.



(a) $y = \tan x$ (b) $y = \cot x$

(c) $y = \csc x$ (d) $y = \sec x$

FIGURE 14

Inverse Trigonometric Functions

▲ Inverse functions are reviewed in Section 1.6.

When we try to find the inverse trigonometric functions, we have a slight difficulty: Because the trigonometric functions are not one-to-one, they don't have inverse functions. The difficulty is overcome by restricting the domains of these functions so that they become one-to-one.

You can see from Figure 15 that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test). But the function $f(x) = \sin x$, $-\pi/2 \leqslant x \leqslant \pi/2$ (see Figure 16), *is* one-to-one. The inverse function of this restricted sine function f exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.

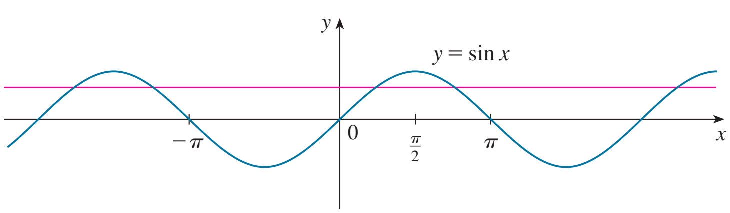


FIGURE 15

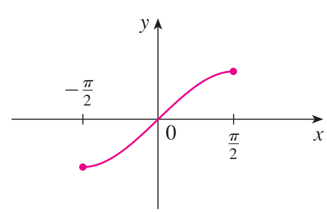


FIGURE 16

Since the definition of an inverse function says that

$$f^{-1}(x) = y \iff f(y) = x$$

we have

$$\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

⊘ $\sin^{-1}x \neq \dfrac{1}{\sin x}$

Thus, if $-1 \leqslant x \leqslant 1$, $\sin^{-1}x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is x.

EXAMPLE 7 Evaluate (a) $\sin^{-1}\left(\frac{1}{2}\right)$ and (b) $\tan\left(\arcsin \frac{1}{3}\right)$.

SOLUTION

(a) We have

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{6}$$

because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

(b) Let $\theta = \arcsin \frac{1}{3}$. Then we can draw a right triangle with angle θ as in Figure 17 and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9-1} = 2\sqrt{2}$. This enables us to read from the triangle that

$$\tan\left(\arcsin \tfrac{1}{3}\right) = \tan\theta = \frac{1}{2\sqrt{2}}$$

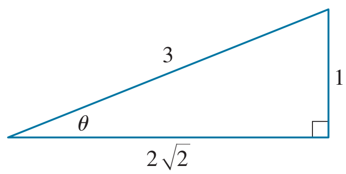


FIGURE 17

The cancellation equations for inverse functions [see (1.6.4)] become, in this case,

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

The inverse sine function, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$, and its graph, shown in Figure 18, is obtained from that of the restricted sine function (Figure 16) by reflection about the line $y = x$.

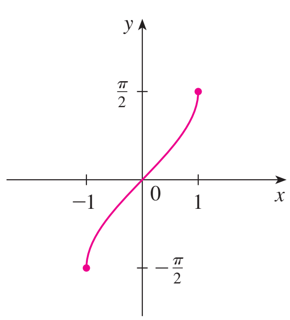


FIGURE 18 $y = \sin^{-1}x$

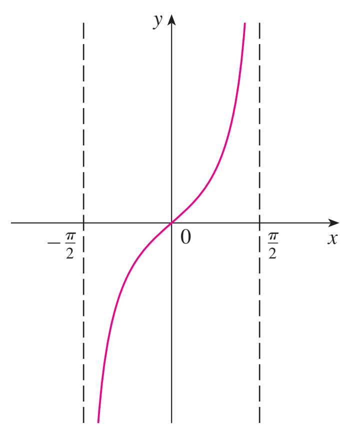


FIGURE 19 $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

The tangent function can be made one-to-one by restricting it to the interval $(-\pi/2, \pi/2)$. Thus, the **inverse tangent function** is defined as the inverse of the function $f(x) = \tan x$, $-\pi/2 < x < \pi/2$. (See Figure 19.) It is denoted by $\tan^{-1}$ or arctan.

$$\tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

EXAMPLE 8 Simplify the expression $\cos(\tan^{-1}x)$.

SOLUTION 1 Let $y = \tan^{-1}x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\sec y = \sqrt{1 + x^2} \qquad \text{(since } \sec y > 0 \text{ for } -\pi/2 < y < \pi/2\text{)}$$

Thus $$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

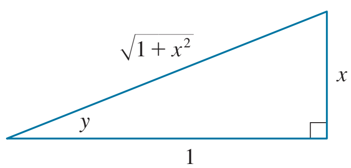


FIGURE 20

SOLUTION 2 Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If $y = \tan^{-1}x$, then $\tan y = x$, and we can read from Figure 20 (which illustrates the case $y > 0$) that

$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$

■

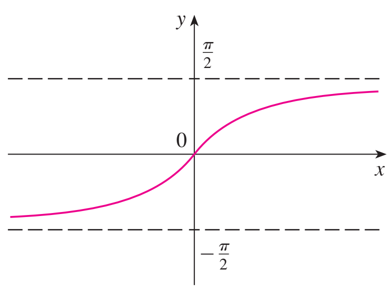


FIGURE 21
$y = \tan^{-1}x = \arctan x$

The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and its range is $(-\pi/2, \pi/2)$. Its graph is shown in Figure 21. We know that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan. Since the graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$.

Of the six inverse trigonometric functions, arcsin and arctan are the ones that are most useful for the purposes of calculus. The inverse cosine function is investigated in Exercise 46. The remaining inverse trigonometric functions don't arise as frequently.

C Exercises

1–2 ■ Convert from degrees to radians.

1. (a) $210°$ (b) $9°$

2. (a) $-315°$ (b) $36°$

3–4 ■ Convert from radians to degrees.

3. (a) 4π (b) $-\dfrac{3\pi}{8}$

4. (a) $-\dfrac{7\pi}{2}$ (b) $\dfrac{8\pi}{3}$

5. Find the length of a circular arc subtended by an angle of $\pi/12$ rad if the radius of the circle is 36 cm.

6. If a circle has radius 10 cm, find the length of the arc subtended by a central angle of $72°$.

7. A circle has radius 1.5 m. What angle is subtended at the center of the circle by an arc 1 m long?

8. Find the radius of a circular sector with angle $3\pi/4$ and arc length 6 cm.

9–10 ■ Draw, in standard position, the angle whose measure is given.

9. (a) $315°$ (b) $-\dfrac{3\pi}{4}$ rad

10. (a) $\dfrac{7\pi}{3}$ rad (b) -3 rad

11–12 ■ Find the exact trigonometric ratios for the angle whose radian measure is given.

11. $\dfrac{3\pi}{4}$ **12.** $\dfrac{4\pi}{3}$

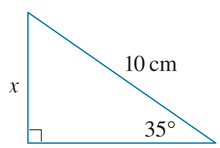
13–14 ■ Find the remaining trigonometric ratios.

13. $\sin\theta = \dfrac{3}{5}, \quad 0 < \theta < \dfrac{\pi}{2}$

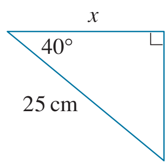
14. $\tan\alpha = 2, \quad 0 < \alpha < \dfrac{\pi}{2}$

15–18 ■ Find, correct to five decimal places, the length of the side labeled x.

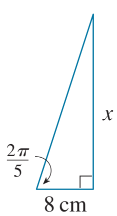
15.



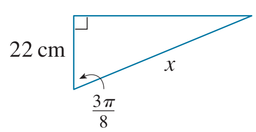
16.



17.



18.



19–20 ■ Prove each equation.

19. (a) Equation 10a (b) Equation 10b

20. (a) Equation 14a (b) Equation 14b

21–26 ■ Prove the identity.

21. $\sin\left(\dfrac{\pi}{2} + x\right) = \cos x$ **22.** $\sin(\pi - x) = \sin x$

23. $\sin\theta\cot\theta = \cos\theta$ **24.** $(\sin x + \cos x)^2 = 1 + \sin 2x$

25. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$ **26.** $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

27–28 ■ If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate the expression.

27. $\sin(x + y)$ **28.** $\cos 2y$

29–32 ■ Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.

29. $2\cos x - 1 = 0$ **30.** $2\sin^2 x = 1$

31. $\sin 2x = \cos x$ **32.** $|\tan x| = 1$

33–36 ■ Find all values of x in the interval $[0, 2\pi]$ that satisfy the inequality.

33. $\sin x \leqslant \frac{1}{2}$ **34.** $2\cos x + 1 > 0$

35. $-1 < \tan x < 1$ **36.** $\sin x > \cos x$

37–40 ■ Graph the function by starting with the graphs in Figures 13 and 14 and applying the transformations of Section 1.3 where appropriate.

37. $y = \cos\left(x - \dfrac{\pi}{3}\right)$ **38.** $y = \tan 2x$

39. $y = \dfrac{1}{3}\tan\left(x - \dfrac{\pi}{2}\right)$ **40.** $y = |\sin x|$

41–44 ■ Find the exact value of each expression.

41. (a) $\sin^{-1}(0.5)$ (b) $\arctan(-1)$

42. (a) $\tan^{-1}\sqrt{3}$ (b) $\arcsin 1$

43. (a) $\sin(\sin^{-1}(0.7))$ (b) $\arcsin\left(\sin\dfrac{5\pi}{4}\right)$

44. (a) $\sec(\arctan 2)$ (b) $\sin\left(2\sin^{-1}\left(\frac{3}{5}\right)\right)$

45. Prove that $\cos(\sin^{-1}x) = \sqrt{1 - x^2}$.

46. The inverse cosine function, $\cos^{-1} = \arccos$, is defined as the inverse of the restricted cosine function $f(x) = \cos x$, $0 \leqslant x \leqslant \pi$.
(a) What are the domain and range of the inverse cosine function?
(b) Sketch the graph of arccos.

47. Find the domain and range of the function

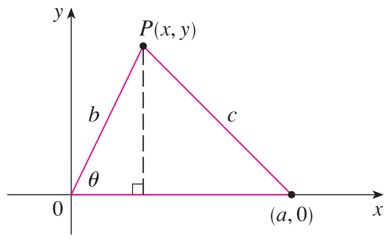
$$g(x) = \sin^{-1}(3x + 1)$$

48. (a) Graph the function $f(x) = \sin(\sin^{-1}x)$ and explain the appearance of the graph.
(b) Graph the function $g(x) = \sin^{-1}(\sin x)$. How do you explain the appearance of this graph?

49. Prove the **Law of Cosines**: If a triangle has sides with lengths a, b, and c, and θ is the angle between the sides with lengths a and b, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

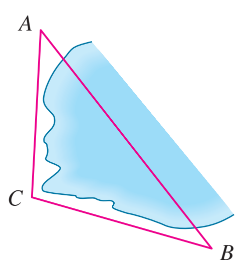
[*Hint:* Introduce a coordinate system so that θ is in standard position as in the figure. Express x and y in terms of θ and then use the distance formula to compute c.]



50. In order to find the distance $|AB|$ across a small inlet, a point C is located as in the figure and the following measurements were recorded:

$$\angle C = 103° \qquad |AC| = 820 \text{ m} \qquad |BC| = 910 \text{ m}$$

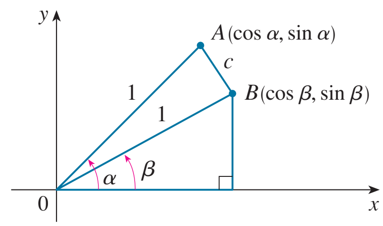
Use the Law of Cosines from Exercise 49 to find the required distance.



51. Use the figure to prove the subtraction formula

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

[*Hint:* Compute c^2 in two ways (using the Law of Cosines from Exercise 49 and also using the distance formula) and compare the two expressions.]



52. Use the formula in Exercise 51 to prove the addition formula for cosine (12b).

53. Use the addition formula for cosine and the identities

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

to prove the subtraction formula for the sine function.

54. (a) Show that the area of a triangle with sides of lengths a and b and with included angle θ is

$$A = \tfrac{1}{2}ab\sin\theta$$

(b) Find the area of triangle ABC, correct to five decimal places, if

$$|AB| = 10 \text{ cm} \qquad |BC| = 3 \text{ cm} \qquad \angle ABC = 107°$$

D Precise Definitions of Limits

The definitions of limits that have been given in this book are appropriate for intuitive understanding of the basic concepts of calculus. For the purposes of deeper understanding and rigorous proofs, however, the precise definitions of this appendix are necessary. In particular, the definition of a limit given here is used in Appendix E to prove that the limit of a sum is the sum of the limits.

When we say that $f(x)$ has a limit L as x approaches a, we mean, according to the intuitive definition in Section 2.2, that we can make $f(x)$ arbitrarily close to L by taking x close enough to a (but not equal to a). A more precise definition is based on the idea of specifying just how small we need to make the distance $|x - a|$ in order to make the distance $|f(x) - L|$ less than some given number. The following example illustrates the idea.

▲ It is traditional to use the Greek letter δ (delta) in this situation.

EXAMPLE 1 Use a graph to find a number δ such that

$$|(x^3 - 5x + 6) - 2| < 0.2 \qquad \text{whenever} \qquad |x - 1| < \delta$$

SOLUTION A graph of $f(x) = x^3 - 5x + 6$ is shown in Figure 1; we are interested in the region near the point $(1, 2)$. Notice that we can rewrite the inequality

$$|(x^3 - 5x + 6) - 2| < 0.2$$

as

$$1.8 < x^3 - 5x + 6 < 2.2$$

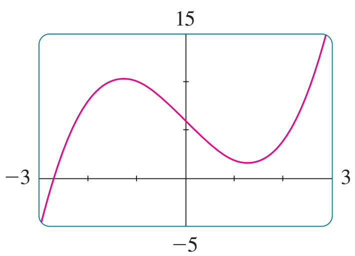


FIGURE 1

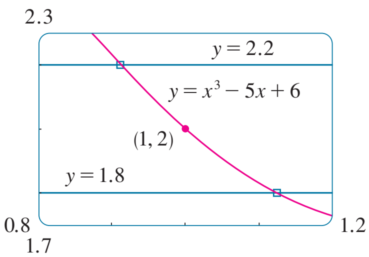


FIGURE 2

So we need to determine the values of x for which the curve $y = x^3 - 5x + 6$ lies between the horizontal lines $y = 1.8$ and $y = 2.2$. Therefore, we graph the curves $y = x^3 - 5x + 6$, $y = 1.8$, and $y = 2.2$ near the point $(1, 2)$ in Figure 2. Then we use the cursor to estimate that the x-coordinate of the point of intersection of the line $y = 2.2$ and the curve $y = x^3 - 5x + 6$ is about 0.911. Similarly, $y = x^3 - 5x + 6$ intersects the line $y = 1.8$ when $x \approx 1.124$. So, rounding to be safe, we can say that

$$1.8 < x^3 - 5x + 6 < 2.2 \qquad \text{whenever} \qquad 0.92 < x < 1.12$$

This interval $(0.92, 1.12)$ is not symmetric about $x = 1$. The distance from $x = 1$ to the left endpoint is $1 - 0.92 = 0.08$ and the distance to the right endpoint is 0.12. We can choose δ to be the smaller of these numbers, that is, $\delta = 0.08$. Then we can rewrite our inequalities in terms of distances as follows:

$$|(x^3 - 5x + 6) - 2| < 0.2 \qquad \text{whenever} \qquad |x - 1| < 0.08$$

This just says that by keeping x within 0.08 of 1, we are able to keep $f(x)$ within 0.2 of 2.

Although we chose $\delta = 0.08$, any smaller positive value of δ would also have worked. ■

Using the same graphical procedure as in Example 1, but replacing the number 0.2 by smaller numbers, we find that

$$|(x^3 - 5x + 6) - 2| < 0.1 \qquad \text{whenever} \qquad |x - 1| < 0.046$$

$$|(x^3 - 5x + 6) - 2| < 0.05 \qquad \text{whenever} \qquad |x - 1| < 0.024$$

$$|(x^3 - 5x + 6) - 2| < 0.01 \qquad \text{whenever} \qquad |x - 1| < 0.004$$

In each case we have found a number δ such that the values of the function

$f(x) = x^3 - 5x + 6$ lie in successively smaller intervals centered at 2 if the distance from x to 1 is less than δ. It turns out that it is always possible to find such a number δ, no matter how small the interval is. In other words, for *any* positive number ε, no matter how small, there exists a positive number δ such that

$$|(x^3 - 5x + 6) - 2| < \varepsilon \qquad \text{whenever} \qquad |x - 1| < \delta$$

This indicates that

$$\lim_{x \to 1} (x^3 - 5x + 6) = 2$$

and suggests a more precise way of defining the limit of a general function.

1 Definition Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - a| < \delta$$

▲ The condition $0 < |x - a|$ is just another way of saying that $x \neq a$.

Definition 1 is illustrated in Figures 3–5. If a number $\varepsilon > 0$ is given, then we draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ and the graph of f. (See Figure 3.) If $\lim_{x \to a} f(x) = L$, then we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ and take $x \neq a$, then the curve $y = f(x)$ lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$. (See Figure 4.) You can see that if such a δ has been found, then any smaller δ will also work.

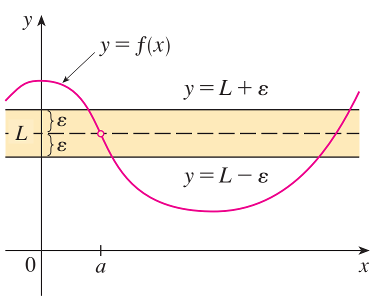


FIGURE 3

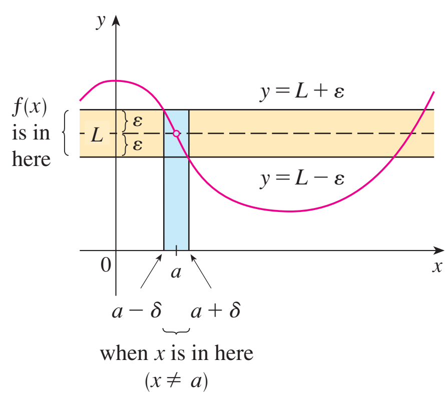


FIGURE 4

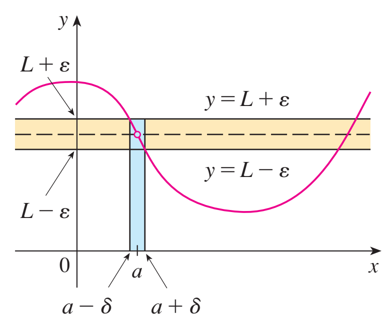


FIGURE 5

It's important to realize that the process illustrated in Figures 3 and 4 must work for *every* positive number ε no matter how small it is chosen. Figure 5 shows that if a smaller ε is chosen, then a smaller δ may be required.

EXAMPLE 2 Use the ε, δ definition to prove that $\lim_{x\to 0} x^2 = 0$.

SOLUTION Let ε be a given positive number. According to Definition 1 with $a = 0$ and $L = 0$, we need to find a number δ such that

$$|x^2 - 0| < \varepsilon \quad \text{whenever} \quad 0 < |x - 0| < \delta$$

that is,
$$x^2 < \varepsilon \quad \text{whenever} \quad 0 < |x| < \delta$$

But, since the square root function is an increasing function, we know that

$$x^2 < \varepsilon \iff \sqrt{x^2} < \sqrt{\varepsilon} \iff |x| < \sqrt{\varepsilon}$$

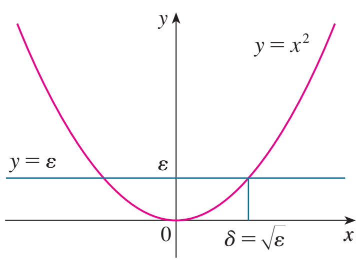


FIGURE 6

So if we choose $\delta = \sqrt{\varepsilon}$, then $x^2 < \varepsilon \iff |x| < \delta$ (see Figure 6). This shows that $\lim_{x\to 0} x^2 = 0$. ■

In proving limit statements it may be helpful to think of the definition of limit as a challenge. First it challenges you with a number ε. Then you must be able to produce a suitable δ. You have to be able to do this for *every* $\varepsilon > 0$, not just a particular ε.

Imagine a contest between two people, A and B, and imagine yourself to be B. Person A stipulates that the fixed number L should be approximated by the values of $f(x)$ to within a degree of accuracy ε (say, 0.01). Person B then responds by finding a number δ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$. Then A may become more exacting and challenge B with a smaller value of ε (say, 0.0001). Again B has to respond by finding a corresponding δ. Usually the smaller the value of ε, the smaller the corresponding value of δ must be. If B always wins, no matter how small A makes ε, then $\lim_{x\to a} f(x) = L$.

EXAMPLE 3 Prove that $\lim_{x\to 3} (4x - 5) = 7$.

SOLUTION

1. *Preliminary analysis of the problem (guessing a value for δ).* Let ε be a given positive number. We want to find a number δ such that

$$|(4x - 5) - 7| < \varepsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta$$

But $|(4x - 5) - 7| = |4x - 12| = |4(x - 3)| = 4|x - 3|$. Therefore, we want

$$4|x - 3| < \varepsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta$$

that is,
$$|x - 3| < \frac{\varepsilon}{4} \quad \text{whenever} \quad 0 < |x - 3| < \delta$$

This suggests that we should choose $\delta = \varepsilon/4$.

2. *Proof (showing that this δ works).* Given $\varepsilon > 0$, choose $\delta = \varepsilon/4$. If $0 < |x - 3| < \delta$, then

$$|(4x - 5) - 7| = |4x - 12| = 4|x - 3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

Thus

$$|(4x - 5) - 7| < \varepsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta$$

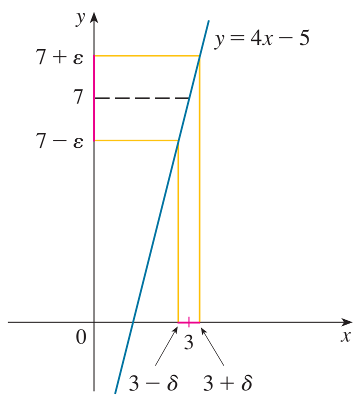


FIGURE 7

Therefore, by the definition of a limit,

$$\lim_{x \to 3} (4x - 5) = 7$$

This example is illustrated by Figure 7. ■

Note that in the solution of Example 2 there were two stages—guessing and proving. We made a preliminary analysis that enabled us to guess a value for δ. But then in the second stage we had to go back and prove in a careful, logical fashion that we had made a correct guess. This procedure is typical of much of mathematics. Sometimes it is necessary to first make an intelligent guess about the answer to a problem and then prove that the guess is correct.

It's not always easy to prove that limit statements are true using the ε, δ definition. For a more complicated function such as $f(x) = (6x^2 - 8x + 9)/(2x^2 - 1)$, a proof would require a great deal of ingenuity. Fortunately, this is not necessary because the Limit Laws stated in Section 2.3 can be proved using Definition 1, and then the limits of complicated functions can be found rigorously from the Limit Laws without resorting to the definition directly.

Limits at Infinity

Infinite limits and limits at infinity can also be defined in a precise way. The following is a precise version of Definition 4 in Section 2.5.

2 Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad x > N$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to L (within a distance ε, where ε is any positive number) by taking x sufficiently large (larger than N, where N depends on ε). Graphically it says that by choosing x large enough (larger than some number N) we can make the graph of f lie between the given horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$ as in Figure 8. This must be true no matter how small we choose ε. If a smaller value of ε is chosen, then a larger value of N may be required.

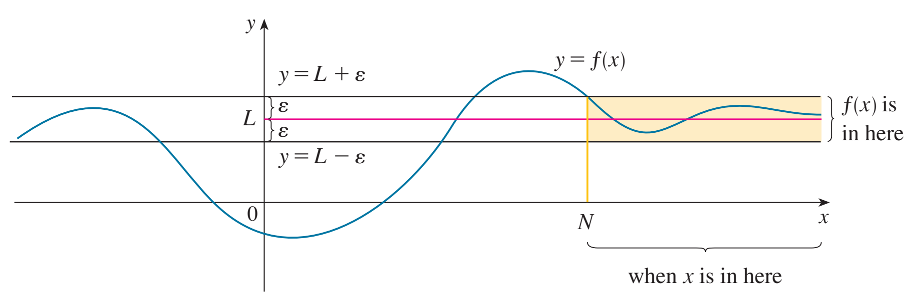


FIGURE 8
$\lim_{x \to \infty} f(x) = L$

In Example 5 in Section 2.5 we calculated that

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

In the next example we use a graphing device to relate this statement to Definition 2 with $L = \frac{3}{5}$ and $\varepsilon = 0.1$.

EXAMPLE 4 Use a graph to find a number N such that

$$\left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1 \qquad \text{whenever} \qquad x > N$$

SOLUTION We rewrite the given inequality as

$$0.5 < \frac{3x^2 - x - 2}{5x^2 + 4x + 1} < 0.7$$

We need to determine the values of x for which the given curve lies between the horizontal lines $y = 0.5$ and $y = 0.7$. So we graph the curve and these lines in Figure 9. Then we use the cursor to estimate that the curve crosses the line $y = 0.5$ when $x \approx 6.7$. To the right of this number the curve stays between the lines $y = 0.5$ and $y = 0.7$. Rounding to be safe, we can say that

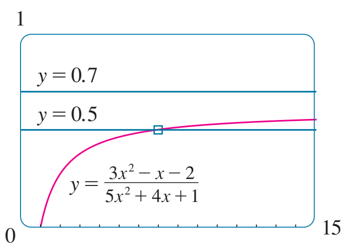


FIGURE 9

$$\left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1 \qquad \text{whenever} \qquad x > 7$$

In other words, for $\varepsilon = 0.1$ we can choose $N = 7$ (or any larger number) in Definition 2. ■

EXAMPLE 5 Use Definition 2 to prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

SOLUTION Let ε be a given positive number. According to Definition 2, we want to find N such that

$$\left| \frac{1}{x} - 0 \right| < \varepsilon \qquad \text{whenever} \qquad x > N$$

In computing the limit we may assume $x > 0$, in which case

$$\left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| = \frac{1}{x}$$

Therefore, we want

$$\frac{1}{x} < \varepsilon \qquad \text{whenever} \qquad x > N$$

that is,
$$x > \frac{1}{\varepsilon} \qquad \text{whenever} \qquad x > N$$

So if we choose $N = 1/\varepsilon$, then $1/x < \varepsilon \iff x > N$. This proves the desired limit.

Figure 10 illustrates the proof by showing some values of ε and the corresponding values of N.

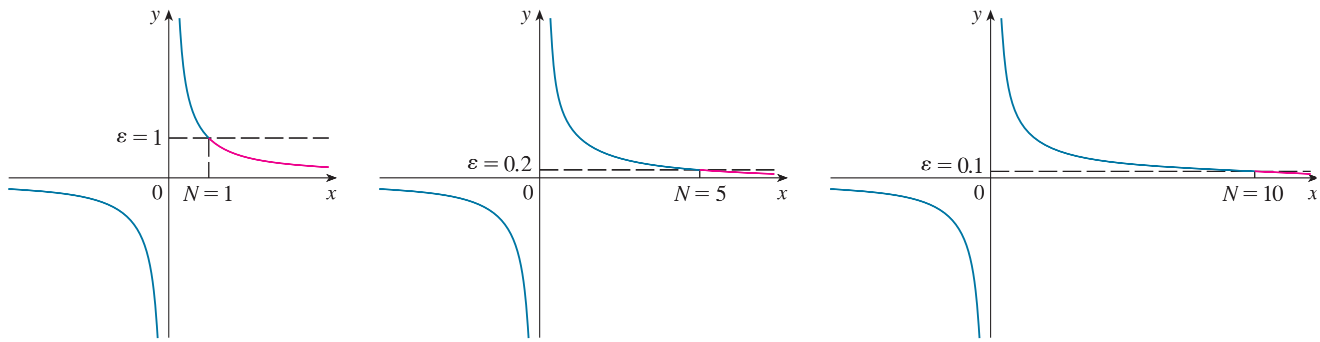


FIGURE 10

Infinite limits can also be formulated precisely. See Exercise 16.

Sequences

In Section 8.1 we used the notation

$$\lim_{n\to\infty} a_n = L$$

to mean that the terms of the sequence $\{a_n\}$ approach L as n becomes large. Notice that the following precise definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity (Definition 2).

3 Definition A sequence $\{a_n\}$ has the **limit** L and we write

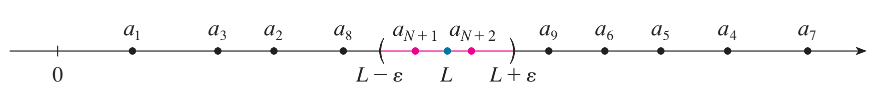
$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N$$

Definition 3 is illustrated by Figure 11, in which the terms $a_1, a_2, a_3, \ldots$ are plotted on a number line. No matter how small an interval $(L - \varepsilon, L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.

FIGURE 11



Another illustration of Definition 3 is given in Figure 12. The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N.

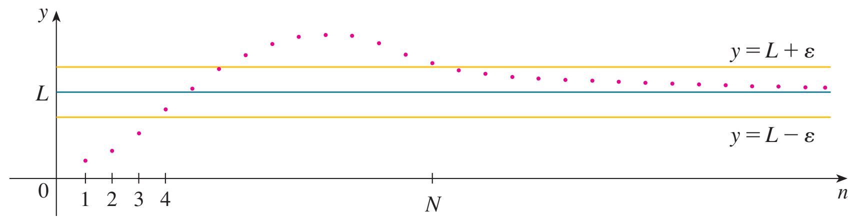


FIGURE 12

Comparison of Definitions 2 and 3 shows that the only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is required to be an integer. The following definition shows how to make precise the idea that $\{a_n\}$ becomes infinite as n becomes infinite.

4 Definition If $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$a_n > M \qquad \text{whenever} \qquad n > N$$

EXAMPLE 6 Prove that $\lim\limits_{n\to\infty} \sqrt{n} = \infty$.

SOLUTION Let M be any positive number. (Think of it as being very large.) Then

$$\sqrt{n} > M \iff n > M^2$$

So if we take $N = M^2$, then Definition 4 shows that $\lim_{n\to\infty} \sqrt{n} = \infty$. ■

Functions of Two Variables

Here is a precise version of Definition 1 in Section 11.2:

5 Definition Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y)\to(a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \text{ whenever } (x, y) \in D \text{ and } 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

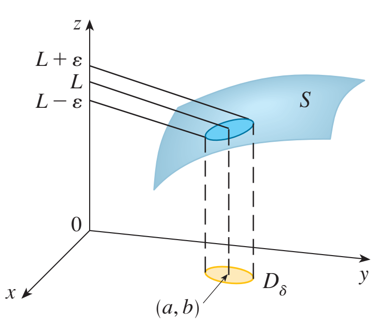


FIGURE 13

Because $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L, and $\sqrt{(x - a)^2 + (y - b)^2}$ is the distance between the point (x, y) and the point (a, b), Definition 5 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). An illustration of Definition 5 is given in Figure 13 where the surface S is the graph of f. If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that if (x, y) is restricted to lie in the disk D_δ with center (a, b) and radius δ, and if $(x, y) \neq (a, b)$, then the corresponding part of S lies between the horizontal planes $z = L - \varepsilon$ and $z = L + \varepsilon$.

EXAMPLE 7 Prove that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

SOLUTION Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta$$

that is,

$$\frac{3x^2|y|}{x^2+y^2} < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta$$

But $x^2 \leqslant x^2 + y^2$ since $y^2 \geqslant 0$, so $x^2/(x^2+y^2) \leqslant 1$ and therefore

$$\frac{3x^2|y|}{x^2+y^2} \leqslant 3|y| = 3\sqrt{y^2} \leqslant 3\sqrt{x^2+y^2}$$

Thus, if we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2+y^2} < \delta$, then

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| \leqslant 3\sqrt{x^2+y^2} \leqslant 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

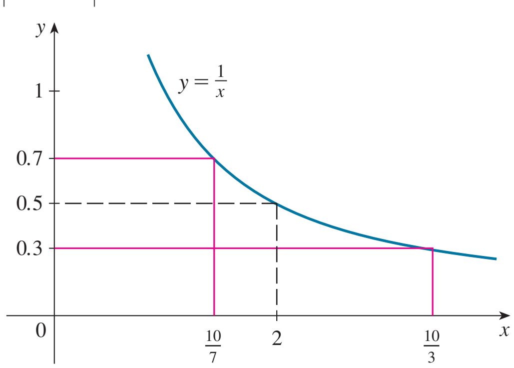
Hence, by Definition 5,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

D Exercises

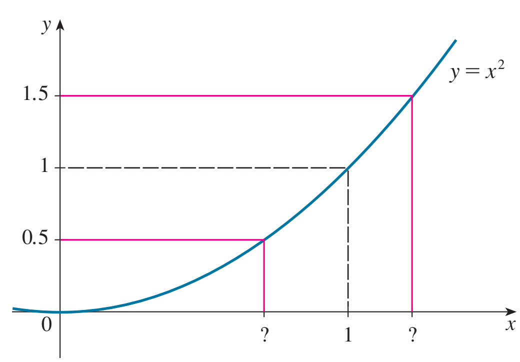
1. Use the given graph of $f(x) = 1/x$ to find a number δ such that

$$\left|\frac{1}{x} - 0.5\right| < 0.2 \quad \text{whenever} \quad |x - 2| < \delta$$



2. Use the given graph of $f(x) = x^2$ to find a number δ such that

$$|x^2 - 1| < \tfrac{1}{2} \quad \text{whenever} \quad |x - 1| < \delta$$



3. Use a graph to find a number δ such that

$$\left|\sqrt{4x+1} - 3\right| < 0.5 \qquad \text{whenever} \qquad |x - 2| < \delta$$

4. Use a graph to find a number δ such that

$$\left|\sin x - \tfrac{1}{2}\right| < 0.1 \qquad \text{whenever} \qquad \left|x - \frac{\pi}{6}\right| < \delta$$

5. For the limit

$$\lim_{x \to 1} (4 + x - 3x^3) = 2$$

illustrate Definition 1 by finding values of δ that correspond to $\varepsilon = 1$ and $\varepsilon = 0.1$.

6. For the limit

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

illustrate Definition 1 by finding values of δ that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

7. Use Definition 1 to prove that $\lim_{x \to 0} x^3 = 0$.

8. (a) How would you formulate an ε, δ definition of the one-sided limit $\lim_{x \to a^+} f(x) = L$?
(b) Use your definition in part (a) to prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

9–10 ■ Prove the statement using the ε, δ definition of limit and illustrate with a diagram like Figure 7.

9. $\lim_{x \to 2} (3x - 2) = 4$ **10.** $\lim_{x \to 4} (5 - 2x) = -3$

11. A machinist is required to manufacture a circular metal disk with area 1000 cm^2.
(a) What radius produces such a disk?
(b) If the machinist is allowed an error tolerance of ± 5 cm^2 in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?
(c) In terms of the ε, δ definition of $\lim_{x \to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

12. A crystal growth furnace is used in research to determine how best to manufacture crystals used in electronic components for the space shuttle. For proper growth of the crystal, the temperature must be controlled accurately by adjusting the input power. Suppose the relationship is given by

$$T(w) = 0.1w^2 + 2.155w + 20$$

where T is the temperature in degrees Celsius and w is the power input in watts.
(a) How much power is needed to maintain the temperature at 200 °C?
(b) If the temperature is allowed to vary from 200 °C by up to ± 1 °C, what range of wattage is allowed for the input power?
(c) In terms of the ε, δ definition of $\lim_{x \to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

13. Use a graph to find a number N such that

$$\left|\frac{6x^2 + 5x - 3}{2x^2 - 1} - 3\right| < 0.2 \qquad \text{whenever} \qquad x > N$$

14. For the limit

$$\lim_{x \to \infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = 2$$

illustrate Definition 2 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

15. (a) Determine how large we have to take x so that

$$\frac{1}{x^2} < 0.0001$$

(b) Use Definition 2 to prove that

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

16. (a) For what values of x is it true that

$$\frac{1}{x^2} > 1{,}000{,}000$$

(b) The precise definition of $\lim_{x \to a} f(x) = \infty$ states that for every positive number M (no matter how large) there is a corresponding positive number δ such that $f(x) > M$ whenever $0 < |x - a| < \delta$. Use this definition to prove that $\lim_{x \to 0} (1/x^2) = \infty$.

17. (a) Use a graph to guess the value of the limit

$$\lim_{n \to \infty} \frac{n^5}{n!}$$

(b) Use a graph of the sequence in part (a) to find the smallest values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.001$ in Definition 3.

18. Use Definition 3 to prove that $\lim_{n \to \infty} r^n = 0$ when $|r| < 1$.

19. Use Definition 3 to prove that if $\lim_{n \to \infty} |a_n| = 0$, then $\lim_{n \to \infty} a_n = 0$.

20. Use Definition 4 to prove that $\lim_{n \to \infty} n^3 = \infty$.

21. Use Definition 5 to prove that $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0$.

E A Few Proofs

In this appendix we present proofs of some theorems that were stated in the main body of the text. We start by proving the Triangle Inequality, which is an important property of absolute value.

> **The Triangle Inequality** If a and b are any real numbers, then
> $$|a + b| \leqslant |a| + |b|$$

Observe that if the numbers a and b are both positive or both negative, then the two sides in the Triangle Inequality are actually equal. But if a and b have opposite signs, the left side involves a subtraction and the right side does not. This makes the Triangle Inequality seem reasonable, but we can prove it as follows.

Notice that

$$-|a| \leqslant a \leqslant |a|$$

is always true because a equals either $|a|$ or $-|a|$. The corresponding statement for b is

$$-|b| \leqslant b \leqslant |b|$$

Adding these inequalities, we get

$$-(|a| + |b|) \leqslant a + b \leqslant |a| + |b|$$

▲ When combined, Properties 4 and 5 of absolute value (see Appendix A) say that

$$|x| \leqslant a \iff -a \leqslant x \leqslant a$$

If we now apply Properties 4 and 5 of absolute value from Appendix A (with x replaced by $a + b$ and a by $|a| + |b|$), we obtain

$$|a + b| \leqslant |a| + |b|$$

which is what we wanted to show. ■

Next we use the Triangle Inequality to prove the Sum Law for limits.

▲ The Sum Law was first stated in Section 2.3.

> **Sum Law** If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ both exist, then
> $$\lim_{x \to a} [f(x) + g(x)] = L + M$$

Proof Let $\varepsilon > 0$ be given. According to Definition 1 in Appendix D, we must find $\delta > 0$ such that

$$|f(x) + g(x) - (L + M)| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - a| < \delta$$

Using the Triangle Inequality we can write

$$\boxed{1} \qquad \begin{aligned} |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leqslant |f(x) - L| + |g(x) - M| \end{aligned}$$

We will make $|f(x) + g(x) - (L + M)|$ less than ε by making each of the terms $|f(x) - L|$ and $|g(x) - M|$ less than $\varepsilon/2$.

Since $\varepsilon/2 > 0$ and $\lim_{x \to a} f(x) = L$, there exists a number $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{2} \qquad \text{whenever} \qquad 0 < |x - a| < \delta_1$$

Similarly, since $\lim_{x \to a} g(x) = M$, there exists a number $\delta_2 > 0$ such that

$$|g(x) - M| < \frac{\varepsilon}{2} \qquad \text{whenever} \qquad 0 < |x - a| < \delta_2$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Notice that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad 0 < |x - a| < \delta_1 \quad \text{and} \quad 0 < |x - a| < \delta_2$$

and so $$|f(x) - L| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |g(x) - M| < \frac{\varepsilon}{2}$$

Therefore, by (1),

$$\begin{aligned} |f(x) + g(x) - (L + M)| &\leqslant |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

To summarize,

$$|f(x) + g(x) - (L + M)| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - a| < \delta$$

Thus, by the definition of a limit,

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$ ■

▲ Fermat's Theorem was discussed in Section 4.2.

Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Proof Suppose, for the sake of definiteness, that f has a local maximum at c. Then, $f(c) \geqslant f(x)$ if x is sufficiently close to c. This implies that if h is sufficiently close to 0, with h being positive or negative, then

$$f(c) \geqslant f(c + h)$$

and therefore

2 $$f(c + h) - f(c) \leqslant 0$$

We can divide both sides of an inequality by a positive number. Thus, if $h > 0$ and h is sufficiently small, we have

$$\frac{f(c + h) - f(c)}{h} \leqslant 0$$

Taking the right-hand limit of both sides of this inequality (using Theorem 2.3.2), we get

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leqslant \lim_{h\to 0^+} 0 = 0$$

But since $f'(c)$ exists, we have

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$$

and so we have shown that $f'(c) \leqslant 0$.

If $h < 0$, then the direction of the inequality (2) is reversed when we divide by h:

$$\frac{f(c+h)-f(c)}{h} \geqslant 0 \qquad h < 0$$

So, taking the left-hand limit, we have

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geqslant 0$$

We have shown that $f'(c) \geqslant 0$ and also that $f'(c) \leqslant 0$. Since both of these inequalities must be true, the only possibility is that $f'(c) = 0$.

We have proved Fermat's Theorem for the case of a local maximum. The case of a local minimum can be proved in a similar manner. ■

▲ Clairaut's Theorem was discussed in Section 11.3.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Proof For small values of h, $h \neq 0$, consider the difference

$$\Delta(h) = [f(a+h, b+h) - f(a+h, b)] - [f(a, b+h) - f(a, b)]$$

Notice that if we let $g(x) = f(x, b+h) - f(x, b)$, then

$$\Delta(h) = g(a+h) - g(a)$$

By the Mean Value Theorem, there is a number c between a and $a + h$ such that

$$g(a+h) - g(a) = g'(c)h = h[f_x(c, b+h) - f_x(c, b)]$$

Applying the Mean Value Theorem again, this time to f_x, we get a number d between b and $b + h$ such that

$$f_x(c, b+h) - f_x(c, b) = f_{xy}(c, d)h$$

Combining these equations, we obtain

$$\Delta(h) = h^2 f_{xy}(c, d)$$

If $h \to 0$, then $(c, d) \to (a, b)$, so the continuity of f_{xy} at (a, b) gives

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = \lim_{(c, d) \to (a, b)} f_{xy}(c, d) = f_{xy}(a, b)$$

Similarly, by writing

$$\Delta(h) = [f(a + h, b + h) - f(a, b + h)] - [f(a + h, b) - f(a, b)]$$

and using the Mean Value Theorem twice and the continuity of f_{yx} at (a, b), we obtain

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a, b)$$

It follows that $f_{xy}(a, b) = f_{yx}(a, b)$. ■

▲ This was stated as Theorem 8 in Section 11.4.

Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Proof Let

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

According to Definition 11.4.7, to prove that f is differentiable at (a, b) we have to show that we can write Δz in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Referring to Figure 1, we write

3 $$\Delta z = [f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y)] + [f(a, b + \Delta y) - f(a, b)]$$

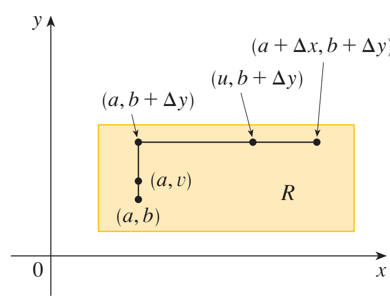


FIGURE 1

Observe that the function of a single variable

$$g(x) = f(x, b + \Delta y)$$

is defined on the interval $[a, a + \Delta x]$ and $g'(x) = f_x(x, b + \Delta y)$. If we apply the Mean Value Theorem to g, we get

$$g(a + \Delta x) - g(a) = g'(u)\,\Delta x$$

where u is some number between a and $a + \Delta x$. In terms of f, this equation becomes

$$f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y) = f_x(u, b + \Delta y)\,\Delta x$$

This gives us an expression for the first part of the right side of Equation 3. For the second part we let $h(y) = f(a, y)$. Then h is a function of a single variable defined on the interval $[b, b + \Delta y]$ and $h'(y) = f_y(a, y)$. A second application of the Mean Value Theorem then gives

$$h(b + \Delta y) - h(b) = h'(v)\,\Delta y$$

where v is some number between b and $b + \Delta y$. In terms of f, this becomes

$$f(a, b + \Delta y) - f(a, b) = f_y(a, v)\,\Delta y$$

We now substitute these expressions into Equation 3 and obtain

$$\begin{aligned}\Delta z &= f_x(u, b + \Delta y)\,\Delta x + f_y(a, v)\,\Delta y \\ &= f_x(a, b)\Delta x + [f_x(u, b + \Delta y) - f_x(a, b)]\,\Delta x + f_y(a, b)\,\Delta y \\ &\qquad + [f_y(a, v) - f_y(a, b)]\,\Delta y \\ &= f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y\end{aligned}$$

where

$$\varepsilon_1 = f_x(u, b + \Delta y) - f_x(a, b)$$

$$\varepsilon_2 = f_y(a, v) - f_y(a, b)$$

Since $(u, b + \Delta y) \to (a, b)$ and $(a, v) \to (a, b)$ as $(\Delta x, \Delta y) \to (0, 0)$ and since f_x and f_y are continuous at (a, b), we see that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Therefore, f is differentiable at (a, b). ■

▲ The Second Derivatives Test was discussed in Section 11.7. Parts (b) and (c) have similar proofs.

Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Proof of part (a) We compute the second-order directional derivative of f in the direction of $\mathbf{u} = \langle h, k \rangle$. The first-order derivative is given by Theorem 11.6.3:

$$D_{\mathbf{u}} f = f_x h + f_y k$$

Applying this theorem a second time, we have

$$\begin{aligned} D_{\mathbf{u}}^2 f = D_{\mathbf{u}}(D_{\mathbf{u}} f) &= \frac{\partial}{\partial x}(D_{\mathbf{u}} f)h + \frac{\partial}{\partial y}(D_{\mathbf{u}} f)k \\ &= (f_{xx}h + f_{yx}k)h + (f_{xy}h + f_{yy}k)k \\ &= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \qquad \text{(by Clairaut's Theorem)} \end{aligned}$$

If we complete the square in this expression, we obtain

4
$$D_{\mathbf{u}}^2 f = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}(f_{xx}f_{yy} - f_{xy}^2)$$

We are given that $f_{xx}(a, b) > 0$ and $D(a, b) > 0$. But f_{xx} and $D = f_{xx}f_{yy} - f_{xy}^2$ are continuous functions, so there is a disk B with center (a, b) and radius $\delta > 0$ such that $f_{xx}(x, y) > 0$ and $D(x, y) > 0$ whenever (x, y) is in B. Therefore, by looking at Equation 4, we see that $D_{\mathbf{u}}^2 f(x, y) > 0$ whenever (x, y) is in B. This means that if C is the curve obtained by intersecting the graph of f with the vertical plane through $P(a, b, f(a, b))$ in the direction of $\mathbf{u}$, then C is concave upward on an interval of length 2δ. This is true in the direction of every vector $\mathbf{u}$, so if we restrict (x, y) to lie in B, the graph of f lies above its horizontal tangent plane at P. Thus, $f(x, y) \geqslant f(a, b)$ whenever (x, y) is in B. This shows that $f(a, b)$ is a local minimum. ■

F Sigma Notation

A convenient way of writing sums uses the Greek letter Σ (capital sigma, corresponding to our letter S) and is called **sigma notation**.

This tells us to end with $i = n$.
This tells us to add.
This tells us to start with $i = m$.

$$\sum_{i=m}^{n} a_i$$

1 Definition If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leqslant n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

With function notation, Definition 1 can be written as

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

Thus, the symbol $\Sigma_{i=m}^{n}$ indicates a summation in which the letter i (called the **index of summation**) takes on consecutive integer values beginning with m and ending with n, that is, $m, m + 1, \ldots, n$. Other letters can also be used as the index of summation.

EXAMPLE 1

(a) $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

(b) $\sum_{i=3}^{n} i = 3 + 4 + 5 + \cdots + (n-1) + n$

(c) $\sum_{j=0}^{5} 2^j = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

(d) $\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

(e) $\sum_{i=1}^{3} \frac{i-1}{i^2+3} = \frac{1-1}{1^2+3} + \frac{2-1}{2^2+3} + \frac{3-1}{3^2+3} = 0 + \frac{1}{7} + \frac{1}{6} = \frac{13}{42}$

(f) $\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 8$

EXAMPLE 2 Write the sum $2^3 + 3^3 + \cdots + n^3$ in sigma notation.

SOLUTION There is no unique way of writing a sum in sigma notation. We could write

$$2^3 + 3^3 + \cdots + n^3 = \sum_{i=2}^{n} i^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{j=1}^{n-1} (j+1)^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{k=0}^{n-2} (k+2)^3$$

The following theorem gives three simple rules for working with sigma notation.

2 Theorem If c is any constant (that is, it does not depend on i), then

(a) $\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i$

(b) $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

(c) $\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$

Proof To see why these rules are true, all we have to do is write both sides in expanded form. Rule (a) is just the distributive property of real numbers:

$$ca_m + ca_{m+1} + \cdots + ca_n = c(a_m + a_{m+1} + \cdots + a_n)$$

Rule (b) follows from the associative and commutative properties:

$$(a_m + b_m) + (a_{m+1} + b_{m+1}) + \cdots + (a_n + b_n)$$
$$= (a_m + a_{m+1} + \cdots + a_n) + (b_m + b_{m+1} + \cdots + b_n)$$

Rule (c) is proved similarly.

EXAMPLE 3 Find $\sum_{i=1}^{n} 1$.

SOLUTION

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n$$

■

EXAMPLE 4 Prove the formula for the sum of the first n positive integers:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

SOLUTION This formula can be proved by mathematical induction (see page 89) or by the following method used by the German mathematician Karl Friedrich Gauss (1777–1855) when he was ten years old.

Write the sum S twice, once in the usual order and once in reverse order:

$$S = 1 + 2 + 3 + \cdots + (n-1) + n$$

$$S = n + (n-1) + (n-2) + \cdots + 2 + 1$$

Adding all columns vertically, we get

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

On the right side there are n terms, each of which is $n + 1$, so

$$2S = n(n+1) \qquad \text{or} \qquad S = \frac{n(n+1)}{2}$$

■

EXAMPLE 5 Prove the formula for the sum of the squares of the first n positive integers:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

SOLUTION 1 Let S be the desired sum. We start with the *telescoping sum* (or collapsing sum):

Most terms cancel in pairs.

$$\sum_{i=1}^{n} [(1+i)^3 - i^3] = (2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + \cdots + [(n+1)^3 - n^3]$$

$$= (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n$$

On the other hand, using Theorem 2 and Examples 3 and 4, we have

$$\sum_{i=1}^{n} [(1+i)^3 - i^3] = \sum_{i=1}^{n} [3i^2 + 3i + 1] = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$= 3S + 3\frac{n(n+1)}{2} + n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n$$

Thus, we have

$$n^3 + 3n^2 + 3n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n$$

Solving this equation for S, we obtain

$$3S = n^3 + \tfrac{3}{2}n^2 + \tfrac{1}{2}n$$

or

$$S = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

▲ **Principle of Mathematical Induction**
Let S_n be a statement involving the positive integer n. Suppose that
1. S_1 is true.
2. If S_k is true, then S_{k+1} is true.
Then S_n is true for all positive integers n.

▲ See pages 89 and 92 for a more thorough discussion of mathematical induction.

SOLUTION 2 Let S_n be the given formula.

1. S_1 is true because $\quad 1^2 = \dfrac{1(1+1)(2 \cdot 1 + 1)}{6}$

2. Assume that S_k is true; that is,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Then

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 &= (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\,\frac{k(2k+1) + 6(k+1)}{6} \\
&= (k+1)\,\frac{2k^2 + 7k + 6}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}
\end{aligned}$$

So S_{k+1} is true.

By the Principle of Mathematical Induction, S_n is true for all n. ■

We list the results of Examples 3, 4, and 5 together with a similar result for cubes (see Exercises 37–40) as Theorem 3. These formulas are needed for finding areas and evaluating integrals in Chapter 5.

3 Theorem Let c be a constant and n a positive integer. Then

(a) $\displaystyle\sum_{i=1}^{n} 1 = n$

(b) $\displaystyle\sum_{i=1}^{n} c = nc$

(c) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(d) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(e) $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

EXAMPLE 6 Evaluate $\sum_{i=1}^{n} i(4i^2 - 3)$.

SOLUTION Using Theorems 2 and 3, we have

$$\begin{aligned}\sum_{i=1}^{n} i(4i^2 - 3) &= \sum_{i=1}^{n} (4i^3 - 3i) = 4\sum_{i=1}^{n} i^3 - 3\sum_{i=1}^{n} i \\ &= 4\left[\frac{n(n+1)}{2}\right]^2 - 3\,\frac{n(n+1)}{2} \\ &= \frac{n(n+1)[2n(n+1) - 3]}{2} \\ &= \frac{n(n+1)(2n^2 + 2n - 3)}{2}\end{aligned}$$

▲ The type of calculation in Example 7 arises in Chapter 5 when we compute areas.

EXAMPLE 7 Find $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right]$.

SOLUTION

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right] &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{3}{n^3} i^2 + \frac{3}{n}\right] \\ &= \lim_{n\to\infty} \left[\frac{3}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} 1\right] \\ &= \lim_{n\to\infty} \left[\frac{3}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\cdot n\right] \\ &= \lim_{n\to\infty} \left[\frac{1}{2}\cdot\frac{n}{n}\cdot\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) + 3\right] \\ &= \lim_{n\to\infty} \left[\frac{1}{2}\cdot 1\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 3\right] \\ &= \tfrac{1}{2}\cdot 1\cdot 1\cdot 2 + 3 = 4\end{aligned}$$

F Exercises

1–10 ■ Write the sum in expanded form.

1. $\sum_{i=1}^{5} \sqrt{i}$

2. $\sum_{i=1}^{6} \frac{1}{i+1}$

3. $\sum_{i=4}^{6} 3^i$

4. $\sum_{i=4}^{6} i^3$

5. $\sum_{k=0}^{4} \frac{2k-1}{2k+1}$

6. $\sum_{k=5}^{8} x^k$

7. $\sum_{i=1}^{n} i^{10}$

8. $\sum_{j=n}^{n+3} j^2$

9. $\sum_{j=0}^{n-1} (-1)^j$

10. $\sum_{i=1}^{n} f(x_i)\,\Delta x_i$

11–20 ■ Write the sum in sigma notation.

11. $1 + 2 + 3 + 4 + \cdots + 10$

12. $\sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7}$

13. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{19}{20}$

14. $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27}$

15. $2 + 4 + 6 + 8 + \cdots + 2n$

16. $1 + 3 + 5 + 7 + \cdots + (2n - 1)$

17. $1 + 2 + 4 + 8 + 16 + 32$

18. $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$

19. $x + x^2 + x^3 + \cdots + x^n$

20. $1 - x + x^2 - x^3 + \cdots + (-1)^n x^n$

21–35 ■ Find the value of the sum.

21. $\sum_{i=4}^{8} (3i - 2)$

22. $\sum_{i=3}^{6} i(i + 2)$

23. $\sum_{j=1}^{6} 3^{j+1}$

24. $\sum_{k=0}^{8} \cos k\pi$

25. $\sum_{n=1}^{20} (-1)^n$

26. $\sum_{i=1}^{100} 4$

27. $\sum_{i=0}^{4} (2^i + i^2)$

28. $\sum_{i=-2}^{4} 2^{3-i}$

29. $\sum_{i=1}^{n} 2i$

30. $\sum_{i=1}^{n} (2 - 5i)$

31. $\sum_{i=1}^{n} (i^2 + 3i + 4)$

32. $\sum_{i=1}^{n} (3 + 2i)^2$

33. $\sum_{i=1}^{n} (i + 1)(i + 2)$

34. $\sum_{i=1}^{n} i(i + 1)(i + 2)$

35. $\sum_{i=1}^{n} (i^3 - i - 2)$

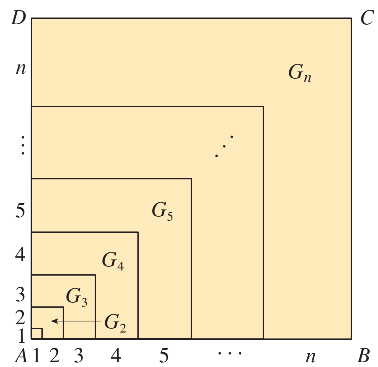
36. Find the number n such that $\sum_{i=1}^{n} i = 78$.

37. Prove formula (b) of Theorem 3.

38. Prove formula (e) of Theorem 3 using mathematical induction.

39. Prove formula (e) of Theorem 3 using a method similar to that of Example 5, Solution 1 [start with $(1 + i)^4 - i^4$].

40. Prove formula (e) of Theorem 3 using the following method published by Abu Bekr Mohammed ibn Alhusain Alkarchi in about A.D. 1010. The figure shows a square $ABCD$ in which sides AB and AD have been divided into segments of lengths $1, 2, 3, \ldots, n$. Thus, the side of the square has length $n(n + 1)/2$ so the area is $[n(n + 1)/2]^2$. But the area is also the sum of the areas of the n "gnomons" $G_1, G_2, \ldots, G_n$ shown in the figure. Show that the area of G_i is i^3 and conclude that formula (e) is true.



41. Evaluate each telescoping sum.

(a) $\sum_{i=1}^{n} [i^4 - (i - 1)^4]$

(b) $\sum_{i=1}^{100} (5^i - 5^{i-1})$

(c) $\sum_{i=3}^{99} \left(\frac{1}{i} - \frac{1}{i + 1}\right)$

(d) $\sum_{i=1}^{n} (a_i - a_{i-1})$

42. Prove the generalized triangle inequality

$$\left| \sum_{i=1}^{n} a_i \right| \leqslant \sum_{i=1}^{n} |a_i|$$

43–46 ■ Find each limit.

43. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2$

44. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^3 + 1\right]$

45. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{2i}{n}\right)^3 + 5\left(\frac{2i}{n}\right)\right]$

46. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(1 + \frac{3i}{n}\right)^3 - 2\left(1 + \frac{3i}{n}\right)\right]$

47. Prove the formula for the sum of a finite geometric series with first term a and common ratio $r \neq 1$:

$$\sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

48. Evaluate $\sum_{i=1}^{n} \frac{3}{2^{i-1}}$.

49. Evaluate $\sum_{i=1}^{n} (2i + 2^i)$.

50. Evaluate $\sum_{i=1}^{m} \left[\sum_{j=1}^{n} (i + j)\right]$.

G Integration of Rational Functions by Partial Fractions

In this appendix we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions,* that we already know how to integrate. To illustrate the method, observe that by taking the fractions $2/(x - 1)$ and $1/(x + 2)$ to a common denominator we obtain

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2) - (x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2 + x - 2}$$

If we now reverse the procedure, we see how to integrate the function on the right side of this equation:

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2} \right) dx$$
$$= 2 \ln|x-1| - \ln|x+2| + C$$

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. It's possible to express f as a sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called *proper.* Recall that if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, then the degree of P is n and we write $\deg(P) = n$.

If f is improper, that is, $\deg(P) \geqslant \deg(Q)$, then we must take the preliminary step of dividing Q into P (by long division) until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$. The division statement is

1 $$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

As the following example illustrates, sometimes this preliminary step is all that is required.

EXAMPLE 1 Find $\displaystyle\int \frac{x^3 + x}{x - 1}\,dx$.

$$\begin{array}{r|l} & x^2 + x + 2 \\ \hline x - 1 & x^3 \qquad\quad + x \\ & x^3 - x^2 \\ & \qquad x^2 + x \\ & \qquad x^2 - x \\ & \qquad\qquad 2x \\ & \qquad\qquad 2x - 2 \\ & \qquad\qquad\quad 2 \end{array}$$

SOLUTION Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division. This enables us to write

$$\int \frac{x^3 + x}{x - 1}\,dx = \int \left(x^2 + x + 2 + \frac{2}{x-1} \right) dx$$
$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2 \ln|x - 1| + C$$ ■

The next step is to factor the denominator $Q(x)$ as far as possible. It can be shown that any polynomial Q can be factored as a product of linear factors (of the form $ax + b$) and irreducible quadratic factors (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$). For instance, if $Q(x) = x^4 - 16$, we could factor it as

$$Q(x) = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$$

The third step is to express the proper rational function $R(x)/Q(x)$ (from Equation 1) as a sum of **partial fractions** of the form

$$\frac{A}{(ax + b)^i} \qquad \text{or} \qquad \frac{Ax + B}{(ax^2 + bx + c)^j}$$

A theorem in algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.

CASE I • The denominator $Q(x)$ is a product of distinct linear factors.
This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

where no factor is repeated. In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

2 $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

These constants can be determined as in the following example.

EXAMPLE 2 Evaluate $\displaystyle\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx$.

SOLUTION Since the degree of the numerator is less than the degree of the denominator, we don't need to divide. We factor the denominator as

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand (2) has the form

3 $$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

▲ Another method for finding A, B, and C is given in the note after this example.

To determine the values of A, B, and C, we multiply both sides of this equation by the product of the denominators, $x(2x - 1)(x + 2)$, obtaining

4 $$x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$$

Expanding the right side of Equation 4 and writing it in the standard form for polynomials, we get

5 $$x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

The polynomials in Equation 5 are identical, so their coefficients must be equal. The coefficient of x^2 on the right side, $2A + B + 2C$, must equal the coefficient of x^2 on the left side—namely, 1. Likewise, the coefficients of x are equal and the constant terms are equal. This gives the following system of equations for A, B, and C:

$$\begin{aligned} 2A + B + 2C &= 1 \\ 3A + 2B - C &= 2 \\ -2A \qquad\qquad &= -1 \end{aligned}$$

▲ Figure 1 shows the graphs of the integrand in Example 2 and its indefinite integral (with $K = 0$). Which is which?

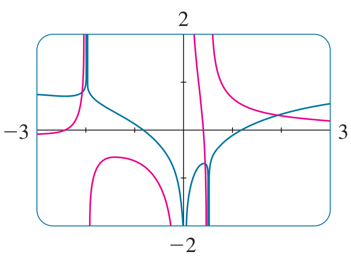


FIGURE 1

Solving, we get $A = \frac{1}{2}$, $B = \frac{1}{5}$, and $C = -\frac{1}{10}$, and so

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx = \int \left[\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x - 1} - \frac{1}{10}\frac{1}{x + 2}\right] dx$$

$$= \tfrac{1}{2}\ln|x| + \tfrac{1}{10}\ln|2x - 1| - \tfrac{1}{10}\ln|x + 2| + K$$

In integrating the middle term we have made the mental substitution $u = 2x - 1$, which gives $du = 2\,dx$ and $dx = du/2$. ■

NOTE • We can use an alternative method to find the coefficients A, B, and C in Example 2. Equation 4 is an identity; it is true for every value of x. Let's choose values of x that simplify the equation. If we put $x = 0$ in Equation 4, then the second and third terms on the right side vanish and the equation then becomes $-2A = -1$, or $A = \frac{1}{2}$. Likewise, $x = \frac{1}{2}$ gives $5B/4 = \frac{1}{4}$ and $x = -2$ gives $10C = -1$, so $B = \frac{1}{5}$ and $C = -\frac{1}{10}$. (You may object that Equation 3 is not valid for $x = 0, \frac{1}{2}$, or -2, so why should Equation 4 be valid for those values? In fact, Equation 4 is true for all values of x, even $x = 0, \frac{1}{2}$, and -2. See Exercise 35 for the reason.)

EXAMPLE 3 Find $\displaystyle\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

SOLUTION The method of partial fractions gives

$$\frac{1}{x^2 - a^2} = \frac{1}{(x - a)(x + a)} = \frac{A}{x - a} + \frac{B}{x + a}$$

and therefore

$$A(x + a) + B(x - a) = 1$$

Using the method of the preceding note, we put $x = a$ in this equation and get $A(2a) = 1$, so $A = 1/(2a)$. If we put $x = -a$, we get $B(-2a) = 1$, so $B = -1/(2a)$. Thus

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\int \left[\frac{1}{x - a} - \frac{1}{x + a}\right] dx$$

$$= \frac{1}{2a}\left[\ln|x - a| - \ln|x + a|\right] + C$$

Since $\ln x - \ln y = \ln(x/y)$, we can write the integral as

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$$

CASE II • **$Q(x)$ is a product of linear factors, some of which are repeated.** Suppose the first linear factor $(a_1 x + b_1)$ is repeated r times; that is, $(a_1 x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1 x + b_1)$ in Equation 2, we would use

6
$$\frac{A_1}{a_1 x + b_1} + \frac{A_2}{(a_1 x + b_1)^2} + \cdots + \frac{A_r}{(a_1 x + b_1)^r}$$

By way of illustration, we could write

$$\frac{x^3 - x + 1}{x^2(x - 1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2} + \frac{E}{(x - 1)^3}$$

but we prefer to work out in detail a simpler example.

EXAMPLE 4 Find $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$.

SOLUTION The first step is to divide. The result of long division is

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

The second step is to factor the denominator $Q(x) = x^3 - x^2 - x + 1$. Since $Q(1) = 0$, we know that $x - 1$ is a factor and we obtain

$$\begin{aligned} x^3 - x^2 - x + 1 &= (x - 1)(x^2 - 1) = (x - 1)(x - 1)(x + 1) \\ &= (x - 1)^2(x + 1) \end{aligned}$$

Since the linear factor $x - 1$ occurs twice, the partial fraction decomposition is

$$\frac{4x}{(x - 1)^2(x + 1)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 1}$$

Multiplying by the least common denominator, $(x - 1)^2(x + 1)$, we get

7
$$\begin{aligned} 4x &= A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2 \\ &= (A + C)x^2 + (B - 2C)x + (-A + B + C) \end{aligned}$$

▲ Another method for finding the coefficients:
Put $x = 1$ in (7): $B = 2$.
Put $x = -1$: $C = -1$.
Put $x = 0$: $A = B + C = 1$.

Now we equate coefficients:

$$\begin{aligned} A \qquad + C &= 0 \\ B - 2C &= 4 \\ -A + B + C &= 0 \end{aligned}$$

Solving, we obtain $A = 1$, $B = 2$, and $C = -1$, so

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right] dx$$

$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + K$$ ■

CASE III • $Q(x)$ contains irreducible quadratic factors, none of which is repeated. If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 6, the expression for $R(x)/Q(x)$ will have a term of the form

8 $$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined. For instance, the function given by $f(x) = x/[(x-2)(x^2+1)(x^2+4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{x^2 + 4}$$

The term given in (8) can be integrated by completing the square and using the formula

9 $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

EXAMPLE 5 Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$.

SOLUTION Since $x^3 + 4x = x(x^2 + 4)$ can't be factored further, we write

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiplying by $x(x^2 + 4)$, we have

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$$= (A + B)x^2 + Cx + 4A$$

Equating coefficients, we obtain

$$A + B = 2 \qquad C = -1 \qquad 4A = 4$$

Thus $A = 1$, $B = 1$, and $C = -1$ and so

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left[\frac{1}{x} + \frac{x - 1}{x^2 + 4} \right] dx$$

In order to integrate the second term we split it into two parts:

$$\int \frac{x-1}{x^2+4}\,dx = \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

We make the substitution $u = x^2 + 4$ in the first of these integrals so that $du = 2x\,dx$. We evaluate the second integral by means of Formula 9 with $a = 2$:

$$\int \frac{2x^2 - x + 4}{x(x^2+4)}\,dx = \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$$= \ln|x| + \tfrac{1}{2}\ln(x^2+4) - \tfrac{1}{2}\tan^{-1}(x/2) + K$$

EXAMPLE 6 Evaluate $\displaystyle\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx$.

SOLUTION Since the degree of the numerator is not less than the degree of the denominator, we first divide and obtain

$$\frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} = 1 + \frac{x-1}{4x^2 - 4x + 3}$$

Notice that the quadratic $4x^2 - 4x + 3$ is irreducible because its discriminant is $b^2 - 4ac = -32 < 0$. This means it can't be factored, so we don't need to use the partial fraction technique.

To integrate the given function we complete the square in the denominator:

$$4x^2 - 4x + 3 = (2x-1)^2 + 2$$

This suggests that we make the substitution $u = 2x - 1$. Then, $du = 2\,dx$ and $x = (u+1)/2$, so

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx = \int \left(1 + \frac{x-1}{4x^2 - 4x + 3}\right) dx$$

$$= x + \tfrac{1}{2}\int \frac{\tfrac{1}{2}(u+1) - 1}{u^2+2}\,du = x + \tfrac{1}{4}\int \frac{u-1}{u^2+2}\,du$$

$$= x + \tfrac{1}{4}\int \frac{u}{u^2+2}\,du - \tfrac{1}{4}\int \frac{1}{u^2+2}\,du$$

$$= x + \tfrac{1}{8}\ln(u^2+2) - \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= x + \tfrac{1}{8}\ln(4x^2 - 4x + 3) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$$

NOTE • Example 6 illustrates the general procedure for integrating a partial fraction of the form

$$\frac{Ax+B}{ax^2+bx+c} \qquad \text{where } b^2 - 4ac < 0$$

We complete the square in the denominator and then make a substitution that brings the integral into the form

$$\int \frac{Cu + D}{u^2 + a^2}\,du = C\int \frac{u}{u^2 + a^2}\,du + D\int \frac{1}{u^2 + a^2}\,du$$

Then the first integral is a logarithm and the second is expressed in terms of $\tan^{-1}$.

CASE IV • $Q(x)$ contains a repeated irreducible quadratic factor.
If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction (8), the sum

10 $$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the terms in (10) can be integrated by first completing the square.

▲ It would be extremely tedious to work out by hand the numerical values of the coefficients in Example 7. Most computer algebra systems, however, can find the numerical values very quickly. For instance, the Maple command

convert(f, parfrac, x)

or the Mathematica command

Apart[f]

gives the following values:

$$A = -1, \quad B = \tfrac{1}{8}, \quad C = D = -1,$$
$$E = \tfrac{15}{8}, \quad F = -\tfrac{1}{8}, \quad G = H = \tfrac{3}{4},$$
$$I = -\tfrac{1}{2}, \quad J = \tfrac{1}{2}$$

EXAMPLE 7 Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x - 1)(x^2 + x + 1)(x^2 + 1)^3}$$

SOLUTION

$$\frac{x^3 + x^2 + 1}{x(x - 1)(x^2 + x + 1)(x^2 + 1)^3}$$
$$= \frac{A}{x} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + x + 1} + \frac{Ex + F}{x^2 + 1} + \frac{Gx + H}{(x^2 + 1)^2} + \frac{Ix + J}{(x^2 + 1)^3}$$ ■

EXAMPLE 8 Evaluate $\displaystyle\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2}\,dx$.

SOLUTION The form of the partial fraction decomposition is

$$\frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

Multiplying by $x(x^2 + 1)^2$, we have

$$\begin{aligned} -x^3 + 2x^2 - x + 1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A \end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0 \qquad C = -1 \qquad 2A + B + D = 2 \qquad C + E = -1 \qquad A = 1$$

which has the solution $A = 1$, $B = -1$, $C = -1$, $D = 1$, and $E = 0$.

Thus

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2}\,dx = \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right) dx$$

$$= \int \frac{dx}{x} - \int \frac{x}{x^2+1}\,dx - \int \frac{dx}{x^2+1} + \int \frac{x\,dx}{(x^2+1)^2}$$

$$= \ln|x| - \tfrac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + K$$

G Exercises

1–10 ■ Write out the form of the partial fraction decomposition of the function (as in Example 7). Do not determine the numerical values of the coefficients.

1. $\dfrac{5}{2x^2 - 3x - 2}$

2. $\dfrac{z^2 - 4z}{(3z+5)^3(z+2)}$

3. $\dfrac{1}{x^4 - x^3}$

4. $\dfrac{x^4 + x^3 - x^2 - x + 1}{x^3 - x}$

5. $\dfrac{x^2 + 1}{x^2 - 1}$

6. $\dfrac{x^3 - 4x^2 + 2}{(x^2+1)(x^2+2)}$

7. $\dfrac{x^2 - 2}{x(x^2+2)}$

8. $\dfrac{x^4 + x^2 + 1}{(x^2+1)(x^2+4)^2}$

9. $\dfrac{x^3 + x^2 + 1}{x^4 + x^3 + 2x^2}$

10. $\dfrac{1}{x^6 - x^3}$

11–28 ■ Evaluate the integral.

11. $\displaystyle\int \frac{x^2 + 2}{x + 2}\,dx$

12. $\displaystyle\int \frac{x}{x - 5}\,dx$

13. $\displaystyle\int_2^4 \frac{4x - 1}{(x-1)(x+2)}\,dx$

14. $\displaystyle\int \frac{1}{(t+4)(t-1)}\,dt$

15. $\displaystyle\int_0^1 \frac{2x + 3}{(x+1)^2}\,dx$

16. $\displaystyle\int_0^2 \frac{x^3 + x^2 - 12x + 1}{x^2 + x - 12}\,dx$

17. $\displaystyle\int_1^2 \frac{4y^2 - 7y - 12}{y(y+2)(y-3)}\,dy$

18. $\displaystyle\int_2^3 \frac{1}{x^3 + x^2 - 2x}\,dx$

19. $\displaystyle\int \frac{1}{(x+5)^2(x-1)}\,dx$

20. $\displaystyle\int \frac{x^2}{(x-3)(x+2)^2}\,dx$

21. $\displaystyle\int_0^1 \frac{x}{x^2 + x + 1}\,dx$

22. $\displaystyle\int_0^1 \frac{x - 1}{x^2 + 2x + 2}\,dx$

23. $\displaystyle\int \frac{3x^2 - 4x + 5}{(x-1)(x^2+1)}\,dx$

24. $\displaystyle\int_1^2 \frac{x^2 + 3}{x^3 + 2x}\,dx$

25. $\displaystyle\int \frac{1}{x^3 - 1}\,dx$

26. $\displaystyle\int \frac{x^4}{x^4 - 1}\,dx$

27. $\displaystyle\int \frac{2t^3 - t^2 + 3t - 1}{(t^2+1)(t^2+2)}\,dt$

28. $\displaystyle\int \frac{x^4 + 1}{x(x^2+1)^2}\,dx$

29. Use a graph of

$$f(x) = \frac{1}{x^2 - 2x - 3}$$

to decide whether $\int_0^2 f(x)\,dx$ is positive or negative. Use the graph to give a rough estimate of the value of the integral and then use partial fractions to find the exact value.

30. Graph both $y = 1/(x^3 - 2x^2)$ and an antiderivative on the same screen.

31. One method of slowing the growth of an insect population without using pesticides is to introduce into the population a number of sterile males that mate with fertile females but produce no offspring. If P represents the number of female insects in a population, S the number of sterile males introduced each generation, and r the population's natural growth rate, then the female population is related to time t by

$$t = \int \frac{P + S}{P[(r-1)P - S]}\,dP$$

Suppose an insect population with 10,000 females grows at a rate of $r = 0.10$ and 900 sterile males are added. Evaluate the integral to give an equation relating the female population to time. (Note that the resulting equation can't be solved explicitly for P.)

32. The region under the curve

$$y = \frac{1}{x^2 + 3x + 2}$$

from $x = 0$ to $x = 1$ is rotated about the x-axis. Find the volume of the resulting solid.

CAS **33.** (a) Use a computer algebra system to find the partial fraction decomposition of the function

$$f(x) = \frac{4x^3 - 27x^2 + 5x - 32}{30x^5 - 13x^4 + 50x^3 - 286x^2 - 299x - 70}$$

(b) Use part (a) to find $\int f(x)\,dx$ (by hand) and compare with the result of using the CAS to integrate f directly. Comment on any discrepancy.

CAS **34.** (a) Find the partial fraction decomposition of the function

$$f(x) = \frac{12x^5 - 7x^3 - 13x^2 + 8}{100x^6 - 80x^5 + 116x^4 - 80x^3 + 41x^2 - 20x + 4}$$

(b) Use part (a) to find $\int f(x)\,dx$ and graph f and its indefinite integral on the same screen.

(c) Use the graph of f to discover the main features of the graph of $\int f(x)\,dx$.

35. Suppose that F, G, and Q are polynomials and

$$\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$$

for all x except when $Q(x) = 0$. Prove that $F(x) = G(x)$ for all x. [*Hint:* Use continuity.]

36. If f is a quadratic function such that $f(0) = 1$ and

$$\int \frac{f(x)}{x^2(x+1)^3}\,dx$$

is a rational function, find the value of $f'(0)$.

H Polar Coordinates

Polar coordinates offer an alternative way of locating points in a plane. They are useful because, for certain types of regions and curves, polar coordinates provide very simple descriptions and equations. The principal applications of this idea occur in multivariable calculus: the evaluation of double integrals and the derivation of Kepler's laws of planetary motion.

H.1 Curves in Polar Coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the **polar coordinate system**, which is more convenient for many purposes.

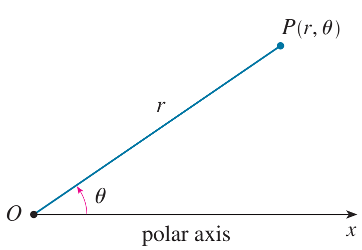


FIGURE 1

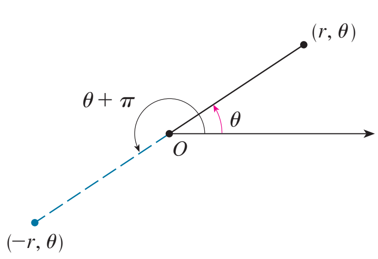


FIGURE 2

We choose a point in the plane that is called the **pole** (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that, as in Figure 2, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O. If $r > 0$, the point (r, θ) lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

EXAMPLE 1 Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$

SOLUTION The points are plotted in Figure 3. In part (d) the point $(-3, 3\pi/4)$ is located three units from the pole in the fourth quadrant because the angle $3\pi/4$ is in the second quadrant and $r = -3$ is negative.

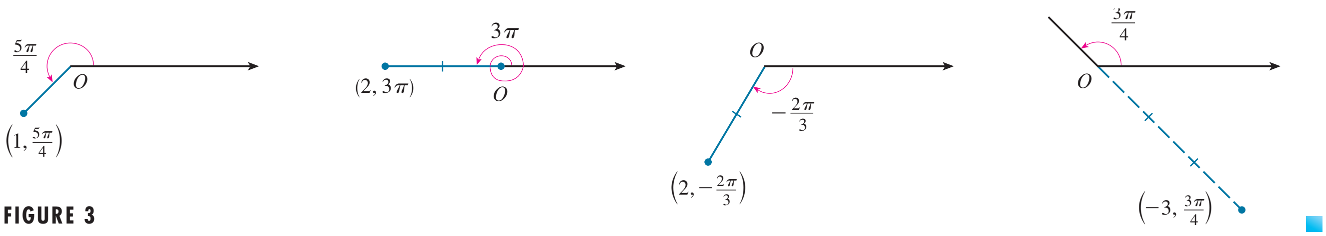


FIGURE 3

In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $(1, 5\pi/4)$ in Example 1(a) could be written as $(1, -3\pi/4)$ or $(1, 13\pi/4)$ or $(-1, \pi/4)$. (See Figure 4.)

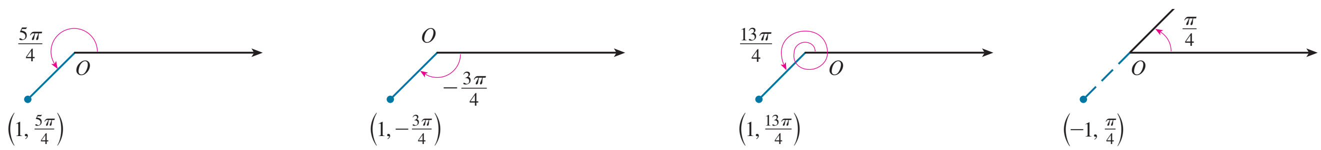


FIGURE 4

In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2n\pi) \qquad \text{and} \qquad (-r, \theta + (2n + 1)\pi)$$

where n is any integer.

The connection between polar and Cartesian coordinates can be seen from Figure 5, in which the pole corresponds to the origin and the polar axis coincides with the positive x-axis. If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure, we have

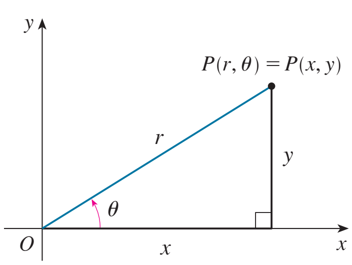


FIGURE 5

$$\cos \theta = \frac{x}{r} \qquad \sin \theta = \frac{y}{r}$$

and so

1 $$x = r \cos \theta \qquad y = r \sin \theta$$

Although Equations 1 were deduced from Figure 5, which illustrates the case where $r > 0$ and $0 < \theta < \pi/2$, these equations are valid for all values of r and θ. (See the general definition of $\sin \theta$ and $\cos \theta$ in Appendix C.)

Equations 1 allow us to find the Cartesian coordinates of a point when the polar coordinates are known. To find r and θ when x and y are known, we use the equations

2 $$r^2 = x^2 + y^2 \qquad \tan \theta = \frac{y}{x}$$

which can be deduced from Equations 1 or simply read from Figure 5.

EXAMPLE 2 Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

SOLUTION Since $r = 2$ and $\theta = \pi/3$, Equations 1 give

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore, the point is $(1, \sqrt{3})$ in Cartesian coordinates. ■

EXAMPLE 3 Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

SOLUTION If we choose r to be positive, then Equations 2 give

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

Since the point $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\pi/4$ or $\theta = 7\pi/4$. Thus, one possible answer is $(\sqrt{2}, -\pi/4)$; another is $(\sqrt{2}, 7\pi/4)$. ■

NOTE • Equations 2 do not uniquely determine θ when x and y are given because, as θ increases through the interval $0 \leqslant \theta < 2\pi$, each value of $\tan\theta$ occurs twice. Therefore, in converting from Cartesian to polar coordinates, it's not good enough just to find r and θ that satisfy Equations 2. As in Example 3, we must choose θ so that the point (r, θ) lies in the correct quadrant.

The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

EXAMPLE 4 What curve is represented by the polar equation $r = 2$?

SOLUTION The curve consists of all points (r, θ) with $r = 2$. Since r represents the distance from the point to the pole, the curve $r = 2$ represents the circle with center O and radius 2. In general, the equation $r = a$ represents a circle with center O and radius $|a|$. (See Figure 6.)

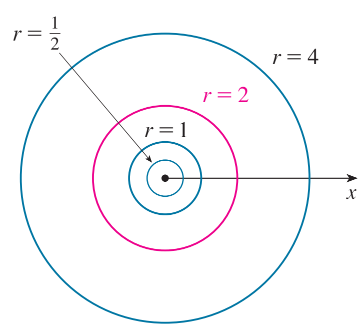


FIGURE 6 ■

EXAMPLE 5 Sketch the polar curve $\theta = 1$.

SOLUTION This curve consists of all points (r, θ) such that the polar angle θ is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the

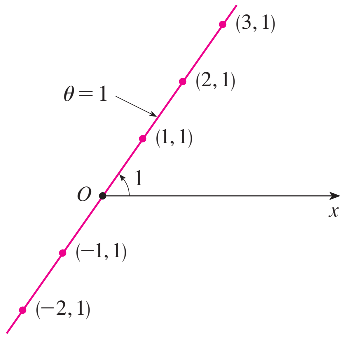


FIGURE 7

polar axis (see Figure 7). Notice that the points $(r, 1)$ on the line with $r > 0$ are in the first quadrant, whereas those with $r < 0$ are in the third quadrant. ■

EXAMPLE 6

(a) Sketch the curve with polar equation $r = 2\cos\theta$.

(b) Find a Cartesian equation for this curve.

SOLUTION

(a) In Figure 8 we find the values of r for some convenient values of θ and plot the corresponding points (r, θ). Then we join these points to sketch the curve, which appears to be a circle. We have used only values of θ between 0 and π, since if we let θ increase beyond π, we obtain the same points again.

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

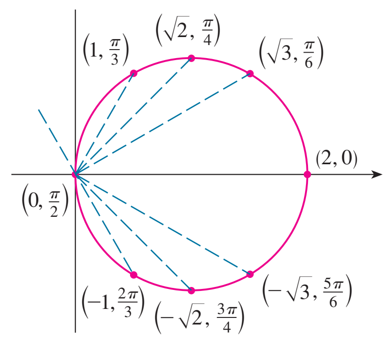


FIGURE 8
Table of values and graph of $r = 2\cos\theta$

(b) To convert the given equation into a Cartesian equation we use Equations 1 and 2. From $x = r\cos\theta$ we have $\cos\theta = x/r$, so the equation $r = 2\cos\theta$ becomes $r = 2x/r$, which gives

$$2x = r^2 = x^2 + y^2 \qquad \text{or} \qquad x^2 + y^2 - 2x = 0$$

Completing the square, we obtain

$$(x - 1)^2 + y^2 = 1$$

which is an equation of a circle with center $(1, 0)$ and radius 1. ■

▲ Figure 9 shows a geometrical illustration that the circle in Example 6 has the equation $r = 2\cos\theta$. The angle OPQ is a right angle (Why?) and so $r/2 = \cos\theta$.

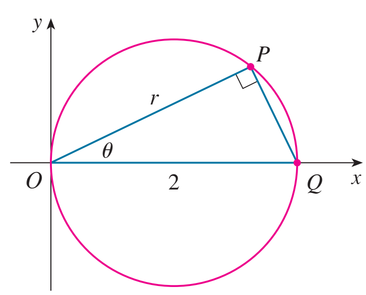


FIGURE 9

EXAMPLE 7 Sketch the curve $r = 1 + \sin\theta$.

SOLUTION Instead of plotting points as in Example 6, we first sketch the graph of $r = 1 + \sin\theta$ in *Cartesian* coordinates in Figure 10 (on page A62) by shifting the

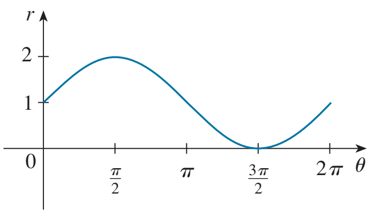


FIGURE 10
$r = 1 + \sin\theta$ in Cartesian coordinates, $0 \leqslant \theta \leqslant 2\pi$

sine curve up one unit. This enables us to read at a glance the values of r that correspond to increasing values of θ. For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) increases from 1 to 2, so we sketch the corresponding part of the polar curve in Figure 11(a). As θ increases from $\pi/2$ to π, Figure 10 shows that r decreases from 2 to 1, so we sketch the next part of the curve as in Figure 11(b). As θ increases from π to $3\pi/2$, r decreases from 1 to 0 as shown in part (c). Finally, as θ increases from $3\pi/2$ to 2π, r increases from 0 to 1 as shown in part (d). If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace our path. Putting together the parts of the curve from Figure 11(a)–(d), we sketch the complete curve in part (e). It is called a **cardioid** because it's shaped like a heart.

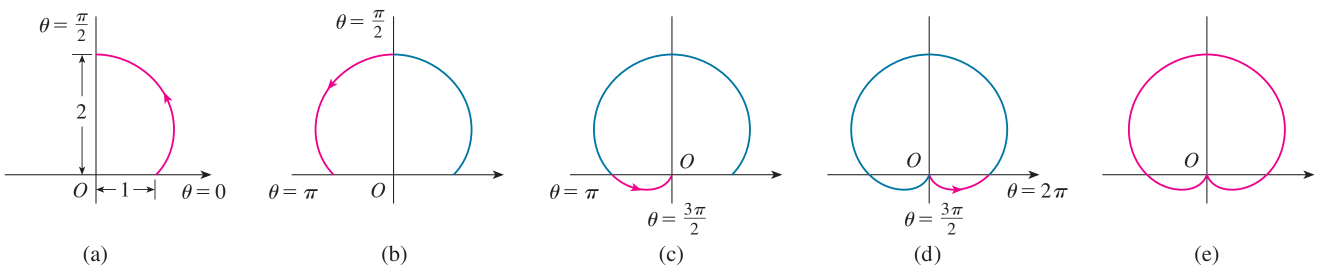


FIGURE 11
Stages in sketching the cardioid $r = 1 + \sin\theta$ ■

TEC Module H helps you see how polar curves are traced out by showing animations similar to Figures 10–13. Tangents to these polar curves can also be visualized as in Figure 15.

EXAMPLE 8 Sketch the curve $r = \cos 2\theta$.

SOLUTION As in Example 7, we first sketch $r = \cos 2\theta$, $0 \leqslant \theta \leqslant 2\pi$, in Cartesian coordinates in Figure 12. As θ increases from 0 to $\pi/4$, Figure 12 shows that r decreases from 1 to 0 and so we draw the corresponding portion of the polar curve in Figure 13 (indicated by ①). As θ increases from $\pi/4$ to $\pi/2$, r goes from 0 to -1. This means that the distance from O increases from 0 to 1, but instead of being in the first quadrant this portion of the polar curve (indicated by ②) lies on the opposite side of the pole in the third quadrant. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in which the portions are traced out. The resulting curve has four loops and is called a **four-leaved rose**.

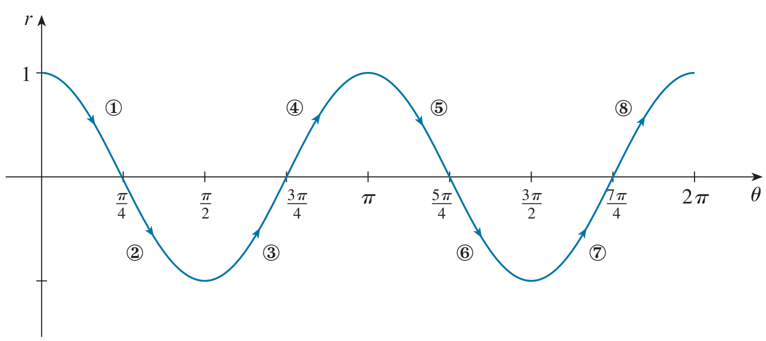


FIGURE 12
$r = \cos 2\theta$ in Cartesian coordinates

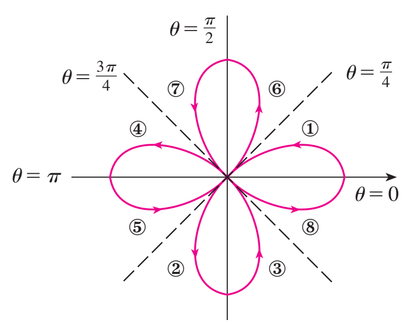


FIGURE 13
Four-leaved rose $r = \cos 2\theta$ ■

When we sketch polar curves it is sometimes helpful to take advantage of symmetry. The following three rules are explained by Figure 14.

(a) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.

(b) If the equation is unchanged when r is replaced by $-r$, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)

(c) If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \pi/2$.

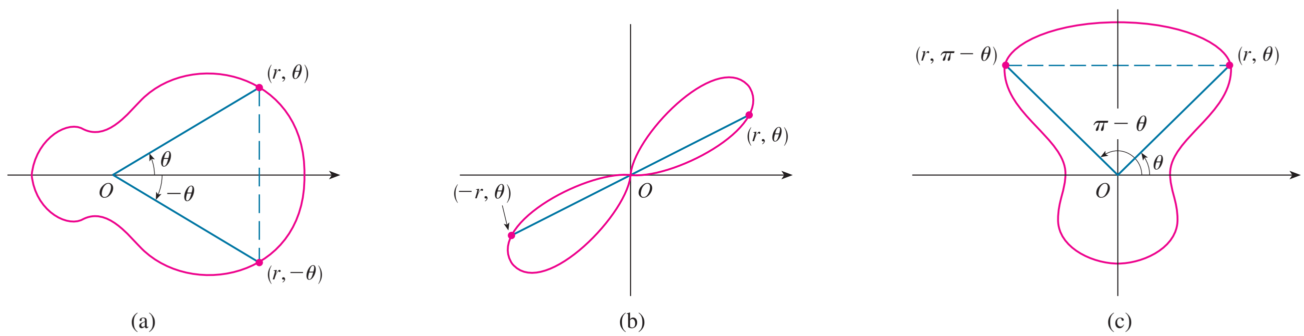


FIGURE 14

The curves sketched in Examples 6 and 8 are symmetric about the polar axis, since $\cos(-\theta) = \cos\theta$. The curves in Examples 7 and 8 are symmetric about $\theta = \pi/2$ because $\sin(\pi - \theta) = \sin\theta$ and $\cos 2(\pi - \theta) = \cos 2\theta$. The four-leaved rose is also symmetric about the pole. These symmetry properties could have been used in sketching the curves. For instance, in Example 6 we need only have plotted points for $0 \leqslant \theta \leqslant \pi/2$ and then reflected about the polar axis to obtain the complete circle.

Tangents to Polar Curves

To find a tangent line to a polar curve $r = f(\theta)$ we regard θ as a parameter and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Then, using the method for finding slopes of parametric curves (Equation 3.5.7) and the Product Rule, we have

3
$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

We locate horizontal tangents by finding the points where $dy/d\theta = 0$ (provided that $dx/d\theta \neq 0$). Likewise, we locate vertical tangents at the points where $dx/d\theta = 0$ (provided that $dy/d\theta \neq 0$).

Notice that if we are looking for tangent lines at the pole, then $r = 0$ and Equation 3 simplifies to

$$\frac{dy}{dx} = \tan\theta \qquad \text{if} \quad \frac{dr}{d\theta} \neq 0$$

For instance, in Example 8 we found that $r = \cos 2\theta = 0$ when $\theta = \pi/4$ or $3\pi/4$. This means that the lines $\theta = \pi/4$ and $\theta = 3\pi/4$ (or $y = x$ and $y = -x$) are tangent lines to $r = \cos 2\theta$ at the origin.

EXAMPLE 9

(a) For the cardioid $r = 1 + \sin\theta$ of Example 7, find the slope of the tangent line when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

SOLUTION Using Equation 3 with $r = 1 + \sin\theta$, we have

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\,\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\,\cos\theta - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1+2\sin\theta)}{1-2\sin^2\theta-\sin\theta} = \frac{\cos\theta\,(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

(a) The slope of the tangent at the point where $\theta = \pi/3$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{\cos(\pi/3)(1+2\sin(\pi/3))}{(1+\sin(\pi/3))(1-2\sin(\pi/3))}$$

$$= \frac{\frac{1}{2}\left(1+\sqrt{3}\right)}{\left(1+\sqrt{3}/2\right)\left(1-\sqrt{3}\right)} = \frac{1+\sqrt{3}}{\left(2+\sqrt{3}\right)\left(1-\sqrt{3}\right)}$$

$$= \frac{1+\sqrt{3}}{-1-\sqrt{3}} = -1$$

(b) Observe that

$$\frac{dy}{d\theta} = \cos\theta\,(1+2\sin\theta) = 0 \qquad \text{when } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} = (1+\sin\theta)(1-2\sin\theta) = 0 \qquad \text{when } \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Therefore, there are horizontal tangents at the points $(2, \pi/2)$, $\left(\frac{1}{2}, 7\pi/6\right)$, $\left(\frac{1}{2}, 11\pi/6\right)$ and vertical tangents at $\left(\frac{3}{2}, \pi/6\right)$ and $\left(\frac{3}{2}, 5\pi/6\right)$. When $\theta = 3\pi/2$, both $dy/d\theta$ and $dx/d\theta$ are 0, so we must be careful. Using l'Hospital's Rule, we have

$$\lim_{\theta\to(3\pi/2)^-}\frac{dy}{dx} = \left(\lim_{\theta\to(3\pi/2)^-}\frac{1+2\sin\theta}{1-2\sin\theta}\right)\left(\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1+\sin\theta}\right)$$

$$= -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1+\sin\theta}$$

$$= -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{-\sin\theta}{\cos\theta} = \infty$$

By symmetry, $$\lim_{\theta\to(3\pi/2)^+}\frac{dy}{dx} = -\infty$$

Thus, there is a vertical tangent line at the pole (see Figure 15). ■

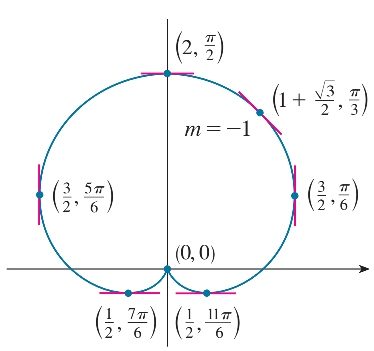


FIGURE 15

Tangent lines for $r = 1 + \sin\theta$

NOTE • Instead of having to remember Equation 3, we could employ the method used to derive it. For instance, in Example 9 we could have written

$$x = r\cos\theta = (1 + \sin\theta)\cos\theta = \cos\theta + \tfrac{1}{2}\sin 2\theta$$

$$y = r\sin\theta = (1 + \sin\theta)\sin\theta = \sin\theta + \sin^2\theta$$

Then we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta}$$

which is equivalent to our previous expression.

Graphing Polar Curves with Graphing Devices

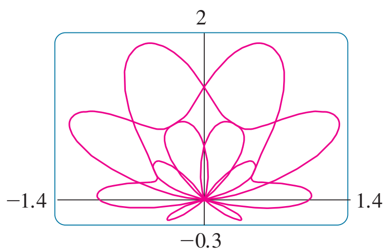


FIGURE 16
$r = \sin\theta + \sin^3(5\theta/2)$

Although it's useful to be able to sketch simple polar curves by hand, we need to use a graphing calculator or computer when we are faced with a curve as complicated as the one shown in Figure 16.

Some graphing devices have commands that enable us to graph polar curves directly. With other machines we need to convert to parametric equations first. In this case we take the polar equation $r = f(\theta)$ and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Some machines require that the parameter be called t rather than θ.

EXAMPLE 10 Graph the curve $r = \sin(8\theta/5)$.

SOLUTION Let's assume that our graphing device doesn't have a built-in polar graphing command. In this case we need to work with the corresponding parametric equations, which are

$$x = r\cos\theta = \sin(8\theta/5)\cos\theta \qquad y = r\sin\theta = \sin(8\theta/5)\sin\theta$$

In any case we need to determine the domain for θ. So we ask ourselves: How many complete rotations are required until the curve starts to repeat itself? If the answer is n, then

$$\sin\frac{8(\theta + 2n\pi)}{5} = \sin\left(\frac{8\theta}{5} + \frac{16n\pi}{5}\right) = \sin\frac{8\theta}{5}$$

and so we require that $16n\pi/5$ be an even multiple of π. This will first occur when $n = 5$. Therefore, we will graph the entire curve if we specify that $0 \leqslant \theta \leqslant 10\pi$. Switching from θ to t, we have the equations

$$x = \sin(8t/5)\cos t \qquad y = \sin(8t/5)\sin t \qquad 0 \leqslant t \leqslant 10\pi$$

and Figure 17 shows the resulting curve. Notice that this rose has 16 loops. ■

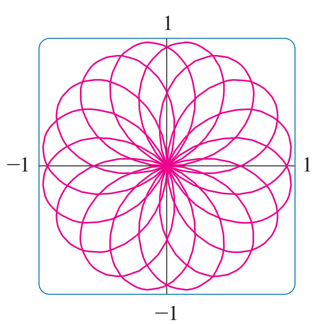


FIGURE 17
$r = \sin(8\theta/5)$

EXAMPLE 11 Investigate the family of polar curves given by $r = 1 + c\sin\theta$. How does the shape change as c changes? (These curves are called **limaçons**, after a French word for snail, because of the shape of the curves for certain values of c.)

SOLUTION Figure 18 shows computer-drawn graphs for various values of c. For $c > 1$ there is a loop that decreases in size as c decreases. When $c = 1$ the loop disappears and the curve becomes the cardioid that we sketched in Example 7. For c between 1 and $\frac{1}{2}$ the cardioid's cusp is smoothed out and becomes a "dimple." When c decreases from $\frac{1}{2}$ to 0, the limaçon is shaped like an oval. This oval becomes more circular as $c \to 0$, and when $c = 0$ the curve is just the circle $r = 1$.

▲ In Exercise 39 you are asked to prove analytically what we have discovered from the graphs in Figure 18.

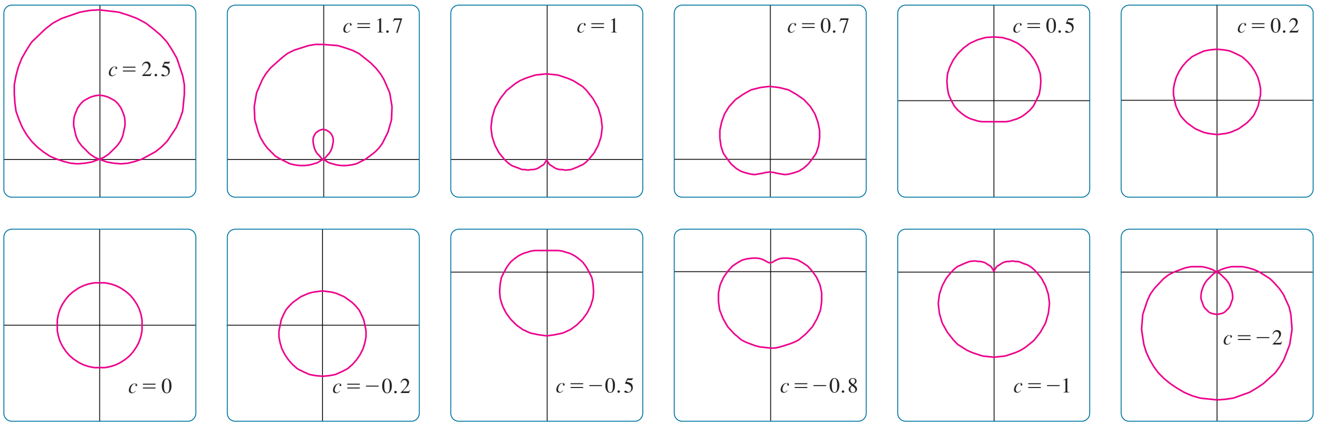


FIGURE 18
Members of the family of limaçons $r = 1 + c \sin \theta$

The remaining parts of Figure 18 show that as c becomes negative, the shapes change in reverse order. In fact, these curves are reflections about the horizontal axis of the corresponding curves with positive c. ■

H.1 Exercises

1–2 ■ Plot the point whose polar coordinates are given. Then find two other pairs of polar coordinates of this point, one with $r > 0$ and one with $r < 0$.

1. (a) $(1, \pi/2)$ (b) $(-2, \pi/4)$ (c) $(3, 2)$

2. (a) $(3, 0)$ (b) $(2, -\pi/7)$ (c) $(-1, -\pi/2)$

3–4 ■ Plot the point whose polar coordinates are given. Then find the Cartesian coordinates of the point.

3. (a) $(3, \pi/2)$ (b) $(2\sqrt{2}, 3\pi/4)$ (c) $(-1, \pi/3)$

4. (a) $(2, 2\pi/3)$ (b) $(4, 3\pi)$ (c) $(-2, -5\pi/6)$

5–6 ■ The Cartesian coordinates of a point are given.
(i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \leqslant \theta < 2\pi$.
(ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \leqslant \theta < 2\pi$.

5. (a) $(1, 1)$ (b) $(2\sqrt{3}, -2)$

6. (a) $(-1, -\sqrt{3})$ (b) $(-2, 3)$

7–12 ■ Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions.

7. $r > 1$

8. $0 \leqslant \theta < \pi/4$

9. $0 \leqslant r \leqslant 2, \quad \pi/2 \leqslant \theta \leqslant \pi$

10. $1 \leqslant r < 3, \quad -\pi/4 \leqslant \theta \leqslant \pi/4$

11. $2 < r < 3, \quad 5\pi/3 \leqslant \theta \leqslant 7\pi/3$

12. $-1 \leqslant r \leqslant 1, \quad \pi/4 \leqslant \theta \leqslant 3\pi/4$

13–16 ■ Find a Cartesian equation for the curve described by the given polar equation.

13. $r = 3 \sin \theta$

14. $r \cos \theta = 1$

15. $r^2 = \sin 2\theta$

16. $r = 1/(1 + 2 \sin \theta)$

17–20 ■ Find a polar equation for the curve represented by the given Cartesian equation.

17. $y = 5$

18. $y = 2x - 1$

19. $x^2 + y^2 = 25$

20. $x^2 = 4y$

21–22 ■ For each of the described curves, decide if the curve would be more easily given by a polar equation or a Cartesian equation. Then write an equation for the curve.

21. (a) A line through the origin that makes an angle of $\pi/6$ with the positive x-axis
(b) A vertical line through the point $(3, 3)$

22. (a) A circle with radius 5 and center (2, 3)
(b) A circle centered at the origin with radius 4

23–34 ■ Sketch the curve with the given polar equation.

23. $r = 5$

24. $\theta = 3\pi/4$

25. $r = \sin\theta$

26. $r = 1 - 3\cos\theta$

27. $r = \theta, \quad \theta \geqslant 0$

28. $r = \sqrt{\theta}$

29. $r = 1 - 2\cos\theta$

30. $r = 2 + \cos\theta$

31. $r = 2\cos 4\theta$

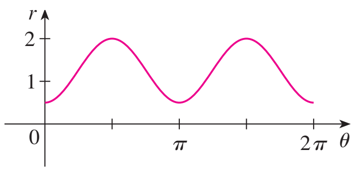
32. $r = \sin 5\theta$

33. $r^2 = 4\cos 2\theta$

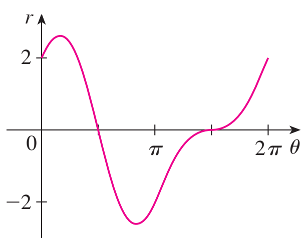
34. $r = 2\cos(3\theta/2)$

35–36 ■ The figure shows the graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.

35.



36.

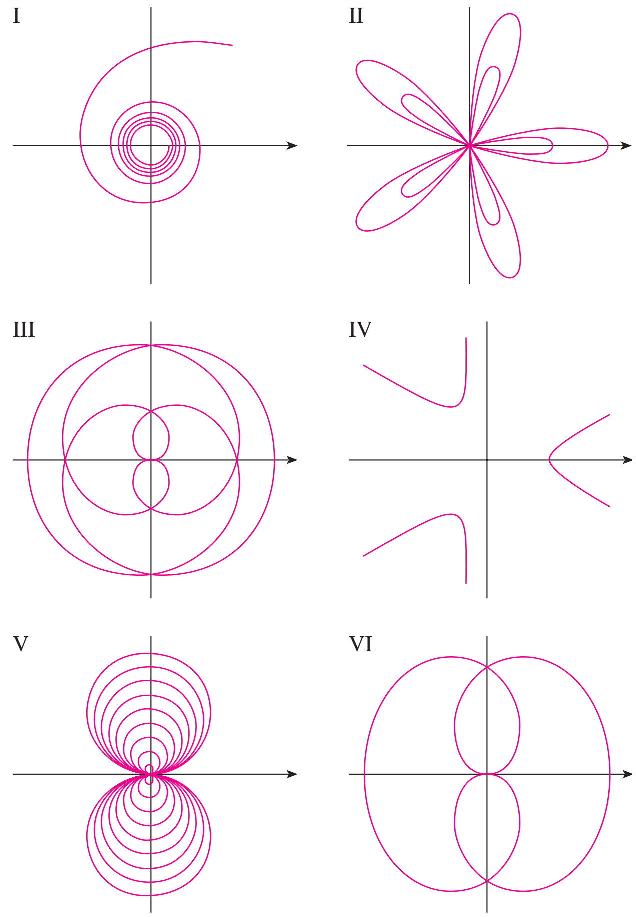


37. Show that the polar curve $r = 4 + 2\sec\theta$ (called a **conchoid**) has the line $x = 2$ as a vertical asymptote by showing that $\lim_{r\to\pm\infty} x = 2$. Use this fact to help sketch the conchoid.

38. Show that the curve $r = \sin\theta\tan\theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote. Show also that the curve lies entirely within the vertical strip $0 \leqslant x < 1$. Use these facts to help sketch the cissoid.

39. (a) In Example 11 the graphs suggest that the limaçon $r = 1 + c\sin\theta$ has an inner loop when $|c| > 1$. Prove that this is true, and find the values of θ that correspond to the inner loop.
(b) From Figure 18 it appears that the limaçon loses its dimple when $c = \frac{1}{2}$. Prove this.

40. Match the polar equations with the graphs labeled I–VI. Give reasons for your choices. (Don't use a graphing device.)
(a) $r = \sin(\theta/2)$
(b) $r = \sin(\theta/4)$
(c) $r = \sec(3\theta)$
(d) $r = \theta\sin\theta$
(e) $r = 1 + 4\cos 5\theta$
(f) $r = 1/\sqrt{\theta}$



41–44 ■ Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

41. $r = 3\cos\theta, \quad \theta = \pi/3$

42. $r = \cos\theta + \sin\theta, \quad \theta = \pi/4$

43. $r = 1 + \cos\theta, \quad \theta = \pi/6$

44. $r = \ln\theta, \quad \theta = e$

45–48 ■ Find the points on the given curve where the tangent line is horizontal or vertical.

45. $r = 3\cos\theta$

46. $r = e^{\theta}$

47. $r = 1 + \cos\theta$

48. $r^2 = \sin 2\theta$

49. Show that the polar equation $r = a\sin\theta + b\cos\theta$, where $ab \neq 0$, represents a circle, and find its center and radius.

50. Show that the curves $r = a\sin\theta$ and $r = a\cos\theta$ intersect at right angles.

51–54 ■ Use a graphing device to graph the polar curve. Choose the parameter interval to make sure that you produce the entire curve.

51. $r = 1 + 2\sin(\theta/2)$ (nephroid of Freeth)

52. $r = \sqrt{1 - 0.8 \sin^2\theta}$ (hippopede)

53. $r = e^{\sin\theta} - 2\cos(4\theta)$ (butterfly curve)

54. $r = \sin^2(4\theta) + \cos(4\theta)$

55. How are the graphs of $r = 1 + \sin(\theta - \pi/6)$ and $r = 1 + \sin(\theta - \pi/3)$ related to the graph of $r = 1 + \sin\theta$? In general, how is the graph of $r = f(\theta - \alpha)$ related to the graph of $r = f(\theta)$?

56. Use a graph to estimate the y-coordinate of the highest points on the curve $r = \sin 2\theta$. Then use calculus to find the exact value.

57. (a) Investigate the family of curves defined by the polar equations $r = \sin n\theta$, where n is a positive integer. How is the number of loops related to n?
(b) What happens if the equation in part (a) is replaced by $r = |\sin n\theta|$?

58. A family of curves is given by the equations $r = 1 + c \sin n\theta$, where c is a real number and n is a positive integer. How does the graph change as n increases? How does it change as c changes? Illustrate by graphing enough members of the family to support your conclusions.

59. A family of curves has polar equations

$$r = \frac{1 - a\cos\theta}{1 + a\cos\theta}$$

Investigate how the graph changes as the number a changes. In particular, you should identify the transitional values of a for which the basic shape of the curve changes.

60. The astronomer Giovanni Cassini (1625–1712) studied the family of curves with polar equations

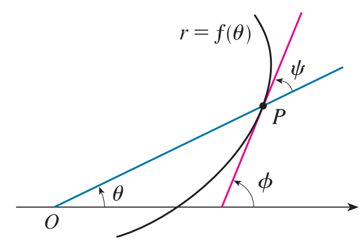
$$r^4 - 2c^2r^2\cos 2\theta + c^4 - a^4 = 0$$

where a and c are positive real numbers. These curves are called the **ovals of Cassini** even though they are oval shaped only for certain values of a and c. (Cassini thought that these curves might represent planetary orbits better than Kepler's ellipses.) Investigate the variety of shapes that these curves may have. In particular, how are a and c related to each other when the curve splits into two parts?

61. Let P be any point (except the origin) on the curve $r = f(\theta)$. If ψ is the angle between the tangent line at P and the radial line OP, show that

$$\tan\psi = \frac{r}{dr/d\theta}$$

[*Hint:* Observe that $\psi = \phi - \theta$ in the figure.]



62. (a) Use Exercise 61 to show that the angle between the tangent line and the radial line is $\psi = \pi/4$ at every point on the curve $r = e^\theta$.
(b) Illustrate part (a) by graphing the curve and the tangent lines at the points where $\theta = 0$ and $\pi/2$.
(c) Prove that any polar curve $r = f(\theta)$ with the property that the angle ψ between the radial line and the tangent line is a constant must be of the form $r = Ce^{k\theta}$, where C and k are constants.

H.2 Areas and Lengths in Polar Coordinates

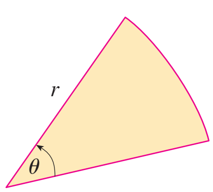


FIGURE 1

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle

$$A = \tfrac{1}{2}r^2\theta \tag{1}$$

where, as in Figure 1, r is the radius and θ is the radian measure of the central angle. Formula 1 follows from the fact that the area of a sector is proportional to its central angle: $A = (\theta/2\pi)\pi r^2 = \frac{1}{2}r^2\theta$.

Let $\mathcal{R}$ be the region, illustrated in Figure 2, bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leqslant 2\pi$. We divide the interval $[a, b]$ into subintervals with endpoints θ_0,

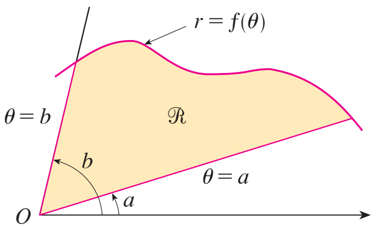


FIGURE 2

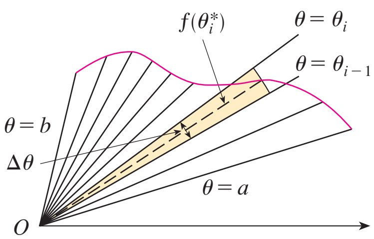


FIGURE 3

$\theta_1, \theta_2, \ldots, \theta_n$ and equal width $\Delta\theta$. The rays $\theta = \theta_i$ then divide $\mathscr{R}$ into n smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$. If we choose θ_i^* in the ith subinterval $[\theta_{i-1}, \theta_i]$, then the area ΔA_i of the ith region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$. (See Figure 3.)

Thus, from Formula 1 we have

$$\Delta A_i \approx \tfrac{1}{2}[f(\theta_i^*)]^2 \, \Delta\theta$$

and so an approximation to the total area A of $\mathscr{R}$ is

2 $$A \approx \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2 \, \Delta\theta$$

It appears from Figure 3 that the approximation in (2) improves as $n \to \infty$. But the sums in (2) are Riemann sums for the function $g(\theta) = \frac{1}{2}[f(\theta)]^2$, so

$$\lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2 \, \Delta\theta = \int_a^b \tfrac{1}{2}[f(\theta)]^2 \, d\theta$$

It therefore appears plausible (and can in fact be proved) that the formula for the area A of the polar region $\mathscr{R}$ is

3 $$A = \int_a^b \tfrac{1}{2}[f(\theta)]^2 \, d\theta$$

Formula 3 is often written as

4 $$A = \int_a^b \tfrac{1}{2} r^2 \, d\theta$$

with the understanding that $r = f(\theta)$. Note the similarity between Formulas 1 and 4.

When we apply Formula 3 or 4 it is helpful to think of the area as being swept out by a rotating ray through O that starts with angle a and ends with angle b.

EXAMPLE 1 Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

SOLUTION The curve $r = \cos 2\theta$ was sketched in Example 8 in Section H.1. Notice from Figure 4 that the region enclosed by the right loop is swept out by a ray that rotates from $\theta = -\pi/4$ to $\theta = \pi/4$. Therefore, Formula 4 gives

$$A = \int_{-\pi/4}^{\pi/4} \tfrac{1}{2} r^2 \, d\theta = \tfrac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta = \int_0^{\pi/4} \cos^2 2\theta \, d\theta$$

We could evaluate the integral using Formula 64 in the Table of Integrals. Or, as in Section 5.7, we could use the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to write

$$A = \int_0^{\pi/4} \tfrac{1}{2}(1 + \cos 4\theta) \, d\theta = \tfrac{1}{2}\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_0^{\pi/4} = \frac{\pi}{8}$$ ■

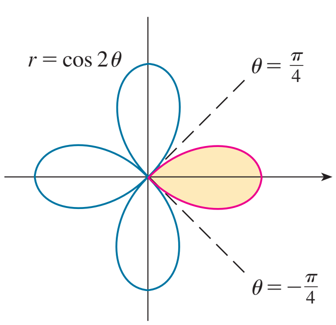


FIGURE 4

EXAMPLE 2 Find the area of the region that lies inside the circle $r = 3 \sin \theta$ and outside the cardioid $r = 1 + \sin \theta$.

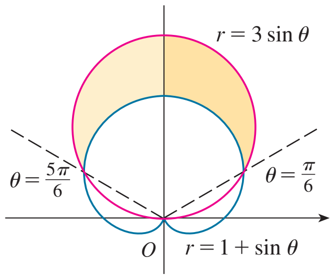


FIGURE 5

SOLUTION The cardioid (see Example 7 in Section H.1) and the circle are sketched in Figure 5 and the desired region is shaded. The values of a and b in Formula 4 are determined by finding the points of intersection of the two curves. They intersect when $3 \sin \theta = 1 + \sin \theta$, which gives $\sin \theta = \frac{1}{2}$, so $\theta = \pi/6, 5\pi/6$. The desired area can be found by subtracting the area inside the cardioid between $\theta = \pi/6$ and $\theta = 5\pi/6$ from the area inside the circle from $\pi/6$ to $5\pi/6$. Thus

$$A = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (3 \sin \theta)^2 \, d\theta - \frac{1}{2} \int_{\pi/6}^{5\pi/6} (1 + \sin \theta)^2 \, d\theta$$

Since the region is symmetric about the vertical axis $\theta = \pi/2$, we can write

$$\begin{aligned} A &= 2\left[\frac{1}{2} \int_{\pi/6}^{\pi/2} 9 \sin^2\theta \, d\theta - \frac{1}{2} \int_{\pi/6}^{\pi/2} (1 + 2 \sin \theta + \sin^2\theta) \, d\theta\right] \\ &= \int_{\pi/6}^{\pi/2} (8 \sin^2\theta - 1 - 2 \sin \theta) \, d\theta \\ &= \int_{\pi/6}^{\pi/2} (3 - 4 \cos 2\theta - 2 \sin \theta) \, d\theta \qquad \left[\text{because } \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)\right] \\ &= 3\theta - 2 \sin 2\theta + 2 \cos \theta \Big]_{\pi/6}^{\pi/2} = \pi \end{aligned}$$

Example 2 illustrates the procedure for finding the area of the region bounded by two polar curves. In general, let $\mathcal{R}$ be a region, as illustrated in Figure 6, that is bounded by curves with polar equations $r = f(\theta)$, $r = g(\theta)$, $\theta = a$, and $\theta = b$, where $f(\theta) \geqslant g(\theta) \geqslant 0$ and $0 < b - a \leqslant 2\pi$. The area A of $\mathcal{R}$ is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$, so using Formula 3 we have

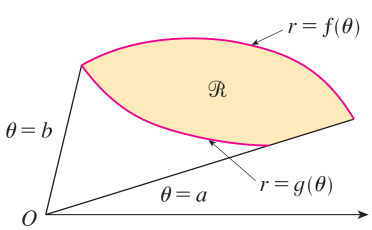


FIGURE 6

$$\begin{aligned} A &= \int_a^b \tfrac{1}{2}[f(\theta)]^2 \, d\theta - \int_a^b \tfrac{1}{2}[g(\theta)]^2 \, d\theta \\ &= \tfrac{1}{2} \int_a^b \left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta \end{aligned}$$

CAUTION • The fact that a single point has many representations in polar coordinates sometimes makes it difficult to find all the points of intersection of two polar curves. For instance, it is obvious from Figure 5 that the circle and the cardioid have three points of intersection; however, in Example 2 we solved the equations $r = 3 \sin \theta$ and $r = 1 + \sin \theta$ and found only two such points, $\left(\frac{3}{2}, \pi/6\right)$ and $\left(\frac{3}{2}, 5\pi/6\right)$. The origin is also a point of intersection, but we can't find it by solving the equations of the curves because the origin has no single representation in polar coordinates that satisfies both equations. Notice that, when represented as $(0, 0)$ or $(0, \pi)$, the origin satisfies $r = 3 \sin \theta$ and so it lies on the circle; when represented as $(0, 3\pi/2)$, it satisfies $r = 1 + \sin \theta$ and so it lies on the cardioid. Think of two points moving along the curves as the parameter value θ increases from 0 to 2π. On one curve the origin is reached at $\theta = 0$ and $\theta = \pi$; on the other curve it is reached at $\theta = 3\pi/2$. The points don't collide at the origin because they reach the origin at different times, but the curves intersect there nonetheless.

Thus, to find *all* points of intersection of two polar curves, it is recommended that you draw the graphs of both curves. It is especially convenient to use a graphing calculator or computer to help with this task.

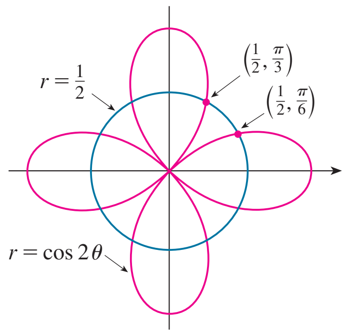


FIGURE 7

EXAMPLE 3 Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

SOLUTION If we solve the equations $r = \cos 2\theta$ and $r = \frac{1}{2}$, we get $\cos 2\theta = \frac{1}{2}$ and, therefore, $2\theta = \pi/3, 5\pi/3, 7\pi/3, 11\pi/3$. Thus, the values of θ between 0 and 2π that satisfy both equations are $\theta = \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$. We have found four points of intersection: $\left(\frac{1}{2}, \pi/6\right)$, $\left(\frac{1}{2}, 5\pi/6\right)$, $\left(\frac{1}{2}, 7\pi/6\right)$, and $\left(\frac{1}{2}, 11\pi/6\right)$.

However, you can see from Figure 7 that the curves have four other points of intersection—namely, $\left(\frac{1}{2}, \pi/3\right)$, $\left(\frac{1}{2}, 2\pi/3\right)$, $\left(\frac{1}{2}, 4\pi/3\right)$, and $\left(\frac{1}{2}, 5\pi/3\right)$. These can be found using symmetry or by noticing that another equation of the circle is $r = -\frac{1}{2}$ and then solving the equations $r = \cos 2\theta$ and $r = -\frac{1}{2}$.

Arc Length

To find the length of a polar curve $r = f(\theta)$, $a \leqslant \theta \leqslant b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

so, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2\end{aligned}$$

Assuming that f' is continuous, we can use Formula 6.3.1 to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Therefore, the length of a curve with polar equation $r = f(\theta)$, $a \leqslant \theta \leqslant b$, is

5
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

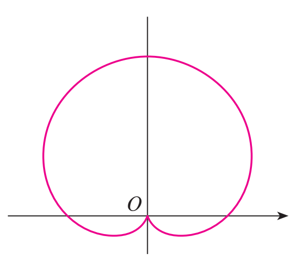


FIGURE 8
$r = 1 + \sin\theta$

EXAMPLE 4 Find the length of the cardioid $r = 1 + \sin\theta$.

SOLUTION The cardioid is shown in Figure 8. (We sketched it in Example 7 in Section H.1.) Its full length is given by the parameter interval $0 \leqslant \theta \leqslant 2\pi$, so

Formula 5 gives

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

We could evaluate this integral by multiplying and dividing the integrand by $\sqrt{2 - 2\sin\theta}$, or we could use a computer algebra system. In any event, we find that the length of the cardioid is $L = 8$. ■

H.2 Exercises

1–4 ■ Find the area of the region that is bounded by the given curve and lies in the specified sector.

1. $r = \sqrt{\theta}$, $0 \leqslant \theta \leqslant \pi/4$

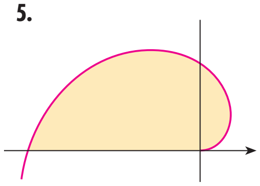
2. $r = e^{\theta/2}$, $\pi \leqslant \theta \leqslant 2\pi$

3. $r = \sin\theta$, $\pi/3 \leqslant \theta \leqslant 2\pi/3$

4. $r = \sqrt{\sin\theta}$, $0 \leqslant \theta \leqslant \pi$

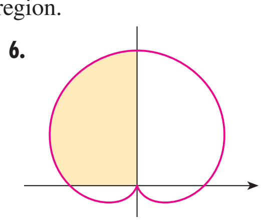
5–8 ■ Find the area of the shaded region.

5.



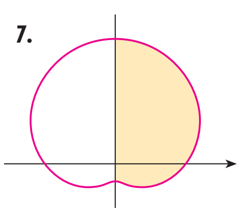
$r = \theta$

6.



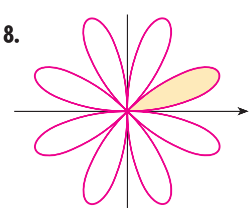
$r = 1 + \sin\theta$

7.



$r = 4 + 3\sin\theta$

8.



$r = \sin 4\theta$

9–12 ■ Sketch the curve and find the area that it encloses.

9. $r^2 = 4\cos 2\theta$

10. $r = 3(1 + \cos\theta)$

11. $r = 4 - \sin\theta$

12. $r = \sin 3\theta$

13. Graph the curve $r = 2 + \cos 6\theta$ and find the area that it encloses.

14. The curve with polar equation $r = 2\sin\theta\cos^2\theta$ is called a **bifolium**. Graph it and find the area that it encloses.

15–18 ■ Find the area of the region enclosed by one loop of the curve.

15. $r = \sin 2\theta$

16. $r = 4\sin 3\theta$

17. $r = 1 + 2\sin\theta$ (inner loop)

18. $r = 2 + 3\cos\theta$ (inner loop)

19–22 ■ Find the area of the region that lies inside the first curve and outside the second curve.

19. $r = 4\sin\theta$, $r = 2$

20. $r = 3\cos\theta$, $r = 2 - \cos\theta$

21. $r = 3\cos\theta$, $r = 1 + \cos\theta$

22. $r = 1 + \cos\theta$, $r = 3\cos\theta$

23–26 ■ Find the area of the region that lies inside both curves.

23. $r = \sin\theta$, $r = \cos\theta$

24. $r = \sin 2\theta$, $r = \sin\theta$

25. $r = \sin 2\theta$, $r = \cos 2\theta$

26. $r^2 = 2\sin 2\theta$, $r = 1$

27. Find the area inside the larger loop and outside the smaller loop of the limaçon $r = \frac{1}{2} + \cos\theta$.

28. Graph the hippopede $r = \sqrt{1 - 0.8\sin^2\theta}$ and the circle $r = \sin\theta$ and find the exact area of the region that lies inside both curves.

29–32 ■ Find all points of intersection of the given curves.

29. $r = \cos\theta$, $r = 1 - \cos\theta$

30. $r = \cos 3\theta$, $r = \sin 3\theta$

31. $r = \sin\theta$, $r = \sin 2\theta$

32. $r^2 = \sin 2\theta$, $r^2 = \cos 2\theta$

33. The points of intersection of the cardioid $r = 1 + \sin\theta$ and the spiral loop $r = 2\theta$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, can't be found exactly. Use a graphing device to find the approximate values of θ at which they intersect. Then use these values to estimate the area that lies inside both curves.

34. Use a graph to estimate the values of θ for which the curves $r = 3 + \sin 5\theta$ and $r = 6 \sin \theta$ intersect. Then estimate the area that lies inside both curves.

35–38 ■ Find the length of the polar curve.

35. $r = 5 \cos \theta, \quad 0 \leqslant \theta \leqslant 3\pi/4$ **36.** $r = e^{2\theta}, \quad 0 \leqslant \theta \leqslant 2\pi$

37. $r = \theta^2, \quad 0 \leqslant \theta \leqslant 2\pi$ **38.** $r = \theta, \quad 0 \leqslant \theta \leqslant 2\pi$

39–40 ■ Use a calculator or computer to find the length of the loop correct to four decimal places.

39. One loop of the four-leaved rose $r = \cos 2\theta$

40. The loop of the conchoid $r = 4 + 2 \sec \theta$

Discovery Project

Conic Sections in Polar Coordinates

In this project we give a unified treatment of all three types of conic sections in terms of a focus and directrix. We will see that if we place the focus at the origin, then a conic section has a simple polar equation. In Chapter 10 we use the polar equation of an ellipse to derive Kepler's laws of planetary motion.

Let F be a fixed point (called the **focus**) and l be a fixed line (called the **directrix**) in a plane. Let e be a fixed positive number (called the **eccentricity**). Let C be the set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e$$

(that is, the ratio of the distance from F to the distance from l is the constant e). Notice that if the eccentricity is $e = 1$, then $|PF| = |Pl|$ and so the given condition simply becomes the definition of a parabola as given in Appendix B.

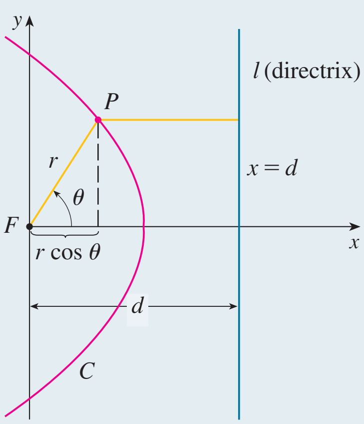


FIGURE 1

1. If we place the focus F at the origin and the directrix parallel to the y-axis and d units to the right, then the directrix has equation $x = d$ and is perpendicular to the polar axis. If the point P has polar coordinates (r, θ), use Figure 1 to show that

$$r = e(d - r \cos \theta)$$

2. By converting the polar equation in Problem 1 to rectangular coordinates, show that the curve C is an ellipse if $e < 1$. (See Appendix B for a discussion of ellipses.)

3. Show that C is a hyperbola if $e > 1$.

4. Show that the polar equation

$$r = \frac{ed}{1 + e \cos \theta}$$

represents an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

5. For each of the following conics, find the eccentricity and directrix. Then identify and sketch the conic.

(a) $r = \dfrac{4}{1 + 3 \cos \theta}$ (b) $r = \dfrac{8}{3 + 3 \cos \theta}$ (c) $r = \dfrac{2}{2 + \cos \theta}$

6. Graph the conics $r = e/(1 - e \cos \theta)$ with $e = 0.4, 0.6, 0.8$, and 1.0 on a common screen. How does the value of e affect the shape of the curve?

7. (a) Show that the polar equation of an ellipse with directrix $x = d$ can be written in the form

$$r = \frac{a(1 - e^2)}{1 - e \cos \theta}$$

(b) Find an approximate polar equation for the elliptical orbit of the planet Earth around the Sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about 2.99×10^8 km.

8. (a) The planets move around the Sun in elliptical orbits with the Sun at one focus. The positions of a planet that are closest to and farthest from the Sun are called its *perihelion* and *aphelion*, respectively. (See Figure 2.) Use Problem 7(a) to show that the perihelion distance from a planet to the Sun is $a(1 - e)$ and the aphelion distance is $a(1 + e)$.

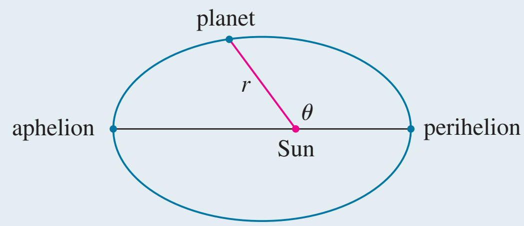


FIGURE 2

(b) Use the data of Problem 7(b) to find the distances from Earth to the Sun at perihelion and at aphelion.

9. (a) The planet Mercury travels in an elliptical orbit with eccentricity 0.206. Its minimum distance from the Sun is 4.6×10^7 km. Use the results of Problem 8(a) to find its maximum distance from the Sun.

(b) Find the distance traveled by the planet Mercury during one complete orbit around the Sun. (Use your calculator or computer algebra system to evaluate the definite integral.)

I Complex Numbers

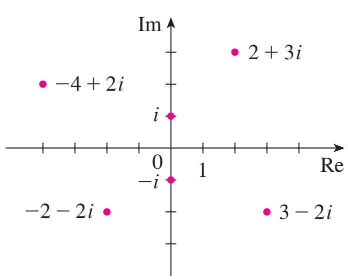


FIGURE 1
Complex numbers as points in the Argand plane

A **complex number** can be represented by an expression of the form $a + bi$, where a and b are real numbers and i is a symbol with the property that $i^2 = -1$. The complex number $a + bi$ can also be represented by the ordered pair (a, b) and plotted as a point in a plane (called the Argand plane) as in Figure 1. Thus, the complex number $i = 0 + 1 \cdot i$ is identified with the point $(0, 1)$.

The **real part** of the complex number $a + bi$ is the real number a and the **imaginary part** is the real number b. Thus, the real part of $4 - 3i$ is 4 and the imaginary part is -3. Two complex numbers $a + bi$ and $c + di$ are **equal** if $a = c$ and $b = d$, that is, their real parts are equal and their imaginary parts are equal. In the Argand plane the x-axis is called the real axis and the y-axis is called the imaginary axis.

The sum and difference of two complex numbers are defined by adding or subtracting their real parts and their imaginary parts:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

For instance,

$$(1 - i) + (4 + 7i) = (1 + 4) + (-1 + 7)i = 5 + 6i$$

The product of complex numbers is defined so that the usual commutative and distributive laws hold:

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + (bi)(c + di)\\ &= ac + adi + bci + bdi^2\end{aligned}$$

Since $i^2 = -1$, this becomes

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

EXAMPLE 1

$$\begin{aligned}(-1 + 3i)(2 - 5i) &= (-1)(2 - 5i) + 3i(2 - 5i)\\ &= -2 + 5i + 6i - 15(-1) = 13 + 11i\end{aligned}$$

Division of complex numbers is much like rationalizing the denominator of a rational expression. For the complex number $z = a + bi$, we define its **complex conjugate** to be $\overline{z} = a - bi$. To find the quotient of two complex numbers we multiply numerator and denominator by the complex conjugate of the denominator.

EXAMPLE 2 Express the number $\dfrac{-1 + 3i}{2 + 5i}$ in the form $a + bi$.

SOLUTION We multiply numerator and denominator by the complex conjugate of $2 + 5i$, namely $2 - 5i$, and we take advantage of the result of Example 1:

$$\frac{-1 + 3i}{2 + 5i} = \frac{-1 + 3i}{2 + 5i} \cdot \frac{2 - 5i}{2 - 5i} = \frac{13 + 11i}{2^2 + 5^2} = \frac{13}{29} + \frac{11}{29}i$$

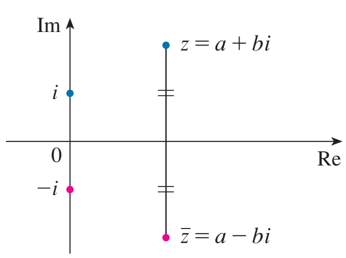


FIGURE 2

The geometric interpretation of the complex conjugate is shown in Figure 2: $\overline{z}$ is the reflection of z in the real axis. We list some of the properties of the complex conjugate in the following box. The proofs follow from the definition and are requested in Exercise 18.

Properties of Conjugates

$$\overline{z + w} = \overline{z} + \overline{w} \qquad \overline{zw} = \overline{z}\,\overline{w} \qquad \overline{z^n} = \overline{z}^n$$

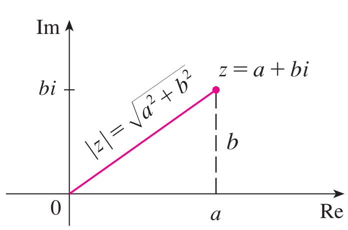


FIGURE 3

The **modulus**, or **absolute value**, $|z|$ of a complex number $z = a + bi$ is its distance from the origin. From Figure 3 we see that if $z = a + bi$, then

$$|z| = \sqrt{a^2 + b^2}$$

Notice that

$$z\overline{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2$$

and so

$$z\overline{z} = |z|^2$$

This explains why the division procedure in Example 2 works in general:

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}$$

Since $i^2 = -1$, we can think of i as a square root of -1. But notice that we also have $(-i)^2 = i^2 = -1$ and so $-i$ is also a square root of -1. We say that i is the **principal square root** of -1 and write $\sqrt{-1} = i$. In general, if c is any positive number, we write

$$\sqrt{-c} = \sqrt{c}\, i$$

With this convention the usual derivation and formula for the roots of the quadratic equation $ax^2 + bx + c = 0$ are valid even when $b^2 - 4ac < 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 3 Find the roots of the equation $x^2 + x + 1 = 0$.

SOLUTION Using the quadratic formula, we have

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$ ■

We observe that the solutions of the equation in Example 3 are complex conjugates of each other. In general, the solutions of any quadratic equation $ax^2 + bx + c = 0$ with real coefficients a, b, and c are always complex conjugates. (If z is real, $\overline{z} = z$, so z is its own conjugate.)

We have seen that if we allow complex numbers as solutions, then every quadratic equation has a solution. More generally, it is true that every polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree at least one has a solution among the complex numbers. This fact is known as the Fundamental Theorem of Algebra and was proved by Gauss.

Polar Form

We know that any complex number $z = a + bi$ can be considered as a point (a, b) and that any such point can be represented by polar coordinates (r, θ) with $r \geqslant 0$. In fact,

$$a = r\cos\theta \qquad b = r\sin\theta$$

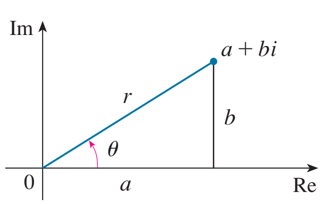


FIGURE 4

as in Figure 4. Therefore, we have

$$z = a + bi = (r\cos\theta) + (r\sin\theta)i$$

Thus, we can write any complex number z in the form

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

where $\qquad r = |z| = \sqrt{a^2 + b^2} \qquad$ and $\qquad \tan\theta = \dfrac{b}{a}$

The angle θ is called the **argument** of z and we write $\theta = \arg(z)$. Note that $\arg(z)$ is not unique; any two arguments of z differ by an integer multiple of 2π.

EXAMPLE 4 Write the following numbers in polar form.
(a) $z = 1 + i$ (b) $w = \sqrt{3} - i$

SOLUTION
(a) We have $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\tan\theta = 1$, so we can take $\theta = \pi/4$. Therefore, the polar form is

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(b) Here we have $r = |w| = \sqrt{3 + 1} = 2$ and $\tan\theta = -1/\sqrt{3}$. Since w lies in the fourth quadrant, we take $\theta = -\pi/6$ and

$$w = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

The numbers z and w are shown in Figure 5. ■

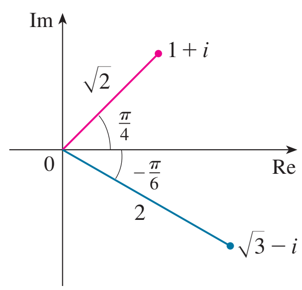


FIGURE 5

The polar form of complex numbers gives insight into multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be two complex numbers written in polar form. Then

$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \end{aligned}$$

Therefore, using the addition formulas for cosine and sine, we have

1 $$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

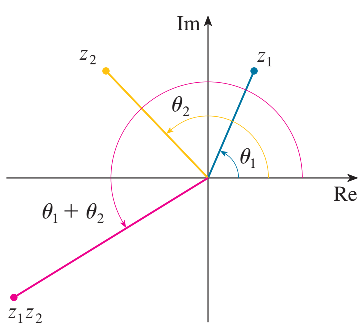


FIGURE 6

This formula says that *to multiply two complex numbers we multiply the moduli and add the arguments.* (See Figure 6.)

A similar argument using the subtraction formulas for sine and cosine shows that *to divide two complex numbers we divide the moduli and subtract the arguments.*

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad z_2 \neq 0$$

In particular, taking $z_1 = 1$ and $z_2 = z$, (and therefore $\theta_1 = 0$ and $\theta_2 = \theta$), we have the following, which is illustrated in Figure 7.

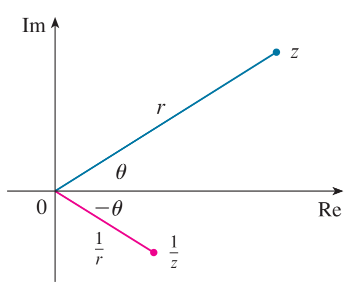


FIGURE 7

$$\text{If} \quad z = r(\cos\theta + i\sin\theta), \quad \text{then} \quad \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta).$$

EXAMPLE 5 Find the product of the complex numbers $1 + i$ and $\sqrt{3} - i$ in polar form.

SOLUTION From Example 4 we have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

and
$$\sqrt{3} - i = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

So, by Equation 1,

$$\begin{aligned}(1 + i)(\sqrt{3} - i) &= 2\sqrt{2}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right] \\ &= 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)\end{aligned}$$

This is illustrated in Figure 8. ■

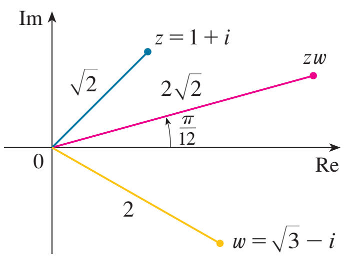


FIGURE 8

Repeated use of Formula 1 shows how to compute powers of a complex number. If

$$z = r(\cos\theta + i\sin\theta)$$

then
$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

and
$$z^3 = zz^2 = r^3(\cos 3\theta + i\sin 3\theta)$$

In general, we obtain the following result, which is named after the French mathematician Abraham De Moivre (1667–1754).

> **2 De Moivre's Theorem** If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then
>
> $$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

This says that *to take the nth power of a complex number we take the nth power of the modulus and multiply the argument by n.*

EXAMPLE 6 Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$.

SOLUTION Since $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$, it follows from Example 4(a) that $\frac{1}{2} + \frac{1}{2}i$ has the polar form

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

So by De Moivre's Theorem,

$$\begin{aligned}\left(\frac{1}{2} + \frac{1}{2}i\right)^{10} &= \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right) \\ &= \frac{2^5}{2^{10}}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \frac{1}{32}i\end{aligned}$$

■

De Moivre's Theorem can also be used to find the nth roots of complex numbers. An nth root of the complex number z is a complex number w such that

$$w^n = z$$

Writing these two numbers in trigonometric form as

$$w = s(\cos\phi + i\sin\phi) \qquad \text{and} \qquad z = r(\cos\theta + i\sin\theta)$$

and using De Moivre's Theorem, we get

$$s^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$

The equality of these two complex numbers shows that

$$s^n = r \qquad \text{or} \qquad s = r^{1/n}$$

and
$$\cos n\phi = \cos\theta \qquad \text{and} \qquad \sin n\phi = \sin\theta$$

From the fact that sine and cosine have period 2π it follows that

$$n\phi = \theta + 2k\pi \qquad \text{or} \qquad \phi = \frac{\theta + 2k\pi}{n}$$

Thus
$$w = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

Since this expression gives a different value of w for $k = 0, 1, 2, \ldots, n - 1$, we have the following.

3 Roots of a Complex Number Let $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer. Then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$.

Notice that each of the nth roots of z has modulus $|w_k| = r^{1/n}$. Thus, all the nth roots of z lie on the circle of radius $r^{1/n}$ in the complex plane. Also, since the argument of each successive nth root exceeds the argument of the previous root by $2\pi/n$, we see that the nth roots of z are equally spaced on this circle.

EXAMPLE 7 Find the six sixth roots of $z = -8$ and graph these roots in the complex plane.

SOLUTION In trigonometric form, $z = 8(\cos\pi + i\sin\pi)$. Applying Equation 3 with $n = 6$, we get

$$w_k = 8^{1/6}\left(\cos\frac{\pi + 2k\pi}{6} + i\sin\frac{\pi + 2k\pi}{6}\right)$$

We get the six sixth roots of -8 by taking $k = 0, 1, 2, 3, 4, 5$ in this formula:

$$w_0 = 8^{1/6}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$w_1 = 8^{1/6}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \sqrt{2}\,i$$

$$w_2 = 8^{1/6}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$w_3 = 8^{1/6}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$w_4 = 8^{1/6}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -\sqrt{2}\,i$$

$$w_5 = 8^{1/6}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

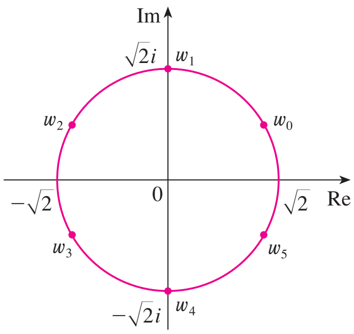


FIGURE 9
The six sixth roots of $z = -8$

All these points lie on the circle of radius $\sqrt{2}$ as shown in Figure 9.

Complex Exponentials

We also need to give a meaning to the expression e^z when $z = x + iy$ is a complex number. The theory of infinite series as developed in Chapter 8 can be extended to the case where the terms are complex numbers. Using the Taylor series for e^x (Equation 8.7.11) as our guide, we define

4 $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

and it turns out that this complex exponential function has the same properties as the real exponential function. In particular, it is true that

5 $$e^{z_1 + z_2} = e^{z_1} e^{z_2}$$

If we put $z = iy$, where y is a real number, in Equation 4, and use the facts that

$$i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = 1, \quad i^5 = i, \quad \ldots$$

we get
$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots \\ &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\ &= \cos y + i\sin y \end{aligned}$$

Here we have used the Taylor series for $\cos y$ and $\sin y$ (Equations 8.7.16 and 8.7.15).

The result is a famous formula called **Euler's formula**:

6
$$e^{iy} = \cos y + i \sin y$$

Combining Euler's formula with Equation 5, we get

7
$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

EXAMPLE 8 Evaluate: (a) $e^{i\pi}$ (b) $e^{-1+i\pi/2}$

SOLUTION

(a) From Euler's formula (6) we have

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1$$

(b) Using Equation 7 we get

$$e^{-1+i\pi/2} = e^{-1}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) = \frac{1}{e}[0 + i(1)] = \frac{i}{e}$$

Finally, we note that Euler's formula provides us with an easier method of proving De Moivre's Theorem:

$$[r(\cos \theta + i \sin \theta)]^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

I Exercises

1–14 ■ Evaluate the expression and write your answer in the form $a + bi$.

1. $(3 + 2i) + (7 - 3i)$

2. $(1 + i) - (2 - 3i)$

3. $(3 - i)(4 + i)$

4. $(4 - 7i)(1 + 3i)$

5. $\overline{12 + 7i}$

6. $\overline{2i(\frac{1}{2} - i)}$

7. $\dfrac{2 + 3i}{1 - 5i}$

8. $\dfrac{5 - i}{3 + 4i}$

9. $\dfrac{1}{1 + i}$

10. $\dfrac{3}{4 - 3i}$

11. i^3

12. i^{100}

13. $\sqrt{-25}$

14. $\sqrt{-3}\sqrt{-12}$

15–17 ■ Find the complex conjugate and the modulus of the given number.

15. $3 + 4i$

16. $\sqrt{3} - i$

17. $-4i$

18. Prove the following properties of complex numbers.
(a) $\overline{z + w} = \overline{z} + \overline{w}$
(b) $\overline{zw} = \overline{z}\,\overline{w}$
(c) $\overline{z^n} = \overline{z}^n$, where n is a positive integer
[*Hint:* Write $z = a + bi$, $w = c + di$.]

19–24 ■ Find all solutions of the equation.

19. $4x^2 + 9 = 0$

20. $x^4 = 1$

21. $x^2 - 8x + 17 = 0$

22. $x^2 - 4x + 5 = 0$

23. $z^2 + z + 2 = 0$

24. $z^2 + \frac{1}{2}z + \frac{1}{4} = 0$

25–28 ■ Write the number in polar form with argument between 0 and 2π.

25. $-3 + 3i$

26. $1 - \sqrt{3}i$

27. $3 + 4i$

28. $8i$

29–32 ■ Find polar forms for zw, z/w, and $1/z$ by first putting z and w into polar form.

29. $z = \sqrt{3} + i$, $\quad w = 1 + \sqrt{3}i$

30. $z = 4\sqrt{3} - 4i$, $\quad w = 8i$

31. $z = 2\sqrt{3} - 2i, \quad w = -1 + i$

32. $z = 4(\sqrt{3} + i), \quad w = -3 - 3i$

33–36 ■ Find the indicated power using De Moivre's Theorem.

33. $(1 + i)^{20}$

34. $(1 - \sqrt{3}i)^5$

35. $(2\sqrt{3} + 2i)^5$

36. $(1 - i)^8$

37–40 ■ Find the indicated roots. Sketch the roots in the complex plane.

37. The eighth roots of 1

38. The fifth roots of 32

39. The cube roots of i

40. The cube roots of $1 + i$

41–46 ■ Write the number in the form $a + bi$.

41. $e^{i\pi/2}$

42. $e^{2\pi i}$

43. $e^{i3\pi/4}$

44. $e^{-i\pi}$

45. $e^{2+i\pi}$

46. e^{1+2i}

47. Use De Moivre's Theorem with $n = 3$ to express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

48. Use Euler's formula to prove the following formulas for $\cos x$ and $\sin x$:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

49. If $u(x) = f(x) + ig(x)$ is a complex-valued function of a real variable x and the real and imaginary parts $f(x)$ and $g(x)$ are differentiable functions of x, then the derivative of u is defined to be $u'(x) = f'(x) + ig'(x)$. Use this together with Equation 7 to prove that if $F(x) = e^{rx}$, then $F'(x) = re^{rx}$ when $r = a + bi$ is a complex number.

50. (a) If u is a complex-valued function of a real variable, its indefinite integral $\int u(x)\,dx$ is an antiderivative of u. Evaluate

$$\int e^{(1+i)x}\,dx$$

(b) By considering the real and imaginary parts of the integral in part (a), evaluate the real integrals

$$\int e^x \cos x\,dx \qquad \text{and} \qquad \int e^x \sin x\,dx$$

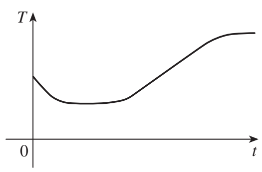
Compare with the method used in Example 4 in Section 5.6.

J Answers to Odd-Numbered Exercises

CHAPTER 1

Exercises 1.1 • page 22

1. (a) -2 (b) 2.8 (c) $-3, 1$ (d) $-2.5, 0.3$
(e) $[-3, 3], [-2, 3]$ (f) $[-1, 3]$
3. $[-85, 115], [-325, 485], [-210, 200]$
5. Yes, $[-3, 2], [-2, 2]$ **7.** No
9. Diet, exercise, or illness
11.



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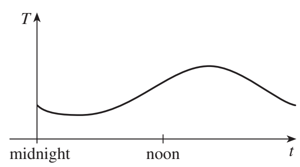

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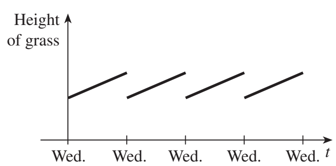
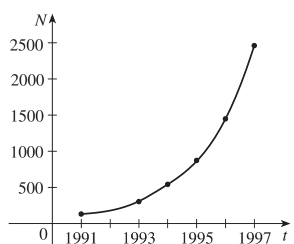
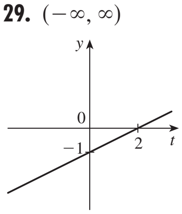


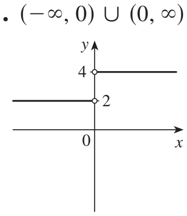
17. (a)



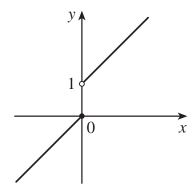
(b) 540, 1450
19. $12, 16, 3a^2 - a + 2, 3a^2 + a + 2, 3a^2 + 5a + 4,$
$6a^2 - 2a + 4, 12a^2 - 2a + 2, 3a^4 - a^2 + 2,$
$9a^4 - 6a^3 + 13a^2 - 4a + 4, 3a^2 + 6ah + 3h^2 - a - h + 2$
21. $-(h^2 + 3h + 2), x + h - x^2 - 2xh - h^2, 1 - 2x - h$
23. $\{x \mid x \neq \frac{1}{3}\} = (-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$
25. $[0, \infty)$ **27.** $(-\infty, 0) \cup (5, \infty)$
29. $(-\infty, \infty)$



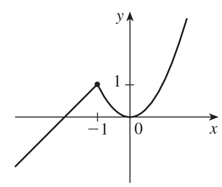
31. $(-\infty, 0) \cup (0, \infty)$



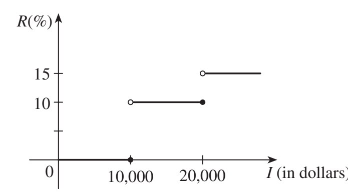
33. $(-\infty, \infty)$



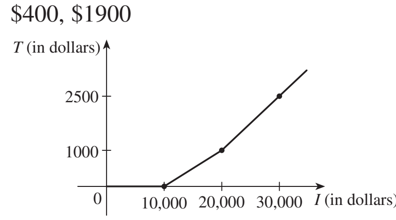
35. $(-\infty, \infty)$



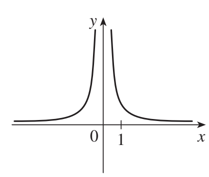
37. $f(x) = -\frac{7}{6}x - \frac{4}{3}, -2 \leqslant x \leqslant 4$
39. $f(x) = 1 - \sqrt{-x}$
41. $f(x) = \begin{cases} x + 1 & \text{if } -1 \leqslant x \leqslant 2 \\ 6 - 1.5x & \text{if } 2 < x \leqslant 4 \end{cases}$
43. $A(L) = 10L - L^2, 0 < L < 10$
45. $A(x) = \sqrt{3}x^2/4, x > 0$ **47.** $S(x) = x^2 + (8/x), x > 0$
49. $V(x) = 4x^3 - 64x^2 + 240x, 0 < x < 6$
51. (a)

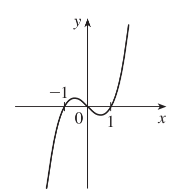


(b) \$400, \$1900
(c)



53. (a) $(-5, 3)$ (b) $(-5, -3)$
55. Even **57.** Neither **59.** Odd

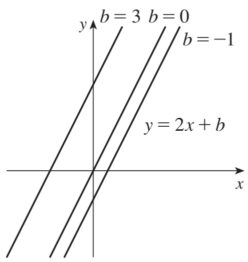




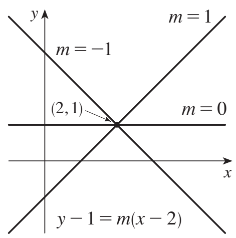
Exercises 1.2 • page 35

1. (a) Root (b) Algebraic (c) Polynomial (degree 9)
(d) Rational (e) Trigonometric (f) Logarithmic
3. (a) h (b) f (c) g

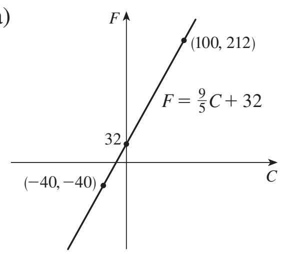
5. (a) $y = 2x + b$, where b is the y-intercept



(b) $y = mx + 1 - 2m$, where m is the slope. See graph at right.
(c) $y = 2x - 3$

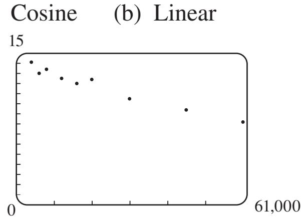


7. (a)



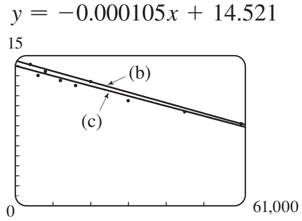
(b) $\frac{9}{5}$, change in °F for every °C change; 32, Fahrenheit temperature corresponding to 0 °C

9. (a) $T = \frac{1}{6}N + \frac{307}{6}$ (b) $\frac{1}{6}$, change in °F for every chirp per minute change (c) 76 °F
11. (a) $P = 0.434d + 15$ (b) 196 ft
13. (a) Cosine (b) Linear
15. (a)

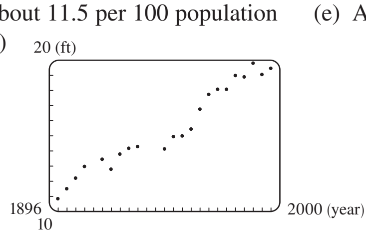


Linear model is appropriate.

(b) $y = -0.000105x + 14.521$

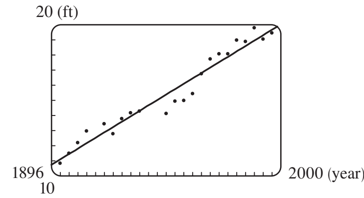


(c) $y = -0.00009979x + 13.951$ [See graph in (b).]
(d) About 11.5 per 100 population (e) About 6% (f) No
17. (a)



Linear model is appropriate.

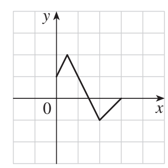
(b) $y = 0.08912x - 158.24$ (c) 20.00 ft (d) No



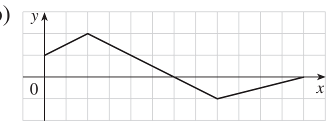
19. $y = 0.00123543x^3 - 6.72226x^2 + 12{,}165x - 7{,}318{,}429$; 1913 million

Exercises 1.3 • page 46

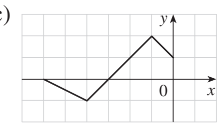
1. (a) $y = f(x) + 3$ (b) $y = f(x) - 3$ (c) $y = f(x - 3)$ (d) $y = f(x + 3)$ (e) $y = -f(x)$ (f) $y = f(-x)$ (g) $y = 3f(x)$ (h) $y = \frac{1}{3}f(x)$
3. (a) 3 (b) 1 (c) 4 (d) 5 (e) 2
5. (a)



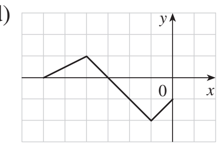
(b)



(c)

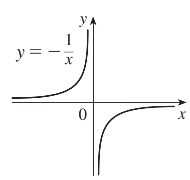


(d)

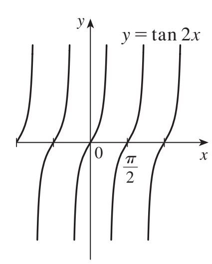


7. $y = -\sqrt{-x^2 - 5x - 4} - 1$

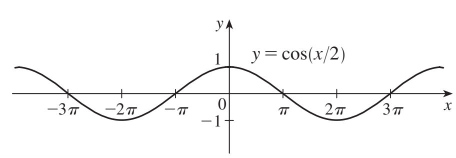
9.



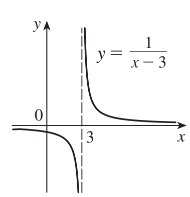
11.



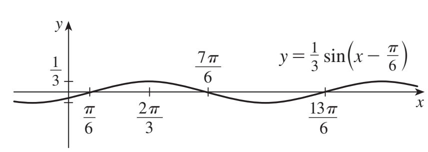
13.



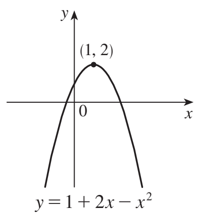
15.



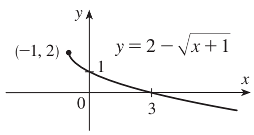
17.



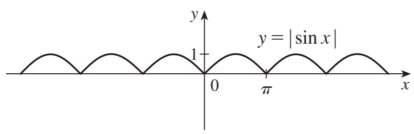
19.



21.



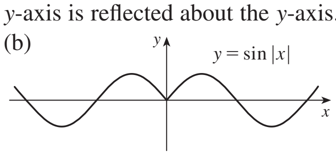
23.



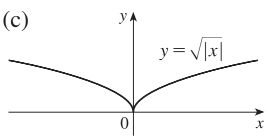
25. $L(t) = 12 + 2\sin\left[\frac{2\pi}{365}(t - 80)\right]$

27. (a) The portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

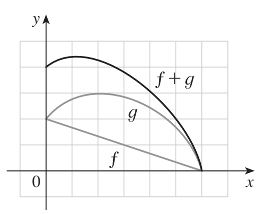
(b)



(c)

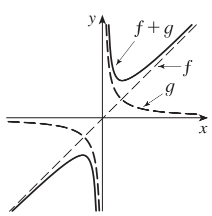


29.



31. $(f + g)(x) = x^3 + 5x^2 - 1$, $(-\infty, \infty)$
$(f - g)(x) = x^3 - x^2 + 1$, $(-\infty, \infty)$
$(fg)(x) = 3x^5 + 6x^4 - x^3 - 2x^2$, $(-\infty, \infty)$
$(f/g)(x) = (x^3 + 2x^2)/(3x^2 - 1)$, $\{x \mid x \neq \pm 1/\sqrt{3}\}$

33.



35. $(f \circ g)(x) = \sin(1 - \sqrt{x})$, $[0, \infty)$
$(g \circ f)(x) = 1 - \sqrt{\sin x}$, $\{x \mid x \in [2n\pi, \pi + 2n\pi], n \text{ an integer}\}$
$(f \circ f)(x) = \sin(\sin x)$, $(-\infty, \infty)$
$(g \circ g)(x) = 1 - \sqrt{1 - \sqrt{x}}$, $[0, 1]$

37. $(f \circ g)(x) = (2x^2 + 6x + 5)/[(x + 2)(x + 1)]$, $\{x \mid x \neq -2, -1\}$
$(g \circ f)(x) = (x^2 + x + 1)/(x + 1)^2$, $\{x \mid x \neq -1, 0\}$
$(f \circ f)(x) = (x^4 + 3x^2 + 1)/[x(x^2 + 1)]$, $\{x \mid x \neq 0\}$
$(g \circ g)(x) = (2x + 3)/(3x + 5)$, $\{x \mid x \neq -2, -\frac{5}{3}\}$

39. $(f \circ g \circ h)(x) = \sqrt{x^2 + 6x + 10}$

41. $g(x) = x^2 + 1$, $f(x) = x^{10}$ **43.** $g(t) = \cos t$, $f(t) = \sqrt{t}$

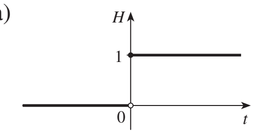
45. $h(x) = x^2$, $g(x) = 3^x$, $f(x) = 1 - x$

47. $h(x) = \sqrt{x}$, $g(x) = \sec x$, $f(x) = x^4$

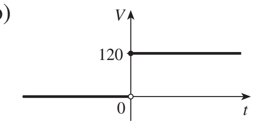
49. (a) 4 (b) 3 (c) 0
(d) Does not exist; $f(6) = 6$ is not in the domain of g.
(e) 4 (f) −2

51. (a) $r(t) = 60t$ (b) $(A \circ r)(t) = 3600\pi t^2$; the area of the circle as a function of time

53. (a)

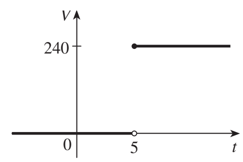


(b)



$V(t) = 120H(t)$

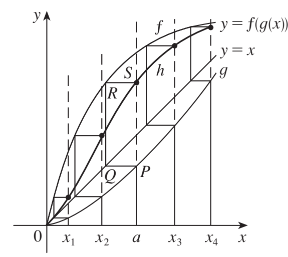
(c)



$V(t) = 240H(t - 5)$

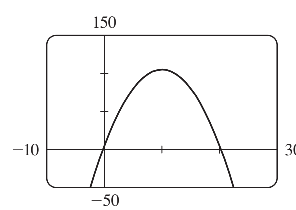
55. (a) $f(x) = x^2 + 6$ (b) $g(x) = x^2 + x - 1$ **57.** Yes

59. (a) $P(a, g(a))$, $Q(g(a), g(a))$ (b) $(g(a), f(g(a)))$
(d)

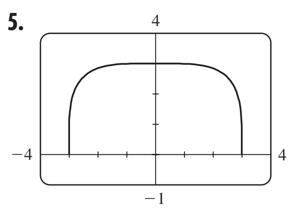


Exercises 1.4 • page 55

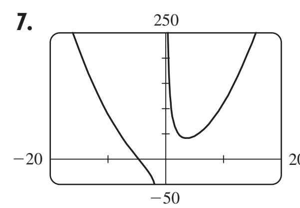
1. (c) **3.**



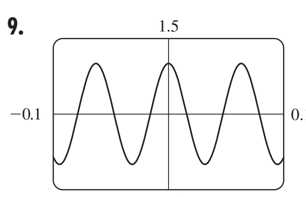
5.



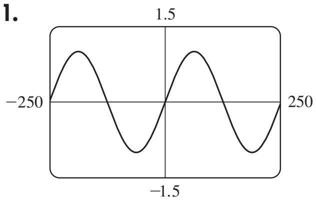
7.



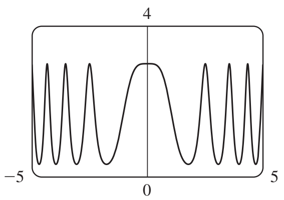
9.



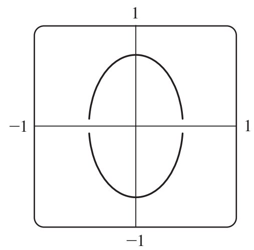
11.



13.

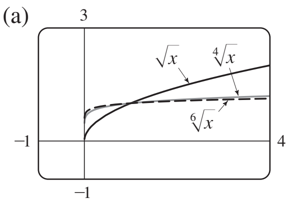


15.

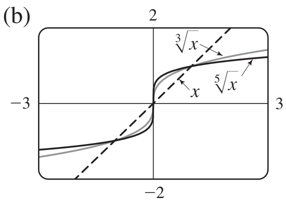


17. 9.05 **19.** 0, 0.88 **21.** g **23.** $-0.85 < x < 0.85$

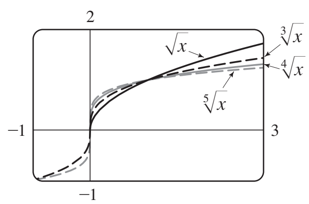
25. (a)



(b)

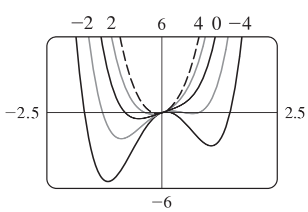


(c)



(d) Graphs of even roots are similar to $\sqrt{x}$, graphs of odd roots are similar to $\sqrt[3]{x}$. As n increases, the graph of $y = \sqrt[n]{x}$ becomes steeper near 0 and flatter for $x > 1$.

27.



If $c < 0$, the graph has three humps: two minimum points and a maximum point. These humps get flatter as c increases until at $c = 0$ two of the humps disappear and there is only one minimum point. This single hump then moves to the right and approaches the origin as c increases.

29. The hump gets larger and moves to the right.

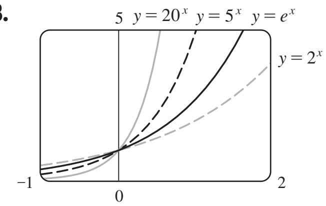
31. If $c < 0$, the loop is to the right of the origin; if $c > 0$, the loop is to the left. The closer c is to 0, the larger the loop.

Exercises 1.5 • page 63

1. (a) $f(x) = a^x, a > 0$ (b) $\mathbb{R}$ (c) $(0, \infty)$

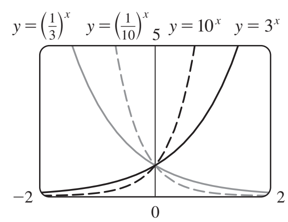
(d) See Figures 4(c), 4(b), and 4(a), respectively.

3.



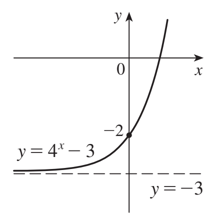
All approach 0 as $x \to -\infty$, all pass through (0, 1), and all are increasing. The larger the base, the faster the rate of increase.

5.

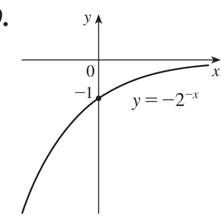


The functions with base greater than 1 are increasing and those with base less than 1 are decreasing. The latter are reflections of the former about the y-axis.

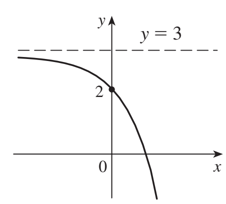
7.



9.



11.



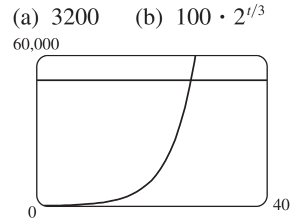
13. (a) $y = e^x - 2$ (b) $y = e^{x-2}$ (c) $y = -e^x$

(d) $y = e^{-x}$ (e) $y = -e^{-x}$

15. $f(x) = 3 \cdot 2^x$ **21.** At $x \approx 35.8$

23. (a) 3200 (b) $100 \cdot 2^{t/3}$ (c) 10,159

(d)



$t \approx 26.9$ h

25. $y = ab^t$, where $a \approx 3.303902537 \times 10^{-12}$ and $b \approx 1.01774077$; 5494 million; 7409 million

Exercises 1.6 • page 73

1. (a) See Definition 1.

(b) It must pass the Horizontal Line Test.

3. No **5.** Yes **7.** No **9.** Yes **11.** No

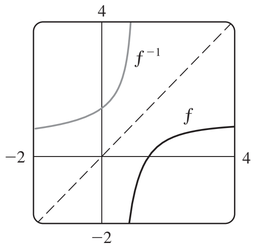
13. No **15.** No **17.** 2 **19.** 0

21. $F = \frac{9}{5}C + 32$; the Fahrenheit temperature as a function of the Celsius temperature; $[-273.15, \infty)$

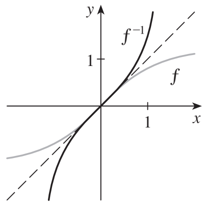
23. $f^{-1}(x) = -\frac{1}{3}x^2 + \frac{10}{3},\ x \geqslant 0$ **25.** $f^{-1}(x) = \sqrt[3]{\ln x}$

27. $y = e^x - 3$

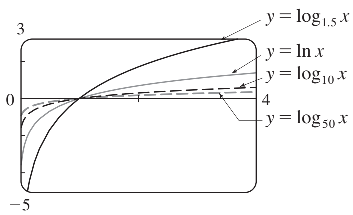
29. $f^{-1}(x) = \sqrt{2/(1-x)}$



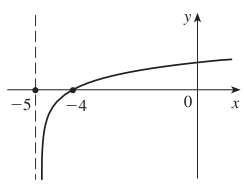
31.



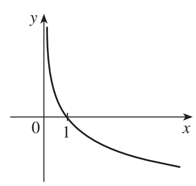
33. (a) It's defined as the inverse of the exponential function with base a, that is, $\log_a x = y \iff a^y = x$.
(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 11.
35. (a) 6 (b) -2
37. (a) 2 (b) 2 **39.** ln 8
41. (a) 2.321928 (b) 2.025563
43.



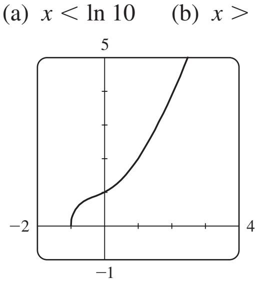
All graphs approach $-\infty$ as $x \to 0^+$, all pass through (1, 0), and all are increasing. The larger the base, the slower the rate of increase.
45. About 1,084,588 mi
47. (a)



(b)



49. (a) $\sqrt{e}$ (b) $-\ln 5$
51. (a) $5 + \log_2 3$ or $5 + (\ln 3)/\ln 2$ (b) $\frac{1}{2}\left(1 + \sqrt{1 + 4e}\right)$
53. (a) $x < \ln 10$ (b) $x > 1/e$
55.

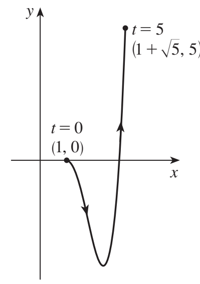


passes the Horizontal Line Test

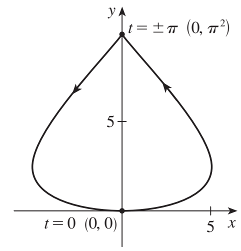
$f^{-1}(x) =$
$-\left(\sqrt[3]{4}/6\right)\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$,
where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$; two of the expressions are complex.
57. (a) $f^{-1}(n) = (3/\ln 2)\ln(n/100)$; the time elapsed when there are n bacteria (b) After about 26.9 h
59. (a) $y = \ln x + 3$ (b) $y = \ln(x + 3)$ (c) $y = -\ln x$
(d) $y = \ln(-x)$ (e) $y = e^x$ (f) $y = e^{-x}$ (g) $y = -e^x$
(h) $y = e^x - 3$

Exercises 1.7 • page 81

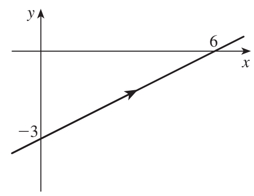
1.



3.

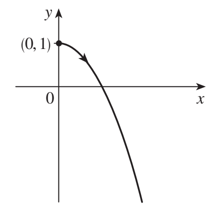


5. (a)



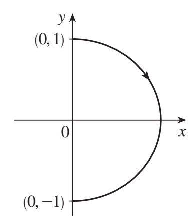
(b) $y = \frac{1}{2}x - 3$

7. (a)

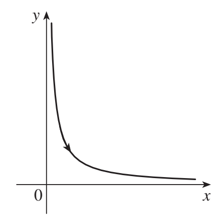


(b) $y = 1 - x^2, x \geqslant 0$

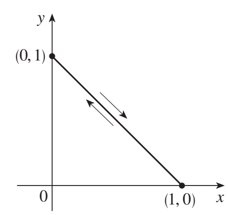
9. (a) $x^2 + y^2 = 1, x \geqslant 0$
(b)



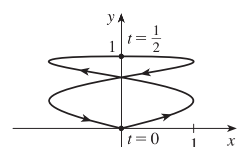
11. (a) $y = 1/x, x > 0$
(b)



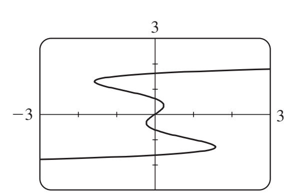
13. (a) $x + y = 1, 0 \leqslant x \leqslant 1$ (b)



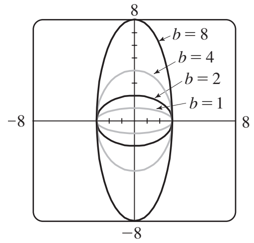
15. Moves counterclockwise along the circle $x^2 + y^2 = 1$ from $(-1, 0)$ to $(1, 0)$
17. Moves once clockwise around the ellipse $(x^2/4) + (y^2/9) = 1$, starting and ending at (0, 3)
19. It is contained in the rectangle described by $1 \leqslant x \leqslant 4$ and $2 \leqslant y \leqslant 3$.
21.



23. (b) $x = -2 + 5t$,
$y = 7 - 8t, 0 \leqslant t \leqslant 1$
25.

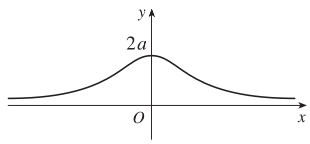


27. (a) $x = 2\cos t,\ y = 1 - 2\sin t,\ 0 \leqslant t \leqslant 2\pi$
(b) $x = 2\cos t,\ y = 1 + 2\sin t,\ 0 \leqslant t \leqslant 6\pi$
(c) $x = 2\cos t,\ y = 1 + 2\sin t,\ \pi/2 \leqslant t \leqslant 3\pi/2$
29. (a) $x = a\sin t,\ y = b\cos t,\ 0 \leqslant t \leqslant 2\pi$
(b)

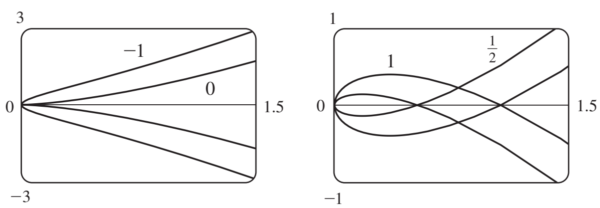


(c) As b increases, the ellipse stretches vertically.

33. $x = a\cos\theta,\ y = b\sin\theta;\ (x^2/a^2) + (y^2/b^2) = 1$, ellipse
35.



37. For $c = 0$, there is a cusp; for $c > 0$, there is a loop whose size increases as c increases.



39. As n increases, the number of oscillations increases; a and b determine the width and height.

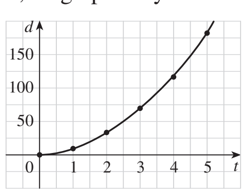
Chapter 1 Review • page 85

True-False Quiz

1. False **3.** False **5.** True **7.** False **9.** True
11. False

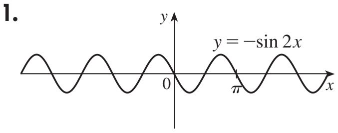
Exercises

1. (a) 2.7 (b) 2.3, 5.6 (c) $[-6, 6]$ (d) $[-4, 4]$
(e) $[-4, 4]$ (f) No; f fails the Horizontal Line Test.
(g) Odd; its graph is symmetric about the origin.
3. (a)

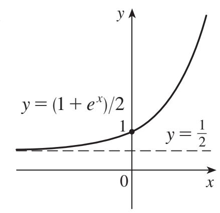


(b) 150 ft

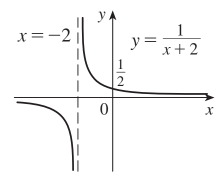
5. $[-2\sqrt{3}/3,\ 2\sqrt{3}/3],\ [0, 2]$ **7.** $\mathbb{R},\ [0, 2]$
9. (a) Shift the graph 8 units upward.
(b) Shift the graph 8 units to the left.
(c) Stretch the graph vertically by a factor of 2, then shift it 1 unit upward.
(d) Shift the graph 2 units to the right and 2 units downward.
(e) Reflect the graph about the x-axis.
(f) Reflect the graph about the line $y = x$ (assuming f is one-to-one).
11.



13.

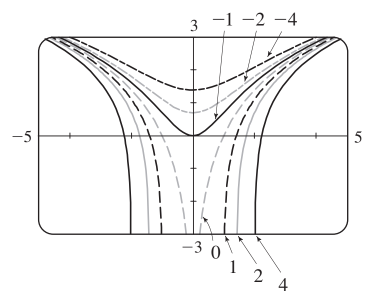


15.

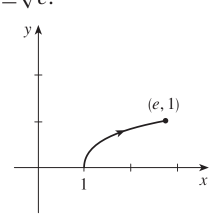


17. (a) Neither
(b) Odd
(c) Even
(d) Neither

19. $(f \circ g)(x) = \ln(x^2 - 9),\ (-\infty, -3) \cup (3, \infty)$
$(g \circ f)(x) = (\ln x)^2 - 9,\ (0, \infty)$
$(f \circ f)(x) = \ln\ln x,\ (1, \infty)$
$(g \circ g)(x) = (x^2 - 9)^2 - 9,\ (-\infty, \infty)$
21. $y = 0.2493x - 423.4818$; about 77.6 years
23. 1 **25.** (a) 9 (b) 2
27. (a) $\frac{1}{16}$ g (b) $m(t) = 2^{-t/4}$
(c) $t(m) = -4\log_2 m$; the time elapsed when there are m grams of ^{100}Pd (d) About 26.6 days
29.

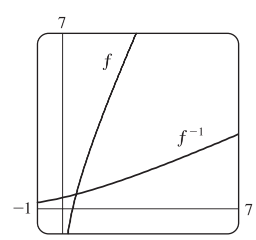


For $c < 0$, f is defined everywhere. As c increases, the dip at $x = 0$ becomes deeper. For $c \geqslant 0$, the graph has asymptotes at $x = \pm\sqrt{c}$.
31. (a)

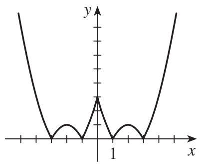
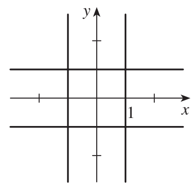
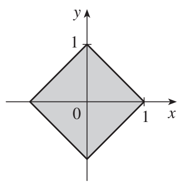


(b) $y = \sqrt{\ln x}$

33.

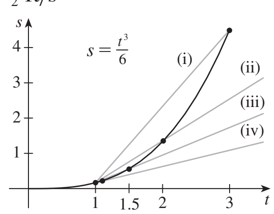
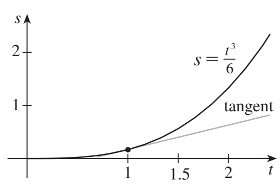


Principles of Problem Solving • page 93

1. $a = 4\sqrt{h^2 - 16}/h$, where a is the length of the altitude and h is the length of the hypotenuse
3. $-\frac{7}{3}, 9$
5.

7.

9.

11. 5
13. $x \in [-1, 1 - \sqrt{3}) \cup (1 + \sqrt{3}, 3]$
15. 40 mi/h
19. $f_n(x) = x^{2^{n+1}}$

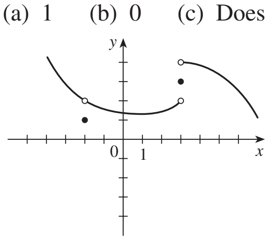
CHAPTER 2

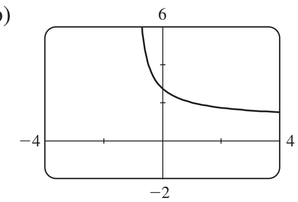
Exercises 2.1 • page 99

1. (a) $-44.4, -38.8, -27.8, -22.2, -16.\overline{6}$
(b) -33.3 (c) $-33\frac{1}{3}$
3. (a) (i) 0.333333 (ii) 0.263158 (iii) 0.251256
(iv) 0.250125 (v) 0.2 (vi) 0.238095 (vii) 0.248756
(viii) 0.249875 (b) $\frac{1}{4}$ (c) $y = \frac{1}{4}x + \frac{1}{4}$
5. (a) (i) -32 ft/s (ii) -25.6 ft/s (iii) -24.8 ft/s
(iv) -24.16 ft/s (b) -24 ft/s
7. (a) (i) $\frac{13}{6}$ ft/s (ii) $\frac{7}{6}$ ft/s (iii) $\frac{19}{24}$ ft/s (iv) $\frac{331}{600}$ ft/s
(b) $\frac{1}{2}$ ft/s
(c)

(d)


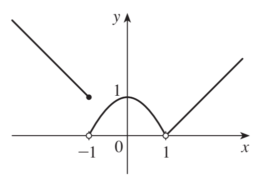
9. (a) 0, 1.7321, -1.0847, -2.7433, 4.3301, -2.8173, 0, -2.1651, -2.6061, -5, 3.4202; no (c) -31.4

Exercises 2.2 • page 108

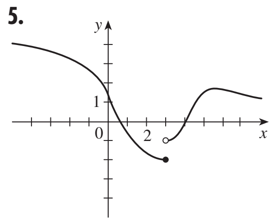
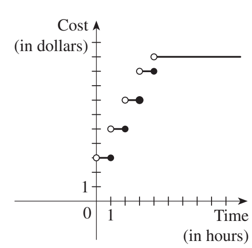
1. Yes **3.** (a) 2 (b) 3 (c) Does not exist
(d) 4 (e) Does not exist
5. (a) -1 (b) -2 (c) Does not exist (d) 2 (e) 0
(f) Does not exist (g) 1 (h) 3
7. (a) 1 (b) 0 (c) Does not exist
9.


11. 0.806452, 0.641026, 0.510204, 0.409836, 0.369004, 0.336689, 0.165563, 0.193798, 0.229358, 0.274725, 0.302115, 0.330022; $\frac{1}{3}$
13. 0.718282, 0.367879, 0.594885, 0.426123, 0.517092, 0.483742, 0.508439, 0.491770, 0.501671, 0.498337; $\frac{1}{2}$
15. (a) 4
17. (a) 2.71828
(b)

19. (a) 0.998000, 0.638259, 0.358484, 0.158680, 0.038851, 0.008928, 0.001465; 0
(b) 0.000572, -0.000614, -0.000907, -0.000978, -0.000993, -0.001000; -0.001
21. Within 0.182; within 0.095

Exercises 2.3 • page 117

1. (a) 5 (b) 9 (c) 2 (d) $-\frac{1}{3}$ (e) $-\frac{3}{8}$ (f) 0
(g) Does not exist (h) $-\frac{6}{11}$
3. 75 **5.** -3 **7.** $\frac{1}{8}$ **9.** 5 **11.** Does not exist
13. $\frac{6}{5}$ **15.** 12 **17.** $\frac{1}{6}$ **19.** $-\frac{1}{16}$
21. (a), (b) $\frac{2}{3}$ **25.** 1 **29.** 0
31. Does not exist
33. (a) (i) 0 (ii) 0 (iii) 1 (iv) 1 (v) 0
(vi) Does not exist
(b)

35. (a) (i) -2 (ii) Does not exist (iii) -3
(b) (i) $n - 1$ (ii) n (c) a is not an integer.
43. 15; -1

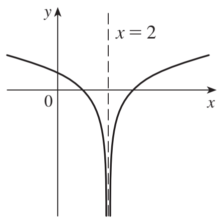
Exercises 2.4 • page 128

1. $\lim_{x\to 4} f(x) = f(4)$
3. (a) -4 (removable), -2 (jump), 2 (jump), 4 (infinite)
(b) -4, neither; -2, left; 2, right; 4, right
5.

7. (a)


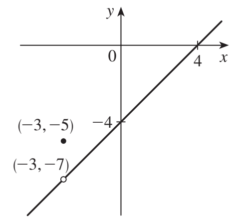
(b) Discontinuous at $t = 1, 2, 3, 4$

9. 6

13. $f(2)$ is not defined

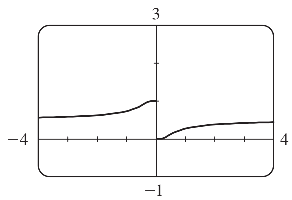


15. $\lim_{x \to -3} f(x) \neq f(-3)$



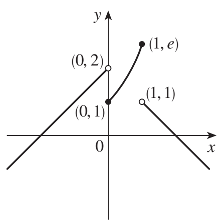
17. $\{x \mid x \neq -3, -2\}$ **19.** $\mathbb{R}$ **21.** $(-\infty, -1) \cup (1, \infty)$

23. $x = 0$



25. $\frac{7}{3}$ **27.** 1

29. 0, right; 1, left



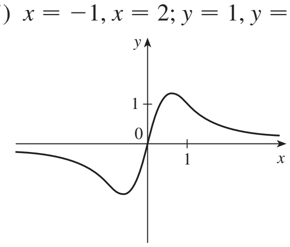
31. $\frac{1}{3}$ **39.** (b) (0.44, 0.45) **41.** (b) 70.347 **45.** Yes

Exercises 2.5 • page 139

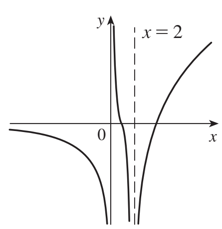
1. (a) As x approaches 2, $f(x)$ becomes large. (b) As x approaches 1 from the right, $f(x)$ becomes large negative. (c) As x becomes large, $f(x)$ approaches 5. (d) As x becomes large negative, $f(x)$ approaches 3.

3. (a) ∞ (b) ∞ (c) $-\infty$ (d) 1 (e) 2 (f) $x = -1, x = 2; y = 1, y = 2$

5.



7.



9. 0 **11.** $x \approx -1.62, x \approx 0.62, x = 1; y = 1$

13. $-\infty$ **15.** ∞ **17.** $-\infty$ **19.** $\frac{1}{2}$ **21.** 2

23. $\frac{1}{6}$ **25.** Does not exist **27.** ∞ **29.** $-\infty$

31. $y = 2; x = -2, x = 1$ **33.** (a), (b) $-\frac{1}{2}$

35. (a) IV (b) III (c) II (d) VI (e) I (f) V

37. $\dfrac{2 - x}{x^2(x - 3)}$ **39.** (a) 0 (b) $\pm\infty$

41. 4 **43.** (b) It approaches the concentration of the brine being pumped into the tank.

45. (b) $x > 23.03$ (c) Yes, $x > 10 \ln 10$

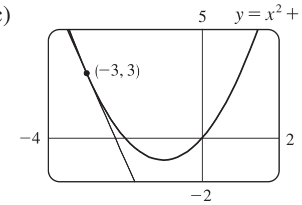
Exercises 2.6 • page 148

1. (a) $\dfrac{f(x) - f(3)}{x - 3}$ (b) $\lim_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$

3. Slopes at D, E, C, A, B

5. (a) (i) -4 (ii) -4 (b) $y = -4x - 9$

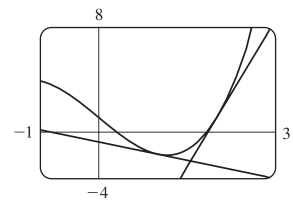
(c)



7. $y = -x + 5$ **9.** $y = \frac{1}{2}x + \frac{1}{2}$

11. (a) $3a^2 - 4$ (b) $y = -x - 1, y = 8x - 15$

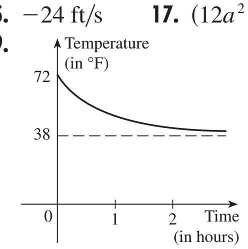
(c)



13. (a) 0 (b) C (c) Speeding up, slowing down, neither (d) The car did not move.

15. -24 ft/s **17.** $(12a^2 + 6)$ m/s, 18 m/s, 54 m/s, 114 m/s

19.



Greater (in magnitude)

21. (a) (i) -1.2 °C/h (ii) -1.25 °C/h (iii) -1.3 °C/h (b) -1.9 °C/h

23. (a) (i) 794 thousand/year (ii) 640 thousand/year (iii) 301 thousand/year (b) 470.5 thousand/year (c) 427.5 thousand/year

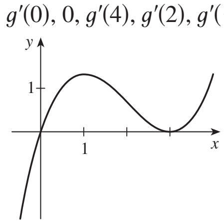
25. (a) (i) \$20.25/unit (ii) \$20.05/unit (b) \$20/unit

Exercises 2.7 • page 155

1. The line from $(2, f(2))$ to $(2 + h, f(2 + h))$

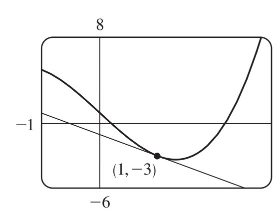
3. $g'(0), 0, g'(4), g'(2), g'(-2)$

5.



7. 7; $y = 7x - 12$

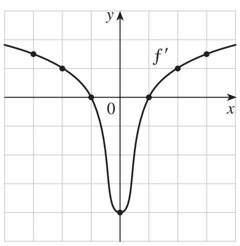
9. (a) -2; $y = -2x - 1$ (b)



11. (a) 3.296 (b) 3.3
13. $-2 + 8a$ **15.** $5/(a + 3)^2$
17. $-1/[2(a + 2)^{3/2}]$ **19.** $f(x) = x^{10}, a = 1$
21. $f(x) = 2^x, a = 5$
23. $f(x) = \cos x, a = \pi$ **25.** -2 m/s
27. (a) The rate at which the cost is changing per ounce of gold produced; dollars per ounce
(b) When the 800th ounce of gold is produced, the cost of production is \$17/oz.
(c) Decrease in the short term; increase in the long term
29. (a) The rate at which the fuel consumption is changing with respect to speed; (gal/h)/(mi/h)
(b) The fuel consumption is decreasing by 0.05 (gal/h)/(mi/h) as the car's speed reaches 20 mi/h.
31. The rate at which the temperature is changing with respect to time when $t = 6$; 1°C/h
33. The rate at which the cash per capita in circulation is changing in dollars per year; \$39.90/year
35. Does not exist

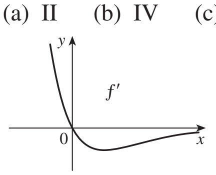
Exercises 2.8 • page 167

1. (a) 1.5 (b) 1 (c) 0 (d) -4 (e) 0 (f) 1 (g) 1.5

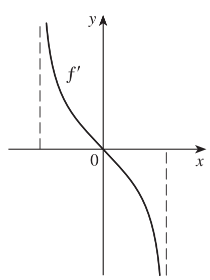


3. (a) II (b) IV (c) I (d) III

5.



7.



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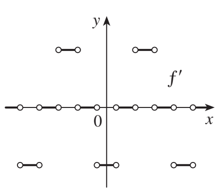

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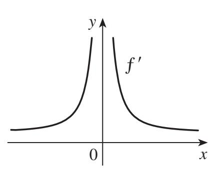
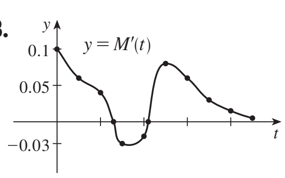
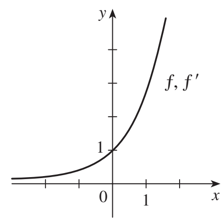


13.



1963 to 1971

15.

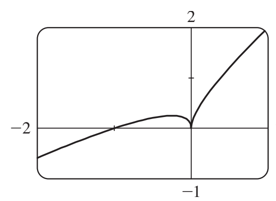


$f'(x) = e^x$

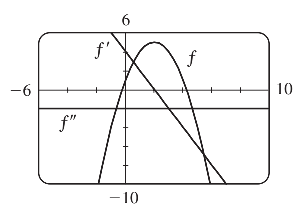
17. (a) 0, 1, 2, 4 (b) $-1, -2, -4$ (c) $f'(x) = 2x$
19. $f'(x) = -7, \mathbb{R}, \mathbb{R}$
21. $f'(x) = 3x^2 - 3, \mathbb{R}, \mathbb{R}$
23. $g'(x) = 1/\sqrt{1 + 2x}, \left[-\frac{1}{2}, \infty\right), \left(-\frac{1}{2}, \infty\right)$
25. $G'(t) = 4/(t + 1)^2, (-\infty, -1) \cup (-1, \infty)$, $(-\infty, -1) \cup (-1, \infty)$
27. (a) $f'(x) = 1 + 2/x^2$
29. (a) The rate at which the unemployment rate is changing, in percent unemployed per year
(b)

t	$U'(t)$	t	$U'(t)$
1989	0.30	1994	-0.65
1990	0.75	1995	-0.35
1991	0.95	1996	-0.35
1992	0.05	1997	-0.45
1993	-0.70	1998	-0.40

31. 4 (discontinuity); 8 (corner); -1, 11 (vertical tangents)
33.

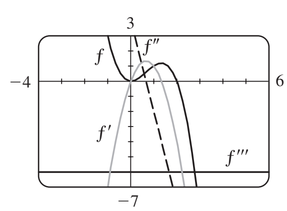


Differentiable at -1; not differentiable at 0
35. $a = f, b = f', c = f''$
37. $a =$ acceleration, $b =$ velocity, $c =$ position
39.



$f'(x) = 4 - 2x$, $f''(x) = -2$

41.

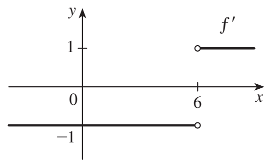


$f'(x) = 4x - 3x^2, f''(x) = 4 - 6x$, $f'''(x) = -6, f^{(4)}(x) = 0$

43. (a) $\frac{1}{3}a^{-2/3}$

45. $f'(x) = \begin{cases} -1 & \text{if } x < 6 \\ 1 & \text{if } x > 6 \end{cases}$

or $f'(x) = \dfrac{x-6}{|x-6|}$

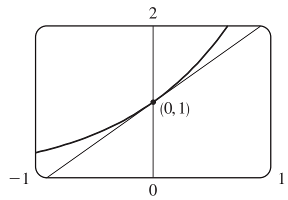


49. 63°

Exercises 2.9 • page 173

1. (a) 1.0986 (b) 1.0549, 1.1099

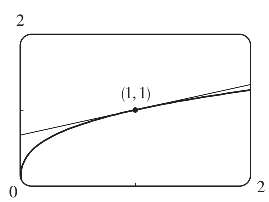
(c)



Less; the tangent line lies below the curve.

3. (a) $\frac{1}{3}$ (b) $\frac{1}{3}x + \frac{2}{3}$ (c) 0.83333, 0.96667, 0.99667, 1.00333, 1.03333, 1.16667, 1.33333; overestimates; those for 0.99 and 1.01

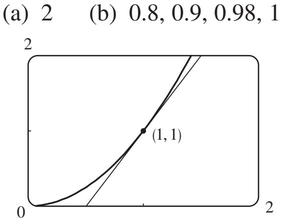
(d)



The tangent lines lie above the curve.

5. (a) 2 (b) 0.8, 0.9, 0.98, 1.02, 1.1, 1.2; underestimates

(c)



7. 148 °F; underestimate **9.** \$1555; underestimate

11. 22.6%, 24.2%; too high; tangent lines lie above the curve

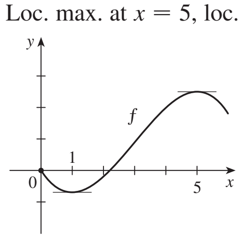
13. (a) 4.8, 5.2 (b) Too large

Exercises 2.10 • page 178

1. (a) Inc. on (1, 5); dec. on (0, 1) and (5, 6)

(b) Loc. max. at $x = 5$, loc. min. at $x = 1$

(c)



3. Inc. on (2, 5); dec. on $(-\infty, 2)$ and $(5, \infty)$

5. If $D(t)$ is the size of the deficit as a function of time, then at the time of the speech $D'(t) > 0$, but $D''(t) < 0$.

7. (a) The rate starts small, grows rapidly, levels off, then decreases and becomes negative.

(b) (1932, 2.5) and (1937, 4.3); the rate of change of population density starts to decrease in 1932 and starts to increase in 1937.

9. $K(3) - K(2)$; CD

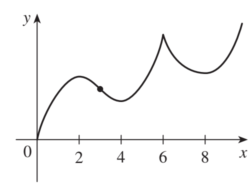
11. (a) Inc. on (0, 2), (4, 6), $(8, \infty)$; dec. on (2, 4), (6, 8)

(b) Loc. max. at $x = 2, 6$; loc. min. at $x = 4, 8$

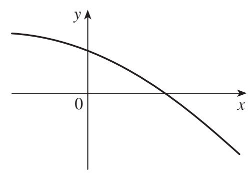
(c) CU on (3, 6), $(6, \infty)$; CD on (0, 3)

(d) 3

(e) See graph at right.



13.



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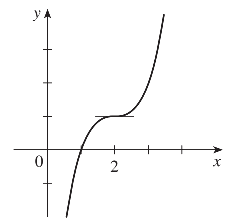
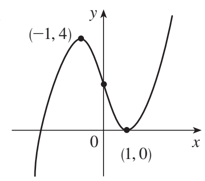
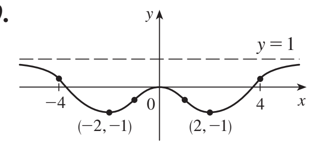


17.



19.



21. (a) Inc. on $(0, \infty)$; dec. on $(-\infty, 0)$

(b) Min. at $x = 0$

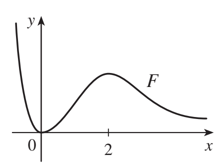
23. (a) Inc. on $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$, $\left(\sqrt{\frac{1}{3}}, \infty\right)$; dec. on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$

(b) CU on $(0, \infty)$; CD on $(-\infty, 0)$

(c) IP at (0, 0)

25. b

27.



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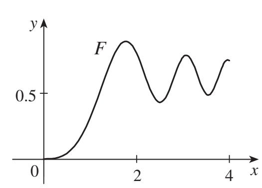


Chapter 2 Review • page 181

True-False Quiz

1. False **3.** True **5.** False **7.** True
9. True **11.** False **13.** True **15.** False
17. False

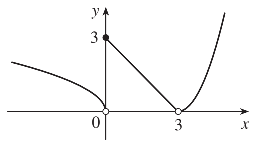
Exercises

1. (a) (i) 3 (ii) 0 (iii) Does not exist (iv) 2 (v) ∞ (vi) $-\infty$ (vii) 4 (viii) -1 (b) $y = 4$, $y = -1$
(c) $x = 0$, $x = 2$ (d) $-3, 0, 2, 4$

3. 1 **5.** $\frac{3}{2}$ **7.** 3 **9.** ∞ **11.** 0 **13.** 0

15. $\sqrt{3}$ **17.** $x = 0$, $y = 0$ **19.** 1

21. (a) (i) 3 (ii) 0 (iii) Does not exist (iv) 0 (v) 0 (vi) 0 (b) At 0 and 3 (c)

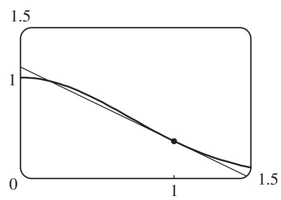


25. (a) (i) 3 m/s (ii) 2.75 m/s (iii) 2.625 m/s (iv) 2.525 m/s (b) 2.5 m/s

27. $f''(5), 0, f'(5), f'(2), 1, f'(3)$

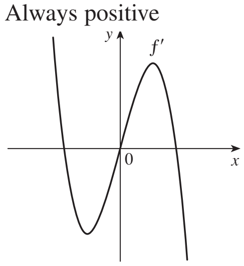
29. (a) -0.736 (b) $y \approx -0.736x + 1.104$

(c)

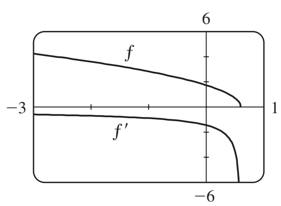


31. (a) The rate at which the cost changes with respect to the interest rate; dollars/(percent per year)
(b) As the interest rate increases past 10%, the cost is increasing at a rate of \$1200/(percent per year).
(c) Always positive

33.



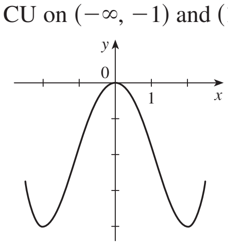
35. (a) $f'(x) = -\frac{5}{2}(3 - 5x)^{-1/2}$
(b) $\left(-\infty, \frac{3}{5}\right], \left(-\infty, \frac{3}{5}\right)$
(c)



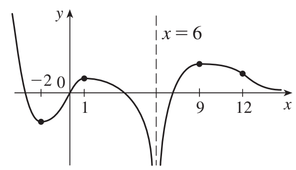
37. -4 (discontinuity), -1 (corner), 2 (discontinuity), 5 (vertical tangent)

39. (a) 1 (b) $x + 1$ (c) 0.8, 0.9, 0.99, 1.01, 1.1, 1.2
(d) Underestimates; those for $e^{-0.01}$ and $e^{0.01}$

41. (a) Inc. on $(-2, 0)$ and $(2, \infty)$; dec. on $(-\infty, -2)$ and $(0, 2)$
(b) Max. at 0; min. at -2 and 2
(c) CU on $(-\infty, -1)$ and $(1, \infty)$; CD on $(-1, 1)$
(d)



43.



45. (a) About 35 ft/s (b) About (8, 180)
(c) The point at which the car's velocity is maximized

Focus on Problem Solving • page 186

1. $\frac{2}{3}$ **3.** -4 **5.** 1 **7.** $a = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$

9. (b) Yes (c) Yes; no **11.** $\left(\pm\sqrt{3}/2, \frac{1}{4}\right)$

13. (a) 0 (b) 1 (c) $f'(x) = x^2 + 1$ **15.** $\frac{3}{4}$

CHAPTER 3

Exercises 3.1 • page 196

1. (a) See Definition of the Number e (page 195).
(b) 0.99, 1.03; $2.7 < e < 2.8$

3. $f'(x) = 5$ **5.** $f'(x) = 36x^3 - 6x$

7. $y' = -\frac{2}{5}x^{-7/5}$

9. $G'(x) = 1/\left(2\sqrt{x}\right) - 2e^x$

11. $V'(r) = 4\pi r^2$

13. $F'(x) = 12{,}288x^2$ **15.** $y' = 0$

17. $y' = \frac{3}{2}\sqrt{x} + \left(2/\sqrt{x}\right) - 3/\left(2x\sqrt{x}\right)$

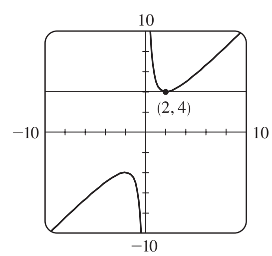
19. $v' = 2t + 3/\left(4t\sqrt[4]{t^3}\right)$

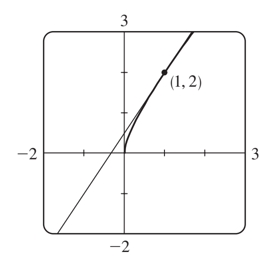
21. $z' = -10A/y^{11} + Be^y$

23. $4x - 4x^3$ **25.** $45x^{14} - 15x^2$ **27.** $1 - x^{-2/3}$

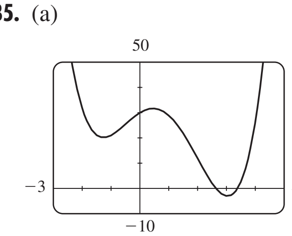
29. (a) 0.264 (b) $2^{2/5}/5 \approx 0.263902$

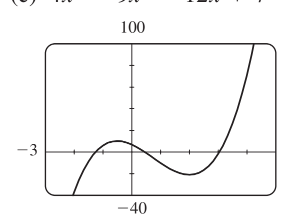
31. $y = 4$ **33.** $y = \frac{3}{2}x + \frac{1}{2}$





35. (a) (c) $4x^3 - 9x^2 - 12x + 7$





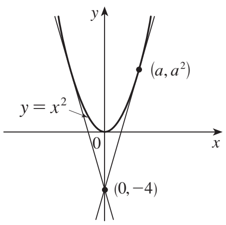
37. $f'(x) = 4x^3 - 9x^2 + 16, f''(x) = 12x^2 - 18x$

39. $f'(x) = 2 - \frac{15}{4}x^{-1/4}, f''(x) = \frac{15}{16}x^{-5/4}$

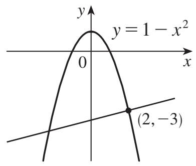
41. (a) $v(t) = 3t^2 - 3, a(t) = 6t$ (b) 12 m/s^2
(c) $a(1) = 6\text{ m/s}^2$

43. $\left(\ln\frac{3}{2}, \infty\right)$ **45.** $(1, 0), \left(-\frac{1}{3}, \frac{32}{27}\right)$

49. $(\pm 2, 4)$



51. $y = \frac{1}{4}x - \frac{7}{2}$



55. $P(x) = x^2 - x + 3$

57. (a) $F(x) = \frac{1}{3}x^3 + C$, C any real number; infinitely many
(b) $F(x) = \frac{1}{4}x^4 + C, \frac{1}{5}x^5 + C$, C any real number
(c) $F(x) = x^{n+1}/(n+1) + C$, C any real number

59. $a = -\frac{1}{2}, b = 2$ **61.** $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$ **63.** 1000

Exercises 3.2 • page 204

1. $y' = 5x^4 + 3x^2 + 2x$ **3.** $f'(x) = x(x + 2)e^x$

5. $y' = (x - 2)e^x/x^3$ **7.** $h'(x) = -3/(x - 1)^2$

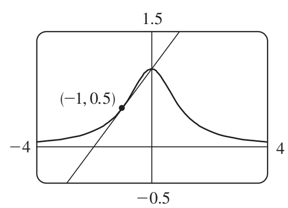
9. $H'(x) = 1 + x^{-2} + 2x^{-3} - 6x^{-4}$

11. $y' = 2t(1 - t)/(3t^2 - 2t + 1)^2$ **13.** $y' = (r^2 - 2)e^r$

15. $y' = 2v - 1/\sqrt{v}$ **17.** $f'(x) = 2cx/(x^2 + c)^2$

19. $y = 2x$

21. (a) $y = \frac{1}{2}x + 1$ (b)



23. (a) $e^x(x - 3)/x^4$ **25.** xe^x, $(x + 1)e^x$

27. (a) -16 (b) $-\frac{20}{9}$ (c) 20 **29.** 7

31. (a) 0 (b) $-\frac{2}{3}$ **33.** \$7.322 billion per year

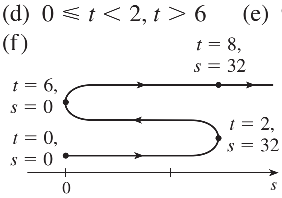
35. $(-3, \infty)$ **37.** Two, $\left(-2 \pm \sqrt{3}, \left(1 \mp \sqrt{3}\right)/2\right)$

39. (c) $3e^{3x}$

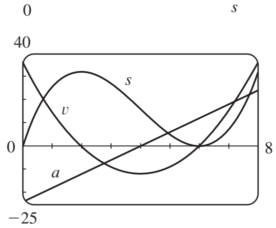
41. $(x^2 + 2x)e^x$, $(x^2 + 4x + 2)e^x$, $(x^2 + 6x + 6)e^x$, $(x^2 + 8x + 12)e^x$, $(x^2 + 10x + 20)e^x$; $f^{(n)}(x) = [x^2 + 2nx + n(n - 1)]e^x$

Exercises 3.3 • page 215

1. (a) $3t^2 - 24t + 36$ (b) -9 m/s (c) $t = 2, 6$
(d) $0 \leqslant t < 2, t > 6$ (e) 96 m
(f)



(g) $6t - 24$; -6 m/s²
(h)



(i) Speeding up when $2 < t < 4$ or $t > 6$; slowing down when $0 \leqslant t < 2$ or $4 < t < 6$

3. (a) $t = 4$ s
(b) $t = 1.5$ s; the velocity has an absolute minimum.

5. (a) 30 mm²/mm; the rate at which the area is increasing with respect to side length as x reaches 15 mm
(b) $\Delta A \approx 2x\,\Delta x$

7. (a) (i) 5π (ii) 4.5π (iii) 4.1π
(b) 4π (c) $\Delta A \approx 2\pi r\,\Delta r$

9. (a) 8π ft²/ft (b) 16π ft²/ft (c) 24π ft²/ft
The rate increases as the radius increases.

11. (a) 6 kg/m (b) 12 kg/m (c) 18 kg/m
At the right end; at the left end

13. (a) 4.75 A (b) 5 A; $t = \frac{2}{3}$ s

15. (a) $dV/dP = -C/P^2$ (b) At the beginning

17. (a) 16 million/year; 78.5 million/year
(b) $P(t) = at^3 + bt^2 + ct + d$, where $a = 0.00123543$, $b = -6.72226$, $c = 12{,}165$, and $d = -7{,}318{,}429$
(c) $P'(t) = 3at^2 + 2bt + c$
(d) 14.5 million/year (smaller); 75.1 million/year (smaller)
(e) 81.3 million/year

19. (a) $a^2k/(akt + 1)^2$ (c) It approaches a moles/L.
(d) It approaches 0. (e) The reaction virtually stops.

21. (a) 0.926 cm/s; 0.694 cm/s; 0
(b) 0; -92.6 (cm/s)/cm; -185.2 (cm/s)/cm
(c) At the center; at the edge

23. (a) $C'(x) = 3 + 0.02x + 0.0006x^2$
(b) \$11/pair, the rate at which the cost is changing as the 100th pair of jeans is being produced; the cost of the 101st pair
(c) \$11.07

25. (a) $[xp'(x) - p(x)]/x^2$; the average productivity increases as new workers are added.

27. -0.2436 K/min

29. (a) 0 and 0 (b) $C = 0$
(c) (0, 0), (500, 50); it is possible for the species to coexist.

Exercises 3.4 • page 223

1. $1 - 3\cos x$ **3.** $3t^2\cos t - t^3\sin t$

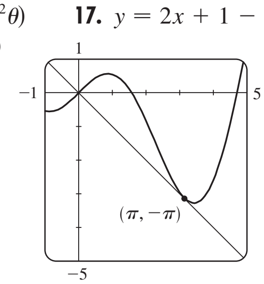
5. $-\csc\theta\cot\theta + e^\theta(\cot\theta - \csc^2\theta)$

7. $(x\sec^2 x - \tan x)/x^2$

9. $(\sin x + \cos x + x\sin x - x\cos x)/(1 + \sin 2x)$

11. $\sec\theta\,(\sec^2\theta + \tan^2\theta)$ **17.** $y = 2x + 1 - \pi/2$

19. (a) $y = -x$ (b)



21. (a) $2 - \csc^2 x$

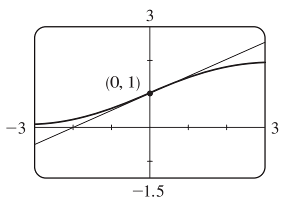
23. $\theta\cos\theta + \sin\theta$; $2\cos\theta - \theta\sin\theta$

25. $(2n + 1)\pi \pm \pi/3$, n an integer **27.** $(\pi/3, 5\pi/3)$

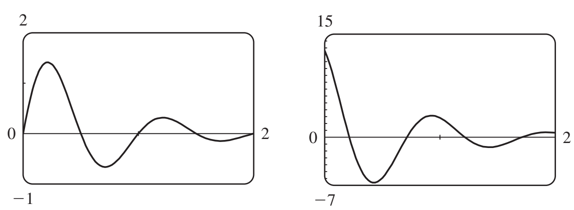
29. (a) $v(t) = 8 \cos t$, $a(t) = -8 \sin t$
(b) $4\sqrt{3}, -4, -4\sqrt{3}$; to the left; speeding up
31. 5 ft/rad **33.** $-\cos x$ **35.** $A = -\frac{3}{10}, B = -\frac{1}{10}$
37. 4 **39.** $\frac{1}{2}$ **41.** 1

Exercises 3.5 • page 233

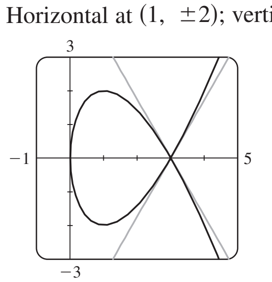
1. $4 \cos 4x$ **3.** $-20x(1 - x^2)^9$ **5.** $e^{\sqrt{x}}/(2\sqrt{x})$
7. $F'(x) = (2 + 3x^2)/[4(1 + 2x + x^3)^{3/4}]$
9. $g'(t) = -12t^3/(t^4 + 1)^4$ **11.** $y' = -3x^2 \sin(a^3 + x^3)$
13. $y' = -me^{-mx}$ **15.** $y' = e^{-x^2}(1 - 2x^2)$
17. $G'(x) = 6(3x - 2)^9(5x^2 - x + 1)^{11}(85x^2 - 51x + 9)$
19. $y' = (\cos x - x \sin x)e^{x \cos x}$
21. $F'(y) = 39(y - 6)^2/(y + 7)^4$
23. $y' = (r^2 + 1)^{-3/2}$ **25.** $y' = 2^{\sin \pi x}(\pi \ln 2) \cos \pi x$
27. $y' = -2 \cos\theta \cot(\sin\theta) \csc^2(\sin\theta)$
29. $y' = \cos\left(\tan\sqrt{\sin x}\right)\left(\sec^2\sqrt{\sin x}\right)\left[1/\left(2\sqrt{\sin x}\right)\right](\cos x)$
31. $y = -x + \pi$
33. (a) $y = \frac{1}{2}x + 1$ (b)



35. (a) $-1/\left(x^2\sqrt{1 - x^2}\right)$ **37.** 28 **39.** (a) 30 (b) 36
41. (a) $\frac{3}{4}$ (b) Does not exist (c) -2 **43.** -17.4
45. (a) $(0, \infty)$ (b) $G'(x) = h'\left(\sqrt{x}\right)/\left(2\sqrt{x}\right)$
47. (a) $F'(x) = e^x f'(e^x)$ (b) $G'(x) = e^{f(x)} f'(x)$
49. $((\pi/2) + 2n\pi, 3)$, $((3\pi/2) + 2n\pi, -1)$, n an integer
53. $-2^{50} \cos 2x$ **55.** $v(t) = (5\pi/2) \cos(10\pi t)$ cm/s
57. (a) $dB/dt = (7\pi/54) \cos(2\pi t/5.4)$ (b) 0.16
59. $v(t) = 2e^{-1.5t}(2\pi \cos 2\pi t - 1.5 \sin 2\pi t)$



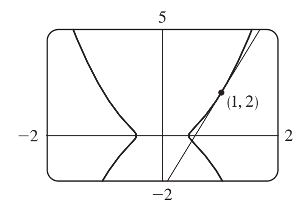
61. dv/dt is the rate of change of velocity with respect to time; dv/ds is the rate of change of velocity with respect to displacement
63. (a) $y \approx 100.012437e^{-10.005531t}$ (b) $-670.63\ \mu$A
65. $y = (1/\pi)x - \pi$
67. (a) $y = \sqrt{3}x - 3\sqrt{3}$, $y = -\sqrt{3}x + 3\sqrt{3}$
(b) Horizontal at $(1, \pm 2)$; vertical at $(0, 0)$
(c)



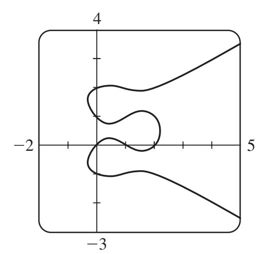
69. (b) The factored form
71. (b) $-n \cos^{n-1}x \sin[(n + 1)x]$

Exercises 3.6 • page 243

1. (a) $y' = -(y + 2 + 6x)/x$
(b) $y = (4/x) - 2 - 3x$, $y' = -(4/x^2) - 3$
3. $y' = -x(3x + 2y)/(x^2 + 8y)$
5. $y' = (3 - 2xy - y^2)/(x^2 + 2xy)$
7. $y' = \left(4xy\sqrt{xy} - y\right)/\left(x - 2x^2\sqrt{xy}\right)$
9. $y' = \tan x \tan y$ **11.** $y' = 1 + e^x(1 + x)/\sin(x - y)$
13. $y = -\frac{5}{4}x - 4$ **15.** $y = x$ **17.** $y = -\frac{9}{13}x + \frac{40}{13}$
19. (a) $y = \frac{9}{2}x - \frac{5}{2}$ (b)

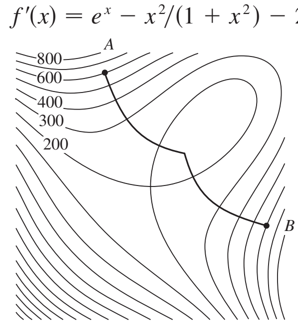


21. (a)

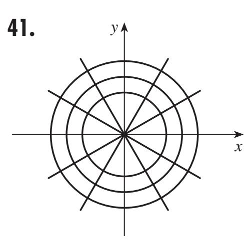


Eight; $x \approx 0.42, 1.58$
(b) $y = -x + 1$,
$y = \frac{1}{3}x + 2$
(c) $1 \mp \sqrt{3}/3$

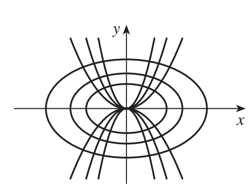
23. $\left(\pm 5\sqrt{3}/4, \pm 5/4\right)$
25. (a) $y' = -x^3/y^3$
(b) $-\dfrac{3x^2y^4 + 3x^6}{y^7}$
27. $y' = 2x/\sqrt{1 - x^4}$
29. $y' = 1/(1 + x) + \left(\tan^{-1}\sqrt{x}\right)/\sqrt{x}$
31. $H'(x) = 1 + 2x \arctan x$ **33.** $y' = \sec^2\theta/\sqrt{1 - \tan^2\theta}$
35. $f'(x) = e^x - x^2/(1 + x^2) - 2x \arctan x$
39.



41.



43.



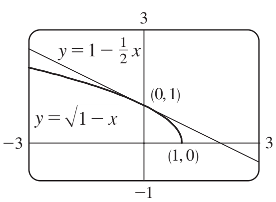
47. $(\pm\sqrt{3}, 0)$ **49.** $(-1, -1), (1, 1)$ **51.** (b) $\frac{3}{2}$
53. (a) 0 (b) $-\frac{1}{2}$

Exercises 3.7 • page 250

1. The differentiation formula is simplest.
3. $f'(\theta) = -\tan\theta$ **5.** $f'(x) = 3/[(3x - 1)\ln 2]$
7. $f'(x) = 1/\left[5x\sqrt[5]{(\ln x)^4}\right]$ **9.** $f'(x) = (2 + \ln x)/(2\sqrt{x})$
11. $F'(t) = 6/(2t + 1) - 12/(3t - 1)$
13. $y' = (1 + x - x\ln x)/[x(1 + x)^2]$
15. $y' = (3x - 2)/[x(x - 1)]$ **17.** $y' = -x/(1 + x)$
19. $e^x[(1/x) + \ln x]$; $e^x[\ln x + (2/x) - (1/x^2)]$
21. $f'(x) = \dfrac{2x - 1 - (x - 1)\ln(x - 1)}{(x - 1)[1 - \ln(x - 1)]^2}$;
$(1, 1 + e) \cup (1 + e, \infty)$
23. $y = 4x - 8$ **25.** (a) $(0, 1/e)$ (b) $(0, \infty)$
27. $y' = (2x + 1)^5(x^4 - 3)^6\left(\dfrac{10}{2x + 1} + \dfrac{24x^3}{x^4 - 3}\right)$
29. $y' = \dfrac{\sin^2 x\tan^4 x}{(x^2 + 1)^2}\left(2\cot x + \dfrac{4\sec^2 x}{\tan x} - \dfrac{4x}{x^2 + 1}\right)$
31. $y' = x^x(1 + \ln x)$ **33.** $y' = x^{\sin x}[(\sin x)/x + \ln x\cos x]$
35. $y' = (\ln x)^x(1/\ln x + \ln\ln x)$
37. $y' = 2x/(x^2 + y^2 - 2y)$
39. $f^{(n)}(x) = (-1)^{n-1}(n - 1)!/(x - 1)^n$

Exercises 3.8 • page 256

1. $L(x) = 3x - 2$ **3.** $L(x) = -x + \frac{\pi}{2}$
5. $\sqrt{1 - x} \approx 1 - \frac{1}{2}x$;
$\sqrt{0.9} \approx 0.95$,
$\sqrt{0.99} \approx 0.995$



7. $-0.69 < x < 1.09$ **9.** $-0.045 < x < 0.055$
15. (b) $-0.344 < x < 0.344$
17. (a) $dy = \frac{1}{10}e^{x/10}\,dx$ (b) 0.01; 0.0101
19. (a) 270 cm^3, 0.01, 1% (b) 36 cm^2, $0.00\overline{6}$, $0.\overline{6}\%$
21. $\frac{5}{8}\pi \approx 2\text{ m}^3$

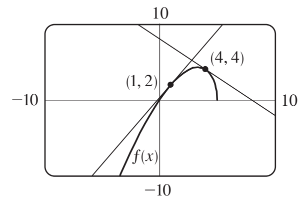
Chapter 3 Review • page 258

True-False Quiz

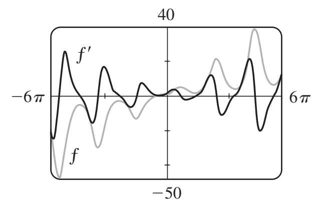
1. True **3.** True **5.** False **7.** False **9.** True
11. True

Exercises

1. $6x(x^4 - 3x^2 + 5)^2(2x^2 - 3)$ **3.** $1/(2\sqrt{x}) - 4/(3\sqrt[3]{x^7})$
5. $2(2x^2 + 1)/\sqrt{x^2 + 1}$ **7.** $2\cos 2\theta\, e^{\sin 2\theta}$
9. $(t^2 + 1)/(1 - t^2)^2$ **11.** $e^{-1/x}(1/x + 1)$
13. $(1 - y^4 - 2xy)/(4xy^3 + x^2 - 3)$
15. $2\sec 2\theta\,(\tan 2\theta - 1)/(1 + \tan 2\theta)^2$
17. $(1 + c^2)e^{cx}\sin x$ **19.** $2/[(1 + 2x)\ln 5]$
21. $\cot x - \sin x\cos x$ **23.** $4x/(1 + 16x^2) + \tan^{-1}(4x)$
25. $5\sec 5x$ **27.** $-6x\csc^2(3x^2 + 5)$
29. $\cos\left(\tan\sqrt{1 + x^3}\right)\left(\sec^2\sqrt{1 + x^3}\right)\left[3x^2/\left(2\sqrt{1 + x^3}\right)\right]$
31. -120 **33.** $2^x(\ln 2)^n$
35. (a) $(10 - 3x)/\left(2\sqrt{5 - x}\right)$
(b) $y = \frac{7}{4}x + \frac{1}{4}$, $y = -x + 8$
(c)



37. $e^{\sin x}(x\cos x + 1)$



The sizes of the oscillations of f and f' are linked.
39. (a) 2 (b) 44
41. $f'(x) = 2xg(x) + x^2g'(x)$
43. $f'(x) = 2g(x)g'(x)$
45. $f'(x) = g'(e^x)e^x$
47. $f'(x) = g'(x)/g(x)$
49. $h'(x) = (f'(x)[g(x)]^2 + g'(x)[f(x)]^2)/[f(x) + g(x)]^2$
51. $(-3, 0)$ **53.** $\left(\pm 2/\sqrt{6}, \mp 1/\sqrt{6}\right)$
55. $v(t) = -Ae^{-ct}[\omega\sin(\omega t + \delta) + c\cos(\omega t + \delta)]$,
$a(t) = Ae^{-ct}[(c^2 - \omega^2)\cos(\omega t + \delta) + 2c\omega\sin(\omega t + \delta)]$
57. 4 kg/m
59. (a) $C'(x) = 2 - 0.04x + 0.00021x^2$
(b) \$0.10/unit; the approximate cost of producing the 101st unit
(c) $C(101) - C(100) = 0.10107$
(d) About 95.24; at this value of x the marginal cost is minimized.
61. (a) $L(x) = 1 + x$, $\sqrt[3]{1 + 3x} \approx 1 + x$, $\sqrt[3]{1.03} \approx 1.01$
(b) $-0.23 < x < 0.40$
63. $(\cos\theta)'\big|_{\theta=\pi/3} = -\sqrt{3}/2$ **65.** $\frac{1}{4}$

Focus on Problem Solving • page 262

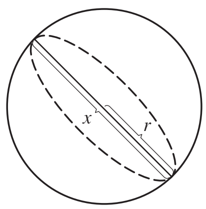
1. $\left(0, \frac{5}{4}\right)$
3. (a) $[-1, 2]$
(b) $-1/\left(8\sqrt{3 - x}\sqrt{2 - \sqrt{3 - x}}\sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}\right)$
5. (a) $4\pi\sqrt{3}/\sqrt{11} \approx 6.56$ rad/s
(b) $40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$ cm
(c) $-480\pi\sin\theta\left(1 + \cos\theta/\sqrt{8 + \cos^2\theta}\right)$ cm/s
9. $x_T \in (3, \infty)$, $y_T \in (2, \infty)$, $x_N \in \left(0, \frac{5}{3}\right)$, $y_N \in \left(-\frac{5}{2}, 0\right)$
11. $f^{(n)}(x) = n!/(1 - x)^{n+1}$
15. (b) (i) 53° (or 127°) (ii) 63° (or 117°)
17. R approaches the midpoint of the radius AO.
19. $(1, -2), (-1, 0)$ **21.** $\sqrt{29}/58$

CHAPTER 4

Exercises 4.1 • page 269

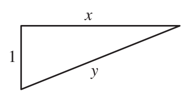
1. $dV/dt = 3x^2\, dx/dt$ **3.** 70

5. (a) The rate of decrease of the surface area is 1 cm^2/min.
(b) The rate of decrease of the diameter when the diameter is 10 cm
(c)



(d) $S = \pi x^2$
(e) $1/(20\pi)$ cm/min

7. (a) The plane's altitude is 1 mi and its speed is 500 mi/h.
(b) The rate at which the distance from the plane to the station is increasing when the plane is 2 mi from the station
(c)



(d) $y^2 = x^2 + 1$ (e) $250\sqrt{3}$ mi/h

9. 65 mi/h **11.** $837/\sqrt{8674} \approx 8.99$ ft/s
13. -1.6 cm/min **15.** $\frac{720}{13} \approx 55.4$ km/h
17. $10/\sqrt{133} \approx 0.87$ ft/s **19.** $\frac{10}{3}$ cm/min
21. $6/(5\pi) \approx 0.38$ ft/min **23.** 0.3 m^2/s **25.** 80 cm^3/min
27. $\frac{107}{810} \approx 0.132\ \Omega/\text{s}$ **29.** (a) 360 ft/s (b) 0.096 rad/s
31. $1650/\sqrt{31} \approx 296$ km/h **33.** $7\sqrt{15}/4 \approx 6.78$ m/s

Exercises 4.2 • page 276

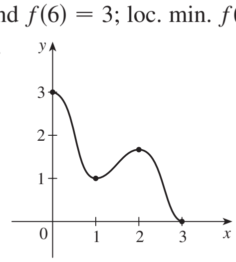
Abbreviations: max., maximum; min., minimum; loc., local; abs., absolute

1. Absolute minimum: smallest function value on the entire domain of the function; local minimum at c: smallest function value when x is near c

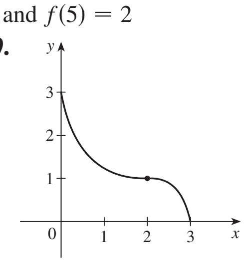
3. Abs. max. at b, loc. max. at b and e, abs. min. at d, loc. min. at d and s

5. Abs. max. $f(4) = 4$; abs. min. $f(7) = 0$; loc. max. $f(4) = 4$ and $f(6) = 3$; loc. min. $f(2) = 1$ and $f(5) = 2$

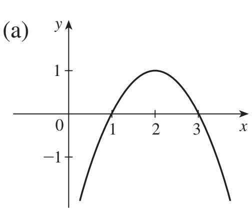
7.



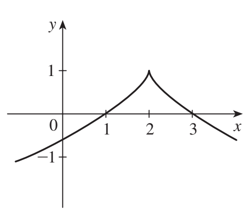
9.



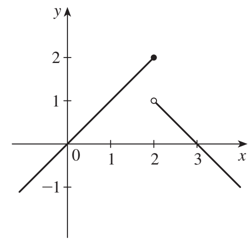
11. (a)



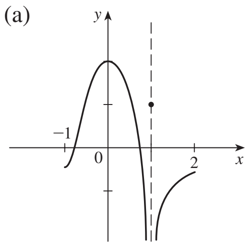
(b)



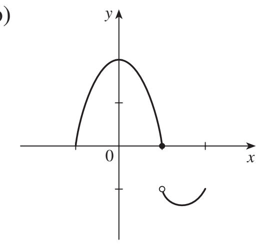
(c)



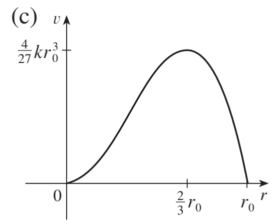
13. (a)



(b)



15. Abs. max. $f(1) = 5$ **17.** None
19. Abs. and loc. max. $f(\pi/2) = f(-3\pi/2) = 1$,
abs. and loc. min. $f(3\pi/2) = f(-\pi/2) = -1$
21. Abs. max. $f(0) = 1$ **23.** $-\frac{2}{5}$ **25.** $0, \left(-1 \pm \sqrt{5}\right)/2$
27. ± 1 **29.** $0, \frac{8}{7}, 4$ **31.** $n\pi/4$ (n an integer)
33. $1/e$ **35.** $f(0) = 5$, $f(2) = -7$
37. $f(3) = 66$, $f(\pm 1) = 2$ **39.** $f(2) = 5$, $f(1) = 3$
41. $f(\pi/4) = \sqrt{2}$, $f(0) = 1$ **43.** $f(1) = 1/e$, $f(0) = 0$
45. $-1.3, 0.2, 1.1$ **47.** (a) 9.71, -7.71 (b) $1 \pm 32\sqrt{6}/9$
49. (a) 0.32, 0 (b) $3\sqrt{3}/16$, 0 **51.** ≈ 3.9665 °C
53. Cheapest, $t = 10$; most expensive, $t \approx 5.1309$
55. (a) $r = \frac{2}{3}r_0$ (b) $v = \frac{4}{27}kr_0^3$ (c)



Exercises 4.3 • page 288

Abbreviations: inc., increasing; dec., decreasing; CD, concave downward; CU, concave upward; HA, horizontal asymptote; VA, vertical asymptote; IP, inflection point

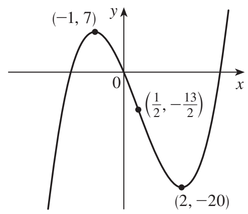
1. 0.8, 3.2, 4.4, 6.1
3. (a) I/D Test (b) Concavity Test
(c) Find points at which the concavity changes.
5. $x = 1, 7$
7. (a) Inc. on $(-\infty, -2)$, $(2, \infty)$; dec. on $(-2, 2)$
(b) Loc. max. $f(-2) = 17$; loc. min. $f(2) = -15$
(c) CU on $(0, \infty)$; CD on $(-\infty, 0)$; IP $(0, 1)$
9. (a) Inc. on $(\pi/3, 5\pi/3)$, $(7\pi/3, 3\pi)$;
dec. on $(0, \pi/3)$, $(5\pi/3, 7\pi/3)$
(b) Loc. max. $f(5\pi/3) = 5\pi/3 + \sqrt{3}$;
loc. min. $f(\pi/3) = \pi/3 - \sqrt{3}$, $f(7\pi/3) = 7\pi/3 - \sqrt{3}$
(c) CU on $(0, \pi)$, $(2\pi, 3\pi)$; CD on $(\pi, 2\pi)$; IP (π, π), $(2\pi, 2\pi)$

11. (a) Inc. on $(-1, \infty)$; dec. on $(-\infty, -1)$
(b) Loc. min. $f(-1) = -1/e$
(c) CU on $(-2, \infty)$; CD on $(-\infty, -2)$; IP $(-2, -2e^{-2})$

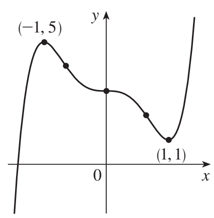
13. (a) Inc. on $(0, e^2)$; dec. on (e^2, ∞)
(b) Loc. max. $f(e^2) = 2/e$
(c) CU on $(e^{8/3}, \infty)$; CD on $(0, e^{8/3})$; IP $\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right)$

15. Loc. max. $f\left(\frac{3}{4}\right) = \frac{5}{4}$

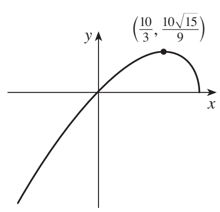
17. (a) Inc. on $(-\infty, -1)$, $(2, \infty)$;
dec. on $(-1, 2)$
(b) Loc. max. $f(-1) = 7$;
loc. min. $f(2) = -20$
(c) CU on $\left(\frac{1}{2}, \infty\right)$; CD on $\left(-\infty, \frac{1}{2}\right)$;
IP $\left(\frac{1}{2}, -\frac{13}{2}\right)$
(d) See graph at right.



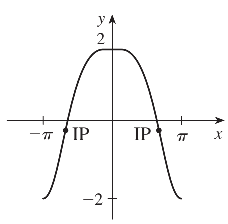
19. (a) Inc. on $(-\infty, -1)$, $(1, \infty)$;
dec. on $(-1, 1)$
(b) Loc. max. $h(-1) = 5$;
loc. min. $h(1) = 1$
(c) CD on $\left(-\infty, -1/\sqrt{2}\right)$, $\left(0, 1/\sqrt{2}\right)$;
CU on $\left(-1/\sqrt{2}, 0\right)$, $\left(1/\sqrt{2}, \infty\right)$;
IP $(0, 3)$, $\left(\pm 1/\sqrt{2}, 3 \mp \frac{7}{8}\sqrt{2}\right)$
(d) See graph at right.



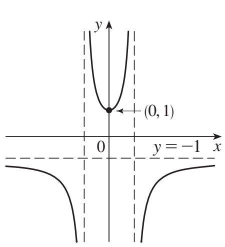
21. (a) Inc. on $\left(-\infty, \frac{10}{3}\right)$; dec. on $\left(\frac{10}{3}, 5\right)$
(b) Loc. max. $f(10/3) = 10\sqrt{15}/9$
(c) CD on $(-\infty, 5)$
(d) See graph at right.



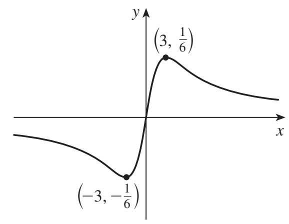
23. (a) Inc. on $(-\pi, 0)$; dec. on $(0, \pi)$
(b) Loc. max. $f(0) = 2$
(c) CD on $(-2\pi/3, 2\pi/3)$;
CU on $(-\pi, -2\pi/3)$, $(2\pi/3, \pi)$;
IP $\left(\pm 2\pi/3, -\frac{1}{4}\right)$
(d) See graph at right.



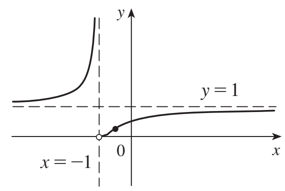
25. (a) VA $x = \pm 1$; HA $y = -1$
(b) Inc. on $(0, 1)$, $(1, \infty)$; dec. on $(-\infty, -1)$, $(-1, 0)$
(c) Loc. min. $f(0) = 1$
(d) CU on $(-1, 1)$; CD on $(-\infty, -1)$, $(1, \infty)$
(e)



27. (a) HA $y = 0$
(b) Inc. on $(-3, 3)$;
dec. on $(-\infty, -3)$, $(3, \infty)$
(c) Loc. min. $f(-3) = -\frac{1}{6}$;
loc. max. $f(3) = \frac{1}{6}$
(d) CU on $\left(-3\sqrt{3}, 0\right)$, $\left(3\sqrt{3}, \infty\right)$;
CD on $\left(-\infty, -3\sqrt{3}\right)$, $\left(0, 3\sqrt{3}\right)$;
IP $(0, 0)$, $\left(\pm 3\sqrt{3}, \pm\sqrt{3}/12\right)$
(e) See graph at right.



29. (a) HA $y = 1$, VA $x = -1$
(b) Inc. on $(-\infty, -1)$, $(-1, \infty)$
(c) None
(d) CU on $(-\infty, -1)$, $\left(-1, -\frac{1}{2}\right)$;
CD on $\left(-\frac{1}{2}, \infty\right)$; IP $\left(-\frac{1}{2}, 1/e^2\right)$
(e) See graph at right.

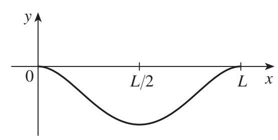


31. (b) CD on $(-\infty, -2.11)$, $(0.25, 1.86)$;
CU on $(-2.11, 0.25)$, $(1.86, \infty)$;
IP at $(-2.11, 386)$, $(0.25, 1.3)$, $(1.86, -87)$

33. (a) Loc. and abs. max. $f(1) = \sqrt{2}$, no min.
(b) $\left(3 - \sqrt{17}\right)/4$

35. CD on $(-\infty, 0.1)$; CU on $(0.1, \infty)$

37.



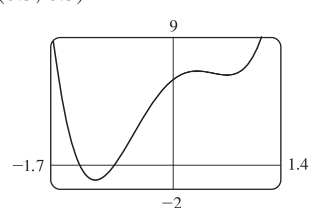
39. When $t \approx 7.17$

41. $f(x) = \frac{1}{9}(2x^3 + 3x^2 - 12x + 7)$

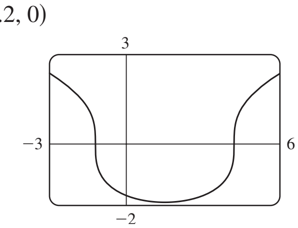
45. 17

Exercises 4.4 • page 297

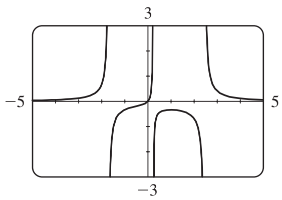
1. Inc. on $(-1.1, 0.3)$, $(0.7, \infty)$; dec. on $(-\infty, -1.1)$, $(0.3, 0.7)$;
loc. max. $f(0.3) \approx 6.6$, loc. min. $f(-1.1) \approx -1.1$,
$f(0.7) \approx 6.3$; CU on $(-\infty, -0.5)$, $(0.5, \infty)$; CD on $(-0.5, 0.5)$;
IP $(-0.5, 2.1)$, $(0.5, 6.5)$



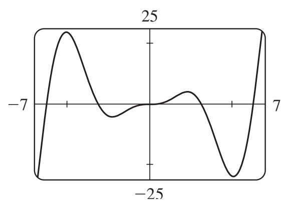
3. Inc. on $(1.5, \infty)$; dec. on $(-\infty, 1.5)$; min. $f(1.5) \approx -1.9$;
CU on $(-1.2, 4.2)$; CD on $(-\infty, -1.2)$, $(4.2, \infty)$;
IP $(-1.2, 0)$, $(4.2, 0)$



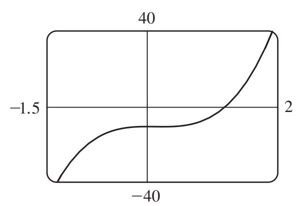
5. Inc. on $(-\infty, -1.7)$, $(-1.7, 0.24)$, $(0.24, 1)$; dec. on $(1, 2.46)$, $(2.46, \infty)$; loc. max. $f(1) = -\frac{1}{3}$; CU on $(-\infty, -1.7)$, $(-0.506, 0.24)$, $(2.46, \infty)$; CD on $(-1.7, -0.506)$, $(0.24, 2.46)$; IP $(-0.506, -0.192)$

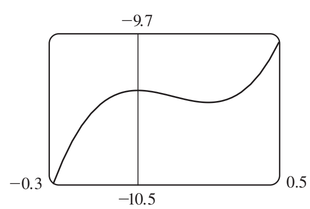


7. Inc. on $(-7, -5.1)$, $(-2.3, 2.3)$, $(5.1, 7)$; dec. on $(-5.1, -2.3)$, $(2.3, 5.1)$; loc. max. $f(-5.1) \approx 24.1$, $f(2.3) \approx 3.9$; loc. min. $f(-2.3) \approx -3.9$, $f(5.1) \approx -24.1$; CU on $(-7, -6.8)$, $(-4.0, -1.5)$, $(0, 1.5)$, $(4.0, 6.8)$; CD on $(-6.8, -4.0)$, $(-1.5, 0)$, $(1.5, 4.0)$, $(6.8, 7)$; IP $(-6.8, -24.4)$, $(-4.0, 12.0)$, $(-1.5, -2.3)$, $(0, 0)$, $(1.5, 2.3)$, $(4.0, -12.0)$, $(6.8, 24.4)$

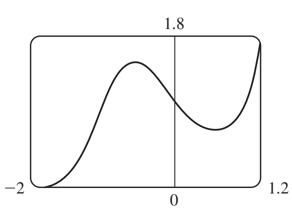


9. Inc. on $(-\infty, 0)$, $\left(\frac{1}{4}, \infty\right)$; dec. on $\left(0, \frac{1}{4}\right)$; loc. max. $f(0) = -10$; loc. min. $f\left(\frac{1}{4}\right) = -\frac{161}{16} \approx -10.1$; CU on $\left(\frac{1}{8}, \infty\right)$; CD on $\left(-\infty, \frac{1}{8}\right)$; IP $\left(\frac{1}{8}, -\frac{321}{32}\right)$

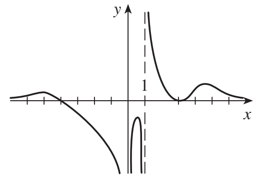


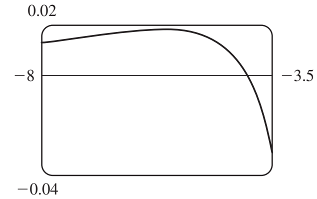


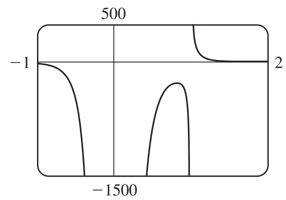
11. Loc. max. $f\left(-1/\sqrt{3}\right) = e^{2\sqrt{3}/9} \approx 1.5$; loc. min. $f\left(1/\sqrt{3}\right) = e^{-2\sqrt{3}/9} \approx 0.7$; IP $(-0.15, 1.15)$, $(-1.09, 0.82)$

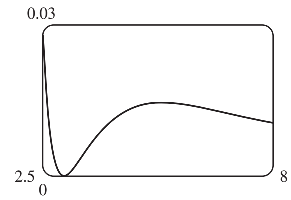


13. Loc. max. $f(-5.6) \approx 0.018$, $f(0.82) \approx -281.5$, $f(5.2) \approx 0.0145$; min. $f(3) = 0$







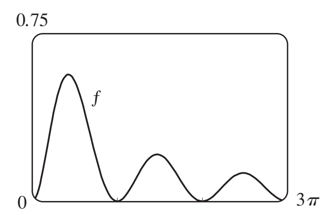


15. $f'(x) = -\dfrac{x(x+1)^2(x^3 + 18x^2 - 44x - 16)}{(x-2)^3(x-4)^5}$

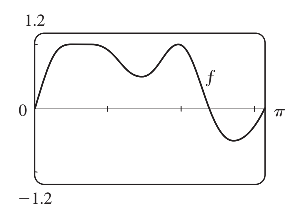
$f''(x) = 2\dfrac{(x+1)(x^6 + 36x^5 + 6x^4 - 628x^3 + 684x^2 + 672x + 64)}{(x-2)^4(x-4)^6}$

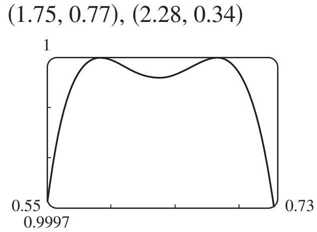
CU on $(-35.3, -5.0)$, $(-1, -0.5)$, $(-0.1, 2)$, $(2, 4)$, $(4, \infty)$; CD on $(-\infty, -35.3)$, $(-5.0, -1)$ $(-0.5, -0.1)$; IP $(-35.3, -0.015)$, $(-5.0, -0.005)$, $(-1, 0)$, $(-0.5, 0.00001)$, $(-0.1, 0.0000066)$

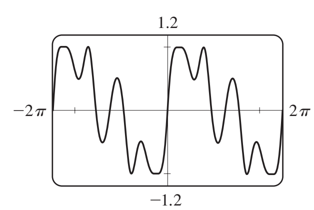
17. Inc. on $(0, 1.3)$, $(\pi, 4.6)$, $(2\pi, 7.8)$; dec. on $(1.3, \pi)$, $(4.6, 2\pi)$, $(7.8, 3\pi)$; loc. max. $f(1.3) \approx 0.6$, $f(4.6) \approx 0.21$, $f(7.8) \approx 0.13$; loc. min. $f(\pi) = f(2\pi) = 0$; CU on $(0, 0.6)$, $(2.1, 3.8)$, $(5.4, 7.0)$, $(8.6, 3\pi)$; CD on $(0.6, 2.1)$, $(3.8, 5.4)$, $(7.0, 8.6)$; IP $(0.6, 0.25)$, $(2.1, 0.31)$, $(3.8, 0.10)$, $(5.4, 0.11)$, $(7.0, 0.061)$, $(8.6, 0.065)$



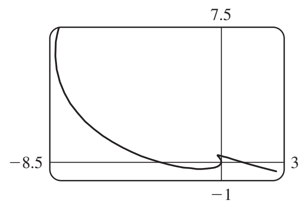
19. Max. $f(0.59) \approx 1$, $f(0.68) \approx 1$, $f(1.96) \approx 1$; min. $f(0.64) \approx 0.99996$, $f(1.46) \approx 0.49$, $f(2.73) \approx -0.51$; IP $(0.61, 0.99998)$, $(0.66, 0.99998)$, $(1.17, 0.72)$, $(1.75, 0.77)$, $(2.28, 0.34)$





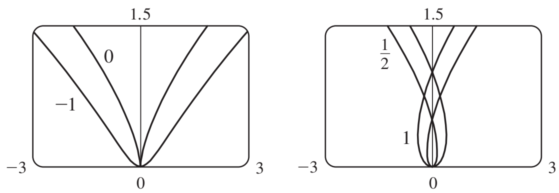


21.

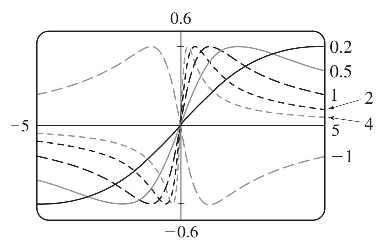


Vertical tangents at (0, 0), $\left(-\frac{3}{16}, \frac{3}{8}\right)$, (−8, 6); horizontal tangents at $\left(-(2\sqrt{3}+5)/9, -2\sqrt{3}/9\right)$, $\left((2\sqrt{3}-5)/9, 2\sqrt{3}/9\right)$

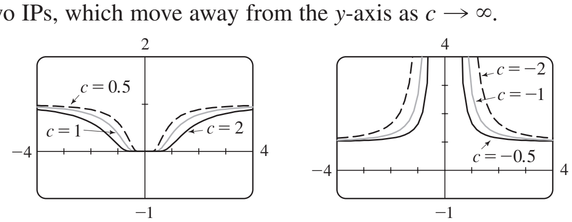
23. For $c = 0$, there is a cusp; for $c > 0$, there is a loop whose size increases as c increases and the curve intersects itself at $(0, c)$; leftmost point $\left(-2c\sqrt{3c}/9, c/3\right)$, rightmost point $\left(2c\sqrt{3c}/9, c/3\right)$



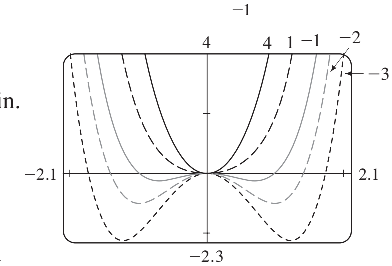
25. For $c > 0$, the maximum and minimum values are always $\pm\frac{1}{2}$, but the extreme points and IPs move closer to the y-axis as c increases. $c = 0$ is a transitional value: when c is replaced by $-c$, the curve is reflected in the x-axis.



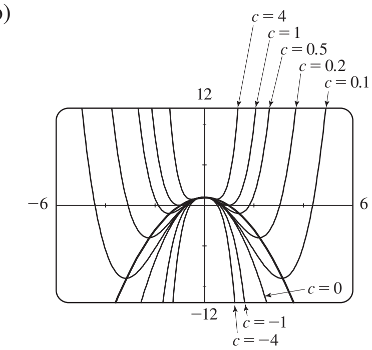
27. There is no maximum or minimum, regardless of the value of c. For $c < 0$, there is a vertical asymptote at $x = 0$, $\lim_{x\to 0} f(x) = \infty$, and $\lim_{x\to\pm\infty} f(x) = 1$.
$c = 0$ is a transitional value at which $f(x) = 1$ for $x \neq 0$.
For $c > 0$, $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to\pm\infty} f(x) = 1$, and there are two IPs, which move away from the y-axis as $c \to \infty$.



29. For $c \geqslant 0$, there is no IP and only one extreme point, the origin. For $c < 0$, there is a maximum point at the origin, two minimum points, and two IPs, which move downward and away from the origin as $c \to -\infty$.



31. (a) Positive (b)

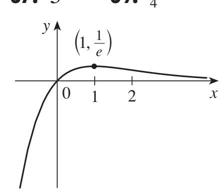


Exercises 4.5 • page 305

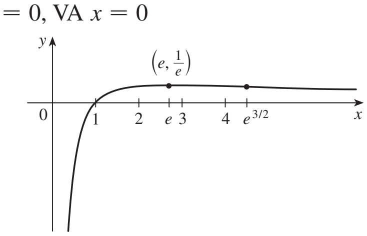
1. (a) Indeterminate (b) 0 (c) 0
(d) ∞, −∞, or does not exist (e) Indeterminate

3. (a) −∞ (b) Indeterminate (c) ∞ **5.** −2 **7.** 1
9. p/q **11.** −∞ **13.** $\ln\frac{5}{3}$ **15.** $\frac{1}{2}$ **17.** 1 **19.** 1
21. 0 **23.** 0 **25.** 0 **27.** 0 **29.** 1 **31.** 1
33. e^{-2} **35.** 1 **37.** 5 **39.** $\frac{1}{4}$

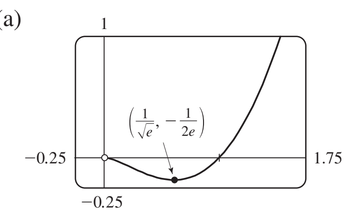
41. HA $y = 0$



43. HA $y = 0$, VA $x = 0$



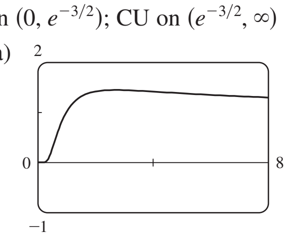
45. (a)



(b) $\lim_{x\to 0^+} f(x) = 0$

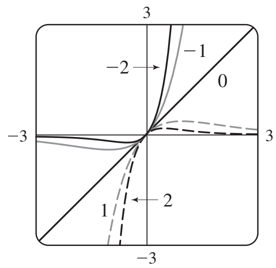
(c) Loc. min. $f(1/\sqrt{e}) = -1/(2e)$;
CD on $(0, e^{-3/2})$; CU on $(e^{-3/2}, \infty)$

47. (a)



(b) $\lim_{x\to 0^+} x^{1/x} = 0$, $\lim_{x\to\infty} x^{1/x} = 1$
(c) Loc. max. $f(e) = e^{1/e}$
(d) IP at $x \approx 0.58, 4.37$

49.

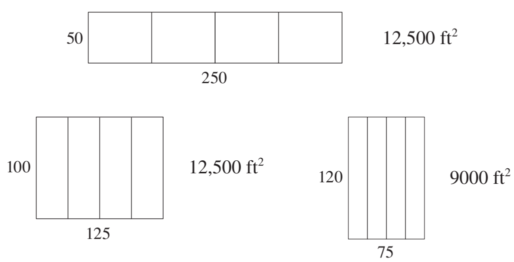


For $c > 0$, $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = -\infty$.
For $c < 0$, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = 0$.
As $|c|$ increases, the maximum and minimum points and the IPs get closer to the origin.

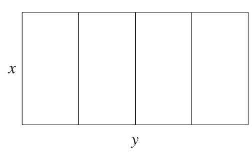
55. $\frac{16}{9}a$

Exercises 4.6 • page 312

1. (a) 11, 12 (b) 11.5, 11.5 **3.** 10, 10
5. 25 m by 25 m
7. (a)



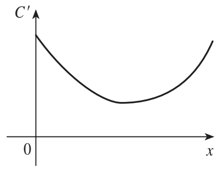
(b)



(c) $A = xy$ (d) $5x + 2y = 750$ (e) $A = 375x - \frac{5}{2}x^2$
(f) 14,062.5 ft^2
9. 4000 cm^3 **13.** $\left(-\frac{28}{17}, \frac{7}{17}\right)$ **15.** $L/2$, $\sqrt{3}\,L/4$
17. $\pi r^2(1 + \sqrt{5})$
19. Width $60/(4 + \pi)$ ft; rectangle height $30/(4 + \pi)$ ft
21. (a) Use all of the wire for the square
(b) $40\sqrt{3}/(9 + 4\sqrt{3})$ m for the square
23. $V = 2\pi R^3/(9\sqrt{3})$
25. (a) $\frac{3}{2}s^2 \csc\theta\,(\csc\theta - \sqrt{3}\cot\theta)$ (b) $\cos^{-1}(1/\sqrt{3}) \approx 55°$
(c) $6s[h + s/(2\sqrt{2})]$
27. $10\sqrt[3]{3}/(1 + \sqrt[3]{3})$ ft from the stronger source
29. $y = -\frac{5}{3}x + 10$ **31.** 9.35 m **35.** $x = 6$ in.
37. $(L + W)^2/2$ **39.** At a distance $5 - 2\sqrt{5}$ from A
41. (a) About 5.1 km from B
(b) C is close to B; C is close to D;
$W/L = \sqrt{25 + x^2}/x$, where $x = |BC|$
(c) ≈1.07; no such value (d) $\sqrt{41}/4 \approx 1.6$
43. (a) $T_1 = D/c_1$, $T_2 = (2h\sec\theta)/c_1 + (D - 2h\tan\theta)/c_2$,
$T_3 = \sqrt{4h^2 + D^2}/c_1$
(c) $c_1 \approx 3.85$, $c_2 \approx 7.66$, $h \approx 0.42$

Exercises 4.7 • page 322

1. (a) $C(0)$ represents fixed costs, which are incurred even when nothing is produced.
(b) The marginal cost is a minimum there.
(c)



3. \$17.40/unit; the cost of producing the 1001st unit is about \$17.40
5. (a) \$1,340,000; \$1340/unit; \$2300/unit
(b) 200 (c) \$700/unit
7. (a) $c(x) = 3700/x + 5 - 0.04x + 0.0003x^2$,
$C'(x) = 5 - 0.08x + 0.0009x^2$
(b) Between 208 and 209 units (c) $c(209) \approx \$27.45$/unit
(d) \$3.22/unit
9. 333 units **11.** 100
13. (a) About 200 yd (b) 192 yd
15. (a) $p(x) = 19 - (x/3000)$ (b) \$9.50
17. (a) $p(x) = 550 - (x/10)$ (b) \$175 (c) \$100

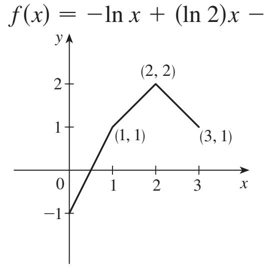
Exercises 4.8 • page 327

1. $x_2 \approx 2.3$, $x_3 \approx 3$ **3.** $\frac{4}{5}$ **5.** 2.1148 **7.** 3.10723251
9. 1.895494 **11.** −1.39194691, 1.07739428, 2.71987822
13. −0.44285440 **15.** 0.15438500, 0.84561500
17. 0.52026899 **19.** (b) 31.622777
25. (0.904557, 1.855277) **27.** 11.28 ft **29.** 0.76286%

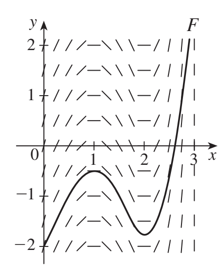
Exercises 4.9 • page 334

1. $2x^3 - 4x^2 + 3x + C$ **3.** $4x^{5/4} - 4x^{7/4} + C$
5. $-5/(4x^8) + C_1$ if $x < 0$; $-5/(4x^8) + C_2$ if $x > 0$
7. $\frac{2}{7}t^{7/2} + \frac{4}{5}t^{5/2} + C$ **9.** $3\sin t + 4\cos t + C$
11. $x^2 + 5\sin^{-1}x + C$ **13.** $x^5 - \frac{1}{3}x^6 + 4$
15. $x^3 + x^4 + Cx + D$ **17.** $\frac{1}{2}x^2 + \frac{25}{126}x^{14/5} + Cx + D$
19. $3\sin x - 5\cos x + 9$ **21.** $\frac{1}{6}x^3 + 2x - 3$
23. $f(x) = -\ln x + (\ln 2)x - \ln 2$ **25.** 10

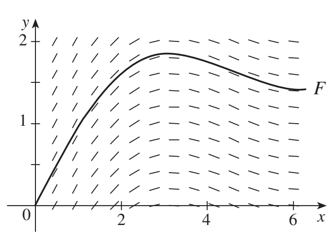
27.



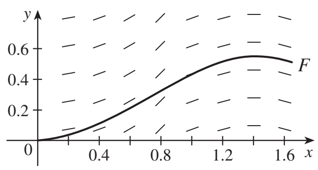
29.



31.



33.



35. $s(t) = 1 - \cos t - \sin t$

37 (a) $s(t) = 450 - 4.9t^2$ (b) $\sqrt{450/4.9} \approx 9.58$ s
(c) $-9.8\sqrt{450/4.9} \approx -93.9$ m/s (d) About 9.09 s

41. \$742.08 **43.** 225 ft **45.** $\frac{88}{15}$ ft/s^2

49. (a) 22.9125 mi (b) 21.675 mi (c) 30 min 33 s
(d) 55.425 mi

Chapter 4 Review • page 336

True-False Quiz

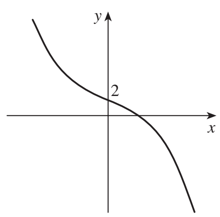
1. False **3.** False **5.** True **7.** False **9.** True
11. True **13.** False

Exercises

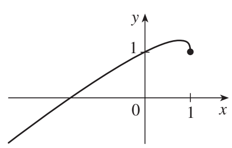
1. Abs. min. $f(0) = 10$; abs. and loc. max. $f(3) = 64$

3. Abs. max. $f(0) = 0$; abs. and loc. min. $f(-1) = -1$

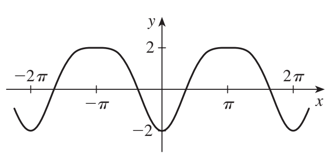
5. (a) None
(b) Dec. on $(-\infty, \infty)$
(c) None
(d) CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP $(0, 2)$
(e) See graph at right.



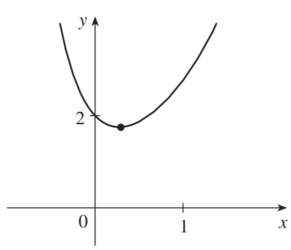
7. (a) None
(b) Inc. on $\left(-\infty, \frac{3}{4}\right)$, dec. on $\left(\frac{3}{4}, 1\right)$
(c) Loc. max. $f\left(\frac{3}{4}\right) = \frac{5}{4}$
(d) CD on $(-\infty, 1)$
(e) See graph at right.



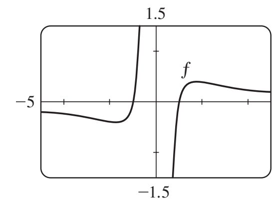
9. (a) None
(b) Inc. on $(2n\pi, (2n + 1)\pi)$, n an integer;
dec. on $((2n + 1)\pi, (2n + 2)\pi)$
(c) Loc. max. $f((2n + 1)\pi) = 2$; loc. min. $f(2n\pi) = -2$
(d) CU on $(2n\pi - (\pi/3), 2n\pi + (\pi/3))$;
CD on $(2n\pi + (\pi/3), 2n\pi + (5\pi/3))$; IP $\left(2n\pi \pm (\pi/3), -\frac{1}{4}\right)$
(e)



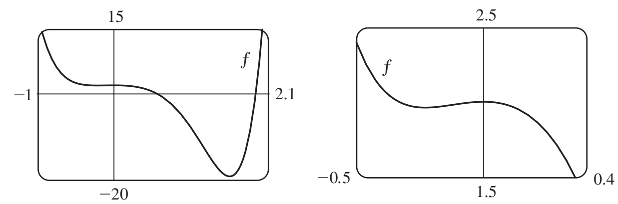
11. (a) None
(b) Inc. on $\left(\frac{1}{4}\ln 3, \infty\right)$,
dec. on $\left(-\infty, \frac{1}{4}\ln 3\right)$
(c) Loc. min.
$f\left(\frac{1}{4}\ln 3\right) = 3^{1/4} + 3^{-3/4}$
(d) CU on $(-\infty, \infty)$
(e) See graph at right.



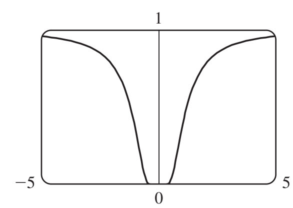
13. Inc. on $\left(-\sqrt{3}, 0\right)$, $\left(0, \sqrt{3}\right)$;
dec. on $\left(-\infty, -\sqrt{3}\right)$, $\left(\sqrt{3}, \infty\right)$;
loc. max. $f\left(\sqrt{3}\right) = 2\sqrt{3}/9$,
loc. min. $f\left(-\sqrt{3}\right) = -2\sqrt{3}/9$;
CU on $\left(-\sqrt{6}, 0\right)$, $\left(\sqrt{6}, \infty\right)$;
CD on $\left(-\infty, -\sqrt{6}\right)$, $\left(0, \sqrt{6}\right)$;
IP $\left(\sqrt{6}, 5\sqrt{6}/36\right)$, $\left(-\sqrt{6}, -5\sqrt{6}/36\right)$



15. Inc. on $(-0.23, 0)$, $(1.62, \infty)$; dec. on $(-\infty, -0.23)$, $(0, 1.62)$; loc. max. $f(0) = 2$; loc. min. $f(-0.23) \approx 1.96$, $f(1.62) \approx -19.2$; CU on $(-\infty, -0.12)$, $(1.24, \infty)$; CD on $(-0.12, 1.24)$; IP $(-0.12, 1.98)$, $(1.2, -12.1)$



17.



$(\pm 0.82, 0.22)$; $\left(\pm\sqrt{2/3}, e^{-3/2}\right)$

19. Max. at $x = 0$, min. at $x \approx \pm 0.87$, IP at $x \approx \pm 0.52$

21. For $C > -1$, f is periodic with period 2π and has local maxima at $2n\pi + \pi/2$, n an integer. For $C \leqslant -1$, f has no graph. For $-1 < C \leqslant 1$, f has vertical asymptotes. For $C > 1$, f is continuous on $\mathbb{R}$. As C increases, f moves upward and its oscillations become less pronounced.

23. $a = -3, b = 7$ **25.** $-1/(2\pi)$ **27.** 0

29. $-\frac{1}{3}$ **31.** $\frac{1}{3}$ **33.** 400 ft/h **35.** 13 ft/s

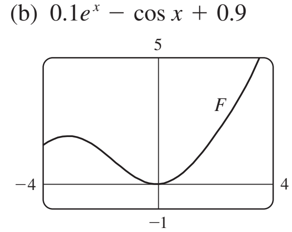
37. 500, 125 **39.** $3\sqrt{3}r^2$ **41.** $4/\sqrt{3}$ cm from D; at C

43. $L = C$ **45.** \$11.50 **47.** -2.063421

49. $F(x) = e^x - 4\sqrt{x} + C$

51. $2\arctan x - 1$ **53.** $\frac{1}{20}x^5 + \frac{1}{6}x^3 + x - 1$

55. (b) $0.1e^x - \cos x + 0.9$
(c)



57. No

59. (b) About 8.5 in. by 2 in. (c) $20/\sqrt{3}$ in., $20\sqrt{2/3}$ in.

61. (a) $20\sqrt{2} \approx 28$ ft
(b) $dI/dt = -480k(h - 4)/[(h - 4)^2 + 1600]^{5/2}$, where k is the constant of proportionality

Focus on Problem Solving • page 341

7. $(-2, 4), (2, -4)$ **9.** $\frac{4}{3}$ **13.** $(m/2, m^2/4)$

15. (a) $-\tan\theta\left[\frac{1}{c}\frac{dc}{dt}+\frac{1}{b}\frac{db}{dt}\right]$

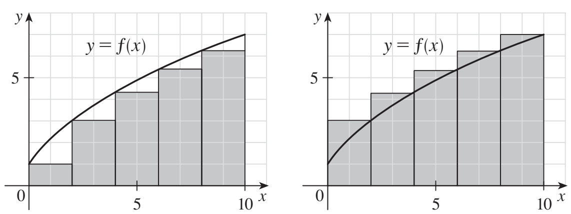
(b) $\dfrac{b\dfrac{db}{dt}+c\dfrac{dc}{dt}-\left(b\dfrac{dc}{dt}+c\dfrac{db}{dt}\right)\sec\theta}{\sqrt{b^2+c^2-2bc\cos\theta}}$

17. (a) $x/(x^2+1)$ (b) $\frac{1}{2}$ **23.** 11.204 cm^3/min

CHAPTER 5

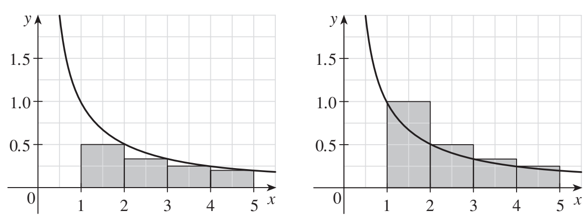
Exercises 5.1 • page 355

1. (a) 40, 52

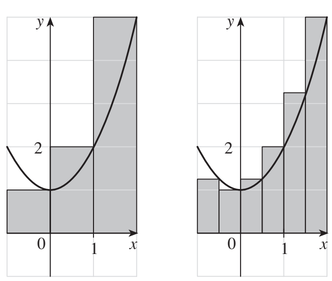


(b) 43.2, 49.2

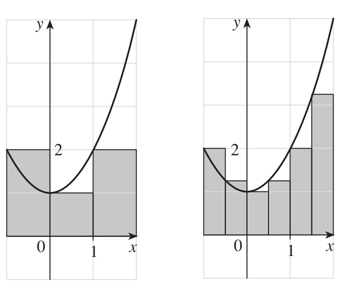
3. (a) $\frac{77}{60}$, underestimate (b) $\frac{25}{12}$, overestimate



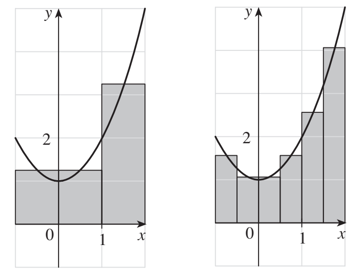
5. (a) 8, 6.875



(b) 5, 5.375



(c) 5.75, 5.9375



(d) M_6

7. 1.9835, 1.9982, 1.9993; 2 **9.** (a) Left: 4.5148, 4.6165, 4.6366; right: 4.8148, 4.7165, 4.6966

11. 34.7 ft, 44.8 ft **13.** 155 ft

15. $\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt[4]{1+15i/n}\cdot(15/n)$

17. The region under the graph of $y=\tan x$ from 0 to $\pi/4$

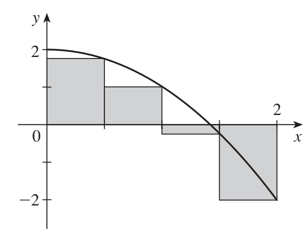
19. (a) $\lim_{n\to\infty}\frac{64}{n^6}\sum_{i=1}^{n}i^5$

(b) $n^2(n+1)^2(2n^2+2n-1)/12$ (c) $\frac{32}{3}$ **21.** $\sin b$, 1

Exercises 5.2 • page 367

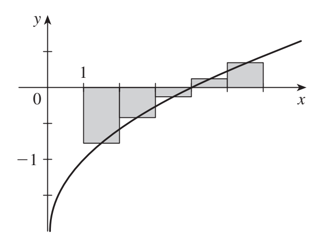
1. 0.25

The Riemann sum represents the sum of the areas of the 2 rectangles above the x-axis minus the sum of the areas of the 2 rectangles below the x-axis.



3. −0.856759

The Riemann sum represents the sum of the areas of the 2 rectangles above the x-axis minus the sum of the areas of the 3 rectangles below the x-axis.



5. (a) 4 (b) 6 (c) 10

7. −475, −85 **9.** 6.4643 **11.** 1.8100

13. 1.81001414, 1.81007263, 1.81008347

15.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

2

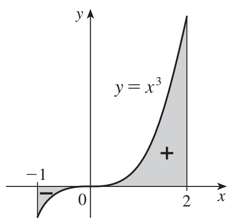
17. $\int_0^{\pi} x\sin x\,dx$ **19.** $\int_0^{1}(2x^2-5x)\,dx$ **21.** 42 **23.** $\frac{4}{3}$

25. 3.75 **27.** $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\sin\frac{5\pi i}{n}\right)\frac{\pi}{n}=\frac{2}{5}$

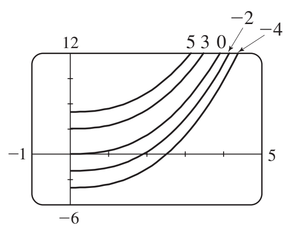
29. (a) 4 (b) 10 (c) -3 (d) 2 **31.** 10
33. $3 + 9\pi/4$ **35.** 0 **37.** $-\frac{38}{3}$ **39.** $\int_1^{12} f(x)\,dx$
41. -0.8 **43.** 3 **45.** $e^5 - e^3$ **49.** $\int_0^1 x^4\,dx$

Exercises 5.3 • page 377

1. The increase in the child's weight (in pounds) between the ages of 5 and 10
3. Number of gallons of oil leaked in the first 2 hours
5. Increase in revenue when production is increased from 1000 to 5000 units
7. Newton-meters (or joules)
9. $\frac{364}{3}$ **11.** 138 **13.** $\frac{16}{3}$ **15.** $-2 + 1/e$ **17.** $\frac{7}{8}$
19. $\frac{18}{5}\sqrt{2} - \frac{12}{5}$ **21.** $(\sqrt{2} - 1)/2$ **23.** $\frac{29}{35}$ **25.** $2\sqrt{3}/3$
27. $\ln 3$ **29.** $2^8/\ln 2$ **31.** $\pi/2$ **33.** $1 + \pi/4$
35. 2 **37.** 0, 1.32; 0.84
39. 3.75



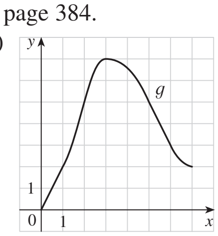
43. $\frac{2}{5}x^{5/2} + C$



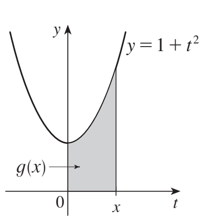
45. $2t - t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + C$ **47.** $\sec x + C$ **49.** $\frac{4}{3}$
51. (a) $-\frac{3}{2}$ m (b) $\frac{41}{6}$ m
53. (a) $v(t) = \frac{1}{2}t^2 + 4t + 5$ m/s (b) $416\frac{2}{3}$ m
55. $46\frac{2}{3}$ kg **57.** 1.4 mi **59.** $58,000
61. (b) At most 40%; $\frac{5}{36}$ **63.** 3

Exercises 5.4 • page 386

1. One process undoes what the other one does. See the Fundamental Theorem of Calculus, page 384.
3. (a) 0, 2, 5, 7, 3
(b) (0, 3)
(c) $x = 3$
(d)

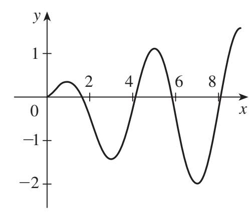


5.



$g'(x) = 1 + x^2$

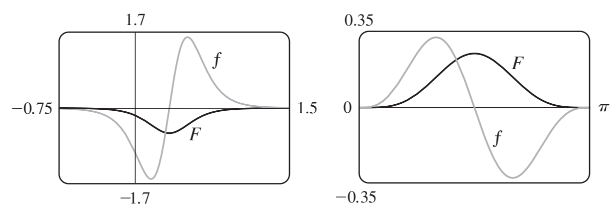
7. $g'(x) = \sqrt{1 + 2x}$ **9.** $g'(y) = y^2 \sin y$
11. $h'(x) = -\arctan(1/x)/x^2$ **13.** $\dfrac{\cos\sqrt{x}}{2x}$
15. $g'(x) = \dfrac{-2(4x^2 - 1)}{4x^2 + 1} + \dfrac{3(9x^2 - 1)}{9x^2 + 1}$ **17.** $\sqrt{257}$
19. (a) Loc. max. at 1 and 5; loc. min. at 3 and 7
(b) 9
(c) $(\frac{1}{2}, 2)$, (4, 6), (8, 9)
(d) See graph at right.



21. (a) $-2\sqrt{n}$, $\sqrt{4n - 2}$, n an integer > 0
(b) (0, 1), $(-\sqrt{4n - 1}, -\sqrt{4n - 3})$, and $(\sqrt{4n - 1}, \sqrt{4n + 1})$, n an integer > 0
(c) 0.74
23. $f(x) = \int_1^x (2^t/t)\,dt$ **25.** $f(x) = x^{3/2}$, $a = 9$
27. (b) Average expenditure over $[0, t]$; minimize average expenditure

Exercises 5.5 • page 395

1. $\frac{1}{3}\sin 3x + C$ **3.** $\frac{2}{9}(x^3 + 1)^{3/2} + C$
5. $-1/(1 + 2x)^2 + C$ **7.** $\frac{1}{5}(x^2 + 3)^5 + C$
9. $\frac{1}{3}(\ln x)^3 + C$ **11.** $\frac{2}{3}(x - 1)^{3/2} + C$
13. $-\frac{1}{3}\ln|5 - 3x| + C$
15. $2\sqrt{1 + x + 2x^2} + C$ **17.** $-2/[5(t + 1)^5] + C$
19. $-\frac{1}{3}\cos 3\theta + C$ **21.** $\frac{2}{3}(1 + e^x)^{3/2} + C$
23. $-\frac{1}{5}\cos^5 x + C$ **25.** $-\frac{2}{3}(\cot x)^{3/2} + C$
27. $x - e^{-x} + C$ **29.** $\frac{1}{3}\sec^3 x + C$
31. $\tan^{-1}x + \frac{1}{2}\ln(1 + x^2) + C$
33. $\dfrac{-1}{6(3x^2 - 2x + 1)^3} + C$ **35.** $\frac{1}{4}\sin^4 x + C$

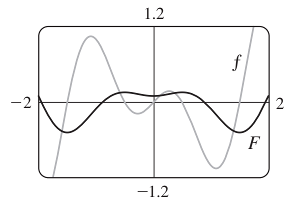


37. 0 **39.** $\frac{182}{9}$ **41.** 0 **43.** $2(e^2 - e)$ **45.** $\frac{16}{15}$
47. 3 **49.** 0 **51.** 2 **53.** $\sqrt{3} - \frac{1}{3}$ **55.** 6π
57. All three areas are equal.
59. $[5/(4\pi)][1 - \cos(2\pi t/5)]$ L **61.** 5

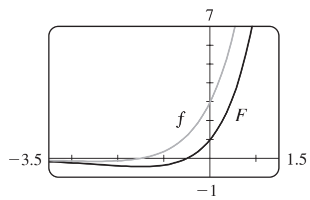
Exercises 5.6 • page 401

1. $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$ **3.** $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$
5. $-\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x + C$
7. $\frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C$
9. $x(\ln x)^2 - 2x\ln x + 2x + C$
11. $\frac{1}{16}r^4(4\ln r - 1) + C$
13. $\frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$

15. $1 - 2/e$ **17.** $-\frac{1}{2}$ **19.** $\frac{1}{12}(\pi - 12 + 6\sqrt{3})$
21. $2 \ln 4 - \frac{3}{2}$ **23.** $\frac{1}{2}(\ln 2 - 1)$
25. $2(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}) + C$ **27.** $-\frac{1}{2} - \pi/4$
29. $(x \sin \pi x)/\pi + (\cos \pi x)/\pi^2 + C$



31. $(2x + 1)e^x + C$



33. (b) $-\frac{1}{4}\cos x \sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$
35. (b) $\frac{2}{3}, \frac{8}{15}$ **39.** $x[(\ln x)^3 - 3(\ln x)^2 + 6 \ln x - 6] + C$
41. $2 - e^{-t}(t^2 + 2t + 2)$ m

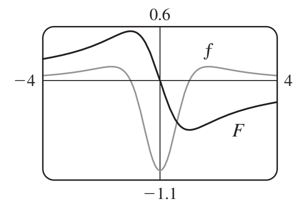
Exercises 5.7 • page 408

1. $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$ **3.** $-\frac{11}{384}$
5. $\frac{3}{8}t + \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + C$
7. $\frac{1}{3}\sec^3 x - \sec x + C$ **9.** $-\dfrac{\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\dfrac{x}{3}\right) + C$
11. $-\dfrac{\sqrt{x^2 + 4}}{4x} + C$ **13.** $\pi/24 + \sqrt{3}/8 - \frac{1}{4}$
15. (a) $\dfrac{A}{x + 4} + \dfrac{B}{x - 1}$ (b) $\dfrac{A}{x - 1} + \dfrac{Bx + C}{x^2 + x + 1}$
17. $2 \ln|x + 5| - \ln|x - 2| + C$ **19.** $\frac{1}{2}\ln\frac{3}{2}$
21. $\ln|x - 1| - \frac{1}{2}\ln(x^2 + 9) - \frac{1}{3}\tan^{-1}(x/3) + C$
23. $\frac{1}{2}\ln(x^2 + 1) + (1/\sqrt{2})\tan^{-1}(x/\sqrt{2}) + C$
25. $\frac{1}{2}x^2 - x + \ln|x + 1| + C$ **27.** $\frac{1}{2}(1 - \ln 2)$
29. $2 + \ln\frac{25}{9}$ **31.** $\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x + 1}{\sqrt{3}}\right) + C$

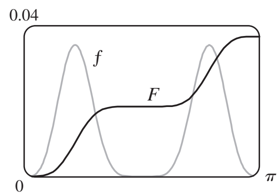
Exercises 5.8 • page 414

1. $\frac{1}{2}x^2 - x - 4\ln(x^2 + 9) + \frac{8}{3}\tan^{-1}(x/3) + C$
3. $(1/(2\pi))\sec(\pi x)\tan(\pi x) + (1/(2\pi))\ln|\sec(\pi x) + \tan(\pi x)| + C$
5. $(-\sqrt{9x^2 - 1}/x) + 3\ln|3x + \sqrt{9x^2 - 1}| + C$
7. $\pi^3 - 6\pi$ **9.** $\frac{1}{2}[x^2\sin^{-1}(x^2) + \sqrt{1 - x^4}] + C$
11. $9\pi/4$ **13.** $\frac{1}{9}\sin^3 x\,[3\ln(\sin x) - 1] + C$ **15.** $\frac{8}{15}$
17. $\frac{1}{5}\ln|x^5 + \sqrt{x^{10} - 2}| + C$
19. $(1 + e^x)\ln(1 + e^x) - e^x + C_1$
21. $\sqrt{e^{2x} - 1} - \cos^{-1}(e^{-x}) + C$
25. $-\frac{1}{4}x(5 - x^2)^{3/2} + \frac{5}{8}x\sqrt{5 - x^2} + \frac{25}{8}\sin^{-1}(x/\sqrt{5}) + C$
27. $-\frac{1}{5}\sin^2 x\cos^3 x - \frac{2}{15}\cos^3 x + C$
29. $\frac{1}{10}(1 + 2x)^{5/2} - \frac{1}{6}(1 + 2x)^{3/2} + C$
31. $-\ln|\cos x| - \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$

33. $\dfrac{2^{x-1}\sqrt{2^{2x} - 1}}{\ln 2} - \dfrac{\ln(\sqrt{2^{2x} - 1} + 2^x)}{2\ln 2} + C$
35. $F(x) = \frac{1}{2}\ln(x^2 - x + 1) - \frac{1}{2}\ln(x^2 + x + 1)$;
max. at -1, min. at 1; IP at -1.7, 0, and 1.7



37. $F(x) = -\frac{1}{10}\sin^3 x\cos^7 x - \frac{3}{80}\sin x\cos^7 x + \frac{1}{160}\cos^5 x\sin x + \frac{1}{128}\cos^3 x\sin x + \frac{3}{256}\cos x\sin x + \frac{3}{256}x$



Exercises 5.9 • page 425

1. (a) $L_2 = 6$, $R_2 = 12$, $M_2 \approx 9.6$
(b) L_2 is an underestimate, R_2 and M_2 are overestimates.
(c) $T_2 = 9 < I$ (d) $L_n < T_n < I < M_n < R_n$
3. (a) $T_4 \approx 0.895759$ (underestimate)
(b) $M_4 \approx 0.908907$ (overestimate)
$T_4 < I < M_4$
5. (a) 5.932957 (b) 5.869247
7. (a) 0.746211 (b) 0.747131 (c) 0.746825
9. (a) 2.031893 (b) 2.014207 (c) 2.020651
11. (a) 0.451948 (b) 0.451991 (c) 0.451976
13. (a) 1.064275 (b) 1.067416 (c) 1.074915
15. (a) $T_{10} \approx 0.881839$, $M_{10} \approx 0.882202$
(b) $|E_T| \leqslant 0.01\overline{3}$, $|E_M| \leqslant 0.00\overline{6}$
(c) $n = 366$ for T_n, $n = 259$ for M_n
17. (a) $T_{10} \approx 1.719713$, $E_T \approx -0.001432$;
$S_{10} \approx 1.718283$, $E_S \approx -0.000001$
(b) $|E_T| \leqslant 0.002266$, $|E_S| \leqslant 0.0000016$
(c) $n = 151$ for T_n, $n = 107$ for M_n, $n = 8$ for S_n
19. (a) 2.8 (b) 7.954926518 (c) 0.2894
(d) 7.954926521 (e) The actual error is much smaller.
(f) 10.9 (g) 7.953789422 (h) 0.0593
(i) The actual error is smaller. (j) $n \geqslant 50$
21.

n	L_n	R_n	T_n	M_n
4	0.140625	0.390625	0.265625	0.242188
8	0.191406	0.316406	0.253906	0.248047
16	0.219727	0.282227	0.250977	0.249512

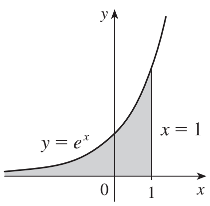
n	E_L	E_R	E_T	E_M
4	0.109375	−0.140625	−0.015625	0.007813
8	0.058594	−0.066406	−0.003906	0.001953
16	0.030273	−0.032227	−0.000977	0.000488

23. (a) 11.5 (b) 12 (c) $11.\overline{6}$ **25.** $37.\overline{73}$ ft/s
27. 10,177 megawatt-hours **29.** (a) 23.44 (b) $0.341\overline{3}$
31. 59.4

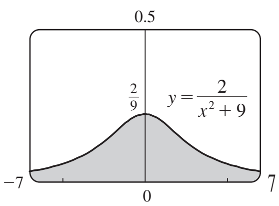
Exercises 5.10 • page 436

Abbreviations: C, convergent; D, divergent

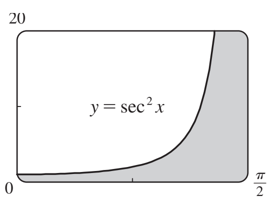
1. (a) Infinite interval (b) Infinite discontinuity
(c) Infinite discontinuity (d) Infinite interval
3. $\frac{1}{2} - 1/(2t^2)$; 0.495, 0.49995, 0.4999995; 0.5
5. $\frac{1}{12}$ **7.** 1 **9.** D **11.** D
13. 0 **15.** D **17.** $e^2/4$
19. D **21.** 1 **23.** $2\sqrt{3}$ **25.** D
27. D **29.** D **31.** $\frac{8}{3}\ln 2 - \frac{8}{9}$
33. e



35. $\frac{2\pi}{3}$



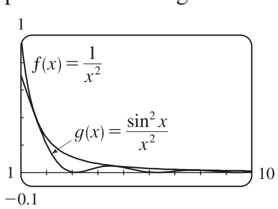
37. Infinite area



39. (a)

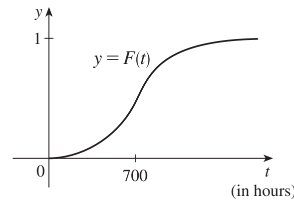
t	$\int_1^t [(\sin^2 x)/x^2]\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1,000	0.672957
10,000	0.673407

It appears that the integral is convergent.
(c)



41. C **43.** C **45.** D **47.** π **49.** $1/(1 - p),\ p < 1$

53. (a)



(b) The rate at which the fraction $F(t)$ increases as t increases
(c) 1; all bulbs burn out eventually
55. 8264.5 years **57.** 1000 **61.** $C = 1$; ln 2

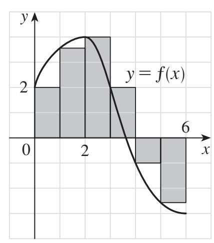
Chapter 5 Review • page 438

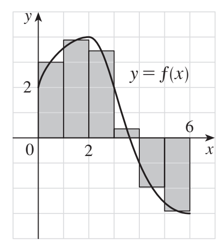
True-False Quiz

1. True **3.** True **5.** False **7.** True **9.** True
11. False **13.** False **15.** False **17.** False

Exercises

1. (a) 8 (b) 5.7





3. $\frac{1}{2} + \pi/4$ **5.** 3 **7.** $f = c,\ f' = b,\ \int_0^x f(t)\,dt = a$
9. 37 **11.** $\frac{9}{10}$ **13.** $\frac{1209}{28}$ **15.** $\frac{1}{2}\ln 2$ **17.** 3480
19. $(1/\pi)(e^{\pi} - 1)$ **21.** $x\sec x - \ln|\sec x + \tan x| + C$
23. $-\sin(1/t) + C$ **25.** $2x - \ln|3x + 2| + C$
27. $-e^{-x}(x^2 + 2x + 2) + C$ **29.** $\frac{1}{2}\ln\left|\frac{t + 2}{t + 4}\right| + C$
31. $\frac{162}{5}$ **33.** $\ln|1 + \sec\theta| + C$
35. $2\sqrt{1 + \sin x} + C$
37. $\frac{64}{5}$ **39.** $F'(x) = \sqrt{1 + x^4}$
41. $y' = (2e^x - e^{\sqrt{x}})/(2x)$
43. $\frac{1}{2}\left[e^x\sqrt{1 - e^{2x}} + \sin^{-1}(e^x)\right] + C$
45. $\frac{1}{4}(2x + 1)\sqrt{x^2 + x + 1} +$
$\frac{3}{8}\ln\left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) + C$
47. (a) 1.090608 (overestimate)
(b) 1.088840 (underestimate)
(c) 1.089429 (unknown)
49. (a) $0.00\overline{6}$, $n \geqslant 259$ (b) $0.00\overline{3}$, $n \geqslant 183$
51. (a) 3.8 (b) 1.7867, 0.000646 (c) $n \geqslant 30$
53. $4 \leqslant \int_1^3 \sqrt{x^2 + 3}\,dx \leqslant 4\sqrt{3}$ **55.** $\frac{1}{36}$ **57.** D
59. 2 **61.** C **63.** (a) $29.1\overline{6}$ m (b) 29.5 m
65. Number of barrels of oil consumed from Jan. 1, 2000, through Jan. 1, 2003
67. $Ce^{-x^2/(4kt)}/\sqrt{4\pi kt}$ **69.** $e^{2x}(1 + 2x)/(1 - e^{-x})$

Focus on Problem Solving • page 444

1. About 1.85 inches from the center **3.** $\pi/2$ **5.** 1
7. e^{-2} **9.** Does not exist **11.** $[-1, 2]$
13. $\sqrt{1 + \sin^4 x}\cos x$ **15.** 0
19. (b) $y = -\sqrt{L^2 - x^2} - L\ln\left(\dfrac{L - \sqrt{L^2 - x^2}}{x}\right)$

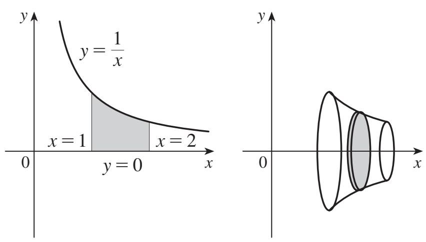
CHAPTER 6

Exercises 6.1 • page 452

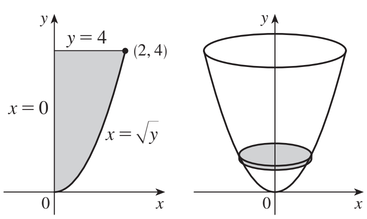
1. $\frac{32}{3}$ **3.** $e - (1/e) + \frac{10}{3}$ **5.** 19.5 **7.** $\frac{1}{6}$ **9.** 4
11. $\frac{32}{3}$ **13.** $\frac{8}{3}$ **15.** $\pi - \frac{2}{3}$
17. $-1.02, 1.02; 2.70$ **19.** 0, 0.70; 0.08 **21.** 118 ft
23. 84 m^2 **25.** $\frac{1}{2}$
27. $r\sqrt{R^2 - r^2} + \pi r^2/2 - R^2 \arcsin(r/R)$ **29.** πab
31. $\frac{1}{2}(e^{\pi/2} - 1)$ **33.** $24\sqrt{3}/5$ **35.** ± 6 **37.** $4^{2/3}$
39. $f(t) = 3t^2$ **41.** $0 < m < 1$; $m - \ln m - 1$

Exercises 6.2 • page 463

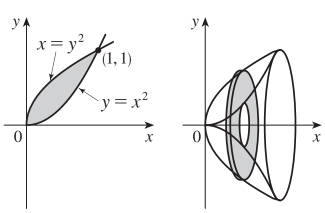
1. $\pi/2$



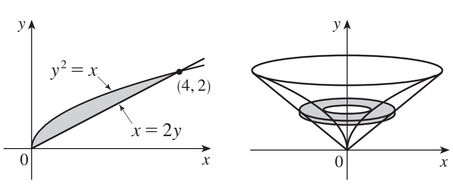
3. 8π



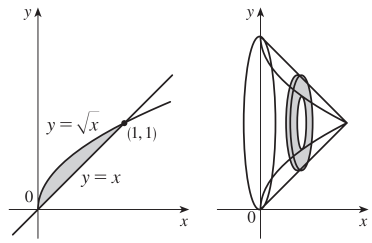
5. $3\pi/10$



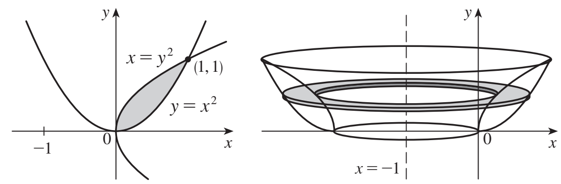
7. $64\pi/15$



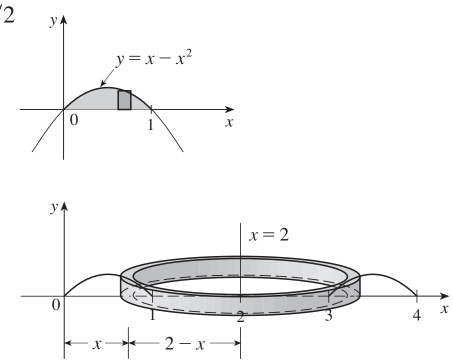
9. $\pi/6$



11. $29\pi/30$

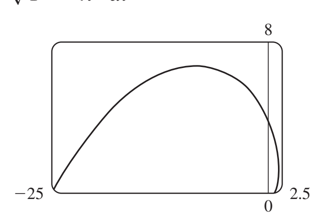


13. $832\pi/21$ **15.** 0, 0.747; 0.132
17. (a) Solid obtained by rotating the region $0 \leqslant y \leqslant \cos x$, $0 \leqslant x \leqslant \pi/2$ about the x-axis
(b) Solid obtained by rotating the region $y^4 \leqslant x \leqslant y^2$, $0 \leqslant y \leqslant 1$ about the y-axis
19. 1110 cm^3 **21.** $\pi r^2 h/3$ **23.** $\pi h^2[r - (h/3)]$
25. $2b^2h/3$ **27.** 10 cm^3 **29.** 24
31. 2 **33.** 3
35. (a) $8\pi R\int_0^r \sqrt{r^2 - y^2}\,dy$ (b) $2\pi^2 r^2 R$
37. (b) $\pi r^2 h$ **39.** $\frac{5}{12}\pi r^3$
41. $8\int_0^r \sqrt{R^2 - y^2}\sqrt{r^2 - y^2}\,dy$
43. $\pi/15$ **45.** 828
47. $\pi/2$

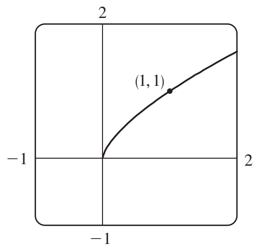


Exercises 6.3 • page 471

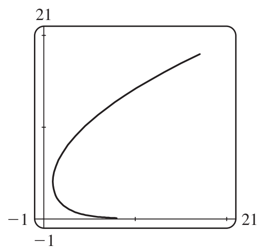
1. $3\sqrt{10}$ **3.** $\int_1^2 \sqrt{1 + 4t^2}\,dt$
5. $\sqrt{2}\,(e^{\pi} - 1)$



7. $(13\sqrt{13} - 8)/27$

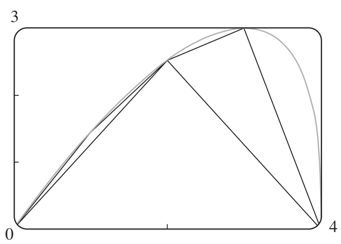


9. $e^3 + 11 - e^{-8}$



11. 0.7314 **13.** 3.820

15. (a), (b)



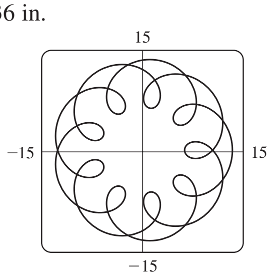
$L_1 = 4$,
$L_2 \approx 6.43$,
$L_4 \approx 7.50$

(c) $\int_0^4 \sqrt{1 + [4(3-x)/(3(4-x)^{2/3})]^2}\,dx$ (d) 7.7988

17. $\frac{205}{128} - \frac{81}{512}\ln 3$ **19.** $\ln(\sqrt{2} + 1)$ **21.** 209.1 m

23. 29.36 in.

27. (a)



$t \in [0, 4\pi]$

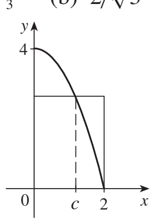
(b) 294

Exercises 6.4 • page 475

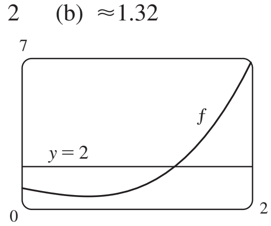
1. $2/\pi$ **3.** $(1 - e^{-25})/10$

5. (a) $\frac{8}{3}$ (b) $2/\sqrt{3}$ **7.** (a) 2 (b) ≈ 1.32

(c)



(c)



11. $(50 + 28/\pi)$ °F ≈ 59 °F

13. (a) $155/\sqrt{2} \approx 110$ V (b) $220\sqrt{2} \approx 311$ V

15. $5/(4\pi) \approx 0.4$ L

Exercises 6.5 • page 485

1. 9 ft-lb **3.** $\frac{15}{4}$ ft-lb **5.** (a) $\frac{25}{24} \approx 1.04$ J (b) 10.8 cm

7. 625 ft-lb **9.** 650,000 ft-lb **11.** 2450 J

13. (a) $\approx 1.06 \times 10^6$ J (b) ≈ 2.0 m

17. (a) $Gm_1m_2[(1/a) - (1/b)]$ (b) $\approx 8.50 \times 10^9$ J

19. $\approx 6.5 \times 10^6$ N **21.** $\approx 3.47 \times 10^4$ lb

23. (a) $\approx 5.63 \times 10^3$ lb (b) $\approx 5.06 \times 10^4$ lb
(c) $\approx 4.88 \times 10^4$ lb (d) $\approx 3.03 \times 10^5$ lb

25. 40, 12, $(1, \frac{10}{3})$ **27.** (1.5, 1.2)

29. $(1/(e-1), (e+1)/4)$ **31.** $\frac{4}{3}$, 0, $(0, \frac{2}{3})$ **33.** (b) $(\frac{1}{2}, \frac{2}{5})$

Exercises 6.6 • page 491

1. \$14,516,000 **3.** \$43,866,933.33 **5.** \$407.25

7. \$4166.67 **9.** 3727; \$37,753

11. $\frac{2}{3}(16\sqrt{2} - 8) \approx$ \$9.75 million **13.** 1.19×10^{-4} cm^3/s

15. $\frac{1}{9}$ L/s

Exercises 6.7 • page 498

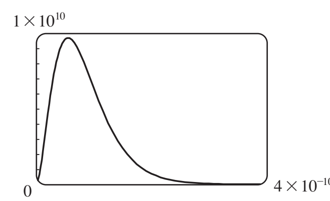
1. (a) The probability that a randomly chosen tire will have a lifetime between 30,000 and 40,000 miles
(b) The probability that a randomly chosen tire will have a lifetime of at least 25,000 miles

3. (a) $f(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (b) 5

7. (a) $e^{-4/2.5} \approx 0.20$ (b) $1 - e^{-2/2.5} \approx 0.55$
(c) If you aren't served within 10 minutes, you get a free hamburger.

9. $\approx 44\%$ **11.** ≈ 0.9545

13. (b) 0; a_0 (c)



(d) $1 - 41e^{-8} \approx 0.986$
(e) $\frac{3}{2}a_0$

Chapter 6 Review • page 500

Exercises

1. $e - \frac{11}{6}$ **3.** 9π **5.** (a) 0.38 (b) 0.87

7. (a) $2\pi/15$ (b) $\pi/6$ (c) $8\pi/15$

9. (a) Solid obtained by rotating the region $0 \leq y \leq \sqrt{2}\cos x$, $0 \leq x \leq \pi/2$ about the x-axis
(b) Solid obtained by rotating the region $2 - \sqrt{x} \leq y \leq 2 - x^2$, $0 \leq x \leq 1$ about the x-axis

11. 36 **13.** $125\sqrt{3}/3$ m^3 **15.** $2(5\sqrt{5} - 1)$

17. 3.2 J **19.** (a) $8000\pi/3 \approx 8378$ ft-lb (b) 2.1 ft

21. ≈ 458 lb **23.** \$7166.67 **25.** $f(x)$

27. (a) $f(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$
(b) ≈ 0.3455 (c) 5, yes

29. (a) $1 - e^{-3/8} \approx 0.31$ (b) $e^{-5/4} \approx 0.29$
(c) $8 \ln 2 \approx 5.55$ min

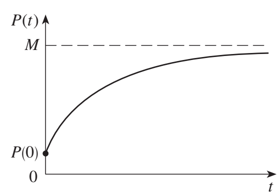
Focus on Problem Solving • page 502

1. $2\pi/3 - \sqrt{3}/2$
3. (a) $V = \int_0^h \pi[f(y)]^2\,dy$
(c) $f(y) = \sqrt{kA/(\pi C)}\,y^{1/4}$
Advantage: the markings on the container are equally spaced.
5. $f(x) = \sqrt{2x/\pi}$
7. (b) 0.2261 (c) 0.6736 m
(d) (i) $1/(105\pi) \approx 0.003$ in/s (ii) $370\pi/3$ s ≈ 6.5 min
11. (a) $P(z) = P_0 + g\int_0^z \rho(x)\,dx$
(b) $(P_0 - \rho_0 gH)(\pi r^2) + \rho_0 gHe^{L/H}\int_{-r}^{r} e^{x/H} \cdot 2\sqrt{r^2 - x^2}\,dx$
13. Height $\sqrt{2}\,b$, volume $\left(\frac{28}{27}\sqrt{6} - 2\right)\pi b^3$
15. $\ln(\pi/2)$ **19.** $2/\pi$, $1/\pi$

CHAPTER 7

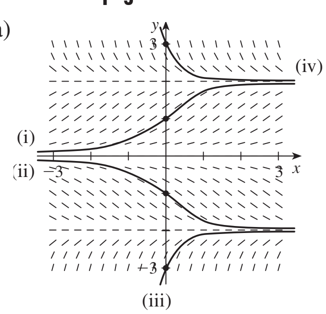
Exercises 7.1 • page 511

3. (a) ± 3 **5.** (b) and (c)
7. (a) It must be either 0 or decreasing
(c) $y = 0$ (d) $y = 1/(x + 2)$
9. (a) $0 < P < 4200$ (b) $P > 4200$
(c) $P = 0$, $P = 4200$
13. (a) At the beginning; stays positive, but decreases
(c)



Exercises 7.2 • page 519

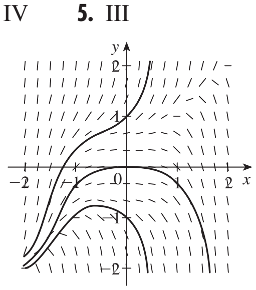
1. (a)



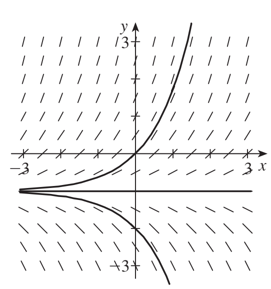
(b) $y = 0$, $y = 2$, $y = -2$

3. IV **5.** III

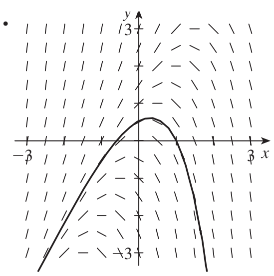
7.



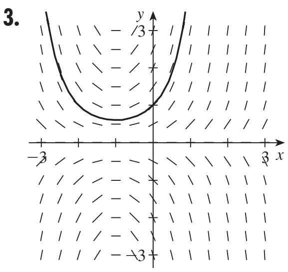
9.



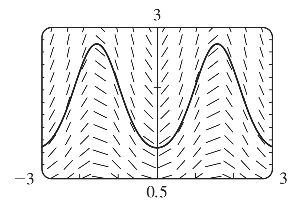
11.



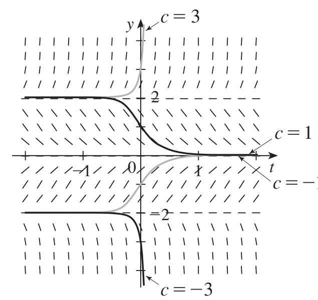
13.



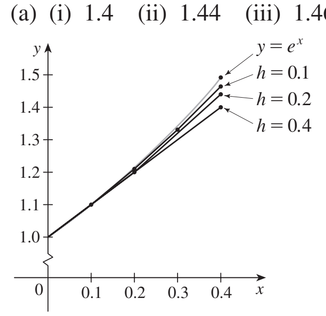
15.



17. $-2 \leqslant c \leqslant 2$; $-2, 0, 2$

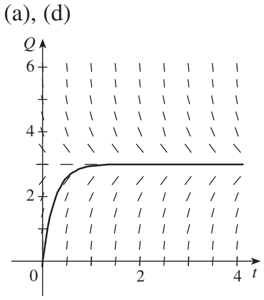


19. (a) (i) 1.4 (ii) 1.44 (iii) 1.4641
(b)



Underestimates

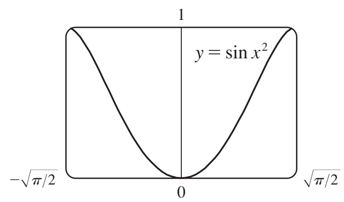
(c) (i) 0.0918 (ii) 0.0518 (iii) 0.0277
It appears that the error is also halved (approximately).
21. $-1, -3, -6.5, -12.25$ **23.** 1.7616
25. (a) (i) 3 (ii) 2.3928 (iii) 2.3701 (iv) 2.3681
(c) (i) -0.6321 (ii) -0.0249 (iii) -0.0022
(iv) -0.0002
It appears that the error is also divided by 10 (approximately).
27. (a), (d)



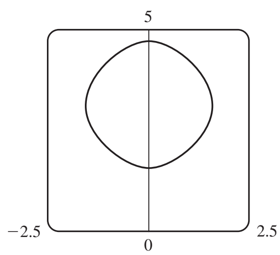
(b) 3
(c) Yes; $Q = 3$
(e) 2.77 C

Exercises 7.3 • page 527

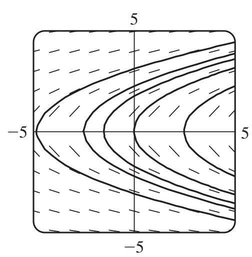
1. $y = -1/(x + C)$ or $y = 0$ **3.** $x^2 - y^2 = C$
5. $y = \pm\sqrt{[3(te^t - e^t + C)]^{2/3} - 1}$ **7.** $u = Ae^{2t+t^2/2} - 1$
9. $y = \tan(x - 1)$ **11.** $x = \sqrt{2(t - 1)e^t + 3}$
13. $u = -\sqrt{t^2 + \tan t + 25}$ **15.** $y = 7e^{x^4}$
17. (a) $\sin^{-1}y = x^2 + C$
(b) $y = \sin(x^2)$,
$-\sqrt{\pi/2} \leqslant x \leqslant \sqrt{\pi/2}$



(c) No
19. $\cos y = \cos x - 1$

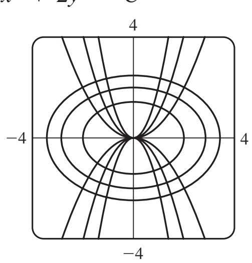


21. (a), (c)

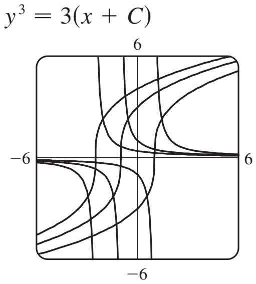


(b) $y = \pm\sqrt{2(x + C)}$

23. $x^2 + 2y^2 = C$



25. $y^3 = 3(x + C)$



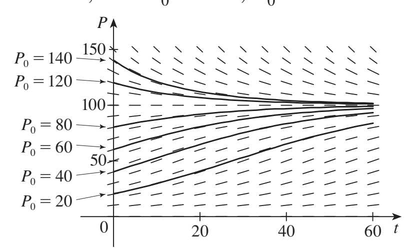
27. $Q(t) = 3 - 3e^{-4t}$; 3 **29.** $P(t) = M - Me^{-kt}$; M
31. (a) $x = a - 4/(kt + 2/\sqrt{a})^2$
(b) $t = \dfrac{2}{k\sqrt{a - b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a - b}} - \tan^{-1}\sqrt{\dfrac{b - x}{a - b}}\right)$
33. (a) $C(t) = (C_0 - r/k)e^{-kt} + r/k$
(b) r/k; the concentration approaches r/k regardless of the value of C_0
35. (a) $15e^{-t/100}$ kg (b) $15e^{-0.2} \approx 12.3$ kg **37.** g/k
39. (a) $dA/dt = k\sqrt{A}\,(M - A)$
(b) $A(t) = M[(Ce^{\sqrt{M}kt} - 1)/(Ce^{\sqrt{M}kt} + 1)]^2$, where $C = (\sqrt{M} + \sqrt{A_0})/(\sqrt{M} - \sqrt{A_0})$ and $A_0 = A(0)$
41. (b) $y(t) = (\sqrt{6} - \frac{1}{144}t)^2$ (c) $144\sqrt{6}$ s $\approx$ 5 min 53 s

Exercises 7.4 • page 538

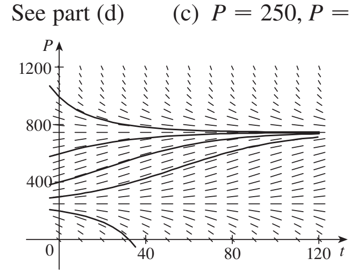
1. About 235
3. (a) $500 \times 16^{t/3}$ (b) $\approx$20,159
(c) 18,631 cells/h (d) $(3 \ln 60)/\ln 16 \approx 4.4$ h
5. (a) 1508 million, 1871 million (b) 2161 million
(c) 3972 million; wars in the first half of century, increased life expectancy in second half
7. (a) $Ce^{-0.0005t}$ (b) $-2000 \ln 0.9 \approx 211$ s
9. (a) $100 \times 2^{-t/30}$ mg (b) ≈ 9.92 mg (c) $\approx$199.3 years
11. $\approx$2500 years
13. (a) $dy/dt = ky$, $y(0) = 110$; $y(t) = 110e^{kt}$
(b) $\approx$137 °F (c) $\approx$116 min
15. (a) $\approx$64.5 kPa (b) $\approx$39.9 kPa
17. (a) (i) \$3828.84 (ii) \$3840.25 (iii) \$3850.08
(iv) \$3851.61 (v) \$3852.01 (vi) \$3852.08
(b) $dA/dt = 0.05A$, $A(0) = 3000$
19. (a) $P(t) = (m/k) + (P_0 - m/k)e^{kt}$ (b) $m < kP_0$
(c) $m = kP_0$, $m > kP_0$ (d) Declining

Exercises 7.5 • page 548

1. (a) 100; 0.05 (b) Where P is close to 0 or 100; on the line $P = 50$; $0 < P_0 < 100$; $P_0 > 100$
(c)

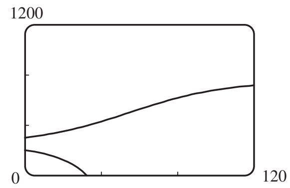


Solutions approach 100; some increase and some decrease, some have an inflection point but others don't; solutions with $P_0 = 20$ and $P_0 = 40$ have inflection points at $P = 50$
(d) $P = 0$, $P = 100$; other solutions move away from $P = 0$ and toward $P = 100$
3. (a) 3.23×10^7 kg (b) $\approx$1.55 years
5. (a) $dP/dt = \frac{1}{265}P(1 - P/100)$, P in billions
(b) 5.49 billion (c) In billions: 7.81, 27.72
(d) In billions: 5.48, 7.61, 22.41
7. (a) $dy/dt = ky(1 - y)$ (b) $y = y_0/[y_0 + (1 - y_0)e^{-kt}]$
(c) 3:36 P.M.
11. (a) Fish are caught at a rate of 15 per week.
(b) See part (d) (c) $P = 250$, $P = 750$
(d)

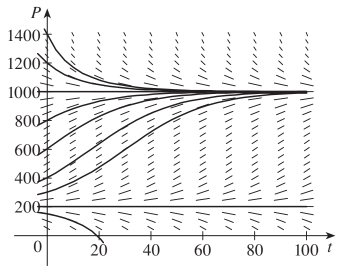


$0 < P_0 < 250$: $P \to 0$; $P_0 = 250$: $P \to 250$;
$P_0 > 250$: $P \to 750$

(e) $P(t) = \dfrac{250 - 750ke^{t/25}}{1 - ke^{t/25}}$, where $k = \frac{1}{11}, -\frac{1}{9}$



13. (b)



$0 < P_0 < 200$: $P \to 0$; $P_0 = 200$: $P \to 200$;
$P_0 > 200$: $P \to 1000$

(c) $P(t) = \dfrac{m(K - P_0) + K(P_0 - m)e^{(K-m)(k/K)t}}{K - P_0 + (P_0 - m)e^{(K-m)(k/K)t}}$

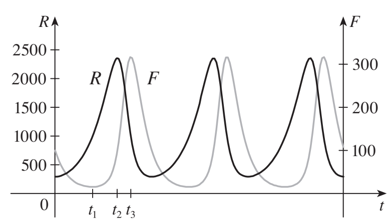
15. (a) $P(t) = P_0 e^{(k/r)[\sin(rt - \phi) + \sin\phi]}$ (b) Does not exist

Exercises 7.6 • page 555

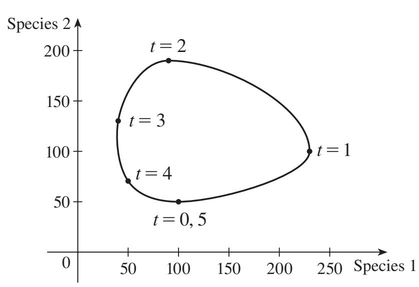
1. (a) x = predators, y = prey; growth is restricted only by predators, which feed only on prey.
(b) x = prey, y = predators; growth is restricted by carrying capacity and by predators, which feed only on prey.

3. (a) The rabbit population starts at about 300, increases to 2400, then decreases back to 300. The fox population starts at 100, decreases to about 20, increases to about 315, decreases to 100, and the cycle starts again.

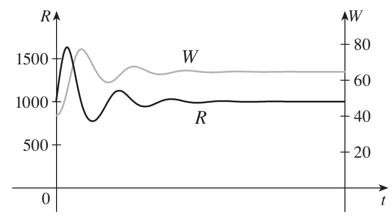
(b)



5.



9. (a) Population stabilizes at 5000.
(b) (i) $W = 0$, $R = 0$: Zero populations
(ii) $W = 0$, $R = 5000$: In the absence of wolves, the rabbit population is always 5000.
(iii) $W = 64$, $R = 1000$: Both populations are stable.
(c) The populations stabilize at 1000 rabbits and 64 wolves.
(d)



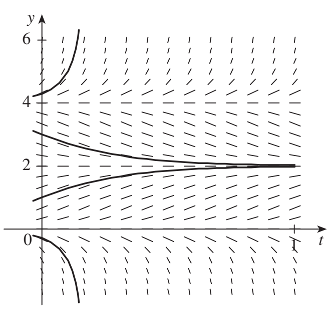
Chapter 7 Review • page 557

True-False Quiz

1. True **3.** False **5.** True

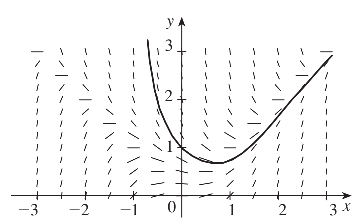
Exercises

1. (a)



(b) $0 \leqslant c \leqslant 4$; $y = 0$, $y = 2$, $y = 4$

3. (a) $y(0.3) \approx 0.8$



(b) 0.75676
(c) $y = x$ and $y = -x$; there is a local maximum or minimum

5. $y^3 + y^2 = \cos x + x \sin x + C$

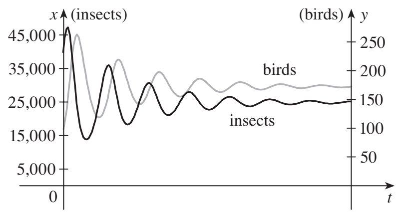
7. $y = \sqrt{(\ln x)^2 + 4}$ **9.** $y^2 - 2\ln|y| + x^2 = C$

11. (a) 1000×3^t (b) 27,000
(c) $27{,}000 \ln 3 \approx 29{,}663$ bacteria per hour
(d) $(\ln 2)/\ln 3 \approx 0.63$ h

13. (a) $C_0 e^{-kt}$ (b) ≈ 100 h

15. (a) $L(t) = L_\infty - [L_\infty - L(0)]e^{-kt}$
(b) $L(t) = 53 - 43e^{-0.2t}$

17. 15 days **19.** $k \ln h + h = (-R/V)t + C$

21. (a) Stabilizes at 200,000
(b) (i) $x = 0$, $y = 0$: Zero populations
(ii) $x = 200{,}000$, $y = 0$: In the absence of birds, the insect population is always 200,000.
(iii) $x = 25{,}000$, $y = 175$: Both populations are stable.
(c) The populations stabilize at 25,000 insects and 175 birds.
(d)


Focus on Problem Solving • page 560

1. $f(x) = \pm 10e^x$ **5.** 20 °C
7. (b) $f(x) = (x^2 - L^2)/(4L) - (L/2)\ln(x/L)$ (c) No
9. (a) 9.8 h (b) $31{,}900\pi \approx 100{,}000\ \text{ft}^2$; $6283\ \text{ft}^2/\text{h}$
(c) 5.1 h
11. $x^2 + (y - 6)^2 = 25$

CHAPTER 8

Exercises 8.1 • page 571

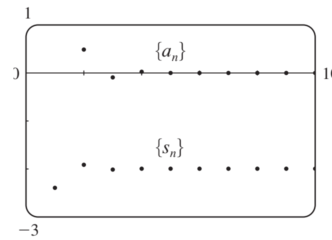
Abbreviation: C, convergent; D, divergent

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.
(b) The terms a_n approach 8 as n becomes large.
(c) The terms a_n become large as n becomes large.
3. $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}$; yes; $\frac{1}{2}$
5. $\left(-\frac{2}{3}\right)^{n-1}$ **7.** $5n - 3$ **9.** D
11. 5 **13.** 0 **15.** 0 **17.** D
19. 0 **21.** 0 **23.** 0 **25.** 0
27. D **29.** $\pi/4$ **31.** 0
33. (a) 1060, 1123.60, 1191.02, 1262.48, 1338.23 (b) D
35. (a) D (b) C **37.** (b) $\left(1 + \sqrt{5}\right)/2$
39. Decreasing; yes
41. Not monotonic; yes
43. Convergent by the Monotonic Sequence Theorem; $5 \leqslant L < 8$
45. $\left(3 + \sqrt{5}\right)/2$ **47.** 62

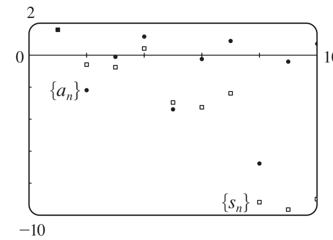
Exercises 8.2 • page 580

1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.
(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

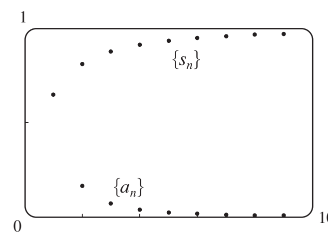
3. $-2.40000, -1.92000$
$-2.01600, -1.99680,$
$-2.00064, -1.99987$
$-2.00003, -1.99999$
$-2.00000, -2.00000$
Convergent, sum $= -2$



5. $1.55741, -0.62763,$
$-0.77018, 0.38764$
$-2.99287, -3.28388$
$-2.41243, -9.21214$
$-9.66446, -9.01610$
Divergent



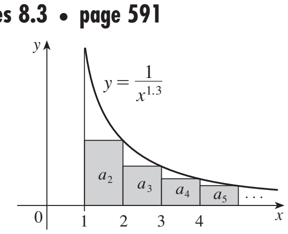
7. 0.64645, 0.80755,
0.87500, 0.91056,
0.93196, 0.94601,
0.95581, 0.96296,
0.96838, 0.97259
Convergent, sum $= 1$



9. (a) C (b) D
11. 3 **13.** 15 **15.** D **17.** D **19.** $\frac{3}{4}$ **21.** $\frac{17}{36}$
23. sin 1 **25.** $\frac{3}{2}$ **27.** D **29.** $\frac{2}{9}$ **31.** $\frac{1138}{333}$
33. $-3 < x < 3$; $x/(3 - x)$ **35.** $|x| > 1$, $x/(x - 1)$
37. $\frac{1}{4}$ **39.** $a_1 = 0$, $a_n = 2/[n(n + 1)]$ for $n > 1$, sum $= 1$
41. (a) $S_n = D(1 - c^n)/(1 - c)$ (b) 5
43. $\left(\sqrt{3} - 1\right)/2$ **45.** $1/[n(n + 1)]$
47. The series is divergent.
51. $\{s_n\}$ is bounded and increasing.
53. (a) $0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1$
55. (a) $\frac{1}{2}, \frac{5}{6}, \frac{23}{24}, \frac{119}{120}$; $[(n + 1)! - 1]/(n + 1)!$ (c) 1

Exercises 8.3 • page 591

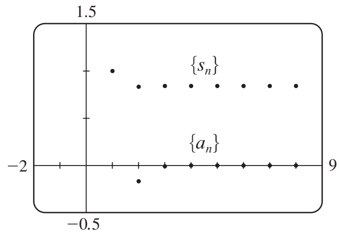
1. C



3. (a) Nothing (b) C
5. p-series; geometric series; $b < -1$; $-1 < b < 1$
7. C **9.** C **11.** C **13.** C **15.** D **17.** C
19. D **21.** C **23.** D **25.** $p > 1$
27. (a) 1.54977, error $\leqslant 0.1$ (b) 1.64522, error $\leqslant 0.005$
(c) $n > 1000$
29. 2.61 **31.** 0.567975, error $\leqslant 0.000\overline{3}$ **37.** Yes

Exercises 8.4 • page 598

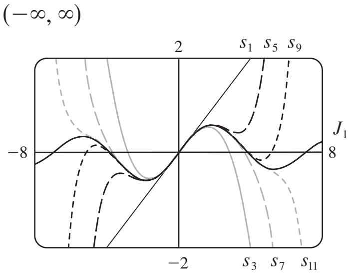
1. (a) A series whose terms are alternately positive and negative
(b) $0 < b_{n+1} \leqslant b_n$ and $\lim_{n\to\infty} b_n = 0$, where $b_n = |a_n|$
(c) $|R_n| \leqslant b_{n+1}$
3. C **5.** C **7.** D
9. An underestimate **11.** $p > 0$ **13.** 7
15. 0.8415



17. 0.6065 **19.** No
21. No **23.** Yes **25.** Yes **27.** Yes **29.** D
31. (a) and (d)
35. (a) $\frac{661}{960} \approx 0.68854$, error < 0.00521 (b) $n \geqslant 11$, 0.693109

Exercises 8.5 • page 604

1. A series of the form $\Sigma_{n=0}^{\infty} c_n(x-a)^n$, where x is a variable and a and the c_n's are constants
3. $1, [-1, 1)$ **5.** $1, (-1, 1)$ **7.** $\infty, (-\infty, \infty)$
9. $\frac{1}{3}, \left[-\frac{1}{3}, \frac{1}{3}\right]$
11. $4, (-4, 4]$ **13.** $1, (0, 2)$ **15.** $2, (-4, 0]$
17. $0, \left\{\frac{1}{2}\right\}$ **19.** (a) Yes (b) No **21.** k^k
23. (a) $(-\infty, \infty)$
(b), (c)



25. $(-1, 1)$, $f(x) = (1+2x)/(1-x^2)$ **27.** 2

Exercises 8.6 • page 610

1. 10 **3.** $\sum_{n=0}^{\infty} (-1)^n x^n, (-1, 1)$ **5.** $\sum_{n=0}^{\infty} x^{3n}, (-1, 1)$

7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{2n}, (-2, 2)$ **9.** $-\sum_{n=0}^{\infty} \frac{1}{5^{n+1}} x^n, (-5, 5)$

11. (a) $\sum_{n=0}^{\infty} (-1)^n (n+1)x^n, R = 1$

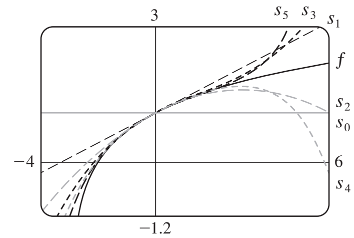
(b) $\frac{1}{2}\sum_{n=0}^{\infty} (-1)^n (n+2)(n+1)x^n, R = 1$

(c) $\frac{1}{2}\sum_{n=2}^{\infty} (-1)^n n(n-1)x^n, R = 1$

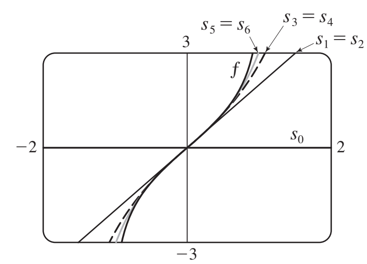
13. $\ln 5 - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}, R = 5$

15. $\sum_{n=3}^{\infty} \frac{n-2}{2^{n-1}} x^n, (-2, 2)$

17. $\ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n3^n} x^n, R = 3$



19. $\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}, R = 1$



21. $C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1}$ **23.** $C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}$
25. 0.199989 **27.** 0.000065 **29.** 0.09531
31. (b) 0.920 **35.** $[-1, 1], [-1, 1), (-1, 1)$

Exercises 8.7 • page 621

1. $b_8 = f^{(8)}(5)/8!$ **3.** $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, R = \infty$

5. $\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} x^n, R = 1$

7. $7 + 5(x-2) + (x-2)^2, R = \infty$

9. $\sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n, R = \infty$ **11.** $\sum_{n=0}^{\infty} (-1)^n (x-1)^n, R = 1$

13. $\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(2n)!}\left(x - \frac{\pi}{4}\right)^{2n} + \frac{1}{(2n+1)!}\left(x - \frac{\pi}{4}\right)^{2n+1}\right]$, $R = \infty$

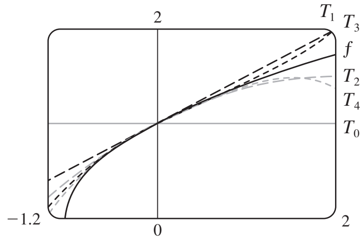
17. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!} x^{2n}, R = \infty$

19. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+2}, R = 1$

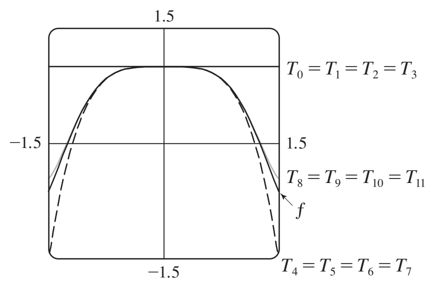
21. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^{n+2}, R = \infty$

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}, R = \infty$

25. $1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n!} x^n$, $R = 1$



27. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{4n}$, $R = \infty$



29. 0.81873

31. $C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$

33. $C + x + \frac{x^4}{8} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n!(3n+1)} x^{3n+1}$

35. 0.310 **37.** 0.09998750 **39.** $\frac{1}{3}$ **41.** $\frac{1}{120}$

43. $1 - \frac{3}{2}x^2 + \frac{25}{24}x^4$ **45.** $-x + \frac{1}{2}x^2 - \frac{1}{3}x^3$ **47.** e^{-x^4}

49. $1/\sqrt{2}$ **51.** $e^3 - 1$

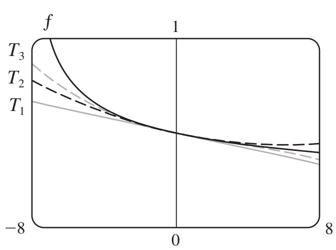
Exercises 8.8 • page 625

1. $1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n!} x^n$, $R = 1$

3. $\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2^{n+4}} x^n$, $R = 2$

5. $\frac{1}{2}x + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!\, 2^{3n+1}} x^{2n+1}$, $R = 2$

7. $\frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-2)}{24^n n!} x^n$, $R = 8$



9. (a) $1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} x^{2n}$

(b) $x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n n!} x^{2n+1}$

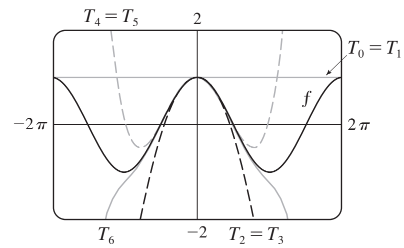
11. (a) $\sum_{n=1}^{\infty} nx^n$ (b) 2

13. (a) $1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n!} x^{2n}$

(b) 99,225

Exercises 8.9 • page 633

1. (a) $T_0(x) = 1 = T_1(x)$, $T_2(x) = 1 - \frac{1}{2}x^2 = T_3(x)$,
$T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 = T_5(x)$,
$T_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$

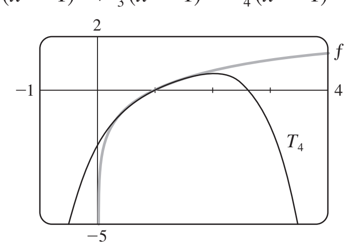


(b)

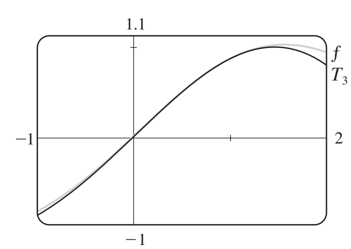
x	f	$T_0 = T_1$	$T_2 = T_3$	$T_4 = T_5$	T_6
$\frac{\pi}{4}$	0.7071	1	0.6916	0.7074	0.7071
$\frac{\pi}{2}$	0	1	−0.2337	0.0200	−0.0009
π	−1	1	−3.9348	0.1239	−1.2114

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

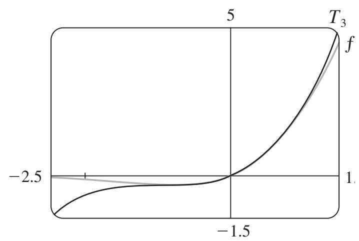
3. $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$



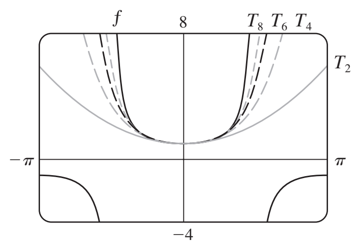
5. $\frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3$



7. $x + x^2 + \frac{1}{3}x^3$



9. $T_8(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8$



11. (a) $2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$ (b) 1.5625×10^{-5}
13. (a) $1 + x^2$ (b) 0.00006
15. (a) $x + \frac{1}{3}x^3$ (b) 0.058 **17.** 0.57358 **19.** 3
21. $-1.037 < x < 1.037$ **23.** 21 m, no

Exercises 8.10 • page 639

1. $c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0 e^x$ **3.** $c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0 e^{x^3/3}$

5. $c_0 \sum_{n=0}^{\infty} \left(-\frac{3}{2}\right)^n \frac{1}{n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-6)^n n!}{(2n+1)!} x^{2n+1}$

7. $\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = e^{x^2/2}$

9. $x + \sum_{n=1}^{\infty} \frac{(-1)^n 2^2 5^2 \cdot \cdots \cdot (3n-1)^2}{(3n+1)!} x^{3n+1}$

Chapter 8 Review • page 640

True-False Quiz

1. False **3.** False **5.** False **7.** False **9.** True
11. True **13.** False **15.** True **17.** False

Exercises

1. $\frac{1}{2}$ **3.** D **5.** D **7.** e^{12} **9.** C **11.** C
13. C **15.** C **17.** C
19. 8 **21.** $\pi/4$ **23.** $\frac{4111}{3330}$ **25.** 0.9721
27. 0.18976224, $|\text{error}| < 6.4 \times 10^{-7}$ **31.** 4, $[-6, 2)$
33. 0.5, $[2.5, 3.5)$

35. $\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(2n)!}\left(x - \frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{(2n+1)!}\left(x - \frac{\pi}{6}\right)^{2n+1} \right]$

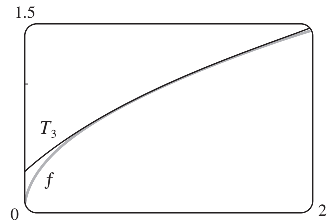
37. $\sum_{n=0}^{\infty} (-1)^n x^{n+2}$, $R = 1$ **39.** $-\sum_{n=1}^{\infty} \frac{x^n}{n}$, $R = 1$

41. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+4}}{(2n+1)!}$, $R = \infty$

43. $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{n!\, 2^{6n+1}} x^n$, $R = 16$

45. $C + \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$

47. (a) $1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$
(b)



(c) 0.000006

49. $-\frac{1}{6}$ **51.** $\sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}$

53. (b) 0 if $x = 0$, $(1/x) - \cot x$ if $x \neq n\pi$, n an integer

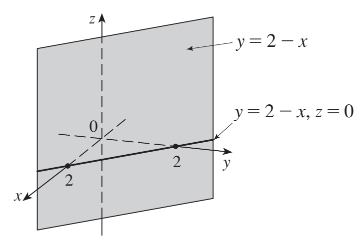
Focus on Problem Solving • page 643

1. $15!/5! = 10{,}897{,}286{,}400$
5. (a) $s_n = 3 \cdot 4^n$, $l_n = 1/3^n$, $p_n = 4^n/3^{n-1}$ (c) $2\sqrt{3}/5$
7. $(-1, 1)$, $(x^3 + 4x^2 + x)/(1 - x)^4$

CHAPTER 9

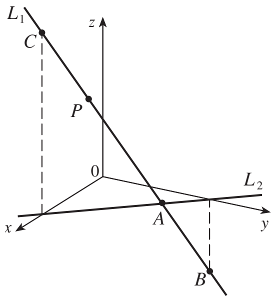
Exercises 9.1 • page 651

1. $(4, 0, -3)$ **3.** Q; R
5. A vertical plane that intersects the xy-plane in the line $y = 2 - x$, $z = 0$ (see graph at right)



7. $|AB| = \sqrt{6}$, $|BC| = \sqrt{33}$, $|CA| = 3\sqrt{3}$; right triangle
9. (a) Yes (b) No
11. $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$
13. $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\sqrt{3}/2$ **15.** (b) $\frac{5}{2}, \frac{1}{2}\sqrt{94}, \frac{1}{2}\sqrt{85}$
17. (a) $(x-2)^2 + (y+3)^2 + (z-6)^2 = 36$
(b) $(x-2)^2 + (y+3)^2 + (z-6)^2 = 4$
(c) $(x-2)^2 + (y+3)^2 + (z-6)^2 = 9$
19. A plane parallel to the xz-plane and 4 units to the left of it
21. A half-space consisting of all points in front of the plane $x = 3$
23. All points on or between the horizontal planes $z = 0$ and $z = 6$
25. All points outside the sphere with radius 1 and center O
27. All points on or inside a circular cylinder of radius 3 with axis the y-axis
29. $y < 0$ **31.** $r^2 < x^2 + y^2 + z^2 < R^2$

33. (a) (2, 1, 4) (b)



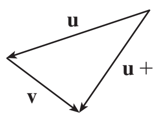
35. $14x - 6y - 10z = 9$, a plane perpendicular to AB

Exercises 9.2 • page 659

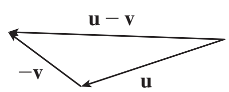
1. (a) Scalar (b) Vector (c) Vector (d) Scalar

3. $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{DA} = \overrightarrow{CB}$, $\overrightarrow{DE} = \overrightarrow{EB}$, $\overrightarrow{EA} = \overrightarrow{CE}$

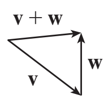
5. (a)



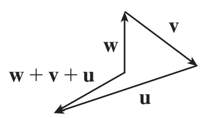
(b)



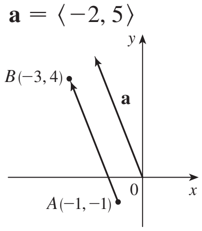
(c)



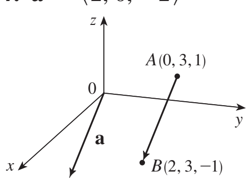
(d)



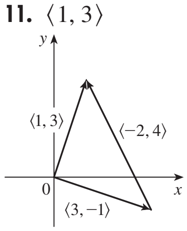
7. $\mathbf{a} = \langle -2, 5 \rangle$



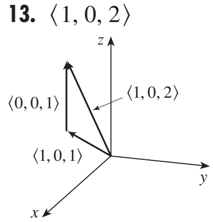
9. $\mathbf{a} = \langle 2, 0, -2 \rangle$



11. $\langle 1, 3 \rangle$



13. $\langle 1, 0, 2 \rangle$



15. $5, \langle 2, 5 \rangle, \langle -10, 1 \rangle, \langle -8, 6 \rangle, \langle 12, 17 \rangle$

17. $\sqrt{6}, \mathbf{i} - \mathbf{j} + 3\mathbf{k}, \mathbf{i} - 3\mathbf{j} - \mathbf{k}, 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $3\mathbf{i} - 2\mathbf{j} + 11\mathbf{k}$

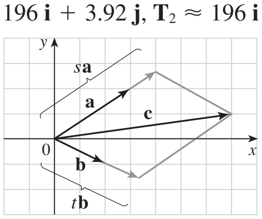
19. $\frac{8}{9}\mathbf{i} - \frac{1}{9}\mathbf{j} + \frac{4}{9}\mathbf{k}$ **21.** $\langle 2, 2\sqrt{3} \rangle$

23. $\mathbf{F} = (6\sqrt{3} - 5\sqrt{2})\mathbf{i} + (6 + 5\sqrt{2})\mathbf{j} \approx 3.32\mathbf{i} + 13.07\mathbf{j}$, $|\mathbf{F}| \approx 13.5$ lb, $\theta \approx 76°$

25. $\sqrt{493} \approx 22.2$ mi/h, N8°W

27. $\mathbf{T}_1 \approx -196\,\mathbf{i} + 3.92\,\mathbf{j}$, $\mathbf{T}_2 \approx 196\,\mathbf{i} + 3.92\,\mathbf{j}$

29. (a), (b)



(d) $s = \frac{9}{7}$, $t = \frac{11}{7}$

31. $\mathbf{a} \approx \langle 0.50, 0.31, 0.81 \rangle$

33. A sphere with radius 1, centered at (x_0, y_0, z_0)

Exercises 9.3 • page 666

1. (b), (c), (d) are meaningful **3.** $90\sqrt{3}$ **5.** -5

7. 32 **9.** $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}, \mathbf{u} \cdot \mathbf{w} = -\frac{1}{2}$

13. $\cos^{-1}\left(\frac{63}{65}\right) \approx 14°$

15. $\cos^{-1}\left(-1/(2\sqrt{7})\right) \approx 101°$

17. (a) Neither (b) Orthogonal (c) Orthogonal (d) Parallel

19. $(\mathbf{i} - \mathbf{j} - \mathbf{k})/\sqrt{3}$ [or $(-\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$]

21. $11/\sqrt{13}, \left\langle \frac{22}{13}, \frac{33}{13} \right\rangle$ **23.** $3/\sqrt{5}, \left\langle \frac{6}{5}, \frac{3}{5}, 0 \right\rangle$

27. $\langle 0, 0, -2\sqrt{10} \rangle$ or any vector of the form $\langle s, t, 3s - 2\sqrt{10} \rangle$, $s, t \in \mathbb{R}$

29. 38 J **31.** $250 \cos 20° \approx 235$ ft-lb **33.** $\frac{13}{5}$

35. $\cos^{-1}\left(1/\sqrt{3}\right) \approx 55°$

Exercises 9.4 • page 674

1. (a) Scalar (b) Meaningless (c) Vector (d) Meaningless (e) Meaningless (f) Scalar

3. 24; into the page **5.** $10.8 \sin 80° \approx 10.6$ J

7. $-\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ **9.** $t^4\mathbf{i} - 2t^3\mathbf{j} + t^2\mathbf{k}$

11. $2\mathbf{i} + 13\mathbf{j} - 8\mathbf{k}$

13. $\langle -2/\sqrt{6}, -1/\sqrt{6}, 1/\sqrt{6} \rangle, \langle 2/\sqrt{6}, 1/\sqrt{6}, -1/\sqrt{6} \rangle$

15. 16 **17.** (a) $\langle 6, 3, 2 \rangle$ (b) $\frac{7}{2}$ **19.** ≈ 417 N

21. 82 **23.** 21 **27.** (b) $\sqrt{97/3}$

33. (a) No (b) No (c) Yes

Exercises 9.5 • page 683

1. (a) True (b) False (c) True (d) False (e) False (f) True (g) False (h) True (i) True (j) False (k) True

3. $\mathbf{r} = (-2\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}) + t(3\mathbf{i} + \mathbf{j} - 8\mathbf{k})$; $x = -2 + 3t, y = 4 + t, z = 10 - 8t$

5. $\mathbf{r} = (\mathbf{i} + 6\mathbf{k}) + t(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$; $x = 1 + t, y = 3t, z = 6 + t$

7. $x = 3, y = 1 + t, z = -1 - 5t$; $x = 3, y - 1 = (z + 1)/(-5)$

9. $x = 2 + 2t, y = 1 + \frac{1}{2}t, z = -3 - 4t$; $(x - 2)/2 = 2y - 2 = (z + 3)/(-4)$

13. (a) $x/2 = (y - 2)/3 = (z + 1)/(-7)$ (b) $\left(-\frac{2}{7}, \frac{11}{7}, 0\right), \left(-\frac{4}{3}, 0, \frac{11}{3}\right), (0, 2, -1)$

15. Skew **17.** Parallel **19.** $-2x + y + 5z = 1$

21. $2x - y + 3z = 0$ **23.** $x + y + z = 2$

25. $33x + 10y + 4z = 190$ **27.** $x - 2y + 4z = -1$

29. $(-3, -1, -2)$ **31.** Neither, 60°

33. Perpendicular

35. (a) $x - 2 = y/(-8) = z/(-7)$ (b) $\cos^{-1}\left(-\sqrt{6}/5\right) \approx 119°$ (or 61°)

37. $(x/a) + (y/b) + (z/c) = 1$

39. $x = 3t, y = 1 - t, z = 2 - 2t$

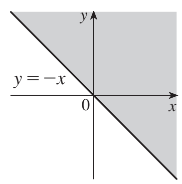
41. P_1 and P_3 are parallel, P_2 and P_4 are identical

43. $\sqrt{22/5}$

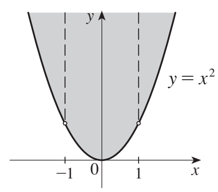
45. $\frac{25}{3}$ **47.** $7\sqrt{6}/18$ **51.** $1/\sqrt{6}$

Exercises 9.6 • page 692

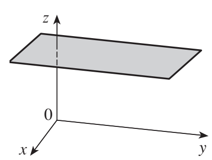
1. (a) 25; a 40-knot wind blowing in the open sea for 15 h will create waves about 25 ft high.
(b) $f(30, t)$ is a function of t giving the wave heights produced by 30-knot winds blowing for t hours.
(c) $f(v, 30)$ is a function of v giving the wave heights produced by winds of speed v blowing for 30 hours.
3. (a) 4 (b) $\mathbb{R}^2$ (c) $[0, \infty)$
5. $\{(x, y) \mid y \geqslant -x\}$



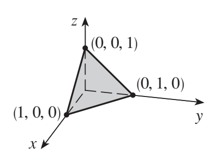
7. $\{(x, y) \mid y \geqslant x^2, x \neq \pm 1\}$



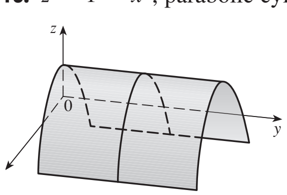
9. $z = 3$, horizontal plane



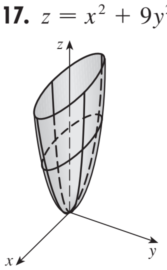
11. $x + y + z = 1$, plane



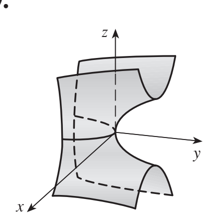
13. $z = 1 - x^2$, parabolic cylinder



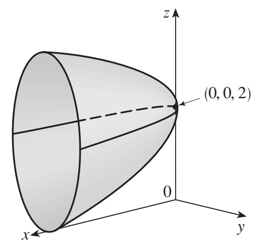
15. (a) VI (b) V (c) I (d) IV (e) II (f) III
17. $z = x^2 + 9y^2$



19.

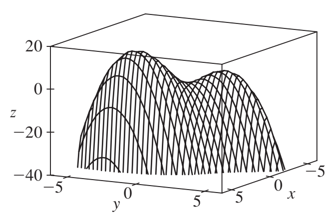


21. elliptic paraboloid



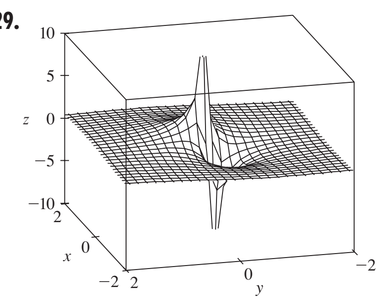
23. (a) A circle of radius 1 centered at the origin
(b) A circular cylinder of radius 1 with axis the z-axis
(c) A circular cylinder of radius 1 with axis the y-axis
25. (a) $x = k$, $y^2 - z^2 = 1 - k^2$, hyperbola $(k \neq \pm 1)$;
$y = k$, $x^2 - z^2 = 1 - k^2$, hyperbola $(k \neq \pm 1)$;
$z = k$, $x^2 + z^2 = 1 + k^2$, circle
(b) The hyperboloid is rotated so that it has axis the y-axis
(c) The hyperboloid is shifted one unit in the negative y-direction

27.



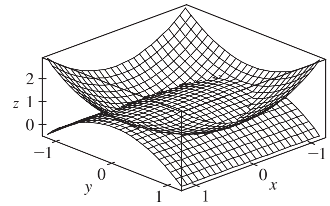
f appears to have a maximum value of about 15. There are two local maximum points but no local minimum point.

29.



The function values approach 0 as x, y become large; as (x, y) approaches the origin, f approaches $\pm\infty$ or 0, depending on the direction of approach.

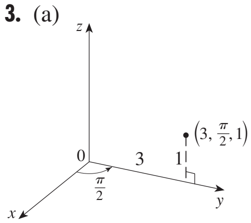
31.



Exercises 9.7 • page 698

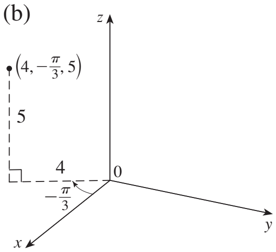
1. See pages 694–695.

3. (a)



(0, 3, 1)

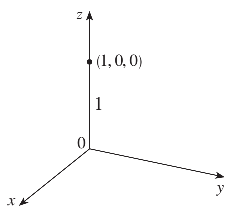
(b)



$(2, -2\sqrt{3}, 5)$

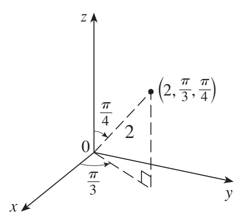
5. (a) $(\sqrt{2}, 7\pi/4, 4)$ (b) $(2, 4\pi/3, 2)$

7. (a)



(0, 0, 1)

(b)



$(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{6}, \sqrt{2})$

9. (a) $(3, \pi, \pi/2)$
(b) $(2\sqrt{2}, \pi/2, 3\pi/4)$

11. Circular cylinder, radius 3, axis the z-axis

13. Half-cone

15. Circular paraboloid

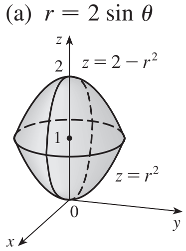
17. Circular cylinder, radius 1, axis parallel to the z-axis

19. Sphere, radius 5, center the origin

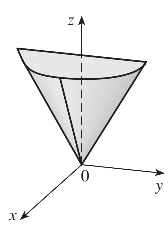
21. (a) $r^2 + z^2 = 16$ (b) $\rho = 4$

23. (a) $r = 2 \sin \theta$ (b) $\rho \sin \phi = 2 \sin \theta$

25.



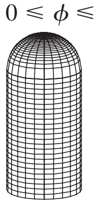
27.



29. Cylindrical coordinates: $6 \leqslant r \leqslant 7$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant z \leqslant 20$

31. $0 \leqslant \phi \leqslant \pi/4$, $0 \leqslant \rho \leqslant \cos \phi$

33.



Chapter 9 Review • page 700

True-False Quiz

1. True **3.** True **5.** True **7.** True **9.** True
11. False **13.** False

Exercises

1. (a) $(x + 1)^2 + (y - 2)^2 + (z - 1)^2 = 69$
(b) $(y - 2)^2 + (z - 1)^2 = 68$, $x = 0$
(c) center $(4, -1, -3)$, radius 5

3. $\mathbf{u} \cdot \mathbf{v} = 3\sqrt{2}$; $|\mathbf{u} \times \mathbf{v}| = 3\sqrt{2}$; out of the page

5. $-2, -4$ **7.** (a) 2 (b) -2 (c) -2 (d) 0

9. $\cos^{-1}(\frac{1}{3}) \approx 71°$ **11.** (a) $\langle 4, -3, 4 \rangle$ (b) $\sqrt{41}/2$

13. 166 N, 114 N **15.** $x = 1 + 2t$, $y = 2 - t$, $z = 4 + 3t$

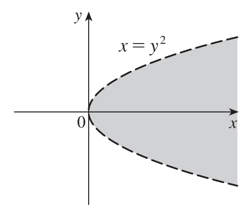
17. $x = 1 + 4t$, $y = -3t$, $z = 1 + 5t$

19. $x + 2y + 5z = 8$

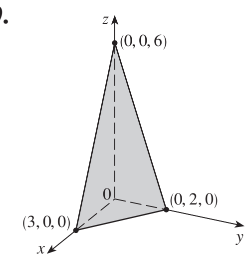
21. $x + y + z = 4$

23. Skew **25.** $22/\sqrt{26}$

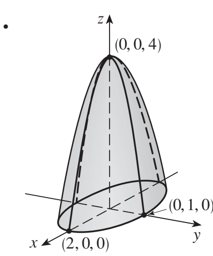
27. $\{(x, y) \mid x > y^2\}$



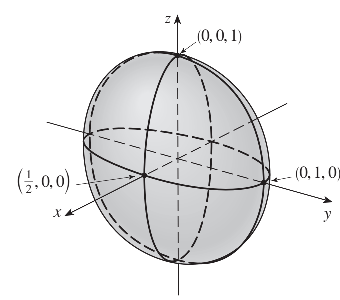
29.



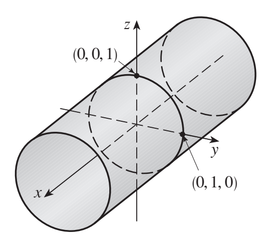
31.



33. Ellipsoid



35. Circular cylinder



37. $(\sqrt{3}, 1, 2)$, $(2\sqrt{2}, \pi/6, \pi/4)$

39. $(1, \sqrt{3}, 2\sqrt{3})$, $(2, \pi/3, 2\sqrt{3})$

41. $r^2 + z^2 = 4$, $\rho = 2$ **43.** $z = 4r^2$

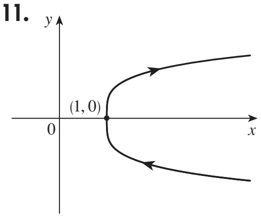
Focus on Problem Solving • page 703

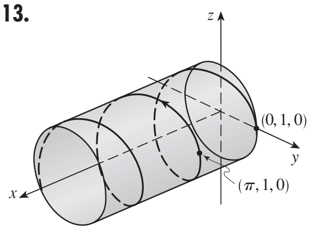
1. $(\sqrt{3} - 1.5)$ m
3. (a) $(x + 1)/(-2c) = (y - c)/(c^2 - 1) = (z - c)/(c^2 + 1)$
(b) $x^2 + y^2 = t^2 + 1, z = t$ (c) $4\pi/3$

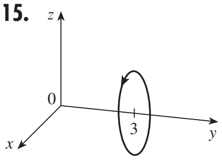
CHAPTER 10

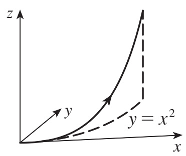
Exercises 10.1 • page 710

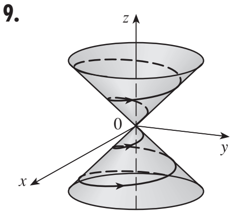
1. [1, 5] **3.** $\langle 1, 0, 0 \rangle$ **5.** VI **7.** IV **9.** V

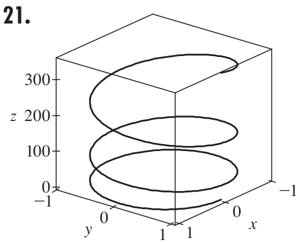
11.


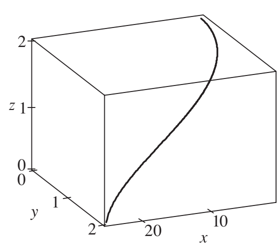
13.


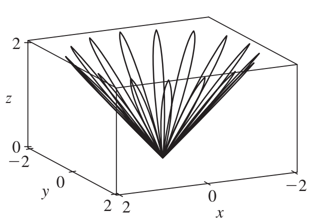
15.


17.


19.


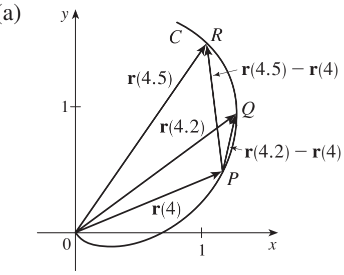
21.


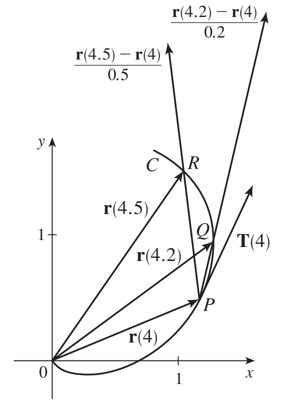
23.


25.


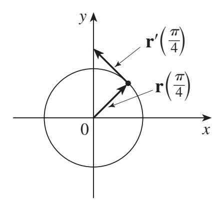
29. $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}(t^2 - 1)\,\mathbf{j} + \frac{1}{2}(t^2 + 1)\,\mathbf{k}$
31. $x = 2\cos t, y = 2\sin t, z = 4\cos^2 t$

Exercises 10.2 • page 716

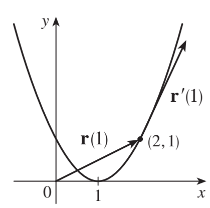
1. (a)


(b), (d)


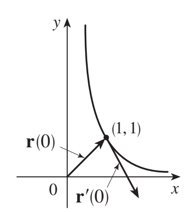
(c) $\mathbf{r}'(4) = \lim_{h \to 0} \dfrac{\mathbf{r}(4 + h) - \mathbf{r}(4)}{h}$; $\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$

3. (a), (c)


(b) $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$

5. (a), (c)


(b) $\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j}$

7. (a), (c)


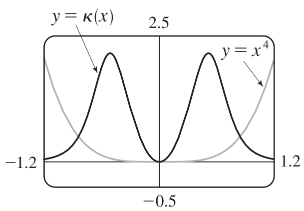
(b) $\mathbf{r}'(t) = e^t\,\mathbf{i} - 2e^{-2t}\,\mathbf{j}$

9. $\mathbf{r}'(t) = \langle 2t, -1, 1/(2\sqrt{t}) \rangle$
11. $\mathbf{r}'(t) = 2te^{t^2}\,\mathbf{i} + [3/(1 + 3t)]\,\mathbf{k}$
13. $\mathbf{r}'(t) = \mathbf{b} + 2t\mathbf{c}$ **15.** $\frac{3}{5}\,\mathbf{j} + \frac{4}{5}\,\mathbf{k}$
17. $\langle 1, 2t, 3t^2 \rangle$, $\langle 1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14} \rangle$, $\langle 0, 2, 6t \rangle$, $\langle 6t^2, -6t, 2 \rangle$
19. $x = 1 + 5t, y = 1 + 4t, z = 1 + 3t$
21. $x = 1 - t, y = t, z = 1 - t$
23. $x = \frac{1}{4}\pi + t, y = 1 - t, z = 1 + t$
25. (a) Not smooth (b) Smooth (c) Not smooth
27. 66° **29.** $4\,\mathbf{i} - 3\,\mathbf{j} + 5\,\mathbf{k}$

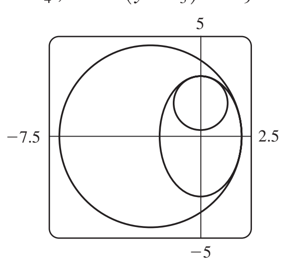
31. $\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \left[(4-\pi)/(4\sqrt{2})\right]\mathbf{k}$
33. $e^t\,\mathbf{i} + t^2\,\mathbf{j} + (t\ln t - t)\mathbf{k} + \mathbf{C}$
35. $\frac{1}{3}t^3\mathbf{i} + (t^4+1)\mathbf{j} - \frac{1}{3}t^3\mathbf{k}$
41. $1 - 4t\cos t + 11t^2\sin t + 3t^3\cos t$

Exercises 10.3 • page 723

1. $20\sqrt{29}$ **3.** $e - e^{-1}$ **5.** 9.5706
7. $\mathbf{r}(t(s)) = (1 + s/\sqrt{2})\sin[\ln(1 + s/\sqrt{2})]\mathbf{i}$
$+ (1 + s/\sqrt{2})\cos[\ln(1 + s/\sqrt{2})]\mathbf{j}$
9. $\mathbf{r}(t(s)) = 3\sin(s/5)\mathbf{i} + (4s/5)\mathbf{j} + 3\cos(s/5)\mathbf{k}$
11. (a) $\langle (2/\sqrt{29})\cos t, 5/\sqrt{29}, (-2/\sqrt{29})\sin t\rangle$,
$\langle -\sin t, 0, -\cos t\rangle$
(b) $\frac{2}{29}$
13. (a) $\langle t^2, 2t, 2\rangle/(t^2+2)$, $\langle 2t, 2-t^2, -2t\rangle/(t^2+2)$
(b) $2/(t^2+2)^2$
15. $2/(4t^2+1)^{3/2}$ **17.** $\sqrt{2}/(1+\cos^2 t)^{3/2}$ **19.** $\sqrt{2}/4$
21. $6|x|/(1+9x^4)^{3/2}$ **23.** $15\sqrt{x}/(1+100x^3)^{3/2}$
25. $(-\frac{1}{2}\ln 2, 1/\sqrt{2})$; approaches 0
27. (a) P (b) 1.3, 0.7
29.



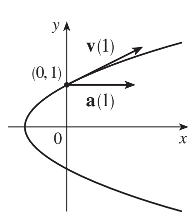
31. a is $y = f(x)$, b is $y = \kappa(x)$
33. $1/(\sqrt{2}e^t)$ **35.** $\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle, \langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\rangle, \langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\rangle$
37. $y = 6x + \pi$, $x + 6y = 6\pi$
39. $(x + \frac{5}{2})^2 + y^2 = \frac{81}{4}$, $x^2 + (y - \frac{5}{3})^2 = \frac{16}{9}$



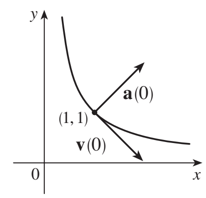
41. $(-1, -3, 1)$ **49.** 2.07×10^{10} Å ≈ 2 m

Exercises 10.4 • page 733

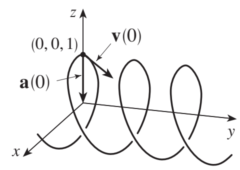
1. (a) $1.8\mathbf{i} - 3.8\mathbf{j} - 0.7\mathbf{k}$, $2.0\mathbf{i} - 2.4\mathbf{j} - 0.6\mathbf{k}$,
$2.8\mathbf{i} + 1.8\mathbf{j} - 0.3\mathbf{k}$, $2.8\mathbf{i} + 0.8\mathbf{j} - 0.4\mathbf{k}$
(b) $2.4\mathbf{i} - 0.8\mathbf{j} - 0.5\mathbf{k}$, 2.58
3. $\mathbf{v}(t) = \langle 2t, 1\rangle$
$\mathbf{a}(t) = \langle 2, 0\rangle$
$|\mathbf{v}(t)| = \sqrt{4t^2+1}$



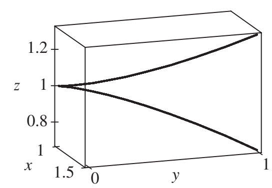
5. $\mathbf{v}(t) = e^t\,\mathbf{i} - e^{-t}\,\mathbf{j}$
$\mathbf{a}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$
$|\mathbf{v}(t)| = \sqrt{e^{2t} + e^{-2t}}$



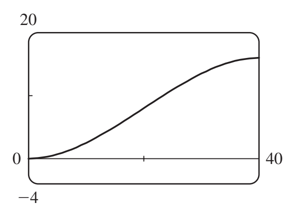
7. $\mathbf{v}(t) = \cos t\,\mathbf{i} + \mathbf{j} - \sin t\,\mathbf{k}$
$\mathbf{a}(t) = -\sin t\,\mathbf{i} - \cos t\,\mathbf{k}$
$|\mathbf{v}(t)| = \sqrt{2}$

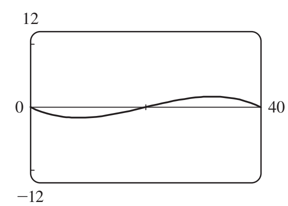


9. $\langle 1, 2t, 3t^2\rangle$, $\langle 0, 2, 6t\rangle$, $\sqrt{1 + 4t^2 + 9t^4}$
11. $\sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}$, $e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$, $e^t + e^{-t}$
13. $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + t\,\mathbf{k}$, $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$
15. (a) $\mathbf{r}(t) = (1 + \frac{1}{2}t^2)\mathbf{i} + t^2\,\mathbf{j} + (1 + \frac{1}{3}t^3)\mathbf{k}$
(b)



17. $t = 4$
19. $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k}$, $|\mathbf{v}(t)| = \sqrt{25t^2 + 2}$
21. (a) ≈ 22 km (b) ≈ 3.2 km (c) 500 m/s
23. 30 m/s
25. $\approx 10.2°$, $\approx 79.8°$
27. (a) 16 m (b) $\approx 23.6°$ upstream

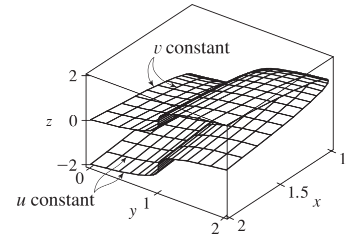




29. $6t$, 6 **31.** 0, 1 **33.** 4.5 cm/s², 9.0 cm/s² **35.** $t = 1$

Exercises 10.5 • page 740

1. Circular paraboloid with axis the z-axis
3. Circular cylinder with axis the x-axis
5.



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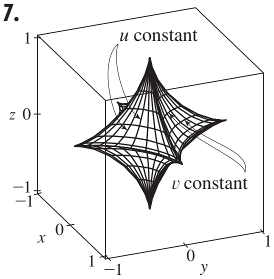

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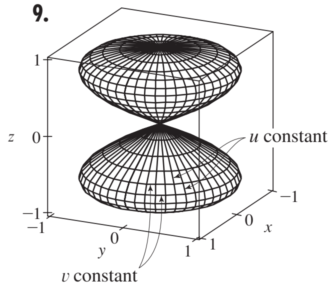


11. IV **13.** I **15.** II

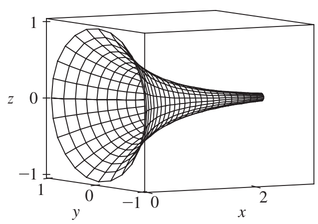
17. $x = 1 + u + v, y = 2 + u - v, z = -3 - u + v$

19. $x = x, z = z, y = \sqrt{1 - x^2 + z^2}$

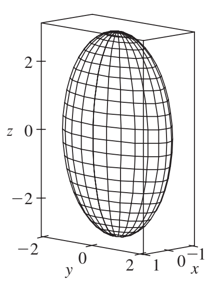
21. $x = 2\sin\phi\cos\theta, y = 2\sin\phi\sin\theta,$
$z = 2\cos\phi, 0 \leqslant \phi \leqslant \pi/4, 0 \leqslant \theta \leqslant 2\pi$
$\left[\text{or } x = x, y = y, z = \sqrt{4 - x^2 - y^2}, 2 \leqslant x^2 + y^2 \leqslant 4\right]$

23. $x = r\cos\theta, y = r\sin\theta, z = 5, 0 \leqslant r \leqslant 4, 0 \leqslant \theta \leqslant 2\pi$
$[\text{or } x = x, y = y, z = 5, x^2 + y^2 \leqslant 16]$

27. $x = x, y = e^{-x}\cos\theta,$
$z = e^{-x}\sin\theta, 0 \leqslant x \leqslant 3,$
$0 \leqslant \theta \leqslant 2\pi$



29. (b)



31. (a) Direction reverses (b) Number of coils doubles

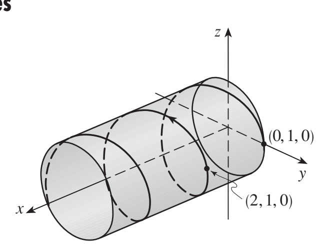
Chapter 10 Review • page 742

True-False Quiz

1. True **3.** False **5.** False **7.** False **9.** True

Exercises

1. (a)



(b) $\mathbf{r}'(t) = \mathbf{i} - \pi\sin\pi t\,\mathbf{j} + \pi\cos\pi t\,\mathbf{k},$
$\mathbf{r}''(t) = -\pi^2\cos\pi t\,\mathbf{j} - \pi^2\sin\pi t\,\mathbf{k}$

3. $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + (5 - 4\cos t)\mathbf{k}, 0 \leqslant t \leqslant 2\pi$

5. $\frac{1}{3}\mathbf{i} - (2/\pi^2)\mathbf{j} + (2/\pi)\mathbf{k}$ **7.** 15.9241 **9.** $\pi/2$

11. (a) $\langle t^2, t, 1\rangle/\sqrt{t^4 + t^2 + 1}$
(b) $\langle 2t, 1 - t^4, -2t^3 - t\rangle/\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}$
(c) $\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}/(t^4 + t^2 + 1)^2$

13. $12/17^{3/2}$ **15.** $x - 2y + 2\pi = 0$

17. $\mathbf{v}(t) = (1 + \ln t)\mathbf{i} + \mathbf{j} - e^{-t}\mathbf{k},$
$|\mathbf{v}(t)| = \sqrt{2 + 2\ln t + (\ln t)^2 + e^{-2t}}, \mathbf{a}(t) = (1/t)\mathbf{i} + e^{-t}\mathbf{k}$

19. (a) About 3.8 ft above the ground, 60.8 ft from the athlete
(b) ≈21.4 ft (c) ≈64.2 ft from the athlete

21. $x = 2\sin\phi\cos\theta, y = 2\sin\phi\sin\theta, z = 2\cos\phi,$
$0 \leqslant \theta \leqslant 2\pi, \pi/3 \leqslant \phi \leqslant 2\pi/3$

23. (c) $-2e^{-t}\mathbf{v}_d + e^{-t}\mathbf{R}$

25. (b) $P(x) = 3x^5 - 8x^4 + 6x^3$; no

Focus on Problem Solving • page 745

1. (a) $\mathbf{v} = \omega R(-\sin\omega t\,\mathbf{i} + \cos\omega t\,\mathbf{j})$ (c) $\mathbf{a} = -\omega^2\mathbf{r}$

3. (a) 90°, $v_0^2/(2g)$ **5.** (b) $\mathbf{R}(t) = (m/k)(1 - e^{-kt/m})\mathbf{v}_0 +$
$(gm/k)[(m/k)(1 - e^{-kt/m}) - t]\mathbf{j}$

7. (a) ≈0.94 ft to the right of the table's edge, ≈15 ft/s
(b) ≈7.6° (c) ≈2.13 ft to the right of the table's edge

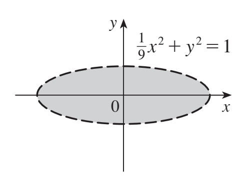
CHAPTER 11

Exercises 11.1 • page 756

1. (a) −7; a temperature of 8 °C with wind blowing at 60 km/h feels equivalent to about −7 °C without wind.
(b) When the temperature is −12 °C, what wind speed gives a wind-chill of −26 °C? 20 km/h
(c) With a wind speed of 80 km/h, what temperature gives a wind-chill of −14 °C? 4 °C
(d) A function of wind speed that gives wind-chill values when the temperature is −4 °C
(e) A function of temperature that gives wind-chill values when the wind speed is 50 km/h

3. Yes

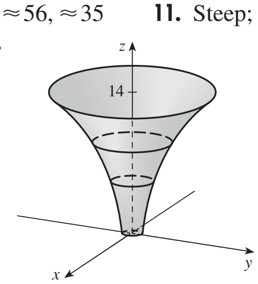
5. $\{(x, y) \mid \frac{1}{9}x^2 + y^2 < 1\}$



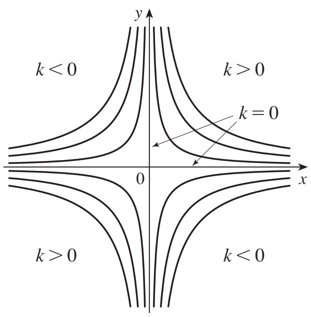
7. (a) e (b) $\{(x, y, z) \mid z \geqslant x^2 + y^2\}$ (c) $[1, \infty)$

9. ≈56, ≈35 **11.** Steep; nearly flat

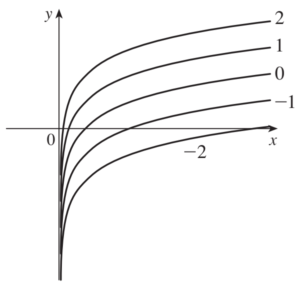
13.



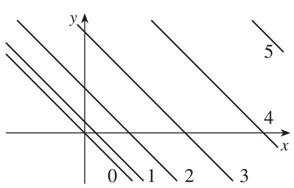
15. $xy = k$



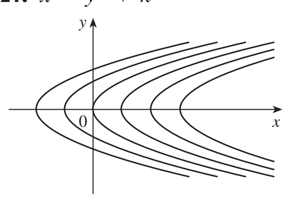
17.



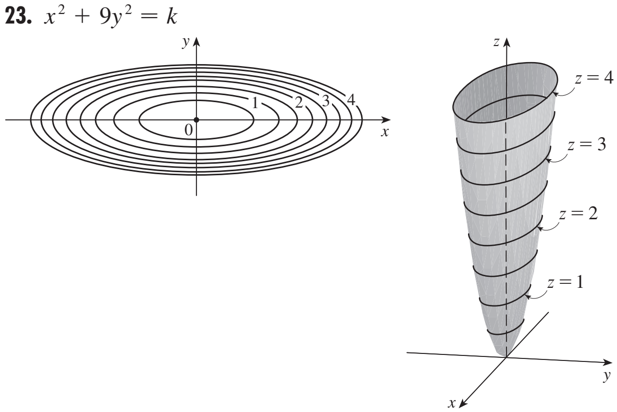
19. $\sqrt{x+y} = k$



21. $x = y^2 + k$



23. $x^2 + 9y^2 = k$



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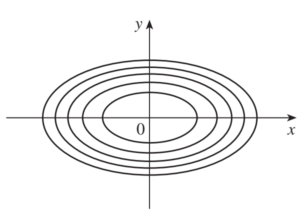

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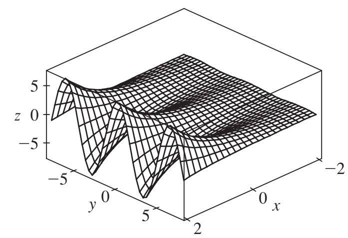

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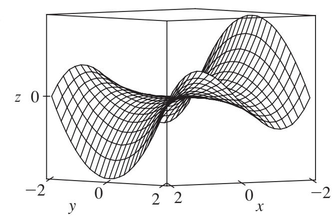


31. (a) B (b) III **33.** (a) F (b) V
35. (a) D (b) IV **37.** Family of parallel planes
39. Family of hyperboloids of one or two sheets with axis the y-axis
41. (a) Shift the graph of f upward 2 units
(b) Stretch the graph of f vertically by a factor of 2
(c) Reflect the graph of f about the xy-plane
(d) Reflect the graph of f about the xy-plane and then shift it upward 2 units
43. If $c = 0$, the graph is a cylindrical surface. For $c > 0$, the level curves are ellipses. The graph curves upward as we leave the origin, and the steepness increases as c increases. For $c < 0$, the level curves are hyperbolas. The graph curves upward in the y-direction and downward, approaching the xy-plane, in the x-direction giving a saddle-shaped appearance near (0, 0, 1).
45. (b) $y = 0.75x + 0.01$

Exercises 11.2 • page 765

1. Nothing; if f is continuous, $f(3, 1) = 6$ **3.** $-\frac{5}{2}$
5. 2025 **7.** Does not exist **9.** Does not exist
11. 0 **13.** Does not exist **15.** 2 **17.** Does not exist
19. The graph shows that the function approaches different numbers along different lines.
21. $h(x, y) = 4x^2 + 9y^2 + 12xy - 24x - 36y + 36 + \sqrt{2x + 3y - 6}$; $\{(x, y) \mid 2x + 3y \geqslant 6\}$
23. Along the line $y = x$ **25.** $\{(x, y) \mid y \neq x^2\}$
27. $\{(x, y) \mid y \geqslant 0\}$ **29.** $\{(x, y, z) \mid z \neq x^2 + y^2\}$
31. $\{(x, y) \mid (x, y) \neq (0, 0)\}$
33. 0 **35.** 0

Exercises 11.3 • page 776

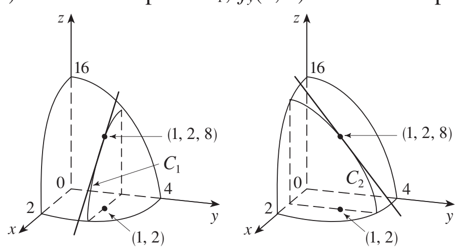
1. (a) The rate of change of temperature as longitude varies, with latitude and time fixed; the rate of change as only latitude varies; the rate of change as only time varies.
(b) Positive, negative, positive
3. (a) $f_T(12, 20) \approx 1.375$; for a temperature of 12 °C and wind speed of 20 km/h, the wind-chill index rises by 1.375 °C for each degree the temperature increases.

$f_v(12, 20) \approx -0.3$; for a temperature of 12 °C and wind speed of 20 km/h, the wind-chill index decreases by 0.3 °C for each km/h the wind speed increases.
(b) Positive, negative (c) 0

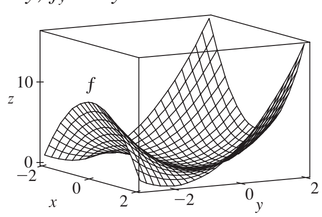
5. (a) Positive (b) Negative

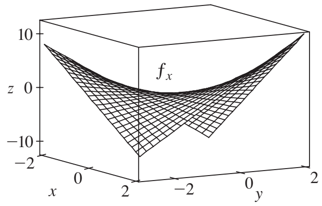
7. $c = f, b = f_x, a = f_y$

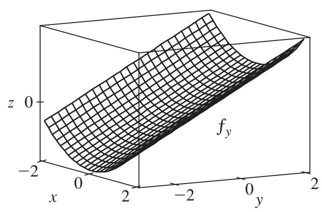
9. $f_x(1, 2) = -8 =$ slope of C_1, $f_y(1, 2) = -4 =$ slope of C_2



11. $f_x = 2x + 2xy, f_y = 2y + x^2$







13. $f_x(x, y) = 3, f_y(x, y) = -8y^3$

15. $\partial z/\partial x = e^{3y}, \partial z/\partial y = 3xe^{3y}$

17. $f_x(x, y) = 2y/(x + y)^2, f_y(x, y) = -2x/(x + y)^2$

19. $\partial w/\partial\alpha = \cos\alpha\cos\beta, \partial w/\partial\beta = -\sin\alpha\sin\beta$

21. $f_u = v/(u^2 + v^2), f_v = -u/(u^2 + v^2)$

23. $\partial z/\partial x = 1/\sqrt{x^2 + y^2}, \partial z/\partial y = y/\left(x^2 + y^2 + x\sqrt{x^2 + y^2}\right)$

25. $f_x = y^2z^3, f_y = 2xyz^3 + 3z, f_z = 3xy^2z^2 + 3y$

27. $\partial w/\partial x = 1/(x + 2y + 3z), \partial w/\partial y = 2/(x + 2y + 3z),$
$\partial w/\partial z = 3/(x + 2y + 3z)$

29. $\partial u/\partial x = e^{-t}\sin\theta, \partial u/\partial t = -xe^{-t}\sin\theta, \partial u/\partial\theta = xe^{-t}\cos\theta$

31. $f_x = 1/(z - t), f_y = 1/(t - z), f_z = (y - x)/(z - t)^2,$
$f_t = (x - y)/(z - t)^2$

33. $\partial u/\partial x_i = x_i/\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ **35.** $\frac{3}{5}$ **37.** $-\frac{1}{3}$

39. $f_x(x, y) = 2x - y, f_y(x, y) = 4y - x$

41. $(y - z)/(x - y), (x + z)/(x - y)$

43. $(x - y - z)/(x + z), (y - x)/(x + z)$

45. (a) $f'(x), g'(y)$ (b) $f'(x + y), f'(x + y)$

47. $f_{xx} = 12x^2 - 6y^3, f_{xy} = -18xy^2 = f_{yx}, f_{yy} = -18x^2y$

49. $u_{ss} = e^{-s}\sin t, u_{st} = -e^{-s}\cos t = u_{ts}, u_{tt} = -e^{-s}\sin t$

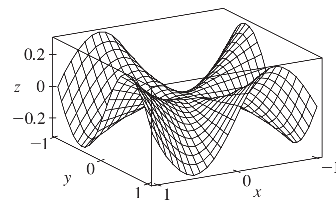
53. $-48xy$ **55.** $48x^3y^3z^2$ **57.** $-\sin y$

59. $\approx 12.2, \approx 16.8, \approx 23.25$

69. R^2/R_1^2 **73.** No **75.** $x = 1 + t, y = 2, z = 2 - 2t$

77. -2

79. (a)



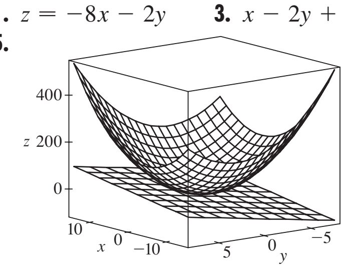
(b) $f_x(x, y) = \dfrac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$, $f_y(x, y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}$

(c) 0, 0 (e) No, since f_{xy} and f_{yx} are not continuous.

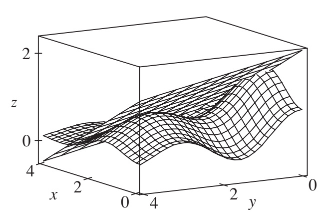
Exercises 11.4 • page 788

1. $z = -8x - 2y$ **3.** $x - 2y + z = 4$

5.



7.



9. $2x + \frac{1}{4}y - 1$ **11.** $\frac{1}{2}x + y + \frac{1}{4}\pi - \frac{1}{2}$

13. $-\frac{2}{3}x - \frac{7}{3}y + \frac{20}{3}; 2.84\overline{6}$ **15.** $\frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z$; 6.9914

17. $4T + H - 329$; 129 °F

19. $du = e^t\sin\theta\,dt + e^t\cos\theta\,d\theta$

21. $dw = (x^2 + y^2 + z^2)^{-1}(x\,dx + y\,dy + z\,dz)$

23. $\Delta z = 0.9225, dz = 0.9$ **25.** 5.4 cm² **27.** 16 cm³

29. 150 **31.** $\frac{1}{17} \approx 0.059\ \Omega$ **33.** $3x - y + 3z = 3$

35. $x = 0$ **37.** $\varepsilon_1 = \Delta x, \varepsilon_2 = \Delta y$

Exercises 11.5 • page 796

1. $\pi\cos x\cos y - (\sin x\sin y)/\left(2\sqrt{t}\right)$

3. $e^{y/z}[2t - (x/z) - (2xy/z^2)]$

5. $\partial z/\partial s = 2x + y + xt + 2yt, \partial z/\partial t = 2x + y + xs + 2ys$

7. $\dfrac{\partial z}{\partial s} = e^r\left(t\cos\theta - \dfrac{s}{\sqrt{s^2 + t^2}}\sin\theta\right),$
$\dfrac{\partial z}{\partial t} = e^r\left(s\cos\theta - \dfrac{t}{\sqrt{s^2 + t^2}}\sin\theta\right)$

9. 62

11. $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r}, \dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s},$
$\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t}$

13. $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial p}\dfrac{\partial p}{\partial x} + \dfrac{\partial v}{\partial q}\dfrac{\partial q}{\partial x} + \dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial x},$
$\dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial p}\dfrac{\partial p}{\partial y} + \dfrac{\partial v}{\partial q}\dfrac{\partial q}{\partial y} + \dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial y},$
$\dfrac{\partial v}{\partial z} = \dfrac{\partial v}{\partial p}\dfrac{\partial p}{\partial z} + \dfrac{\partial v}{\partial q}\dfrac{\partial q}{\partial z} + \dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial z}$

15. 2, 0 **17.** 0, 0, 4

19. $\partial u/\partial p = 2(z - x)/(y + z)^2 = -t/p^2$, $\partial u/\partial r = 0$,
$\partial u/\partial t = 2/(y + z) = 1/p$

21. $\dfrac{\sin(x - y) + e^y}{\sin(x - y) - xe^y}$

23. $-\dfrac{y^2 + 2xz}{2yz + x^2}, -\dfrac{2xy + z^2}{2yz + x^2}$

25. $-(e^y + ze^x)/(y + e^x), -(xe^y + z)/(y + e^x)$

27. 2 °C/s

29. ≈ -0.33 m/s per minute

31. (a) 6 m^3/s (b) 10 m^2/s (c) 0 m/s **33.** -0.27 L/s

35. (a) $\partial z/\partial r = (\partial z/\partial x)\cos\theta + (\partial z/\partial y)\sin\theta$,
$\partial z/\partial\theta = -(\partial z/\partial x)r\sin\theta + (\partial z/\partial y)r\cos\theta$

41. $4rs\,\partial^2 z/\partial x^2 + (4r^2 + 4s^2)\partial^2 z/\partial x\,\partial y + 4rs\,\partial^2 z/\partial y^2 + 2\,\partial z/\partial y$

Exercises 11.6 • page 808

1. ≈ -0.1 millibar/mi **3.** 0.83 **5.** $\frac{5}{16}\sqrt{3} + \frac{1}{4}$

7. (a) $\nabla f(x, y) = \langle 5y^2 - 12x^2y, 10xy - 4x^3\rangle$
(b) $\langle -4, 16\rangle$ (c) 172/13

9. (a) $\nabla f(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$
(b) $\langle 4, -4, 12\rangle$ (c) $20/\sqrt{3}$

11. 23/10 **13.** 4/9 **15.** $-\pi/(4\sqrt{3})$ **17.** 2/5

19. 1, $\langle 0, 1\rangle$ **21.** $\sqrt{11}$, $\langle 1, -1, -3\rangle$

23. (b) $\langle -12, 92\rangle$ **25.** All points on the line $y = x + 1$

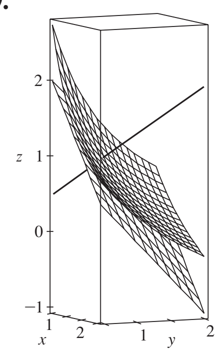
27. (a) $-40/(3\sqrt{3})$

29. (a) $32/\sqrt{3}$ (b) $\langle 38, 6, 12\rangle$ (c) $2\sqrt{406}$ **31.** $\frac{327}{13}$

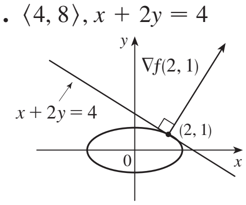
35. (a) $4x - 2y + 3z = 21$ (b) $\dfrac{x - 4}{8} = \dfrac{y + 1}{-4} = \dfrac{z - 1}{6}$

37. (a) $x + y - z = 1$ (b) $x - 1 = y = -z$

39.



41. $\langle 4, 8\rangle$, $x + 2y = 4$



45. $(\pm\sqrt{6}/3, \mp 2\sqrt{6}/3, \pm\sqrt{6}/2)$

49. $x = -1 - 10t, y = 1 - 16t, z = 2 - 12t$

53. If $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, then $af_x + bf_y$ and $cf_x + df_y$ are known, so we solve linear equations for f_x and f_y.

Exercises 11.7 • page 818

1. (a) f has a local minimum at (1, 1).
(b) f has a saddle point at (1, 1).

3. Local minimum at (1, 1), saddle point at (0, 0)

5. Maximum $f(-1, \frac{1}{2}) = 11$

7. Minimum $f(0, 0) = 4$, saddle points $(\pm\sqrt{2}, -1)$

9. Saddle point (1, 2) **11.** None

13. Saddle points $(0, n\pi)$, n an integer

15. Maximum $f(0, 0) = 2$, minimum $f(0, 2) = -2$, saddle points $(\pm 1, 1)$

17. Maximum $f(\pi/3, \pi/3) = 3\sqrt{3}/2$, minimum $f(5\pi/3, 5\pi/3) = -3\sqrt{3}/2$

19. Minima $f(-1.714, 0) \approx -9.200$, $f(1.402, 0) \approx 0.242$, saddle point (0.312, 0), lowest point $(-1.714, 0, -9.200)$

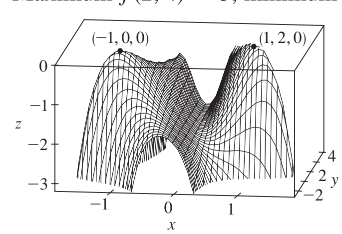
21. Maxima $f(-1.267, 0) \approx 1.310$, $f(1.629, \pm 1.063) \approx 8.105$, saddle points $(-0.259, 0)$, (1.526, 0), highest points $(1.629, \pm 1.063, 8.105)$

23. Maximum $f(2, 0) = 9$, minimum $f(0, 3) = -14$

25. Maximum $f(\pm 1, 1) = 7$, minimum $f(0, 0) = 4$

27. Maximum $f(2, 4) = 3$, minimum $f(-2, 4) = -9$

29.



31. $\sqrt{3}$ **33.** (0, 0, 1), (0, 0, −1) **35.** $\frac{100}{3}, \frac{100}{3}, \frac{100}{3}$

37. $16/\sqrt{3}$ **39.** $\frac{4}{3}$ **41.** Cube, edge length $c/12$

43. Square base of side 40 cm, height 20 cm

Exercises 11.8 • page 827

1. $\approx$59, 30 **3.** Maxima $f(\pm 1, 0) = 1$, minima $f(0, \pm 1) = -1$

5. Maxima $f(\pm 2, 1) = 4$, minima $f(\pm 2, -1) = -4$

7. Maximum $f(1, 3, 5) = 70$, minimum $f(-1, -3, -5) = -70$

9. Maximum $2/\sqrt{3}$, minimum $-2/\sqrt{3}$

11. Maximum $\sqrt{3}$, minimum 1

13. Maximum $f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 2$, minimum $f(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) = -2$

15. Maximum $f(1, \sqrt{2}, -\sqrt{2}) = 1 + 2\sqrt{2}$, minimum $f(1, -\sqrt{2}, \sqrt{2}) = 1 - 2\sqrt{2}$

17. Maximum $\frac{3}{2}$, minimum $\frac{1}{2}$

19. Maxima $f(\pm 1/\sqrt{2}, \mp 1/(2\sqrt{2})) = e^{1/4}$, minima $f(\pm 1/\sqrt{2}, \pm 1/(2\sqrt{2})) = e^{-1/4}$

25–35. See Exercises 31–41 in Section 11.7.
37. Nearest $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, farthest $(-1, -1, 2)$
39. Maximum ≈ 9.7938, minimum ≈ -5.3506
41. (a) c/n (b) When $x_1 = x_2 = \cdots = x_n$

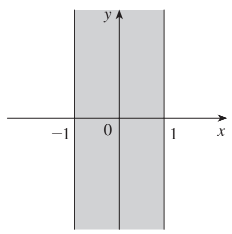
Chapter 11 Review • page 832

True-False Quiz

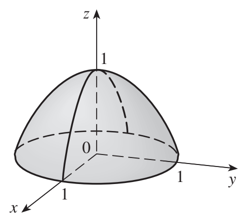
1. True **3.** False **5.** False **7.** True **9.** False
11. True

Exercises

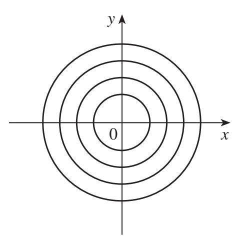
1. $\{(x, y) \mid -1 \leqslant x \leqslant 1\}$



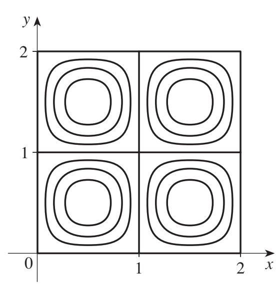
3.



5.



7.



9. $\frac{2}{3}$
11. (a) ≈ 3.5 °C/m, -3.0 °C/m
(b) ≈ 0.35 °C/m by Equation 11.6.9
(Definition 11.6.2 gives ≈ 1.1 °C/m.)
(c) -0.25
13. $f_x = 1/\sqrt{2x + y^2}$, $f_y = y/\sqrt{2x + y^2}$
15. $g_u = \tan^{-1}v$, $g_v = u/(1 + v^2)$
17. $T_p = \ln(q + e^r)$, $T_q = p/(q + e^r)$, $T_r = pe^r/(q + e^r)$
19. $f_{xx} = 24x$, $f_{xy} = -2y = f_{yx}$, $f_{yy} = -2x$
21. $f_{xx} = k(k-1)x^{k-2}y^l z^m$, $f_{xy} = klx^{k-1}y^{l-1}z^m = f_{yx}$,
$f_{xz} = kmx^{k-1}y^l z^{m-1} = f_{zx}$, $f_{yy} = l(l-1)x^k y^{l-2}z^m$,
$f_{yz} = lmx^k y^{l-1}z^{m-1} = f_{zy}$, $f_{zz} = m(m-1)x^k y^l z^{m-2}$
25. (a) $z = 8x + 4y + 1$ (b) $\dfrac{x-1}{8} = \dfrac{y+2}{4} = 1 - z$
27. (a) $2x - 2y - 3z = 3$ (b) $\dfrac{x-2}{4} = \dfrac{y+1}{-4} = \dfrac{z-1}{-6}$
29. (a) $4x - y - 2z = 6$
(b) $x = 3 + 8t$, $y = 4 - 2t$, $z = 1 - 4t$
31. $\left(\pm\sqrt{2/7}, \pm 1/\sqrt{14}, \mp 3/\sqrt{14}\right)$
33. $60x + \frac{24}{5}y + \frac{32}{5}z - 120$; 38.656
35. $e^t + 2(y/z)(3t^2 + 4) - 2t(y^2/z^2)$ **37.** -47, 108
43. $ze^{x\sqrt{y}}\langle z\sqrt{y}, xz/(2\sqrt{y}), 2\rangle$ **45.** $\frac{43}{5}$ **47.** $\sqrt{145}/2$, $\langle 4, \frac{9}{2}\rangle$
49. $\approx \frac{5}{8}$ knot/mi **51.** Minimum $f(-4, 1) = -11$
53. Maximum $f(1, 1) = 1$; saddle points $(0, 0)$, $(0, 3)$, $(3, 0)$
55. Maximum $f(1, 2) = 4$, minimum $f(2, 4) = -64$
57. Maximum $f(-1, 0) = 2$, minima $f(1, \pm 1) = -3$,
saddle points $(-1, \pm 1)$, $(1, 0)$
59. Maximum $f\left(\pm\sqrt{2/3}, 1/\sqrt{3}\right) = 2/\left(3\sqrt{3}\right)$,
minimum $f\left(\pm\sqrt{2/3}, -1/\sqrt{3}\right) = -2/\left(3\sqrt{3}\right)$
61. Maximum 1, minimum -1
63. $\left(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)$, $\left(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)$
65. $P\left(2 - \sqrt{3}\right)$, $P\left(3 - \sqrt{3}\right)/6$, $P\left(2\sqrt{3} - 3\right)/3$

Focus on Problem Solving • page 836

1. L^2W^2, $\frac{1}{4}L^2W^2$ **3.** (a) $x = w/3$, base $= w/3$ (b) Yes
9. $\sqrt{6}/2$, $3\sqrt{2}/2$

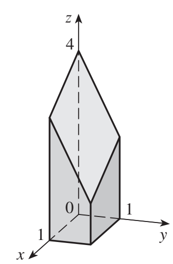
CHAPTER 12

Exercises 12.1 • page 847

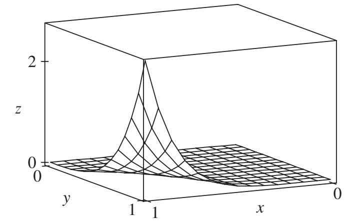
1. (a) -17.75 (b) -15.75 (c) -8.75 (d) -6.75
3. (a) 288 (b) 144 **5.** (a) -6 (b) -3.5
7. $U < V < L$ **9.** (a) ≈ 248 (b) 15.5 **11.** 60
13. 3 **15.** 0.6065, 0.5694, 0.5606, 0.5585, 0.5579, 0.5578

Exercises 12.2 • page 853

1. $9 + 27y$, $8x + 24x^2$ **3.** 10 **5.** $\frac{4}{15}\left(31 - 9\sqrt{3}\right)$
7. $\frac{21}{2}\ln 2$ **9.** 6 **11.** $\frac{21}{2}$ **13.** $9\ln 2$
15. $\left[\left(\sqrt{3} - 1\right)/2\right] - (\pi/12)$
17.



19. 37.5
21. $\frac{166}{27}$
23. $\frac{4}{15}\left(2\sqrt{2} - 1\right)$
25. 36
27. $21e - 57$

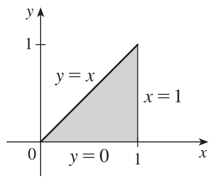


29. $\frac{5}{6}$
31. Fubini's Theorem does not apply. The integrand has an infinite discontinuity at the origin.

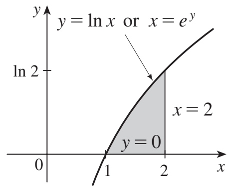
Exercises 12.3 • page 861

1. $\frac{9}{20}$ **3.** $\frac{4}{9}e^{3/2} - \frac{32}{45}$ **5.** $e - 1$ **7.** $\frac{256}{21}$ **9.** $\frac{1}{2}\ln 2$
11. $(1 - \cos 1)/2$ **13.** $\frac{147}{20}$ **15.** 0
17. $\frac{6}{35}$ **19.** $\frac{31}{8}$ **21.** $\frac{1}{6}$ **23.** $\frac{1}{3}$ **25.** 0, 1.213, 0.713
27. 13,984,735,616/14,549,535

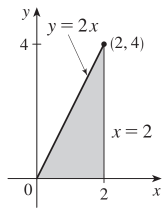
29. $\int_0^1 \int_y^1 f(x, y)\, dx\, dy$



31. $\int_0^{\ln 2} \int_{e^y}^2 f(x, y)\, dx\, dy$



33. $\int_0^1 \int_y^1 f(x, y)\, dx\, dy$



35. $(e^9 - 1)/6$ **37.** $\frac{1}{4}\sin 81$ **39.** $(2\sqrt{2} - 1)/3$ **41.** 1

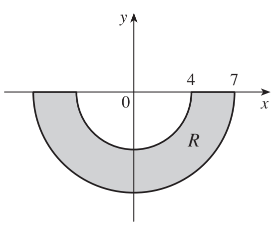
43. $0 \leqslant \iint_D \sqrt{x^3 + y^3}\, dA \leqslant \sqrt{2}$ **47.** 8π **49.** $2\pi/3$

Exercises 12.4 • page 867

1. $\int_0^{2\pi} \int_0^2 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$ **3.** $\int_{-2}^2 \int_x^2 f(x, y)\, dy\, dx$

5. $\int_0^{2\pi} \int_2^5 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

7. $33\pi/2$



9. 0 **11.** $(\pi/2)(1 - e^{-4})$ **13.** 0

15. $81\pi/2$ **17.** $\frac{4}{3}\pi a^3$ **19.** $(2\pi/3)[1 - (1/\sqrt{2})]$

21. $(8\pi/3)(64 - 24\sqrt{3})$ **23.** $\pi/12$ **25.** $(\pi/4)(e - 1)$

27. $4\pi/3$ **29.** $1800\pi\ \text{ft}^3$ **31.** $\frac{15}{16}$

33. (a) $\sqrt{\pi}/4$ (b) $\sqrt{\pi}/2$

Exercises 12.5 • page 877

1. $\frac{64}{3}$ C **3.** $\frac{4}{3}, (\frac{4}{3}, 0)$ **5.** $6, (\frac{3}{4}, \frac{3}{2})$

7. $\frac{27}{2}, (\frac{8}{5}, \frac{1}{2})$ **9.** $(\frac{3}{8}, 3\pi/16)$

11. $(2a/5, 2a/5)$ if vertex is (0, 0) and sides are along positive axes

13. $\frac{4}{5}, \frac{8}{3}, \frac{52}{15}$ **15.** $\frac{189}{20}, \frac{1269}{28}, \frac{1917}{35}$

17. $m = \pi^2/8$, $(\bar{x}, \bar{y}) = \left(\dfrac{2\pi}{3} - \dfrac{1}{\pi}, \dfrac{16}{9\pi}\right)$, $I_x = 3\pi^2/64$, $I_y = (\pi^4 - 3\pi^2)/16$, $I_0 = \pi^4/16 - 9\pi^2/64$

19. (a) $\frac{1}{2}$ (b) 0.375 (c) $\frac{5}{48} \approx 0.1042$

21. (b) (i) $e^{-0.2} \approx 0.8187$

(ii) $1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481$ (c) 2, 5

23. (a) ≈ 0.500 (b) ≈ 0.632

25. (a) $\iint_D (k/20)\left[20 - \sqrt{(x - x_0)^2 + (y - y_0)^2}\right] dA$, where D is the disk with radius 10 mi centered at the center of the city

(b) $200\pi k/3 \approx 209k$, $200(\pi/2 - \frac{8}{9})k \approx 136k$, on the edge

Exercises 12.6 • page 881

1. $15\sqrt{26}$ **3.** $3\sqrt{14}$ **5.** $(\pi/6)(17\sqrt{17} - 5\sqrt{5})$

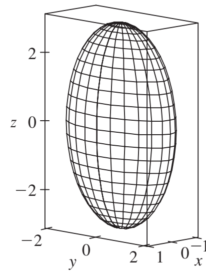
7. $\pi(2\sqrt{6} - \frac{8}{3})$ **9.** $(\sqrt{21}/2) + \frac{17}{4}[\ln(2 + \sqrt{21}) - \ln\sqrt{17}]$

11. $(2\pi/3)(2\sqrt{2} - 1)$ **13.** (a) ≈ 1.83 (b) ≈ 1.8616

15. 4.4506

17. $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left[(11\sqrt{5} + 3\sqrt{70})/(3\sqrt{5} + \sqrt{70})\right]$

19. (b)



(c) $\int_0^{2\pi} \int_0^{\pi} \sqrt{36\sin^4 u\cos^2 v + 9\sin^4 u\sin^2 v + 4\cos^2 u\sin^2 u}\; du\, dv$

21. 4π **25.** $\pi(37\sqrt{37} - 17\sqrt{17})/6$

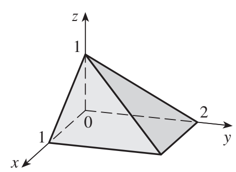
Exercises 12.7 • page 890

3. 1 **5.** $\frac{1}{3}(e^3 - 1)$ **7.** 4 **9.** $\frac{65}{28}$ **11.** $\frac{1}{12}$

13. $16\pi/3$ **15.** $\frac{16}{3}$ **17.** $\frac{8}{15}$

19. (a) $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\, dy\, dx$ (b) $\frac{1}{4}\pi - \frac{1}{3}$

21. 60.533 **23.**



25. $\int_{-2}^2 \int_0^6 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y, z)\, dz\, dy\, dx$

$= \int_0^6 \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y, z)\, dz\, dx\, dy$

$= \int_{-2}^2 \int_0^6 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x, y, z)\, dx\, dy\, dz$

$= \int_0^6 \int_{-2}^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x, y, z)\, dx\, dz\, dy$

$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^6 f(x, y, z)\, dy\, dz\, dx$

$= \int_{-2}^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_0^6 f(x, y, z)\, dy\, dx\, dz$

27. $\int_{-1}^1 \int_0^{1-x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$

$= \int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \int_0^y f(x, y, z)\, dz\, dx\, dy = \int_0^1 \int_z^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x, y, z)\, dx\, dy\, dz$

$= \int_0^1 \int_0^y \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x, y, z)\, dx\, dz\, dy$

$= \int_{-1}^1 \int_0^{1-x^2} \int_z^{1-x^2} f(x, y, z)\, dy\, dz\, dx$

$= \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_z^{1-x^2} f(x, y, z)\, dy\, dx\, dz$

29. $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx = \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x, y, z)\, dz\, dx\, dy$

$= \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x, y, z)\, dx\, dy\, dz = \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z)\, dx\, dz\, dy$

$= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dz\, dx$

$= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dx\, dz$

31. $\int_0^1 \int_0^x \int_0^y f(x, y, z)\, dz\, dy\, dx = \int_0^1 \int_z^1 \int_y^1 f(x, y, z)\, dx\, dy\, dz$

$= \int_0^1 \int_0^y \int_y^1 f(x, y, z)\, dx\, dz\, dy = \int_0^1 \int_0^x \int_z^x f(x, y, z)\, dy\, dz\, dx$

$= \int_0^1 \int_z^1 \int_z^x f(x, y, z)\, dy\, dx\, dz$

33. $\frac{79}{30}, \left(\frac{358}{553}, \frac{33}{79}, \frac{571}{553}\right)$ **35.** a^5, $(7a/12, 7a/12, 7a/12)$

37. (a) $m = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{4y^2+4z^2}^{4} (x^2 + y^2 + z^2)\, dx\, dz\, dy$

(b) $(\bar{x}, \bar{y}, \bar{z})$, where

$\bar{x} = (1/m) \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{4y^2+4z^2}^{4} x(x^2 + y^2 + z^2)\, dx\, dz\, dy$

$\bar{y} = (1/m) \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{4y^2+4z^2}^{4} y(x^2 + y^2 + z^2)\, dx\, dz\, dy$

$\bar{z} = (1/m) \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{4y^2+4z^2}^{4} z(x^2 + y^2 + z^2)\, dx\, dz\, dy$

(c) $\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{4y^2+4z^2}^{4} (x^2 + y^2)(x^2 + y^2 + z^2)\, dx\, dz\, dy$

39. (a) $\frac{3}{32}\pi + \frac{11}{24}$ (b) $(\bar{x}, \bar{y}, \bar{z})$, where $\bar{x} = 28/(9\pi + 44)$,
$\bar{y} = 2(15\pi + 64)/[5(9\pi + 44)]$,
$\bar{z} = (45\pi + 208)/[15(9\pi + 44)]$
(c) $(68 + 15\pi)/240$

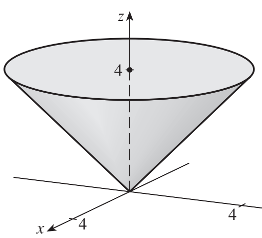
41. $I_x = I_y = I_z = \frac{2}{3}kL^5$ **43.** (a) $\frac{1}{8}$ (b) $\frac{1}{64}$ (c) $\frac{1}{5760}$

45. $L^3/8$

47. The region bounded by the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$

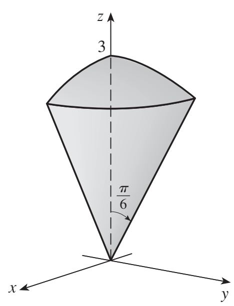
Exercises 12.8 • page 898

1.



$64\pi/3$

3.



$(9\pi/4)(2 - \sqrt{3})$

5. $\int_0^{\pi/2} \int_0^3 \int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$ **7.** 384π

9. 0 **11.** $2\pi/5$ **13.** $\pi K a^2/8$, $(0, 0, 2a/3)$

15. $4\pi/5$ **17.** $15\pi/16$ **19.** $4\pi(2 - \sqrt{3})$

21. (a) 10π (b) $(0, 0, 2.1)$

23. (a) $\left(0, 0, \frac{3}{8}a\right)$ (b) $4K\pi a^5/15$

25. $(2\pi/3)[1 - (1/\sqrt{2})]$, $(0, 0, 3/[8(2 - \sqrt{2})])$ **27.** $5\pi/6$

29. $8\pi/35$ **31.** $136\pi/99$

33. (a) $\iiint_C h(P)g(P)\, dV$, where C is the cone
(b) $\approx 3.1 \times 10^{19}$ ft-lb

Exercises 12.9 • page 909

1. -14 **3.** 0 **5.** $2uvw$

7. The parallelogram with vertices $(0, 0)$, $(6, 3)$, $(12, 1)$, $(6, -2)$

9. The region bounded by the line $y = 1$, the y-axis, and $y = \sqrt{x}$

11. $\frac{11}{3}$ **13.** 6π **15.** $2 \ln 3$

17. (a) $\frac{4}{3}\pi abc$ (b) 1.083×10^{12} km^3 **19.** $-\frac{66}{125}$

21. $\frac{3}{2}\sin 1$ **23.** $e - e^{-1}$

Chapter 12 Review • page 911

True-False Quiz

1. True **3.** True **5.** False

Exercises

1. ≈ 64.0 **3.** $4e^2 - 4e + 3$ **5.** $\frac{1}{2}\sin 1$ **7.** $\frac{2}{3}$

9. $\int_0^{\pi} \int_2^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

11. The region outside the circle $r = 1$ and inside the cardioid $r = 1 + \sin\theta$

13. $(e - 1)/2$ **15.** $\ln \frac{3}{2}$ **17.** $\frac{1}{40}$ **19.** $\frac{41}{30}$ **21.** $81\pi/5$

23. $\frac{32}{3}$ **25.** $\pi/96$ **27.** $\frac{64}{15}$ **29.** 176 **31.** $\frac{2}{3}$

33. $2ma^3/9$ **35.** (a) $\frac{1}{4}$ (b) $\left(\frac{1}{3}, \frac{8}{15}\right)$ (c) $I_x = \frac{1}{12}$, $I_y = \frac{1}{24}$

37. (a) $(0, 0, h/4)$ (b) $\pi a^4 h/10$

39. $\ln(\sqrt{2} + \sqrt{3}) + \sqrt{2}/3$

41. $(\pi/8)\ln 5$ **43.** 0.0512 **45.** (a) $\frac{1}{15}$ (b) $\frac{1}{3}$ (c) $\frac{1}{45}$

47. $\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dy\, dz$ **49.** $-\ln 2$ **51.** 0

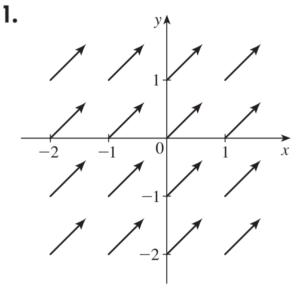
Focus on Problem Solving • page 914

1. 30 **3.** $\frac{1}{2}\sin 1$ **7.** (b) 0.90

CHAPTER 13

Exercises 13.1 • page 922

1.



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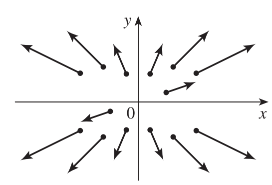

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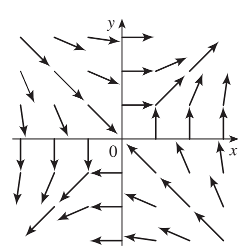

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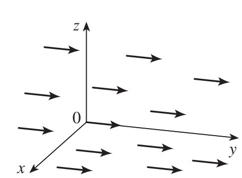

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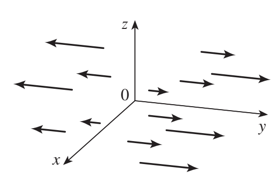
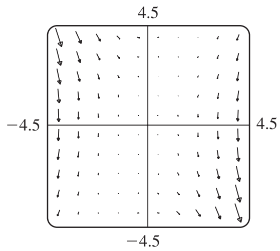


11. III **13.** II **15.** IV **17.** III

19. 

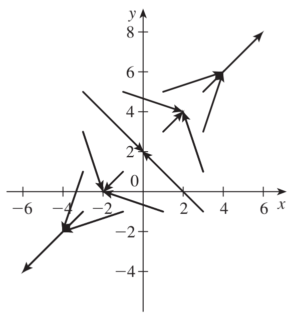
The line $y = 2x$

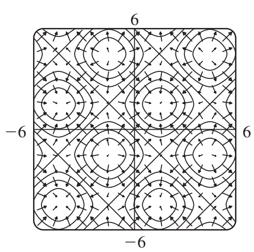
21. $\nabla f(x, y) = \dfrac{1}{x + 2y}\,\mathbf{i} + \dfrac{2}{x + 2y}\,\mathbf{j}$

23. $\nabla f(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{i}$

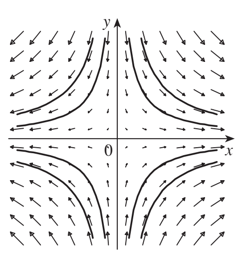
$+ \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{k}$

25. $\nabla f(x, y) = (y - 2)\,\mathbf{i} + x\,\mathbf{j}$



27. 

29. IV **31.** II

33. (a) 

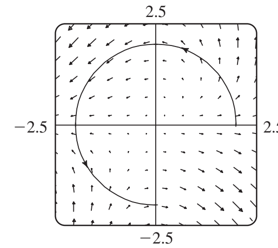
$y = C/x$

(b) $y = 1/x,\ x > 0$

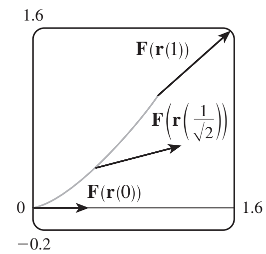
Exercises 13.2 • page 934

1. $(17\sqrt{17} - 1)/12$ **3.** 1638.4 **5.** $\frac{17}{3}$ **7.** 320
9. $\sqrt{14}(e^6 - 1)/12$ **11.** $\frac{77}{6}$
13. (a) Positive (b) Negative **15.** $-\frac{59}{105}$
17. $\frac{6}{5} - \cos 1 - \sin 1$

19. $3\pi + \frac{2}{3}$



21. (a) $\frac{11}{8} - 1/e$ (b)



23. $\frac{945}{16,777,216}\,\pi$ **25.** $2\pi k$, $(4/\pi, 0)$
27. (a) $\bar{x} = (1/m)\int_C x\rho(x, y, z)\,ds$,
$\bar{y} = (1/m)\int_C y\rho(x, y, z)\,ds$,
$\bar{z} = (1/m)\int_C z\rho(x, y, z)\,ds$, where $m = \int_C \rho(x, y, z)\,ds$
(b) $2\sqrt{13}\,k\pi$, $(0, 0, 3\pi)$
29. $I_x = k\big((\pi/2) - \frac{4}{3}\big)$, $I_y = k\big((\pi/2) - \frac{2}{3}\big)$
31. $2\pi^2$ **33.** $\frac{23}{88}$ **35.** 1.67×10^4 ft-lb **37.** ≈22 J

Exercises 13.3 • page 943

1. 40 **3.** $f(x, y) = 3x^2 + 5xy + 2y^2 + K$
5. Not conservative **7.** $f(x, y) = x^2 \cos y - y \sin x + K$
9. $f(x, y) = ye^x + x \sin y + K$ **11.** (b) 16
13. (a) $f(x, y) = \frac{1}{4}x^4y^4$ (b) 4
15. (a) $f(x, y, z) = xyz + z^2$ (b) 77
17. (a) $f(x, y, z) = xy^2 \cos z$ (b) 0
19. $25 \sin 1 - 1$ **21.** $\frac{8}{3}$ **23.** No **25.** No
29. (a) Yes (b) Yes (c) Yes
31. (a) Yes (b) Yes (c) No

Exercises 13.4 • page 951

1. 6 **3.** $\frac{2}{3}$ **7.** $e - 1$ **9.** $\frac{1}{3}$ **11.** -24π **13.** 0
15. $\pi + \frac{16}{3}\big[(1/\sqrt{2}) - 1\big]$ **17.** $-\frac{1}{12}$ **19.** $3\pi/8$
21. (c) $\frac{9}{2}$ **23.** $\left(\frac{1}{3}, \frac{1}{3}\right)$

Exercises 13.5 • page 958

1. (a) $-y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}$ (b) $x + y + z$
3. (a) $-x^2\,\mathbf{i} + 3xy\,\mathbf{j} - xz\,\mathbf{k}$ (b) yz **5.** (a) $\mathbf{0}$ (b) 1
7. (a) Negative (b) curl $\mathbf{F} = \mathbf{0}$
9. (a) Zero (b) curl $\mathbf{F}$ points in the negative z-direction
11. $f(x, y, z) = xyz + K$ **13.** $f(x, y, z) = x^2y + y^2z + K$
15. Not conservative **17.** No

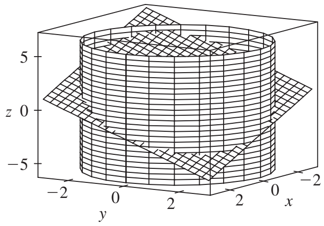
Exercises 13.6 • page 969

1. $8\big(1 + \sqrt{2} + \sqrt{3}\big) \approx 33.17$ **3.** 900π **5.** 0
7. $171\sqrt{14}$ **9.** $\sqrt{3}/24$ **11.** $\big(33\sqrt{33} - 17\sqrt{17}\big)/6$

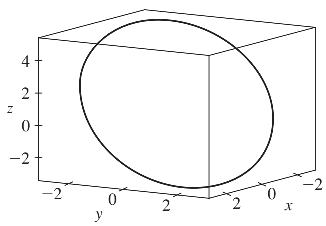
13. $\pi\sqrt{2}/4$ **15.** 16π **17.** 16π **19.** $\frac{713}{180}$
21. $-\frac{1}{6}$ **23.** 108π **25.** 0 **27.** 48 **29.** 3.4895
31. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D [P(\partial h/\partial x) - Q + R(\partial h/\partial z)]\,dA$, where D = projection on xz-plane
33. $(0, 0, a/2)$
35. (a) $I_z = \iint_S (x^2 + y^2)\rho(x, y, z)\,dS$ (b) $4329\sqrt{2}\,\pi/5$
37. $194{,}400\pi$ **39.** $8\pi a^3\varepsilon_0/3$ **41.** 1248π

Exercises 13.7 • page 975

3. 0 **5.** 0 **7.** -1 **9.** -4π
11. (a) $81\pi/2$ (b)



(c) $x = 3\cos t$, $y = 3\sin t$, $z = 1 - 3(\cos t + \sin t)$, $0 \leqslant t \leqslant 2\pi$



17. 16

Exercises 13.8 • page 982

1. Negative at P_1, positive at P_2 **7.** 2 **9.** $9\pi/2$
11. 0 **13.** $12\pi/5$ **15.** $341\sqrt{2}/60 + \frac{81}{20}\arcsin(\sqrt{3}/3)$
17. $13\pi/20$

Chapter 13 Review • page 985

True-False Quiz

1. False **3.** True **5.** False **7.** True

Exercises

1. (a) Negative (b) Positive **3.** $4\sqrt{5}$ **5.** $-\pi$
7. $\frac{17}{2}$ **9.** 5 **11.** $f(x, y) = e^y + xe^{xy}$ **13.** 0
17. -8π **25.** $\pi(391\sqrt{17} + 1)/60$ **27.** $-64\pi/3$
31. $-\frac{1}{2}$ **35.** -4 **37.** 21

APPENDIXES

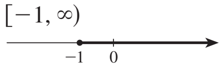
Exercises A • page A6

1. 18 **3.** $5 - \sqrt{5}$ **5.** $2 - x$

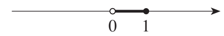
7. $|x + 1| = \begin{cases} x + 1 & \text{for } x \geqslant -1 \\ -x - 1 & \text{for } x < -1 \end{cases}$ **9.** $x^2 + 1$

11. $(-2, \infty)$

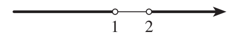
13. $[-1, \infty)$



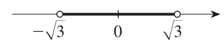
15. $(0, 1]$



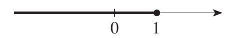
17. $(-\infty, 1) \cup (2, \infty)$



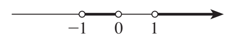
19. $(-\sqrt{3}, \sqrt{3})$



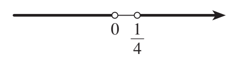
21. $(-\infty, 1]$



23. $(-1, 0) \cup (1, \infty)$



25. $(-\infty, 0) \cup (\frac{1}{4}, \infty)$

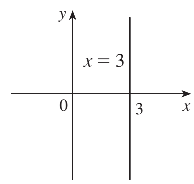


27. $10 \leqslant C \leqslant 35$ **29.** (a) $T = 20 - 10h$, $0 \leqslant h \leqslant 12$
(b) $-30\,^\circ\text{C} \leqslant T \leqslant 20\,^\circ\text{C}$ **31.** $2, -\frac{4}{3}$
33. $(-3, 3)$ **35.** $(3, 5)$ **37.** $(-\infty, -7] \cup [-3, \infty)$
39. $[1.3, 1.7]$ **41.** $x \geqslant (a + b)c/(ab)$

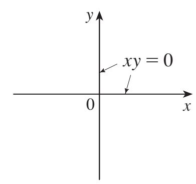
Exercises B • page A16

1. 5 **3.** $-\frac{9}{2}$

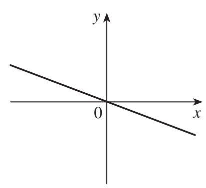
7.



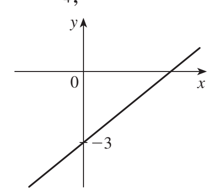
9.



11. $y = 6x - 15$ **13.** $y = -5x + 11$ **15.** $y = 3x - 2$
17. $y = 3x - 3$ **19.** $y = 5$ **21.** $y = -\frac{1}{2}x - \frac{11}{2}$
23. $y = \frac{5}{2}x + \frac{1}{2}$
25. $m = -\frac{1}{3}$, $b = 0$



27. $m = \frac{3}{4}$, $b = -3$



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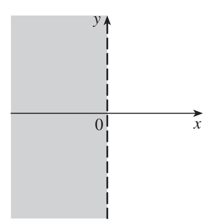

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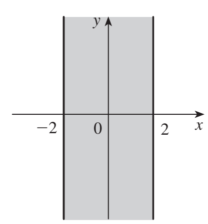

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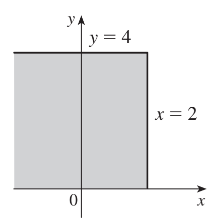

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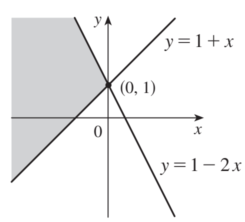
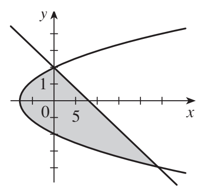


37. $(x - 3)^2 + (y + 1)^2 = 25$ **39.** $(2, -5)$, 4
41. $(1, -2)$ **45.** $y = x - 3$

53.

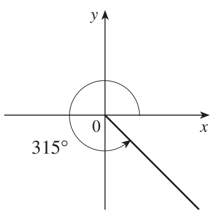


Exercises C • page A27

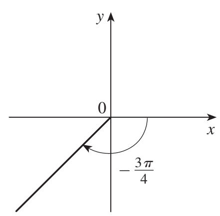
1. (a) $7\pi/6$ (b) $\pi/20$ **3.** (a) $720°$ (b) $-67.5°$

5. 3π cm **7.** $\frac{2}{3}$ rad $= (120/\pi)°$

9. (a)



(b)



11. $\sin(3\pi/4) = 1/\sqrt{2}$, $\cos(3\pi/4) = -1/\sqrt{2}$, $\tan(3\pi/4) = -1$, $\csc(3\pi/4) = \sqrt{2}$, $\sec(3\pi/4) = -\sqrt{2}$, $\cot(3\pi/4) = -1$

13. $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\csc\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = \frac{4}{3}$

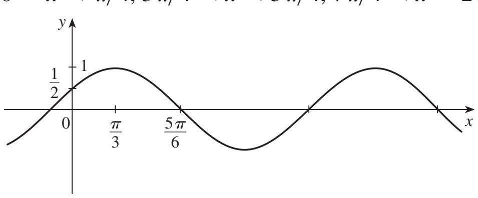
15. 5.73576 cm **17.** 24.62147 cm **27.** $(4 + 6\sqrt{2})/15$

29. $\pi/3, 5\pi/3$ **31.** $\pi/6, \pi/2, 5\pi/6, 3\pi/2$

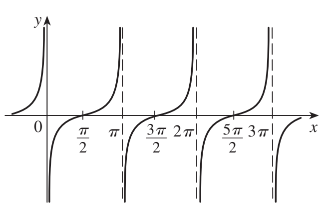
33. $0 \leqslant x \leqslant \pi/6$ and $5\pi/6 \leqslant x \leqslant 2\pi$

35. $0 \leqslant x < \pi/4, 3\pi/4 < x < 5\pi/4, 7\pi/4 < x \leqslant 2\pi$

37.



39.



41. (a) $\pi/6$ (b) $-\pi/4$ **43.** (a) 0.7 (b) $-\pi/4$

47. (a) $\left[-\frac{2}{3}, 0\right]$ (b) $[-\pi/2, \pi/2]$

Exercises D • page A37

1. $\frac{4}{7}$ (or any smaller positive number)

3. 0.6875 (or any smaller positive number)

5. 0.11, 0.012 (or smaller positive numbers)

11. (a) $\sqrt{1000/\pi}$ cm (b) Within approximately 0.0445 cm (c) Radius; area; $\sqrt{1000/\pi}$; 1000; 5; ≈ 0.0445

13. $N \geqslant 13$ **15.** (a) $x > 100$ **17** (a) 0 (b) 9, 11

Exercises F • page A48

1. $\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}$ **3.** $3^4 + 3^5 + 3^6$

5. $-1 + \frac{1}{3} + \frac{3}{5} + \frac{5}{7} + \frac{7}{9}$ **7.** $1^{10} + 2^{10} + 3^{10} + \cdots + n^{10}$

9. $1 - 1 + 1 - 1 + \cdots + (-1)^{n-1}$ **11.** $\sum_{i=1}^{10} i$

13. $\sum_{i=1}^{19} \frac{i}{i+1}$ **15.** $\sum_{i=1}^{n} 2i$ **17.** $\sum_{i=0}^{5} 2^i$ **19.** $\sum_{i=1}^{n} x^i$

21. 80 **23.** 3276 **25.** 0 **27.** 61 **29.** $n(n + 1)$

31. $n(n^2 + 6n + 17)/3$ **33.** $n(n^2 + 6n + 11)/3$

35. $n(n^3 + 2n^2 - n - 10)/4$

41. (a) n^4 (b) $5^{100} - 1$ (c) $\frac{97}{300}$ (d) $a_n - a_0$

43. $\frac{1}{3}$ **45.** 14 **49.** $2^{n+1} + n^2 + n - 2$

Exercises G • page A57

1. $\dfrac{A}{2x+1} + \dfrac{B}{x-2}$ **3.** $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x-1}$

5. $1 + \dfrac{A}{x-1} + \dfrac{B}{x+1}$ **7.** $\dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 2}$

9. $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx + D}{x^2 + x + 2}$

11. $\frac{1}{2}x^2 - 2x + 6\ln|x + 2| + C$

13. $\ln 3 + 3\ln 6 - 3\ln 4 = \ln\frac{81}{8}$ **15.** $2\ln 2 + \frac{1}{2}$

17. $\frac{27}{5}\ln 2 - \frac{9}{5}\ln 3$ $\left(\text{or } \frac{9}{5}\ln\frac{8}{3}\right)$

19. $-\dfrac{1}{36}\ln|x + 5| + \dfrac{1}{6}\dfrac{1}{x+5} + \dfrac{1}{36}\ln|x - 1| + C$

21. $\ln\sqrt{3} - (\sqrt{3}\pi/18)$

23. $\ln(x - 1)^2 + \ln\sqrt{x^2 + 1} - 3\tan^{-1}x + C$

25. $\frac{1}{3}\ln|x - 1| - \frac{1}{6}\ln(x^2 + x + 1) - \dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x + 1}{\sqrt{3}} + C$

27. $\frac{1}{2}\ln(t^2 + 1) + \frac{1}{2}\ln(t^2 + 2) - (1/\sqrt{2})\tan^{-1}(t/\sqrt{2}) + C$

29. $-\frac{1}{2}\ln 3 \approx -0.55$

31. $t = -\ln P - \frac{1}{9}\ln(0.9P + 900) + C$, where $C \approx 10.23$

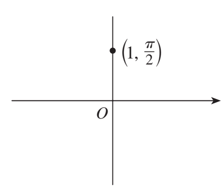
33. (a) $\dfrac{24{,}110}{4879}\dfrac{1}{5x+2} - \dfrac{668}{323}\dfrac{1}{2x+1} - \dfrac{9438}{80{,}155}\dfrac{1}{3x-7} + \dfrac{1}{260{,}015}\dfrac{22{,}098x + 48{,}935}{x^2 + x + 5}$

(b) $\dfrac{4822}{4879}\ln|5x + 2| - \dfrac{334}{323}\ln|2x + 1| - \dfrac{3146}{80{,}155}\ln|3x - 7| + \dfrac{11{,}049}{260{,}015}\ln(x^2 + x + 5) + \dfrac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\dfrac{2x + 1}{\sqrt{19}} + C$

The CAS omits the absolute value signs and the constant of integration.

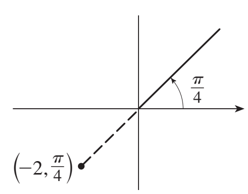
Exercises H.1 • page A66

1. (a)



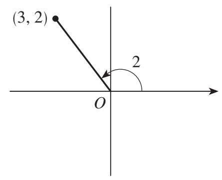
$(1, 5\pi/2), (-1, 3\pi/2)$

(b)



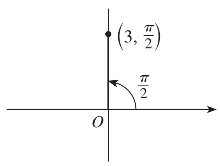
$(2, 5\pi/4), (-2, 9\pi/4)$

(c)



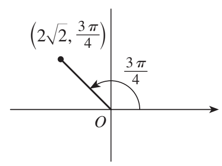
$(3, 2 + 2\pi), (-3, 2 + \pi)$

3. (a)



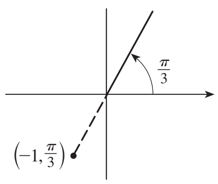
$(0, 3)$

(b)



$(-2, 2)$

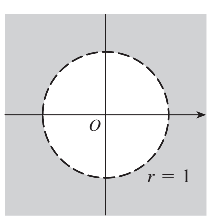
(c)



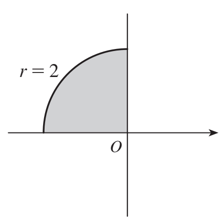
$\left(-\frac{1}{2}, -\sqrt{3}/2\right)$

5. (a) (i) $\left(\sqrt{2}, \pi/4\right)$ (ii) $\left(-\sqrt{2}, 5\pi/4\right)$
(b) (i) $(4, 11\pi/6)$ (ii) $(-4, 5\pi/6)$

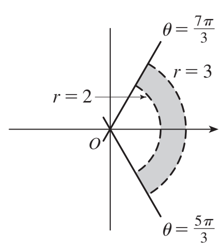
7.



9.



11.

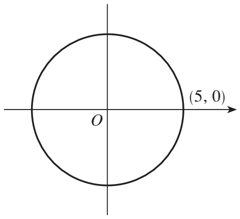


13. $x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$ **15.** $(x^2 + y^2)^2 = 2xy$

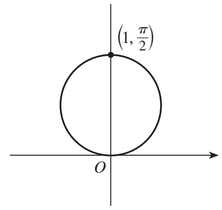
17. $r \sin\theta = 5$ **19.** $r = 5$

21. (a) $\theta = \pi/6$ (b) $x = 3$

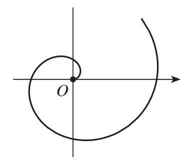
23.



25.



27.



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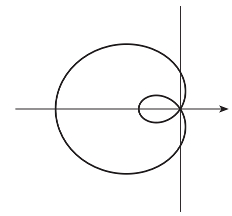

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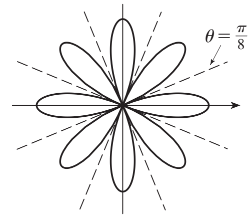
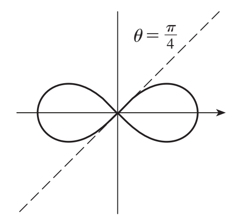


33.



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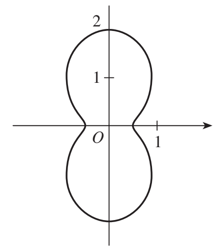
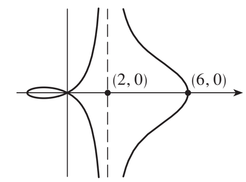


37.



39. (a) For $c < -1$, the loop begins at $\theta = \sin^{-1}(-1/c)$ and ends at $\theta = \pi - \sin^{-1}(-1/c)$; for $c > 1$, it begins at $\theta = \pi + \sin^{-1}(1/c)$ and ends at $\theta = 2\pi - \sin^{-1}(1/c)$.

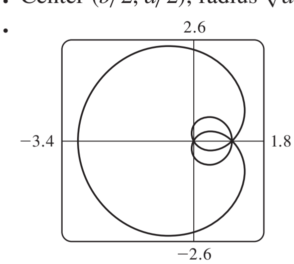
41. $1/\sqrt{3}$ **43.** -1

45. Horizontal at $\left(3/\sqrt{2}, \pi/4\right), \left(-3/\sqrt{2}, 3\pi/4\right)$; vertical at $(3, 0), (0, \pi/2)$

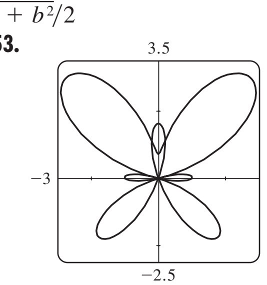
47. Horizontal at $\left(\frac{3}{2}, \pi/3\right), \left(\frac{3}{2}, 5\pi/3\right)$, and the pole; vertical at $(2, 0), \left(\frac{1}{2}, 2\pi/3\right), \left(\frac{1}{2}, 4\pi/3\right)$

49. Center $(b/2, a/2)$, radius $\sqrt{a^2 + b^2}/2$

51.



53.



55. By counterclockwise rotation through angle $\pi/6$, $\pi/3$, or α about the origin

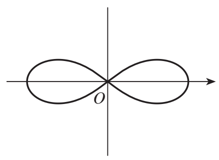
57. (a) A rose with n loops if n is odd and $2n$ loops if n is even
(b) Number of loops is always $2n$

59. For $0 < a < 1$, the curve is an oval, which develops a dimple as $a \to 1^-$. When $a > 1$, the curve splits into two parts, one of which has a loop.

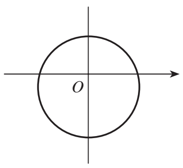
Exercises H.2 • page A72

1. $\pi^2/64$ **3.** $\pi/12 + \sqrt{3}/8$ **5.** $\pi^3/6$ **7.** $41\pi/4$

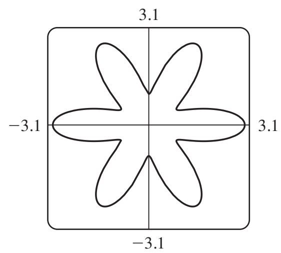
9. 4



11. $33\pi/2$



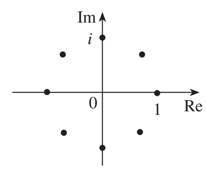
13. $9\pi/2$

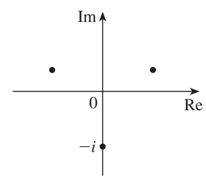


15. $\pi/8$ **17.** $\pi - (3\sqrt{3}/2)$ **19.** $(4\pi/3) + 2\sqrt{3}$
21. π **23.** $(\pi - 2)/8$ **25.** $(\pi/2) - 1$
27. $(\pi + 3\sqrt{3})/4$ **29.** $(\frac{1}{2}, \pi/3), (\frac{1}{2}, 5\pi/3)$, and the pole
31. $(\sqrt{3}/2, \pi/3), (\sqrt{3}/2, 2\pi/3)$, and the pole
33. Intersection at $\theta \approx 0.89, 2.25$; area ≈ 3.46 **35.** $15\pi/4$
37. $\frac{8}{3}[(\pi^2 + 1)^{3/2} - 1]$ **39.** 2.422

Exercises I • page A81

1. $10 - i$ **3.** $13 - i$ **5.** $12 - 7i$ **7.** $-\frac{1}{2} + \frac{1}{2}i$
9. $\frac{1}{2} - \frac{1}{2}i$ **11.** $-i$ **13.** $5i$ **15.** $3 - 4i, 5$
17. $4i, 4$ **19.** $\pm\frac{3}{2}i$ **21.** $4 \pm i$
23. $-\frac{1}{2} \pm (\sqrt{7}/2)i$ **25.** $3\sqrt{2}\,[\cos(3\pi/4) + i\sin(3\pi/4)]$
27. $5\left[\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right) + i\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)\right]$
29. $4[\cos(\pi/2) + i\sin(\pi/2)]$, $\cos(-\pi/6) + i\sin(-\pi/6)$, $\frac{1}{2}[\cos(-\pi/6) + i\sin(-\pi/6)]$
31. $4\sqrt{2}\,[\cos(7\pi/12) + i\sin(7\pi/12)]$, $(2\sqrt{2})[\cos(13\pi/12) + i\sin(13\pi/12)]$, $\frac{1}{4}[\cos(\pi/6) + i\sin(\pi/6)]$
33. -1024 **35.** $-512\sqrt{3} + 512i$
37. $\pm 1, \pm i, (1/\sqrt{2})(\pm 1 \pm i)$ **39.** $\pm(\sqrt{3}/2) + \frac{1}{2}i, -i$





41. i **43.** $(-1/\sqrt{2}) + (1/\sqrt{2})i$ **45.** $-e^2$
47. $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$,
$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

ALGEBRA

ARITHMETIC OPERATIONS

$$a(b+c) = ab + ac \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$\frac{a+c}{b} = \frac{a}{b} + \frac{c}{b} \qquad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

EXPONENTS AND RADICALS

$$x^m x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn} \qquad x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{1/n} = \sqrt[n]{x} \qquad x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y} \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

FACTORING SPECIAL POLYNOMIALS

$$x^2 - y^2 = (x+y)(x-y)$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

BINOMIAL THEOREM

$$(x+y)^2 = x^2 + 2xy + y^2 \qquad (x-y)^2 = x^2 - 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot k}$

QUADRATIC FORMULA

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

INEQUALITIES AND ABSOLUTE VALUE

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

If $a < b$ and $c < 0$, then $ca > cb$.

If $a > 0$, then

$|x| = a$ means $x = a$ or $x = -a$

$|x| < a$ means $-a < x < a$

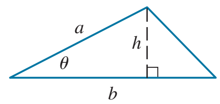
$|x| > a$ means $x > a$ or $x < -a$

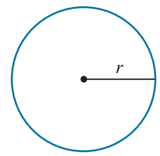
GEOMETRY

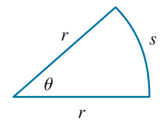
GEOMETRIC FORMULAS

Formulas for area A, circumference C, and volume V:

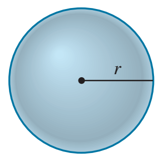
Triangle	Circle	Sector of Circle
$A = \frac{1}{2}bh$	$A = \pi r^2$	$A = \frac{1}{2}r^2\theta$
$= \frac{1}{2}ab\sin\theta$	$C = 2\pi r$	$s = r\theta$ (θ in radians)

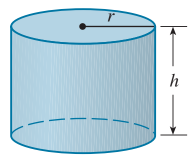


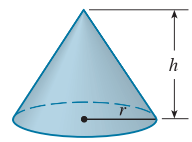




Sphere	Cylinder	Cone
$V = \frac{4}{3}\pi r^3$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$
$A = 4\pi r^2$		







DISTANCE AND MIDPOINT FORMULAS

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $\overline{P_1P_2}$: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

LINES

Slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope equation of line through $P_1(x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

CIRCLES

Equation of the circle with center (h, k) and radius r:

$$(x-h)^2 + (y-k)^2 = r^2$$

TRIGONOMETRY

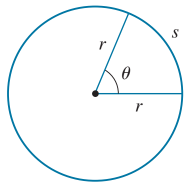
ANGLE MEASUREMENT

π radians $= 180°$

$1° = \dfrac{\pi}{180}$ rad $\qquad$ 1 rad $= \dfrac{180°}{\pi}$

$s = r\theta$

(θ in radians)

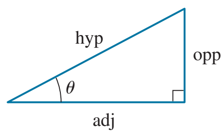


RIGHT ANGLE TRIGONOMETRY

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}} \qquad \cot\theta = \dfrac{\text{adj}}{\text{opp}}$

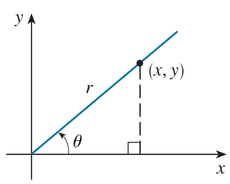


TRIGONOMETRIC FUNCTIONS

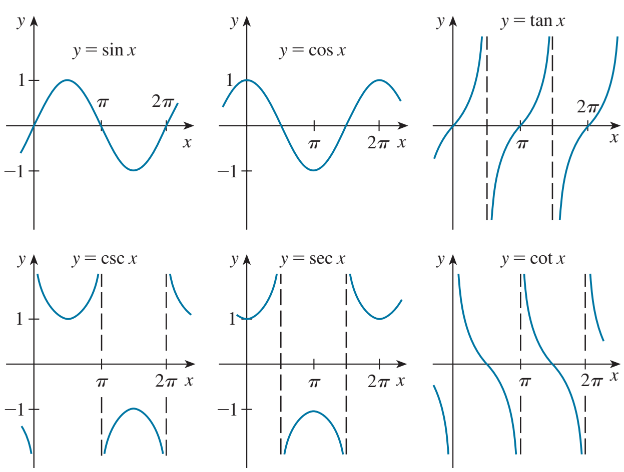
$\sin\theta = \dfrac{y}{r} \qquad \csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r} \qquad \sec\theta = \dfrac{r}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$



GRAPHS OF THE TRIGONOMETRIC FUNCTIONS



TRIGONOMETRIC FUNCTIONS OF IMPORTANT ANGLES

θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	—

FUNDAMENTAL IDENTITIES

$\csc\theta = \dfrac{1}{\sin\theta} \qquad \sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} \qquad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\cot\theta = \dfrac{1}{\tan\theta} \qquad \sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$

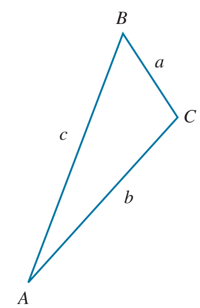
$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta \qquad \sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta \qquad \tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta$

THE LAW OF SINES

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



THE LAW OF COSINES

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

ADDITION AND SUBTRACTION FORMULAS

$\sin(x + y) = \sin x\cos y + \cos x\sin y$

$\sin(x - y) = \sin x\cos y - \cos x\sin y$

$\cos(x + y) = \cos x\cos y - \sin x\sin y$

$\cos(x - y) = \cos x\cos y + \sin x\sin y$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x\tan y}$

DOUBLE-ANGLE FORMULAS

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

HALF-ANGLE FORMULAS

$\sin^2 x = \dfrac{1 - \cos 2x}{2} \qquad \cos^2 x = \dfrac{1 + \cos 2x}{2}$

DIFFERENTIATION RULES

GENERAL FORMULAS

1. $\dfrac{d}{dx}(c) = 0$

2. $\dfrac{d}{dx}[cf(x)] = cf'(x)$

3. $\dfrac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

4. $\dfrac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

5. $\dfrac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$ (Product Rule)

6. $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ (Quotient Rule)

7. $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ (Chain Rule)

8. $\dfrac{d}{dx}(x^n) = nx^{n-1}$ (Power Rule)

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

9. $\dfrac{d}{dx}(e^x) = e^x$

10. $\dfrac{d}{dx}(a^x) = a^x \ln a$

11. $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$

12. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$

TRIGONOMETRIC FUNCTIONS

13. $\dfrac{d}{dx}(\sin x) = \cos x$

14. $\dfrac{d}{dx}(\cos x) = -\sin x$

15. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

16. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

17. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

18. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

INVERSE TRIGONOMETRIC FUNCTIONS

18. $\dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{1 - x^2}}$

20. $\dfrac{d}{dx}(\cos^{-1}x) = -\dfrac{1}{\sqrt{1 - x^2}}$

21. $\dfrac{d}{dx}(\tan^{-1}x) = \dfrac{1}{1 + x^2}$

22. $\dfrac{d}{dx}(\csc^{-1}x) = -\dfrac{1}{x\sqrt{x^2 - 1}}$

23. $\dfrac{d}{dx}(\sec^{-1}x) = \dfrac{1}{x\sqrt{x^2 - 1}}$

24. $\dfrac{d}{dx}(\cot^{-1}x) = -\dfrac{1}{1 + x^2}$

HYPERBOLIC FUNCTIONS

25. $\dfrac{d}{dx}(\sinh x) = \cosh x$

26. $\dfrac{d}{dx}(\cosh x) = \sinh x$

27. $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

28. $\dfrac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

29. $\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

30. $\dfrac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

INVERSE HYPERBOLIC FUNCTIONS

31. $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1 + x^2}}$

32. $\dfrac{d}{dx}(\cosh^{-1}x) = \dfrac{1}{\sqrt{x^2 - 1}}$

33. $\dfrac{d}{dx}(\tanh^{-1}x) = \dfrac{1}{1 - x^2}$

34. $\dfrac{d}{dx}(\operatorname{csch}^{-1}x) = -\dfrac{1}{|x|\sqrt{x^2 + 1}}$

35. $\dfrac{d}{dx}(\operatorname{sech}^{-1}x) = -\dfrac{1}{x\sqrt{1 - x^2}}$

36. $\dfrac{d}{dx}(\coth^{-1}x) = \dfrac{1}{1 - x^2}$

TABLE OF INTEGRALS

BASIC FORMS

1. $\int u\,dv = uv - \int v\,du$

2. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\int \frac{du}{u} = \ln|u| + C$

4. $\int e^u\,du = e^u + C$

5. $\int a^u\,du = \frac{a^u}{\ln a} + C$

6. $\int \sin u\,du = -\cos u + C$

7. $\int \cos u\,du = \sin u + C$

8. $\int \sec^2 u\,du = \tan u + C$

9. $\int \csc^2 u\,du = -\cot u + C$

10. $\int \sec u \tan u\,du = \sec u + C$

11. $\int \csc u \cot u\,du = -\csc u + C$

12. $\int \tan u\,du = \ln|\sec u| + C$

13. $\int \cot u\,du = \ln|\sin u| + C$

14. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

15. $\int \csc u\,du = \ln|\csc u - \cot u| + C$

16. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$

17. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

18. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

19. $\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

20. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

FORMS INVOLVING $\sqrt{a^2 + u^2}$, $a > 0$

21. $\int \sqrt{a^2 + u^2}\,du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

22. $\int u^2\sqrt{a^2 + u^2}\,du = \frac{u}{8}(a^2 + 2u^2)\sqrt{a^2 + u^2} - \frac{a^4}{8}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

23. $\int \frac{\sqrt{a^2 + u^2}}{u}\,du = \sqrt{a^2 + u^2} - a\ln\left|\frac{a + \sqrt{a^2 + u^2}}{u}\right| + C$

24. $\int \frac{\sqrt{a^2 + u^2}}{u^2}\,du = -\frac{\sqrt{a^2 + u^2}}{u} + \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

25. $\int \frac{du}{\sqrt{a^2 + u^2}} = \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

26. $\int \frac{u^2\,du}{\sqrt{a^2 + u^2}} = \frac{u}{2}\sqrt{a^2 + u^2} - \frac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

27. $\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2 + u^2} + a}{u}\right| + C$

28. $\int \frac{du}{u^2\sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2 u} + C$

29. $\int \frac{du}{(a^2 + u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 + u^2}} + C$

TABLE OF INTEGRALS

FORMS INVOLVING $\sqrt{a^2 - u^2}$, $a > 0$

30. $\displaystyle \int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

31. $\displaystyle \int u^2\sqrt{a^2 - u^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

32. $\displaystyle \int \frac{\sqrt{a^2 - u^2}}{u}\, du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

33. $\displaystyle \int \frac{\sqrt{a^2 - u^2}}{u^2}\, du = -\frac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\frac{u}{a} + C$

34. $\displaystyle \int \frac{u^2\, du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

35. $\displaystyle \int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

36. $\displaystyle \int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2 u}\sqrt{a^2 - u^2} + C$

37. $\displaystyle \int (a^2 - u^2)^{3/2}\, du = -\frac{u}{8}(2u^2 - 5a^2)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

38. $\displaystyle \int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 - u^2}} + C$

FORMS INVOLVING $\sqrt{u^2 - a^2}$, $a > 0$

39. $\displaystyle \int \sqrt{u^2 - a^2}\, du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

40. $\displaystyle \int u^2\sqrt{u^2 - a^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

41. $\displaystyle \int \frac{\sqrt{u^2 - a^2}}{u}\, du = \sqrt{u^2 - a^2} - a\cos^{-1}\frac{a}{|u|} + C$

42. $\displaystyle \int \frac{\sqrt{u^2 - a^2}}{u^2}\, du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

43. $\displaystyle \int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

44. $\displaystyle \int \frac{u^2\, du}{\sqrt{u^2 - a^2}} = \frac{u}{2}\sqrt{u^2 - a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

45. $\displaystyle \int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$

46. $\displaystyle \int \frac{du}{(u^2 - a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2 - a^2}} + C$

TABLE OF INTEGRALS

FORMS INVOLVING $a + bu$

47. $$\int \frac{u\,du}{a+bu} = \frac{1}{b^2}\left(a + bu - a\ln|a+bu|\right) + C$$

48. $$\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$$

49. $$\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$$

50. $$\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

51. $$\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$$

52. $$\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

53. $$\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left(a + bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$$

54. $$\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu - 2a)(a+bu)^{3/2} + C$$

55. $$\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a+bu} + C$$

56. $$\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}(8a^2 + 3b^2u^2 - 4abu)\sqrt{a+bu} + C$$

57. $$\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad \text{if } a > 0$$
$$= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, \quad \text{if } a < 0$$

58. $$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$$

59. $$\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$$

60. $$\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$$

61. $$\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$$

62. $$\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

TABLE OF INTEGRALS

TRIGONOMETRIC FORMS

63. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

64. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

65. $\int \tan^2 u\,du = \tan u - u + C$

66. $\int \cot^2 u\,du = -\cot u - u + C$

67. $\int \sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$

68. $\int \cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$

69. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

70. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

71. $\int \sec^3 u\,du = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

72. $\int \csc^3 u\,du = -\frac{1}{2}\csc u\cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

73. $\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u\,du$

74. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u\,du$

75. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1}u - \int \tan^{n-2}u\,du$

76. $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1}u - \int \cot^{n-2}u\,du$

77. $\int \sec^n u\,du = \frac{1}{n-1}\tan u\sec^{n-2}u + \frac{n-2}{n-1}\int \sec^{n-2}u\,du$

78. $\int \csc^n u\,du = \frac{-1}{n-1}\cot u\csc^{n-2}u + \frac{n-2}{n-1}\int \csc^{n-2}u\,du$

79. $\int \sin au\,\sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

80. $\int \cos au\,\cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

81. $\int \sin au\,\cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

82. $\int u\sin u\,du = \sin u - u\cos u + C$

83. $\int u\cos u\,du = \cos u + u\sin u + C$

84. $\int u^n\sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$

85. $\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

86. $$\int \sin^n u\,\cos^m u\,du = -\frac{\sin^{n-1}u\,\cos^{m+1}u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2}u\,\cos^m u\,du$$
$$= \frac{\sin^{n+1}u\,\cos^{m-1}u}{n+m} + \frac{m-1}{n+m}\int \sin^n u\,\cos^{m-2}u\,du$$

INVERSE TRIGONOMETRIC FORMS

87. $\int \sin^{-1}u\,du = u\sin^{-1}u + \sqrt{1-u^2} + C$

88. $\int \cos^{-1}u\,du = u\cos^{-1}u - \sqrt{1-u^2} + C$

89. $\int \tan^{-1}u\,du = u\tan^{-1}u - \frac{1}{2}\ln(1+u^2) + C$

90. $\int u\sin^{-1}u\,du = \frac{2u^2-1}{4}\sin^{-1}u + \frac{u\sqrt{1-u^2}}{4} + C$

91. $\int u\cos^{-1}u\,du = \frac{2u^2-1}{4}\cos^{-1}u - \frac{u\sqrt{1-u^2}}{4} + C$

92. $\int u\tan^{-1}u\,du = \frac{u^2+1}{2}\tan^{-1}u - \frac{u}{2} + C$

93. $\int u^n\sin^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1}u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

94. $\int u^n\cos^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1}u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

95. $\int u^n\tan^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1}u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

TABLE OF INTEGRALS

EXPONENTIAL AND LOGARITHMIC FORMS

96. $\displaystyle\int ue^{au}\,du = \frac{1}{a^2}(au - 1)e^{au} + C$

97. $\displaystyle\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

98. $\displaystyle\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a\sin bu - b\cos bu) + C$

99. $\displaystyle\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a\cos bu + b\sin bu) + C$

100. $\displaystyle\int \ln u\,du = u\ln u - u + C$

101. $\displaystyle\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

102. $\displaystyle\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

HYPERBOLIC FORMS

103. $\displaystyle\int \sinh u\,du = \cosh u + C$

104. $\displaystyle\int \cosh u\,du = \sinh u + C$

105. $\displaystyle\int \tanh u\,du = \ln\cosh u + C$

106. $\displaystyle\int \coth u\,du = \ln|\sinh u| + C$

107. $\displaystyle\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$

108. $\displaystyle\int \operatorname{csch} u\,du = \ln\left|\tanh\tfrac{1}{2}u\right| + C$

109. $\displaystyle\int \operatorname{sech}^2 u\,du = \tanh u + C$

110. $\displaystyle\int \operatorname{csch}^2 u\,du = -\coth u + C$

111. $\displaystyle\int \operatorname{sech} u\tanh u\,du = -\operatorname{sech} u + C$

112. $\displaystyle\int \operatorname{csch} u\coth u\,du = -\operatorname{csch} u + C$

FORMS INVOLVING $\sqrt{2au - u^2}$, $a > 0$

113. $\displaystyle\int \sqrt{2au - u^2}\,du = \frac{u-a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

114. $\displaystyle\int u\sqrt{2au - u^2}\,du = \frac{2u^2 - au - 3a^2}{6}\sqrt{2au - u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

115. $\displaystyle\int \frac{\sqrt{2au - u^2}}{u}\,du = \sqrt{2au - u^2} + a\cos^{-1}\left(\frac{a-u}{a}\right) + C$

116. $\displaystyle\int \frac{\sqrt{2au - u^2}}{u^2}\,du = -\frac{2\sqrt{2au - u^2}}{u} - \cos^{-1}\left(\frac{a-u}{a}\right) + C$

117. $\displaystyle\int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$

118. $\displaystyle\int \frac{u\,du}{\sqrt{2au - u^2}} = -\sqrt{2au - u^2} + a\cos^{-1}\left(\frac{a-u}{a}\right) + C$

119. $\displaystyle\int \frac{u^2\,du}{\sqrt{2au - u^2}} = -\frac{(u+3a)}{2}\sqrt{2au - u^2} + \frac{3a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

120. $\displaystyle\int \frac{du}{u\sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$