Vasile Cîrtoaje

ALGEBRAIC INEQUALITIES

Old and New Methods

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Warm-up problem set

1.1 Applications

1. Let a, b, c, d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$a^3 + b^3 + c^3 + d^3 \le 8.$$

2. If a, b, c are non-negative numbers, then

$$a^3 + b^3 + c^3 - 3abc \ge 2\left(\frac{b+c}{2} - a\right)^3.$$

3. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{a+b+c}{3} \ge \sqrt[5]{\frac{a^2+b^2+c^2}{3}}.$$

4. Let a, b, c be non-negative numbers such that $a^3 + b^3 + c^3 = 3$. Prove that

$$a^4b^4 + b^4c^4 + c^4a^4 \le 3.$$

(*Vasile Cîrtoaje*, GM-A, 1, 2003)

5. If a, b, c are non-negative numbers, then

$$a^2 + b^2 + c^2 + 2abc + 1 \ge 2(ab + bc + ca).$$

(*Darij Grinberg*, MS, 2004)

6. If a, b, c are distinct real numbers, then

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} \ge 2.$$

7. If a, b, c are non-negative numbers, then

$$(a^2 - bc)\sqrt{b+c} + (b^2 - ca)\sqrt{c+a} + (c^2 - ab)\sqrt{a+b} \geq 0.$$

8. If a, b, c, d are non-negative real numbers, then

$$\frac{a-b}{a+2b+c} + \frac{b-c}{b+2c+d} + \frac{c-d}{c+2d+a} + \frac{d-a}{d+2a+b} \geq 0$$

9. Let a, b, c be non-negative numbers such that

$$a^2 + b^2 + c^2 = a + b + c$$

Prove that

$$a^2b^2 + b^2c^2 + c^2a^2 \leq ab + bc + ca.$$

(*Vasile Cîrtoaje*, MS, 2006)

10. Let a, b, c be non-negative numbers, no two of them are zero. Then,

$$\frac{a^2}{a^2 + ab + b^2} + \frac{b^2}{b^2 + bc + c^2} + \frac{c^2}{c^2 + ca + a^2} \geq 1$$

11. If a, b, c are non-negative numbers, then

$$\sqrt{\frac{a^3}{a^3 + (b+c)^3}} + \sqrt{\frac{b^3}{b^3 + (c+a)^3}} + \sqrt{\frac{c^3}{c^3 + (a+b)^3}} \geq 1.$$

12. Let a, b, c be positive numbers and let

$$E(a, b, c) = a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b).$$

Prove that

a) $(a+b+c)E(a,b,c) \geq ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2$,

b) $2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)E(a,b,c) \geq (a-b)^2 + (b-c)^2 + (c-a)^2$.

(*Vasile Cîrtoaje*, MS, 2005)

13. Let a, b, c and x, y, z be real numbers such that $a + x \geq b + y \geq c + z \geq 0$ and $a + b + c = x + y + z$. Prove that

$$ay + bx \geq ac + xz.$$

14. Let $a, b, c \in \left[\frac{1}{3}, 3\right]$ Prove that

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \geq \frac{7}{5}.$$

15. Let a, b, c and x, y, z be non-negative numbers such that

$$a+b+c = x+y+z.$$

Prove that

$$ax(a+x) + by(b+y) + cz(c+z) \geq 3(abc + xyz).$$

(*Vasile Cîrtoaje*, MS, 2005)

16. If a, b, c are non-negative numbers, then

$$4(a+b+c)^3 \geq 27(ab^2 + bc^2 + ca^2 + abc).$$

17. Let a, b, c be non-negative numbers such that $a+b+c=3$. Prove that

$$\frac{1}{2ab^2+1} + \frac{1}{2bc^2+1} + \frac{1}{2ca^2+1} \geq 1.$$

18. If a, b, c, d are positive numbers, then

$$\frac{1}{a^2+ab} + \frac{1}{b^2+bc} + \frac{1}{c^2+cd} + \frac{1}{d^2+da} \geq \frac{4}{ac+bd}$$

19. If $a, b, c \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$, then

$$\frac{3}{a+2b} + \frac{3}{b+2c} + \frac{3}{c+2a} \geq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a}.$$

20. Let a, b, c be non-negative numbers such that $ab+bc+ca=3$. Prove that

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq 1.$$

21. Let a, b, c be non-negative real numbers such that $ab+bc+ca=3$. Prove that

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} \geq \frac{3}{2}.$$

(*Vasile Cîrtoaje*, MS, 2005)

22. Let a, b, c be non-negative numbers such that $a^2 + b^2 + c^2 = 3$ Prove that

$$\frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} \le 1$$

(*Vasile Cîrtoaje*, MS, 2005)

23. Let a, b, c be positive numbers such that $abc = 1$. Prove that

a) $\dfrac{a-1}{b} + \dfrac{b-1}{c} + \dfrac{c-1}{a} \ge 0,$

b) $\dfrac{a-1}{b+c} + \dfrac{b-1}{c+a} + \dfrac{c-1}{a+b} \ge 0$

24. Let a, b, c, d be non-negative numbers such that $a^2 - ab + b^2 = c^2 - cd + d^2$. Prove that

$$(a+b)(c+d) \ge 2(ab+cd).$$

25. Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 a_2 \ldots a_n = 1$. Prove that

$$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \cdots + \frac{1}{1+(n-1)a_n} \ge 1$$

(*Vasile Cîrtoaje*, GM-B, 10, 1991)

26. Let a, b, c, d be non-negative real numbers such that $a^2 + b^2 + c^2 + d^2 = 1$ Prove that

$$(1-a)(1-b)(1-c)(1-d) \ge abcd.$$

(*Vasile Cîrtoaje*, GM-B, 9-10, 2001)

27. If a, b, c are positive real numbers, then

$$\sqrt{\frac{2a}{a+b}} + \sqrt{\frac{2b}{b+c}} + \sqrt{\frac{2c}{c+a}} \le 3.$$

(*Vasile Cîrtoaje*, GM-B, 7-8, 1992)

28. If a, b, c, d are positive real numbers, then

$$\left(\frac{a}{a+b}\right)^2 + \left(\frac{b}{b+c}\right)^2 + \left(\frac{c}{c+d}\right)^2 + \left(\frac{d}{d+a}\right)^2 \ge 1.$$

(*Vasile Cîrtoaje*, GM-B, 6, 1995)

29. Let a, b, c be positive numbers such that $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. If $a \le b \le c$, then

$$ab^2c^3 \ge 1.$$

(*Vasile Cîrtoaje*, GM-B, 11, 1998)

30. Let a, b, c be non-negative numbers, no two of them are zero Then

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2} \ge \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

(*Vasile Cîrtoaje*, GM-B, 10, 2002)

31. If a, b, c are non-negative numbers, then

$$2(a^2+1)(b^2+1)(c^2+1) \ge (a+1)(b+1)(c+1)(abc+1).$$

(*Vasile Cîrtoaje*, GM-A, 2, 2001)

32. If a, b, c are non-negative numbers, then

$$3(1-a+a^2)(1-b+b^2)(1-c+c^2) \ge 1+abc+a^2b^2c^2.$$

(*Vasile Cîrtoaje*, *Mircea Lascu*, RMT, 1-2, 1989)

33. If a, b, c, d are non-negative numbers, then

$$(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2) \ge \left(\frac{1+abcd}{2}\right)^2.$$

(*Vasile Cîrtoaje*, GM-B, 1, 1992)

34. If a, b, c are non-negative numbers, then

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge (ab+bc+ca)^3.$$

(*Vasile Cîrtoaje*, *Mircea Lascu*, ONI, 1995)

35. Let a, b, c, d be positive numbers such that $abcd = 1$. Prove that

$$\frac{1}{1+ab+bc+ca}+\frac{1}{1+bc+cd+db}+\frac{1}{1+cd+da+ac}+\frac{1}{1+da+ab+bd} \le 1.$$

36. If a, b, c and x, y, z are real numbers, then

$$4(a^2+x^2)(b^2+y^2)(c^2+z^2) \geq 3(bcx+cay+abz)^2.$$

(*Vasile Cîrtoaje*, MS, 2004)

37. If $a \geq b \geq c \geq d \geq e$, then

$$(a+b+c+d+e)^2 \geq 8(ac+bd+ce).$$

For $e \geq 0$, determine when equality occurs.

(*Vasile Cîrtoaje*, MS, 2005)

38. If a, b, c, d are real numbers, then

$$6(a^2+b^2+c^2+d^2)+(a+b+c+d)^2 \geq 12(ab+bc+cd)$$

(*Vasile Cîrtoaje*, MS, 2005)

39. If a, b, c are positive numbers, then

$$\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq 1+\sqrt{1+\sqrt{(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}.$$

(*Vasile Cîrtoaje*, GM-B, 11, 2002)

40. If a, b, c, d are positive numbers, then

$$5+\sqrt{2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-2} \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

(*Vasile Cîrtoaje*, GM-B, 5, 2004)

41. If a, b, c, d are positive numbers, then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b} \geq 0.$$

42. If $a, b, c > -1$, then

$$\frac{1+a^2}{1+b+c^2}+\frac{1+b^2}{1+c+a^2}+\frac{1+c^2}{1+a+b} \geq 2.$$

(*Laurenţiu Panaitopol*, Junior BMO, 2003)

43. Let a, b, c and x, y, z be positive real numbers such that

$$(a+b+c)(x+y+z) = (a^2+b^2+c^2)(x^2+y^2+z^2) = 4.$$

Prove that

$$abcxyz < \frac{1}{36}.$$

(*Vasile Cîrtoaje, Mircea Lascu*, ONI, 1996)

44. Let a, b, c be positive numbers such that $a^2+b^2+c^2=3$ Prove that

$$\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} \geq 3.$$

(*Cezar Lupu*, MS, 2005)

45. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+bc} + \frac{1}{b^2+ca} + \frac{1}{c^2+ab} \geq \frac{3}{ab+bc+ca}.$$

(*Vasile Cîrtoaje*, MS, 2005)

46. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{1}{b^2-bc+c^2} + \frac{1}{c^2-ca+a^2} + \frac{1}{a^2-ab+b^2} \geq \frac{3}{ab+bc+ca}.$$

47. Let a, b, c be positive numbers such that $a+b+c=3$ Prove that

$$abc + \frac{12}{ab+bc+ca} \geq 5$$

48. Let a, b, c be non-negative numbers such that $a^2+b^2+c^2=3$ Prove that

$$12 + 9abc \geq 7(ab+bc+ca).$$

(*Vasile Cîrtoaje*, MS, 2005)

49. Let a, b, c be non-negative numbers such that $ab+bc+ca=3$. Prove that

$$a^3+b^3+c^3+7abc \geq 10.$$

(*Vasile Cîrtoaje*, MS, 2005)

50. If a, b, c are positive numbers such that $abc = 1$, then

$$(a+b)(b+c)(c+a) + 7 \geq 5(a+b+c)$$

(*Vasile Cîrtoaje*, MS, 2005)

51. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{a^3}{(2a^2+b^2)(2a^2+c^2)} + \frac{b^3}{(2b^2+c^2)(2b^2+a^2)} + \frac{c^3}{(2c^2+a^2)(2c^2+b^2)} \leq \frac{1}{a+b+c}$$

(*Vasile Cîrtoaje*, MS, 2005)

52. Let a, b, c be non-negative numbers such that $a+b+c \geq 3$. Prove that

$$\frac{1}{a^2+b+c} + \frac{1}{a+b^2+c} + \frac{1}{a+b+c^2} \leq 1$$

53. Let a, b, c be non-negative numbers such that $ab+bc+ca = 3$. If $r \geq 1$, then

$$\frac{1}{r+a^2+b^2} + \frac{1}{r+b^2+c^2} + \frac{1}{r+c^2+a^2} \leq \frac{3}{r+2}.$$

(*Pham Van Thuan*, MS, 2005)

54. Let a, b, c be positive numbers such that $abc = 1$ Prove that

$$\frac{1}{(1+a)^3} + \frac{1}{(1+b)^3} + \frac{1}{(1+c)^3} + \frac{5}{(1+a)(1+b)(1+c)} \geq 1.$$

(*Pham Kim Hung*, MS, 2006)

55. Let a, b, c be positive numbers such that $abc = 1$ Prove that

$$\frac{2}{a+b+c} + \frac{1}{3} \geq \frac{3}{ab+bc+ca}$$

56. If a, b, c are real numbers, then

$$2(1+abc) + \sqrt{2(1+a^2)(1+b^2)(1+c^2)} \geq (1+a)(1+b)(1+c)$$

(*Wolfgang Berndt*, MS, 2006)

57. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{a(b+c)}{a^2+bc} + \frac{b(c+a)}{b^2+ca} + \frac{c(a+b)}{c^2+ab} \geq 2$$

(*Pham Kim Hung*, MS, 2006)

58. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\sqrt{\frac{a(b+c)}{a^2+bc}} + \sqrt{\frac{b(c+a)}{b^2+ca}} + \sqrt{\frac{c(a+b)}{c^2+ab}} \geq 2.$$

(*Vasile Cîrtoaje*, MS, 2006)

59. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \geq \frac{a}{a^2+bc} + \frac{b}{b^2+ca} + \frac{c}{c^2+ab}.$$

60. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \geq \frac{2a}{3a^2+bc} + \frac{2b}{3b^2+ca} + \frac{2c}{3c^2+ab}$$

(*Vasile Cîrtoaje*, MS, 2005)

61. Let a, b, c be positive numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$5(a+b+c) + \frac{3}{abc} \geq 18.$$

(*Vasile Cîrtoaje*, MS, 2005)

62. Let a, b, c be non-negative numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{6-ab} + \frac{1}{6-bc} + \frac{1}{6-ca} \leq \frac{3}{5}$$

63. Let $n \geq 4$ and let $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n \quad \text{and} \quad a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that

$$\max\{a_1, a_2, \ldots, a_n\} \geq 2.$$

(*Titu Andreescu*, USAMO, 1999)

64. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{13}{6} - \frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)}.$$

(*Vasile Cîrtoaje*, MS, 2006)

65. Let a, b, c be non-negative numbers, no two of which are zero. Prove that
$$\frac{a^2(b+c)}{b^2+c^2}+\frac{b^2(c+a)}{c^2+a^2}+\frac{c^2(a+b)}{a^2+b^2}\ge a+b+c$$

(*Darij Grinberg*, MS, 2004)

66. Let a, b, c be non-negative numbers such that
$$(a+b)(b+c)(c+a)=2.$$

Prove that
$$(a^2+bc)(b^2+ca)(c^2+ab)\le 1.$$

(*Vasile Cîrtoaje*, MS, 2005)

1.2 Solutions

1. *Let a, b, c, d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that*

$$a^3 + b^3 + c^3 + d^3 \leq 8.$$

Solution. From $a^2 + b^2 + c^2 + d^2 = 4$ we get $a^2 \leq 4$, $a \leq 2$, $a^2(a-2) \leq 0$, $a^3 \leq 2a^2$. Similarly, $b^3 \leq 2b^2$, $c^3 \leq 2c^2$, $d^3 \leq 2d^2$. Thus,

$$a^3 + b^3 + c^3 + d^3 \leq 2(a^2 + b^2 + c^2 + d^2) = 8.$$

Equality occurs for $(a, b, c, d) = (2, 0, 0, 0)$ or any cyclic permutation.

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2. *If a, b, c are non-negative numbers, then*

$$a^3 + b^3 + c^3 - 3abc \geq 2\left(\frac{b+c}{2} - a\right)^3.$$

Solution. By the AM-GM Inequality we have $a^3 + b^3 + c^3 \geq 3abc$. If $\frac{b+c}{2} - a \leq 0$, the inequality is trivial. Consider now $\frac{b+c}{2} - a > 0$. Let

$$E = a^3 + b^3 + c^3 - 3abc - 2\left(\frac{b+c}{2} - a\right)^3.$$

Setting $b = a + 2x$ and $c = a + 2y$, we obtain

$$E = 12a(x^2 - xy + y^2) + 6(x+y)(x-y)^2 \geq$$
$$\geq 6(x+y)(x-y)^2 = \frac{3}{2}\left(\frac{b+c}{2} - a\right)(b-c)^2 \geq 0.$$

Equality occurs when either $(a, b, c) \sim (1, 1, 1)$ or $(a, b, c) \sim (0, 1, 1)$

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3. *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{a+b+c}{3} \geq \sqrt[5]{\frac{a^2 + b^2 + c^2}{3}}.$$

First Solution Write the inequality in the homogeneous form

$$(a+b+c)^5 \geq 81abc(a^2+b^2+c^2)$$

In order to eliminate the product abc, we can use the known inequality

$$(ab+bc+ca)^2 \geq 3abc(a+b+c),$$

which is equivalent to

$$a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2 \geq 0$$

Thus, we still have to show that

$$(a+b+c)^6 \geq 27(ab+bc+ca)^2(a^2+b^2+c^2).$$

Setting $S=a+b+c$ and $Q=ab+bc+ca$ yields

$$(a+b+c)^6-27(ab+bc+ca)^2(a^2+b^2+c^2)=$$
$$=S^6-27Q^2(S^2-2Q)=(S^2-3Q)^2(S^2+6Q) \geq 0.$$

Equality occurs for $a=b=c=1$

Second Solution In order to prove the homogeneous inequality

$$(a+b+c)^5 \geq 81abc(a^2+b^2+c^2),$$

we may give up the constraint $abc=1$ and assume that $a+b+c=3$. For $a+b+c=3$, we must show that the expression

$$E(a,b,c)=abc(a^2+b^2+c^2)$$

is maximal for $a=b=c=1$. For the sake of contradiction, assume that $E(a,b,c)$ is maximal for a triple (a,b,c) with $b \neq c$. To finish the proof it suffices to show that

$$E(a,b,c)<E\left(a,\frac{b+c}{2},\frac{b+c}{2}\right)$$

Indeed, we have

$$E\left(a,\frac{b+c}{2},\frac{b+c}{2}\right)-E(a,b,c)=a^3\left[\left(\frac{b+c}{2}\right)^2-bc\right]+a\left[2\left(\frac{b+c}{2}\right)^4-bc(b^2+c^2)\right]=$$
$$=\frac{1}{4}a^3(b-c)^2+\frac{1}{8}a(b-c)^4>0$$

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4. *Let a, b, c be non-negative numbers such that $a^3 + b^3 + c^3 = 3$. Prove that*

$$a^4b^4 + b^4c^4 + c^4a^4 \le 3.$$

Solution (by *Gabriel Dospinescu*). By the AM-GM Inequality, we have

$$bc \le \frac{b^3 + c^3 + 1}{3} = \frac{4 - a^3}{3}.$$

Then,

$$b^4c^4 \le \frac{4b^3c^3 - a^3b^3c^3}{3}$$

and, similarly,

$$c^4a^4 \le \frac{4c^3a^3 - a^3b^3c^3}{3}, \quad a^4b^4 \le \frac{4a^3b^3 - a^3b^3c^3}{3}.$$

Summing up these inequalities yields

$$a^4b^4 + b^4c^4 + c^4a^4 \le \frac{4(a^3b^3 + b^3c^3 + c^3a^3)}{3} - a^3b^3c^3.$$

Thus, it suffices to show that

$$4(a^3b^3 + b^3c^3 + c^3a^3) - 3a^3b^3c^3 \le 9,$$

which is just the third degree Schur's Inequality

$$4(xy + yz + zx)(x + y + z) - 9xyz \le (x + y + z)^3$$

for $x = a^3$, $y = b^3$, $z = c^3$. Equality occurs for $a = b = c = 1$.

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5. *If a, b, c are non-negative numbers, then*

$$a^2 + b^2 + c^2 + 2abc + 1 \ge 2(ab + bc + ca).$$

Solution. Among the numbers $1 - a$, $1 - b$ and $1 - c$ there are always two of them with the same sign; let us say $(1 - b)(1 - c) \ge 0$. We have

$$a^2 + b^2 + c^2 + 2abc + 1 - 2(ab + bc + ca) =$$
$$= (a - 1)^2 + (b - c)^2 + 2a + 2abc - 2(ab + ca) =$$
$$= (a - 1)^2 + (b - c)^2 + 2a(1 - b)(1 - c) \ge 0.$$

Equality occurs for $a = b = c = 1$.

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6. *If a, b, c are distinct real numbers, then*

$$\frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2}\geq 2.$$

Solution. Using the well-known identity

$$\frac{bc}{(a-b)(a-c)}+\frac{ca}{(b-c)(b-a)}+\frac{ab}{(c-a)(c-b)}=1,$$

we have

$$\frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2}=\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)^2+$$
$$+\frac{2bc}{(a-b)(a-c)}+\frac{2ca}{(b-c)(b-a)}+\frac{2ab}{(c-a)(c-b)}=$$
$$=\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)^2+2\geq 2$$

The equality occurs only in the case

$$\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}=0.$$

★

7. *If a, b, c are non-negative numbers, then*

$$(a^2-bc)\sqrt{b+c}+(b^2-ca)\sqrt{c+a}+(c^2-ab)\sqrt{a+b}\geq 0.$$

First Solution Letting $b+c=2x^2$, $c+a=2y^2$ and $a+b=2z^2$ ($x\geq 0$, $y\geq 0$, $z\geq 0$) yields

$$a=-x^2+y^2+z^2,\ b=x^2-y^2+z^2,\ c=x^2+y^2-z^2.$$

The inequality transforms into

$$xy(x^3+y^3)+yz(y^3+z^3)+zx(z^3+x^3)\geq x^2y^2(x+y)+y^2z^2(y+z)+z^2x^2(z+x).$$

Since $xy(x^3+y^3)-x^2y^2(x+y)=xy(x+y)(x-y)^2$, we may write the inequality in the form

$$xy(x+y)(x-y)^2+yz(y+z)(y-z)^2+zx(z+x)(z-x)^2\geq 0,$$

which is clearly true. For $a \geq b \geq c$, equality occurs when either $(a,b,c) \sim (1,1,1)$ or $(a,b,c) \sim (1,0,0)$.

Second Solution. If two of a, b, c are zero, then the inequality becomes equality. Otherwise, we write the inequality in the form

$$\frac{(a^2-bc)(b+c)}{\sqrt{b+c}} + \frac{(b^2-ca)(c+a)}{\sqrt{c+a}} + \frac{(c^2-ab)(a+b)}{\sqrt{a+b}} \geq 0,$$

or

$$\frac{X}{\sqrt{b+c}} + \frac{Y}{\sqrt{c+a}} + \frac{Z}{\sqrt{a+b}} \geq 0,$$

where

$$X = (a^2-bc)(b+c),\ Y = (b^2-ca)(c+a),\ Z = (c^2-ab)(a+b).$$

Consider now, without loss of generality, that $a \geq b \geq c$. It is easy to check that $X+Y+Z=0$, $X \geq 0$ and $Z \leq 0$ Therefore,

$$\frac{X}{\sqrt{b+c}} + \frac{Y}{\sqrt{c+a}} + \frac{Z}{\sqrt{a+b}} = \frac{X}{\sqrt{b+c}} - \frac{X+Z}{\sqrt{c+a}} + \frac{Z}{\sqrt{a+b}} =$$
$$= X\left(\frac{1}{\sqrt{b+c}} - \frac{1}{\sqrt{c+a}}\right) + (-Z)\left(\frac{1}{\sqrt{c+a}} - \frac{1}{\sqrt{a+b}}\right) \geq 0.$$

Third Solution. Write the inequality as

$$A(a^2-bc) + B(b^2-ca) + C(c^2-ab) \geq 0,$$

where $A = \sqrt{b+c}$, $B = \sqrt{c+a}$ and $C = \sqrt{a+b}$ We have

$$\begin{aligned} 2\sum A(a^2-bc) &= \sum A\left[(a-b)(a+c) + (a-c)(a+b)\right] = \\ &= \sum A(a-b)(a+c) + \sum B(b-a)(b+c) = \\ &= \sum (a-b)\left[A(a+c) - B(b+c)\right] = \\ &= \sum (a-b)\frac{A^2(a+c)^2 - B^2(b+c)^2}{A(a+c)+B(b+c)} = \\ &= \sum \frac{(a-b)^2(a+c)(b+c)}{A(a+c)+B(b+c)} \geq 0, \end{aligned}$$

where $\sum$ is cyclic over a, b, c.

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8. *If a, b, c, d are non-negative real numbers, then*

$$\frac{a-b}{a+2b+c}+\frac{b-c}{b+2c+d}+\frac{c-d}{c+2d+a}+\frac{d-a}{d+2a+b}\geq 0.$$

Solution. Write first the inequality as

$$\sum\left(\frac{a-b}{a+2b+c}+\frac{1}{2}\right)\geq 2$$

or

$$\sum\frac{3a+c}{a+2b+c}\geq 4.$$

By the Cauchy-Schwarz Inequality, we have

$$\sum\frac{3a+c}{a+2b+c}\geq\frac{\left[\sum(3a+c)\right]^2}{\sum(3a+c)(a+2b+c)}.$$

Since

$$\sum(3a+c)(a+2b+c)=4(a+b+c+d)^2$$

and

$$\left[\sum(3a+c)\right]^2=16(a+b+c+d)^2,$$

we get

$$\sum\frac{3a+c}{a+2b+c}\geq 4.$$

Equality occurs for $a=c$ and $b=d$.

★

9. *Let a, b, c be non-negative numbers such that*

$$a^2+b^2+c^2=a+b+c.$$

Prove that

$$a^2b^2+b^2c^2+c^2a^2\leq ab+bc+ca.$$

Solution (by *Michael Rozenberg*) By squaring, from the hypothesis condition we get

$$a^4+b^4+c^4-a^2-b^2-c^2=2(ab+bc+ca-a^2b^2+b^2c^2+c^2a^2).$$

Therefore, the required inequality is equivalent to

$$a^4+b^4+c^4\geq a^2+b^2+c^2.$$

The homogeneous form of this inequality,

$$(a+b+c)^2(a^4+b^4+c^4) \geq (a^2+b^2+c^2)^3,$$

follows immediately from Hölder's Inequality.

Equality occurs for $(a,b,c) = (1,1,1)$, for $(a,b,c) = (0,0,0)$, for $(a,b,c) = (0,1,1)$ or any cyclic permutation, and also for $(a,b,c) = (1,0,0)$ or any cyclic permutation

10. *Let a, b, c be non-negative numbers, no two of them are zero. Then,*

$$\frac{a^2}{a^2+ab+b^2} + \frac{b^2}{b^2+bc+c^2} + \frac{c^2}{c^2+ca+a^2} \geq 1$$

Solution. Let $A = a^2+ab+b^2$, $B = b^2+bc+c^2$ and $C = c^2+ca+a^2$ We have

$$\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)\left(\frac{a^2}{A}+\frac{b^2}{B}+\frac{c^2}{C}-1\right) =$$

$$= \sum \frac{a^2}{A^2} + \sum \frac{b^2+c^2}{BC} - \sum \frac{1}{A} =$$

$$= \sum \left(\frac{a^2}{A^2} - \frac{bc}{BC}\right) + \sum \frac{b^2+bc+c^2}{BC} - \sum \frac{1}{A} =$$

$$= \sum \left(\frac{a^2}{A^2} - \frac{bc}{BC}\right) = \frac{1}{2} \sum \left(\frac{b}{B} - \frac{c}{C}\right)^2 \geq 0,$$

from which the desired inequality follows. Equality occurs only for $a = b = c$.

★

11. *If a, b, c are non-negative numbers, then*

$$\sqrt{\frac{a^3}{a^3+(b+c)^3}} + \sqrt{\frac{b^3}{b^3+(c+a)^3}} + \sqrt{\frac{c^3}{c^3+(a+b)^3}} \geq 1.$$

Solution. By the AM-GM Inequality, for $x \geq 0$, we have

$$\sqrt{1+x^3} = \sqrt{(1+x)(1-x+x^2)} \leq \frac{(1+x)+(1-x+x^2)}{2} = 1 + \frac{x^2}{2}$$

Consequently, for $a > 0$ we get

$$\sqrt{\frac{a^3}{a^3+(b+c)^3}} = \frac{1}{\sqrt{1+\left(\frac{b+c}{a}\right)^3}} \geq \frac{1}{1+\frac{1}{2}\left(\frac{b+c}{a}\right)^2} \geq \frac{1}{1+\frac{b^2+c^2}{a^2}} =$$

$$= \frac{a^2}{a^2 \quad b^2+c^2}.$$

The obtained inequality is clearly true for $a = 0$ as well. Analogously,

$$\sqrt{\frac{b^3}{b^3+(c+a)^3}} \geq \frac{b^2}{a^2+b^2+c^2}, \quad \sqrt{\frac{c^3}{c^3+(a+b)^3}} \geq \frac{c^2}{a^2+b^2+c^2}.$$

Adding up these inequalities, the conclusion follows Equality occurs only for $a = b = c$.

★

12. *Let a, b, c be positive numbers and let*

$$E(a,b,c) = a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b).$$

Prove that:

a) $(a+b+c)E(a,b,c) \geq ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2$;

b) $2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)E(a,b,c) \geq (a-b)^2 + (b-c)^2 + (c-a)^2$.

Solution. a) Using the Schur's Inequality $\sum a^2(a-b)(a-c) \geq 0$, we have

$$(a+b+c)E(a,b,c) = \sum a^2(a-b)(a-c) + \sum a(b+c)(a-b)(a-c) \geq$$
$$\geq \sum a(b+c)(a-b)(a-c) = \sum ab(a-b)(a-c) + \sum ca(a-b)(a-c) =$$
$$= \sum ab(a-b)(a-c) + \sum ab(b-c)(b-a) = \sum ab(a-b)^2$$

b) Since

$$(ab+bc+ca)\sum a(a-b)(a-c) =$$
$$= \sum abc(a-b)(a-c) + \sum (ab+ac)a(a-b)(a-c) =$$
$$= abc(a^2+b^2+c^2-ab-bc-ca) + \sum bc\left[b(b-c)(b-a)+c(c-a)(c-b)\right] =$$
$$= \frac{1}{2}abc\sum(b-c)^2 + \sum bc(b+c-a)(b-c)^2,$$

the inequality is equivalent to

$$\sum bc(b+c-a)(b-c)^2 \geq 0$$

Without loss of generality, assume that $a \geq b \geq c$ Then,

$$\sum bc(b+c-a)(b-c)^2 \geq bc(b+c-a)(b-c)^2 + ac(a+c-b)(a-c)^2 \geq$$
$$\geq bc(b+c-a)(b-c)^2 + ac(a+c-b)(b-c)^2 =$$
$$= c(b-c)^2\left[(a-b)^2 + c(a+b)\right] \geq 0$$

The both inequalities become equality for $(a,b,c) \sim (1,1,1)$. Notice that the first inequality is valid for any non-negative a,b,c and becomes again equality for $(a,b,c) \sim (0,1,1)$ or any cyclic permutation.

★

13. *Let a,b,c and x,y,z be real numbers such that $a+x \geq b+y \geq c+z \geq 0$ and $a+b+c = x+y+z$. Prove that*

$$ay + bx \geq ac + xz.$$

Solution. We have

$$\begin{aligned} ay + bx - ac - xz &= a(y-c) + x(b-z) = a(a+b-x-z) + x(b-z) = \\ &= a(a-x) + (a+x)(b-z) = \\ &= \frac{1}{2}(a-x)^2 + \frac{1}{2}(a^2-x^2) + (a+x)(b-z) = \\ &= \frac{1}{2}(a-x)^2 + \frac{1}{2}(a+x)(a+2b-x-2z) = \\ &= \frac{1}{2}(a-x)^2 + \frac{1}{2}(a+x)(b-c+y-z) \geq 0, \end{aligned}$$

from which the required inequality follows. Equality occurs for $a = x$, $b = z$, $c = y$ and $2x \geq y+z \geq 0$.

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14. *Let $a,b,c \in \left[\frac{1}{3}, 3\right]$. Prove that*

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \geq \frac{7}{5}.$$

Solution. Denote

$$E(a,b,c) = \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a},$$

and assume, without loss of generality, that $a = \max\{a,b,c\}$. We will show that

$$E(a,b,c) \geq E\left(a,b,\sqrt{ab}\right) \geq \frac{7}{5}.$$

We have

$$E(a,b,c) - E\left(a,b,\sqrt{ab}\right) = \frac{b}{b+c} + \frac{c}{c+a} - \frac{2\sqrt{b}}{\sqrt{a}+\sqrt{b}} =$$
$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{ab}-c\right)^2}{\left(\sqrt{a}+\sqrt{b}\right)(b+c)(c+a)} \geq 0.$$

Let now $x = \sqrt{\frac{a}{b}}$. From $a,b,c \in \left[\frac{1}{3},3\right]$, we get $x \leq 3$. Hence,

$$E\left(a,b,\sqrt{ab}\right) - \frac{7}{5} = \frac{a}{a+b} + \frac{2\sqrt{b}}{\sqrt{a}+\sqrt{b}} - \frac{7}{5} = \frac{x^2}{x^2+1} + \frac{2}{x+1} - \frac{7}{5} =$$
$$= \frac{3-7x+8x^2-2x^3}{5(x+1)(x^2+1)} = \frac{(3-x)\left[x^2+(1-x)^2\right]}{5(x+1)(x^2+1)} \geq 0$$

Equality occurs for $(a,b,c) = \left(3,\frac{1}{3},1\right)$ or any cyclic permutation.

★

15. *Let a,b,c and x,y,z be non-negative numbers such that*

$$a+b+c = x+y+z.$$

Prove that

$$ax(a+x) + by(b+y) + cz(c+z) \geq 3(abc+xyz).$$

Solution. Applying the Cauchy-Schwarz Inequality to the triples

$$\left(a\sqrt{x}, b\sqrt{y}, c\sqrt{z}\right) \quad \text{and} \quad \left(\sqrt{yz}, \sqrt{zx}, \sqrt{xy}\right),$$

we get

$$(a^2x+b^2y+c^2z)(yz+zx+xy) \geq xyz(a+b+c)^2$$

This implies together with

$$(a+b+c)^2 = (x+y+z)^2 \geq 3(xy+yz+zx),$$

that

$$a^2x+b^2y+c^2z \geq 3xyz$$

Similarly,

$$ax^2+by^2+cz^2 \geq 3abc$$

Adding up these inequalities yields the desired result.

★

16. *If a, b, c are non-negative numbers, then*

$$4(a+b+c)^3 \geq 27(ab^2+bc^2+ca^2+abc)$$

Solution. Without loss of generality, suppose that $a = \min\{a, b, c\}$. Setting $b = a + x$ and $c = a + y$ $(x \geq 0, y \geq 0)$, the inequality reduces to

$$9(x^2 - xy + y^2)a + (2x - y)^2(x + 4y) \geq 0,$$

which is obviously true. Equality occurs for $(a, b, c) \sim (1, 1, 1)$, and also for $(a, b, c) \sim (0, 1, 2)$ or any cyclic permutation

★

17. *Let a, b, c be non-negative numbers such that $a + b + c = 3$. Prove that*

$$\frac{1}{2ab^2+1} + \frac{1}{2bc^2+1} + \frac{1}{2ca^2+1} \geq 1.$$

Solution. The inequality is equivalent to

$$ab^2 + bc^2 + ca^2 + 1 \geq 4a^3b^3c^3.$$

By the AM-GM Inequality, we have

$$ab^2 + bc^2 + ca^2 \geq 3abc,$$

and

$$1 = \left(\frac{a+b+c}{3}\right)^3 \geq abc.$$

Then,

$$ab^2 + bc^2 + ca^2 + 1 - 4a^3b^3c^3 \geq 3abc + 1 - 4a^3b^3c^3 = (1 - abc)(1 + 2abc)^2 \geq 0.$$

Equality occurs for $a = b = c = 1$.

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18. *If a, b, c, d are positive numbers, then*

$$\frac{1}{a^2+ab} + \frac{1}{b^2+bc} + \frac{1}{c^2+cd} + \frac{1}{d^2+da} \geq \frac{4}{ac+bd}.$$

Solution. Write the inequality as follows

$$\sum\left(\frac{ac+bd}{a^2+ab}+1\right)\geq 8,$$

$$\sum\left[\frac{c+a}{a+b}+\frac{b(d+a)}{a(a+b)}\right]\geq 8,$$

$$\sum\frac{c+a}{a+b}+\sum\frac{b(d+a)}{a(a+b)}\geq 8.$$

By the AM-GM Inequality, we get

$$\sum\frac{b(d+a)}{a(a+b)}=\frac{b(d+a)}{a(a+b)}+\frac{c(a+b)}{b(b+c)}+\frac{d(b+c)}{c(c+d)}+\frac{a(c+d)}{d(d+a)}\geq 4$$

Therefore, it remains to show that

$$\sum\frac{c+a}{a+b}\geq 4$$

We have

$$\sum\frac{c+a}{a+b}=\frac{c+a}{a+b}+\frac{d+b}{b+c}+\frac{a+c}{c+d}+\frac{b+d}{d+a}=$$
$$=(a+c)\left(\frac{1}{a+b}+\frac{1}{c+d}\right)+(b+d)\left(\frac{1}{a+d}+\frac{1}{b+c}\right).$$

Since $\frac{1}{a+b}+\frac{1}{c+d}\geq\frac{4}{(a+b)+(c+d)}$ and $\frac{1}{a+d}+\frac{1}{b+c}\geq\frac{4}{(a+d)+(b+c)}$ we get

$$\sum\frac{c+a}{a+b}\geq\frac{4(a+c)}{a+b+c+d}+\frac{4(b+d)}{a+b+c+d}=4.$$

Equality occurs for $a=b=c=d$

★

19. *If* $a,b,c\in\left[\frac{1}{\sqrt{2}},\sqrt{2}\right]$, *then*

$$\frac{3}{a+2b}+\frac{3}{b+2c}+\frac{3}{c+2a}\geq\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}.$$

Solution. Write the inequality as follows

$$\sum\left(\frac{3}{a+2b}-\frac{2}{a+b}+\frac{1}{6a}-\frac{1}{6b}\right)\geq 0,$$

$$\sum\frac{(a-b)^2(2b-a)}{6ab(a+2b)(a+b)}\geq 0.$$

Since

$$2b - a \geq \frac{2}{\sqrt{2}} - \sqrt{2} = 0,$$

the inequality is obviously true Equality occurs for $a = b = c$

★

20. *Let a, b, c be non-negative numbers such that $ab + bc + ca = 3$. Prove that*

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq 1.$$

Solution. The inequality is equivalent to

$$a^2b^2 + b^2c^2 + c^2a^2 + a^2b^2c^2 \geq 4.$$

By setting $bc = x$, $ca = y$ and $ab = x$, we have to show that

$$x^2 + y^2 + z^2 + xyz \geq 4,$$

for $x, y, z \geq 0$ such that $x + y + z = 3$. Assuming that $x = \min\{x, y, z\}$, $x \leq 1$, we have

$$\begin{aligned} x^2 + y^2 + z^2 + xyz - 4 &= x^2 + (y+z)^2 + yz(x-2) - 4 \geq \\ &\geq x^2 + (y+z)^2 + \frac{1}{4}(y+z)^2(x-2) - 4 = \\ &= x^2 + \frac{x+2}{4}(y+z)^2 - 4 = x^2 + \frac{x+2}{4}(3-x)^2 - 4 \\ &= \frac{1}{4}(x-1)^2(x+2) \geq 0 \end{aligned}$$

Equality occurs for $a = b = c = 1$.

★

21. *Let a, b, c be non-negative real numbers such that $ab + bc + ca = 3$. Prove that*

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} \geq \frac{3}{2}.$$

First Solution. By expanding, the inequality becomes

$$a^2 + b^2 + c^2 + 3 \geq a^2b^2 + b^2c^2 + c^2a^2 + 3a^2b^2c^2.$$

By the AM-GM Inequality, we have

$$(a + b + c)(ab + bc + ca \geq 9abc,$$

that is

$$a + b + c \geq 3abc$$

Thus, it suffices to show that

$$a^2 + b^2 + c^2 + 3 \geq a^2b^2 + b^2c^2 + c^2a^2 + abc(a + b + c)$$

This inequality is equivalent to the homogeneous inequality

$$(ab+bc+ca)(a^2+b^2+c^2)+(ab+bc+ca)^2 \geq 3(a^2b^2+b^2c^2+c^2a^2)+3abc(a+b+c).$$

We may reduce this inequality to

$$ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) \geq 2(a^2b^2 + b^2c^2 + c^2a^2),$$

or

$$ab(a - b)^2 + bc(b - c)^2 + ca(a - b)^2 \geq 0,$$

which is clearly true Equality occurs for $(a, b, c) = (1, 1, 1)$, and also for $(a, b, c) = \left(0, \sqrt{3}, \sqrt{3}\right)$ or any cyclic permutation

Second Solution (by *Ho Chung Siu*). Without loss of generality, assume that $a = \min\{a, b, c\}$ From $ab + bc + ca = 3$, we get $bc \geq 1$ On the other hand, from the known inequality

$$(ab + bc + ca)\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) \geq 9,$$

we obtain $a + b + c \geq 3abc$ The desired inequality follows now by summing up the following inequalities.

$$\frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} \geq \frac{2}{bc + 1},$$
$$\frac{1}{a^2 + 1} + \frac{1}{bc + 1} \geq \frac{3}{2}.$$

We have

$$\frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} - \frac{2}{bc + 1} = \frac{b(c - b)}{(b^2 + 1)(bc + 1)} + \frac{c(b - c)}{(c^2 + 1)(bc + 1)} =$$
$$= \frac{(b - c)^2(bc - 1)}{(b^2 + 1)(c^2 + 1)(bc + 1)} \geq 0$$

and

$$\frac{1}{a^2 + 1} + \frac{1}{bc + 1} - \frac{3}{2} = \frac{a^2 - bc + 3 - 3a^2bc}{2(a^2 + 1)(bc + 1)} = \frac{a(a + b + c - 3abc)}{2(a^2 + 1)(bc + 1)} \geq 0.$$

★

22. *Let a, b, c be non-negative numbers such that $a^2+b^2+c^2=3$. Prove that*

$$\frac{a}{b+2}+\frac{b}{c+2}+\frac{c}{a+2}\le 1.$$

Solution. By expanding, the inequality becomes

$$ab^2+bc^2+ca^2\le abc+2.$$

Without loss of generality, assume that

$$\min\{a,b,c\}\le b\le\max\{a,b,c\}$$

Under this assumption, we have

$$\begin{aligned}2-ab^2-bc^2-ca^2+abc&=2-ab^2-b(3-a^2-b^2)-ca^2+abc=\\&=(b^3-3b+2)-a(b^2-ab+ca-bc)=\\&=(b-1)^2(b+2)-a(b-a)(b-c)\ge 0.\end{aligned}$$

Equality occurs for $(a,b,c)=(1,1,1)$, and also for $(a,b,c)=\left(0,1,\sqrt{2}\right)$ or any cyclic permutation.

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23. *Let a, b, c be positive numbers such that $abc=1$. Prove that*

a) $\dfrac{a-1}{b}+\dfrac{b-1}{c}+\dfrac{c-1}{a}\ge 0;$

b) $\dfrac{a-1}{b+c}+\dfrac{b-1}{c+a}+\dfrac{c-1}{a+b}\ge 0.$

Solution. a) Write the inequality as

$$ab^2+bc^2+ca^2\ge a+b+c.$$

Applying the AM-GM Inequality, we get

$$\begin{aligned}3(ab^2+bc^2+ca^2)&=(2ab^2+bc^2)+(2bc^2+ca^2)+(2ca^2+ab^2)\ge\\&\ge 3\sqrt[3]{a^2b^5c^2}+3\sqrt[3]{a^2b^2c^5}+3\sqrt[3]{a^5b^2c^2}=3(b+c+a).\end{aligned}$$

We have equality for $a=b=c=1$.

b) Write the inequality as follows

$$\sum(a-1)\left[a^2+(ab+bc+ca)\right]\geq 0,$$
$$\sum a^3-\sum a^2+(a+b+c-3)(ab+bc+ca)\geq 0$$

Since $a+b+c\geq 3$ (by the AM-GM Inequality), it remains to show that $\sum a^3-\sum a^2\geq 0$ We can obtain this inequality applying the AM-GM Inequality in this manner

$$9\sum a^3=\sum(7a^3+b^3+c^3)\geq\sum 9\sqrt[9]{a^{21}b^3c^3}=9\sum a^2$$

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24. *Let a,b,c,d be non-negative numbers such that $a^2-ab+b^2=c^2-cd+d^2$. Prove that*

$$(a+b)(c+d)\geq 2(ab+cd).$$

Solution. Let $x=a^2-ab+b^2=c^2-cd+d^2$ Without loss of generality, suppose that $ab\geq cd$ We have $x\geq ab\geq cd$ and

$$(a+b)^2=x+3ab,\quad (c+d)^2=x+3cd.$$

By squaring, the desired inequality becomes

$$(x+3ab)(x+3cd)\geq 4(ab+cd)^2$$

Since $x\geq ab$, we get

$$(x+3ab)(x+3cd)-4(ab+cd)^2\geq 4ab(ab+3cd)-4(ab+cd)^2=4cd(ab-cd)\geq 0.$$

Equality occurs for $(a,b,c,d)\sim(1,1,1,1)$, and also for $(a,b,c,d)\sim(0,1,1,1)$ or any cyclic permutation.

★

25. *Let $a_1,a_2,\quad,a_n$ be positive numbers such that $a_1a_2\quad a_n=1$. Prove that*

$$\frac{1}{1+(n-1)a_1}+\frac{1}{1+(n-1)a_2}+\cdot\ \cdot+\frac{1}{1+(n-1)a_n}\geq 1.$$

First Solution. Let $r=\dfrac{n-1}{n}$ The inequality can be obtained by summing up the below inequalities for $i=1,2,\ldots,n$.

$$\frac{1}{1+(n-1)a_i}\geq\frac{a_i^{-r}}{a_1^{-r}+a_2^{-r}+\cdot\quad+a_n^{-r}}$$

This inequality is equivalent to

$$a_1^{-r} + \cdot + a_{i-1}^{-r} + a_{i+1}^{-r} + \cdots + a_n^{-r} \geq (n-1)a_i^{1-r},$$

which follows immediately from the AM-GM Inequality. Equality occurs when $a_1 = a_2 = \cdot \quad = a_n = 1.$

Second Solution. Let

$$E(a_1, a_2, \quad ., a_n) = \frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \cdots + \frac{1}{1+(n-1)a_n}$$

We will consider two cases.

Case $1-(n-1)^2 a_i a_j < 0$ *for all* $i \neq j$. Since the expression E is symmetric and $a_1 a_2 . \quad a_n = 1$, it suffices to show that

$$E(a_1, a_2, a_3, .. \ , a_n) \geq E(1, a_1 a_2, a_3, \ . \ , a_n)$$

for $a_1 \leq 1$ and $a_2 \geq 1$. If this assertion is valid, then it is easy to prove (by contrapositive way) that

$$E(a_1, a_2, . \ ., a_n) \geq E(1, 1, \ldots, 1) = 1$$

We have

$$E(a_1, a_2, \quad ., a_n) - E(1, a_1 a_2, \ . \ , a_n) =$$
$$= \frac{n-1}{n} \cdot \frac{1-a_1}{1+(n-1)a_1} \cdot \frac{1-a_2}{1+(n-1)a_2} \cdot \frac{1-(n-1)^2 a_1 a_2}{1+(n-1)a_1 a_2} \geq 0$$

Case $1-(n-1)^2 a_i a_j \geq 0$ *for a couple* (i,j) *with* $i \neq j$. It suffices to show that

$$\frac{1}{1+(n-1)a_i} + \frac{1}{1+(n-1)a_j} \geq 1.$$

This inequality is equivalent to $1-(n-1)^2 a_i a_j \geq 0$.

Third Solution Using the substitution $a_i = \dfrac{1}{x_i}$ for all i, the inequality becomes

$$\frac{x_1}{x_1+n-1} + \frac{x_2}{x_2+n-1} + \cdots + \frac{x_n}{x_n+n-1} \geq 1,$$

where $x_1, x_2, .. \ , x_n$ are positive numbers such that $x_1 x_2 . \ . x_n = 1$. By the Cauchy-Schwarz Inequality, we have

$$\sum \frac{x_1}{x_1+n-1} \geq \frac{\left(\sqrt{x_1} + \sqrt{x_2} + \ \cdot\cdot + \sqrt{x_n}\right)^2}{\sum(x_1+n-1)}.$$

Thus, we still have to show

$$(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n})^2 \ge n(n-1)+\sum x_1,$$

which is equivalent to

$$\sum_{1\le i<j\le n} \sqrt{x_i x_j} \ge \frac{n(n-1)}{2}$$

Since $x_1 x_2 \cdots x_n = 1$, the inequality follows immediately from the AM-GM Inequality

Fourth Solution Using the substitution $a_i = \frac{x_{i+1}}{x_i}$ for all i, where $x_1, x_2, \ldots, x_n$ are positive numbers ($x_{n+1} = x_1$), the inequality becomes

$$\frac{x_1}{x_1+(n-1)x_2}+\frac{x_2}{x_2+(n-1)x_3}+\cdots+\frac{x_n}{x_n+(n-1)x_1} \ge 1,$$

or

$$\frac{x_1-x_2}{x_1+(n-1)x_2}+\frac{x_2-x_3}{x_2+(n-1)x_3}+\cdots+\frac{x_n-x_1}{x_n+(n-1)x_1} \ge 0$$

We will prove, by induction over n, a slightly more general statement: *if* $m \ge n-1$, *then*

$$\frac{x_1-x_2}{x_1+mx_2}+\frac{x_2-x_3}{x_2+mx_3}+\cdots+\frac{x_n-x_1}{x_n+mx_1} \ge 0$$

For $n=2$, we have

$$\frac{x_1-x_2}{x_1+mx_2}+\frac{x_2-x_1}{x_2+mx_1} = \frac{(m-1)(x_1-x_2)^2}{(x_1+mx_2)(x_2+mx_1)} \ge 0.$$

Suppose now that the inequality is true for n numbers ($n \ge 2$), and let us prove it for $n+1$ numbers. We have to show that

$$\frac{x_1-x_2}{x_1+mx_2}+\frac{x_2-x_3}{x_2+mx_3}+\cdots+\frac{x_n-x_{n+1}}{x_n+mx_{n+1}}+\frac{x_{n+1}-x_1}{x_{n+1}+mx_1} \ge 0 \quad (1)$$

for $m \ge n$

Without loss of generality, consider that $x_{n+1} = \max\{x_1, x_2, \ldots, x_{n+1}\}$ Since $m \ge n$ implies $m \ge n-1$, we may use the inductive hypothesis. So, we still have to prove the inequality

$$\frac{x_n-x_{n+1}}{x_n+mx_{n+1}}+\frac{x_{n+1}-x_1}{x_{n+1}+mx_1} \ge \frac{x_n-x_1}{x_n+mx_1},$$

which is equivalent to

$$(x_{n+1} - x_1)(x_{n+1} - x_n)(m^2 x_1 - x_n) \geq 0.$$

Since this inequality is true for $m^2 x_1 \geq x_n$, it suffices to prove (1) for $m^2 x_1 < x_n$. We write (1) in the form

$$\frac{x_1}{x_1 + mx_2} + \frac{x_2}{x_2 + mx_3} + \cdots + \frac{x_n}{x_n + mx_{n+1}} + \frac{x_{n+1}}{x_{n+1} + mx_1} \geq \frac{n+1}{m+1},$$

and see that

$$\begin{aligned} &\frac{x_n}{x_n + mx_{n+1}} + \frac{x_{n+1}}{x_{n+1} + mx_1} = \\ &= 1 + \frac{x_{n+1}(x_n - m^2 x_1)}{(x_n + mx_{n+1})(x_{n+1} + mx_1)} > 1 \geq \frac{n+1}{m+1}. \end{aligned}$$

Fifth Solution Suppose that the desired inequality is false, and then show that the hypothesis $a_1 a_2 \ldots a_n = 1$ does not hold. Actually, we will prove that if

$$\frac{1}{1 + (n-1)a_1} + \frac{1}{1 + (n-1)a_2} + \cdots + \frac{1}{1 + (n-1)a_n} < 1,$$

then $a_1 a_2 \ldots a_n > 1$. To do this, let $x_i = \dfrac{1}{1 + (n-1)a_i}$ for $i = 1, 2, \ldots, n$. Note that $0 < x_i < 1$ and $a_i = \dfrac{1 - x_i}{(n-1)x_i}$ for all i. So we have to show that $x_1 + x_2 + \cdots + x_n < 1$ implies

$$(1 - x_1)(1 - x_2) \ldots (1 - x_n) > (n-1)^n x_1 x_2 \ldots x_n.$$

We can easily prove this inequality using the AM-GM Inequality Indeed, for all $k = 1, 2, \ldots, n$, we have

$$1 - x_k > \sum_{j \neq k} x_j \geq (n-1) \sqrt[n-1]{\prod_{j \neq k} x_j}.$$

Multiplying these inequalities, the conclusion follows.

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26. *Let a, b, c, d be non-negative real numbers such that $a^2+b^2+c^2+d^2=1$. Prove that*

$$(1-a)(1-b)(1-c)(1-d) \geq abcd.$$

Solution. The desired inequality follows by multiplying the inequalities

$$(1-a)(1-b) \geq cd,$$
$$(1-c)(1-d) \geq ab$$

With regard to the first inequality, we have

$$2cd \leq c^2+d^2 = 1-a^2-b^2,$$

and hence,

$$2(1-a)(1-b)-2cd \geq 2(1-a)(1-b)-1+a^2+b^2 =$$
$$=(1-a-b)^2 \geq 0$$

The second inequality can be proven similarly.

The given inequality becomes equality for $(a,b,c,d)=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$, and also for $(a,b,c,d)=(1,0,0,0)$ or any cyclic permutation

★

27. *If a, b, c are positive real numbers, then*

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+a}} \leq 3.$$

First Solution Setting $x=\sqrt{\frac{b}{a}}$, $y=\sqrt{\frac{c}{b}}$ and $z=\sqrt{\frac{a}{c}}$, the problem reduces to show that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}} \leq \frac{3\sqrt{2}}{2},$$

where x, y, z are positive numbers such that $xyz=1$. Assuming that $x=\max\{x,y,z\}$, which implies $yz \leq 1$, the inequality can be obtained by summing up the inequalities

$$\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}} \leq \frac{2}{\sqrt{1+yz}},$$
$$\frac{1}{\sqrt{1+x^2}}+\frac{2}{\sqrt{1+yz}} \leq \frac{3\sqrt{2}}{2}.$$

The first inequality can be proven as follows·

$$\frac{1}{2}\left(\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\right)^2 \le \frac{1}{1+y^2}+\frac{1}{1+z^2} = 1+\frac{1-y^2z^2}{(1+y^2)(1+z^2)} \le$$

$$\le 1+\frac{1-y^2z^2}{(1+yz)^2} = \frac{2}{1+yz}.$$

With regard to the second inequality, since

$$\frac{1}{\sqrt{1+x^2}} \le \frac{\sqrt{2}}{1+x},$$

it suffices to show that

$$\frac{1}{1+x}+\sqrt{\frac{2}{1+yz}} \le \frac{3}{2}.$$

We have

$$\frac{3}{2}-\frac{1}{1+x}-\sqrt{\frac{2}{1+yz}} = \frac{1+3x}{2(1+x)}-\sqrt{\frac{2x}{1+x}} = \frac{1+3x-2\sqrt{2x(1+x)}}{2(1+x)} =$$

$$= \frac{\left(\sqrt{1+x}-\sqrt{2x}\right)^2}{2(1+x)} \ge 0$$

This completes the proof. Equality occurs for $a=b=c=1$

Second Solution (by *Mikhail Leptchinski*). Applying the Cauchy-Schwarz Inequality, for any positive numbers x, y, z we have

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+a}} \le \sqrt{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\left(\frac{2ax}{a+b}+\frac{2by}{b+c}+\frac{2cz}{c+a}\right)}$$

Thus, it suffices to show that

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\left(\frac{2ax}{a+b}+\frac{2by}{b+c}+\frac{2cz}{c+a}\right) \le 9.$$

Choosing $x = \dfrac{1}{a+c}$, $y = \dfrac{1}{b+a}$ and $z = \dfrac{1}{c+b}$, the inequality becomes as follows

$$\frac{a}{(a+b)(a+c)}+\frac{b}{(b+c)(b+a)}+\frac{c}{(c+a)(c+b)} \le \frac{9}{4(a+b+c)},$$

$$a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2) \ge 6abc,$$

$$a(b-c)^2+b(c-a)^2+c(a-b)^2 \ge 0,$$

the last being clearly true.

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28. *If a, b, c, d are positive real numbers, then*

$$\left(\frac{a}{a+b}\right)^2 + \left(\frac{b}{b+c}\right)^2 + \left(\frac{c}{c+d}\right)^2 + \left(\frac{d}{d+a}\right)^2 \geq 1.$$

Solution. Setting $x = \frac{b}{a}$, $y = \frac{c}{b}$, $z = \frac{d}{c}$ and $t = \frac{a}{d}$, the inequality becomes

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} + \frac{1}{(1+z)^2} + \frac{1}{(1+t)^2} \geq 1,$$

where x, y, z, t are positive numbers such that $xyzt = 1$. This inequality follows by summing the inequalities.

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} \geq \frac{1}{1+xy},$$
$$\frac{1}{(1+z)^2} + \frac{1}{(1+t)^2} \geq \frac{1}{1+zt} = \frac{xy}{1+xy}.$$

We have

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} - \frac{1}{1+xy} = \frac{xy(x^2+y^2) - x^2y^2 - 2xy + 1}{(1+x)^2(1+y)^2(1+xy)} =$$
$$= \frac{xy(x-y)^2 + (1-xy)^2}{(1+x)^2(1+y)^2(1+xy)} \geq 0$$

and similarly,

$$\frac{1}{(1+z)^2} + \frac{1}{(1+t)^2} - \frac{1}{1+zt} = \frac{zt(z-t)^2 + (1-zt)^2}{(1+z)^2(1+t)^2(1+zt)} \geq 0.$$

Equality occurs for $a = b = c = d$.

★

29. *Let a, b, c be positive numbers such that $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. If $a \leq b \leq c$, then*

$$ab^2c^3 \geq 1$$

Solution. First we will show that $a \leq 1$. Indeed, if $a > 1$, then $1 < a \leq b \leq c$ and

$$a + b + c - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} = \frac{1-a^2}{a} + \frac{1-b^2}{b} + \frac{1-c^2}{c} < 0,$$

which is false On the other hand, from $a \le 1$ and

$$a - \frac{1}{a} = (b+c)\left(\frac{1}{bc} - 1\right),$$

it follows that $bc \ge 1$. Similarly, we can show that $c \ge 1$ and $ab \le 1$.

Since $bc \ge 1$, it suffices to show that $abc^2 \ge 1$. Taking account of $ab \le 1$, we have

$$c - \frac{1}{c} = (a+b)\left(\frac{1}{ab} - 1\right) \ge 2\sqrt{ab}\left(\frac{1}{ab} - 1\right) = 2\left(\frac{1}{\sqrt{ab}} - \sqrt{ab}\right) \ge \frac{1}{\sqrt{ab}} - \sqrt{ab},$$

and hence

$$\left(c - \frac{1}{\sqrt{ab}}\right)\left(1 + \frac{\sqrt{ab}}{c}\right) \ge 0,$$

which gives us $abc^2 \ge 1$. Equality occurs for $a = b = c = 1$.

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30. *Let a, b, c be non-negative numbers, no two of them are zero. Then*

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} \ge \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

Solution. Adding up the identities

$$\frac{a^2}{b^2+c^2} - \frac{a}{b+c} = \frac{ab(a-b) + ac(a-c)}{(b^2+c^2)(b+c)},$$

$$\frac{b^2}{c^2+a^2} - \frac{b}{c+a} = \frac{bc(b-c) + ba(b-a)}{(c^2+a^2)(c+a)},$$

$$\frac{c^2}{a^2+b^2} - \frac{c}{a+b} = \frac{ca(c-a) + cb(c-b)}{(a^2+b^2)(a+b)}$$

yields

$$\sum \frac{a^2}{b^2+c^2} - \sum \frac{a}{b+c} =$$

$$= \sum bc(b-c)\left[\frac{1}{(c^2+a^2)(c+a)} - \frac{1}{(a^2+b^2)(a+b)}\right] =$$

$$= (a^2+b^2+c^2+ab+bc+ca)\sum \frac{bc(b-c)^2}{(a^2+b^2)(a^2+c^2)(a+b)(a+c)} \ge 0.$$

Equality occurs for $(a, b, c) \sim (1, 1, 1)$, and also for $(a, b, c) \sim (0, 1, 1)$ or any cyclic permutation.

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31. *If a, b, c are non-negative numbers, then*

$$2(a^2+1)(b^2+1)(c^2+1) \geq (a+1)(b+1)(c+1)(abc+1)$$

First Solution. For $a=b=c$, the inequality reduces to

$$2(a^2+1)^3 \geq (a+1)^3(a^3+1)$$

This inequality is true since

$$2(a^2+1)^3 - (a+1)^3(a^3+1) = (a-1)^4(a^2+a+1) \geq 0$$

Multiplying now the inequalities

$$2(a^2+1)^3 \geq (a+1)^3(a^3+1),$$
$$2(b^2+1)^3 \geq (b+1)^3(b^3+1),$$
$$2(c^2+1)^3 \geq (c+1)^3(c^3+1),$$

we get

$$8(a^2+1)^3(b^2+1)^3(c^2+1)^3 \geq (a+1)^3(b+1)^3(c+1)^3(a^3+1)(b^3+1)(c^3+1).$$

Using this result, we still have to show that

$$(a^3+1)(b^3+1)(c^3+1) \geq (abc+1)^3$$

This inequality follows by Hölder's Inequality

$$(a^3+1)(b^3+1)(c^3+1) \geq \left(\sqrt[3]{a^3b^3c^3} + \sqrt[3]{1 \quad 1 \cdot 1}\right)^3 = (abc+1)^3,$$

but it can be also invoking the AM-GM Inequality Write the inequality as

$$(a^3b^3 + b^3c^3 + c^3a^3 - 3a^2b^2c^2) + (a^3+b^3+c^3-3abc) \geq 0,$$

and notice that $a^3b^3+b^3c^3+c^3a^3 \geq 3a^2b^2c^2$ and $a^3+b^3+c^3 \geq 3abc$ Equality occurs for $a=b=c=1$.

Second Solution (by *Marian Tetiva*). We will use the substitution

$$a = \frac{1-x}{1+x},\ b = \frac{1-y}{1+y},\ c = \frac{1-z}{1+z},$$

where $-1 < x, y, z \le 1$. Since

$$\frac{a^2+1}{a+1} = \frac{x^2+1}{x+1}, \quad \frac{b^2+1}{b+1} = \frac{y^2+1}{y+1}, \quad \frac{c^2+1}{c+1} = \frac{z^2+1}{z+1}$$

and

$$abc + 1 = \frac{2(xy + yz + zx + 1)}{(x+1)(y+1)(z+1)},$$

the inequality becomes

$$(x^2+1)(y^2+1)(z^2+1) \ge xy + yz + zx + 1,$$
$$x^2y^2 + y^2z^2 + z^2x^2 + x^2 + y^2 + z^2 \ge xy + yz + zx,$$
$$x^2y^2 + y^2z^2 + z^2x^2 + \frac{1}{2}(x-y)^2 + \frac{1}{2}(y-z)^2 + \frac{1}{2}(z-x)^2 \ge 0.$$

The last form is clearly true for any real numbers x, y, z. Consequently, the given inequality is also valid for any real numbers a, b, c.

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32. *If a, b, c are non-negative numbers, then*

$$3(1-a+a^2)(1-b+b^2)(1-c+c^2) \ge 1 + abc + a^2b^2c^2.$$

Solution. From the identity

$$2(1-a+a^2)(1-b+b^2) = 1 + a^2b^2 + (a-b)^2 + (1-a)^2(1-b)^2,$$

the inequality follows

$$2(1-a+a^2)(1-b+b^2) \ge 1 + a^2b^2.$$

Thus, it is enough to prove that

$$3(1+a^2b^2)(1-c+c^2) \ge 2(1 + abc + a^2b^2c^2).$$

This inequality is equivalent to

$$(3+a^2b^2)c^2 - (3 + 2ab + 3a^2b^2)c + 1 + 3a^2b^2 \ge 0$$

It is true because the quadratic in c has the discriminant

$$D = -3(1-ab)^4 \le 0.$$

Equality occurs for $a = b = c = 1$.

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33. *If a, b, c, d are non-negative numbers, then*

$$(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2) \geq \left(\frac{1+abcd}{2}\right)^2.$$

Solution. For $a = b = c = d$, the inequality reduces to

$$2(1-a+a^2)^2 \geq 1+a^4$$

This inequality is valid since

$$2(1-a+a^2)^2 - 1 - a^4 = (1-a)^4 \geq 0.$$

Using this result, we have

$$4(1-a+a^2)^2(1-b+b^2)^2 \geq (1+a^4)(1+b^4).$$

Since $(1+a^4)(1+b^4) \geq (1+a^2b^2)^2$, we get

$$2(1-a+a^2)(1-b+b^2) \geq 1+a^2b^2.$$

The desired inequality follows now by multiplying the inequalities

$$2(1-a+a^2)(1-b+b^2) \geq 1+a^2b^2,$$
$$2(1-c+c^2)(1-d+d^2) \geq 1+c^2d^2,$$
$$(1+a^2b^2)(1+c^2d^2) \geq (1+abcd)^2$$

Equality occurs for $a = b = c = d = 1$

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34. *If a, b, c are non-negative numbers, then*

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \geq (ab+bc+ca)^3.$$

Solution. We have

$$4(a^2+ab+b^2) - 3(a+b)^2 = (a-b)^2 \geq 0.$$

Multiplying the inequalities

$$4(a^2+ab+b^2) \geq 3(a+b)^2,$$
$$4(b^2+bc+c^2) \geq 3(b+c)^2,$$
$$4(c^2+ca+a^2) \geq 3(c+a)^2,$$

we get

$$64(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \geq 27(a+b)^2(b+c)^2(c+a)^2.$$

Thus, it suffices to show that

$$27(a+b)^2(b+c)^2(c+a)^2 \geq 64(ab+bc+ca)^3.$$

Since $3(ab+bc+ca) \leq (a+b+c)^2$, it is enough to prove that

$$81(a+b)^2(b+c)^2(c+a)^2 \geq 64(a+b+c)^2(ab+bc+ca)^2.$$

This inequality is equivalent to

$$9(a+b)(b+c)(c+a) \geq 8(a+b+c)(ab+bc+ca),$$

which reduces to the obvious inequality

$$a(b-c)^2+b(c-a)^2+c(a-b)^2 \geq 0.$$

Equality occurs for $(a,b,c) \sim (1,1,1)$, and also for $(a,b,c) \sim (1,0,0)$ or any cyclic permutation.

Remark *Kee-Wai Lau* found out the following nice identity·

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)-(ab+bc+ca)^2 =$$
$$= \frac{1}{3}(ab+bc+ca)^2\sum(b-c)^2+\frac{1}{6}(a+b+c)^2\sum a^2(b-c)^2,$$

which shows that the given inequality holds for any real numbers a, b, c

★

35. *Let a, b, c, d be positive numbers such that $abcd = 1$. Prove that*

$$\frac{1}{1+ab+bc+ca}+\frac{1}{1+bc+cd+db}+\frac{1}{1+cd+da+ac}+\frac{1}{1+da+ab+bd} \leq 1.$$

Solution. We have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}} = \sqrt{d}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right),$$

whence

$$ab+bc+ca \geq \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{d}}$$

and

$$\frac{1}{1+ab+bc+ca} \leq \frac{\sqrt{d}}{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}}.$$

Similarly,

$$\frac{1}{1+bc+cd+dc} \leq \frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}},$$
$$\frac{1}{1+cd+da+ac} \leq \frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}},$$
$$\frac{1}{1+da+ab+bd} \leq \frac{\sqrt{c}}{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}}.$$

Adding up these inequalities yields the conclusion. Equality occurs for $a = b = c = d = 1$.

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36. *If a, b, c and x, y, z are real numbers, then*

$$4(a^2+x^2)(b^2+y^2)(c^2+z^2) \geq 3(bcx+cay+abz)^2.$$

Solution. By the Cauchy-Schwarz Inequality, we have

$$(a^2+x^2)\left[(cy+bz)^2+b^2c^2\right] \geq [a(cy+bz)+bcx]^2.$$

Thus, we still have to show that

$$4(b^2+y^2)(c^2+z^2) \geq 3\left[(cy+bz)^2+b^2c^2\right].$$

This inequality reduces to

$$(cy-bz)^2+(bc-2yz)^2 \geq 0,$$

which is clearly true. In the case $abc \neq 0$, equality holds for

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{\sqrt{2}}{2}.$$

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37. *If $a \geq b \geq c \geq d \geq e$, then*

$$(a+b+c+d+e)^2 \geq 8(ac+bd+ce).$$

For $e \geq 0$, determine when equality occurs.

Solution. We have

$$(a+b+c+d+e)^2 - 8(ac+bd+ce) =$$
$$= (a+b+c+d+e-4c)^2 + 8(a+b+c+d+e)c - 16c^2 - 8(ac+bd+ce) =$$
$$= (a+b+c+d+e-4c)^2 + 8(b-c)(c-d) \geq 0.$$

From here the desired inequality follows. Equality occurs for either $b = c = \dfrac{a+d+e}{2}$ or $c = d = \dfrac{a+b+e}{2}$ For $e \geq 0$, the equality conditions $c = d = \dfrac{a+b+e}{2}$ yield $e = 0$ and $a = b = c = d$ Since this case is included in the first equality case, we can conclude that equality occurs only for $b = c = \dfrac{a+d+e}{2}$.

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38. *If a, b, c, d are real numbers, then*

$$6(a^2+b^2+c^2+d^2) + (a+b+c+d)^2 \geq 12(ab+bc+cd).$$

First Solution. Let

$$E(a,b,c,d) = 6(a^2+b^2+c^2+d^2) + (a+b+c+d)^2 - 12(ab+bc+cd).$$

We have

$$E(x+a, x+b, x+c, x+d) = 4x^2 + 4(2a-b-c+2d)x+$$
$$+ 7(a^2+b^2+c^2+d^2) + 2(ac+ad+bd) - 10(ab+bc+cd) =$$
$$= (2x+2a-b-c+2d)^2+$$
$$+ 3(a^2+2b^2+2c^2+d^2-2ab+2ac-2ad-4bc+2bd-2cd) =$$
$$= (2x+2a-b-c+2d)^2 + 3(b-c)^2 + 3(a-b+c-d)^2.$$

For $x = 0$, we get

$$E(a,b,c,d) = (2a-b-c+2d)^2 + 3(b-c)^2 + 3(a-b+c-d)^2 \geq 0.$$

Equality occurs for $2a = b = c = 2d$

Second solution Let $a = b + x$ and $d = c + y$. We have

$$\begin{aligned} E(a,b,c,d) &= 6(x^2+y^2) + [x+y+2(b+c)^2]^2 - 12bc = \\ &= 3(x-y)^2 + 4(x+y)^2 + 4(x+y)(b+c) + (b+c)^2 + 3(b-c)^2 = \\ &= 3(x-y)^2 + (2x+2y+b+c)^2 + 3(b-c)^2 \geq 0. \end{aligned}$$

★

39. *If a, b, c are positive numbers, then*

$$\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq 1 + \sqrt{1 + \sqrt{(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}.$$

Solution. (by *Gabriel Dospinescu*). Using the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} \left(\sum a\right)\left(\sum \frac{1}{a}\right) &= \sqrt{\left(\sum a^2 + 2\sum bc\right)\left(\sum \frac{1}{a^2} + 2\sum \frac{1}{bc}\right)} \geq \\ &\geq \sqrt{\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right)} + 2\sqrt{\left(\sum bc\right)\left(\sum \frac{1}{bc}\right)} = \\ &= \sqrt{\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right)} + 2\sqrt{\left(\sum a\right)\left(\sum \frac{1}{a}\right)}, \end{aligned}$$

and hence

$$\left(\sqrt{\left(\sum a\right)\left(\sum \frac{1}{a}\right)} - 1\right)^2 \geq 1 + \sqrt{\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right)}.$$

From this inequality, the conclusion immediately follows. Equality occurs if and only if

$$\left(\sum a^2\right)\left(\sum \frac{1}{bc}\right) = \left(\sum \frac{1}{a^2}\right)\left(\sum bc\right),$$

which is equivalent to

$$(a^2 - bc)(b^2 - ca)(c^2 - ab) = 0$$

Consequently, equality occurs for $a^2 = bc$, or $b^2 = ca$, or $c^2 = ab$

★

40. *If a, b, c, d are positive numbers, then*

$$5 + \sqrt{2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-2} \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Solution. Let $x = \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ and $y = \frac{b}{a}+\frac{c}{b}+\frac{a}{c}$. We have

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = x+y+3$$

and

$$2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-2 =$$
$$= 2\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)+2\left(\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\right)+4 =$$
$$= 2(x^2-2y)+2(y^2-2x)+4 = (x+y-2)^2+(x-y)^2 \geq (x+y-2)^2$$

Therefore,

$$\sqrt{2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-2} \geq x+y-2 = (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-5$$

Equality occurs if and only if $a=b$, or $b=c$, or $c=a$

★

41. *If a, b, c, d are positive numbers, then*

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b} \geq 0.$$

Solution. We have

$$\frac{a-b}{b+c}+\frac{c-d}{d+a} = \frac{a+c}{b+c}+\frac{a+c}{d+a}-2 = (a+c)\left(\frac{1}{b+c}+\frac{1}{d+a}\right)-2.$$

Since

$$\frac{1}{b+c}+\frac{1}{d+a} \geq \frac{4}{(b+c)+(d+a)},$$

we get

$$\frac{a-b}{b+c}+\frac{c-d}{d+a} \geq \frac{4(a+c)}{a+b+c+d}-2$$

Adding this inequality to the similar inequality

$$\frac{b-c}{c+d}+\frac{d-a}{a+b} \geq \frac{4(b+d)}{a+b+c+d}-2,$$

we find the desired inequality Equality holds if and only if $a = c$ and $b = d$.

Conjecture. If a, b, c, d are positive numbers, then

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+e} + \frac{d-e}{e+a} + \frac{e-a}{a+b} \geq 0.$$

★

42. *If $a, b, c > -1$, then*

$$\frac{1+a^2}{1+b+c^2} + \frac{1+b^2}{1+c+a^2} + \frac{1+c^2}{1+a+b} \geq 2.$$

Solution. We have $1+b+c^2 \geq 1+b > 0$, $1+b+c^2 \leq \frac{1+b^2}{2} + 1 + c^2$ and hence

$$\frac{1+a^2}{1+b+c^2} \geq \frac{2(1+a^2)}{1+b^2+2(1+c^2)}.$$

Setting $x = 1+a^2$, $y = 1+b^2$, $z = 1+c^2$, it suffices to show that

$$\frac{x}{y+2z} + \frac{y}{z+2x} + \frac{z}{x+2y} \geq 1$$

Using the Cauchy-Schwarz Inequality, we have

$$\frac{x}{y+2z} + \frac{y}{z+2x} + \frac{z}{x+2y} \geq \frac{(x+y+z)^2}{x(y+2z)+y(z+2x)+z(x+2y)} =$$
$$= \frac{(x+y+z)^2}{3(xy+yz+zx)} \geq 1.$$

Equality occurs if and only if $a = b = c = 1$.

★

43. *Let a, b, c and x, y, z be positive real numbers such that*

$$(a+b+c)(x+y+z) = (a^2+b^2+c^2)(x^2+y^2+z^2) = 4$$

Prove that

$$abcxyz < \frac{1}{36}$$

Solution. Using the given relations and the AM-GM Inequality, we have

$$\begin{aligned}
&4(ab+bc+ca)(xy+yz+zx) = \\
&= \left[(a+b+c)^2-(a^2+b^2+c^2)\right]\left[(x+y+z)^2-(x^2+y^2+z^2)\right] = \\
&= 20-(a+b+c)^2(x^2+y^2+z^2)-(x+y+z)^2(a^2+b^2+c^2) \le \\
&\le 20-2(a+b+c)(x+y+z)\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)} = 4,
\end{aligned}$$

therefore

$$(ab+bc+ca)(xy+yz+zx) \le 1.$$

On the other hand, multiplying the well-known inequalities

$$(ab+bc+ca)^2 \ge 3abc(a+b+c),$$
$$(xy+yz+zx)^2 \ge 3xyz(x+y+z),$$

we get

$$(ab+bc+ca)^2(xy+yz+zx)^2 \ge 36abcxyz.$$

Thus,

$$1 \ge (ab+bc+ca)^2(xy+yz+zx)^2 \ge 36abcxyz.$$

To have $1 = 36abcxyz$, it is necessary to have $(ab+bc+ca)^2 = 3abc(a+b+c)$ and $(xy+yz+zx)^2 = 3xyz(x+y+z)$. But these equalities imply $a=b=c$ and $x=y=z$, which contradict the hypothesis

$$(a+b+c)(x+y+z) = (a^2+b^2+c^2)(x^2+y^2+z^2) = 4$$

Consequently, we have $1 > 36abcxyz$

★

44. *Let a, b, c be positive numbers such that $a^2+b^2+c^2=3$. Prove that*

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \ge 3.$$

Solution. Write the inequality as follows

$$\sum\left(\frac{b^2+c^2}{b+c}-\frac{b+c}{2}\right) \ge \sqrt{3(a^2+b^2+c^2)}-a-b-c,$$

$$\sum\frac{(b-c)^2}{2(b+c)} \ge \frac{(a-b)^2+(b-c)^2+(c-a)^2}{\sqrt{3(a^2+b^2+c^2}+a+b+c}.$$

Since $\sqrt{3(a^2+b^2+c^2)} \geq a+b+c$, it suffices to show that

$$\sum \frac{(b-c)^2}{2(b+c)} \geq \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(a+b+c)}.$$

This inequality is equivalent to

$$\sum \frac{a}{b+c}(b-c)^2 \geq 0,$$

which is clearly true Equality occurs for $a=b=c=1$.

★

45. *Let a, b, c be non-negative numbers, no two of which are zero. Prove that*

$$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab} \geq \frac{3}{ab+bc+ca}.$$

Solution. Since

$$\frac{ab+bc+ca}{a^2+bc} = 1+\frac{a(b+c-a)}{a^2+bc},$$

we may write the inequality as

$$\frac{a(b+c-a)}{a^2+bc}+\frac{b(c+a-b)}{b^2+ca}+\frac{c(a+b-c)}{c^2+ab} \geq 0.$$

Assume that $a \leq b \leq c$. Since $b+c-a>0$, it suffices to show that

$$\frac{b(c+a-b)}{b^2+ca}+\frac{c(a+b-c)}{c^2+ab} \geq 0.$$

This inequality is equivalent to

$$(b^2+c^2)a^2-(b+c)(b^2-3bc+c^2)a+bc(b-c)^2 \geq 0.$$

It is true because

$$(b^2+c^2)a^2-(b+c)(b^2-3bc+c^2)a+bc(b-c)^2 =$$
$$= (b^2+c^2-2bc)a^2-(b+c)(b^2-2bc+c^2)a+bc(b-c)^2+abc(2a+b+c) =$$
$$= (b-c)^2(a-b)(a-c)+abc(2a+b+c) \geq 0$$

Equality occurs for $(a,b,c) \sim (0,1,1)$ or any cyclic permutation.

★

46. *Let a, b, c be non-negative numbers, no two of which are zero. Prove that*

$$\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}+\frac{1}{a^2-ab+b^2}\geq\frac{3}{ab+bc+ca}.$$

Solution. Denote

$$E(a,b,c)=\frac{ab+bc+ca}{b^2-bc+c^2}+\frac{ab+bc+ca}{c^2-ca+a^2}+\frac{ab+bc+ca}{a^2-ab+b^2}.$$

We first assume that $a \leq b \leq c$, and then show that

$$E(a,b,c)\geq E(0,b,c)\geq 0.$$

We have

$$E(a,b,c)-E(0,b,c)=\frac{a(b+c)}{b^2-bc+c^2}+\frac{a(c^2+2bc-ab)}{c^2-ca+a^2}+\frac{a(b^2+2bc-ac)}{a^2-ab+b^2}\geq$$

$$\geq\frac{a(b+c)}{b^2-bc+c^2}+\frac{a(bc-ab)}{c^2-ca+a^2}+\frac{a(bc-ac)}{a^2-ab+b^2}\geq 0$$

and

$$E(0,b,c)-3=\frac{bc}{b^2-bc+c^2}+\frac{b}{c}+\frac{c}{b}-3=\frac{(b-c)^4}{bc(b^2-bc+c^2)}\geq 0$$

Equality occurs for $(a,b,c)\sim(0,1,1)$ or any cyclic permutation.

★

47. *Let a, b, c be positive numbers such that $a+b+c=3$. Prove that*

$$abc+\frac{12}{ab+bc+ca}\geq 5.$$

Solution. By the third degree Schur's Inequality

$$(a+b+c)^3+9abc\geq 4(a+b+c)(ab+bc+ca),$$

we get $3abc \geq 4(ab+bc+ca)-9$. Thus, it suffices to show that

$$4(ab+bc+ca)-9+\frac{36}{ab+bc+ca}\geq 15.$$

This inequality is equivalent to

$$(ab+bc+ca-3)^2\geq 0,$$

which is clearly true. Equality occurs for $(a,b,c)=(1,1,1$.

★

48. *Let a, b, c be non-negative numbers such that $a^2+b^2+c^2=3$. Prove that*

$$12+9abc \geq 7(ab+bc+ca).$$

Solution. Let $s=a+b+c$. Since

$$ab+bc+ca=\frac{(a+b+c)^2-(a^2+b^2+c^2)}{2}=\frac{s^2-3}{2},$$

the inequality becomes

$$45+18abc-7s^2 \geq 0.$$

On the other hand, by Schur's Inequality

$$(a+b+c)^3+9abc \geq 4(a+b+c)(ab+bc+ca),$$

we get

$$s^3+9abc \geq 2s(s^2-3),$$

that is

$$9abc \geq s^3-6s.$$

Then,

$$45+18abc-7s^2 \geq 45+2(s^3-6s)-7s^2=(s-3)^2(2s+5) \geq 0.$$

Equality holds if and only if $(a,b,c)=(1,1,1)$.

Remark. From the proof above, the identity follows for $a^2+b^2+c^2=3$:

$$12+9abc-7(ab+bc+ca)=\sum a(a-b)(a-c)+(a+b+c-3)^2\left(a+b+c+\frac{5}{2}\right).$$

★

49. *Let a, b, c be non-negative numbers such that $ab+bc+ca=3$. Prove that*

$$a^3+b^3+c^3+7abc \geq 10.$$

Solution. Let $s=a+b+c$. From the well-known inequality

$$(a+b+c)^2 \geq 3(ab+bc+ca),$$

we get $s \geq 3$. Since

$$a^3 + b^3 + c^3 = 3abc + (a+b+c)^3 - 3(ab+bc+ca)(a+b+c) =$$
$$= 3abc + s^3 - 9s,$$

the inequality becomes

$$10abc + s^3 - 9s - 10 \geq 0.$$

This inequality is true for $s \geq 4$, because

$$s^3 - 9s - 10 \geq 16s - 9s - 10 = 7s - 10 > 0$$

Consider now that $3 \leq s < 4$. By Schur's Inequality

$$(a+b+c)^3 + 9abc \geq 4(ab+bc+ca)(a+b+c),$$

we obtain

$$9abc \geq 12s - s^3.$$

Thus, we have

$$10abc + s^3 - 9s - 10 \geq \frac{10(12s - s^3)}{9} + s^3 - 9s - 10 =$$
$$= \frac{-s^3 + 39s - 90}{9} = \frac{(s-3)(30 - s^2 - 3s)}{9} =$$
$$= \frac{(s-3)\left[(16 - s^2) + 3(4 - s) + 2\right]}{9} \geq 0,$$

which completes the proof. Equality occurs if and only if $a = b = c = 1$.

★

50. *If* a, b, c *are positive numbers such that* $abc = 1$, *then*

$$(a+b)(b+c)(c+a) + 7 \geq 5(a+b+c).$$

Solution. Assume that $a = \max\{a, b, c\}$ and denote $b + c = x$. We have $a \geq 1$, $x \geq 2\sqrt{bc} = \dfrac{2}{\sqrt{a}}$ and

$$E = (a+b)(b+c)(c+a) + 7 - 5(a+b+c) =$$
$$= x(ax + a^2 + bc) + 7 - 5a - 5x = ax^2 + (a^2 + bc - 5)x + 7 - 5a =$$
$$= a\left(x + \frac{a^2 + bc - 5}{2a}\right)^2 - \frac{(a^2 + bc - 5)^2}{4a} + 7 - 5a$$

Since

$$x + \frac{a^2+bc-5}{2a} \ge \frac{2}{\sqrt{a}} + \frac{a^2+bc-5}{2a} \ge \frac{2}{a} + \frac{a^2+\frac{1}{a}-5}{2a} =$$
$$= \frac{1}{2a}\left(a^2+\frac{1}{a}-1\right) > 0,$$

it suffices to consider $x = \frac{2}{\sqrt{a}}$. In this case, we have

$$E = ax^2 + (a^2+bc-5)x + 7 - 5a = 2\left(a^2+\frac{1}{a}-5\right)\frac{1}{\sqrt{a}} + 11 - 5a$$

Setting $t = \sqrt{a}$, $t \ge 1$, yields

$$E \ge 2\left(t^3+\frac{1}{t^3}-\frac{5}{t}\right) + 11 - 5t^2 = \frac{2t^6-5t^5+11t^3-10t^2+2}{t^3} =$$
$$= \frac{(t-1)^2(2t^4-t^3-4t^2+4t+2)}{t^3} \ge \frac{(t-1)^2(2t^4-t^3-4t^2+3t)}{t^3} =$$
$$= \frac{(t-1)^4(2t+3)}{t^2} \ge 0.$$

Equality occurs if and only if $a = b = c = 1$.

★

51. *Let a, b, c be non-negative numbers, no two of which are zero. Prove that*

$$\frac{a^3}{(2a^2+b^2)(2a^2+c^2)} + \frac{b^3}{(2b^2+c^2)(2b^2+a^2)} + \frac{c^3}{(2c^2+a^2)(2c^2+b^2)} \le \frac{1}{a+b+c}$$

Solution. The inequality follows by summing the inequalities

$$\frac{a^2}{(2a^2+b^2)(2a^2+c^2)} \le \frac{1}{(a+b+c)^2},$$
$$\frac{b^2}{(2b^2+c^2)(2b^2+a^2)} \le \frac{1}{(a+b+c)^2},$$
$$\frac{c^2}{(2c^2+a^2)(2c^2+b^2)} \le \frac{1}{(a+b+c)^2},$$

multiplied by a, b and c, respectively These inequalities directly follow by the Cauchy-Schwarz Inequality. For example, from

$$(a^2+a^2+b^2)(c^2+a^2+a^2) \ge (ac+a^2+ba)^2,$$

the first inequality follows. Equality occurs if and only if $a = b = c$

★

52. *Let a, b, c be non-negative numbers such that $a+b+c \geq 3$. Prove that*

$$\frac{1}{a^2+b+c}+\frac{1}{a+b^2+c}+\frac{1}{a+b+c^2} \leq 1.$$

Solution. It is easy to check that it suffices to consider $a+b+c=3$. In this case, we may write the inequality in the form

$$\frac{1}{a^2-a+3}+\frac{1}{b^2-b+3}+\frac{1}{c^2-c+3} \leq 1.$$

We can prove this inequality by adding the inequalities

$$\frac{1}{a^2-a+3} \leq \frac{4-a}{9}, \quad \frac{1}{b^2-b+3} \leq \frac{4-b}{9}, \quad \frac{1}{c^2-c+3} \leq \frac{4-c}{9}.$$

We notice that

$$\frac{4-a}{9}-\frac{1}{a^2-a+3}=\frac{(a-1)^2(3-a)}{9(a^2-a+3)}=\frac{(a-1)^2(b+c)}{9(a^2-a+3)} \geq 0.$$

Equality occurs if and only if $a=b=c=1$.

★

53. *Let a, b, c be non-negative numbers such that $ab+bc+ca=3$. If $r \geq 1$, then*

$$\frac{1}{r+a^2+b^2}+\frac{1}{r+b^2+c^2}+\frac{1}{r+c^2+a^2} \leq \frac{3}{r+2}.$$

Solution (by *Pham Kim Hung*). Since

$$\frac{r}{r+b^2+c^2}=1-\frac{b^2+c^2}{r+b^2+c^2},$$

we may write the inequality as

$$\sum \frac{b^2+c^2}{r+b^2+c^2} \geq \frac{6}{r+2}.$$

On the other hand

$$b^2+c^2 \geq \frac{(b+c)^2}{2}$$

and

$$\frac{b^2+c^2}{r+b^2+c^2} \geq \frac{(b+c)^2}{2r+(b+c)^2}.$$

Thus, it suffices to show that

$$\sum \frac{(b+c)^2}{2r+(b+c)^2} \geq \frac{6}{r+2}$$

By the Cauchy-Schwarz Inequality, we have

$$\sum \frac{(b+c)^2}{2r+(b+c)^2} \geq \frac{4(a+b+c)^2}{6r+\sum(b+c)^2} =$$
$$= \frac{2(a+b+c)^2}{a^2+b^2+c^2+(r+1)(ab+bc+ca)} =$$
$$= \frac{6}{r+2} + \frac{r-1}{r+2} \cdot \frac{2(a^2+b^2+c^2-ab-bc-ca)}{a^2+b^2+c^2+(r+1)(ab+bc+ca)} \geq$$
$$\geq \frac{6}{r+2}$$

Equality occurs if and only if $a=b=c=1$.

★

54. *Let a, b, c be positive numbers such that $abc=1$. Prove that*

$$\frac{1}{(1+a)^3} + \frac{1}{(1+b)^3} + \frac{1}{(1+c)^3} + \frac{5}{(1+a)(1+b)(1+c)} \geq 1.$$

Solution. Set $x = \frac{1}{1+a}$, $y = \frac{1}{1+b}$, $z = \frac{1}{1+c}$, $S = x+y+z$ and $Q = xy+yz+zx$, where $0 < x, y, z < 1$. The hypothesis $abc = 1$ becomes $xyz = (1-x)(1-y)(1-z)$, that is $2xyz = 1-S+Q$, while the required inequality transforms into $x^3+y^3+z^3+5xyz \geq 1$. That is

$$8xyz + S^3 - 3SQ \geq 1,$$

or

$$S^3 - 4S + 3 \geq (3S-4)Q.$$

We have to prove the last inequality for $S - 1 < Q \leq \frac{S^2}{3}$. The left hand side condition follows from $2xyz = 1-S+Q$, while the right hand side condition is well-known. We will consider three cases.

Case $S \leq 1$ We have

$$S^3 - 4S + 3 = (1-S)(3-S-S^2) \geq 0 > (3S-4)Q$$

Case $1 < S < \frac{4}{3}$. We have

$$S^3 - 4S + 3 - (3S-4)Q > S^3 - 4S + 3 - (3S-4)(S-1) = (S-1)^3 > 0.$$

Case $S \geq \frac{4}{3}$. We have

$$S^3 - 4S + 3 - (3S-4)Q \geq S^3 - 4S + 3 - (3S-4)\frac{S^2}{3} = \frac{(2S-3)^2}{3} \geq 0.$$

Equality occurs if and only if $a = b = c = 1$.

★

55. *Let* a, b, c *be positive numbers such that* $abc = 1$. *Prove that*

$$\frac{2}{a+b+c} + \frac{1}{3} \geq \frac{3}{ab+bc+ca}.$$

Solution. Let $u = \frac{ab+bc+ca}{3}$ and $s = \frac{a+b+c}{3}$. By the AM-GM Inequality, we get

$$u \geq \sqrt[3]{ab \cdot bc \cdot ca} = 1.$$

On the other hand, the third degree Schur's Inequality states

$$(x+y+z)^3 + 9xyz \geq 4(x+y+z)(xy+yz+zx)$$

for any non-negative numbers x, y, z. Substituting x, y, z by bc, ca, ab, respectively, we get

$$(ab+bc+ca)^3 + 9 \geq 4(ab+bc+ca)(a+b+c),$$

which is equivalent to

$$3u^3 + 1 \geq 4us.$$

Therefore,

$$\frac{6}{a+b+c} + 1 - \frac{9}{ab+bc+ca} = \frac{2}{s} + 1 - \frac{3}{u} \geq \frac{8u}{3u^3+1} + 1 - \frac{3}{u} =$$
$$= \frac{3u^4 - 9u^3 + 8u^2 + u - 3}{u(3u^3+1)} = \frac{(u-1)(3u^3 - 6u^2 + 2u + 3)}{u(3u^3+1)}.$$

Since $u \geq 1$, we have to show that $3u^3 - 6u^2 + 2u + 3 \geq 0$. For $u \geq 2$, we have

$$3u^3 - 6u^2 + 2u + 3 > 3u^3 - 6u^2 = 3u^2(u-2) \geq 0,$$

and for $1 \leq u < 2$, we have

$$3u^3 - 6u^2 + 2u + 3 = 3u(u-1)^2 + 3 - u > 0.$$

Equality occurs if and only if $a = b = c = 1$

★

56. *If a, b, c are real numbers, then*

$$2(1 + abc) + \sqrt{2(1 + a^2)(1 + b^2)(1 + c^2)} \geq (1 + a)(1 + b)(1 + c).$$

Solution. Using the substitution $u = a+b+c$, $v = ab+bc+ca$ and $w = abc$, the inequality becomes

$$\sqrt{2(u^2 + v^2 + w^2 - 2wu - 2v + 1)} \geq u + v - w - 1$$

It suffices to show that

$$2(u^2 + v^2 + w^2 - 2wu - 2v + 1) \geq (u + v - w - 1)^2$$

This inequality is equivalent to

$$u^2 + v^2 + w^2 - 2uv + 2vw - 2wu + 2u - 2v - 2w + 1 \geq 0,$$

or

$$(u - v - w + 1)^2 \geq 0.$$

Equality occurs if and only if $u - v - w + 1 = 0$ and $u + v - w - 1 \geq 0$.

★

57. *Let a, b, c be non-negative numbers, no two of which are zero. Prove that*

$$\frac{a(b+c)}{a^2 + bc} + \frac{b(c+a)}{b^2 + ca} + \frac{c(a+b)}{c^2 + ab} \geq 2.$$

Solution (by *Pham Van Thuam*). Assume that $a \geq b \geq c$ and write the inequality as

$$\frac{b(c+a)}{b^2+ca} \geq \frac{(a-b)(a-c)}{a^2+bc} + \frac{(a-c)(b-c)}{c^2+ab}$$

Since

$$\frac{(a-b)(a-c)}{a^2+bc} \leq \frac{(a-b)a}{a^2+bc} \leq \frac{a-b}{a}$$

and

$$\frac{(a-c)(b-c)}{c^2+ab} \leq \frac{a(b-c)}{c^2+ab} \leq \frac{b-c}{b},$$

it suffices to show that

$$\frac{b(c+a)}{b^2+ca} \geq \frac{a-b}{a} + \frac{b-c}{b}.$$

This inequality is equivalent to

$$b^2(a-b)^2 - 2abc(a-b) + a^2c^2 + ab^2c \geq 0$$

or

$$(ab-b^2-ac)^2 + ab^2c \geq 0.$$

Under the assumption $a \geq b \geq c$, equality occurs if and only if $a = b$ and $c = 0$.

★

58. *Let a, b, c be non-negative numbers, no two of which are zero. Prove that*

$$\sqrt{\frac{a(b+c)}{a^2+bc}} + \sqrt{\frac{b(c+a)}{b^2+ca}} + \sqrt{\frac{c(a+b)}{c^2+ab}} \geq 2.$$

First Solution By squaring, the inequality becomes

$$\sum \frac{a(b+c)}{a^2+bc} + 2\sum \sqrt{\frac{bc(a+b)(a+c)}{(b^2+ca)(c^2+ab)}} \geq 4.$$

Taking into account the preceding inequality, it suffices to show that

$$\sum \sqrt{\frac{bc(a+b)(a+c)}{(b^2+ca)(c^2+ab)}} \geq 1$$

Squaring again, it is enough to prove that

$$\sum \frac{bc(a+b)(a+c)}{(b^2+ca)(c^2+ab)} \geq 1$$

We have

$$\sum \frac{bc(a+b)(a+c)}{(b^2+ca)(c^2+ab)} \geq \sum \frac{bc(a^2+bc)}{(b^2+ca)(c^2+ab)} =$$
$$= 1 + \frac{4a^2b^2c^2}{(a^2+bc)(b^2+ca)(c^2+ab)} \geq 1.$$

Under the assumption $a \geq b \geq c$, equality occurs if and only if $a = b$ and $c = 0$

Second Solution (by *Minh Can*). Using the AM-GM Inequality, we have

$$\sqrt{\frac{a(b+c)}{a^2+bc}} = \frac{a(b+c)}{\sqrt{(a^2+bc)(ab+bc)}} \geq \frac{2a(b+c)}{(a^2+bc)+(ab+bc)} =$$
$$= \frac{2a(b+c)}{(a+b)(c+a)}$$

Thus, it suffices to show that

$$a(b+c)^2 + b(c+a)^2 + c(a+b)^2 \geq (a+b)(b+c)(c+a)$$

This inequality is true, because it reduces to $4abc \geq 0$.

★

59. *Let a, b, c be non-negative numbers, no two of which are zero. Prove that*

$$\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \geq \frac{a}{a^2+bc} + \frac{b}{b^2+ca} + \frac{c}{c^2+ab}$$

First Solution (by *Michael Rozenberg*) Without loss of generality, assume that $a = \min\{a, b, c\}$ We have

$$\sum \frac{1}{b+c} - \sum \frac{a}{a^2+bc} = \sum \left(\frac{1}{b+c} - \frac{a}{a^2+bc} \right) = \sum \frac{(a-b)(a-c)}{(b+c)(a^2+bc)}$$

Since $(a-b)(a-c) \geq 0$, it suffices to show that

$$\frac{(b-c)(b-a)}{(c+a)(b^2+ca)} + \frac{(c-a)(c-b)}{(a+b)(c^2+ab)} \geq 0$$

This inequality is equivalent to

$$(b-c)\left[(b^2-a^2)(c^2+ab)+(a^2-c^2)(b^2+ca)\right]\geq 0$$

or

$$a(b-c)^2(b^2+c^2-a^2+ab+bc+ca)\geq 0$$

The last inequality is clearly true for $a=\min\{a,b,c\}$ Equality occurs if and only if $a=b=c$.

Second Solution (by *Darij Grinberg*) According to the identity

$$\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}-\frac{1}{1+xy}=\frac{xy(x-y)^2+(1-xy)^2}{(1+x)^2(1+y)^2(1+xy)}$$

(used also in the proof of problem 28), we have

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}-\frac{1}{a^2+bc}=\frac{bc(b-c)^2+(a^2-bc)^2}{(a+b)^2(a+c)^2(a^2+bc)}\geq 0$$

Using this inequality, we get

$$\sum\frac{1}{b+c}=\sum\left[\frac{b}{(b+c)^2}+\frac{c}{(b+c)^2}\right]=\sum\frac{a}{(a+b)^2}+\sum\frac{a}{(c+a)^2}=$$
$$=\sum a\left[\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}\right]\geq\sum\frac{a}{a^2+bc}.$$

★

60. *Let a,b,c be non-negative numbers, no two of which are zero. Prove that*

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\geq\frac{2a}{3a^2+bc}+\frac{2b}{3b^2+ca}+\frac{2c}{3c^2+ab}.$$

Solution. Since

$$\sum\frac{1}{b+c}-\sum\frac{2a}{3a^2+bc}=\sum\left(\frac{1}{b+c}-\frac{2a}{3a^2+bc}\right)=$$
$$=\sum\frac{(a-b)(a-c)+a(2a-b-c)}{(b+c)(3a^2+bc)}=$$
$$=\sum\frac{(a-b)(a-c)}{(b+c)(3a^2+bc)}+\sum\frac{a(2a-b-c)}{(b+c)(3a^2+bc)},$$

we can obtain the desired inequality by summing the inequalities

$$\sum\frac{(a-b)(a-c)}{(b+c\ (3a^2+bc)}\geq 0$$

and

$$\sum \frac{a(2a-b-c)}{(b+c)(3a^2+bc)} \geq 0.$$

To prove the first inequality, assume that $a = \min\{a,b,c\}$. Since $(a-b)(a-c) \geq 0$, it suffices to show that

$$\frac{(b-c)(b-a)}{(c+a)(3b^2+ca)} + \frac{(c-a)(c-b)}{(a+b)(3c^2+ab)} \geq 0.$$

This inequality is equivalent to

$$(b-c)\left[(b^2-a^2)(3c^2+ab)+(a^2-c^2)(3b^2+ca)\right] \geq 0$$

or

$$a(b-c)^2(b^2+c^2-a^2+3ab+bc+3ca) \geq 0$$

The last inequality clearly occurs for $a = \min\{a,b,c\}$

To prove the second inequality, we have

$$\sum \frac{a(2a-b-c)}{(b+c)(3a^2+bc)} = \sum \frac{a(a-b)}{(b+c)(3a^2+bc)} + \sum \frac{a(a-c)}{(b+c)(3a^2+bc)} =$$

$$= \sum \frac{a(a-b)}{(b+c)(3a^2+bc)} + \sum \frac{b(b-a)}{(c+a)(3b^2+ca)} =$$

$$= \sum (a-b)\left[\frac{a}{(b+c)(3a^2+bc)} - \frac{b}{(c+a)(3b^2+ca)}\right] =$$

$$= \sum \frac{c(a-b)^2\left[(a-b)^2+c(a+b)\right]}{(b+c)(c+a)(3a^2+bc)(3b^2+ca)} \geq 0.$$

Equality occurs if and only if $a=b=c$

★

61. *Let a,b,c be positive numbers such that $a^2+b^2+c^2=3$. Prove that*

$$5(a+b+c)+\frac{3}{abc} \geq 18.$$

Solution. Let $p = a+b+c$ and $q = ab+bc+ca$. From $a^2+b^2+c^2=3$ we get $p^2 = 2q+3$, $p > \sqrt{3}$, while from the well-known inequality

$$(ab+bc+ca)^2 \geq 3abc(a+b+c)$$

we obtain

$$\frac{1}{abc} \geq \frac{3p}{q^2}$$

Thus, it suffices to show that

$$5p + \frac{9p}{q^2} \geq 18$$

Since

$$5p + \frac{9p}{q^2} - 18 = 5p + \frac{36p}{(p^2-3)^2} - 18 =$$
$$= \frac{5p^5 - 18p^4 - 30p^3 + 108p^2 + 81p - 162}{(p^2-3)^2} =$$
$$= \frac{(p-3)^2(5p^3 + 12p^2 - 3p - 18)}{(p^2-3)^2},$$

we still have to show that $5p^3 + 12p^2 - 3p - 18 \geq 0$ Taking into account that $p > \sqrt{3}$, we get

$$5p^3 + 12p^2 - 3p - 18 = p^2\left(5p + 12 - \frac{3}{p} - \frac{18}{p^2}\right) >$$
$$> p^2\left(5\sqrt{3} + 12 - \sqrt{3} - 6\right) > 0$$

Equality occurs if and only if $a = b = c$

★

62. *Let a, b, c be non-negative numbers such that $a + b + c = 3$. Prove that*

$$\frac{1}{6-ab} + \frac{1}{6-bc} + \frac{1}{6-ca} \leq \frac{3}{5}$$

Solution. By expanding, the inequality becomes

$$108 - 48(ab + bc + ca) + 13abc(a + b + c) - 3a^2b^2c^2 \geq 0,$$

or

$$4\left[9 - 4(ab + bc + ca) + 3abc\right] + abc(1 - abc) \geq 0.$$

By the AM-GM Inequality, we have

$$1 = \left(\frac{a+b+c}{3}\right)^3 \geq abc$$

Consequently, it suffices to show that

$$9 - 4(ab + bc + ca) + 3abc \geq 0$$

This inequality has the homogeneous form

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

which is just Schur's Inequality of third degree.

Equality occurs for $a=b=c=1$, as well as for $a=0$ and $b=c=\frac{3}{2}$, $b=0$ and $c=a=\frac{3}{2}$, $c=0$ and $a=b=\frac{3}{2}$

Remark Actually, the following inequality holds

$$\frac{1}{p-ab} + \frac{1}{p-bc} + \frac{1}{p-ca} \leq \frac{3}{p-1}$$

for a, b, c non-negative numbers such that $a+b+c=3$ and $p \geq 6$. This inequality is equivalent to

$$p\,[3p-(p+2)(ab+bc+ca)+6abc] + 3abc(1-abc) \geq 0.$$

Since $1-abc \geq 0$, the inequality is true if

$$3p-(p+2)(ab+bc+ca)+6abc \geq 0$$

or

$$(p-6)(3-ab-bc-ca) + 18 - 8(ab+bc+ca) + 6abc \geq 0.$$

Since

$$3-ab-bc-ca = \frac{(a+b+c)^2}{3} - ab - bc - ca = \\ = \frac{(a-b)^2+(b-c)^2+(c-a)^2}{6} \geq 0$$

and

$$9 - 4(ab+bc+ca) + 3abc \geq 0,$$

the conclusion follows. For $p > 6$, equality occurs if and only if $a=b=c=1$.

63. *Let $n \geq 4$ and let $a_1, a_2, \ldots, a_n$ be real numbers such that*

$$a_1 + a_2 + \cdots + a_n \geq n \quad \text{and} \quad a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that

$$\max\{a_1, a_2, \ldots, a_n\} \geq 2$$

Solution. For the sake of contradiction, assume that $a_i < 2$ for all i. Let $x_i = 2 - a_i > 0$ for all i, and let $S = x_1 + x_2 + \cdots + x_n$, $S > 0$ From

$$n \le a_1 + a_2 + \cdots + a_n = 2n - S,$$

we get $S \le n$, and from

$$n^2 \le a_1^2 + a_2^2 + \cdots + a_n^2 = \sum_{i=1}^{n}(2 - x_i)^2 =$$

$$= 4n - 4S + \sum_{i=1}^{n} x_i^2 < 4n - 4S + S^2 = 4n - 4 + (S-2)^2,$$

we get $(S-2)^2 > (n-2)^2$. For $S \ge 2$, $(S-2)^2 > (n-2)^2$ implies $S > n$, which contradicts $S \le n$. For $S < 2$, $(S-2)^2 > (n-2)^2$ implies $2 - S > n - 2$, and hence $S < 4 - n \le 0$, which contradicts $S > 0$.

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64. *Let a, b, c be non-negative numbers, no two of which are zero Prove that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{13}{6} - \frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)}$$

Solution (by *Pham Huu Duc*) Rewrite the inequality as

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \ge \frac{2}{3}\left(1 - \frac{ab+bc+ca}{a^2+b^2+c^2}\right)$$

Since

$$\sum\left(\frac{a}{b+c} - \frac{1}{2}\right) = \sum \frac{(a-b)+(a-c)}{2(b+c)} = \sum \frac{a-b}{2(b+c)} + \sum \frac{b-a}{2(c+a)} =$$

$$= \sum \frac{a-b}{2}\left(\frac{1}{b+c} - \frac{1}{c+a}\right) = \sum \frac{(a-b)^2}{2(b+c)(c+a)}$$

and

$$\frac{2}{3}\left(1 - \frac{ab+bc+ca}{a^2+b^2+c^2}\right) = \sum \frac{(a-b)^2}{3(a^2+b^2+c^2)},$$

the inequality becomes

$$\sum (a-b)^2 \left[\frac{1}{2(b+c)(c+a)} - \frac{1}{3(a^2+b^2+c^2)}\right] \ge 0$$

It is true because

$$3(a^2+b^2+c^2) - 2(b+c)(c+a) = (a+b-c)^2 + 2(a-b)^2 \ge 0.$$

Equality holds if and only if $a = b = c$.

★

65. *Let a, b, c be non-negative numbers, no two of which are zero. Prove that*

$$\frac{a^2(b+c)}{b^2+c^2}+\frac{b^2(c+a)}{c^2+a^2}+\frac{c^2(a+b)}{a^2+b^2}\geq a+b+c$$

First Solution (by *Gabriel Dospinescu*). We have

$$\sum \frac{a^2(b+c)}{b^2+c^2}-\sum a=\sum\left[\frac{a^2(b+c)}{b^2+c^2}-a\right]=\sum \frac{ab(a-b)+ac(a-c)}{b^2+c^2}=$$

$$=\sum \frac{ab(a-b)}{b^2+c^2}+\sum \frac{ba(b-a)}{c^2+a^2}=\sum \frac{ab(a+b)(a-b)^2}{(b^2+c^2)(c^2+a^2)}\geq 0$$

Equality occurs for $a=b=c$, as well as for $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$.

Second Solution. By the Cauchy-Schwarz Inequality, we have

$$\sum \frac{a^2(b+c)}{b^2+c^2}\geq \frac{\left[\sum a^2(b+c)\right]^2}{\sum a^2(b+c)(b^2+c^2)}.$$

Then, it suffices to show that

$$\left[\sum a^2(b+c)\right]^2\geq \left(\sum a\right)\left[\sum a^2(b+c)(b^2+c^2)\right]$$

Let $p=a+b+c$ and $q=ab+bc+ca$. Since

$$\left[\sum a^2(b+c)\right]^2=(pq-3abc)^2=p^2q^2-6abcpq+9a^2b^2c^2$$

and

$$\sum a^2(b+c)(b^2+c^2)=\sum (b+c)\left[(a^2b^2+b^2c^2+c^2a^2)-b^2c^2\right]=$$
$$=(a^2b^2+b^2c^2+c^2a^2)\sum (b+c)-\sum (p-a)b^2c^2=$$
$$=p(a^2b^2+b^2c^2+c^2a^2)+abcq=p(q^2-2abcp)+abcq,$$

the inequality becomes

$$abc(2p^3+9abc-7pq)\geq 0$$

This inequality immediately follows by the third degree Schur's Inequality

$$p^3+9abc\geq 4pq$$

and the known inequality $p^2-3q\geq 0$.

★

66. *Let a, b, c be non-negative numbers such that*

$$(a+b)(b+c)(c+a) = 2.$$

Prove that

$$(a^2+bc)(b^2+ca)(c^2+ab) \le 1$$

Solution. We have to prove the homogeneous inequality

$$4(a^2+bc)(b^2+ca)(c^2+ab) \le (a+b)^2(b+c)^2(c+a)^2.$$

Without loss of generality, assume that $a \ge b \ge c$. Since

$$a^2+bc \le (a+c)^2$$

and

$$4(b^2+ca)(c^2+ab) \le (b^2+ca+c^2+ab)^2,$$

it suffices to show that

$$b^2+c^2+ab+ac \le (a+b)(b+c)$$

This inequality is equivalent to $c(c-b) \le 0$, which is clearly true. Equality occurs if and only if $a=0$ and $b=c=1$, $b=0$ and $c=a=1$, $c=0$ and $a=b=1$.

Remark *Michael Rozenberg* noticed that the above homogeneous inequality is equivalent to

$$(a-b)^2(b-c)^2(c-a)^2 + 4abc\sum bc(b+c) + 8a^2b^2c^2 \ge 0.$$

Chapter 2

Starting from some special fourth degree inequalities

2.1 Main results

1. If x, y, z are real numbers, then

$$(x^2+y^2+z^2)^2 \geq 3(x^3y+y^3z+z^3x)$$

(*Vasile Cîrtoaje*, GM-B, 7-8, 1992)

2. If x, y, z and r are real numbers, then

$$\sum x^4 + (3r^2-1)\sum x^2y^2 + 3r(1-r)xyz\sum x \geq 3r\sum x^3y.$$

(*Vasile Cîrtoaje*, MS, 2005)

3. If x, y, z are real numbers, then

$$x^4+y^4+z^4+xy^3+yz^3+zx^3 \geq 2(x^3y+y^3z+z^3x).$$

(*Vasile Cîrtoaje*, GM-B, 10, 1998)

4. If x, y, z are non-negative real numbers, then

$$x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2 \geq 2(x^3y+y^3z+z^3x-xy^3-yz^3-zx^3)$$

5. If x, y, z and r are real numbers, then

$$\sum(x-ry)(x-rz)(x-y)(x-z) \geq 0,$$

where $\sum$ is cyclic over x, y, z.

(*Vasile Cîrtoaje*, MS, 2005)

6. Let x, y, z be non-negative numbers, and let $S_i = \sum x^i(x-y)(x-z)$. For any real numbers p, q satisfying $pq > 0$, the inequality holds

$$S_0 \cdot S_{p+q} \geq S_p \quad S_q$$

(*Vasile Cîrtoaje*, MS, 2005)

7. Let x, y, z be non-negative real numbers such that $x + y + z = 3$. If $m \doteq \dfrac{\ln 3}{\ln 9 - \ln 4} \approx 1\ 355$ and $0 < r \leq m$, then

$$x^r y^r + y^r z^r + z^r x^r \leq 3.$$

(*Vasile Cîrtoaje*, CM, 1, 2004)

8. Let x, y, z be non-negative real numbers such that $x + y + z = 2$. If $2 \leq r \leq 3$, then

$$x^r(y+z) + y^r(z+x) + z^r(x+y) \leq 2$$

9. Let x, y, z be non-negative real numbers satisfying $x + y + z = 1$. If $p > 0$ and $q \leq \dfrac{(p-1)(2p+1)}{4}$, then

$$\frac{yz+q}{x+p} + \frac{zx+q}{y+p} + \frac{xy+q}{z+p} \leq \frac{1+9q}{1+3p}.$$

(*Vasile Cîrtoaje*, MS, 2005)

10. Let x, y, z be positive real numbers If $1 \leq r \leq 3$, then

$$x^r y^{4-r} + y^r z^{4-r} + z^r x^{4-r} \leq \frac{1}{3}(x^2 + y^2 + z^2)^2.$$

11. Let x, y, z be positive real numbers.

a) If $x + y + z = 3$ and $0 < r \leq \dfrac{1}{2}$, then

$$x^{1+r}y^r + y^{1+r}z^r + z^{1+r}x^r \leq 3;$$

b) If $x + y + z = 1 + 2r$ and $r \geq 1$, then

$$x^{1+r}y^r + y^{1+r}z^r + z^{1+r}x^r \leq r^r(1+r)^{1+r}$$

12. Let x, y, z be positive real numbers.

a) If $x + y + z = 3$ and $0 < r \le \frac{3}{2}$, then

$$x^r y + y^r z + z^r x \le 3;$$

b) If $x + y + z = r + 1$ and $r \ge 2$, then

$$x^r y + y^r z + z^r x \le r^r.$$

13. Let $m > n > 0$, and let x, y, z be positive real numbers such that

$$x^{m+n} + y^{m+n} + z^{m+n} = 3.$$

Then

$$\frac{x^m}{y^n} + \frac{y^m}{z^n} + \frac{z^m}{x^n} \ge 3.$$

(*Vasile Cîrtoaje*, MS, 2005)

14. Let a, b, c, d be non-negative real numbers. If $p > 0$, then

$$\left(1 + p\frac{a}{b+c}\right)\left(1 + p\frac{b}{c+d}\right)\left(1 + p\frac{c}{d+a}\right)\left(1 + p\frac{d}{a+b}\right) \ge (1+p)^2.$$

(*Vasile Cîrtoaje*, MS, 2004)

15. If a, b, c are positive real numbers, then

$$\frac{1}{4a} + \frac{1}{4b} + \frac{1}{4c} + \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge 3\left(\frac{1}{3a+b} + \frac{1}{3b+c} + \frac{1}{3c+a}\right).$$

(*Gabriel Dospinescu*, MS, 2004)

16. If x, y, z are non-negative real numbers satisfying $x + y + z = 3$, then

$$\frac{x}{xy+1} + \frac{y}{yz+1} + \frac{z}{zx+1} \ge \frac{3}{2}.$$

17. If x, y, z are non-negative real numbers satisfying $x + y + z = 3$, then

$$\frac{x}{y^2+3} + \frac{y}{z^2+3} + \frac{z}{x^2+3} \ge \frac{3}{4}.$$

18. If a, b, c are positive numbers satisfying $abc = 1$, then

$$\sqrt{\frac{a}{b+8}} + \sqrt{\frac{b}{c+8}} + \sqrt{\frac{c}{a+8}} \ge 1.$$

19. If a, b, c are the side-lengths of a triangle, then

$$a) \qquad 3(a^3b + b^3c + c^3a) \ge (ab + bc + ca)(a^2 + b^2 + c^2),$$

$$b) \qquad 9(ab + bc + ca)(a^2 + b^2 + c^2) \ge (a + b + c)^4$$

20. Let a, b, c be the side-lengths of a triangle. If $r \ge 2$, then

$$3(a^rb + b^rc + c^ra) \ge (a + b + c)(a^{r-1}b + b^{r-1}c + c^{r-1}a).$$

21. Let a, b, c be the side-lengths of a triangle. If $r \ge 2$, then

$$a^rb(a - b) + b^rc(b - c) + c^ra(c - a) \ge 0$$

(*Vasile Cîrtoaje*, GM-B, 4, 1986)

22. Let a, b, c be the side-lengths of a triangle. If $0 < r \le 1$, then

$$a^2b(a^r - b^r) + b^2c(b^r - c^r) + c^2a(c^r - a^r) \ge 0.$$

(*Vasile Cîrtoaje*, MS, 2005)

23. Let a, b, c be the side-lengths of a triangle. If x, y, z are real numbers, then

$$(ya^2+zb^2+xc^2)(za^2+xb^2+yc^2) \ge (xy+yz+zx)(a^2b^2+b^2c^2+c^2a^2)$$

(*Vasile Cîrtoaje*, GM-A, 2, 2001)

2.2 Solutions

1. *If x, y, z are real numbers, then*

$$(x^2 + y^2 + z^2)^2 \ge 3(x^3y + y^3z + z^3x). \qquad (1)$$

Proof. A way to prove (1) would be a suitable arrangement of the variables Let

$$E(x, y, z) = (x^2 + y^2 + z^2)^2 - 3(x^3y + y^3z + z^3x)$$

First we write $E(x, y, z)$ in the form

$$E = \sum \left[rx^4 + (1 - r)y^4 + 2x^2y^2 - 3x^3y\right],$$

where r is a real number and $\sum$ is cyclic over x, y, z (this convention will be used along all the book), then try to find a suitable number r, $0 \leq r \leq 1$, such that

$$rx^4 + (1-r)y^4 + 2x^2y^2 - 3x^3y \geq 0$$

for any real numbers x and y. We can't find such a number r, since the left side of the inequality divides by $x-y$ for any r, but divides by $(x-y)^2$ only for $r = \frac{5}{4} > 1$. Thus this method fails for our inequality.

Under the circumstances, we will use *the substitution method* Setting

$$y = x + p, \quad z = x + q,$$

inequality (1) can be written as

$$E_1 - 2E_2 \geq 0,$$

where

$$\begin{aligned} E_1 &= \sum x^3(x-y) = -px^3 + (p-q)y^3 + qz^3 = \\ &= p(y-x)(y^2+yx+x^2) + q(z-y)(z^2+zy+y^2) = \\ &= 3(p^2-pq+q^2)x^2 + 3(p^3-p^2q+q^3)x + p^4 - p^3q + q^4, \end{aligned}$$

$$\begin{aligned} E_2 &= \sum x^2y(x-y) = -px^2y + (p-q)y^2z + qz^2x = \\ &= py(yz-x^2) + qz(zx-y^2) = \\ &= (p^2-pq+q^2)x^2 + (p^3+p^2q-2pq^2+q^3)x + p^3q - p^2q^2 \end{aligned}$$

The inequality is equivalent to

$$\alpha x^2 + \beta x + \gamma \geq 0,$$

where

$$\begin{aligned} \alpha &= p^2 - pq + q^2, \\ \beta &= p^3 - 5p^2q + 4pq^2 + q^3, \\ \gamma &= p^4 - 3p^3q + 2p^2q^2 + q^4. \end{aligned}$$

For $p = q = 0$, we have $\alpha x^2 + \beta x + \gamma = 0$. Otherwise, we have $\alpha > 0$, and it is enough to show that the discriminant δ of the quadratic function

$\alpha x^2 + \beta x + \gamma$ is less than or equal to zero. Indeed, we have

$$\delta = \beta^2 - 4\alpha\gamma = -3(p^6 - 2p^5 q - 3p^4 q^2 + 6p^3 q^3 + 2p^2 q^4 - 4pq^5 + q^6),$$
$$\delta = -3(p^3 - p^2 q - 2pq^2 + q^3)^2 \le 0.$$

We observe that equality in (1) occurs for $(x, y, z) \sim (1, 1, 1)$ Besides, equality occurs for

$$(x, y, z) \sim \left(\sin^2 \frac{4\pi}{7}, \sin^2 \frac{2\pi}{7}, \sin^2 \frac{\pi}{7}\right)$$

or any cyclic permutation thereof. The last equality points can be derived from the equality equations

$$p^3 - p^2 q - 2pq^2 + q^3 = 0,$$
$$x = \frac{-(p^3 - 5p^2 q + 4pq^2 + q^3)}{2(p^2 - pq + q^2)},$$

taking into account that $y = x + p$, $z = x + q$. □

Remark 1. Starting from the obvious relation

$$4\alpha(\alpha x^2 + \beta x + \gamma) = (2\alpha x + \beta)^2 - \delta,$$

we can deduce the following identity

$$4F \cdot E(x, y, z) = (A - 5B + 4C)^2 + 3(A - B - 2C + 2D)^2,$$

where

$$F = x^2 + y^2 + z^2 - xy - yz - zx = \frac{(x-y)^2 + (y-z)^2 + (z-x)^2}{2},$$
$$A = x^3 + y^3 + z^3,\ B = x^2 y + y^2 z + z^2 x,\ C = xy^2 + yz^2 + zx^2,\ D = 3xyz$$

Remark 2. We can also prove (1) using the special identities

$$(x^2 + y^2 + z^2)^2 - 3(x^3 y + y^3 z + z^3 x) = \frac{1}{2} \sum (x^2 - y^2 - xy + 2yz - zx)^2 \quad (2)$$

and

$$(x^2 + y^2 + z^2)^2 - 3(x^3 y + y^3 z + z^3 x) = \frac{1}{6} \sum (2x^2 - y^2 - z^2 - 3xy + 3yz)^2 \quad (3)$$

Remark 3. Inequality (1) can be rewritten as

$$x^2(x-y)(x-2y)+y^2(y-z)(y-2z)+z^2(z-x)(z-2x)\geq 0.$$

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2. *If x, y, z and r are real numbers, then*

$$\sum x^4+(3r^2-1)\sum x^2y^2+3r(1-r)xyz\sum x\geq 3r\sum x^3y. \qquad (4)$$

Proof. We first notice that (4) is a generalization of (1). Indeed, for $r=1$, the inequality (4) turns into (1).

Let $y=x+p$ and $z=x+q$. We see that (4) is equivalent to

$$E_1+(1-3r)E_2+3r(r-1)E_3\geq 0,$$

where E_1 and E_2 are the previous expressions, and

$$\begin{aligned}E_3&=\sum x^2y^2-xyz\sum x=\frac{1}{2}\sum x^2(y-z)^2=\\&=(p^2-pq+q^2)x^2+(p^2q+pq^2)x+p^2q^2.\end{aligned}$$

Thus, inequality (4) reduces to

$$\alpha x^2+\beta x+\gamma\geq 0,$$

where

$$\begin{aligned}\alpha&=(3r^2-6r+4)(p^2-pq+q^2),\\ \beta&=(4-3r)p^3+(3r^2-6r-2)p^2q+(3r^2+3r-2)pq^2+(4-3r)q^3,\\ \gamma&=p^4-3rp^3q+(3r^2-1)p^2q^2+q^4.\end{aligned}$$

For $p=q=0$, we have $\alpha x^2+\beta x+\gamma=0$ Otherwise, we have $\alpha>0$, and

$$\delta=\beta^2-4\alpha\gamma=-3\left[rp^3-(3r^2-2)p^2q+(3r^2-3r-2)pq^2+rq^3\right]^2\leq 0.$$

Another proof of (4) is the following We write the inequality in the form

$$\begin{aligned}3\left(\sum x^2y^2-xyz\sum x\right)r^2-3\left(\sum x^3y-xyz\sum x\right)r+\\+\sum x^4-\sum x^2y^2\geq 0 \qquad (5)\end{aligned}$$

Since

$$3\left(\sum x^2y^2 - xyz\sum x\right) = \frac{1}{2}\sum(xy - 2yz + zx)^2,$$
$$3\left(\sum x^3y - xyz\sum x\right) = -3\sum yz(x^2 - y^2) =$$
$$= -3\sum yz(x^2 - y^2) + \sum(xy + yz + zx)(x^2 - y^2) =$$
$$= \sum(x^2 - y^2)(xy - 2yz + zx),$$
$$\sum x^4 - \sum x^2y^2 = \frac{1}{2}\sum(x^2 - y^2)^2,$$

the inequality becomes as follows:

$$\frac{1}{2}r^2\sum(xy-2yz+zx)^2 - r\sum(x^2-y^2)(xy-2yz+zx) + \frac{1}{2}\sum(x^2-y^2)^2 \geq 0,$$

or

$$\frac{1}{2}\sum(x^2 - y^2 - rxy + 2ryz - rzx)^2 \geq 0,$$

which is clearly true

Equality in (4) occurs for $(x, y, z) \sim (1, 1, 1)$ For $r \geq \frac{1}{\sqrt{2}}$, we claim that equality again occurs for a triple $(x, y, z) \sim (x_1, y_1, 1)$ with $x_1 \geq 0$, $y_1 \geq 0$ and $(x_1, y_1, 1) \neq (1, 1, 1)$ For example, in the case $r = \frac{1}{\sqrt{2}}$, equality occurs for $(x, y, z) \sim (0, \sqrt{2}, 1)$. □

Remark 1. For $r = \frac{2}{3}$ and $r = \frac{-1}{3}$, inequality (4) becomes

$$3\sum x^4 + \left(\sum xy\right)^2 \geq 6\sum x^3y$$

and

$$3\sum x^4 + 3\sum x^3y \geq 2\left(\sum xy\right)^2,$$

respectively. Equality occurs in both inequalities for $(x, y, z) \sim (1, 1, 1)$ The first inequality becomes again equality for $(x, y, z) \sim (1, y_1, z_1)$ with $y_1 \approx -25\ 65$ and $z_1 \approx -18\ 35$, while the second inequality becomes again equality for $(x, y, z) \sim (1, y_2, z_2)$ with $y_2 \approx -0.4874$ and $z_2 \approx -0.9115$

Remark 2. We can also write inequality (4) as a sum of squares, as follows

$$\sum(2x^2 - y^2 - z^2 - 3rxy + 3ryz)^2 \geq 0.$$

Remark 3. The following statements is valid:

If x, y, z are real numbers, then

$$4\left(\sum x^4-\sum y^2z^2\right)\left(\sum y^2z^2-xyz\sum x\right)\ge 3\left(\sum x^3y-xyz\sum x\right)^2. \quad (6)$$

(*Vasile Cîrtoaje*, MS, 2005)

We note that (6) is equivalent to $\delta \le 0$, where δ is the discriminant of the non-negative quadratic of r from the left hand side of (5).
Surprisingly, *Thomas Mildorf* noticed that (6) is equivalent to the following obvious inequality

$$\left[\sum x^2(xy+yz-2zx)\right]^2\ge 0$$

Equality in (6) occurs for $(x, y, z) \sim (1, 1, 1)$, but also for many other triples (x, y, z).

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3. *If x, y, z are real numbers, then*

$$x^4+y^4+z^4+xy^3+yz^3+zx^3\ge 2(x^3y+y^3z+z^3x). \quad (7)$$

Proof. Setting $y = x + p$ and $z = x + q$, the inequality turns into

$$Ax^2+Bx+C\ge 0,$$

where

$$A=3(p^2-pq+q^2),\ B=3(p^3-2p^2q+pq^2+q^3),\ C=p^4-2p^3q+pq^3+q^4$$

Since the discriminant of the quadratic $Ax^2 + Bx + C$ is non-positive,

$$B^2-4AC=-3(p^6-6p^4q^2+2p^3q^3+9p^2q^4-6pq^5+q^6)=$$
$$=-3(p^3-3pq^2+q^3)^2\le 0,$$

the conclusion follows.

We have equality for $(x, y, z) \sim (1, 1, 1)$. Besides, equality again holds for $(x,y,z)\sim\left(\sin\frac{\pi}{9},\sin\frac{2\pi}{9}-\sin\frac{\pi}{3},\sin\frac{2\pi}{9}\right)$ or any cyclic permutation. □

Remark 1. Inequality (7) is more interesting in the case $xyz \le 0$. If x, y, z are positive numbers, then inequality (7) is less sharp than inequality (1), because (7) can be obtained by adding (1) to

$$xy(x-y)^2+yz(y-z)^2+zx(z-x)^2\ge 0$$

Remark 2. From the proof above, we can derive the following identity

$$M \cdot F(x,y,z) = (A - 3C + 2D)^2 + 3(A - 2B + C)^2, \tag{8}$$

where

$$F(x,y,z) = x^4 + y^4 + z^4 + xy^3 + yz^3 + zx^3 - 2(x^3y + y^3z + z^3x),$$
$$M = 4(x^2 + y^2 + z^2 - xy - yz - zx) = 2(x-y)^2 + 2(y-z)^2 + 2(z-x)^2,$$
$$A = x^3 + y^3 + z^3,\ B = x^2y + y^2z + z^2x,\ C = xy^2 + yz^2 + zx^2,\ D = 3xyz$$

Remark 3. Inequality (7) is a direct consequence of the identity

$$x^4 + y^4 + z^4 + xy^3 + yz^3 + zx^3 - 2(x^3y + y^3z + z^3x) =$$
$$= \frac{1}{2}\sum(x^2 - y^2 + yz - xy)^2. \tag{9}$$

Remark 4. By identity (9), it follows that (7) becomes equality if and only if

$$x(x-y) = y(y-z) = z(z-x)$$

Assuming that

$$x(x-y) = y(y-z) = z(z-x) = s,\ s \neq 0,$$

we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{x-y}{s} + \frac{y-z}{s} + \frac{z-x}{s} = 0$$

This result yields the following nice statement:

If x, y, z are distinct real numbers such that

$$x^4 + y^4 + z^4 + xy^3 + yz^3 + zx^3 = 2(x^3y + y^3z + z^3x),$$

then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0.$

Remark 5. Inequality (7) is equivalent to either of the inequalities

$$(x-y)(2x^3 + y^3) + (y-z)(2y^3 + z^3) + (z-x)(2z^3 + x^3) \geq 0,$$
$$(x-y)(x^3 + 2z^3) + (y-z)(y^3 + 2x^3) + (z-x)(z^3 + 2y^3) \geq 0$$

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4. *If x, y, z are non-negative real numbers, then*

$$x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2 \geq 2(x^3y+y^3z+z^3x-xy^3-yz^3-zx^3). \quad (10)$$

Proof. Write first the inequality in the form

$$\frac{1}{2}(x^2-y^2)^2+\frac{1}{2}(y^2-z^2)^2+\frac{1}{2}(z^2-x^2)^2+ \\ +2(x-y)(y-z)(z-x)(x+y+z) \geq 0.$$

Due to symmetry, we may consider that $x = \min\{x, y, z\}$. Using the substitution $y = x+p$, $z = x+q$ ($x \geq 0$, $p \geq 0$, $q \geq 0$), the inequality reduces to

$$2Ax^2+4Bx+C \geq 0,$$

where

$$A = p^2+(p-q)^2+q^2,\ B = p(p-q)^2+q^3,$$
$$C = p^4-2p^3q-p^2q^2+2pq^3+q^4 = (p^2-pq-q^2)^2.$$

Since $A \geq 0$, $B \geq 0$ and $C \geq 0$, the inequality is obviously true. Equality occurs for $(x, y, z) \sim (1,1,1)$, and again for $(x, y, z) \sim \left(0, \frac{1+\sqrt{5}}{2}, 1\right)$ or any cyclic permutation. □

Remark. Inequality (10) is equivalent to

$$x(x^2-y^2)(x-2y)+y(y^2-z^2)(y-2z)+z(z^2-x^2)(z-2x) \geq 0.$$

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5. *If x, y, z and r are real numbers, then*

$$\sum(x-ry)(x-rz)(x-y)(x-z) \geq 0, \quad (11)$$

where $\sum$ is cyclic over x, y, z.

Proof. Let $y = x+p$ and $z = x+q$. We can rewrite the inequality in the form

$$Au^2+Bu+C \geq 0,$$

where $u = (1-r)x$, $s = 2+r$ and

$$A = p^2 - pq + q^2,$$
$$B = (p+q)(2A - spq),$$
$$C = (p+q)^2 A - spq(p+q)^2 + s^2p^2q^2$$

The quadratic $Au^2 + Bu + C$ has the discriminant

$$D = B^2 - 4AC = -3s^2p^2q^2(p-q)^2$$

Except for the trivial case $p = q = 0$, we have $A > 0$ and $D \le 0$, and the conclusion follows

We have equality in (11) for $(x, y, z) \sim (1, 1, 1)$. Additionally, equality again occurs for $(x, y, z) \sim (r, 1, 1)$ or any cyclic permutation. □

Remark 1. Setting $r = 0$ in (10) yields Schur's Inequality of fourth degree

$$\sum x^2(x-y)(x-z) \ge 0$$

which is equivalent to each of the following inequalities

$$x^4 + y^4 + z^4 + xyz(x+y+z) \ge \sum yz(y^2+z^2),$$
$$x^4 + y^4 + z^4 + 2xyz(x+y+z) \ge (xy+yz+zx)(x^2+y^2+z^2),$$
$$\sum (y-z)^2(y+z-x)^2 \ge 0$$

and

$$6xyz \ge \frac{\left(S_1^2 - S_2\right)\left(4S_2 - S_1^2\right)}{S_1},$$

where $S_1 = x+y+z$ and $S_2 = xy+yz+zx$.

Remark 2. Inequality (11) is equivalent to each of the inequalities

$$\sum x^4 + r(r+2)\sum y^2z^2 + (1-r^2)xyz\sum x \ge (r+1)\sum yz(y^2+z^2) \quad (12)$$

and

$$3(r-1)(r+2)xyz \le \frac{S_1^4 - (r+5)S_1^2S_2 + (r+2)^2S_2^2}{S_1}, \quad (13)$$

where $S_1 = x+y+z$ and $S_2 = xy+yz+zx$ For $r = 1$ and $r = 2$, from (12) we get the inequalities

$$x^4 + y^4 + z^4 + 3(x^2y^2 + y^2z^2 + z^2x^2) \ge 2\sum yz(y^2+z^2),$$
$$x^4 + y^4 + z^4 + 8(x^2y^2 + y^2z^2 + z^2x^2) \ge 3(xy+yz+zx)(x^2+y^2+z^2),$$

respectively We have equality when $(x,y,z) \sim (1,1,1)$. For the last inequality, equality again occurs when $(x,y,z) \sim (2,1,1)$ or any cyclic permutation. Notice that the first inequality can be written as

$$(x-y)^4 + (y-z)^4 + (z-x)^4 \geq 0.$$

Remark 3. We can also prove (11) using the identity

$$\sum(x-ry)(x-rz)(x-y)(x-z) = \frac{1}{2}\sum(y-z)^2(y+z-x-rx)^2 \quad (14)$$

Remark 4. From the proof above, we can deduce the following identity

$$4M\sum(x-ry)(x-rz)(x-y)(x-z) = p^2 + 3Q^2, \quad (15)$$

where

$$M = x^2+y^2+z^2-xy-yz-zx = \sum(x-y)(x-z),$$
$$P = 2\sum x(x-y)(x-z) - r\sum x(y-z)^2,$$
$$Q = (r+2)^2(x-y)^2(y-z)^2(z-x)^2.$$

For $r = 0$, we get the identity

$$M\sum x^2(x-y)(x-z) = \left(\sum x(x-y)(x-z)\right)^2 + 3(x-y)^2(y-z)^2(z-x)^2 \quad (16)$$

Denoting $S_i = \sum x^i(x-y)(x-z)$, identity (16) yields the following inequality

$$S_0 \quad S_2 \geq S_1^2,$$

with equality if and only if two the numbers x, y, z are equal.

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6. *Let x, y, z be non-negative numbers, and let $S_i = \sum x^i(x-y)(x-z)$. For any real numbers p, q satisfying $pq > 0$, the inequality holds*

$$S_0 \cdot S_{p+q} \geq S_p \cdot S_q. \quad (17)$$

Proof. If two of x, y, z are equal, then $S_0 = S_p = S_q = S_{p+q} = 0$ Consider now, without loss of generality, that $x > y > z$ Dividing by

$$(x-y)^2(y-z)^2(z-x^{\ 2},$$

the inequality becomes successively as follows·

$$\left(\sum \frac{1}{y-z}\right)\left(\sum \frac{x^{p+q}}{y-z}\right) \geq \left(\sum \frac{x^p}{y-z}\right)\left(\sum \frac{x^q}{y-z}\right),$$

$$\sum \frac{y^{p+q}+z^{p+q}-y^p z^q - y^q z^p}{(x-y)(z-x)} \geq 0,$$

$$\sum (y-z)(y^p-z^p)(y^q-z^q) \leq 0,$$

$$(y-z)(y^p-z^p)(y^q-z^q)+(x-y)(x^p-y^p)(x^q-y^q) \leq (x-z)(x^p-z^p)(x^q-z^q).$$

Since $(y^p-z^p)(y^q-z^q) \geq 0$ and $(x^p-y^p)(x^q-y^q) \geq 0$, we thus have

$$(y-z)(y^p-z^p)(y^q-z^q) \leq (x-z)(y^p-z^p)(y^q-z^q)$$

and

$$(x-y)(x^p-y^p)(x^q-y^q) \leq (x-z)(x^p-y^p)(x^q-y^q).$$

Thus, it suffices to show that

$$(y^p-z^p)(y^q-z^q)+(x^p-y^p)(x^q-y^q) \leq (x^p-z^p)(x^q-z^q).$$

This inequality reduces to

$$(y^p-x^p)(y^q-z^q)+(y^p-z^p)(y^q-x^q) \leq 0,$$

which is true for all real numbers p, q with $pq > 0$. This completes the proof. We have equality if and only if two of the numbers x, y, z are equal □

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7. *Let x, y, z be non-negative real numbers such that $x+y+z=3$. If $m = \dfrac{\ln 3}{\ln 9 - \ln 4} \approx 1.355$ and $0 < r \leq m$, then*

$$x^r y^r + y^r z^r + z^r x^r \leq 3. \tag{18}$$

Proof Let $E_r(x,y,z) = x^r y^r + y^r z^r + z^r x^r$. By the Power-Mean Inequality, we have

$$\left(\frac{E_r}{3}\right)^{\frac{1}{r}} \leq \left(\frac{E_m}{3}\right)^{\frac{1}{m}}.$$

Thus, it suffices to show that $E_m \leq 3$. To prove this, suppose that $x = \min\{x,y,z\}$ and denote $t = \dfrac{y+z}{2}$ (hence $x + 2t = 3$, $t \geq x$).
We will show that

$$E_m(x,y,z) \leq E_m(x,t,t) \leq E_m(1,1,1) \tag{19}$$

The left inequality of (19) can be written as

$$x^m y^m + y^m z^m + z^m x^m \le 2x^m t^m + t^{2m},$$

or

$$t^{2m} \ge x^m(y^m + z^m - 2t^m) + y^m z^m.$$

By Jensen's Inequality, we have $y^m + z^m - 2t^m \ge 0$. On the other hand, from $x = \min\{x, y, z\}$ we have $x^m \le \sqrt{y^m z^m}$ Therefore,

$$x^m(y^m + z^m - 2t^m) \le \sqrt{y^m z^m}(y^m + z^m - 2t^m)$$

Thus, it suffices to show that

$$t^{2m} \ge \sqrt{y^m z^m}(y^m + z^m - 2t^m) + y^m z^m$$

This inequality is equivalent to each of the following inequalities:

$$\left(t^m + \sqrt{y^m z^m}\right)^2 \ge \sqrt{y^m z^m}\left(\sqrt{y^m} + \sqrt{z^m}\right)^2,$$

$$t^m + \sqrt{y^m z^m} \ge \sqrt[4]{y^m z^m}\left(\sqrt{y^m} + \sqrt{z^m}\right),$$

$$t^m - \left(\frac{\sqrt{y^m} + \sqrt{z^m}}{2}\right)^2 + \left(\frac{\sqrt{y^m} + \sqrt{z^m}}{2} - \sqrt[4]{y^m z^m}\right)^2 \ge 0$$

Since $t^m - \left(\frac{\sqrt{y^m} + \sqrt{z^m}}{2}\right)^2 \ge 0$ (by the Power-Mean Inequality), the inequality is clearly true.

The right inequality of (19) can be written in the homogeneous form

$$\frac{2x^m t^m + t^{2m}}{3} \le \left(\frac{x + 2t}{3}\right)^{2m} \tag{20}$$

For $t = 0$, the inequality is trivial Otherwise, we may set $t = 1$, which implies $x \le 1$ Taking logarithms yield

$$\ln \frac{2x^m + 1}{3} \le 2m \ln \frac{x + 2}{3}.$$

To prove this inequality, we consider the function

$$f(x) = 2m \ln \frac{x + 2}{3} - \ln \frac{2x^m + 1}{3}.$$

We have to show that $f(x) \geq 0$ for $0 \leq x \leq 1$. The derivative

$$f'(x) = \frac{2m}{x+2} - \frac{2mx^{m-1}}{2x^m+1} = \frac{2m(x^m - 2x^{m-1} + 1)}{(x+2)(2x^m+1)}$$

has the same sign as $g(x) = x^m - 2x^{m-1} + 1$, and the derivative

$$g'(x) = mx^{m-1} - \frac{2(m-1)}{x^{2-m}}$$

is zero for $x = x_1 = \dfrac{2(m-1)}{m} \approx 0\,524$. Since $g'(x) < 0$ for $x \in (0, x_1)$ and $g'(x) > 0$ for $x \in (x_1, 1]$, the function $g(x)$ is strictly decreasing for $x \in [0, x_1]$, and strictly increasing for $x \in [x_1, 1]$. Since $g(0) = 1$ and $g(1) = 0$, there exists $x_2 \in (0, x_1)$ such that $g(x_2) = 0$, $g(x) > 0$ for $x \in [0, x_2)$, and $g(x) < 0$ for $x \in (x_2, 1)$ Hence, $f'(x_2) = 0$, $f'(x) > 0$ for $x \in [0, x_2)$, and $f'(x) < 0$ for $x \in (x_2, 1)$ Therefore, the function $f(x)$ is strictly increasing for $x \in [0, x_2]$, and strictly decreasing for $x \in [x_2, 1]$ As a consequence,

$$f(x) \geq \min\{f(0), f(1)\}$$

Since $f(0) = f(1) = 0$, we get $f(x) \geq 0$, establishing the desired result.

We have equality in (18) for $(x, y, z) = (1, 1, 1)$. In the case $r = m$, equality again occurs for $(x, y, z) = \left(0, \frac{3}{2}, \frac{3}{2}\right)$ or any cyclic permutation □

Remark 1. For $r = \dfrac{4}{3}$, we obtain the following nice statement

If x, y, z are non-negative real numbers such that $x + y + z = 3$, then

$$(xy)^{\frac{4}{3}} + (yz)^{\frac{4}{3}} + (zx)^{\frac{4}{3}} \leq 3 \tag{21}$$

(*Vasile Cîrtoaje*, GM-A, 1, 2003)

Remark 2. An interesting extension of inequality (18) is the following.

Let x, y, z be non-negative real numbers such that $x + y + z = 2$. If $r \geq m = \dfrac{\ln 3}{\ln 9 - \ln 4} \approx 1.355$, then

$$x^r y^r + y^r z^r + z^r x^r \leq 1 \tag{22}$$

Let $p = \dfrac{r}{m}$, $p \geq 1$, and let $a = y^m z^m$, $b = z^m x^m$, $c = x^m y^m$. From

$$\left(\frac{a}{a+b+c}\right)^p + \left(\frac{b}{a+b+c}\right)^p + \left(\frac{c}{a+b+c}\right)^p \leq$$
$$\leq \frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c} = 1,$$

we get

$$a^p + b^p + c^p \leq (a+b+c)^p$$

Hence

$$x^r y^r + y^r z^r + z^r x^r = (x^m y^m)^p + (y^m z^m)^p + (z^m x^m)^p \leq$$
$$\leq (x^m y^m + y^m z^m + z^m x^m)^p.$$

Consequently, it suffices to show that $x^m y^m + y^m z^m + z^m x^m \leq 1$. According to (18) – case $r = m$, we have

$$x^m y^m + y^m z^m + z^m x^m \leq 3\left(\frac{x+y+z}{3}\right)^{2m} = 2\left(\frac{2}{3}\right)^{2m} = 1,$$

and the proof is complete.

Equality in (22) occurs for $(x, y, z) = (0, 1, 1)$ or any cyclic permutation. In the case $r = m$, equality occurs once again for $(x, y, z) = \left(\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right)$.

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8. *Let x, y, z be non-negative real numbers such that $x + y + z = 2$. If $2 \leq r \leq 3$, then*

$$x^r(y+z) + y^r(z+x) + z^r(x+y) \leq 2. \tag{23}$$

Proof. Let $E_r(x, y, z)$ be the left hand side of the inequality. Assume, without loss of generality, that $x \leq y \leq z$, and then show that

$$E_r(x, y, z) \leq E_r(0, x+y, z) \leq 2.$$

The inequality $E_r(x, y, z) \leq E_r(0, x+y, z)$ is equivalent to

$$\frac{xy}{z}\left(x^{r-1} + y^{r-1}\right) \leq (x+y)^r - x^r - y^r.$$

Since the left hand side is decreasing with regard to z, it is enough to consider that $z = y$ In this case, the inequality reduces to

$$2x^r + y^{r-1}(x+y) \le (x+y)^r.$$

Since $2x^r \le x^{r-1}(x+y)$, it suffices to show that

$$x^{r-1} + y^{r-1} \le (x+y)^{r-1}$$

This inequality is true, because

$$\frac{x^{r-1}+y^{r-1}}{(x+y)^{r-1}} = \left(\frac{x}{x+y}\right)^{r-1} + \left(\frac{y}{x+y}\right)^{r-1} \le \frac{x}{x+y} + \frac{y}{x+y} = 1$$

Notice that the inequality $E_r(x,y,z) \le E_r(0,x+y,z)$ is valid for any real $r \ge 2$.

Setting now $t = x+y$ (hence $t+z=2$), the inequality $E_r(0,x+y,z) \le 2$ becomes

$$tz(t^{r-1} + z^{r-1}) \le 2$$

By Power-Mean Inequality, for $r \le 3$, we have

$$\left(\frac{t^{r-1}+z^{r-1}}{2}\right)^{\frac{1}{r-1}} \le \left(\frac{t^2+z^2}{2}\right)^{\frac{1}{2}},$$

so that

$$t^{r-1} + z^{r-1} \le 2\left(\frac{t^2+z^2}{2}\right)^{\frac{r-1}{2}}.$$

Thus, it suffices to show that

$$tz\left(\frac{t^2+z^2}{2}\right)^{\frac{r-1}{2}} \le 1$$

Since $t+z=2$, this inequality is equivalent to

$$tz(2-tz)^{\frac{r-1}{2}} \le 1,$$

or

$$\left(\frac{1}{tz}\right)^{\frac{2}{r-1}} \ge 2 - tz.$$

Let $u = \frac{1}{tz}$, $u \geq 1$, and let $p = \frac{2}{r-1}$, $1 \leq p \leq 2$. Using Bernoulli's Inequality, we get

$$\left(\frac{1}{tz}\right)^{\frac{2}{r-1}} - 2 + tz = [1 + (u-1)]^p - 2 + \frac{1}{u} \geq 1 + p(u-1) - 2 + \frac{1}{u} =$$
$$= (u-1)\left(p - \frac{1}{u}\right) \geq 0.$$

Equality in (23) occurs for $(x, y, z) = (0, 1, 1)$ or any cyclic permutation. □

Remark 1. For $r = 3$, the inequality has the form

$$x^3(y+z) + y^3(z+x) + z^3(x+y) \leq \frac{1}{8}(x+y+z)^4 \tag{24}$$

We can prove this inequality using the assumption $x = \max\{x, y, z\}$ and the identity

$$\frac{1}{8}(x+y+z)^4 = \frac{1}{8}(-x+y+z)^4 + x^3(y+z) + x(y+z)^3.$$

The inequality becomes

$$yz(y^2 + z^2 - 3xy - 3xz) \leq (-x+y+z)^4,$$

and it is true, because left hand side is less than or equal to zero:

$$y^2 + z^2 - 3xy - 3xz \leq y^2 + z^2 - 3y^2 - 3z^2 \leq 0.$$

Remark 2. Inequality (23) is not valid for $r > 3$. However, as shown above, if the numbers x, y, z sum to a constant value, then the expression $E_r(x, y, z)$ with $r \geq 2$ attains its maximum value when one of x, y, z is zero. For $r = 4$, we have the following nice statement:

If x, y, z are non-negative real numbers, then

$$x^4(y+z) + y^4(z+x) + z^4(x+y) \leq \frac{1}{12}(x+y+z)^5. \tag{25}$$

(*Vasile Cîrtoaje*, MS, 2005)

On the assumption that $x \leq y \leq z$, this inequality follows from

$$E_4(x, y, z) \leq E_4(0, x+y, z) \leq \frac{1}{12}(x+y+z)^5.$$

We have

$$E_4(0,x+y,z)-E_4(x,y,z)=z(x+y)^4-x^4(y+z)-y^4(z+x)=$$
$$=xy\left[2z(2x^2+2y^2+3xy)-x^3-y^3\right]\ge$$
$$\ge xy\left[(x+y)(2x^2+2y^2+3xy)-x^3-y^3\right]\ge 0,$$

and

$$(x+y+z)^5-12E_4(0,x+y,z)=(x+y+z)(x^2+y^2+z^2+2xy-4yz-4zx)^2\ge 0$$

Equality in (25) occurs for $(x,y,z)\sim\left(0,1,2+\sqrt{3}\right)$ or any symmetrical permutation.

Remark 3. We will show in chapter 5 that inequality (23) is valid for the larger range $r_0\le r\le 3$, where

$$r_0=\frac{\ln 2}{\ln 3-\ln 2}\approx 1\ 71$$

On the other hand, we will show in chapter 3 that for $0<r\le r_0$ and $x+y+z=3$, the inequality holds

$$x^r(y+z)+y^r(z+x)+z^r(x+y)\le 6$$

All these results solve the problem posted on Mathlinks Inequalities Forum in 2005, by *Pham Kim Hung.*

Let $x\le y\le z$ be non-negative numbers such that $x+y+z=3$. For $r>0$, when the expression $E_r(x,y,z)$ attains its maximum value?

The answer to this problem is the following·

a) $E(x,y,z)\le E(1,1,1)$, for $0<r<r_0$,

b) $E(x,y,z)\le E(1,1,1)=E\left(0,\frac{3}{3},\frac{3}{2}\right)$, for $r=r_0$;

c) $E(x,y,z)\le E\left(0,\frac{3}{2},\frac{3}{2}\right)$, for $r_0<r<3$,

d) $E(x,y,z)\le \max\limits_{y+z=3}E(0,y,z)=\max\limits_{y+z=3}yz(y^{r-1}+z^{r-1})$, for $r>3$

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9. *Let x,y,z be non-negative real numbers satisfying $x+y+z=1$. If $p>0$ and $q\le\dfrac{(p-1)(2p+1)}{4}$, then*

$$\frac{yz+q}{x+p}+\frac{zx+q}{y+p}+\frac{xy+q}{z+p}\le\frac{1+9q}{1+3p}. \tag{26}$$

Proof. We write the inequality in the form

$$\frac{yz}{x+p}+\frac{zx}{y+p}+\frac{xy}{z+p}-\frac{1}{1+3p}+q\left(\frac{1}{x+p}+\frac{1}{y+p}+\frac{1}{z+p}-\frac{9}{1+3p}\right)\le 0.$$

By the AM-HM Inequality, we have

$$\frac{1}{x+p}+\frac{1}{y+p}+\frac{1}{z+p}\ge\frac{9}{(x+p)+(y+p)+(z+p)}=\frac{9}{1+3p}.$$

Thus, it suffices to prove the inequality for $q=\dfrac{(p-1)(2p+1)}{4}$. In this case, the inequality becomes

$$\frac{yz+q}{x+p}+\frac{zx+q}{y+p}+\frac{xy+q}{z+p}\le\frac{6p-5}{4},$$

or

$$(6p-5)(x+p)(y+p)(z+p)\ge 4\sum(yz+q)(y+p)(z+p).$$

Let $t=xy+yz+zx$. By the well-known inequality

$$(x+y+z)^2\ge 3(xy+yz+zx),$$

we get $t\le\dfrac{1}{3}$. Since

$$(6p-5)(x+p)(y+p)(z+p)=(6p-5)(xyz+pt+p^2+p^3)$$

and

$$\begin{aligned}4\sum(yz+q)(y+p)(z+p)&=\sum(4yz+2p^2-p-1)(yz-px+p+p^2)=\\&=4\sum y^2z^2+(6p^2+3p-1)t+p(3p+2)(2p^2-p-1)-12pxyz=\\&=4t^2+(6p^2+3p-1)t+p(3p+2)(2p^2-p-1)-4(3p+2)xyz,\end{aligned}$$

the inequality reduces to

$$(1-4t)(2p+t)+3(6p+1)xyz\ge 0.$$

For $t<\dfrac{1}{4}$ the inequality is clearly true. Consider now $\dfrac{1}{4}<t\le\dfrac{1}{3}$. By the third degree Schur's Inequality

$$(x+y+z)^3+9xyz\ge 4(x+y+z)(xy+yz+zx),$$

we get

$$xyz \geq \frac{4t-1}{9}.$$

Thus,

$$(1-4t)(2p+t)+3(6p+1)xyz \geq (1-4t)(2p+t)+\frac{(6p+1)(4t-1)}{3} = \\ = \frac{(4t-1)(1-3t)}{3} \geq 0.$$

In the original inequality, equality occurs for $(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ In the special case $q=\frac{(p-1)(2p+1)}{4}$, equality also occurs for $(x,y,z)=\left(0,\frac{1}{2},\frac{1}{2}\right)$ and any cyclic permutation. □

Remark In the particular cases $p=1$, $q=0$ and $p=\frac{5}{6}$, $q=-\frac{1}{9}$, from (26) we find the following inequalities

$$\frac{yz}{x+1}+\frac{zx}{y+1}+\frac{xy}{z+1} \leq \frac{1}{4},$$

$$\frac{9yz-1}{6x+5}+\frac{9zx-1}{6y+5}+\frac{9xy-1}{6z+5} \leq 0,$$

respectively. Equality occurs for $(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, as well as for $(x,y,z)=\left(0,\frac{1}{2},\frac{1}{2}\right)$ and any cyclic permutation

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10. *Let x,y,z be positive real numbers. If $1 \leq r \leq 3$, then*

$$x^r y^{4-r}+y^r z^{4-r}+z^r x^{4-r} \leq \frac{1}{3}(x^2+y^2+z^2)^2 \qquad (27)$$

Proof. We notice that for $r=1$ and $r=3$, this inequality becomes of type (1) Let $E_r = x^r y^{4-r}+y^r z^{4-r}+z^r x^{4-r}$ For $1<r<3$, we apply Jensen's Inequality to the concave function $f(t)=t^{\frac{r-1}{2}}$ to get

$$E_r = xy^3\left(\frac{x^2}{y^2}\right)^{\frac{r-1}{2}}+yz^3\left(\frac{y^2}{z^2}\right)^{\frac{r-1}{2}}+zx^3\left(\frac{z^2}{x^2}\right)^{\frac{r-1}{2}} \leq E_1\left(\frac{E_3}{E_1}\right)^{\frac{r-1}{2}} = \\ = E_1^{\frac{3-r}{2}} E_3^{\frac{r-1}{2}}$$

According to (1), $E_1 \leq \frac{1}{3}(x^2+y^2+z^2)^2$ and $E_3 \leq \frac{1}{3}(x^2+y^2+z^2)^2$, and hence $E_r \leq \frac{1}{3}(x^2+y^2+z^2)^2$ There is equality in (27) for $(x,y,z) \sim (1,1,1)$. In the case $r=3$, equality again occurs for $(x,y,z) \sim \left(\sin^2\frac{4\pi}{7}, \sin^2\frac{2\pi}{7}, \sin^2\frac{\pi}{7}\right)$ or any cyclic permutation. Also, in the case $r=1$, equality occurs occurs again for $(x,y,z) \sim \left(\sin^2\frac{\pi}{7}, \sin^2\frac{2\pi}{7}, \sin^2\frac{4\pi}{7}\right)$ or any cyclic permutation. □

Remark. Replacing x, y, z with $\sqrt{x}$, $\sqrt{y}$, $\sqrt{z}$, respectively, and r with $2r$, we get the following equivalent statement.

Let x, y, z be positive real numbers such that $x+y+z=3$. If $\frac{1}{2} \leq r \leq \frac{3}{2}$, then

$$x^r y^{2-r} + y^r z^{2-r} + z^r x^{2-r} \leq 3.$$

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11. *Let x, y, z be positive real numbers.*

a) If $x+y+z=3$ and $0 < r \leq \frac{1}{2}$, then

$$x^{1+r}y^r + y^{1+r}z^r + z^{1+r}x^r \leq 3; \tag{28}$$

b) If $x+y+z = 1+2r$ and $r \geq 1$, then

$$x^{1+r}y^r + y^{1+r}z^r + z^{1+r}x^r \leq r^r(1+r)^{1+r}. \tag{29}$$

Proof. Let $F_r(x,y,z) = x^{1+r}y^r + y^{1+r}z^r + z^{1+r}x^r$.

a) For $r = \frac{1}{2}$, the inequality $F_{\frac{1}{2}} \leq 3$ is just of type (1). For $0 < r < \frac{1}{2}$, applying Jensen's Inequality to the concave function $f(t) = t^{2r}$ yields

$$F_r = x\left(\sqrt{xy}\right)^{2r} + y\left(\sqrt{yz}\right)^{2r} + z\left(\sqrt{zx}\right)^{2r} \leq$$
$$\leq (x+y+z)\left(\frac{F_{\frac{1}{2}}}{x+y+z}\right)^{2r} \leq 3$$

For $0 < r < \frac{1}{2}$, equality in (28) occurs if and only if $(x,y,z) = (1,1,1)$.

b) There are two cases to consider.

Case $x \le z \le y$. We will show that

$$F_r(x,y,z) \le F_r(0,x+y,z) \le F(0,1+r,r)$$

We have

$$F_r(0,x+y,z) - F_r(x,y,z) = (x+y)^{1+r}z^r - x^{1+r}y^r - y^{1+r}z^r - z^{1+r}x^r.$$

Since

$$(x+y)^{1+r} \ge (x+y)(x^r+y^r) \ge xy^r + x^r y + y^{1+r},$$

we get

$$\begin{aligned} F_r(0,x+y,z) - F_r(x,y,z) &\ge xy^r z^r + x^r y z^r - x^{1+r}y^r - x^r z^{1+r} = \\ &= xy^r(z^r - x^r) + x^r z^r (y-z) \ge 0. \end{aligned}$$

Setting now $x+y=t$ ($t>0$, $t+z=1+2r$), the right inequality becomes

$$F_r(0,t,z) \le F_r(0,1+r,r),$$

or

$$\left(\frac{t}{1+r}\right)^{1+r}\left(\frac{z}{r}\right)^r \le 1.$$

This inequality follows by the weighted AM-GM Inequality, as follows

$$\left(\frac{t}{1+r}\right)^{1+r}\left(\frac{z}{r}\right)^r \le \frac{1+r}{1+2r}\,\frac{t}{1+r} + \frac{r}{1+2r}\,\frac{z}{r} = \frac{t+z}{1+2r} = 1$$

Case $x \le y \le z$. We will show that

$$F_r(x,y,z) \le F_r(0,x+z,y) \le F_r(0,1+r,r).$$

Since the right inequality is similar to the above one, we will prove only the left inequality. We have

$$F_r(0,x+z,y) - F_r(x,y,z) = (x+z)^{1+r}y^r - x^{1+r}y^r - y^{1+r}z^r - z^{1+r}x^r.$$

Since

$$(x+z)^{1+r} \ge (x+z)(x^r+z^r) \ge x^{1+r} + xz^r + z^{1+r},$$

we get

$$\begin{aligned} F_r(0,x+z,y) - F_r(x,y,z) &\ge xy^r z^r + y^r z^{1+r} - y^{1+r}z^r - x^r z^{1+r} = \\ &= y^r z^r (x-y) + z^{1+r}(y^r - x^r) \ge y^r z^r (x-y) + yz^r(y^r - x^r) = \\ &= xyz^r(y^{r-1} - x^{r-1}) \ge 0. \end{aligned}$$

Equality in (29) occurs for $(x,y,z) = (0,1+r,r)$ or any cyclic permutation. □

Remark Inequalities (28) and (29) are not valid for $\frac{1}{2} < r < 1$

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12. *Let x, y, z be positive real numbers.*

a) If $x + y + z = 3$ and $0 < r \le \frac{3}{2}$, then

$$x^r y + y^r z + z^r x \le 3, \tag{30}$$

b) If $x + y + z = r + 1$ and $r \ge 2$, then

$$x^r y + y^r z + z^r x \le r^r. \tag{31}$$

Proof. Let $G_r(x, y, z) = x^r y + y^r z + z^r x$.

a) Since the function $f(t) = t^{\frac{2r}{3}}$ is concave on $(0, \infty)$, by Jensen's Inequality we get

$$G_r = y\left(x^{\frac{3}{2}}\right)^{\frac{2r}{3}} + z\left(y^{\frac{3}{2}}\right)^{\frac{2r}{3}} + x\left(z^{\frac{3}{2}}\right)^{\frac{2r}{3}} \le$$

$$\le (y + z + x)\left(\frac{yx^{\frac{3}{2}} + zy^{\frac{3}{2}} + xz^{\frac{3}{2}}}{y + z + x}\right)^{\frac{2r}{3}} = 3\left(\frac{G_{\frac{3}{2}}}{3}\right)^{\frac{2r}{3}}.$$

Thus, it is enough to show that $G_{\frac{3}{2}} \le 3$ Since the function $f(t) = \sqrt{t}$ is concave, by Jensen's Inequality we get

$$G_{\frac{3}{2}} = xy\sqrt{x} + yz\sqrt{y} + zx\sqrt{z} \le (xy + yz + zx)\sqrt{\frac{x^2 y + y^2 z + z^2 x}{xy + yz + zx}}.$$

We still have to show that

$$(xy + yz + zx)(x^2 y + y^2 z + z^2 x) \le 9.$$

Write this inequality in the homogeneous form

$$27(xy + yz + zx)(x^2 y + y^2 z + z^2 x) \le (x + y + z)^5. \tag{32}$$

Suppose that $x = \min\{x, y, z\}$. Setting $y = x + p$ and $z = x + q$ ($p \ge 0$, $q \ge 0$), inequality (32) becomes

$$27(p^2 - pq + q^2)x^3 + 9Bx^2 + 3(p + q)Cx + D \ge 0,$$

where

$$B = 4p^3-6p^2q+3pq^2+4q^3,\ C = 5p^3-12p^2q+6pq^2+5q^3,\ D = (p+q)^5-27p^3q^2$$

The last inequality is true since

$$B = \left(\frac{1}{2}p - q\right)^2(7p+4q) + \frac{9}{4}p^3 \geq 0,$$
$$C = (p-2q)^2(p+q) + 6p\left(\frac{3}{4}p - q\right)^2 + \frac{5}{8}p^3 + q^3 \geq 0,$$
$$D = \left(\frac{p}{3}+\frac{p}{3}+\frac{p}{3}+\frac{q}{2}+\frac{q}{2}\right)^5 - 27p^3q^2 \geq \left[5\sqrt[5]{\left(\frac{p}{3}\right)^3\left(\frac{q}{2}\right)^2}\right]^5 - 27p^3q^2 =$$
$$= \frac{209}{108}p^3q^2 \geq 0$$

Equality in (30) and (32) occurs for $(x,y,z) = (1,1,1)$.

b) We will present an elegant solution posted on Mathlinks Inequalities Forum by *Gabriel Dospinescu*. Using the assumption $x = \max\{x,y,z\}$, he proved that

$$G_r(x,y,z) \leq G_r\left(x+\frac{z}{2}, y+\frac{z}{2}, 0\right) \leq G_r(r,1,0). \tag{33}$$

The left inequality of (33), namely

$$\left(x+\frac{z}{2}\right)^r\left(y+\frac{z}{2}\right) \geq x^r y + y^r z + z^r x,$$

can be obtained by adding up the below inequalities multiplied by y and $\frac{z}{2}$, respectively:

$$\left(x+\frac{z}{2}\right)^r \geq x^r + y^{r-1}z,$$
$$\left(x+\frac{z}{2}\right)^r \geq 2xz^{r-1}.$$

To prove these two inequalities, we notice that

$$\left(x+\frac{z}{2}\right)^r = x^r\left(1+\frac{z}{2x}\right)^r \geq x^r\left(1+\frac{z}{2x}\right)^2 \geq x^r\left(1+\frac{z}{x}\right) = x^r + x^{r-1}z.$$

Since

$$x^r + x^{r-1}z \geq x^r + y^{r-1}z$$

and

$$x^r + x^{r-1}z \geq xz^{r-1} + xz^{r-1} = 2xz^{r-1},$$

the conclusion follows.

The right inequality of (33) has the homogeneous form

$$r^r \left(\frac{x+y+z}{r+1}\right)^{r+1} \geq \left(x+\frac{z}{2}\right)^r \left(y+\frac{z}{2}\right)$$

Using the substitution $t = \left(y+\frac{z}{2}\right)/\left(x+\frac{z}{2}\right)$, reduces it to

$$\left(\frac{rt+r}{r+1}\right)^{r+1} \geq rt.$$

By Bernoulli's Inequality, we have

$$\left(\frac{rt+r}{r+1}\right)^{r+1} = \left(1+\frac{rt-1}{r+1}\right)^{r+1} \geq 1+(r+1)\frac{rt-1}{r+1} = rt,$$

and the conclusion follows. Equality in (31) occurs for $(x,y,z) = (r,1,0)$ or any cyclic permutation . □

Remark 1. In the following section of this chapter (problem 33), we will show that inequality (30) holds for $0 < r \leq r_1$, where $r_1 \approx 1.558$ is a root of the equation

$$(1+r)^{1+r} = (3r)^r. \tag{34}$$

Remark 2. Inequality (31) was published in Vietnamese journal "Mathematics and Youth", 1996 On the assumption $x = \max\{x,y,z\}$, we can prove that inequalities (33) are valid for $r \geq r_0$, where

$$r_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71.$$

Then, we can say that (31) is valid for any $r \geq r_0$. Moreover, we *conjecture* that (31) holds true for $r \geq r_1$, where $r_1 \approx 1.558$ is a root of equation (34).

★

13. *Let $m > n > 0$, and let x,y,z be positive real numbers such that $x^{m+n} + y^{m+n} + z^{m+n} = 3$. Then*

$$\frac{x^m}{y^n} + \frac{y^m}{z^n} + \frac{z^m}{x^n} \geq 3. \tag{35}$$

Proof. Using the substitutions $p = \dfrac{2n}{m+n}$, $a = x^{\frac{m+n}{2}}$, $b = y^{\frac{m+n}{2}}$ and $c = z^{\frac{m+n}{2}}$, we have to show that $a^2+b^2+c^2=3$ yields

$$\frac{a^{2-p}}{b^p}+\frac{b^{2-p}}{c^p}+\frac{c^{2-p}}{a^p}\geq 3.$$

Write this inequality as

$$\frac{a^2}{(ab)^p}+\frac{b^2}{(bc)^p}+\frac{c^2}{(ca)^p}\geq 3$$

Applying Jensen's Inequality to the convex function $f(u)=\dfrac{1}{u^p}$, we get

$$\frac{a^2}{(ab)^p}+\frac{b^2}{(bc)^p}+\frac{c^2}{(ca)^p}\geq \frac{a^2+b^2+c^2}{\left(\frac{a^2\cdot ab+b^2\cdot bc+c^2\cdot ca}{a^2+b^2+c^2}\right)^p}=\frac{3^{1+p}}{(a^3b+b^3c+c^3a)^p}.$$

To end the proof, it suffices to show that $a^3b+b^3c+c^3a\leq 3$. This inequality immediately follows from

$$(a^2+b^2+c^2)^2\geq 3(a^3b+b^3c+c^3a),$$

which is just (1). Equality occurs for $(x,y,z)=(1,1,1)$. □

Remark. *The above inequality is a generalization of the below statement, posted on Mathlinks Inequalities Forum by* Michael Rozenberg.

• If n,x,y,z are positive numbers such that $x^{2n+1}+y^{2n+1}+z^{2n+1}=3$, then

$$\frac{x^{n+1}}{y^n}+\frac{y^{n+1}}{z^n}+\frac{z^{n+1}}{x^n}\geq 3$$

★

14. *Let a,b,c,d be non-negative real numbers. If $p>0$, then*

$$\left(1+p\frac{a}{b+c}\right)\left(1+p\frac{b}{c+d}\right)\left(1+p\frac{c}{d+a}\right)\left(1+p\frac{d}{a+b}\right)\geq(1+p)^2$$

Proof. This inequality is well known for $p=1$; that is

$$(a+b+c)(b+c+d)(c+d+a)(d+a+b)\geq 4(a+b)(b+c)(c+d)(d+a)$$

Since $(a+b+c)^2 \geq (2a+b)(2c+b)$ and $(2a+b)(2b+a) \geq 2(a+b)^2$, we have

$$\prod(a+b+c)^2 \geq \prod(2a+b)(2c+b) = \prod(2a+b)(2b+a) \geq 2^4 \prod(a+b)^2,$$

and the above inequality follows ($\prod$ is cyclic over a, b, c)

Another proof of the same particular case is based on the inequalities:

$$(a+b+c)(b+c+d) \geq (b+c)(a+b+c+d) \geq 2(b+c)\sqrt{(a+b)(c+d)}$$

Then,

$$\prod(a+b+c)^2 = \prod(a+b+c)(b+c+d) \geq$$
$$\geq 2^4 \prod(b+c)\sqrt{(a+b)(c+d)} = 2^4 \prod(a+b)^2$$

In order to prove the original inequality, denote $x = \dfrac{a}{b+c}$, $y = \dfrac{b}{c+d}$, $z = \dfrac{c}{d+a}$ and $t = \dfrac{d}{a+b}$. Since

$$\prod(1+px) \geq 1 + p(x+y+x+t) + p^2(xy+yz+zt+tx+xz+yt),$$

it suffice to show that

$$x+y+z+t \geq 2$$

and

$$xy+yz+zt+tx+xz+yt \geq 1.$$

The inequality $x+y+z+t \geq 2$ is the well-known Shapiro's Inequality for 4 positive numbers. It can be derived as follows

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geq \frac{(a+b+c+d)^2}{a(b+c)+b(c+d)+c(d+a)+d(a+b)} \geq 2.$$

The left inequality follows by the Cauchy-Schwarz Inequality, whereas the right inequality reduces to the obvious inequality $(a-c)^2+(b-d)^2 \geq 0$.

The inequality $xy+yz+zt+tx+xz+yt \geq 1$ can be derived using the inequalities

$$\frac{x+z}{2} - xz = \frac{bc+da+(a-c)^2}{2(b+c)(d+a)} \geq 0, \quad \frac{y+t}{2} - yt = \frac{ab+cd+(b-d)^2}{2(a+b)(c+d)} \geq 0,$$

and the identity

$$xz(1+y+t) + y(1+x+z) = 1.$$

Indeed, we have

$$xy + yz + zt + tx + xz + yt = \frac{x+z}{2}(y+t) + \frac{y+t}{2}(x+z) + xz + yt \geq$$
$$\geq xz(y+t) + yt(x+z) + xz + yt = xz(1+y+t) + yt(1+x+z) = 1,$$

and the proof is finished
There is equality for either $a = c = 0$ or $b = d = 0$. □

★

15. *If a, b, c are positive real numbers, then*

$$\frac{1}{4a} + \frac{1}{4b} + \frac{1}{4c} + \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq 3\left(\frac{1}{3a+b} + \frac{1}{3b+c} + \frac{1}{3c+a}\right). \quad (36)$$

Proof. We present the author's solution, which emphasizes that (36) is an ingenious consequence of the special inequality (1).

Actually, we will prove that for any positive t, the following more general inequality holds

$$\frac{t^{4a}}{4a} + \frac{t^{4b}}{4b} + \frac{t^{4c}}{4c} + \frac{t^{2(a+b)}}{a+b} + \frac{t^{2(b+c)}}{b+c} + \frac{t^{2(c+a)}}{c+a} - 3\left(\frac{t^{3a+b}}{3a+b} + \frac{t^{3b+c}}{3b+c} + \frac{t^{3c+a}}{3c+a}\right) \geq 0$$

For $t = 1$, this inequality turns into (36) Denoting the left hand side by $f(t)$, the inequality becomes $f(t) \geq f(0)$ We see that it suffices to show that $f'(t) \geq 0$ for $t > 0$ Indeed,

$$f'(t) = t^{4a-1} + t^{4b-1} + t^{4c-1} + 2\left(t^{2a+2b-1} + t^{2b+2c-1} + t^{2c+2a-1}\right) -$$
$$- 3\left(t^{3a+b-1} + t^{3b+c-1} + t^{3c+a-1}\right),$$

and letting $x = t^{a-\frac{1}{4}}$, $y = t^{b-\frac{1}{4}}$, $z = t^{c-\frac{1}{4}}$, the inequality $f'(t) \geq 0$ reduces to

$$x^4 + y^4 + z^4 + 2\left(x^2y^2 + y^2z^2 + z^2x^2\right) \geq 3\left(x^3y + y^3z + z^3x\right),$$

which is just (1). Equality occurs for $a = b = c$. □

Remark Another similar problem is the following

If a, b, c are positive real numbers, then

$$\frac{1}{4a} + \frac{1}{4b} + \frac{1}{4c} + \frac{1}{a+3b} + \frac{1}{b+3c} + \frac{1}{c+3a} \geq 2\left(\frac{1}{3a+b} + \frac{1}{3b+c} + \frac{1}{3c+a}\right) \quad (37)$$

(*Vasile Cîrtoaje*, MS, 2005)

This inequality is a particular case ($t = 1$) of the inequality $g(t) \geq 0$, where

$$g(t) = \frac{t^{4a}}{4a} + \frac{t^{4b}}{4b} + \frac{t^{4c}}{4c} + \frac{t^{a+3b}}{a+3b} + \frac{t^{b+3c}}{b+3c} + \frac{t^{c+3a}}{c+3a} - \\ -2\left(\frac{t^{3a+b}}{3a+b} + \frac{t^{3b+c}}{3b+c} + \frac{t^{3c+a}}{3c+a}\right).$$

In order to prove that $g(t) \geq g(0)$, it suffices to show that $g'(t) \geq 0$ for $t > 0$ We have

$$g'(t) = t^{4a-1} + t^{4b-1} + t^{4c-1} + t^{a+3b-1} + t^{b+3c-1} + t^{c+3a-1} - \\ -2\left(t^{3a+b-1} + t^{3b+c-1} + t^{3c+a-1}\right)$$

Denoting $x = t^{a-\frac{1}{4}}$, $y = t^{b-\frac{1}{4}}$ and $z = t^{c-\frac{1}{4}}$, the inequality $g'(t) \geq 0$ becomes

$$x^4 + y^4 + z^4 + xy^3 + yz^3 + zx^3 \geq 2(x^3y + y^3z + z^3x),$$

which is just (7). We have equality for $a = b = c$.

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16. *If x, y, z are non-negative real numbers satisfying $x + y + z = 3$, then*

$$\frac{x}{xy+1} + \frac{y}{yz+1} + \frac{z}{zx+1} \geq \frac{3}{2}. \tag{38}$$

Proof. Since the Cauchy-Schwarz method fails in proving this inequality, we have to choose between expanding and using a suitable hint. We will approach the second way The hint is using the relations

$$\frac{x}{xy+1} = x - \frac{x^2y}{xy+1}, \quad \frac{y}{yz+1} = y - \frac{y^2z}{yz+1}, \quad \frac{z}{zx+1} = z - \frac{z^2x}{zx+1},$$

to transform (38) into

$$\frac{x^2y}{xy+1} + \frac{y^2z}{yz+1} + \frac{z^2x}{zx+1} \leq \frac{3}{2}.$$

By the AM-GM Inequality, we have

$$xy + 1 \geq 2\sqrt{xy}, \quad yz + 1 \geq 2\sqrt{yz}, \quad zx + 1 \geq 2\ \ zx.$$

Consequently, it suffices to show that

$$\frac{x^2y}{2\sqrt{xy}} + \frac{y^2z}{2\sqrt{yz}} + \frac{z^2x}{2\sqrt{zx}} \le \frac{3}{2};$$

that is

$$x\sqrt{xy} + y\sqrt{yz} + z\sqrt{zx} \le 3$$

This inequality has the homogeneous form

$$(x+y+z)^2 \ge 3\left(x\sqrt{xy} + y\sqrt{yz} + z\sqrt{zx}\right).$$

Replacing x, y, z by x^2, y^2, z^2, respectively, we get just inequality (1) Equality occurs in (38) only for $(x, y, z) = (1, 1, 1)$ □

Remark 1. A slightly more general statement is the following

Let x, y, z be non-negative real numbers satisfying $x + y + z = 3$. If $0 < p \le 1$, then

$$\frac{x}{xy+p} + \frac{y}{yz+p} + \frac{z}{zx+p} \ge \frac{3}{1+p} \tag{39}$$

Proceeding as before, we can rewrite the inequality as

$$\frac{x^2y}{xy+p} + \frac{y^2z}{yz+p} + \frac{z^2x}{zx+p} \le \frac{3}{1+p}.$$

By the weighted AM-GM Inequality, we have

$$xy + p = 1 \cdot xy + p \cdot 1 \ge (1+p)(xy)^{\frac{1}{1+p}}\ 1^{\frac{p}{1+p}} = (1+p)(xy)^{\frac{1}{1+p}}.$$

Hence,

$$\frac{x^2y}{xy+p} \le \frac{1}{1+p}x(xy)^{\frac{p}{1+p}},$$

and similarly,

$$\frac{y^2z}{yz+p} \le \frac{1}{1+p}y(yz)^{\frac{p}{1+p}}, \quad \frac{z^2x}{zx+p} \le \frac{1}{1+p}z(zx)^{\frac{p}{1+p}}.$$

Thus, it suffices to show that

$$x(xy)^{\frac{p}{1+p}} + y(yz)^{\frac{p}{1+p}} + z(zx)^{\frac{p}{1+p}} \le 3$$

Since $0 < \dfrac{p}{1+p} \le \dfrac{1}{2}$, this inequality coincides with (28)

We *conjecture* that inequality (39) is valid for $0 < p \le p_0$, where $p_0 \approx 1.5874$. Replacing the triple $(x,y,z) = \left(0, \frac{9}{4}, \frac{3}{4}\right)$ in (39) yields the necessary condition $p \le \frac{27}{17} \approx 1.588$.

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17. *If x, y, z are non-negative real numbers satisfying $x+y+z=3$, then*

$$\frac{x}{y^2+3} + \frac{y}{z^2+3} + \frac{z}{x^2+3} \ge \frac{3}{4}. \tag{40}$$

Proof. By the AM-GM Inequality, we have

$$y^2+3 = y^2+1+1+1 \ge 4\sqrt{y}.$$

Hence,

$$\frac{3x}{y^2+3} = x - \frac{xy^2}{y^2+3} \ge x - \frac{xy^2}{4\sqrt{y}} = x - \frac{1}{4}\,xy^{\frac{3}{2}},$$

and similarly,

$$\frac{3y}{z^2+3} \ge y - \frac{1}{4}\,yz^{\frac{3}{2}}, \quad \frac{3z}{x^2+3} \ge z - \frac{1}{4}\,zx^{\frac{3}{2}}.$$

Using these results, it suffices to show that

$$xy^{\frac{3}{2}} + yz^{\frac{3}{2}} + zx^{\frac{3}{2}} \le 3.$$

This inequality is just (30) for $r = \frac{3}{2}$. Equality occurs in (40) only for $(x,y,z) = (1,1,1)$. □

Remark. The following more general statement is valid:

Let x, y, z be non-negative real numbers satisfying $x+y+z=3$. If $0 < p \le 3$, then

$$\frac{x}{y^2+p} + \frac{y}{z^2+p} + \frac{z}{x^2+p} \ge \frac{3}{1+p} \tag{41}$$

By the weighted AM-GM Inequality, we have

$$y^2+p \ge (1+p\;y^{\frac{2}{1+p}}.$$

Hence,

$$\frac{px}{y^2+p} = x - \frac{xy^2}{y^2+p} \geq x - \frac{xy^2}{(1+p)y^{\frac{2}{1+p}}} = x - \frac{1}{1+p}\, xy^{\frac{2p}{1+p}},$$

and similarly,

$$\frac{py}{z^2+p} \geq y - \frac{1}{1+p}\, yz^{\frac{2p}{1+p}}, \quad \frac{pz}{1+px^2} \geq z - \frac{1}{1+p}\, zx^{\frac{2p}{1+p}}.$$

Consequently, if the below inequality is true,

$$x \cdot y^{\frac{2p}{1+p}} + y \cdot z^{\frac{2p}{1+p}} + z \cdot x^{\frac{2p}{1+p}} \leq 3,$$

then (41) is also true. Since $0 < \dfrac{2p}{1+p} \leq \dfrac{3}{2}$, this inequality follows from (30).

We *conjecture* that inequality (41) holds for $0 < p \leq 3+2\sqrt{3}$ Replacing the triple $(x,y,z) = \left(0, 3-\sqrt{3}, \sqrt{3}\right)$ in (41), we get the necessary condition $p \leq 3+2\sqrt{3}$.

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18. *If a,b,c are positive numbers satisfying $abc = 1$, then*

$$\sqrt{\frac{a}{b+8}} + \sqrt{\frac{b}{c+8}} + \sqrt{\frac{c}{a+8}} \geq 1. \tag{42}$$

Proof. By Bernoulli's Inequality, we get

$$\frac{\sqrt{b+8}}{3} = \sqrt{1+\frac{b-1}{9}} \leq 1 + \frac{b-1}{18} = \frac{b+17}{18}.$$

Then

$$\sqrt{\frac{a}{b+8}} \geq \frac{6\sqrt{a}}{b+17}, \quad \sqrt{\frac{b}{c+8}} \geq \frac{6\sqrt{b}}{c+17}, \quad \sqrt{\frac{c}{a+8}} \geq \frac{6\sqrt{c}}{a+17},$$

and it suffices to show that

$$\frac{\sqrt{a}}{b+17} + \frac{\sqrt{b}}{c+17} + \frac{\sqrt{c}}{a+17} \geq \frac{1}{6}.$$

Substituting $\dfrac{x}{y}$ for $\sqrt{a}$, $\dfrac{z}{x}$ for $\sqrt{b}$, and $\dfrac{y}{z}$ for $\sqrt{c}$ to obtain $abc = 1$ $(x,y,z > 0)$, the inequality becomes

$$\frac{x^3}{y(17x^2+z^2)} + \frac{z^3}{x(17z^2+y^2)} + \frac{y^3}{z(17y^2+x^2)} \geq \frac{1}{6}$$

By the Cauchy-Schwarz Inequality, we have

$$\frac{x^3}{y(17x^2+z^2)}+\frac{y^3}{z(17y^2+x^2)}+\frac{z^3}{x(17z^2+y^2)} \geq$$
$$\geq \frac{(x^2+y^2+z^2)^2}{xy(17x^2+z^2)+yz(17y^2+x^2)+zx(17z^2+y^2)}.$$

Therefore, it is enough to show that

$$6(x^2+y^2+z^2)^2 \geq 17(x^3y+y^3z+z^3x)+xyz(x+y+z),$$

which follows by combining (1) and

$$(x^2+y^2+z^2)^2 \geq 3xyz(x+y+z).$$

The last inequality can be obtained as follows

$$(x^2+y^2+z^2)^2 \geq (xy+yz+zx)^2 \geq 3xyz(x+y+z).$$

Equality occurs in (42) only for $(a,b,c)=(1,1,1)$. □

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19. *If a,b,c are the side-lengths of a triangle, then*

a) $$3(a^3b+b^3c+c^3a) \geq (ab+bc+ca)(a^2+b^2+c^2), \tag{43}$$

b) $$9(ab+bc+ca)(a^2+b^2+c^2) \geq (a+b+c)^4 \tag{44}$$

Proof. In order to prove (43), we write it as cyclic sum

$$\sum ab(2a^2-b^2-c^2) \geq 0.$$

Since

$$\sum ab(2a^2-b^2-c^2) = \sum ab(a^2-b^2) - \sum ab(c^2-a^2) =$$
$$= \sum ab(a^2-b^2) - \sum bc(a^2-b^2) = \sum (a^2-b^2)(a-c)b,$$

the inequality becomes

$$\sum (a^2-b^2)(a-c)b \geq 0.$$

Using now the classical substitution $a = y + z$, $b = z + x$, $c = x + y$ $(x, y, z > 0)$, we have

$$a^2 - b^2 = (a - b)(a + b) = (y - x)(y + x + 2z) = y^2 - x^2 + 2z(y - x),$$
$$(a - c)b = (z - x)(z + x) = z^2 - x^2,$$
$$\sum(a^2 - b^2)(a - c)b = \sum(y^2 - x^2)(z^2 - x^2) + 2\sum z(y - x)(z^2 - x^2).$$

Since

$$\sum(y^2 - x^2)(z^2 - x^2) = \sum(x^4 - x^2y^2 + y^2z^2 - z^2x^2) = \sum(x^4 - x^2y^2)$$

and

$$\sum z(y-x)(z^2-x^2) = \sum(yz^3 - x^2yz - xz^3 + x^3z) = \sum(2x^3z - x^2yz - xz^3),$$

the inequality transforms into

$$\sum(x^4 - x^2y^2 + 4x^3z - 2x^2yz - 2xz^3) \geq 0.$$

We can find this inequality by adding the below inequalities

$$\sum(x^4 - x^2y^2 + 2x^3z - 2xz^3) \geq 0,$$
$$2\sum(x^3z - x^2yz) \geq 0.$$

First inequality is just (10), while the second inequality follows by the Cauchy-Schwarz Inequality applied to the triples $\left(x\sqrt{xz}, y\sqrt{yx}, z\sqrt{zy}\right)$ and $\left(\sqrt{y}, \sqrt{z}, \sqrt{x}\right)$.

$$(x^3z + y^3x + z^3y)(y + z + x) \geq xyz(x + y + z)^2.$$

To prove (44), denote $A = a^2 + b^2 + c^2$ and $B = ab + bc + ca$. Since

$$9(ab + bc + ca)(a^2 + b^2 + c^2) - (a + b + c)^4 = 9AB - (A + 2B)^2 = (A - B)(4B - A)$$

and

$$A - B = \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{2} \geq 0,$$

we still have to show that $4B - A \geq 0$ Indeed, we have

$$4B - A > 2(ab + bc + ca) - a^2 - b^2 - c^2 =$$
$$= \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(-\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a}-\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right) > 0.$$

Equality occurs in both (43) and (44) only for an equilateral triangle □

Remark. From (43) and (44) we get

$$\frac{a^3b+b^3c+c^3a}{3} \geq \left(\frac{a+b+c}{3}\right)^4. \tag{45}$$

The following more general statement is valid·

If a, b, c are the side lengths of a triangle and $r \geq 3$, then

$$\frac{a^rb+b^rc+c^ra}{3} \geq \left(\frac{a+b+c}{3}\right)^{r+1}. \tag{46}$$

Indeed, from the weighted Power-Mean Inequality and (45), we have

$$\left(\frac{a^rb+b^rc+c^ra}{b+c+a}\right)^{\frac{1}{r}} \geq \left(\frac{a^3b+b^3c+c^3a}{b+c+a}\right)^{\frac{1}{3}} \geq \frac{a+b+c}{3},$$

and from here, (46) follows

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20. *Let a, b, c be the side-lengths of a triangle. If $r \geq 2$, then*

$$3(a^rb+b^rc+c^ra) \geq (a+b+c)(a^{r-1}b+b^{r-1}c+c^{r-1}a). \tag{47}$$

Proof. By the weighted Power-Mean Inequality, we have

$$\left(\frac{a^rb+b^rc+c^ra}{a+b+c}\right)^{\frac{r-1}{r}} \geq \frac{a^{r-1}b+b^{r-1}c+c^{r-1}a}{a+b+c}.$$

Thus, it suffices to show that

$$\frac{3(a^rb+b^rc+c^ra)}{(a+b+c)^2} \geq \left(\frac{a^rb+b^rc+c^ra}{a+b+c}\right)^{\frac{r-1}{r}},$$

which is just (46). Since (46) is valid for $r \geq 3$, it follows that (47) is also valid for $r \geq 3$

For $r = 2$, (47) reduces to

$$2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3 \tag{48}$$

Assuming $a = \min\{a, b, c\}$ and using the substitution $b = x + \frac{a+c}{2}$, inequality (48) becomes

$$(2c-a)x^2 + \left(x+\frac{3a}{4}\right)(a-c)^2 \geq 0$$

This inequality is true, because $2c - a > 0$ and

$$4x + 3a = a + 4b - 2c = 2(a + b - c) + (2b - a) > 0$$

Inequality (48) becomes equality only for $a = b = c$

To prove (47) for all $r \geq 2$, we rewrite it in the form

$$a^{r-1}b(2a - b - c) + b^{r-1}c(2b - c - a) + c^{r-1}a(2c - a - b) \geq 0. \qquad (49)$$

We claim that the following more general statement holds.

If a, b, c are the side-lengths of a triangle and $f(x)$ is an increasing positive function on $(0, \infty)$, then

$$ab(2a - b - c)f(a) + bc(2b - c - a)f(b) + ca(2c - a - b)f(c) \geq 0 \qquad (50)$$

First notice that for $f(x) = x^{r-2}$, $r \geq 2$, (50) turns into (49) In order to prove (50), denote its left side by $E(a, b, c)$, and then consider two cases. $a \geq b \geq c$ and $a \geq c \geq b$

Case $a \geq b \geq c$ Since $f(a) \geq f(b) \geq f(c)$, we have

$$\begin{aligned} E(a,b,c) &\geq ab(2a - b - c)f(b) + bc(2b - c - a)f(b) + ca(2c - a - b)f(c) = \\ &= b\left[2(a-b)(a-c) + ab - c^2\right] f(b) + ca(2c - a - b)f(c) \geq \\ &\geq b\left[2(a-b)(a-c) + ab - c^2\right] f(c) + ca(2c - a - b)f(c) = \\ &= abc\left[2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) - \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) - 3\right] f(c). \end{aligned}$$

Taking account of (48), we get $E(a, b, c) \geq 0$

Case $a \geq c \geq b$. Since $f(a) \geq f(c) \geq f(b)$, we have

$$\begin{aligned} E(a,b,c) &\geq ab(2a - b - c)f(c) + bc(2b - c - a)f(b) + ca(2c - a - b)f(c) = \\ &= a\left[(c-b)(2c-a) + b(a-b)\right] f(c) + bc(2b - c - a)f(b). \end{aligned}$$

Since

$$(c - b)(2c - a) + b(a - b) \geq (c - b)(b + c - a) + b(a - b) \geq 0,$$

we get

$$\begin{aligned} E(a,b,c) &\geq a\left[(c-b)(2c-a) + b(a-b)\right] f(b) + bc(2b - c - a)f(b) = \\ &= abc\left[2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) - \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) - 3\right] f(b) \geq 0 \end{aligned}$$

□

Remark. The following inequality is sharper than (48).

$$3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3 \tag{51}$$

(*Vasile Cîrtoaje*, MS, 2004)

Proceeding in the same manner as in the proof of (48), we obtain the inequality

$$(3c-2a)(2b-a-c)^2+(4b+2a-3c)(a-c)^2\geq 0$$

It is true, because $3c-2a>0$ and $4b+2a-3c=3(a+b-c)+(b-a)>0$.

★

21. *Let a,b,c be the side-lengths of a triangle. If $r\geq 2$, then*

$$a^rb(a-b)+b^rc(b-c)+c^ra(c-a)\geq 0. \tag{52}$$

Proof. For $r=2$, the inequality turns into the well-known inequality

$$a^3b+b^3c+c^3a\geq a^2b^2+b^2c^2+c^2a^2. \tag{53}$$

Using the substitution $a=y+z$, $b=z+x$, $c=x+y$ $(x,y,z>0)$, this inequality reduces to

$$xy^3+yz^3+zx^3\geq xyz(x+y+z),$$

which follows by the Cauchy-Schwarz inequality

$$(x^3z+y^3x+z^3y)(y+z+x)\geq xyz(x+y+z)^2$$

Let us denote now by $E(a,b,c)$ the left side of (52) and assume, without loss of generality, that $c=\max\{a,b,c\}$. We have

$$E(a,b,a)=ab(a-b)(a^{r-1}-b^{r-1}),$$
$$E(a,b,c)-E(a,b,a)=(c-a)\left[a(c^r-b^r)-(c-b)b^r\right],$$

whence

$$E(a,b,c)=ab(a-b)(a^{r-1}-b^{r-1})+(c-a)\left[a(c^r-b^r)-(c-b)b^r\right]$$

Writing now the product ab as

$$ab = c(a+b-c) + (c-a)(c-b),$$

we get

$$E(a,b,c) = c(a+b-c)(a-b)(a^{r-1}-b^{r-1}) + \\ + (c-a)\left[(c-b)(a-b)(a^{r-1}-b^{r-1}) + a(c^r-b^r) - (c-b)b^r\right],$$

$$E(a,b,c) = c(a+b-c)(a-b)(a^{r-1}-b^{r-1}) + \\ + a(c-a)\left[(a-b)(c-b)a^{r-2} + c(c^{r-1}-b^{r-1})\right].$$

Since

$$(a-b)(c-b)a^{r-2} + c(c^{r-1}-b^{r-1}) = \\ = (a-b+c)(c-b)a^{r-2} + c\left[-(c-b)a^{r-2} + c^{r-1} - b^{r-1}\right] = \\ = (a-b+c)(c-b)a^{r-2} + c(c-b)(c^{r-2}-a^{r-2}) + bc(c^{r-2}-b^{r-2}),$$

we obtain the final form

$$E(a,b,c) = c(a+b-c)(a-b)(a^{r-1}-b^{r-1}) + (c-a)(c-b)(a-b+c)a^{r-1} + \\ + ac(c-a)(c-b)(c^{r-2}-a^{r-2}) + abc(c-a)(c^{r-2}-b^{r-2}). \quad (54)$$

For $r = 2$, this identity has the form

$$a^3b + b^3c + c^3a - a^2b^2 - b^2c^2 - c^2a^2 = \\ = c(a+b-c)(a-b)^2 + a(a-b+c)(c-a)(c-b)$$

From (54) it is clear that $r \geq 2$ together with $c = \max\{a,b,c\}$ imply $E(a,b,c) \geq 0$. Equality occurs only for an equilateral triangle

Another interesting solution was posted on Mathlinks Inequalities Forum by *Mikhail Leptchinski* It two of a, b, c are equal, then (52) is valid. Otherwise, consider that $c = \max\{a,b,c\}$. On the other hand, since the inequality is homogeneous, we may assume that $b = 1$ This implies either $a < 1 < c$ or $1 < a < c$ Let

$$f(x) = a^x b(a-b) + b^x c(b-c) + c^x a(c-a) = a^x(a-1) + c(1-c) + c^x a(c-a).$$

According to (53), we have $f(2) \geq 0$. Therefore, it suffices to prove that $f(x) \geq f(2)$ for $x \geq 2$. We have

$$f'(x) = a^x(a-1)\ln a + c^x a(c-a)\ln c$$

Since $(a-1)\ln a > 0$ and $(c-a)\ln c > 0$, it follows that $f'(x) > 0$ Therefore, $f(x)$ is strictly increasing and hence $f(x) \geq f(2)$ □

Remark An interesting generalization of (52) is the following:

Let a, b, c be the side-lengths of a triangle If $f(x)$ is an increasing positive function on $(0, \infty)$, then

$$a^2 f(a)b(a-b) + b^2 f(b)c(b-c) + c^2 f(c)a(c-a) \geq 0. \tag{55}$$

(*Darij Grinberg*, MS, 2005)

For $f(x) = x^{r-2}$, the inequality turns into (52). To prove (55), denote the left side by $E(a,b,c)$, and then consider the following two cases $a \geq b \geq c$ and $a \geq c \geq b$.

Case $a \geq b \geq c$. Since $f(a) \geq f(b) \geq f(c)$, we have

$$\begin{aligned} E(a,b,c) &\geq a^2 f(c)b(a-b) + b^2 f(c)c(b-c) + c^2 f(c)a(c-a) = \\ &= f(c)\left[a^2 b(a-b) + b^2 c(b-c) + c^2 a(c-a)\right] \geq 0. \end{aligned}$$

Case $a \geq c \geq b$. Since $f(a) \geq f(c) \geq f(b)$, we have

$$\begin{aligned} E(a,b,c) &\geq a^2 f(a)b(a-b) + b^2 f(a)c(b-c) + c^2 f(a)a(c-a) = \\ &= f(a)\left[a^2 b(a-b) + b^2 c(b-c) + c^2 a(c-a)\right] \geq 0 \end{aligned}$$

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22. *Let a, b, c be the side-lengths of a triangle. If $0 < r \leq 1$, then*

$$a^2 b(a^r - b^r) + b^2 c(b^r - c^r) + c^2 a(c^r - a^r) \geq 0. \tag{56}$$

Proof. We observe that for $r = 1$, the inequality transforms to the well-known inequality

$$a^3 b + b^3 c + c^3 a \geq a^2 b^2 + b^2 c^2 + c^2 a^2$$

On the other hand, we see that the inequality is true if two of a, b, c are equal For example, if $a = b$, the inequality reduces to

$$ac(a-c)(a^r - c^r) \geq 0,$$

which is clearly true We will consider now that a, b, c have distinct values and $a = \min\{a, b, c\}$ Rewrite the inequality in the form

$$a^{r+1}(ab - c^2) + b^{r+1}(bc - a^2) + c^{r+1}(ca - b^2) \geq 0 \tag{57}$$

Since the inequality is homogeneous, we may consider $a = 1$
This assumption yields either $b < c < 1$ or $c < b < 1$ Let

$$f(x) = a^{x+1}(ab - c^2) + b^{x+1}(bc - a^2) + c^{x+1}(ca - b^2) =$$
$$= b - c^2 + b^{x+1}(bc - 1) + c^{x+1}(c - b^2).$$

We must prove that $f(x) \geq 0$ for $0 \leq x \leq 1$ Note that $f(0) = 0$ and, in accordance with (53), $f(1) > 0$ The function $f(x)$ has the derivative

$$\frac{f'(x)}{c^{x+1}} = \left(\frac{b}{c}\right)^{x+1}(bc - 1)\ln b + (c - b^2)\ln c.$$

Case $b < c < 1$ Since $(bc - 1)\ln b > 0$ and $0 < \frac{b}{c} < 1$, the function $f'(x)$ is strictly decreasing We claim that $f'(0) > 0$ Indeed, if $f'(0) \leq 0$, then $f'(x) < 0$ for $0 < x \leq 1$, the function $f(x)$ is strictly decreasing on $[0, 1]$, and therefore $f(1) < f(0) = 0$, which is not true. Hence $f'(0) > 0$, as claimed Since $f'(x)$ is strictly decreasing and $f'(0) > 0$, two cases are possible either $f'(x) \geq 0$ for $0 \leq x \leq 1$, or there exists $x_1 \in (0, 1)$ such that $f'(x_1) = 0$, $f'(x) > 0$ for $x \in [0, x_1)$ and $f'(x) < 0$ for $x \in (x_1, 1]$ In the first case, $f(x)$ is strictly increasing on $[0, 1]$, and hence $f(x) \geq f(0) = 0$ In the second case, $f(x)$ is strictly increasing on $[0, x_1]$ and strictly decreasing on $[x_1, 1]$ Consequently, $f(x) \geq \min\{f(0), f(1)\} = f(0) = 0$

Case $c < b < 1$. Let us show that $f'(0) > 0$ We have

$$f'(0) = b(bc - 1)\ln b + c(c - b^2)\ln c.$$

If $c - b^2 \leq 0$, then $f'(0) > 0$, because $(bc - 1)\ln b > 0$ and $(c - b^2)\ln c \geq 0$
If $c - b^2 > 0$, that is $\ln c > 2\ln b$, then

$$f'(0) > b(bc - 1)\ln b + 2c(c - b^2)\ln b = (2c^2 - b^2c - b)\ln b$$

Since $\ln b < 0$ and $2c^2 - b^2c - b \leq 2c^2 - c^3 - c = -c(c - 1)^2 < 0$, it follows that $f'(0) > 0$, as claimed. To finish the proof, we observe that the function $f'(x)$ is strictly increasing, because $(bc - 1)\ln b > 0$ and $\frac{b}{c} > 1$ Therefore, $f'(x) \geq f'(0) > 0$, $f(x)$ is strictly increasing, and hence $f(x) \geq f(0) = 0$ for $0 \leq x \leq 1$ Equality occurs only for an equilateral triangle ☐

★

23. *Let a, b, c be the side-lengths of a triangle. If x, y, z are real numbers, then*

$$(ya^2+zb^2+xc^2)(za^2+xb^2+yc^2) \geq (xy+yz+zx)(a^2b^2+b^2c^2+c^2a^2) \tag{58}$$

Proof. We write the inequality as follows:

$$x^2b^2c^2 + y^2c^2a^2 + z^2a^2b^2 \geq yza^2(b^2+c^2-a^2) + zxb^2(c^2+a^2-b^2) + $$
$$+ xyc^2(a^2+b^2-c^2),$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \geq \frac{yz(b^2+c^2-a^2)}{b^2c^2} + \frac{zx(c^2+a^2-b^2}{c^2a^2} + \frac{xy(a^2+b^2-c^2)}{a^2b^2},$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \geq \frac{2yz\cos A}{b^2c^2} + \frac{2zx\cos B}{c^2a^2} + \frac{2xy\cos C}{a^2b^2},$$

$$\left(\frac{x}{a} - \frac{y}{b}\cos C - \frac{z}{c}\cos B\right)^2 + \left(\frac{y}{b}\sin C - \frac{z}{c}\sin B\right)^2 \geq 0$$

Since the last inequality is clearly true, the proof is complete. Equality occurs if and only if $\frac{x}{a^2} = \frac{y}{b^2} = \frac{z}{c^2}$. □

Remark 1. For $x = \frac{1}{b}$, $y = \frac{1}{c}$ and $z = \frac{1}{a}$, from (58) we get again the well-known inequality

$$a^3b + b^3c + c^3a \geq a^2b^2 + b^2c^2 + c^2a^2.$$

Remark 2. For $x = \frac{1}{c^2}$, $y = \frac{1}{a^2}$ and $z = \frac{1}{b^2}$, from (58) we obtain the elegant asymmetric inequality of *Walker* (Math. Mag. 43, 1970):

$$3\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) \geq (a^2+b^2+c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

2.3 Another related inequalities

1. Let x, y, z be non-negative numbers. If $0 \leq r \leq \sqrt{2}$, then

$$\sqrt{x^4+y^4+z^4} + r\sqrt{x^2y^2+y^2z^2+z^2x^2} \geq (1+r)\sqrt{x^3y+y^3z+z^3x}.$$

(*Vasile Cîrtoaje*, MS, 2004)

2. Let x, y, z be real numbers. If $-1 \leq r \leq 2$, then

$$x^2(x-y)(x-ry) + y^2(y-z)(y-rz) + z^2(z-x)(z-rx) \geq 0.$$

3. Let x, y, z be non-negative numbers. If $-2 \le r \le 2$, then

$$x(x-y)(x^2-ry^2)+y(y-z)(y^2-rz^2)+z(z-x)(z^2-rx^2) \ge 0$$

4. If x, y, z are real numbers, then

$$(x-y)(2x+y)^3+(y-z)(2y+z)^3+(z-x)(2z+x)^3 \ge 0.$$

5. If $x_1, x_2, \ldots, x_n$ are real numbers, then

$$(x_1-x_2)(3x_1+x_2)^3+(x_2-x_3)(3x_2+x_3)^3+\cdots+(x_n-x_1)(3x_n+x_1)^3 \ge 0.$$

6. If x, y, z are non-negative numbers, then

$$(x-y)(3x+2y)^3+(y-z)(3y+2z)^3+(z-x)(3z+2x)^3 \ge 0$$

7. Let $x_1, x_2, \ldots, x_n$ be non-negative numbers. If $r \ge \dfrac{1}{\sqrt[3]{4}-1} \approx 1.7024$, then

$$(x_1-x_2)(rx_1+x_2)^3+(x_2-x_3)(rx_2+x_3)^3+\cdots+(x_n-x_1)(rx_n+x_1)^3 \ge 0.$$

8. If x, y, z are real numbers, then

$$(x-y)\sqrt[3]{2x+y}+(y-z)\sqrt[3]{2y+z}+(z-x)\sqrt[3]{2z+x} \ge 0.$$

9. If x, y, z are real numbers, then

$$(x-y)(x+2z)^3+(y-z)(y+2x)^3+(z-x)(z+2y)^3 \ge 0$$

10. If x, y, z are real numbers, then

$$(x-y)\sqrt[3]{x+2z}+(y-z)\sqrt[3]{y+2x}+(z-x)\sqrt[3]{z+2y} \ge 0$$

11. Let $x_1, x_2, \ldots, x_n$ be real numbers. If $0 \le r \le \dfrac{\sqrt{3}-1}{2}$, then

$$x_1^4+x_2^4+\cdots+x_n^4+r\left(x_1x_2^3+x_2x_3^3+\cdots+x_nx_1^3\right) \ge$$
$$\ge (1+r)\left(x_1^3x_2+x_2^3x_3+\cdots+x_n^3x_1\right).$$

12. If $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$x_1^4+x_2^4+\cdots+x_n^4+\frac{1}{2}\left(x_1x_2^3+x_2x_3^3+\cdots+x_nx_1^3\right) \ge$$
$$\ge \frac{3}{2}\left(x_1^3x_2+x_2^3x_3+\cdots+x_n^3x_1\right).$$

13. If x, y, z are real numbers, then

$$x(x+y)^3 + y(y+z)^3 + z(z+x)^3 \geq 0$$

(*Vasile Cîrtoaje*, GM-B, 11-12, 1989)

14. If a, b, c are positive numbers, then

$$\frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c} - \frac{1}{a+b} - \frac{1}{b+c} - \frac{1}{c+a} \geq$$
$$\geq 4\left(\frac{1}{3a+b} + \frac{1}{3b+c} + \frac{1}{3c+a} - \frac{1}{a+3b} - \frac{1}{b+3c} - \frac{1}{c+3a}\right).$$

15. If $x, y, z \in \left[\frac{1}{2}, 2\right]$, then

$$8\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right) \geq 5\left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right) + 9.$$

16. Let $p = \sqrt{4 + 3\sqrt{2}}$, and let $x, y, z \in \left[\frac{1}{p}, p\right]$ Prove that

$$9(xy + yz + zx)(x^2 + y^2 + z^2) \geq (x + y + z)^4.$$

(*Vasile Cîrtoaje*, Moldova TST, 2005)

17. Let $x, y, z \geq \frac{2}{3}$ such that $x + y + z = 3$. Prove that

$$x^2y^2 + y^2z^2 + z^2x^2 \geq xy + yz + zx.$$

18. If x, y, z are real numbers, then

$$3(x^4+y^4+z^4-x^3y-y^3z-z^3x) \geq x^2(y-z)^2+y^2(z-x)^2+z^2(x-y)^2.$$

(*Vasile Cîrtoaje*, MS, 2005)

19. If x, y, z are real numbers, then

$$x^4+y^4+z^4-xyz(x+y+z) \geq 2\sqrt{2}(x^3y+y^3z+z^3x-xy^3-yz^3-zx^3).$$

(*Pham Kim Hung*, MS, 2006)

20. If x, y, z are non-negative numbers, then

$$x^4+y^4+z^4+17(x^2y^2+y^2z^2+z^2x^2) \geq 6(x+y+z)(x^2y+y^2z+z^2x).$$

21. If x, y, z are non-negative numbers, then

$$\sum (x^2 - yz)^2 \geq \sqrt{6} \sum xy(z-x)^2$$

22. If x, y, z are non-negative numbers, then

$$x^4 + y^4 + z^4 + 5(x^3y + y^3z + z^3x) \geq 6(x^2y^2 + y^2z^2 + z^2x^2)$$

23. Let x, y, z be non-negative numbers, no two of them are zero. Prove that

$$\frac{x^2 - yz}{x+y} + \frac{y^2 - zx}{y+z} + \frac{z^2 - xy}{z+x} \geq 0.$$

24. If x, y, z are real numbers, then

$$3(x^4 + y^4 + z^4) + 4(x^3y + y^3z + z^3x) \geq 0.$$

(*Vasile Cîrtoaje*, MS, 2005)

25. Let x, y, z be positive numbers such that $x + y + z = 3$. Prove that

$$\frac{x}{1+y^3} + \frac{y}{1+z^3} + \frac{z}{1+x^3} \geq \frac{3}{2}.$$

(*Bin Zhao*, MS, 2006)

26. Let a, b, c, d be non-negative numbers such that $a + b + c + d = 4$. Prove that

$$3(a^2 + b^2 + c^2 + d^2) + 4abcd \geq 16.$$

(*Vasile Cîrtoaje*, MS, 2004)

27. Let a, b, c, d be positive real numbers such that $a + b + c + d = 4$ Prove that

$$\frac{a}{1+b^2} + \frac{b}{1+c^2} + \frac{c}{1+d^2} + \frac{d}{1+a^2} \geq 2$$

(Russian Winter Olympiad, 2006)

28. Let a, b, c be non-negative numbers such that $a + b + c = 1$ Prove that

$$\frac{2bc+3}{a+1} + \frac{2ca+3}{b+1} + \frac{2ab+3}{c+1} \leq \frac{15}{2}$$

(*Vasile Cîrtoaje*, MS, 2005)

29. If a, b, c are the side lengths of a triangle, then

$$a^2(a+b)(b-c)+b^2(b+c)(c-a)+c^2(c+a)(a-b) \geq 0.$$

30. If a, b, c are the side lengths of a non-equilateral triangle, then

$$\frac{a^3b+b^3c+c^3a-a^2b^2-b^2c^2-c^2a^2}{a^2b+b^2c+c^2a-3abc} \geq \min\{b+c-a, c+a-b, a+b-c\}$$

31. Let a, b, c be the side lengths of a triangle. If x, y, z are real numbers such that $x+y+z=0$, then

$$yza(b+c-a)+zxb(c+a-b)+xyc(a+b-c) \leq 0.$$

32. If a, b, c are the side lengths of a triangle, then

$$(2a^2-bc)(b-c)^2+(2b^2-ca)(c-a)^2+(2c^2-ab)(a-b)^2 \geq 0.$$

33. Let x, y, z be non-negative real numbers. If $0 < r \leq m$, where $m \approx 1.558$ is a root of the equation

$$(1+m)^{1+m} = (3m)^m,$$

then

$$\frac{x^ry+y^rz+z^rx}{3} \leq \left(\frac{x+y+z}{3}\right)^{r+1}.$$

(*Vasile Cîrtoaje*, MS, 2005)

2.4 Solutions

1. *Let x, y, z be non-negative numbers. If $0 \leq r \leq \sqrt{2}$, then*

$$\sqrt{x^4+y^4+z^4}+r\sqrt{x^2y^2+y^2z^2+z^2x^2} \geq (1+r)\sqrt{x^3y+y^3z+z^3x}.$$

Solution. Squaring transforms the inequality into

$$\sum x^4+r^2\sum y^2z^2+2r\sqrt{\left(\sum x^4\right)\left(\sum x^2y^2\right)} \geq (1+r)^2\sum x^3y.$$

By the Cauchy-Schwarz Inequality, we have

$$\sqrt{\left(\sum x^4\right)\left(\sum x^2y^2\right)} \geq \sum x^3y$$

Thus, it suffices to show that

$$\sum x^4 + r^2 \sum y^2 z^2 \geq (1+r^2) \sum x^3 y.$$

For $r = 0$, this inequality is true since

$$x^4 + y^4 + z^4 - x^3y - y^3z - z^3x = \frac{1}{4}\sum(3x^4 + y^4 - 4x^3y) =$$
$$-\frac{1}{4}\sum(x-y)^2(3x^2 + 2xy + y^2) \geq 0.$$

For $0 < r \leq \sqrt{2}$, the inequality can be rewritten as

$$\frac{1}{r^2}\left(\sum x^4 - \sum x^3y\right) \geq \sum x^3y - \sum y^2z^2$$

Since $\sum x^4 - \sum x^3y \geq 0$, it suffices to consider $r = \sqrt{2}$ In this case, the inequality is equivalent to (1) There is equality if and only if $x = y = z$.

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2. *Let x, y, z be real numbers. If $-1 \leq r \leq 2$, then*

$$x^2(x-y)(x-ry) + y^2(y-z)(y-rz) + z^2(z-x)(z-rx) \geq 0.$$

Solution. Denote by E the left hand side of the inequality There are three cases to consider.

Case $r = 0$ We have

$$E = x^4 + y^4 + z^4 - x^3y - y^3z - z^3x = \frac{1}{4}\sum(x-y)^2\left[2x^2 + (x+y)^2\right] \geq 0.$$

Case $0 < r \leq 2$ We have

$$\frac{1}{r}E = \frac{1}{r}(x^4 + y^4 + z^4 - x^3y - y^3z - z^3x) - x^2y(x-y) -$$
$$- y^2z(y-z) - z^2x(z-x)$$

Since $x^4+y^4+z^4-x^3y-y^3z-z^3x \geq 0$, it suffices to prove the inequality for $r = 2$, when it becomes just (1)

Case $-1 \leq r < 0$. We have

$$\frac{-1}{r}E = \frac{-1}{r}(x^4 + y^4 + z^4 - x^3y - y^3z - z^3x) +$$
$$+ x^2y(x-y) + y^2z(y-z) + z^2(z-x)$$

Since $x^4 + y^4 + z^4 - x^3y - y^3z - z^3x \geq 0$, it suffices to consider $r = -1$ In this case, we get

$$E = x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2 = \frac{(x^2-y^2)^2 + (y^2-z^2)^2 + (z^2-x^2)^2}{2} \geq 0.$$

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3. *Let x, y, z be non-negative numbers. If $-2 \leq r \leq 2$, then*

$$x(x-y)(x^2-ry^2) + y(y-z)(y^2-rz^2) + z(z-x)(z^2-rx^2) \geq 0$$

Solution. For $r = 0$, this inequality is true (as shown above).

Case $0 < r \leq 2$. We write the inequality in the form

$$\frac{1}{r}(x^4+y^4+z^4-x^3y-y^3z-z^3x) \geq xy^2(x-y) + yz^2(y-z) + zx^2(z-x)$$

Since $x^4 + y^4 + z^4 - x^3y - y^3z - z^3x \geq 0$, it suffices to prove the inequality for $r = 2$; that is

$$x^4+y^4+z^4+2(xy^3+yz^3+zx^3) \geq 2(x^2y^2+y^2z^2+z^2x^2) + x^3y+y^3z+z^3x.$$

Summing up the inequalities

$$(x^2+y^2+z^2)^2 \geq 3(x^3y+y^3z+z^3x)$$

and

$$2(x^3y+y^3z+z^3x) + 2(xy^3+yz^3+zx^3) \geq 4(x^2y^2+y^2z^2+z^2x^2),$$

we get the desired inequality. The first inequality is just (1), while the second is equivalent to the obvious inequality

$$xy(x-y)^2 + yz(y-z)^2 + zx(z-x)^2 \geq 0.$$

Case $-2 \leq r < 0$ We write the inequality as

$$\frac{-1}{r}(x^4+y^4+z^4-x^3y-y^3z-z^3x) + xy^2(x-y) + yz^2(y-z) + zx^2(z-x) \geq 0.$$

Since $x^4 + y^4 + z^4 - x^3y - y^3z - z^3x \geq 0$, it suffices to prove the inequality for $r = -2$, that is

$$(x^2+y^2+z^2) \geq x^3y + y^3z + z^3x + 2(xy^3 + yz^3 + zx^3 \ .$$

According to (1), we have

$$(x^2+y^2+z^2)^2 \geq 3(x^3y+y^3z+z^3x)$$

and

$$(x^2+y^2+z^2)^2 \geq 3(xy^3+yz^3+zx^3),$$

whence the conclusion follows.

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4. *If x, y, z are real numbers, then*

$$(x-y)(2x+y)^3+(y-z)(2y+z)^3+(z-x)(2z+x)^3 \geq 0$$

Solution. Using the substitution $2x+y=a$, $2y+z=b$, $2z+x=c$, which is equivalent to

$$x=\frac{4a-2b+c}{9}, \quad y=\frac{4b-2c+a}{9}, \quad z=\frac{4c-2a+b}{9},$$

the inequality reduces to (7):

$$a^4+b^4+c^4+ab^3+bc^3+ca^3 \geq 2(a^3b+b^3c+c^3a).$$

★

5. *If $x_1, x_2, \ldots, x_n$ are real numbers, then*

$$(x_1-x_2)(3x_1+x_2)^3+(x_2-x_3)(3x_2+x_3)^3+\cdots+(x_n-x_1)(3x_n+x_1)^3 \geq 0$$

Solution. We will show that there is a real number q such that the following inequality holds for any real numbers x and y

$$(x-y)(3x+y)^3 \geq q(x^4-y^4).$$

Since

$$(x-y)(3x+y)^3-q(x^4-y^4)=$$
$$=(x-y)\left[(3x+y)^3-q(x^3+x^2y+xy^2+y^3)\right],$$

we will choose $q=16$. For this value of q, we have

$$(x-y)(3x+y)^3-16(x^4-y^4)=(x-y)^2\left[11(x+y)^2+4y^2\right] \geq 0$$

Using now this result, we have (for $x_{n+1}=x_1$)

$$\sum_{i=1}^{n}(x_i-x_{i+1})(3x_i+x_{i+1})^3 \geq \sum_{i=1}^{n} q\left(x_i^4-x_{i+1}^4\right)=0,$$

which completes the proof Equality occurs only for $x_1=x_2=\cdots=x_n$.

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6. *If x, y, z are non-negative numbers, then*

$$(x-y)(3x+2y)^3+(y-z)(3y+2z)^3+(z-x)(3z+2x)^3 \geq 0.$$

Solution. By expanding, the inequality becomes

$$19(x^4+y^4+z^4)+27(x^3y+y^3z+z^3x) \geq$$
$$\geq 28(xy^3+yz^3+zx^3)+18(x^2y^2+y^2z^2+z^2x^2).$$

Since $x^4+y^4+z^4 \geq xy^3+yz^3+zx^3$, it is enough to show that

$$18(x^4+y^4+z^4)+27(x^3y+y^3z+z^3x) \geq$$
$$\geq 27(xy^3+yz^3+zx^3)+18(x^2y^2+y^2z^2+z^2x^2),$$

that is

$$2(x^4+y^4+z^4)+3(x^3y+y^3z+z^3x) \geq$$
$$\geq 3(xy^3+yz^3+zx^3)+2(x^2y^2+y^2z^2+z^2x^2)$$

This inequality follows by adding up the inequalities

$$2(x^4+y^4+z^4)+4(x^2y^2+y^2z^2+z^2x^2) \geq 6(xy^3+yz^3+zx^3)$$

and

$$3(x^3y+y^3z+z^3x)+3(xy^3+yz^3+zx^3) \geq 6(x^2y^2+y^2z^2+z^2x^2).$$

First inequality is equivalent to (1), while the second inequality is equivalent to

$$xy(x-y)^2+yz(y-z)^2+zx(z-x)^2 \geq 0.$$

Also, we can prove the above inequality by adding the inequalities

$$2(x^4+y^4+z^4)+2(x^3y+y^3z+z^3x) \geq 4(xy^3+yz^3+zx^3)$$

and

$$(x^3y+y^3z+z^3x)+(xy^3+yz^3+zx^3) \geq 2(x^2y^2+y^2z^2+z^2x^2).$$

Notice that the first inequality is of type (7).

Equality in the original inequality occurs only for $x=y=z$.

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7. *Let $x_1, x_2, \ldots, x_n$ be non-negative numbers. If $r \geq \frac{1}{\sqrt[3]{4}-1} \approx 1.7024$, then*

$$(x_1-x_2)(rx_1+x_2)^3+(x_2-x_3)(rx_2+x_3)^3+\cdots+(x_n-x_1)(rx_n+x_1)^3 \geq 0$$

Solution. As above, it suffices to show that there is a real number q such that the following inequality holds for $r \geq \frac{1}{\sqrt[3]{4}-1}$ and any positive numbers x and y:

$$(x-y)(rx+y)^3 \geq q(x^4-y^4).$$

Since

$$\begin{aligned}&(x-y)(rx+y)^3 - q(x^4-y^4) = \\ &= (x-y)\left[(rx+y)^3 - q(x^3+x^2y+xy^2+y^3)\right],\end{aligned}$$

choosing $q = \frac{(r+1)^3}{4}$, we get

$$(x-y)(rx+y)^3 - q(x^4-y^4) = \frac{1}{4}(x-y)^2(Ax^2+Bxy+Cy^2),$$

where

$$A = 4r^3-(r+1)^3, \quad B = 2(r-1)(r^2+4r+1), \quad C = (r+1)^3-4$$

For $r \geq \frac{1}{\sqrt[3]{4}-1}$, we have $A \geq 0$, $B > 0$ and $C > 0$. Hence

$$(x-y)(rx+y)^3 - q(x^4-y^4) \geq 0$$

Equality occurs only for $x_1 = x_2 = \cdots = x_n$.

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8. *If x, y, z are real numbers, then*

$$(x-y)\sqrt[3]{2x+y} + (y-z)\sqrt[3]{2y+z} + (z-x)\sqrt[3]{2z+x} \geq 0.$$

Solution. Let $2x+y=a^3$, $2y+z=b^3$ and $2z+x=c^3$. We obtain

$$x = \frac{4a^3-2b^3+c^3}{9}, \quad y = \frac{4b^3-2c^3+a^3}{9}, \quad z = \frac{4c^3-2a^3+b^3}{9},$$

and the inequality transforms into

$$a^4+b^4+c^4+a^3b+b^3c+c^3a \ge 2(ab^3+bc^3+ca^3),$$

which is of type (7).

★

9. *If x,y,z are real numbers, then*

$$(x-y)(x+2z)^3+(y-z)(y+2x)^3+(z-x)(z+2y)^3 \ge 0.$$

Solution. Let $x+2z=a$, $y+2x=b$ and $z+2y=c$. We have

$$x=\frac{a+4b-2c}{9}, \quad y=\frac{b+4c-2a}{9}, \quad z=\frac{c+4a-2b}{9},$$

and the inequality reduces to

$$a^4+b^4+c^4+a^3b+b^3c+c^3a \ge 2(ab^3+bc^3+ca^3),$$

which is also of type (7).

★

10. *If x,y,z are real numbers, then*

$$(x-y)\sqrt[3]{x+2z}+(y-z)\sqrt[3]{y+2x}+(z-x)\sqrt[3]{z+2y} \ge 0.$$

Solution. Setting $x+2z=a^3$, $y+2x=b^3$ and $z+2y=c^3$, we have

$$x=\frac{a^3+4b^3-2c^3}{9}, \ y=\frac{b^3+4c^3-2a^3}{9}, \ z=\frac{c^3+4a^3-2b^3}{9},$$

and the inequality reduces to (7)

$$a^4+b^4+c^4+ab^3+bc^3+ca^3 \ge 2(a^3b+b^3c+c^3a).$$

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11. *Let $x_1,x_2,\dots,x_n$ be real numbers. If $0 \le r \le \dfrac{\sqrt{3}-1}{2}$, then*

$$x_1^4+x_2^4+\cdots+x_n^4+r\left(x_1x_2^3+x_2x_3^3+\cdots+x_nx_1^3\right) \ge$$
$$\ge (1+r)\left(x_1^3x_2+x_2^3x_3+\cdots+x_n^3x_1\right).$$

Solution. We will show that there is a real number q such that the following inequality holds for $0 \le r \le \frac{\sqrt{3}-1}{2}$ and any real numbers x and y:

$$x^4 + rxy^3 - (r+1)x^3y \ge q(x^4 - y^4)$$

If this claim is true, we get the given inequality as follows (for $x_{n+1} = x_1$)

$$\sum_{i=1}^{n} \left[x_i^4 + rx_ix_{i+1}^3 - (r+1)x_i^3x_{i+1}\right] \ge \sum_{i=1}^{n} q\left(x_i^4 - x_{i+1}^4\right) = 0$$

We have

$$x^4 + rxy^3 - (r+1)x^3y - q(x^4 - y^4) =$$
$$= (x-y)\left[x^3 - rxy(x+y) - q(x^3 + x^2y + xy^2 + y^3)\right].$$

Choosing $q = \frac{1-2r}{4}$, we get

$$x^4 + rxy^3 - (r+1)x^3y - q(x^4 - y^4) =$$
$$= \frac{1}{4}(x-y)^2\left[(2r+3)x^2 - 2xy + (1-2r)y^2\right] =$$
$$= \frac{1}{4}(x-y)^2\left[(2r+3)\left(x - \frac{1}{2r+3}y\right)^2 + \frac{2(1-2r-2r^2)}{2r+3}\right] \ge 0$$

Equality occurs only for $x_1 = x_2 = \cdots = x_n$.

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12. *If $x_1, x_2, \ldots, x_n$ are positive numbers, then*

$$x_1^4 + x_2^4 + \cdots + x_n^4 + \frac{1}{2}\left(x_1x_2^3 + x_2x_3^3 + \cdots + x_nx_1^3\right) \ge$$
$$\ge \frac{3}{2}\left(x_1^3x_2 + x_2^3x_3 + \cdots + x_n^3x_1\right)$$

Solution. Write the inequality as

$$\sum_{i=1}^{n}\left(2x_i^4 + x_ix_{i+1}^3 - 3x_i^3x_{i+1}\right) \ge 0,$$

or

$$\sum_{i=1}^{n} x_i(x_i - x_{i+1})^2(2x_i + x_{i+1}) \ge 0.$$

Equality occurs only for $x_1 = x_2 = \cdots = x_n$.

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13. *If x, y, z are real numbers, then*

$$x(x+y)^3 + y(y+z)^3 + z(z+x)^3 \geq 0.$$

Solution. Using the substitution $y+z=2a$, $z+x=2b$ and $x+y=2c$, the inequality becomes

$$a^4+b^4+c^4+ab^3+bc^3+ca^3 \geq a^3b+b^3c+c^3a.$$

This inequality is equivalent to

$$(a^2-ab-b^2)^2+(b^2-bc-c^2)^2+(c^2-ca-a^2)^2+a^2b^2+b^2c^2+c^2a^2 \geq 0,$$

which is clearly true. Equality occurs if and only if $x=y=z=0$

Remark. Similarly, we can show that the following inequality holds for any real numbers x, y, z.

$$x(x+y)^5 + y(y+z)^5 + z(z+x)^5 \geq 0.$$

Using the same substitution, the inequality transforms into

$$a^6+b^6+c^6+ab^5+bc^5+ca^5 \geq a^5b+b^5c+c^5a,$$

which is equivalent to

$$\sum(a^6+2ab^5-2a^5b+b^6) \geq 0,$$

or

$$\sum(a^2+b^2)(a^2-ab-b^2)^2 \geq 0.$$

Equality occurs if and only if $x=y=z=0$.

★

14. *If a, b, c are positive numbers, then*

$$\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}-\frac{1}{a+b}-\frac{1}{b+c}-\frac{1}{c+a} \geq$$
$$\geq 4\left(\frac{1}{3a+b}+\frac{1}{3b+c}+\frac{1}{3c+a}-\frac{1}{a+3b}-\frac{1}{b+3c}-\frac{1}{c+3a}\right).$$

Solution. We will prove that the following more general inequality holds for any positive t:

$$f(t)=\frac{t^{4a}}{2a}+\frac{t^{4b}}{2b}+\frac{t^{4c}}{2a}-\frac{t^{2(a+b)}}{a+b}-\frac{t^{2(b+c)}}{b+c}-\frac{t^{2(c+a)}}{c+a}-$$
$$-4\left(\frac{t^{3a+b}}{3a+b}+\frac{t^{3b+c}}{3b+c}+\frac{t^{3c+a}}{3c+a}-\frac{t^{a+3b}}{a+3b}-\frac{t^{b+3c}}{b+3c}-\frac{t^{c+3a}}{c+3a}\right)\geq 0.$$

Since $f(0)=0$, it suffices to show that $f'(t)\geq 0$ for $t>0$. We have

$$\frac{1}{2}f'(t)=t^{4a-1}+t^{4b-1}+t^{4c-1}-t^{2(a+b)-1}-t^{2(b+c)-1}-t^{2(c+a)-1}-$$
$$-2\left(t^{3a+b-1}+t^{3b+c-1}+t^{3c+a-1}-t^{a+3b-1}-t^{b+3c-1}-t^{c+3a-1}\right).$$

Denoting $x=t^{a-\frac{1}{4}}$, $y=t^{b-\frac{1}{4}}$ and $z=t^{c-\frac{1}{4}}$, the inequality $f'(t)\geq 0$ reduces to

$$x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2\geq 2(x^3y+y^3z+z^3x-xy^3-yz^3-zx^3),$$

which is just (10).

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15. *If* $x,y,z\in\left[\frac{1}{2},2\right]$, *then*

$$8\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\geq 5\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)+9.$$

Solution. Let

$$E(x,y,z)=8\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)-5\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)-9.$$

Without loss of generality, assume that $x=\max\{x,y,z\}$ We will show that

$$E(x,y,z)\geq E\left(x,\sqrt{xz},z\right)\geq 0$$

We have

$$E(x,y,z)-E\left(x,\sqrt{xz},z\right)=8\left(\frac{x}{y}+\frac{y}{z}-2\sqrt{\frac{x}{z}}\right)-5\left(\frac{y}{x}+\frac{z}{y}-2\sqrt{\frac{z}{x}}\right)=$$
$$=\frac{\left(y-\sqrt{xz}\right)^2(8x-5z)}{xyz}\geq 0.$$

Let now $t=\sqrt{\frac{x}{z}}$, $1\le t\le 2$ We get

$$\begin{aligned}E\left(x,\sqrt{xz},z\right)&=8\left(2\sqrt{\frac{x}{z}}+\frac{z}{x}-3\right)-5\left(2\sqrt{\frac{z}{x}}+\frac{x}{z}-3\right)=\\&=8\left(2t+\frac{1}{t^2}-3\right)-5\left(\frac{2}{t}+t^2-3\right)=\\&=\frac{8}{t^2}(t-1)^2(2t+1)-\frac{5}{t}(t-1)^2(t+2)=\\&=\frac{(t-1)^2(8+6t-5t^2)}{t^2}=\frac{(t-1)^2(4+5t)(2-t)}{t^2}\ge 0.\end{aligned}$$

This completes the proof. In the assumption $x=\max\{x,y,z\}$, equality occurs only for $x=y=z$ and $(x,y,z)=\left(2,1,\frac{1}{2}\right)$.

Remark. Using the same way, we can prove the following more general statement·

If $p>1$ *and* $x,y,z\in\left[\frac{1}{p},p\right]$, *then*

$$p(p+2)\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\ge(2p+1)\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)+3(p^2-1).$$

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16. *Let* $p=\sqrt{4+3\sqrt{2}}$, *and let* $x,y,z\in\left[\frac{1}{p},p\right]$. *Prove that*

$$9(xy+yz+zx)(x^2+y^2+z^2)\ge(x+y+z)^4.$$

Solution. Let $A=x^2+y^2+z^2$ and $B=xy+yz+zx$. Since

$$\begin{aligned}&9(xy+yz+zx)(x^2+y^2+z^2)-(x+y+z)^4=\\&=9AB-(A+2B)^2=(A-B)(4B-A)\end{aligned}$$

and

$$2(A-B)=(x-y)^2+(y-z)^2+(z-x)^2\ge 0,$$

we have show that $4B-A\ge 0$. This inequality is equivalent to

$$E(x,y,z)\le 0,$$

where

$$E(x,y,z)=x^2+y^2+z^2-4(xy+yz+zx).$$

We will show that the expression $E(x,y,z)$ is maximal for $x,y,z \in \left\{\frac{1}{p}, p\right\}$. Assume that this assertion is not true. Then, there exists a triple $\{x,y,z\}$ with $\frac{1}{p} < x < p$ such that

$$E(x,y,z) \geq \max\left\{E\left(\frac{1}{p}, y, z\right),\ E(p,y,z)\right\}.$$

From

$$E(x,y,z) - E\left(\frac{1}{p}, y, z\right) = \left(x - \frac{1}{p}\right)\left(x + \frac{1}{p} - 4y - 4z\right) \geq 0,$$

we get

$$4(y+z) - x \leq \frac{1}{p},$$

and from

$$E(x,y,z) - E(p,y,z) = (x-p)(x+p-4y-4z) \geq 0,$$

we get

$$4(y+z) - x \geq p$$

These results imply $p \leq \frac{1}{p}$, which is false. Consequently, the expression $E(x,y,z)$ is maximal for $x,y,z \in \left\{\frac{1}{p}, p\right\}$. Since $E(x,y,z)$ is a symmetric expression, we have

$$E(x,y,z) \leq \max\left\{E\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}\right), E\left(\frac{1}{p}, \frac{1}{p}, p\right), E\left(\frac{1}{p}, p, p\right), E(p,p,p)\right\} =$$

$$= \max\left\{\frac{-9}{p^2}, -9p^2, p^2 - \frac{2}{p^2} - 8, \frac{1}{p^2} - 2p^2 - 8\right\} = p^2 - \frac{2}{p^2} - 8 = 0.$$

Equality occurs when $x = y = z$, and also for $(x,y,z) = \left(\frac{1}{p}, \frac{1}{p}, p\right)$ or any cyclic permutation

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17. *Let $x,y,z \geq \frac{2}{3}$ such that $x+y+z=3$. Prove that*

$$x^2y^2 + y^2z^2 + z^2x^2 \geq xy + yz + zx.$$

Solution. Without loss of generality, assume that $x \geq y \geq z$. Let

$$E(x,y,z) = x^2y^2 + y^2z^2 + z^2x^2 - xy - yz - zx$$

and $t = \dfrac{y+z}{2}$. From $z,y,z \geq \dfrac{2}{3}$, $x \geq y \geq z$ and $x+y+z=3$, it follows immediately that $\dfrac{2}{3} \leq t \leq 1$. We will prove that

$$E(x,y,z) \geq E(x,t,t) \geq 0.$$

We have

$$\begin{aligned} E(x,y,z) - E(x,t,t) &= x^2(y^2+z^2-2t^2) - (t^2-yz)(t^2+yz-1) = \\ &= \frac{1}{4}(y-z)^2\left[x^2 + (x^2-yz) + (1-t^2)\right] \geq 0, \end{aligned}$$

and

$$E(x,t,t) = 2x^2t^2 + t^4 - 2xt - t^2.$$

Since $x + 2t = 3$, we get

$$\begin{aligned} 9E(x,t,t) &= 18x^2t^2 + 9t^4 - (2xt+t^2)(x+2t)^2 = \\ &= t(5t^3 - 12xt^2 + 9x^2t - 2x^3) = \\ &= t(t-x)^2(5t-2x) = 3t(t-x)^2(3t-2) \geq 0 \end{aligned}$$

and the proof is completed. Equality occurs for $(x,y,z) = (1,1,1)$, and also for $(x,y,z) = \left(\dfrac{5}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right)$ or any cyclic permutation.

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18. *If x, y, z are real numbers, then*

$$3(x^4+y^4+z^4-x^3y-y^3z-z^3x) \geq x^2(y-z)^2+y^2(z-x)^2+z^2(x-y)^2.$$

Solution. The inequality can be rewritten as

$$3x^3(x-y)+3y^3(y-z)+3z^3(z-x) \geq x^2(y-z)^2+y^2(z-x)^2+z^2(x-y)^2.$$

Using the substitution $y = x + p$ and $z = x + q$, the inequality becomes as follows

$$\begin{aligned} 7(p^2-pq+q^2)x^2 + (9p^3 - 11p^2q - 2pq^2 + 9q^3)x + \\ + 3p^4 - 3p^3q - 2p^2q^2 + 3q^4 \geq 0. \end{aligned}$$

The left hand side is a quadratic of x For $p = q = 0$, the inequality becomes equality. Otherwise, we have $p^2 - pq + q^2 > 0$, and it remains to show that the discriminant is non-positive. Indeed, we have

$$\begin{aligned}\Delta &= (9p^3 - 11p^2q - 2pq^2 + 9q^3)^2 - \\ &- 28(p^2 - pq + q^2)(3p^4 - 3p^3q - 2p^2q^2 + 3q^4) = \\ &= -3(p^6 + 10p^5q + 9p^4q^2 - 78p^3q^3 + 74p^2q^4 - 16pq^5 + q^6) = \\ &= -3(p^3 + 5p^2q - 8pq^2 + q^3)^2 \leq 0.\end{aligned}$$

Remark 1. We can rewrite the inequality as a sum of squares, as follows:

$$\sum(2x^2 - y^2 - z^2 - xy + yz)^2 \geq 0,$$
$$\sum(3x^2 - 3y^2 - xy + 2yz - zx)^2 \geq 0$$

Remark 2. Using the substitution $x = 2a - b$, $y = 2b - c$ and $z = 2c - a$, the inequality transforms into

$$(a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^3a),$$

which is just (1)

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19. *If x, y, z are non-negative real numbers, then*

$$x^4 + y^4 + z^4 - xyz(x + y + z) \geq 2\sqrt{2}(x^3y + y^3z + z^3x - xy^3 - yz^3 - zx^3).$$

Solution. First write the inequality as

$$\begin{aligned}&x^2(x^2 - yz) + y^2(y^2 - zx) + z^2(z^2 - xy) \geq \\ &\geq 2\sqrt{2}(x + y + z)(x - y)(y - z)(z - x)\end{aligned}$$

Without loss of generality, assume that $x = \min\{x, y, z\}$ Using the substitution $y = x + p$ and $z = x + q$ $(p \geq 0, q \geq 0)$, the inequality becomes

$$\begin{aligned}&5(p^2 - pq + q^2)x^2 + (4p^3 - p^2q - pq^2 + 4q^3)x + p^4 + q^4 \geq \\ &\geq 2\sqrt{2}pq(p - q)(3x + p + q)\end{aligned}$$

or

$$\begin{aligned}&5(p^2 - pq + q^2)x^2 + \left[4p^3 - \left(6\sqrt{2} + 1\right)p^2q + \left(6\sqrt{2} - 1\right)pq^2 + 4q^3\right]x + \\ &+ p^4 + q^4 + 2\sqrt{2}pq(q^2 - p^2) \geq 0\end{aligned}$$

Since $p^2-pq+q^2 \geq 0$ and $p^4+q^4+2\sqrt{2}pq(q^2-p^2) = \left(p^2-\sqrt{2}pq-q^2\right)^2 \geq 0$, it is enough to show that

$$4p^3-\left(6\sqrt{2}+1\right)p^2q+\left(6\sqrt{2}-1\right)pq^2+4q^3 \geq 0.$$

Indeed, we have

$$4p^3-\left(6\sqrt{2}+1\right)p^2q+\left(6\sqrt{2}-1\right)pq^2+4q^3 \geq$$
$$\geq 4p^3-10p^2q+\frac{25}{4}pq^2+4q^3 = p\left(2p-\frac{5q}{2}\right)^2+4q^3 \geq 0.$$

Equality occurs for $(x,y,z) \sim (1,1,1)$, and also for $(x,y,z) \sim \left(0,\sqrt{2}+\sqrt{6},2\right)$ or any cyclic permutation.

★

20. *If x, y, z are non-negative numbers, then*

$$x^4+y^4+z^4+17(x^2y^2+y^2z^2+z^2x^2) \geq 6(x+y+z)(x^2y+y^2z+z^2x)$$

Solution. Write the inequality as

$$\left(\sum x^4-\sum y^2z^2\right)+12\left(\sum y^2z^2-xyz\sum x\right)-6\left(\sum x^3y-xyz\sum x\right) \geq 0.$$

Since

$$\sum x^4-\sum y^2z^2 = \frac{1}{2}\sum(x^2-y^2)^2,$$
$$6\left(\sum y^2z^2-xyz\sum x\right) = \sum(xy-2yz+zx)^2,$$
$$3\left(\sum x^3y-xyz\sum x\right) = -3\sum yz(x^2-y^2) =$$
$$= -3\sum yz(x^2-y^2)+\sum(xy+yz+zx)(x^2-y^2) =$$
$$= \sum(x^2-y^2)(xy-2yz+zx),$$

the inequality becomes as follows:

$$\frac{1}{2}\sum(x^2-y^2)^2+2\sum(xy-2yz+zx)^2-$$
$$-2\sum(x^2-y^2)(xy-2yz+zx) \geq 0,$$

or

$$\frac{1}{2}\sum(x^2-y^2-2xy+4yz-2zx)^2 \geq 0.$$

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21. *If x, y, z are non-negative numbers, then*

$$\sum (x^2 - yz)^2 \ge \sqrt{6} \sum xy(z-x)^2$$

Solution. First write the inequality in the form

$$\left(\sum x^4 - \sum y^2z^2\right) + 2\left(\sum y^2z^2 - xyz\sum x\right) - \sqrt{6}\left(\sum x^3y - xyz\sum x\right) \ge 0$$

Since

$$\begin{aligned}
\sum x^4 - \sum y^2z^2 &= \frac{1}{2}\sum (x^2-y^2)^2, \\
6\left(\sum y^2z^2 - xyz\sum x\right) &= \sum (xy - 2yz + zx)^2, \\
3\left(\sum x^3y - xyz\sum x\right) &= -3\sum yz(x^2-y^2) = \\
&= -3\sum yz(x^2-y^2) + \sum (xy+yz+zx)(x^2-y^2) = \\
&= \sum (x^2-y^2)(xy-2yz+zx),
\end{aligned}$$

the inequality becomes as follows:

$$\frac{1}{2}\sum (x^2-y^2)^2 + \frac{1}{3}\sum (xy-2yz+zx)^2 - \\ - \sqrt{\frac{2}{3}}\sum (x^2-y^2)(xy-2yz+zx) \ge 0,$$

or

$$\frac{1}{2}\sum \left[x^2 - y^2 - \sqrt{\frac{2}{3}}\,(xy-2yz+zx)\right]^2 \ge 0.$$

★

22. *If x, y, z are non-negative numbers, then*

$$x^4 + y^4 + z^4 + 5(x^3y + y^3z + z^3x) \ge 6(x^2y^2 + y^2z^2 + z^2x^2).$$

Solution. Without loss of generality, assume that $x = \max\{x, y, z\}$ Using the substitution $y = x + p$ and $z = x + q$ ($p \ge 0$, $q \ge 0$), the inequality becomes

$$9(p^2 - pq + q^2)x^2 + 3(3p^3 + p^2q - 4pq^2 + 3q^3)x + \\ + p^4 + 5p^3q - 6p^2q^2 + q^4 \ge 0.$$

This inequality is true, since

$$p^2 - pq + q^2 \geq 0,$$
$$3p^3 + p^2q - 4pq^2 + 3q^3 = 3p(p-q)^2 + q(7p^2 - 7pq + 3q^2) \geq 0,$$
$$p^4 + 5p^3q - 6p^2q^2 + q^4 = (p-q)^4 + pq(3p - 2q)^2 \geq 0$$

Equality occurs for $x = y = z$.

★

23. *Let x, y, z be non-negative numbers, no two of them are zero. Prove that*

$$\frac{x^2 - yz}{x+y} + \frac{y^2 - zx}{y+z} + \frac{z^2 - xy}{z+x} \geq 0$$

First Solution. Since

$$\frac{x^2 - yz}{x+y} = \frac{x(x+z)}{x+y} - z,$$

we can write the inequality as

$$\frac{x(x+z)}{x+y} + \frac{y(y+x)}{y+z} + \frac{z(z+y)}{z+x} \geq x + y + z$$

Applying the Cauchy-Schwarz Inequality, we get

$$\frac{x(x+z)}{x+y} + \frac{y(y+x)}{y+z} + \frac{z(z+y)}{z+x} \geq$$
$$\geq \frac{\left[\sum x(x+z)\right]^2}{\sum x(x+y)(x+z)} = \frac{\left(\sum x^2 + \sum yz\right)^2}{\sum x^3 + \left(\sum x\right)\left(\sum yz\right)}.$$

Then, it is enough to show that

$$\left(\sum x^2 + \sum yz\right)^2 \geq \left(\sum x\right)\left(\sum x^3\right) + \left(\sum x\right)^2\left(\sum yz\right).$$

Since

$$\left(\sum x^2 + \sum yz\right)^2 = \left(\sum x^2\right)^2 + +2\left(\sum x^2\right)\left(\sum yz\right) + \left(\sum yz\right)^2$$

and

$$\left(\sum x\right)^2\left(\sum yz\right) = \left(\sum x^2\right)\left(\sum yz\right) + 2\left(\sum yz\right)^2,$$

the inequality becomes

$$\left(\sum x^2\right)^2+\left(\sum x^2\right)\left(\sum yz\right)\geq\left(\sum x\right)\left(\sum x^3\right)+\left(\sum yz\right)^2.$$

This inequality reduces to

$$\sum y^2z^2\geq xyz\sum x,$$

which is true because

$$\sum y^2z^2-xyz\sum x=\frac{1}{2}\sum x^2(y-z)^2$$

We have equality if and only if $x=y=z$

Second Solution By expanding, the inequality becomes as follows

$$\sum(y^2-xz)\left(x^2+\sum yz\right)\geq 0,$$
$$\sum x^2y^2-\sum xy^3+\left(\sum x^2\right)\left(\sum yz\right)-\left(\sum yz\right)^2\geq 0,$$
$$\sum x^3y\geq xyz\sum x.$$

The last inequality follows immediately from the Cauchy-Schwarz Inequality

$$\left(\sum x^3y\right)\left(\sum z\right)\geq xyz\left(\sum x\right)^2$$

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24. *If x,y,z are real numbers, then*

$$3(x^4+y^4+z^4)+4(x^3y+y^3z+z^3x)\geq 0$$

Solution. If x,y,z are non-negative numbers, then the inequality is trivial. Since the inequality remains unchanged by replacing x,y,z with $-x,-y,-z$, respectively, it suffices to consider the case when only one of x,y,z is negative, let $z<0$. Replacing now z with $-z$, the inequality becomes

$$3(x^4+y^4+z^4)+4x^3y\geq 4(y^3z+z^3x),$$

where $x\geq 0$, $y\geq 0$, $z>0$ It is enough to show that

$$3(x^4+y^4+x^3y)\geq 4(y^3z+z^3x)$$

Case $x\leq y$ Since $3x^3y\geq 3x^4$, it suffices to show that

$$6x^4+3y^4+3z^4\geq 4(y^3z+xz^3)$$

Using the AM-GM Inequality, we have

$$3y^4 + z^4 \geq 4\sqrt[4]{y^{12}z^4} = 4y^3z$$

and

$$3x^4 + z^4 = 3x^4 + \frac{1}{3}z^4 + \frac{1}{3}z^4 + \frac{1}{3}z^4 \geq 4\sqrt[4]{\frac{x^4z^{12}}{9}} = \frac{4}{\sqrt{3}}xz^3 \geq 2xz^3.$$

Adding the first inequality to the second inequality multiplied by 2, the desired inequality follows.

Case $x \geq y$ Since $3x^3y \geq 3y^4$, it suffices to show that

$$3x^4 + 6y^4 + 3z^4 \geq 4(y^3z + z^3x).$$

Since

$$6y^4 + \frac{z^4}{8} = 2y^4 + 2y^4 + 2y^4 + \frac{z^4}{8} \geq 4\sqrt[4]{y^{12}z^4} = 4y^3z,$$

we still to show that $3x^4 + \frac{23}{8}z^4 \geq 4xz^3$. We will prove that the following sharper inequality holds

$$3x^4 + \frac{5}{2}z^4 \geq 4xz^3$$

Indeed, we havc

$$3x^4 + \frac{5}{2}z^4 = 3x^4 + \frac{5}{6}z^4 + \frac{5}{6}z^4 + \frac{5}{6}z^4 \geq 4\sqrt[4]{\frac{125x^4z^{12}}{72}} \geq 4xz^3.$$

Equality holds only for $x = y = z = 0$.

★

25. *Let x, y, z be positive numbers such that $x + y + z = 3$. Prove that*

$$\frac{x}{1+y^3} + \frac{y}{1+z^3} + \frac{z}{1+x^3} \geq \frac{3}{2}.$$

Solution. Using the AM-GM Inequality, we have

$$\frac{x}{1+y^3} = x - \frac{xy^3}{1+y^3} \geq x - \frac{xy^3}{2y^{3/2}} = x - \frac{xy^{3/2}}{2},$$

and, similarly,

$$\frac{y}{1+z^3} \geq y - \frac{yz^{3/2}}{2}, \quad \frac{z}{1+x^3} \geq z - \frac{zx^{3/2}}{2}.$$

Thus, it suffices to show that

$$xy^{3/2} + yz^{3/2} + zx^{3/2} \le 3.$$

This inequality follows immediately from (30). Equality occurs if and only if $x = y = z = 1$

★

26. *Let a, b, c, d be non-negative numbers such that $a + b + c + d = 4$. Prove that*

$$3(a^2 + b^2 + c^2 + d^2) + 4abcd \ge 16$$

Solution (by *Gabriel Dospinescu*). Setting $x = \dfrac{b+c+d}{3}$ yields $x \le \dfrac{4}{3}$ and $a + 3x = 4$ Without loss of generality, assume that $a = \min\{a, b, c, d\}$, $a \le 1$ We will show that

$$E(a, b, c, d) \ge E(a, x, x, x) \ge 0,$$

where

$$E(a, b, c, d) = 3(a^2 + b^2 + c^2 + d^2) + 4abcd - 16.$$

Assume that $a = \min\{a, b, c, d\}$, $a \le 1$. We will show that

$$E(a, b, c, d) \ge E(a, x, x, x) \ge 0.$$

The left side inequality is equivalent to

$$3(3x^2 - S) \ge 2a(x^3 - bcd),$$

where $S = bc + cd + db$ By Schur's Inequality

$$(b + c + d)^3 + 9bcd \ge 4(b + c + d)(bc + cd + db),$$

we find that

$$x^3 - bcd \le \frac{4x}{3}(3x^2 - S)$$

Thus, it enough to show that

$$3(3x^2 - S) \ge \frac{8ax}{3}(3x^2 - S),$$

that is

$$(3x^2 - S)(9 - 8ax) \ge 0.$$

The inequality is true since

$$6(3x^2 - S) = (b-c)^2 + (c-d)^2 + (d-a)^2 \geq 0$$

and

$$3(9 - 8ax) = 27 - 8a(4-a) = 8(1-a)^2 + 16(1-a) + 3 > 0.$$

With regard to the inequality $E(a,x,x,x) \geq 0$, we have

$$\begin{aligned} E(a,x,x,x) &= 3(a^2 + 3x^2) + 4ax^3 - 16 = \\ &= 3(4-3x)^2 + 9x^2 + 4(4-3x)x^3 - 16 = \\ &= 4(8 - 18x + 9x^2 + 4x^3 - 3x^4) = \\ &= 4(1-x)^2(2+x)(4-3x) \geq 0. \end{aligned}$$

This completes the proof. Equality occurs for $(a,b,c,d) = (1,1,1,1)$, and also for $(a,b,c,d) = \left(0, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$ or any cyclic permutation.

★

27. *Let a,b,c,d be positive real numbers such that $a+b+c+d=4$. Prove that*

$$\frac{a}{1+b^2} + \frac{b}{1+c^2} + \frac{c}{1+d^2} + \frac{d}{1+a^2} \geq 2.$$

Solution. Using the AM-GM Inequality, we have

$$\frac{a}{1+b^2} = a - \frac{ab^2}{1+b^2} \geq a - \frac{ab^2}{2b} = a - \frac{ab}{2},$$

and, similarly,

$$\frac{b}{1+c^2} \geq b - \frac{bc}{2}, \quad \frac{c}{1+d^2} \geq c - \frac{cd}{2}, \quad \frac{d}{1+a^2} \geq d - \frac{da}{2}.$$

Thus, it suffices to show that

$$ab + bc + cd + da \leq 4$$

Indeed, we have

$$\begin{aligned} 16 - 4(ab+bc+cd+da) &= (a+b+c+d)^2 - 4(ab+bc+cd+da) = \\ &= (a-b+c-d)^2 \geq 0 \end{aligned}$$

Equality occurs if and only if $a = b = c = d = 1$.

★

28. *Let a, b, c be non-negative numbers such that $a+b+c=1$. Prove that*

$$\frac{2bc+3}{a+1}+\frac{2ca+3}{b+1}+\frac{2ab+3}{c+1}\le\frac{15}{2}.$$

Solution. Let $x=ab+bc+ca$. By the known inequality

$$(a+b+c)^2\ge 3(ab+bc+ca),$$

we get $x\le\frac{1}{3}$, and by Schur's Inequality

$$(a+b+c)^3+9abc\ge 4(a+b+c)(ab+bc+ca),$$

we get

$$1+9abc\ge 4x$$

The required inequality is equivalent to

$$15(a+1)(b+1)(c+1)\ge 2\sum(2bc+3)(b+1)(c+1)$$

Since

$$15(a+1)(b+1)(c+1)=15(abc+x+2)$$

and

$$2\sum(2bc+3)(b+1)(c+1)=\sum(4bc+6)(bc-a+2)=$$
$$=4\sum b^2c^2+14x+30-12abc=4x^2+14x+30-20abc,$$

the inequality reduces to

$$x(1-4x)+35abc\ge 0$$

For $x\le\frac{1}{4}$, the inequality is clearly true. Consider now $\frac{1}{4}<x\le\frac{1}{3}$. Since $abc\ge\frac{4x-1}{9}$, we have

$$x(1-4x)+35abc\ge x(1-4x)+\frac{35(4x-1)}{9}=\frac{(4x-1)(35-9x)}{9}>0$$

For $a\le b\le c$, equality in the original inequality occurs when

$$(a,b,c)=(0,0,1)\ \text{ and }\ (a,b,c)=\left(0,\frac{1}{2},\frac{1}{2}\right).$$

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29. *If a, b, c are the side lengths of a triangle, then*

$$a^2(a+b)(b-c)+b^2(b+c)(c-a)+c^2(c+a)(a-b)\geq 0$$

First Solution We write the inequality as

$$a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)\geq ab^3+bc^3+ca^3-a^3b-b^3c-c^3a,$$

or

$$a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2\geq 2(a+b+c)(a-b)(b-c)(c-a)$$

Using now the substitution $a=y+z$, $b=z+x$, $c=x+y$ $(x,y,z>0)$, we have

$$a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2=(y^2-z^2)^2+(z^2-x^2)^2+(x^2-z^2)^2=$$
$$=2(x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2)$$

and

$$2(a+b+c)(a-b)(b-c)(c-a)=4(x+y+z)(y-x)(z-y)(x-z)=$$
$$=4(x^3y+y^3z+z^3x-xy^3-yz^3-zx^3).$$

Thus, the inequality reduces to

$$x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2\geq$$
$$\geq 2(x^3y+y^3z+z^3x-xy^3-yz^3-zx^3),$$

which is just (10). Equality holds only for an equilateral triangle.

Second Solution. Write the inequality as follows

$$b^2c^2+c^2a^2+a^2b^2\geq ab(b^2+c^2-a^2)+bc(c^2+a^2-b^2)+$$
$$+ca(a^2+b^2-c^2),$$

$$\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\geq 2b\cos A+2c\cos B+2a\cos C.$$

Making the substitution $x=\sqrt{\frac{ca}{b}}$, $y=\sqrt{\frac{ab}{c}}$, $z=\sqrt{\frac{bc}{a}}$, the inequality transforms into the well-known inequality

$$x^2+y^2+z^2\geq 2yz\cos A+2zx\cos B+2xy\cos C,$$

which is equivalent to

$$(x-y\cos C-z\cos B)^2+(y\sin C-z\sin B)^2\geq 0.$$

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30. *If a, b, c are the side lengths of a non-equilateral triangle, then*

$$\frac{a^3b+b^3c+c^3a-a^2b^2-b^2c^2-c^2a^2}{a^2b+b^2c+c^2a-3abc} \geq \min\{b+c-a, c+a-b, a+b-c\}$$

Solution. By the AM-GM Inequality, we have

$$a^2b+b^2c+c^2a-3abc > 0$$

Let us assume now that $c = \max\{a, b, c\}$ Then

$$\min\{b+c-a, c+a-b, a+b-c\} = a+b-c,$$

and the inequality follows from the identity

$$a^3b+b^3c+c^3a-a^2b^2-b^2c^2-c^2a^2 =$$
$$= (a^2b+b^2c+c^2a-3abc)(a+b-c)+2a(c-a)(b-c)^2$$

On the assumption $c = \max\{a, b, c\}$, equality occurs for either $c = a$ or $c = b$

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31. *Let a, b, c be the side lengths of a triangle. If x, y, z are real numbers such that $x+y+z=0$, then*

$$yza(b+c-a)+zxb(c+a-b)+xyc(a+b-c) \leq 0$$

Solution. Multiplying the inequality by $\dfrac{a+b+c}{abc}$, it becomes as follows:

$$yz\left(\frac{b^2+c^2-a^2}{bc}+2\right)+zx\left(\frac{c^2+a^2-b^2}{ca}+2\right)+xy\left(\frac{a^2+b^2-c^2}{ab}+2\right) \leq 0,$$
$$2yz\cos A+2zx\cos B+2xy\cos C+2(xy+yz+zx) \leq 0,$$
$$x^2+y^2+z^2-2yz\cos A-2zx\cos B-2xy\cos C \geq 0,$$
$$(x-y\cos C-z\cos B)^2+(y\sin C-z\sin B)^2 \geq 0$$

The last inequality is obviously true. Equality occurs if and only if $x+y+z=0$ and $\dfrac{x}{\sin A}=\dfrac{y}{\sin B}=\dfrac{z}{\sin C}$; that is for $x=y=z=0$

Remark For $x=a-c$, $y=b-a$ and $z=c-b$, we get the known inequality

$$a^2b^2+b^2c^2+c^2a^2 \leq a^3b+b^3c+c^3a$$

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32. *If a, b, c are the side lengths of a triangle, then*

$$(2a^2 - bc)(b-c)^2 + (2b^2 - ca)(c-a)^2 + (2c^2 - ab)(a-b)^2 \geq 0$$

First Solution Using the substitution $a = y+z$, $b = z+x$, $c = x+y$ $(x, y, z > 0)$, the inequality becomes

$$2\sum x^4 + 2xyz\sum x \geq \sum yz(y^2+z^2) + 2\sum y^2z^2.$$

This inequality follows by summing Schur's Inequality below multiplied by 2

$$\sum x^4 + xyz\sum x \geq \sum yz(y^2+z^2)$$

to the obvious inequality

$$\sum yz(y-z)^2 \geq 0.$$

Equality occurs for an equilateral triangle.

Second Solution. Without loss of generality, assume that $a \geq b \geq c$. Since $2a^2 - bc > 0$, it suffices to show that

$$(2b^2 - ca)(a-c)^2 + (2c^2 - ab)(a-b)^2 \geq 0.$$

Since $(a-c)^2 \geq (a-b)^2$ and $2b^2 - ca > 2b^2 - c(b+c) = (b-c)(2b+c) \geq 0$, it is enough to prove that

$$2b^2 - ca + 2c^2 - ab \geq 0.$$

Indeed, we have

$$2b^2 + 2c^2 - ab - ac = (b-c)^2 + (b+c)^2 - ab - ac =$$
$$= (b-c)^2 + (b+c)(b+c-a) > 0.$$

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33. *Let x, y, z be non-negative real numbers. If $0 < r \leq m$, where $m \approx 1.558$ is a root of the equation*

$$(1+m^{\ 1+m} = \ 3m)^m,$$

then

$$\frac{x^r y + y^r z + z^r x}{3} \le \left(\frac{x+y+z}{3}\right)^{r+1}$$

Solution. Since the function $f(t) = t^{r/m}$ is concave, by Jensen's Inequality we get

$$x^r y + y^r z + z^r x = y(x^m)^{r/m} + z(y^m)^{r/m} + x(z^m)^{r/m} \le$$
$$\le (y+z+x)\left(\frac{yx^m + zy^m + xz^m}{y+z+x}\right)^{r/m}$$

Thus, it is enough to prove the inequality for $r = m$ Without loss of generality, assume that $x = \min\{x, y, z\}$ There are two cases to consider $x \le z \le y$ and $x \le y < z$.

I. *Case* $x \le z \le y$. If $z = 0$, then $x = 0$, and the inequality is trivial. Otherwise, for fixed y and z $(y \ge z > 0)$, let us denote

$$f(x) = 3\left(\frac{x+y+z}{3}\right)^{m+1} - x^m y - y^m z - z^m x, \quad x \in [0, z].$$

We will show that

$$f(x) \ge \min\{f(0), f(z)\} \ge 0. \tag{1}$$

Let us show that $f(x) \ge \min\{f(0), f(z)\}$. We have

$$f'(x) = (m+1)\left(\frac{x+y+z}{3}\right)^m - mx^{m-1}y - z^m,$$
$$\frac{f''(x)}{m} = \frac{m+1}{3}\left(\frac{x+y+z}{3}\right)^{m-1} - \frac{(m-1)y}{x^{2-m}}.$$

Since $f''(x)$ is strictly increasing and $\lim_{x\to 0} f''(x) = -\infty$, two cases are possible·

a) $f''(x) \le 0$, for $0 < x \le z$;

b) there exists $x_1 \in (0, z)$ such that $f''(x_1) = 0$, $f''(x) < 0$ for $x \in (0, x_1)$ and $f''(x) > 0$ for $x \in (x_1, z]$, the point x_1 satisfies the relation

$$(m+1)\left(\frac{x_1+y+z}{3}\right)^{m-1} = 3(m-1)x_1^{m-2}y \tag{2}$$

Case (a). The function $f(x)$ is concave on $[0, z]$, and hence

$$f(x) \ge \min\{f(0), f(z)\}$$

Case (b) The derivative $f'(x)$ is strictly decreasing on $[0, x_1]$ and strictly increasing on $[x_1, z]$. We have

$$f'(0) = (m+1)\left(\frac{y+z}{3}\right)^m - z^m \geq \left[(m+1)\left(\frac{2}{3}\right)^m - 1\right] z^m.$$

By Bernoulli's Inequality, $\left(\frac{2}{3}\right)^m = \left(1 - \frac{1}{3}\right)^m > 1 - \frac{m}{3}$ Then,

$$(m+1)\left(\frac{2}{3}\right)^m - 1 > (m+1)\left(1 - \frac{m}{3}\right) - 1 = \frac{m(2-m)}{3} > 0,$$

and hence, $f'(0) > 0$ There are two possible cases to consider $f'(z) \leq 0$ or $f'(z) > 0$.

Sub-case $f'(z) \leq 0$ There exists $x_2 \in (0, z)$ such that $f'(x_2) = 0$, $f'(x) > 0$ for $x \in [0, x_2)$ and $f'(x) < 0$ for $x \in (x_2, z)$ The function $f(x)$ is strictly increasing on $[0, x_2]$ and strictly decreasing on $[x_2, z]$. Hence, $f(x) \geq \min\{f(0), f(z)\}$

Sub-case $f'(z) > 0$ We claim that $f'(x_1) > 0$. If our claim is true, then $f'(x) \geq f'(x_1) > 0$ for $x \in [0, z]$, the function $f(x)$ is strictly increasing on $[0, z]$ and $f(x) \geq f(0) \geq \min\{f(0), f(z)\}$ To show that $f'(x_1) > 0$, taking into account (2), we have

$$\begin{aligned} f'(x_1) &= (m+1)\left(\frac{x_1+y+z}{3}\right)^m - m x_1^{m-1} y - z^m = \\ &= (m-1) x_1^{m-2} y (x_1 + y + z) - m x_1^{m-1} y - z^m = \\ &= x_1^{m-2}\left[(m-1) y (x_1 + y + z) - m x_1 y - x_1^{2-m} z^m\right] = \\ &= x_1^{m-2}\left[(m-1) y (y+z) - x_1 y - x_1^{2-m} z^m\right] > \\ &> x_1^{m-2}\left[(m-1) y (y+z) - zy - z^2\right] = x_1^{m-2}(y+z)[(m-1)y - z]. \end{aligned}$$

Thus, it suffices to show that $(m-1)y \geq z$. To prove this, we will show that $(m-1)y < z$ implies $f'(z) \leq 0$, which is a contradiction We have

$$f'(z) = (m+1)\left(\frac{y+2z}{3}\right)^m - m z^{m-1} y - z^m.$$

For fixed z, consider the function

$$h(y) = (m+1)\left(\frac{y+2z}{3}\right)^m - m z^{m-1} y - z^m.$$

We must prove that $h(y) \le 0$ for $y \in \left[z, \dfrac{z}{m-1}\right)$. Since $h(z) = 0$, it suffices to show that $h'(y) \le 0$ on $\left[z, \dfrac{z}{m-1}\right]$ Indeed, since the derivative

$$h'(y) = \frac{m(m+1)}{3}\left(\frac{y+2z}{3}\right)^{m-1} - mz^{m-1}$$

is strictly increasing, we have

$$h'(y) \le h'\left(\frac{z}{m-1}\right) = m\left[\frac{m+1}{3}\left(\frac{2m-1}{3m-3}\right)^{m-1} - 1\right] z^{m-1} \approx$$
$$\approx -0.0282\, mz^{m-1} < 0.$$

This completes the proof of the left inequality (1). To prove the right inequality (1), we will show that $f(0) \ge 0$ and $f(z) \ge 0$ Since

$$f(0) = 3\left(\frac{y+z}{3}\right)^{m+1} - y^m z = m^m\left(\frac{y+z}{m+1}\right)^{m+1} - y^m z,$$

the inequality $f(0) \ge 0$ is equivalent to

$$\left(\frac{y}{m}\right)^m z \le \left(\frac{y+z}{m+1}\right)^{m+1}$$

This inequality follows from either the weighted AM-GM Inequality or Jensen's Inequality bellow applied to the concave function $f(x) = \ln x$.

$$mf\left(\frac{y}{m}\right) + f(z) \le (m+1)f\left(\frac{y+z}{m+1}\right).$$

Since

$$f(z) = 3\left(\frac{y+2z}{3}\right)^{m+1} - y^m z - yz^m - z^{m+1},$$

the required inequality $f(z) \ge 0$ can be rewritten as $g(t) \ge g(1)$, where $t = \dfrac{y}{z} \ge 1$ and

$$g(t) = 3\left(\frac{t+2}{3}\right)^{m+1} - t^m - t - 1.$$

We have

$$g'(t) = (m+1)\left(\frac{t+2}{3}\right)^m - mt^{m-1} - 1,$$

$$g''(t) = \frac{m(m+1)}{3}\left(\frac{t+2}{3}\right)^{m-1} - \frac{m(m-1)}{t^{2-m}}.$$

Since the function $g''(t)$ is increasing, we get

$$g''(t) \geq g''(1) = \frac{2m(2-m)}{3} > 0$$

Then, the derivative $g'(t)$ is strictly increasing on $[1, \infty)$, and hence $g'(t) \geq g'(1) = 0$. Consequently, the function $g(t)$ is also strictly increasing on $[1, \infty)$, and therefore $g(t) \geq g(1)$.

II. *Case* $x \leq y < z$. Let $F(x, y, z) = x^m y + y^m z + z^m x$. We will show that

$$F(x, y, z) \leq F(x, z, y) \leq 3\left(\frac{x+y+z}{3}\right)^{m+1}$$

Since the right inequality is true (as shown in the preceding case), it is enough to prove the left inequality $F(x, y, z) \leq F(x, z, y)$. For $x = y$, the inequality becomes equality, while $x < y < z$ it is equivalent to

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(y) - f(z)}{y - z}$$

or

$$\frac{f(x)}{(x-y)(x-z)} + \frac{f(y)}{(y-z)(y-x)} + \frac{f(z)}{(z-x)(z-y)} \geq 0,$$

where $f(t) = t^m$. These inequality are true because the function f is convex.

The original inequality becomes equality for $(x, y, z) \sim (1, 1, 1)$. Moreover, in the special case $r = m$, equality occurs again for $(x, y, z) \sim (0, m, 1)$ or any cyclic permutation.

Chapter 3

Inequalities with right convex and left concave functions

Let f be a function defined on an interval $\mathbb{I} \subset \mathbb{R}$. The function f is said to be right convex on f if there is $s \in \mathbb{I}$ such that f is convex for $x \geq s$. Similarly, f is said to be left concave on $\mathbb{I}$ if there is $s \in \mathbb{I}$ such that f is concave for $x \leq s$ [7] The following two theorems and their corollaries are useful to prove a large class of Jensen's type inequalities for right convex and left concave functions.

3.1 Inequalities with right convex functions

Right Convex Function Theorem (RCF–Theorem) *Let $f(u)$ be a function defined on an interval $\mathbb{I} \subset \mathbb{R}$ and convex for $u \geq s$, $s \in \mathbb{I}$. If*

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \tag{1}$$

for all $x_1, x_2, \quad, x_n \in \mathbb{I}$ such that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = s \ \textit{and} \ x_2 = x_3 = \cdots = x_n \geq s,$$

then (1) *holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ such that $\frac{x_1+x_2+\cdots+x_n}{n} \geq s$.*

Proof. Without loss of generality, assume that $x_1 \le x_2 \le \cdots \le x_n$. If $x_1 \ge s$, then the desired inequality is just Jensen's Inequality for convex functions. Assume now that $x_1 < s$ Since $x_1 + x_2 + \cdots + x_n \ge ns$, there exists $k \in \{1, 2, \ldots, n-1\}$ such that

$$x_1 \le \cdots \le x_k < s \le x_{k+1} \le \cdots \le x_n.$$

Setting

$$S = \frac{x_1 + x_2 + \cdots + x_n}{n}, \quad z = \frac{x_1 + \cdots + x_k}{k}, \quad t = \frac{x_{k+1} + \cdots + x_n}{n-k},$$

we have $z \in \mathbb{I}$, $t \in \mathbb{I}$, $kz + (n-k)t = nS$ and

$$z < s \le S < t$$

By Jensen's Inequality, we get

$$f(x_{k+1}) + \cdots + f(x_n) \ge (n-k)f(t)$$

Then, we still have to show that

$$f(x_1) + \cdots + f(x_k) + (n-k)f(t) \ge nf(S)$$

Denote now $y_i = \dfrac{ns - x_i}{n-1}$ for $i = 1, 2, \ldots, k$. Let us show that $s < y_i \le t$ The left side inequality reduces to $x_i < s$, which is true for $i = 1, 2, \ldots, k$. In addition, we have

$$y_i \le y_1 = \frac{ns - x_1}{n-1} \le \frac{nS - x_1}{n-1} = \frac{x_2 + \cdots + x_n}{n-1} \le \frac{x_{k+1} + \cdots + x_n}{n-k} = t.$$

Thus, according to the hypothesis, the inequality holds

$$f(x_i) + (n-1)f(y_i) \ge nf(s)$$

Summing all these inequalities for $i = 1, 2, \ldots, k$, we get

$$f(x_1) + \cdots + f(x_k) \ge knf(s) - (n-1)\left[f(y_1) + \cdots + f(y_k)\right],$$

and we still have to show that

$$knf(s) + (n-k)f(t) \ge nf(S) + (n-1)\left[f(y_1) + \cdots + f(y_k)\right].$$

Let $s_1 = \dfrac{(n+k-1)s - kz}{n-1}$. We have

$$s < s_1 \le \frac{ns + (k-1)S - kz}{n-1} = \frac{(k-1)s + (n-k)t}{n-1} \le t.$$

We will apply now the Karamata Majorization Inequality, which states the following.

• *If f is a convex function on $\mathbb{I}$, and a vector $\vec{A} = (a_1, a_2, \ldots, a_k)$ with $a_i \in \mathbb{I}$ majorizes a vector $\vec{B} = (b_1, b_2, \ldots, b_k)$ with $b_i \in \mathbb{I}$, then*

$$f(a_1) + f(a_2) + \cdots + f(a_k) \ge f(b_1) + f(b_2) + \cdots + f(b_k).$$

We say that $\vec{A} = (a_1, a_2, \ldots, a_n)$ with $a_1 \ge a_2 \ge \cdots \ge a_n$ majorizes $\vec{B} = (b_1, b_2, \ldots, b_n)$ with $b_1 \ge b_2 \ge \cdots \ge b_n$, and write it as $\vec{A} \succ \vec{B}$, if

$$\begin{gathered} a_1 \ge b_1, \\ a_1 + a_2 \ge b_1 + b_2 \\ \ldots \qquad \ldots\ldots \\ a_1 + a_2 + \cdots + a_{n-1} \ge b_1 + b_2 + \cdots + b_{n-1}, \\ a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n. \end{gathered}$$

In our case, the vector $\vec{A} = (s_1, s, \ldots, s)$ majorizes the vector $\vec{B} = (y_k, y_{k-1}, \ldots, y_1)$, since $(k-1)s + s_1 = y_1 + \cdots + y_k$ and $s \le y_k \le y_{k-1} \le \cdots \le y_1$ Consequently, by Karamata's Majorization Inequality we have

$$f(y_1) + \cdots + f(y_k) \le (k-1)f(s) + f(s_1).$$

Therefore, it suffices to show that

$$(n+k-1)f(s) + (n-k)f(t) \ge nf(S) + (n-1)f(s_1).$$

This inequality can be obtained by summing the following Jensen's inequalities multiplied by n and $n-1$ respectively.

$$\frac{t-S}{t-s} f(s) + \frac{S-s}{t-s} f(t) \ge f(S),$$
$$\frac{t-s_1}{t-s} f(s) + \frac{s_1-s}{t-s} f(t) \ge f(s_1).$$

□

Remark 1. The theorem hypothesis is equivalent to the condition·

$$f(x)+(n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ *such that* $x \leq s \leq y$ *and* $x+(n-1)y=ns$.

Remark 2. Let $g(t)=\dfrac{f(t)-f(s)}{t-s}$ For $x<s<y$ and $x+(n-1)y=ns$, we get

$$f(x)+(n-1)f(y)-nf(s)=f(x)-f(s)+(n-1)\,[f(y)-f(s)]=$$
$$=(x-s)g(x)+(n-1)(y-s)g(y)=(s-x)[g(y)-g(x)].$$

Thus, the theorem hypothesis is equivalent to the condition
$g(x) \leq g(y)$ *for all* $x, y \in \mathbb{I}$ *such that* $x \leq s \leq y$ *and* $x+(n-1)y=ns$.

Remark 3. Assume that f is differentiable on $\mathbb{I}$ Then, the RCF–Theorem holds valid by replacing the theorem hypothesis with the more restrictive condition
$f'(x) \leq f'(y)$ *for all* $x, y \in \mathbb{I}$ *such that* $x \leq s \leq y$ *and* $x+(n-1)y=ns$.

To prove this claim, let us denote

$$F(x)=f(x)+(n-1)f\left(\frac{ns-x}{n-1}\right)-nf(s)$$

Since

$$F'(x)=f'(x)-f'(y) \leq 0,$$

the function $F(x)$ is decreasing for $x \leq s$ Therefore, $F(x) \geq F(s)=0$ for $x \leq s$, and hence $f(x)+(n-1)f(y)-nf(s) \geq 0$.

Right Convex Function Corollary (RCF–Corollary) *Let f be a function defined on $(0,\infty)$, and let $r>0$. If the function $f_1(u)=f(e^u)$ is convex for $u \geq \ln r$, and*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf\left(\sqrt[n]{a_1a_2 \cdots a_n}\right) \tag{2}$$

for all $a_1, a_2, \ldots, a_n>0$ such that

$$\sqrt[n]{a_1a_2\ldots a_n}=r \;\text{ and }\; a_2=a_3=\cdots=a_n \geq r,$$

then (2) holds for all $a_1, a_2, \ldots, a_n>0$ such that $\sqrt[n]{a_1a_2\ldots a_n} \geq r$.

Proof. Apply RCF–Theorem to the function $f(e^u)$. Moreover, replace s by $\ln r$, and x_i by $\ln a_i$ for all indices i. □

Remark 4. The corollary hypothesis is equivalent to the condition·

$$f(x) + (n-1)f(y) \geq nf(r)$$

for all $x, y > 0$ *such that* $x \leq r \leq y$ *and* $xy^{n-1} = r^n$.

Remark 5. Assume that f is differentiable on $\mathbb{I}$ The RCF–Corollary holds valid by replacing the theorem hypothesis with a more restrictive condition

$xf'(x) \leq yf'(y)$ *for all* $x, y \in \mathbb{I}$ *such that* $x \leq r \leq y$ *and* $xy^{n-1} = r^n$.

To prove this, let us denote

$$F(x) = f(x) + (n-1)f\left(r \sqrt[n-1]{\frac{r}{x}}\right) - nf(r).$$

Since

$$F'(x) = f'(x) - \frac{r}{n} \sqrt[n-1]{\frac{r}{x}} f'(y) = \frac{xf'(x) - yf'(y)}{x} \leq 0,$$

the function $f(x)$ is decreasing for $x \leq r$ Therefore, $F(x) \geq F(r) = 0$ for $x \leq r$, and hence $f(x) + (n-1)f(y) - nf(r) \geq 0$

3.2 Inequalities with left concave functions

Left Concave Function Theorem (LCF–Theorem). *Let $f(u)$ be a function defined on an interval $\mathbb{I} \subset \mathbb{R}$ and concave for $u \leq s$, $s \in \mathbb{I}$. If*

$$f(x_1) + f(x_2) + \cdots + f(x_n) \leq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \tag{3}$$

for all $x_1, x_2, .., x_n \in \mathbb{I}$ *such that*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = s \ \textit{and} \ x_1 = x_2 = \cdots = x_{n-1} \leq s,$$

then (3) *holds for all* $x_1, x_2, .., x_n \in \mathbb{I}$ *such that* $\dfrac{x_1 + x_2 + \cdots + x_n}{n} \leq s$.

Proof. To prove this theorem we proceed as in RCF-Theorem proof. □

Remark 6. The theorem hypothesis is equivalent to the condition.

$$(n-1)f(x) + f(y) \leq nf(s)$$

for all $x, y \in \mathbb{I}$ *such that* $x \leq s \leq y$ *and* $(n-1)x + y = ns$.

Remark 7. Let $g(t) = \dfrac{f(t) - f(s)}{t - s}$ The theorem hypothesis is equivalent to the condition

$g(x) \geq g(y)$ *for all* $x, y \in \mathbb{I}$ *such that* $x \leq s \leq y$ *and* $(n-1)x + y = ns$.

Remark 8. If f is differentiable on $\mathbb{I}$, then the LCF-Theorem holds valid by replacing the theorem hypothesis with a more restrictive condition

$f'(x) \geq f'(y)$ *for all* $x, y \in \mathbb{I}$ *such that* $x \leq s \leq y$ *and* $(n-1)x + y = ns$

Left Concave Function Corollary (LCF-Corollary). *Let f be a continuous function on $(0, \infty)$, and let $r > 0$. If the function $f_1(u) = f(e^u)$ is concave for $u \leq \ln r$, and*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \leq nf\left(\sqrt[n]{a_1 a_2 \cdots a_n}\right) \tag{4}$$

for all $a_1, a_2, \ldots, a_n > 0$ such that

$$\sqrt[n]{a_1 a_2 \ldots a_n} = r \quad \text{and} \quad a_1 = a_2 = \cdots = a_{n-1} \leq r,$$

then (4) *holds for all $a_1, a_2, \ldots, a_n > 0$ such that $\sqrt[n]{a_1 a_2 \cdots a_n} \leq r$.*

Remark 9. The corollary hypothesis is equivalent to the condition:

$(n-1)f(x) + f(y) \leq ng(r)$ *for all* $x, y > 0$ *such that* $x \leq r \leq y$ *and* $x^{n-1}y = r^n$

Remark 10. If f is differentiable on $\mathbb{I}$, then the LCF-Corollary holds valid by replacing the theorem hypothesis with a more restrictive condition

$xf'(x) \geq yf'(y)$ *for all* $x, y \in \mathbb{I}$ *such that* $x \leq r \leq y$ *and* $x^{n-1}y = r^n$

3.3 Inequalities with left concave-right convex functions

Left Concave - Right Convex Function Theorem (LCRCF - Theorem). *Let $a < c$ be real numbers, and let f be a continuous function on $\mathbb{I} = [a, \infty)$, concave on $[a, c]$ and convex on $[c, \infty)$. If $x_1, x_2, \ldots, x_n \in \mathbb{I}$ such that*

$$x_1 + x_2 + \cdots + x_n = S = \text{constant},$$

then the expression

$$E = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is maximal for $x_1 = x_2 = \cdots = x_{n-1} \leq x_n$.

Proof. Without loss of generality, assume that $x_1 \le x_2 \le \cdots \le x_n$. If $x_n \le c$, then by Jensen's Inequality for concave function we have

$$f(x_1) + f(x_2) + \cdots + f(x_n) \le nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right).$$

Therefore, the expression E is maximal for $x_1 = x_2 = \cdots = x_n$. If $x_n > c$, there exists $k \in \{0, 1, \ldots, n-1\}$ such that

$$a \le x_1 \le \cdots \le x_k \le c < x_{k+1} \le \cdots \le x_n.$$

By Karamata's Majorization Inequality for convex function and Jensen's Inequality for concave function, we have

$$f(x_{k+1}) + \cdots + f(x_n) \le (n-k-1)f(c) + f\,(x_{k+1} + \cdots + x_n - (n-k-1)c)$$

and

$$(n-k-1)f(c) + f(x_1) + \cdots + f(x_k) \le (n-1)f\left(\frac{(n-k-1)c + x_1 + \cdots + x_k}{n-1}\right),$$

respectively. Summing up these inequalities yields

$$f(x_1) + f(x_2) + \cdots + f(x_n) \le (n-1)f(x) + f(y),$$

where

$$x = \frac{(n-k-1)c + x_1 + \cdots + x_k}{n-1}, \quad y = x_{k+1} + \cdots + x_n - (n-k-1)c.$$

It easy to check that $(n-1)x + y = x_1 + x_2 + \cdots + x_n$ and $x \le y$. According to the last inequality, the expression E is maximal for $x_1 = \cdots = x_{n-1} = x$ and $x_n = y$. □

Remark 11. Theorem 1 is also valid in the case $\mathbb{I} = (a, \infty)$ and $\lim_{x \to a} f(x) = -\infty$. In addition, Theorem 1 is still valid if $a < c < b$, $S < (n-1)c + b$ and f is a continuous function on $\mathbb{I} = [a, b)$, concave on $[a, c]$ and convex on $[c, b)$.

In a similar manner we can prove that the following statement.

Single Inflexion Point Theorem (SIP - Theorem). *Let f be a twice differentiable function on $\mathbb{R}$ with a single inflexion point, let S be a fixed real number and let*

$$g(x) = f(x) + (n-1)f\left(\frac{S-x}{n-1}\right).$$

If $x_1, x_2, \ldots, x_n$ are real numbers such that $x_1 + x_2 + \cdots + x_n = S$, then

$$\inf_{x \in \mathbb{R}} g(x) \le f(x_1) + f(x_2) + \cdots + f(x_n) \le \sup_{x \in \mathbb{R}} g(x).$$

3.4 Applications

1. If $x_1, x_2, \ldots, x_n$ are non-negative real numbers such that

$$x_1 + x_2 + \cdots + x_n = n,$$

then

$$(n-1)\left(x_1^3 + x_2^3 + \cdots + x_n^3\right) + n^2 \geq (2n-1)\left(x_1^2 + x_2^2 + \cdots + x_n^2\right).$$

(*Vasile Cîrtoaje*, GM-A, 2, 2002)

2. If $x_1, x_2, \ldots, x_n$ are non-negative real numbers such that

$$x_1 + x_2 + \cdots + x_n = n,$$

then

$$x_1^3 + x_2^3 + \cdots + x_n^3 + n^2 \leq (n+1)\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)$$

(*Vasile Cîrtoaje*, MS, 2004)

3. If $x_1, x_2, \ldots, x_n$ are non-negative numbers such that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = r \geq \sqrt{\frac{n-1}{n}},$$

then

$$\frac{1}{1+x_1^2} + \frac{1}{1+x_2^2} + \cdots + \frac{1}{1+x_n^2} \geq \frac{n}{1+r^2}.$$

(*Vasile Cîrtoaje*, GM-A, 2, 2005)

4. If $x_1, x_2, \ldots, x_n$ are non-negative real numbers such that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = r \leq \sqrt{\frac{n-1}{n^2-n+1}},$$

then

$$\frac{1}{1+x_1^2} + \frac{1}{1+x_2^2} + \cdots + \frac{1}{1+x_n^2} \leq \frac{n}{1+r^2}.$$

(*Vasile Cîrtoaje*, GM-A, 2, 2005)

5. If $x_1, x_2, \ldots, x_n$ are positive real numbers such that $x_1 + x_2 + \cdots + x_n = 1$, then

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \geq (n-2)^2 + 4n(n-1)\left(x_1^2 + x_2^2 + \cdots + x_n^2\right).$$

(*Vasile Cîrtoaje*, MS, 2004)

6. If $x_1, x_2, \ldots, x_n$ are non-negative real numbers such that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = r \le \frac{n-1}{\left(n+\sqrt{n-1}\right)^2},$$

then

$$\frac{1}{1-\sqrt{x_1}} + \frac{1}{1-\sqrt{x_2}} + \cdots + \frac{1}{1-\sqrt{x_n}} \le \frac{n}{1-\sqrt{r}}$$

7. Let $0 \le x_1, x_2, \ldots, x_n < 1$ such that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = r \ge \frac{n-1}{\left(\sqrt{n}+\sqrt{n-1}\right)^2}.$$

Then

$$\frac{1}{1-\sqrt{x_1}} + \frac{1}{1-\sqrt{x_2}} + \cdots + \frac{1}{1-\sqrt{x_n}} \ge \frac{n}{1-\sqrt{r}}$$

8. If $x_1, x_2, \ldots, x_n$ are positive real numbers such that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = r \le 1 + \frac{2\sqrt{n-1}}{n},$$

then

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\cdots\left(x_n + \frac{1}{x_n}\right) \ge \left(r + \frac{1}{r}\right)^n.$$

9. If $x_1, x_2, \ldots, x_n$ $(n \ge 3)$ are positivc real numbers such that

$$x_1 + x_2 + \cdots + x_n = 1,$$

then

$$\left(\frac{1}{\sqrt{x_1}} - \sqrt{x_1}\right)\left(\frac{1}{\sqrt{x_2}} - \sqrt{x_2}\right)\cdots\left(\frac{1}{\sqrt{x_n}} - \sqrt{x_n}\right) \ge \left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^n.$$

10. If x, y, z are non-negative real numbers, no two of them are zero, then

$$\sqrt{1 + \frac{48x}{y+z}} + \sqrt{1 + \frac{48y}{z+x}} + \sqrt{1 + \frac{48z}{x+y}} \ge 15.$$

(*Vasile Cîrtoaje*, CM, 6, 2005)

11. Let x, y, z be non-negative real numbers, no two of them are zero. If $r \ge r_0$, where $r_0 = \dfrac{\ln 3}{\ln 2} - 1 \approx 0\,585$, then

$$\left(\frac{2x}{y+z}\right)^r + \left(\frac{2y}{z+x}\right)^r + \left(\frac{2z}{x+y}\right)^r \ge 3.$$

(*Vasile Cîrtoaje*, CM, 6, 2005)

12. Let x, y, z be non-negative real numbers such that $x+y+z=3$. If $0<r\le r_0$, where $r_0=\dfrac{\ln 2}{\ln 3-\ln 2}\approx 1.71$, then

$$x^r(y+z)+y^r(z+x)+z^r(x+y)\le 6.$$

13. If $x_1, x_2, \ldots, x_n<1$ are non-negative real numbers such that

$$\frac{x_1+x_2+\cdots+x_n}{n}=r\ge\frac{1}{3},$$

then

$$\frac{\sqrt{x_1}}{1-x_1}+\frac{\sqrt{x_2}}{1-x_2}+\cdots+\frac{\sqrt{x_n}}{1-x_n}\ge\frac{n\sqrt{r}}{1-r}.$$

(*Vasile Cîrtoaje*, CM, 7, 2004)

14. If a, b, c are non-negative real numbers such that $a+b+c=3$, then

$$(1-a+a^2)(1-b+b^2)(1-c+c^2)\ge 1$$

15. If $x_1, x_2, \ldots, x_n$ are non-negative numbers such that $x_1+x_2+\cdots+x_n=n$, then

$$\frac{1}{n-x_1+x_1^2}+\frac{1}{n-x_2+x_2^2}+\cdots+\frac{1}{n-x_n+x_n^2}\le 1.$$

(*Vasile Cîrtoaje*, MS, 2005)

16. If a, b, c are positive real numbers such that $abc=1$, then

$$1+a+b+c\ge 2\sqrt{1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

17. If a, b, c, d are positive real numbers such that $abcd=1$, then

$$(a-1)(a-2)+(b-1)(b-2)+(c-1)(c-2)+(d-1)(d-2)\ge 0$$

18. If $a_1, a_2, \ldots, a_n$ $(n\ge 4)$ are positive real numbers such that $a_1a_2\cdots a_n=1$, then

$$(n-1)\left(a_1^2+a_2^2+\cdots+a_n^2\right)+n(n+3)\ge(2n+2)(a_1+a_2+\cdots+a_n)$$

(*Vasile Cîrtoaje*, MS, 2005)

19. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1a_2\cdots a_n=1$, then

$$a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1}+n(n-2)\ge(n-1)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)$$

20. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \ldots a_n = 1$. If $m \geq n$, then

$$a_1^m + a_2^m + \cdots + a_n^m + mn \geq (m+1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

(*Vasile Cîrtoaje*, MS, 2006)

21. If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are positive real numbers such that

$$\sqrt[n]{a_1 a_2 \ldots a_n} = p \geq \sqrt{n} - 1,$$

then

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} \geq \frac{n}{(1+p)^2}$$

22. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$\sqrt[n]{a_1 a_2 \ldots a_n} = p \geq n^2 - 1,$$

then

$$\frac{1}{\sqrt{1+a_1}} + \frac{1}{\sqrt{1+a_2}} + \cdots + \frac{1}{\sqrt{1+a_n}} \geq \frac{n}{\sqrt{1+p}}.$$

23. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$\sqrt[n]{a_1 a_2 \ldots a_n} = p \leq \sqrt{\frac{n}{n-1}} - 1,$$

then

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} \leq \frac{n}{(1+p)^2}.$$

24. If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are positive real numbers such that

$$\sqrt[n]{a_1 a_2 \ldots a_n} = p \leq \frac{2n-1}{(n-1)^2},$$

then

$$\frac{1}{\sqrt{1+a_1}} + \frac{1}{\sqrt{1+a_2}} + \cdots + \frac{1}{\sqrt{1+a_n}} \leq \frac{n}{\sqrt{1+p}}.$$

25. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $\sqrt[n]{a_1 a_2 \ldots a_n} = p \geq 1$, then

$$\frac{1}{1+a_1+\cdots+a_1^{n-1}} + \frac{1}{1+a_2+\cdots+a_2^{n-1}} + \cdots + \frac{1}{1+a_n+\cdots+a_n^{n-1}} \geq$$
$$\geq \frac{n}{1+p+\cdots+p^{n-1}}$$

(*Vasile Cîrtoaje*, GM-A, 2, 2005)

26. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \ldots a_n \geq 1$, then

$$a_1 + a_2 + \cdots + a_n - \sqrt[n]{a_1 a_2 \ldots a_n} \geq \frac{1}{2n^2} \sum_{1 \leq i \leq j \leq n} (\ln a_i - \ln a_j)^2$$

(*Marian Tetiva*)

27. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \ldots a_n = 1$, then

$$\left(1 - \frac{1}{n}\right)^{a_1} + \left(1 - \frac{1}{n}\right)^{a_2} + \cdots + \left(1 - \frac{1}{n}\right)^{a_n} \leq n - 1$$

(*Vasile Cîrtoaje*, GM-A, 3, 2004)

28. If $x_1, x_2, \ldots, x_n$ are non-negative real numbers such that

$$x_1 + x_2 + \cdots + x_n = n,$$

then

$$n^{-x_1^2} + n^{-x_2^2} + \cdots + n^{-x_n^2} \geq 1.$$

(*Pham Kim Hung*, MS, 2006)

29. Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers such that

$$x_1 + x_2 + \cdots + x_n = n.$$

Prove that

$$2\left(x_1^3 + x_2^3 + \cdots + x_n^3\right) + n^2 \leq (2n+1)\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)$$

30. Let x, y, z be positive real numbers such that $x + y + z = 3$ Prove that

$$8\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + 9 \geq 10(x^2 + y^2 + z^2).$$

(*Vasile Cîrtoaje*, MS, 2006)

3.5 Solutions

1. *If $x_1, x_2, \ldots, x_n$ are non-negative real numbers such that*

$$x_1 + x_2 + \cdots + x_n = n,$$

then

$$(n-1)\left(x_1^3 + x_2^3 + \cdots + x_n^3\right) + n^2 \geq (2n-1)\left(x_1^2 + x_2^2 + \cdots + x_n^2\right).$$

Proof. We may write the inequality in the form

$$f(x_1)+f(x_2)+\ \cdot+f(x_n)\geq nf\left(\frac{x_1+x_2+\ \ \cdot+x_n}{n}\right),$$

where $f(u)=(n-1)u^3-(2n-1)u^2$, $u\geq 0$. The second derivative

$$f''(u)=6(n-1)u-2(2n-1)$$

shows that f is convex for $u\geq\dfrac{2n-1}{3(n-1)}$, and hence for $u\geq s$, where

$$s=\frac{x_1+x_2+\cdots+x_n}{n}=1\geq\frac{2n-1}{3(n-1)}.$$

By RCF–Theorem, it is sufficient to prove the inequality for

$$x_1\leq 1\leq x_2=x_3=\cdots=x_n.$$

According to Remark 2, we have to show that $g(x)\leq g(y)$ for $0\leq x\leq 1\leq y$ and $x+(n-1)y=n$, where

$$g(t)=\frac{f(t)-f(1)}{t-1}=(n-1)(t^2+t+1)-(2n-1)(t+1)$$

Indeed, we have

$$g(x)-g(y)=(x-y)[(n-1)(x+y+1)-2n+1]=(n-2)x(x-y)\leq 0$$

For $n=2$, our inequality becomes equality. For $n\geq 3$, equality occurs when either $x_1=x_2=\ \cdot\cdot=x_n=1$, or one of x_i is equal to 0 and the other ones are equal to $\dfrac{n}{n-1}$. □

★

2. *If $x_1,x_2,\ldots,x_n$ are non-negative real numbers such that*

$$x_1+x_2+\ \cdot\cdot+x_n=n,$$

then

$$x_1^3+x_2^3+\cdots+x_n^3+n^2\leq(n+1)\left(x_1^2+x_2^2+\cdots+x_n^2\right).$$

Proof We may write the inequality in the form

$$f(x_1)+f(x_2)+\cdots+f(x_n) \le nf\left(\frac{x_1+x_2+\cdots+x_n}{n}\right),$$

where $f(u) = u^3-(n+1)u^2$, $u \ge 0$. This function is concave for $0 \le u \le \dfrac{n+1}{3}$, and hence for $0 \le u \le s$, where

$$s = \frac{x_1+x_2+\cdots+x_n}{n} = 1 \le \frac{n+1}{3}$$

By LCF–Theorem it is suffices to prove the inequality for

$$x_1 = x_2 = \cdots = x_{n-1} \le 1 \le x_n.$$

Taking into account Remark 7, we have to show that $g(x) \ge g(y)$, for

$$0 \le x \le 1 \le y \text{ and } (n-1)x+y=n.$$

We have

$$g(t) = \frac{f(t)-f(1)}{t-1} = t^2-nt-n$$

and

$$g(x)-g(y) = (x-y)(x+y-n) = (n-2)x(y-x) \ge 0.$$

For $n=2$, the original inequality becomes equality For $n \ge 3$, equality occurs when either $x_1 = x_2 = \cdots = x_n = 1$, or one of x_i is equal to n and the other ones are equal to 0. □

★

3. *If $x_1, x_2, \ldots, x_n$ are non-negative numbers such that*

$$\frac{x_1+x_2+\cdots+x_n}{n} = r \ge \sqrt{\frac{n-1}{n}},$$

then

$$\frac{1}{1+x_1^2}+\frac{1}{1+x_2^2}+\cdots+\frac{1}{1+x_n^2} \ge \frac{n}{1+r^2}.$$

Proof. Apply RCF–Theorem to the function $f(u) = \dfrac{1}{1+u^2}$, $u \ge 0$. From

$$f''(u) = \frac{2(3u^2-1)}{(1+u^2)^3},$$

it follows that f is convex on $\left[\frac{1}{\sqrt{3}},\infty\right)$. Since $s=\sqrt{\frac{n-1}{n}}>\frac{1}{\sqrt{3}}$, the function f is convex on $[s,\infty)$ By RCF–Theorem and Remark 2, we have to show that $g(x)\le g(y)$ for $0\le x\le s\le y$ and $x+(n-1)y=ns$, where

$$g(t)=\frac{f(t)-f(s)}{t-s}=\frac{-t-s}{(1+s^2)(1+t^2)}$$

Since

$$g(x)-g(y)=\frac{(x-y)[s(x+y)+xy-1]}{(1+s^2)(1+x^2)(1+y^2)},$$

we still have to show that $s(x+y)+xy-1\ge 0$ Indeed,

$$s(x+y)+xy-1=\frac{ns^2-n+1+x[2(n-1)s-x]}{n+1}\ge\frac{ns^2-n+1}{n+1}=0.$$

Equality occurs for $x_1=x_2=\cdots=x_n=r$. In the case $r=\sqrt{\frac{n-1}{n}}$, equality occurs again when one of x_i is equal to 0 and the others equal $\sqrt{\frac{n}{n-1}}$. □

★

4. *If $x_1,x_2,\ldots,x_n$ are non-negative real numbers such that*

$$\frac{x_1+x_2+\cdots+x_n}{n}=r\le\sqrt{\frac{n-1}{n^2-n+1}},$$

then

$$\frac{1}{1+x_1^2}+\frac{1}{1+x_2^2}+\cdots+\frac{1}{1+x_n^2}\le\frac{n}{1+r^2}$$

Proof. Apply LCF–Theorem to the function $f(u)=\frac{1}{1+u^2}$, $u\ge 0$ Since f is concave on $\left[0,\frac{1}{\sqrt{3}}\right]$ and $s=\sqrt{\frac{n-1}{n^2-n+1}}\le\frac{1}{\sqrt{3}}$, it follows that f is concave on $[0,s]$. According to LCF–Theorem and Remark 7, we have to show that $g(x)\ge g(y)$ for $0\le x\le s\le y$ and $(n-1)x+y=ns$, where

$$g(t)=\frac{f(t)-f(s)}{t-s}=\frac{-t-s}{(1+s^2)(1+t^2)}.$$

Since

$$g(x)-g(y)=\frac{(x-y)[s(x+y)+xy-1]}{(1+s^2)(1+x^2)(1+y^2)}=$$
$$=\frac{(x-y)\left[ns^2-1+2sx-(n-1)x^2\right]}{(1+s^2)(1+x^2)(1+y^2)},$$

we have to show that $ns^2-1+2sx-(n-1)x^2\le 0$ Indeed,

$$ns^2-1+2sx-(n-1)x^2=\frac{(n^2-n+1)s^2-n+1-[(n-1)x-s]^2}{n-1}=$$
$$=\frac{-[(n-1)x-s]^2}{n-1}\le 0$$

Equality occurs for $x_1=x_2=\cdots=x_n=r$ In the case $r=\sqrt{\frac{n-1}{n^2-n+1}}$, equality occurs again when one of x_i is equal to $(n-1)r$, and the other ones are equal to $\frac{r}{n-1}$ □

★

5. *If $x_1,x_2,\ldots,x_n$ are positive real numbers such that $x_1+x_2+\cdots+x_n=1$, then*

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\ge(n-2)^2+4n(n-1)\left(x_1^2+x_2^2+\cdots+x_n^2\right)$$

Proof We may write the inequality in the form

$$f(x_1)+f(x_2)+\cdots+f(x_n)\le nf\left(\frac{x_1+x_2+\cdots+x_n}{n}\right),$$

where $f(u)=4n(n-1)u^2-\frac{1}{u}$, $u>0$. We see that the function $f(u)$ is concave for $0<u\le\sqrt[3]{\frac{1}{4n(n-1)}}$ Since $s=\frac{1}{n}\le\sqrt[3]{\frac{1}{4n(n-1)}}$, the function $f(u)$ is concave on $(0,s]$

According to LCF-Theorem and Remark 7, it is enough to show that

$$g(x)\ge g(y)\ \text{ for }\ 0<x\le\frac{1}{n}\le y\ \text{ and }\ (n-1)x+y=1.$$

Indeed, we have

$$g(t)=\frac{f(t)-f(s)}{t-s}=4n(n-1)(t+s)+\frac{1}{st}=4(n-1)(nt+1)+\frac{n}{t}$$

and

$$g(x)-g(y)=n(x-y)\left(4n-4-\frac{1}{xy}\right)=\frac{n(y-x)(2nx-2x-1)^2}{xy}\geq 0$$

This completes the proof. Equality occurs for $x_1=x_2=\cdots=x_n=\frac{1}{n}$, as well as when one of x_i is equal to $\frac{1}{2}$, and the others are equal to $\frac{1}{2n-2}$. □

★

6. *If* $x_1,x_2,\ \ ,x_n$ *are non-negative real numbers such that*

$$\frac{x_1+x_2+\ \cdot\ +x_n}{n}=r\leq\frac{n-1}{\left(n+\sqrt{n-1}\right)^2},\cdot$$

then

$$\frac{1}{1-\sqrt{x_1}}+\frac{1}{1-\sqrt{x_2}}+\ \ \cdot+\frac{1}{1-\sqrt{x_n}}\leq\frac{n}{1-\sqrt{r}}$$

Proof. Since $\sqrt{x_1+x_2+\cdots+x_n}\leq\frac{\sqrt{n(n-1)}}{n+\sqrt{n-1}}<1$, we have $x_i<1$ for all i.

We will apply LCF-Theorem to the function $f(u)=\frac{1}{1-\sqrt{u}}$, $0\leq u<1$. From $f''(u)=\frac{3\sqrt{u}-1}{4u\sqrt{u}\,(1-\sqrt{u})^3}$, it follows that f is concave on $\left[0,\frac{1}{9}\right]$, and hence on $[0,s]$, where $s=\frac{n-1}{\left(n+\sqrt{n-1}\right)^2}$ By LCF-Theorem, it suffices to consider the case $x_1=x_2=\cdot\ \cdot=x_{n-1}\leq s\leq x_n$. Taking into account Remark 7, we have to show that $g(x)\geq g(y)$, for $0\leq x\leq s\leq y$ and $(n-1)x+y=ns$. Since

$$g(t)=\frac{f(t)-f(s)}{t-s}=\frac{1}{(1-\sqrt{s})\left(1-\sqrt{t}\right)\left(\sqrt{s}+\sqrt{t}\right)}$$

and

$$g(x)-g(y)=\frac{\left(\sqrt{y}-\sqrt{x}\right)\left(1-\sqrt{s}-\sqrt{x}-\sqrt{y}\right)}{(1-\sqrt{s})\,(1-\sqrt{x})\,(\sqrt{s}+\sqrt{x})\left(1-\sqrt{y}\ \ (\sqrt{s}+\sqrt{y}\right)},$$

we still have to show that $1-\sqrt{s} \geq \sqrt{x}+\sqrt{y}$ By the Cauchy-Schwarz Inequality, we have

$$\left(\frac{1}{n-1}+1\right)[(n-1)x+y] \geq \left(\sqrt{x}+\sqrt{y}\right)^2,$$

or, equivalently

$$n\sqrt{\frac{s}{n-1}} \geq \sqrt{x}+\sqrt{y}.$$

Therefore,

$$1-\sqrt{s}-\sqrt{x}-\sqrt{y} \geq 1-\left(1+\frac{n}{\sqrt{n-1}}\right)\sqrt{s}=0.$$

Equality occurs for $x_1=x_2=\cdots=x_n=r$ In the case $r=\dfrac{n-1}{\left(n+\sqrt{n-1}\right)^2}$, equality occurs again when one of x_i is equal to $(n-1)r$, and the other ones are equal to $\dfrac{r}{n-1}$ □

★

7. *Let* $0 \leq x_1, x_2, \ldots, x_n < 1$ *such that*

$$\frac{x_1+x_2+\cdots+x_n}{n}=r \geq \frac{n-1}{\left(\sqrt{n}+\sqrt{n-1}\right)^2}.$$

Then

$$\frac{1}{1-\sqrt{x_1}}+\frac{1}{1-\sqrt{x_2}}+\cdots+\frac{1}{1-\sqrt{x_n}} \geq \frac{n}{1-\sqrt{r}}.$$

Proof We will apply RCF-Theorem to the function $f(u)=\dfrac{1}{1-\sqrt{u}}$, $0 \leq u < 1$ From $f''(u)=\dfrac{3\sqrt{u}-1}{4u\sqrt{u}\left(1-\sqrt{u}\right)^3}$, it follows that f is convex on $\left[\dfrac{1}{9},1\right)$, and hence on $[s,1)$, where $s=\dfrac{n-1}{\left(\sqrt{n}+\sqrt{n-1}\right)^2}$

By RCF-Theorem, it suffices to consider the case

$$x_1 \leq s \leq x_2=\cdots=x_{n-1}=x_n.$$

Taking into account Remark 2, we have to show that $g(x) \le g(y)$, for $0 \le x \le s \le y < 1$ and $x + (n-1)y = ns$ Since

$$g(t) = \frac{f(t)-f(s)}{t-s} = \frac{1}{(1-\sqrt{s})\left(1-\sqrt{t}\right)\left(\sqrt{s}+\sqrt{t}\right)}$$

and

$$g(x)-g(y) = \frac{\left(\sqrt{y}-\sqrt{x}\right)\left(1-\sqrt{s}-\sqrt{x}-\sqrt{y}\right)}{(1-\sqrt{s})(1-\sqrt{x})(\sqrt{s}+\sqrt{x})\left(1-\sqrt{y}\right)\left(\sqrt{s}+\sqrt{y}\right)},$$

we still have to show that $1-\sqrt{s} \le \sqrt{x}+\sqrt{y}$ Indeed, we have

$$\sqrt{x}+\sqrt{y}+\sqrt{s}-1 \ge \sqrt{\frac{x}{n-1}+y}+\sqrt{s}-1 = \sqrt{\frac{ns}{n-1}}+\sqrt{s}-1 = 0$$

Equality occurs for $x_1 = x_2 = \cdots = x_n = r$. In the case $r = \dfrac{n-1}{\left(\sqrt{n}+\sqrt{n-1}\right)^2}$ equality occurs again when one of x_i is 0, and the other ones are equal to $\dfrac{nr}{n-1}$ □

★

8. *If* $x_1, x_2, \ldots, x_n$ *are positive real numbers such that*

$$\frac{x_1+x_2+\cdots+x_n}{n} = r \le 1 + \frac{2\sqrt{n-1}}{n},$$

then

$$\left(x_1+\frac{1}{x_1}\right)\left(x_2+\frac{1}{x_2}\right) \cdots \left(x_n+\frac{1}{x_n}\right) \ge \left(r+\frac{1}{r}\right)^n.$$

Proof. Apply LCF–Theorem to the function $f(u) = -\ln\left(u+\dfrac{1}{u}\right)$, $u > 0$. The first two derivatives of f are given by

$$f'(u) = \frac{1-u^2}{u(u^2+1)} \quad \text{and} \quad f''(x) = \frac{u^4-4u^2-1}{u^2(u^2+1)^2}$$

From the second derivative, it follows that f is concave for $0 < u \le \sqrt{2+\sqrt{5}}$. Since $s = 1 + \dfrac{2\sqrt{n-1}}{n} \le 2 < \sqrt{2+\sqrt{5}}$, $f(u)$ is concave for $0 < u \le s$. By LCF–Theorem and Remark 8, it suffices to show that $f'(x) \ge f'(y$ for

$0 < x \le s \le y$ and $(n-1)x + y = ns$
Since

$$f'(x)-f'(y) = (y-x)\frac{1+(x+y)^2-x^2y^2}{xy(1+x^2)(1+y^2)} \ge (y-x)\frac{(x+y)^2-x^2y^2}{xy(1+x^2)(1+y^2)},$$

it is enough to show that $x + y \ge xy$ Indeed, we have

$$x+y-xy = x+(1-x)\left[n+2\sqrt{n-1}-(n-1)x\right] = \left(\sqrt{n-1}x-1-\sqrt{n-1}\right)^2 \ge 0$$

Equality occurs only for $x_1 = x_2 = \cdots = x_n = r$. □

★

9. *If* $x_1, x_2, \ldots, x_n$ $(n \ge 3)$ *are positive real numbers such that*

$$x_1 + x_2 + \cdots + x_n = 1,$$

then

$$\left(\frac{1}{\sqrt{x_1}}-\sqrt{x_1}\right)\left(\frac{1}{\sqrt{x_2}}-\sqrt{x_2}\right)\cdots\left(\frac{1}{\sqrt{x_n}}-\sqrt{x_n}\right) \ge \left(\sqrt{n}-\frac{1}{\sqrt{n}}\right)^n \tag{5}$$

Proof. We will apply LCF–Theorem to the function $f(u) = -\ln\left(\frac{1}{\sqrt{u}}-\sqrt{u}\right)$, $0 < u < 1$ We have

$$f'(u) = \frac{1}{1-u} + \frac{1}{2u}, \quad f''(u) = \frac{u^2+2u-1}{2u^2(1-u)^2}.$$

Since f is concave on $\left(0, \sqrt{2}-1\right]$ and $s = \frac{x_1+x_2+\cdots+x_n}{n} = \frac{1}{n} < \sqrt{2}-1$ (for $n \ge 3$), f is also concave on $[0, s]$ By LCF–Theorem, it is enough to show that $(n-1)f(x)+f(y) \le nf\left(\frac{1}{n}\right)$ for $0 < x \le \frac{1}{n} \le y$ and $(n-1)x+y = 1$. Write this inequality as

$$n^{\frac{n}{2}}(1-x)^{n-1} \ge (n-1)^{n-1}x^{\frac{n-3}{2}}y^{\frac{1}{2}}.$$

By squaring, it becomes

$$(2-2x)^{2n-2} \ge (2n-2)^{2n-2}\frac{1}{n^n}\,x^{n-3}y$$

Since

$$2-2x = n\frac{1}{n} + (n-3)x + y,$$

this inequality follows from the AM-GM Inequality. Equality in the original inequality occurs for $x_1 = x_2 = \cdots = x_n = \frac{1}{n}$ □

Remark 1. According to Remark 8, inequality (5) holds if $f'(x) \geq f'(y)$ for $0 < x \leq \frac{1}{n} \leq y$ and $(n-1)x + y = 1$ Indeed, we have

$$f'(x) - f'(y) = \frac{1}{2x} + \frac{1}{1-x} - \frac{1}{2y} - \frac{1}{1-y} = \frac{(y-x)(1-x-y-xy)}{2xy(1-x)(1-y)} =$$
$$= \frac{x(y-x)(n-2-y)}{2xy(1-x)(1-y)} \geq \frac{x(y-x)(1-y)}{2xy(1-x)(1-y)} =$$
$$= \frac{(n-1)x^2(y-x)}{2xy(1-x)(1-y)} \geq 0.$$

Remark 2. Inequality (5) can be written as

$$\prod_{i=1}^{n}\left(\frac{1}{\sqrt{x_i}} - 1\right)\prod_{i=1}^{n}(1+\sqrt{x_i}) \geq \left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^n$$

On the other hand, by the AM-GM Inequality and Jensen's Inequality, we have

$$\prod_{i=1}^{n}(1+\sqrt{x_i}) \leq \left(1+\frac{1}{n}\sum_{i=1}^{n}\sqrt{x_i}\right)^n \leq \left(1+\sqrt{\frac{1}{n}\sum_{i=1}^{n}x_i}\right)^n = \left(1+\frac{1}{\sqrt{n}}\right)^n.$$

Thus, the following result follows·

- *If* $x_1, x_2, \ldots, x_n$ $(n \geq 3)$ *are positive numbers such that*

$$x_1 + x_2 + \cdots + x_n = 1,$$

then

$$\left(\frac{1}{\sqrt{x_1}} - 1\right)\left(\frac{1}{\sqrt{x_2}} - 1\right) \cdots \left(\frac{1}{\sqrt{x_n}} - 1\right) \geq \left(\sqrt{n} - 1\right)^n.$$

Remark 3. By squaring, inequality (5) becomes

$$\prod_{i=1}^{n}\left(\frac{1}{x_i} + x_i - 2\right) \geq \left(\frac{1}{n} + n - 2\right)^n. \tag{6}$$

Since the function $f(x) = \ln\frac{1+x}{1-x}$ is convex for $0 < x < 1$, by Jensen's Inequality we get

$$\prod_{i=1}^{n}\frac{1+x_i}{1-x_i} \geq \left(\frac{1+\dfrac{x_1+x_2+\ldots x_n}{n}}{1-\dfrac{x_1+x_2+\ldots x_n}{n}}\right)^n = \left(\frac{n+1}{n-1}\right)^n.$$

Multiplying this inequality and (6), we obtain the inequality of *Kee-Wai Lau* (Crux Mathematicorum, 2000):

- *If* $x_1, x_2, \ldots, x_n$ $(n \geq 3)$ *are positive numbers such that*

$$x_1 + x_2 + \cdots + x_n = 1,$$

then

$$\left(\frac{1}{x_1} - x_1\right)\left(\frac{1}{x_2} - x_2\right)\cdots\left(\frac{1}{x_n} - x_n\right) \geq \left(n - \frac{1}{n}\right)^n$$

★

10. *If* x, y, z *are non-negative real numbers, no two of them are zero, then*

$$\sqrt{1 + \frac{48x}{y+z}} + \sqrt{1 + \frac{48y}{z+x}} + \sqrt{1 + \frac{48z}{x+y}} \geq 15.$$

Proof. Since the inequality is homogeneous, we may assume that $x+y+z=1$. Under this supposition, the inequality becomes

$$\sqrt{\frac{1+47x}{1-x}} + \sqrt{\frac{1+47y}{1-y}} + \sqrt{\frac{1+47z}{1-z}} \geq 15$$

To prove this inequality, we will apply RCF–Theorem to the function $f(u) = \sqrt{\frac{1+47u}{1-u}}$, $0 \leq u < 1$. From the second derivative

$$f''(u) = \frac{48(47u - 11)}{\sqrt{(1-u)^5(1+47u)^3}},$$

it follows that f is convex on $\left[\frac{11}{47}, 1\right)$. Therefore, f is convex on $[s, 1)$, where $s = \frac{x+y+z}{3} = \frac{1}{3}$. By RCF–Theorem, it suffices to consider $x \leq y = z$. In this case, the problem reduces to show that $0 \leq x \leq \frac{1}{3}$ implies

$$\sqrt{\frac{1+47x}{1-x}} + 2\sqrt{\frac{49-47x}{1+x}} \geq 15$$

Setting $t = \sqrt{\frac{49-47x}{1+x}}$ $(5 \leq t \leq 7)$, the inequality transforms into

$$\sqrt{\frac{1175 - 23t^2}{t^2 - 1}} \geq 15 - 2t.$$

By squaring, the inequality becomes

$$350 - 15t - 61t^2 + 15t^3 - t^4 \geq 0,$$

or

$$(t-5)^2(t+2)(7-t) \geq 0,$$

which is clearly true.
Equality occurs when $(x,y,z) \sim (1,1,1)$, and also when $(x,y,z) \sim (0,1,1)$ or any cyclic permutation. . □

★

11. *Let x, y, z be non-negative real numbers, no two of them are zero. If $r \geq r_0$, where $r_0 = \frac{\ln 3}{\ln 2} - 1 \approx 0.585$, then*

$$\left(\frac{2x}{y+z}\right)^r + \left(\frac{2y}{z+x}\right)^r + \left(\frac{2z}{x+y}\right)^r \geq 3.$$

Proof. We distinguish three cases.

Case $r = 1$. The inequality reduces to the well-known inequality

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}.$$

Case $r > 1$. The inequality follows by Jensen's Inequality applied to the concave function $f(u) = u^r$

$$\left(\frac{2x}{y+z}\right)^r + \left(\frac{2y}{z+x}\right)^r + \left(\frac{2z}{x+y}\right)^r \geq 3\left(\frac{\frac{2x}{y+z} + \frac{2y}{z+x} + \frac{2z}{x+y}}{3}\right)^r \geq 3.$$

Case $r_0 \leq r < 1$. Since the inequality is homogeneous, we may assume that $x + y + z = 1$ and write the inequality in the form

$$f(x) + f(y) + f(z) \geq 3f\left(\frac{x+y+z}{3}\right),$$

where $f(u) = \left(\frac{2u}{1-u}\right)^r$, $0 \leq u < 1$. From the second derivative

$$f''(u) = \frac{4r}{(1-u)^4}\left(\frac{2u}{1-u}\right)^{r-2}(2u + r - 1),$$

it follows that f is convex on $\left[\frac{1-r}{2}, 1\right)$ Therefore, f is convex on $[s, 1)$, where $s = \frac{x+y+z}{3} = \frac{1}{3} > \frac{1-r}{2}$. By RCF–Theorem, it suffices to consider $x \le y = z$ It is convenient to return to the original inequality (leaving aside the constraint $x + y + z = 1$) and to consider $y = z = 1$ (which implies $0 \le x \le 1$) Thus, the problem reduces to show that $0 \le x \le 1$ implies $h(x) \ge 3$, where

$$h(x) = x^r + 2\left(\frac{2}{x+1}\right)^r$$

The derivative

$$h'(x) = rx^{r-1} - r\left(\frac{2}{x+1}\right)^{r+1}$$

has for $0 < x \le 1$ the same sign as the function

$$g(x) = (r-1)\ln x - (r+1)\ln\frac{2}{x+1}$$

From $g'(x) = \frac{2rx + r - 1}{x(x+1)}$, it follows that $g'(x) = 0$ for $x_0 = \frac{1-r}{2r} < 1$, $g'(x) > 0$ for $x \in (0, x_0)$ and $g'(x) > 0$ for $x \in (x_0, 1]$. Then, the function $g(x)$ is strictly decreasing for $x \in (0, x_0]$ and strictly increasing for $x \in [x_0, 1]$. Since $\lim_{x\to 0} g(x) = \infty$ and $g(1) = 0$, there exists $x_1 \in (0, x_0)$ such that $g(x_1) = 0$, $g(x) > 0$ for $x \in (0, x_1)$ and $g(x) < 0$ for $x \in (x_1, 1)$, hence, $h'(x_1) = 0$, $h'(1) = 0$, $h'(x) > 0$ for $x \in (0, x_1)$ and $h'(x) < 0$ for $x \in (x_1, 1)$ Therefore, the function $h(x)$ is strictly increasing for $x \in [0, x_1]$ and strictly decreasing for $x \in [x_1, 1]$ Since $h(0) = 2^{r+1} \ge 2^{r_0+1} = 3$ and $h(1) = 3$, it follows that $h(x) \ge 3$ for $0 \le x \le 1$

Equality occurs when $(x, y, z) \sim (1, 1, 1)$ Moreover, for $r = r_0$, equality holds again when $(x, y, z) \sim (0, 1, 1)$ or any cyclic permutation □

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12. *Let x, y, z be non-negative real numbers such that $x + y + z = 3$ If $0 < r \le r_0$, where $r_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1\ 71$, then*

$$x^r(y+z) + y^r(z+x) + z^r(x+y) \le 6.$$

Proof. We have three case to consider.

Case $r = 1$ The inequality reduces to the well-known inequality

$$3(xy + yz + zx) \leq (x + y + z)^2$$

Case $0 < r < 1$ The inequality follows by Jensen's Inequality applied to the concave function $f(u) = \sqrt{u}$

$$\begin{aligned}
&(y+z)x^r + (z+x)y^r + (x+y)z^r \leq \\
&\leq 2(x+y+z)\left[\frac{(y+z)x + (z+x)y + (x+y)z}{2(x+y+z)}\right]^r = \\
&= 6\left(\frac{xy+yz+zx}{3}\right)^r \leq 6\left(\frac{x+y+z}{3}\right)^{2r} = 6.
\end{aligned}$$

Case $1 < r \leq r_0$. We may write the inequality in the form

$$f(x) + f(y) + f(z) \geq 3f\left(\frac{x+y+z}{3}\right),$$

where $f(u) = u^r(u-3)$, $0 \leq u \leq 3$ From

$$f''(u) = ru^{r-2}\left[(r+1)u - 3(r-1)\right],$$

it follows that f is convex on $\left[\frac{3r-3}{r+1}, 3\right]$. Since

$$s = \frac{x+y+z}{3} = 1 > \frac{3r-3}{r+1},$$

f is also convex on $[s, 3]$ By RCF–Theorem, it is enough to consider $x \leq y = z$. It is convenient to write the inequality in the homogeneous form

$$6\left(\frac{x+y+z}{3}\right)^{r+1} \geq x^r(y+z) + y^r(z+x) + z^r(x+y),$$

to leave aside the constraint $x + y + z = 3$ and to consider $y = z = 1$ (which implies $0 \leq x \leq 1$) The inequality reduces to $g(x) \geq 0$, where

$$g(x) = 3\left(\frac{x+2}{3}\right)^{r+1} - x^r - x - 1.$$

We have

$$g'(x) = (r+1)\left(\frac{x+2}{3}\right)^r - rx^{r-1} - 1,$$

$$\frac{1}{r}g''(x) = \frac{r+1}{3}\left(\frac{x+2}{3}\right)^{r-1} - \frac{r-1}{x^{2-r}}.$$

Since g'' is strictly increasing on $(0,1]$, $g''(0)=-\infty$ and $\frac{1}{r}g''(1)=\frac{2(2-r)}{3}>0$, there exists $x_1 \in (0,1)$ such that $g''(x_1)=0$, $g''(x)<0$ for $x \in (0,x_1)$, and $g''(x)>0$ for $x \in (x_1,1]$. Therefore, the function $g'(x)$ is strictly decreasing for $x \in [0,x_1]$, and strictly increasing for $x \in [x_1,1]$. Since

$$g'(0)=(r+1)\left(\frac{2}{3}\right)^r-1 \geq (r+1)\left(\frac{2}{3}\right)^{r_0}-1=\frac{r+1}{2}-1=\frac{r-1}{2}>0$$

and $g'(1)=0$, there exists $x_2 \in (0,x_1)$ such that $g'(x_2)=0$, $g'(x)>0$ for $x \in [0,x_2)$, and $g'(x)<0$ for $x \in (x_2,1)$. Thus, the function $g(x)$ is strictly increasing for $x \in [0,x_2]$, and strictly decreasing for $x \in [x_2,1]$. Since $g(0)=3\left(\frac{2}{3}\right)^{r+1}-1=2\left(\frac{2}{3}\right)^r-1 \geq 2\left(\frac{2}{3}\right)^{r_0}-1=0$ and $g(1)=0$, it follows that $g(x) \geq 0$ for $0 \leq x \leq 1$.

Equality occurs when $(x,y,z)=(1,1,1)$. Moreover, for $r=r_0$, equality holds again when $(x,y,z)=\left(0,\frac{3}{2},\frac{3}{2}\right)$ or any cyclic permutation. □

★

13. *If $x_1, x_2, \ldots, x_n < 1$ are non-negative real numbers such that*

$$\frac{x_1+x_2+\cdots+x_n}{n}=r \geq \frac{1}{3},$$

then

$$\frac{\sqrt{x_1}}{1-x_1}+\frac{\sqrt{x_2}}{1-x_2}+\cdots+\frac{\sqrt{x_n}}{1-x_n} \geq \frac{n\sqrt{r}}{1-r}.$$

Proof. Apply RCF-Theorem to the function $f(u)=\frac{\sqrt{u}}{1-u}$, $0 \leq u < 1$. From

$$f''(u)=\frac{3u^2+6u-1}{4u\sqrt{u}(1-u)^3},$$

it follows that f is convex on $\left[\frac{2}{\sqrt{3}}-1,1\right)$. Since $s=\frac{1}{3}>\frac{2}{\sqrt{3}}-1$, the function f is convex on $[s,1)$. By RCF-Theorem and Remark 2, it suffices to show that $g(x) \leq g(y)$ for $0 \leq x \leq s \leq y < 1$ and $x+(n-1)y=ns$, where $g(t)=\frac{f(t)-f(s)}{t-s}$. For convenience, let $a=\sqrt{x}$, $b=\sqrt{y}$ and $c=\sqrt{s}$. We have

$$g(t^2)=\frac{f(t^2)-f(c^2)}{t^2-c^2}=\frac{1+ct}{(1-c^2)(1-t^2)(t+c)},$$

and

$$g(x) - g(y) = g(a^2) - g(b^2) =$$
$$= (a^2 - b^2)\frac{a^2+b^2+c(a+b)+c^2-1+ab(1+c^2)+abc(a+b)}{(1-c^2)(1-a^2)(1-b^2)(a+c)(b+c)}.$$

Since

$$a^2 + b^2 + c(a+b) + c^2 - 1 + ab(1+c^2) + abc(a+b) \geq$$
$$\geq a^2 + b^2 + c(a+b) + c^2 - 1 \geq$$
$$\geq a^2 + b^2 + c\sqrt{a^2+b^2} + c^2 - 1,$$

it is enough to show that

$$x + y + \sqrt{s(x_y)} + s - 1 \geq 0.$$

Indeed, we have

$$x + y = \frac{ns + (n-2)x}{n-1} \geq \frac{ns}{n-1},$$

and therefore,

$$x+y+\sqrt{s(x+y)}+s-1 \geq \left(\frac{n}{n-1} + \sqrt{\frac{n}{n-1}+1}\right) s-1 > 3s-1 = 0.$$

Equality occurs only for $x_1 = x_2 = \cdots = x_n = r$. □

Remark. From the final part of the proof it follows that the inequality holds for the larger condition

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = r \geq \frac{1}{\dfrac{n}{n-1} + \sqrt{\dfrac{n}{n-1}+1}}.$$

In the case $r = \dfrac{1}{\dfrac{n}{n-1} + \sqrt{\dfrac{n}{n-1}+1}}$, equality occurs again when one of x_1 is equal to 0 and the other ones are equal to $\dfrac{nr}{n-1}$.

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14. *If a, b, c are non-negative real numbers such that $a+b+c=3$, then*

$$(1-a+a^2)(1-b+b^2)(1-c+c^2) \geq 1$$

Proof. We may write the inequality in the form

$$f(a)+f(b)+f(c) \leq 3f\left(\frac{a+b+c}{3}\right),$$

where $f(u) = -\ln(1-u+u^2)$, $0 \leq u \leq 3$ We have

$$f'(u) = \frac{1-2u}{1-u+u^2}, \quad f''(u) = \frac{2u^2-2u-1}{(1-u+u^2)^2}$$

Since f is concave on $\left[0, \frac{1+\sqrt{3}}{2}\right]$ and $s = \frac{a+b+c}{3} = 1$, f is also concave on $[0, s]$ Therefore, according to LCF–Theorem and Remark 8, it is enough to show that $f'(x) \geq f'(y)$ for $0 \leq x \leq 1 \leq y$ and $2x+y=3$ Indeed, we have

$$f'(x)-f'(y) = \frac{(y-x)(1+x+y-2xy)}{(1-x-x^2)(1-y-y^2)} = \frac{(y-x)(4x^2-7x+4)}{(1-x-x^2)(1-y-y^2)} \geq$$
$$\geq \frac{(y-x)(4x^2-8x+4)}{(1-x-x^2)(1-y-y^2)} = \frac{4(y-x)(x-1)^2}{(1-x-x^2)(1-y-y^2)} \geq 0.$$

This completes the proof Equality occurs only for $a=b=c=1$ □

Remark 1. *Marian Tetiva* found for this inequality a nice elementary solution He noticed that among the numbers a, b, c always exist two (let b and c) which are either less or equal to 1, or larger or equal to 1; that is $(b-1)(c-1) \geq 0$ Thus,

$$(1-b+b^2)(1-c+c^2) \geq (b^2-b)(c^2-c)+b^2+c^2-b-c+1 \geq$$
$$\geq b^2+c^2-b-c+1 \geq$$
$$\geq \frac{1}{2}(b+c)^2-(b+c)+1 = \frac{a^2-4a+5}{2}$$

and hence,

$$(1-a+a^2)(1-b+b^2)(1-c+c^2)-1 \geq \frac{(a^2-a+1)(a^2-4a+5)}{2}-1 =$$
$$= \frac{(a-1)^2(a^2-3a+3)}{2} \geq 0$$

Remark 2. Actually, the following more general statement holds.

- *Let $x_1, x_2, \ldots, x_n$ be non-negative numbers such that*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = r \geq 1$$

If $n \leq 13$, then

$$\left(1 - x_1 + x_1^2\right)\left(1 - x_2 + x_2^2\right)\cdots\left(1 - x_n + x_n^2\right) \geq (1 - r + r^2)^n.$$

We can prove this statement for $n \leq 10$ by following the same way as for $n = 3$. We must only show that $1 + x + y - 2xy \geq 0$ for $0 \leq x \leq 1 \leq y$ and $(n-1)x + y = n$. Indeed, for $0 \leq x \leq \frac{1}{2}$ we have

$$1 + x + y - 2xy = 1 + x + y(1 - 2x) > 0,$$

and for $\frac{1}{2} < x \leq 1$ we have

$$\begin{aligned} 1 + x + y - 2xy &= 1 + 2x - 2x^2 - n(2x - 1)(1 - x) \geq \\ &\geq 1 + 2x - 2x^2 - 10(2x - 1)(1 - x) = \\ &= 18x^2 - 28x + 11 = 2\left(3x - \frac{7}{3}\right)^2 + \frac{1}{9} > 0 \end{aligned}$$

Remark 3. *Bin Zhao* posted on Mathlink Site, in November 2005, the following *conjecture*:

- *If a, b, c are non-negative numbers such that $a + b + c = 3$, then*

$$(1 - a + a^p)(1 - b + b^p)(1 - c + c^p) \geq 1$$

for any $p > 1$.

★

15. *If $x_1, x_2, \ldots, x_n$ are non-negative numbers such that $x_1 + x_2 + \cdots + x_n = n$, then*

$$\frac{1}{n - x_1 + x_1^2} + \frac{1}{n - x_2 + x_2^2} + \cdots + \frac{1}{n - x_n + x_n^2} \leq 1.$$

Proof. We may write the inequality in the form

$$f(x_1) + f(x_2) + \cdots + f(x_n) \leq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right),$$

where $f(u) = \dfrac{1}{n-u+u^2}$, $u \geq 0$ We have

$$f'(u) = \frac{1-2u}{(n-u+u^2)^2} \quad \text{and} \quad f''(u) = \frac{6u(u-1)+2(1-n)}{(n-u+u^2)^3}$$

Since $f''(u) < 0$ for $0 \leq u \leq 1$, it follows that the function $f(u)$ is concave on $[0, s]$, where $s = \dfrac{x_1+x_2+\cdots+x_n}{n} = 1$ According to LCF–Theorem and Remark 7, it is enough to show that $g(x) \geq g(y)$ for $0 < x \leq 1 \leq y$ and $(n-1)x + y = n$, where $g(t) = \dfrac{f(t)-f(1)}{t-1}$
Indeed, we have

$$g(t) = \frac{-t}{n(n-t+t^2)}$$

and

$$g(x) - g(y) = \frac{(y-x)(n-xy)}{n(n-x+x^2)(n-y+y^2)} \geq 0,$$

because $n - xy \geq n - y = (n-1)x \geq 0$. This completes the proof.
Equality occurs for $x_1 = x_2 = \cdots = x_n = 1$ □

Conjecture *If $x_1, x_2, \ldots, x_n$ are non-negative numbers such that*

$$x_1 + x_2 + \cdots + x_n = n,$$

then for any $p > 1$ the inequality holds

$$\frac{1}{n-x_1+x_1^p} + \frac{1}{n-x_2+x_2^p} + \cdots + \frac{1}{n-x_n+x_n^p} \leq 1$$

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16. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$1 + a + b + c \geq 2\sqrt{1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}}.$$

Proof. By squaring, the inequality becomes

$$a^2 + b^2 + c^2 + 2(a+b+c) \geq 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + 3,$$

or

$$f(a) + f(b) + f(c) \geq 3f\left(\sqrt[3]{abc}\right),$$

where $f(t) = t^2 + 2t - \frac{2}{t}$, $t > 0$. To prove this inequality, we will apply RCF–Corollary for $r = 1$ Let

$$f_1(u) = f(e^u) = e^{2u} + 2e^u - 2e^{-u}.$$

From the second derivative $f_1''(u) = 2e^{-u}\left(2e^{3u} + e^{2u} - 1\right)$, it follows that $f_1(u)$ is convex for $u \geq \ln r = 0$. According to RCF–Corollary, we need to show that $f(x) + 2f(y) \geq 3f(1)$ for $0 < x \leq 1 \leq y$ and $xy^2 = 1$. This inequality is equivalent to each of the following

$$x^2 + 2x - \frac{2}{x} + 2y^2 + 4y - \frac{4}{y} \geq 3,$$

$$4y^5 - 3y^4 - 4y^3 + 2y^2 + 1 \geq 0,$$

$$(y-1)^2(y+1)(4y^2 + y + 1) \geq 0.$$

The last inequality is clearly true.
Equality occurs if and only if $a = b = c = 1$. □

Remark *Marian Tetiva* noticed that

$$a^2 + b^2 + c^2 + 2(a+b+c) - 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 3 =$$
$$= a^2 + b^2 + c^2 + 2(a+b+c) - 2(ab + bc + ca) - 3 =$$
$$= (b-c)^2 + (a-1)^2 + 2(1-a)(b+c-2) \geq 0,$$

because the allowable assumption $a \leq b \leq c$ yields $1 - a \geq 0$ and

$$b + c - 2 \geq 2\sqrt{bc} - 2 = 2\left(\frac{1}{\sqrt{a}} - 1\right) \geq 0.$$

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17. *If a, b, c, d are positive real numbers such that $abcd = 1$, then*

$$(a-1)(a-2) + (b-1)(b-2) + (c-1)(c-2) + (d-1)(d-2) \geq 0.$$

Proof. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f\left(\sqrt[4]{abcd}\right),$$

where $f(t) = (t-1)(t-2)$, $t > 0$, and apply RCF–Corollary for $r = 1$ Let

$$f_1(u) = f(e^u) = (e^u - 1 \quad e^u - 2 .$$

From the second derivative $f_1''(u) = e^u(4e^u - 3)$, it follows that $f_1(u)$ is convex for $u \ge \ln r = 0$. According to RCF–Corollary, we need to show that $f(x) + 3f(y) \ge 4f(1)$ for $x \le 1 \le y$ and $xy^3 = 1$. This inequality is equivalent to

$$\left(\frac{1}{y^3} - 1\right)\left(\frac{1}{y^3} - 2\right) + 3(y-1)(y-2) \ge 0$$

We may write it as

$$(y-1)^2\left[y^3(y-1)(3y^2-1) + 3y^2 + 2y + 1\right] \ge 0,$$

which is clearly true. Equality occurs if and only if $a = b = c = d = 1$ □

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18. *If $a_1, a_2, \ldots, a_n$ $(n \ge 4)$ are positive real numbers such that $a_1 a_2 \ldots a_n = 1$, then*

$$(n-1)\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) + n(n+3) \ge (2n+2)(a_1 + a_2 + \cdots + a_n).$$

Proof. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\sqrt[n]{a_1 a_2 \cdots a_n}\right),$$

where $f(t) = (n-1)t^2 - (2n+2)t + n + 3$, $t > 0$, and apply RCF–Corollary for $r = 1$ Let

$$f_1(u) = f(e^u) = (n-1)e^{2u} - (2n+2)e^u + n + 3.$$

From the second derivative $f_1''(u) = 2e^u\left[(2n-2)e^u - n - 1\right]$, it follows that $f_1(u)$ is convex for $u \ge \ln r = 0$. According to RCF–Corollary and Remark 5, it suffices to show that $xf'(x) \le yf'(y)$ for $x \le 1 \le y$ and $xy^{n-1} = 1$. Since

$$\begin{aligned} xf'(x) - yf'(y) &= 2(n-1)x^2 - (2n+2)x - 2(n-1)y^2 + (2n+2)y = \\ &= 2(x-y)[(n-1)(x+y) - n - 1], \end{aligned}$$

we need to show that $x + y \ge \dfrac{n+1}{n-1}$. By the AM-GM Inequality, we have

$$x + y = x + \frac{y}{n-1} + \cdots + \frac{y}{n-1} \ge n\sqrt[n]{\frac{xy^{n-1}}{(n-1)^{n-1}}} = \frac{n\sqrt[n]{n-1}}{n-1}.$$

Therefore, it suffices to show that

$$n\sqrt[n]{n-1} \ge n+1$$

This inequality is equivalent to

$$n-1 \ge \left(1+\frac{1}{n}\right)^n$$

It is true because for $n \ge 4$ we have

$$n-1 \ge 3 > \left(1+\frac{1}{n}\right)^n$$

Equality occurs for $a_1 = a_2 = \cdots = a_n = 1$. □

Remark Using the same way, we can prove the following sharper statement

- *If $a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1 a_2 \ldots a_n = 1$, then*

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \ge \frac{2n\sqrt[n]{n-1}}{n-1}(a_1 + a_2 + \cdots + a_n - n).$$

(*Gabriel Dospinescu* and *Călin Popa*)

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19. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \ldots a_n = 1$, then*

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \ge (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

Proof. We write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\sqrt[n]{a_1 a_2 \ldots a_n}\right),$$

where $f(t) = t^{n-1} - \dfrac{n-1}{t}$, $t > 0$ Let

$$f_1(u) = f(e^u) = e^{(n-1)u} - (n-1)e^{-u}.$$

From the second derivative

$$f_1''(u) = (n-1)^2 e^{(n-1)u} - (n-1)e^{-u} = (n-1)e^{-u}\left[(n-1)e^{nu} - 1\right],$$

it follows that $f_1(u)$ is convex for $u \ge \ln r = 0$, where $r = 1$.
By RCF–Corollary and Remark 5, it suffices to show that $xf'(x) \le yf'(y)$ for $0 < x \le 1 \le y$ and $xy^{n-1} = 1$. We have

$$tf'(t) = (n-1)t^{n-1} + \frac{n-1}{t},$$

and

$$yf'(y) - xf'(x) = (n-1)y^{n-1} + \frac{n-1}{y} - (n-1)x^{n-1} - \frac{n-1}{x} =$$

$$= \frac{n-1}{y} - (n-1)x^{n-1} = \frac{(n-1)\left(y^{n^2-2n}-1\right)}{y^{(n-1)^2}} \geq 0.$$

Equality occurs for $a_1 = a_2 = \cdots = a_n = 1$. □

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20. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \ldots a_n = 1$. If $m \geq n$, then*

$$a_1^m + a_2^m + \cdots + a_n^m + mn \geq (m+1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

Proof. We write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\sqrt[n]{a_1 a_2 \ldots a_n}\right),$$

where $f(t) = t^m - \dfrac{m+1}{t}$, $t > 0$. Let

$$f_1(u) = f(e^u) = e^{mu} - (m+1)e^{-u}.$$

From the second derivative

$$f_1''(u) = m^2 e^{mu} - (m+1)e^{-u} = e^{-u}\left[m^2 e^{(m+1)u} - m - 1\right],$$

it follows that $f_1(u)$ is convex for $u \geq \ln r = 0$, where $r = 1$, because

$$m^2 e^{(m+1)u} - m - 1 \geq m^2 - m - 1 > 0.$$

According to LCF–Corollary, it suffices to show that the given inequality is true for $a_2 = a_3 = \cdots = a_n \geq 1$, that is to prove that

$$x^m + (n-1)y^m + mn - \frac{m+1}{x} - \frac{(m+1)(n-1)}{y} \geq 0$$

for $0 < x \leq 1 \leq y$ and $xy^{n-1} = 1$. By the weighted AM-GM Inequality, we have

$$x^m + (mn - m - 1) \geq m(n-1)\sqrt[n-1]{x} = \frac{m(n-1)}{y}.$$

Then, we still have to show that

$$(n-1)\left(y^m - \frac{1}{y}\right) - (m+1)\left(\frac{1}{x} - 1\right) \ge 0.$$

This inequality is equivalent to $h(y) \ge 0$ for $y \ge 1$, where

$$h(y) = (n-1)(y^{m+1} - 1) - (m+1)(y^n - y).$$

Since

$$\begin{aligned}\frac{h'(y)}{m+1} &= (n-1)y^m - ny^{n-1} + 1 \ge (n-1)y^n - ny^{n-1} + 1 = \\ &= ny^{n-1}(y-1) - (y^n - 1) = \\ &= (y-1)\left[(y^{n-1} - y^{n-2}) + (y^{n-1} - y^{n-3}) + \cdots + (y^{n-1} - 1)\right] \ge 0,\end{aligned}$$

the function $h(y)$ is increasing. Therefore, $h(y) \ge h(1) = 0$. Equality occurs for $a_1 = a_2 = \cdots = a_n = 1$. □

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21. *If* $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ *are positive real numbers such that*

$$\sqrt[n]{a_1 a_2 \cdots a_n} = p \ge \sqrt{n} - 1,$$

then

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} \ge \frac{n}{(1+p)^2}. \tag{7}$$

Proof. We will apply RCF-Corollary to the function $f(t) = \dfrac{1}{(1+t)^2}$, $t > 0$. First we must show that the function $f_1(u) = f(e^u) = \dfrac{1}{(1+e^u)^2}$ is convex for $u \ge \ln r$, where $r = \sqrt{n} - 1$. Indeed, the second derivative is given by

$$f_1''(u) = \frac{2e^u(2e^u - 1)}{(1+e^u)^4},$$

and for $u \ge \ln r$, we have

$$2e^u - 1 \ge 2r - 1 = 2 \qquad - 3 > 0.$$

Therefore, we have to show that (7) is true for $a_2 = a_3 = \cdot \quad = a_n \geq r$ and $a_1 a_2 \quad a_n = r^n$; that is to prove that $h(x) \geq h(r)$ for $x \geq r$, where

$$h(x) = \frac{x^{2n-2}}{(x^{n-1}+r^n)^2} + \frac{n-1}{(1+x)^2}.$$

The derivative

$$h'(x) = \frac{2(n-1)r^n x^{2n-3}}{(x^{n-1}+r^n)^3} - \frac{2(n-1)}{(x+1)^3},$$

has the same sign as the function

$$h_1(x) = r^{\frac{n}{3}} x^{\frac{2n}{3}-1}(x+1) - x^{n-1} - r^n$$

Let $m = \dfrac{n}{3}$, $m \geq 1$ We see that

$$h_1(x) = r^m\left(x^{2m}+x^{2m-1}\right) - x^{3m-1} - r^{3m} =$$
$$= (x^m - r^m)\left(r^m x^m + r^{2m} - x^{2m-1}\right) = x^m(x^m - r^m)\, h_2(x),$$

where

$$h_2(x) = r^m + \frac{r^{2m}}{x^m} - x^{m-1}.$$

Since $h_2(x)$ is strictly decreasing for $x \geq r$,

$$h_2(r) = r^{m-1}(2r-1) = r^{m-1}\left(2\sqrt{n}-3\right) > 0$$

and $h_2(\infty) < 0$ ($h_2(\infty) = r - 1 = \sqrt{3} - 2$ for $m = 1$, and $h_2(\infty) = -\infty$ for $m > 1$), there exists $x_1 > r$ such that $h_2(x_1) = 0$, $h_2(x) > 0$ for $r \leq x < x_1$, and $h_2(x) < 0$ for $x > x_1$ Since the functions $h_1(x)$ and $h'(x)$ have the same sign as $h_2(x)$ for $x > r$, we may say that the continuous function $h(x)$ is strictly increasing for $r \leq x \leq x_1$, and strictly decreasing for $x \geq x_1$; consequently, $h(x) \geq \min\{h(r), h(\infty)\}$ Since $h(r) = h(\infty) = 1$, we get $h(x) \geq h(r)$ for $x \geq r$, and the proof is complete Equality occurs for $a_1 = a_2 = \quad = a_n = p$ □

Remark We can rewrite inequality (7) as follows.

• *Let $a_1, a_2, \ \ ., a_n$ ($n \geq 3$) be positive numbers such that $a_1 a_2 \ ..\, a_n = 1$, and let $p \geq \sqrt{n} - 1$. Then*

$$\frac{1}{(1+pa_1)^2} + \frac{1}{(1+pa_2)^2} + \cdots + \frac{1}{(1+pa_n)^2} \geq \frac{n}{(1+p)^2}.$$

For $n = 4$ and $p = 1$, we get the well-known statement:

- *If a, b, c, d are positive numbers such that $abcd = 1$, then*

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2}\geq 1$$

(*Vasile Cîrtoaje*, GM-B, 11, 1999)

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22. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$\sqrt[n]{a_1a_2\ldots a_n}=p\geq n^2-1,$$

then

$$\frac{1}{\sqrt{1+a_1}}+\frac{1}{\sqrt{1+a_2}}+\cdots+\frac{1}{\sqrt{1+a_n}}\geq\frac{n}{\sqrt{1+p}} \tag{8}$$

Proof. We will apply RCF-Corollary to the function $f(t)=\dfrac{1}{\sqrt{1+t}}$, $t>0$.

First we must show that the function $f_1(u) = f(e^u) = \dfrac{1}{\sqrt{1+e^u}}$ is convex for $u \geq \ln r$, where $r = n^2 - 1$ Indeed, the second derivative is given by

$$f_1''(u)=\frac{e^u(e^u-2)}{4(1+e^u)^{\frac{5}{2}}},$$

and for $u \geq \ln r$, we get

$$e^u-2\geq r-2=n^2-3>0.$$

We have now to show that (8) is true for $a_2 = a_3 = \cdots = a_n \geq r$ and $a_1a_2\ldots a_n = r^n$; that is to prove that $h(x) \geq h(r)$ for $x \geq r$, where

$$h(x)=\sqrt{\frac{x^{n+1}}{x^{n-1}+r^n}}+\frac{n-1}{\sqrt{1+x}}$$

The derivative

$$h'(x)=\frac{(n-1)r^nx^{\frac{n-3}{2}}}{2(x^{n-1}+r^n)^{\frac{3}{2}}}-\frac{n-1}{2(x+1)^{\frac{3}{2}}},$$

has the same sign as the function

$$h_1(x)=r^{\frac{2n}{3}}x^{\frac{n}{3}-1}(x+1\ -x^{n-1}-r^n.$$

Let $m = \frac{n}{3}$, $m \geq \frac{2}{3}$ We see that

$$h_1(x) = r^{2m}(x^m + x^{m-1}) - x^{3m-1} - r^{3m} =$$
$$= r^{2m}(x^m - r^m) + x^{m-1}(r^{2m} - x^{2m}) =$$
$$= (x^m - r^m)h_2(x)$$

where

$$h_2(x) = r^{2m} - r^m x^{m-1} - x^{2m-1}$$

We see that $h_2(x)$ is strictly decreasing for $x \geq r$,

$$h_2(r) = r^{2m-1}(r-2) = r^{2m-1}(n^2-3) > 0$$

and $h_2(\infty) < 0$ Then, there exists $x_1 > r$ such that $h_2(x_1) = 0$, $h_2(x) > 0$ for $r \leq x < x_1$, and $h_2(x) < 0$ for $x > x_1$ Since the functions $h_1(x)$ and $h'(x)$ have the same sign as $h_2(x)$ for $x > r$, the function $h(x)$ is strictly increasing for $r \leq x \leq x_1$, and strictly decreasing for $x \geq x_1$, consequently, $h(x) \geq \min\{h(r), h(\infty)\}$ Since $h(r) = h(\infty) = 1$, we get $h(x) \geq h(r)$ for $x \geq r$, and the proof is complete Equality occurs for $a_1 = a_2 = \quad = a_n = p$ □

Remark. Inequalities (7) and (8) are special cases of the more general statement

• *Let $n \geq 2$ be an integer, and let $k \leq n-1$ be a positive number. If $a_1, a_2, \quad, a_n$ are positive numbers satisfying $\sqrt[n]{a_1 a_2 \quad a_n} = p \geq n^{\frac{1}{k}} - 1$, then*

$$\frac{1}{(1+a_1)^k} + \frac{1}{(1+a_2)^k} + \cdot + \frac{1}{(1+a_n)^k} \geq \frac{n}{(1+p)^k}$$

(*Vasile Cîrtoaje*, GM-A, 2, 2005)

We can rewrite this statement as follows

• *Let $n \geq 2$ be an integer, and let $0 < k \leq n-1$ and $p \geq n^{\frac{1}{k}} - 1$. If $a_1, a_2, \quad, a_n$ are positive numbers satisfying $a_1 a_2 \quad a_n = 1$, then*

$$\frac{1}{(1+pa_1)^k} + \frac{1}{(1+pa_2)^k} + \quad \cdot + \frac{1}{(1+pa_n)^k} \geq \frac{n}{(1+p)^k}$$

An interesting corollary is the following

• *Let $n \geq 2$ be an integer, and let $0 < k \leq n-1$ and $p = n^{\frac{1}{k}} - 1$ If $a_1, a_2, \quad, a_n$ are positive numbers such that $a_1 a_2 \quad a_n = 1$, then*

$$\frac{1}{(1+pa_1)^k} + \frac{1}{(1+pa_2)^k} + \quad + \frac{1}{(1+pa_n)^k} \geq 1$$

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23. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$\sqrt[n]{a_1 a_2 \ldots a_n} = p \le \sqrt{\frac{n}{n-1}} - 1,$$

then

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} \le \frac{n}{(1+p)^2}. \tag{9}$$

Proof. We will apply LCF–Corollary to the function $g(t) = \dfrac{1}{(1+t)^2}$, $t > 0$. First we have to show that the function $f(u) = g(e^u) = \dfrac{1}{(1+e^u)^2}$ is concave for $u \le \ln r$, where $r = \sqrt{\dfrac{n}{n-1}} - 1$. Indeed, we have

$$f''(u) = \frac{2e^u(2e^u - 1)}{(1+e^u)^4},$$

and, for $u \le \ln r$,

$$2e^u - 1 \le 2r - 1 = 2\sqrt{\frac{n}{n-1}} - 3 \le 2\sqrt{2} - 3 < 0.$$

We need now to show that (9) is true for $a_1 = a_2 = \cdots = a_{n-1} \le r$ and $a_1 a_2 \ldots a_n = r^n$; that is to prove that $h(x) \le h(r)$ for $0 < x \le r$, where

$$h(x) = \frac{n-1}{(1+x)^2} + \frac{x^{2n-2}}{(x^{n-1}+r^n)^2}.$$

The derivative

$$h'(x) = \frac{2(n-1)r^n x^{2n-3}}{(x^{n-1}+r^n)^3} - \frac{2(n-1)}{(x+1)^3}$$

has the same sign as the function

$$h_1(x) = r^{\frac{n}{3}} x^{\frac{2n}{3}-1}(x+1) - x^{n-1} - r^n.$$

Let $m = \dfrac{n}{3}$, $m \ge \dfrac{2}{3}$. We have

$$h_1(x) = (r^m - x^m)(x^{2m-1} - r^m x^m - r^{2m}) = x^m(r^m - x^m)h_2(x),$$

where

$$h_2(x) = x^{m-1} - r^m - \frac{r^{2m}}{x^m}$$

Notice that $\lim_{x\to 0} h_2(x) = -\infty$ and

$$h_2(r) = r^{m-1}(1-2r) = r^{m-1}\left(3 - 2\sqrt{\frac{n}{n-1}}\right) > 0$$

In the case $n \geq 3$ $(m \geq 1)$, the function $h_2(x)$ is clearly strictly increasing for $0 < x \leq r$ It can be readily checked that this property is also valid for $n = 2$ $(m = \frac{2}{3})$ Thus, there is $x_1 \in (0, r)$ such that $h_2(x_1) = 0$, $h_2(x) < 0$ for $0 < x < x_1$, and $h_2(x) > 0$ for $x_1 < x \leq r$ Since the functions $h_1(x)$ and $h'(x)$ have the same sign as $h_2(x)$ for $0 < x < r$, the continuous function $h(x)$ is strictly decreasing for $0 \leq x \leq x_1$, and strictly increasing for $x_1 \leq x \leq r$, consequently, $h(x) \leq \max\{h(0), h(r)\}$ From $h(0) = h(r) = n - 1$, we obtain $h(x) \leq h(r)$ for $0 \leq x \leq r$, and the proof is finished
Equality occurs for $a_1 = a_2 = \quad = a_n = p$. □

Remark We can rewrite inequality (9) as follows:

- *Let $a_1, a_2, .\ ., a_n$ be positive numbers such that $a_1 a_2 \quad a_n = 1$, and let $p \leq \sqrt{\frac{n}{n-1}} - 1$. Then*

$$\frac{1}{(1+pa_1)^2} + \frac{1}{(1+pa_2)^2} + \cdots + \frac{1}{(1+pa_n)^2} \leq \frac{n}{(1+p)^2}$$

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24. *If $a_1, a_2, \quad , a_n$ $(n \geq 3)$ are positive real numbers such that*

$$\sqrt[n]{a_1 a_2 \ldots a_n} = p \leq \frac{2n-1}{(n-1)^2},$$

then

$$\frac{1}{\sqrt{1+a_1}} + \frac{1}{\sqrt{1+a_2}} + \cdot \quad + \frac{1}{\sqrt{1+a_n}} \leq \frac{n}{\sqrt{1+p}}. \tag{10}$$

Proof We will apply LCF-Corollary to the function $f(t) = \frac{1}{\sqrt{1+t}}$, $t > 0$

First we have to show that the function $f_1(u) = f(e^u) = \frac{1}{\sqrt{1+e^u}}$ is concave for $u \leq \ln r$, where $r = \dfrac{2n-1}{(n-1)^2}$. Indeed, the second derivative is given by

$$f_1''(u) = \frac{e^u(e^u - 2)}{4(1+e^u)^{\frac{5}{2}}},$$

and for $u \leq \ln r$, we get

$$e^u - 2 \leq r - 2 = \frac{-2n^2 + 6n - 3}{(n-1)^2} < \frac{2n(3-n)}{(n-1)^2} \leq 0.$$

We need now to show that (10) is true for $a_1 = a_2 = \cdots = a_{n-1} \leq r$ and $a_1 a_2 \ldots a_n = r^n$, that is to prove that $h(x) \leq h(r)$ for $0 < x \leq r$, where

$$h(x) = \frac{n-1}{\sqrt{1+x}} + \sqrt{\frac{x^{n-1}}{x^{n-1} + r^n}}$$

The derivative

$$h'(x) = \frac{(n-1)r^n x^{\frac{n-3}{2}}}{2(x^{n-1} + r^n)^{\frac{3}{2}}} - \frac{n-1}{2(x+1)^{\frac{3}{2}}},$$

has the same sign as the function

$$h_1(x) = r^{\frac{2n}{3}} x^{\frac{n}{3} - 1}(x+1) - x^{n-1} - r^n$$

Let $m = \dfrac{n}{3}$, $m \geq 1$. We see that

$$\begin{aligned} h_1(x) &= r^{2m}(x^m + x^{m-1}) - x^{3m-1} - r^{3m} = \\ &= r^{2m}(x^m - r^m) + x^{m-1}(r^{2m} - x^{2m}) = \\ &= (r^m - x^m)h_2(x) \end{aligned}$$

where

$$h_2(x) = x^{2m-1} + r^m x^{m-1} - r^{2m}.$$

Notice that $h_2(x)$ is strictly increasing for $0 \leq x \leq r$, $h_2(0) < 0$ and

$$h_2(r) = r^{2m-1}(2-r) > 0$$

Therefore, there exists $x_1 \in (0, r)$ such that $h_2(x_1) = 0$, $h_2(x) < 0$ for $0 \leq x < x_1$, and $h_2(x) > 0$ for $x_1 < x \leq r$. Since the functions $h_1(x)$ and $h'(x)$ have the same sign as $h_2(x)$ for $0 < x < r$, the function $h(x)$ is strictly decreasing for $0 \leq x \leq x_1$, and strictly increasing for $x_1 \leq x \leq r$; consequently, $h(x) \leq \max\{h(0), h(r)\}$. From $h(0) = h(r) = n - 1$, we obtain $h(x) \leq h(r)$ for $0 \leq x \leq r$, and the proof is finished. Equality occurs for $a_1 = a_2 = \cdots = a_n = p$ □

Remark Inequalities (9) and (10) are special cases of the more general statement:

• *Let $n \ge 2$ be an integer, and let $k \ge \frac{1}{n-1}$ be a positive number. If $a_1, a_2, \ldots, a_n$ are positive numbers satisfying*

$$\sqrt[n]{a_1 a_2 \ldots a_n} = p \le \left(\frac{n}{n-1}\right)^{\frac{1}{k}} - 1,$$

then the inequality holds

$$\frac{1}{(1+a_1)^k} + \frac{1}{(1+a_2)^k} + \cdots + \frac{1}{(1+a_n)^k} \le \frac{n}{(1+p)^k}$$

(*Vasile Cîrtoaje*, GM-A, 2, 2005)

We can rewrite this statement as follows:

• *Let $n \ge 2$ be an integer, and let $k \ge \frac{1}{n-1}$ and $0 < p \le \left(\frac{n}{n-1}\right)^{\frac{1}{k}} - 1$. If $a_1, a_2, \ldots, a_n$ are positive numbers satisfying $a_1 a_2 \ldots a_n = 1$, then*

$$\frac{1}{(1+pa_1)^k} + \frac{1}{(1+pa_2)^k} + \cdots + \frac{1}{(1+pa_n)^k} \le \frac{n}{(1+p)^k}$$

An interesting corollary is the following:

• *Let $n \ge 2$ be an integer, and let $k \ge \frac{1}{n-1}$ and $p = \left(\frac{n}{n-1}\right)^{\frac{1}{k}} - 1$. If $a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1 a_2 \ldots a_n = 1$, then*

$$\frac{1}{(1+pa_1)^k} + \frac{1}{(1+pa_2)^k} + \cdots + \frac{1}{(1+pa_n)^k} \le n-1$$

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25. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $\sqrt[n]{a_1 a_2 \ldots a_n} = p \ge 1$, then*

$$\frac{1}{1+a_1+\cdots+a_1^{n-1}} + \frac{1}{1+a_2+\cdots+a_2^{n-1}} + \cdots + \frac{1}{1+a_n+\cdots+a_n^{n-1}} \ge$$
$$\ge \frac{n}{1+p+\cdots+p^{n-1}}. \tag{11}$$

Proof We will apply RCF-Corollary to the function

$$f(t) = \frac{1}{1+t+\cdots+t^{n-1}},\ t > 0$$

First we have to show that the function $f_1(u) = f(e^u) = \dfrac{1}{1+e^u+\ \ ..+e^{(n-1)u}}$ is convex for $u \geq \ln r$, where $r = 1$; that is for $u \geq 0$ Setting $y = e^u$ $(y \geq 1)$, the necessary condition $f''(u) \geq 0$ reduces to

$$2\left[y + 2y^2 + \cdot\ \ \cdot + (n-1)y^{n-1}\right]^2 \geq$$
$$\geq \left[y + 2y^2 + \cdots + (n-1)^2 y^{n-1}\right]\left[1 + y + \cdot\cdot\ \ + y^{n-1}\right].$$

We will prove this inequality by induction over n. For $n = 2$, the inequality becomes $y(y-1) \geq 0$, which is clearly true. Suppose now that the inequality is true for n and prove it for $n+1$, $n \geq 2$. Using the inductive hypothesis, we still have to show that

$$n^2(y^n - 1) + a_1 y + a_2 y^2 + \ \ \cdot\cdot + a_{n-1}y^{n-1} \geq 0,$$

where $a_i = 3n^2 - (2n-i)^2$ Since

$$a_1 < a_2 < \ \ \cdot\cdot < a_{n-1} \text{ and } y \leq y^2 \leq \cdot\cdot\ \ \leq y^{n-1},$$

by Chebyshev's Inequality we get

$$n(a_1y + a_2y^2 + \ \ \cdot\cdot + a_{n-1}y^{n-1}) \geq (a_1 + a_2 + \cdot\cdot\ \ + a_{n-1})(y + y^2 + \cdot\ \ \cdot + y^{n-1}).$$

Thus, it is enough to show that $a_1 + a_2 + \cdots + a_{n-1} \geq 0$. Indeed, we have

$$a_1 + a_2 + \qquad + a_{n-1} = \frac{n(10n^2 - 15n - 1)}{6} > 0$$

Finally, it remains to show that (11) is true for $a_2 = a_3 = \ \ \cdot\cdot = a_n \geq 1$ and $a_1 a_2 \cdot\cdot\ \ a_n = 1$; that is to prove that

$$f(x) + (n-1)f(y) \geq 1,$$

for $0 < x \leq 1 \leq y$ and $xy^{n-1} = 1$. Setting $k = n-1$, $k \geq 1$, the inequality is equivalent to

$$h(y) \geq h(1),$$

where

$$h(y) = \frac{y^{k^2}}{1 + y^k + \cdots + y^{k^2}} + \frac{k}{1 + y + \cdots + y^k}.$$

For the nontrivial case $y > 1$, we write successively the inequality $h(y) \geq h(1)$ as follows:

$$\frac{k}{1+y+\cdots+y^k} \geq \frac{1+y^k+\cdots+y^{(k-1)k}}{1+y^k+\cdots+y^{k^2}},$$

$$\frac{k(y-1)}{y^{k+1}-1} \geq \frac{y^{k^2}-1}{y^k-1} \cdot \frac{y^k-1}{y^{(k+1)k}-1},$$

$$\frac{k(y-1)}{y^{k+1}-1} \geq \frac{y^{k^2}-1}{y^{(k+1)k}-1},$$

$$k\frac{y^{k(k+1)}-1}{y^{k+1}-1} \geq \frac{y^{k^2}-1}{y-1},$$

$$k\left[1+y^{k+1}+y^{2(k+1)}+\cdots+y^{(k-1)(k+1)}\right] \geq 1+y+y^2+\cdots+y^{(k-1)(k+1)},$$

$$k\left[1\cdot 1+y\cdot y^k+y^2\cdot y^{2k}+\cdots+y^{k-1}y^{(k-1)k}\right] \geq$$
$$\geq (1+y+y^2+\cdots+y^{k-1})\left[1+y^k+y^{2k}+\cdots+y^{(k-1)k}\right].$$

Since $1 < y < y^2 < \cdots < y^{k-1}$ and $1 < y^k < y^{2k} < \cdots < y^{(k-1)k}$, the last inequality is Chebyshev's Inequality applied to the k-tuples

$$(1, y, \ldots, y^{k-1}) \text{ and } (1, y^k, \ldots, y^{(k-1)k})$$

This completes the proof. For $n \geq 3$, equality occurs is and only if $a_1 = a_2 = \cdots = a_n$. □

Remark For $p = 1$, we obtain the following nice statement:

- *If $a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1a_2\ldots a_n = 1$, then*

$$\frac{1}{1+a_1+\cdots+a_1^{n-1}}+\frac{1}{1+a_2+\cdots+a_2^{n-1}}+\cdots+\frac{1}{1+a_n+\cdots+a_n^{n-1}} \geq 1$$

In the case $n = 4$, the well-known statements follows

- *If a, b, c, d are positive numbers such that $abcd = 1$, then*

$$\frac{1}{(1+a)(1+a^2)}+\frac{1}{(1+b)(1+b^2)}+\frac{1}{(1+c)(1+c^2)}+\frac{1}{(1+d)(1+d^2)} \geq 1.$$

(*Vasile Cîrtoaje*, GM-B, 11, 1999)

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26. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \ldots a_n \ge 1$, then*

$$a_1 + a_2 + \cdots + a_n - \sqrt[n]{a_1 a_2 \ldots a_n} \ge \frac{1}{2n^2} \sum_{1 \le i \le j \le n} (\ln a_i - \ln a_j)^2$$

Proof Since

$$\sum_{1 \le i \le j \le n} (\ln a_i - \ln a_j)^2 = n \sum_{i=1}^{n} \ln^2 a_i - \left(\sum_{i=1}^{n} \ln a_i \right)^2 =$$
$$= n \sum_{i=1}^{n} \ln^2 a_i - \ln^2 (a_1 a_2 \ldots a_n),$$

we may write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge n f\left(\sqrt[n]{a_1 a_2 \ldots a_n} \right),$$

where $f(t) = t - \dfrac{1}{2n} \ln^2 t$, $t > 0$ The function

$$f_1(u) = f(e^u) = e^u - \frac{1}{2n} u^2$$

has the derivative

$$f_1''(u) = e^u - \frac{1}{n}.$$

Since $f_1''(u) > 0$ for $u \ge 0$, the function $f_1(u)$ is convex for $u \ge \ln r$, where $r = 1$ By RCF–Corollary and Remark 5, it suffices to show that

$$x f'(x) \le y f'(y) \text{ for } 0 < x \le 1 \le y \text{ and } x y^{n-1} = 1$$

We have

$$t f'(t) = t - \frac{1}{n} \ln t,$$

and

$$y f'(y) - x f'(x) = y - \frac{1}{n} \ln y - x + \frac{1}{n} \ln x = y - x - \ln y = y - \frac{1}{y^{n-1}} - \ln y.$$

Let $h(y) = y - \dfrac{1}{y^{n-1}} - \ln y$. Since

$$h'(y) = 1 + \frac{n-1}{y^n} - \frac{1}{y} \ge \frac{n-1}{y^n} > 0,$$

the function $h(y)$ is strictly increasing for $y \ge 1$. Therefore, $h(y) \ge h(1) = 0$, and hence $y f'(y) - x f'(x) \ge 0$.

Equality occurs for $a_1 = a_2 = \cdots = a_n = 1$. □

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27. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \ldots a_n = 1$, then*

$$\left(1-\frac{1}{n}\right)^{a_1} + \left(1-\frac{1}{n}\right)^{a_2} + \cdots + \left(1-\frac{1}{n}\right)^{a_n} \le n-1$$

Proof. Setting $a_i = \dfrac{x_i}{\ln n - \ln(n-1)}$ for each $i \in \{1, 2, \ldots, n\}$, the statement becomes as follows:

- *If $x_1, x_2, \ldots, x_n$ are positive numbers such that*

$$\sqrt[n]{x_1 x_2 \ldots x_n} = r = \ln\frac{n}{n-1},$$

then

$$e^{-x_1} + e^{-x_2} + \cdots + e^{-x_n} \le ne^{-r}.$$

We may write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \le nf\left(\sqrt[n]{a_1 a_2 \ldots a_n}\right),$$

where $f(t) = e^{-t}$, $t > 0$. The function

$$f_1(u) = f(e^u) = e^{-e^u}$$

has the second derivative

$$f_1''(u) = (e^u - 1)e^{u-e^u}.$$

Since $f_1''(u) < 0$ for $u < 0$, the function $f_1(u)$ is concave for

$$u \le \ln r = \ln\ln\frac{n}{n-1} < 0$$

According to LCF-Corollary, it suffices to show that

$$(n-1)e^{-x} + e^{-y} \le ne^{-r}$$

for $0 < x \le r \le y$ and $x^{n-1}y = r^n$. That is $g(x) \le g(r)$ for $0 < x \le r$, where

$$g(x) = (n-1)e^{-x} + e^{-y}, \quad \text{with} \quad y = \frac{r^n}{x^{n-1}}$$

Since

$$\frac{x^n e^y}{n-1} g'(x) = r^n - x^n e^{y-x},$$

it follows that the derivative g' has the same sign as the function

$$g_1(x) = r^n - x^n e^{y-x}$$

From

$$e^{x-y} g_1'(x) = x^n - nx^{n-1} + (n-1)r^n,$$

we find that $g_1'(x)$ has the same sign as the function

$$h(x) = x^n - nx^{n-1} + (n-1)r^n$$

The derivative of $h(x)$ is given by $h'(x) = nx^{n-2}(x-n+1)$. Since $h'(x) < 0$ for $0 < x \le r$, the function $h(x)$ is strictly decreasing In addition, since $h(0) = (n-1)r^n > 0$ and $h(r) = nr^{n-1}(r-1) < 0$, it exists $x_1 \in (0, r)$ such that $h(x) > 0$ for $x \in [0, x_1)$, $h(x_1) = 0$ and $h(x) < 0$ for $x \in (x_1, r]$. Therefore, the function $g_1(x)$ is strictly increasing on $(0, x_1]$ and strictly decreasing on $[x_1, r]$ Since $g_1(0_+) = -\infty$ and $g_1(r) = 0$, it exists $x_2 \in (0, x_1)$ such that $g_1(x) < 0$ for $x \in (0, x_2)$, $g_1(x_2) = 0$ and $g_1(x) > 0$ for $x \in (x_2, r)$ Consequently, the function $g(x)$ is strictly decreasing on $(0, x_2]$ and strictly increasing on $[x_2, r]$. Since $g(0_+) = n-1$ and $g(r) = ne^{-r} = n-1 = g(0_+)$, we get $g(x) \le g(r)$ for $0 < x \le r$ Equality occurs for $a_1 = a_2 = \cdots = a_n = 1$ □

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28. *If $x_1, x_2, \ldots, x_n$ are non-negative real numbers such that*

$$x_1 + x_2 + \cdots + x_n = n,$$

then

$$n^{-x_1^2} + n^{-x_2^2} + \cdots + n^{-x_n^2} \ge 1.$$

Proof We may write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right),$$

where $f(u) = n^{-u^2}$, $u \ge 0$. from the expression of the second derivative

$$f''(u) = 2n^{-u^2}(2u^2 \ln n - 1) \ln n,$$

it follows that f is convex for $u \ge 1$, and also for $u \ge s = \dfrac{x_1 + x_2 + \cdots + x_n}{n} = 1$. By RCF-Theorem, it suffices to prove the inequality for

$$x_1 \le 1 \le x_2 = x_3 = \cdots = x_n$$

Let

$$g(x) = n^{-x^2} + (n-1)n^{-y^2} - 1,$$

where $x + (n-1)y = n$ and $0 \le x \le 1 \le y$. We have to show that $g(x) \ge 0$ for $0 \le x \le 1$ Taking into account that $y' = \dfrac{-1}{n-1}$, we get

$$g'(x) = 2\left(yn^{-y^2} - xn^{-x^2}\right)\ln n$$

The derivative g' has the same sign as the funtion

$$g_1(x) = \ln\left(yn^{-y^2}\right) - \ln\left(xn^{-x^2}\right) = \ln y - \ln x + (x^2 - y^2)\ln n$$

From

$$g_1'(x) = \frac{-1}{(n-1)y} - \frac{1}{x} + 2\left(x + \frac{y}{n-1}\right)\ln n =$$
$$= n\left[\frac{-1}{x(n-x)} + \frac{2+2(n-2)x}{(n-1)^2}\ln n\right],$$

we see that $g_1'(x)$ has for $0 < x \le 1$ the same sign as the function

$$h(x) = \frac{-(n-1)^2}{2\ln n} + x(n-x)\left[1 + (n-2)x\right]$$

The derivative of $h(x)$ is given by

$$h'(x) = n + 2(n^2 - 2n - 1)x - 3(n-2)x^2.$$

Since

$$h'(x) = n + 2(n^2 - 2n - 1)x - 3(n-2)x^2 \ge$$
$$\ge nx + 2(n^2 - 2n - 1)x - 3(n-2)x =$$
$$= 2(n-1)(n-2)x > 0$$

for $0 < x \le 1$, the function $h(x)$ is strictly increasing Since $h(0) < 0$ and $h(1) = (n-1)^2\left(1 - \dfrac{1}{2\ln n}\right) > 0$, it exists $x_1 \in (0,1)$ such that $h(x) < 0$ for $x \in [0, x_1)$, $h(x_1) = 0$ and $h(x) > 0$ for $x \in (x_1, 1]$ Therefore, the function $g_1(x)$ is strictly decreasing on $(0, x_1]$ and strictly increasing on $[x_1, 1]$ Since $g_1(0_+) = +\infty$ and $g_1(1) = 0$, it exists $x_2 \in (0, x_1)$ such that $g_1(x) > 0$ for $x \in (0, x_2)$, $g_1(x_2) = 0$ and $g_1(x) < 0$ for $x \in (x_2, 1)$ Consequently, the function $g(x)$ is strictly increasing on $[0, x_2]$ and strictly decreasing on $[x_2, 1]$ Since $g(0) = (n-1)n^{-\left(\frac{n}{n-1}\right)^2} > 0$ and $g(1) = 0$, it follows that $g(x) \ge 0$ for $0 \le x \le 1$ Equality occurs if and only if $x_1 = x_2 = \quad = x_n = 1$ □

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29. *Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers such that*

$$x_1 + x_2 + \cdots + x_n = n.$$

Prove that

$$2\left(x_1^3 + x_2^3 + \cdots + x_n^3\right) + n^2 \le (2n+1)\left(x_1^2 + x_2^2 + \cdots + x_n^2\right).$$

Proof Write the inequality in the form

$$f(x_1) + f(x_2) + \cdots + f(x_n) \le 0,$$

where $f(x) = 2x^3 - (2n+1)x^2 + n$. Taking into account the second derivative

$$f''(x) = 2(6x - 2n - 1),$$

it follows that f is concave on $\left[0, \frac{2n+1}{6}\right]$ and convex on $\left[\frac{2n+1}{6}, \infty\right)$ By LCRCF-Theorem, the sum $E = f(x_1) + f(x_2) + \cdots + f(x_n)$ is maximal for $x_1 = x_2 = \cdots = x_{n-1} \le x_n$. Therefore, it suffices to prove the inequality

$$(n-1)f(x) + f(y) \le 0,$$

for $0 \le x \le 1 \le y$ and $(n-1)x + y = n$. The inequality is equivalent to

$$n(n-1)x\left[2(n-2)x^2 - (4n-7)x + 2n - 2\right] \ge 0.$$

It is true because

$$2(n-2)x^2 - (4n-7)x + 2n-2 = 2(n-2)(x-1)^2 + 2 - x \ge 0.$$

Equality occurs if one of x_i is equal to n and the other ones are 0. □

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30. *Let x, y, z be positive real numbers such that $x + y + z = 3$. Prove that*

$$8\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + 9 \ge 10(x^2 + y^2 + z^2).$$

Proof. Write the inequality in the form

$$f(x) + f(y) + f(z) \leq 9,$$

where $f(t) = 10t^2 - \frac{8}{t}$. According to the second derivative $f''(t) = \frac{4(5t^3 - 4)}{t^3}$, the function f is concave on $\left[0, \sqrt[3]{\frac{4}{5}}\right]$ and convex on $\left[\sqrt[3]{\frac{4}{5}}, \infty\right)$.
By LCRCF-Theorem, the sum $E = f(x) + f(y) + f(z)$ is maximal for $x = y \leq z$ Therefore, it suffices to prove the inequality

$$2f(x) + f(z) \leq 9,$$

for $0 \leq x \leq 1 \leq z$ and $2x + z = 3$. The inequality is equivalent to

$$40x^4 - 140x^3 + 174x^2 - 89x + 16 \geq 0,$$

or

$$(2x - 1)^2(10x^2 - 25x + 16) \geq 0$$

Because

$$10x^2 - 25x + 16 = 10(x - 1)^2 + 6 - 5x > 0,$$

the inequality is clearly true. Equality occurs if and only if two of x, y, z are equal to $\frac{1}{2}$, and the other one is equal to 2 □

Chapter 4

On Popoviciu's inequality

4.1 Introduction

In 1965 the Romanian mathematician *T. Popoviciu* proved the following inequality

$$f(x)+f(y)+f(z)+3f\left(\frac{x+y+z}{3}\right)\geq$$
$$\geq 2f\left(\frac{x+y}{2}\right)+2f\left(\frac{y+z}{2}\right)+2f\left(\frac{z+x}{2}\right),$$

where f is a convex function on an interval I and $x,y,z\in I$.

A. Lupaş generalized this in 1982, in the following form (where p,q and r are positive numbers):

$$pf(x)+qf(y)+rf(z)+(p+q+r)f\left(\frac{px+qr+rz}{p+q+r}\right)\geq$$
$$\geq(p+q)f\left(\frac{px+qy}{p+q}\right)+(q+r)f\left(\frac{qy+rz}{q+r}\right)+(r+p)f\left(\frac{rz+px}{r+p}\right).$$

In 2002 and 2004, we extended Popoviciu's Inequality to n variables [5, 6], as follows

Theorem 1 (Generalized Popoviciu's Inequality) *If f is a convex function on an interval $\mathbb{I}$ and $a_1,a_2,\ldots,a_n\in\mathbb{I}$, then*

$$f(a_1)+f(a_2)+\cdots+f(a_n)+n(n-2)f(a)\geq(n-1)\left[f(b_1)+f(b_2)+\cdots+f(b_n)\right],$$

where $a=\dfrac{a_1+a_2+\cdots+a_n}{n}$, *and* $b_i=\dfrac{1}{n-1}\sum_{j\neq i}a_j$ *for all i.*

Theorem 2. *If f is a convex function on an interval $\mathbb{I}$ and $a_1, a_2, \ldots, a_n \in \mathbb{I}$, then*

$$f(a_1) + f(a_2) + \cdots + f(a_n) + \frac{n}{n-2} f(a) \geq \frac{2}{n-2} \sum_{1 \leq i < j \leq n} f\left(\frac{a_i + a_j}{2}\right),$$

where $a = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$.

Soon after these inequalities were posted on Mathlinks Inequalities Forum, *Bill Zhao* conjectured the following general statement

Theorem 3 *If f is a convex function on an interval $\mathbb{I}$ and $a_1, a_2, \ldots, a_n \in \mathbb{I}$, then*

$$\binom{n-2}{m-1} [f(a_1) + f(a_2) + \cdots + f(a_n)] + n\binom{n-2}{m-2} f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \geq$$
$$\geq m \sum_{1 \leq i_1 < \cdots < i_m \leq n} f\left(\frac{a_{i1} + a_{i2} + \cdots + a_{im}}{m}\right)$$

Darij Grinberg posted in 2005 on Mathlinks Inequalities Forum a long proof of this inequality by induction over n.

In this section, we will prove the first two theorems, and then will give some applications of these Our proof relies on Karamata's Inequality for convex functions, which we now recall. We say that a vector $\vec{A} = (a_1, a_2, \ldots, a_n)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ majorizes a vector $\vec{B} = (b_1, b_2, \ldots, b_n)$ with $b_1 \geq b_2 \geq \cdots \geq b_n$, and write it as $\vec{A} \geq \vec{B}$, if

$$\begin{gathered}
a_1 \geq b_1, \\
a_1 + a_2 \geq b_1 + b_2, \\
\cdots \\
a_1 + a_2 + \cdots + a_{n-1} \geq b_1 + b_2 + \cdots + b_{n-1}, \\
a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n
\end{gathered}$$

Karamata's Inequality states that for any convex function and $\vec{A} \geq \vec{B}$, the following inequality holds:

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(b_1) + f(b_2) + \cdots + f(b_n)$$

Proof of Theorem 1 Without loss of generality, we may assume that $n \geq 3$ and $a_1 \leq a_2 \leq \cdots \leq a_n$. Then there is an integer m with $1 \leq m \leq n-1$, such that

$$a_1 \le \cdot \quad \le a_m \le a \le a_{m+1} \le \quad \le a_n$$

and

$$b_1 \ge \ \cdot\cdot \ge b_m \ge a \ge b_{m+1} \ge \cdots \ge b_n,$$

where $a = \dfrac{a_1 + a_2 + \cdot \quad + a_n}{n}$. It is clear that the required inequality that we are trying to prove is the sum of the following inequalities

$$f(a_1) + \cdot\cdot \ + f(a_m) + n(n-m-1)f(a) \ge (n-1)\left[f(b_{m+1}) + \ \cdot \ + f(b_n)\right], \quad (1)$$

$$f(a_{m+1}) + \ \cdot\cdot + f(a_n) + n(m-1)f(a) \ge (n-1)\left[f(b_1) + \cdot \ \cdot + f(b_m)\right] \quad (2)$$

In order to prove (1), we apply Jensen's Inequality to get

$$f(a_1) + \cdots + f(a_m) + (n-m-1)f(a) \ge (n-1)f(b),$$

where

$$b = \frac{a_1 + \ \cdot\cdot + a_m + (n-m-1)a}{n-1}$$

Thus, we still have to show that

$$(n-m-1)f(a) + f(b) \ge f(b_{m+1}) + \ \cdot \ + f(b_n).$$

Since $a \ge b_{m+1} \ge \ \cdot \ \ge b_n$ and $(n-m-1)a + b = b_{m+1} + \cdots + b_n$, we see that $\vec{A}_{n-m} = (a, \cdot \ ., a, b)$ majorizes $\vec{B}_{n-m} = (b_{m+1}, b_{m+2}, \cdot\cdot \ , b_n)$ Consequently, the inequality follows by Karamata's Inequality for convex functions.

Similarly, we can prove inequality (2) adding Jensen's Inequality

$$\frac{f(a_{m+1}) + \ \cdot \ + f(a_n) + (m-1)f(a)}{n-1} \ge f(c)$$

and the inequality

$$f(c) + (m-1)f(a) \ge f(b_1) + \cdot \ \cdot + f(b_m),$$

where

$$c = \frac{a_{m+1} + \cdots + a_n + (m-1)a}{n-1}.$$

The last inequality follows from Karamata's Inequality, because

$$b_1 \ge \cdots \ge b_m \ge a \quad \text{and} \quad c + (m-1\ a = b_1 + \cdot \ \cdot + b_m,$$

and therefore $\vec{C}_m = (c, a, \ldots, a)$ majorizes $\vec{D}_m = (b_1, b_2, \ldots, b_m)$ □

Proof of Theorem 2 We will prove this by induction over n For $n = 2$, one has equality Suppose now that $n \geq 3$ and the inequality is valid for $n-1$ We will show that it holds for n Let $a = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$ and let $x = \dfrac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}$. According to the induction hypothesis, we have

$$(n-3)\left[f(a_1)+f(a_2)+\cdots+f(a_{n-1})\right]+(n-1)f(x) \geq 2 \sum_{1\leq i<j\leq n-1} f\left(\frac{a_i+a_j}{2}\right).$$

Thus, it suffices to show that

$$f(a_1) + f(a_2) + \cdots + f(a_{n-1}) + (n-2)f(a_n) + nf(a) \geq$$
$$\geq (n-1)f(x) + 2\sum_{i=1}^{n-1} f\left(\frac{a_i + a_n}{2}\right).$$

By Jensen's Inequality, we have

$$f(a_1) + f(a_2) + \cdots + f(a_{n-1}) \geq (n-1)f(x).$$

Hence, we just have to show that

$$(n-2)f(a_n) + nf(a) \geq 2\sum_{i=1}^{n-1} f\left(\frac{a_i + a_n}{2}\right).$$

Since $(n-2)a_n + na = 2\displaystyle\sum_{i=1}^{n-1} \frac{a_i + a_n}{2}$, we will again use Karamata's Inequality for two cases.

Case $2a \geq \min\{a_1, a_2, \ldots, a_n\} + \max\{a_1, a_2, \ldots, a_n\}$ Without loss of generality, assume that

$$a_1 = \max\{a_1, a_2, \ldots, a_n\}, \; a_n = \min\{a_1, a_2, \ldots, a_n\}.$$

Then, $2a \geq a_1 + a_n$. According to Karamata's Inequality, it is enough to show that

$$a_n \leq \min\left\{\frac{a_1 + a_n}{2}, \frac{a_2 + a_n}{2}, \ldots, \frac{a_{n-1} + a_n}{2}\right\}$$

and

$$a \geq \max\left\{\frac{a_1 + a_n}{2}, \frac{a_2 + a_n}{2}, \ldots, \frac{a_{n-1} + a_n}{2}\right\}$$

The first condition is clearly true, while and the second condition reduces to $a \geq \frac{a_1 + a_n}{2}$

Case $2a \leq \min\{a_1, a_2, \ldots, a_n\} + \max\{a_1, a_2, \ldots, a_n\}$. Without loss of generality, assume that

$$a_1 = \min\{a_1, a_2, \ldots, a_n\},\ a_n = \max\{a_1, a_2, \ldots, a_n\}.$$

Then, $2a \leq a_1 + a_n$ According to Karamata's Inequality, it is enough to show that

$$a \leq \min\left\{\frac{a_1 + a_n}{2}, \frac{a_2 + a_n}{2}, \ldots, \frac{a_{n-1} + a_n}{2}\right\}$$

and

$$a_n \geq \max\left\{\frac{a_1 + a_n}{2}, \frac{a_2 + a_n}{2}, \ldots, \frac{a_{n-1} + a_n}{2}\right\}$$

The first condition reduces to $a \leq \frac{a_1 + a_n}{2}$, while the second condition is clearly true □

Remark The generalized Popoviciu's Inequality may be rewritten in the following form

$$E_n(a_1, a_2, \ldots, a_n) = \frac{f(a_1) + f(a_2) + \cdots + f(a_n) - nf(a)}{f(b_1) + f(b_2) + \cdots + f(b_n) - nf(a)} \geq n - 1$$

For some convex functions, the greatest lower bound of E_n is just $n-1$, but for other functions, the greatest lower bound of E_n is greater than $n-1$. In this last case, the generalized Popoviciu's Inequality may be improved For instance, for the convex function $f(x) = x^2$, the equality holds

$$\frac{a_1^2 + a_2^2 + \cdots + a_n^2 - na^2}{b_1^2 + b_2^2 + \cdots + b_n^2 - na^2} = (n-1)^2,$$

while for the convex function $f(x) = x^3$, $x \geq 0$, the greatest lower bound of E_n is

$$\frac{(2n-1)(n-1)^3}{3n^2 - 5n + 1}$$

Therefore, if $a_1, a_2, \ldots, a_n$ are non-negative numbers, then

$$\frac{a_1^3 + a_2^3 + \cdots + a_n^3 - na^3}{b_1^3 + b_2^3 + \cdots + b_n^3 - na^3} \geq \frac{(2n-1)(n-1)^3}{3n^2 - 5n + 1}.$$

On the assumption that $a_1 + a_2 + \cdots + a_n = n$, this inequality is equivalent to the first inequality from the section 3 4.

$$(n-1)\left(a_1^3 + a_2^3 + \cdots + a_n^3\right) + n^2 \geq (2n-1)\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)$$

For $n \geq 3$, equality holds when either $a_1 = a_2 = \cdots = a_n = 1$, or one of a_i equals zero and the others equal $\frac{n}{n-1}$.

4.2 Applications

1. If $a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1 a_2 \ldots a_n = 1$, then

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \geq (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

2. If $a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1 a_2 \ldots a_n = 1$, then

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \geq$$
$$\geq \frac{n-1}{2}\left(a_1 + a_2 + \cdots + a_n + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

(*Bin Zhao*, MS, 2005)

3. If $a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1 + a_2 + \cdots + a_n = n$, then

$$(n-a_1)(n-a_2)\ldots(n-a_n) \geq (n-1)^n \sqrt[n-1]{a_1 a_2 \ldots a_n}$$

4. If $a_1, a_2, \ldots, a_n$ are positive numbers, and $b_i = \frac{1}{n-1}\sum_{j \neq i} a_j$ for all i, then

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \geq \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}.$$

5. If $x_1, x_2, \ldots, x_n$ are positive numbers such that

$$x_1 + x_2 + \cdots + x_n = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n},$$

then

a) $$\frac{1}{1+(n-1)x_1} + \frac{1}{1+(n-1)x_2} + \cdots + \frac{1}{1+(n-1)x_n} \geq 1,$$

b) $$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \leq 1$$

(*Vasile Cîrtoaje*, A M M , 1996)

6. If $a_1, a_2, ..., a_n$ $(n \geq 3)$ are positive numbers such that $a_1+a_2+\cdots+a_n = 1$, then

$$\left(a_1 + \frac{1}{a_1} - 2\right)\left(a_2 + \frac{1}{a_2} - 2\right)\ldots\left(a_n + \frac{1}{a_n} - 2\right) \geq \left(n + \frac{1}{n} - 2\right)^n.$$

7. If $x_1, x_2, \ldots, x_n$ are positive numbers such that

$$x_1 + x_2 + \cdots + x_n = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = ns,$$

then

$$\frac{1}{x_1+n-1} + \frac{1}{x_2+n-1} + \cdots + \frac{1}{x_n+n-1} \geq$$
$$\geq \frac{1}{ns-x_1+1} + \frac{1}{ns-x_2+1} + \cdots + \frac{1}{ns-x_n+1}$$

(*Gabriel Dospinescu*, MC, 2004)

8. Let $x_1, x_2, \ldots, x_n$ $(n \geq 3)$ be positive numbers satisfying $x_1x_2\ldots x_n = 1$ If $0 < p \leq \frac{2n-1}{(n-1)^2}$, then

$$\frac{1}{\sqrt{1+px_1}} + \frac{1}{\sqrt{1+px_2}} + \cdots + \frac{1}{\sqrt{1+px_n}} \leq \frac{n}{\sqrt{1+p}}.$$

(*Vasile Cîrtoaje*, and *Gabriel Dospinescu*)

9. If $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$(n-1)\left(x_1^2 + x_2^2 + \cdots + x_n^2\right) + n\sqrt[n]{x_1^2x_2^2\ldots x_n^2} \geq (x_1 + x_2 + \cdots + x_n)^2.$$

(*F. Shleifer*, Kvant, No 3, 1979)

10. If a, b, c, d are positive numbers such that $ab + bc + cd + da = 4$, then

$$\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{d}\right)\left(1 + \frac{d}{a}\right) \geq (a+b+c+d)^2.$$

4.3 Solutions

1. *If* $a_1, a_2, \ldots, a_n$ *are positive numbers such that* $a_1a_2\ldots a_n = 1$, *then*

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \geq (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right). \quad (3)$$

Proof. The inequality follows from generalized Popoviciu's Inequality (Theorem 1) considering the convex function $f(x) = e^x$ and replacing $a_1, a_2, \ldots, a_n$ with $(n-1)\ln a_1, (n-1)\ln a_2, \ldots, (n-1)\ln a_n$, respectively.

For $n \geq 3$, one has equality if and only if $a_1 = a_2 = \cdots = a_n = 1$. □

Remark For $n = 3$ and $a_1 = \frac{x^2}{yz}$, $a_2 = \frac{y^2}{zx}$, $a_3 = \frac{z^2}{xy}$, one obtains the known inequality

$$x^6 + y^6 + z^6 + 3(xyz)^2 \geq 2(y^3z^3 + z^3x^3 + x^3y^3).$$

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2. *If $a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1a_2 \cdots a_n = 1$, then*

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \geq$$
$$\geq \frac{n-1}{2}\left(a_1 + a_2 + \cdots + a_n + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

Proof. We can get this inequality by adding (3) to the inequality

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \geq (n-1)(a_1 + a_2 + \cdots + a_n)$$

The last inequality follows by adding up the inequalities

$$a_i^{n-1} + n - 2 \geq (n-1)a_i$$

for all i. We have

$$a_i^{n-1} + n-2-(n-1)a_i = a_i^{n-1} - 1 - (n-1)(a_i - 1) =$$
$$= (a_i - 1)\left[(a_i^{n-2} - 1) + (a_i^{n-3} - 1) + \cdots + (a_i - 1)\right] \geq 0.$$

For $n \geq 3$, equality occurs if and only if $a_1 = a_2 = \cdots = a_n = 1$. □

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3. *If $a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$(n-a_1)(n-a_2)\cdots(n-a_n) \geq (n-1)^n \sqrt[n-1]{a_1a_2\cdots a_n}$$

Proof. We apply Theorem 1 to the convex function $f(x) = -\ln x$ for $x > 0$.
For $n \geq 3$, one has equality if and only if $a_1 = a_2 = \cdots = a_n = 1$. □

Remark Since $a_1 + a_2 + \cdots + a_n = n$ implies $a_1 a_2 \ldots a_n \le 1$ (by the AM-GM Inequality), the above inequality is sharper than the inequality

$$(n - a_1)(n - a_2) \ldots (n - a_n) \ge (n-1)^n a_1 a_2 \ldots a_n,$$

which easily follows by multiplying the inequalities

$$n - a_1 = a_2 + \cdots + a_n \ge (n-1)\sqrt[n-1]{a_2 \ldots a_n},$$

$$\vdots$$

$$n - a_n = a_1 + \cdots + a_{n-1} \ge (n-1)\sqrt[n-1]{a_1 \ldots a_{n-1}}.$$

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4. *If $a_1, a_2, \ldots, a_n$ are positive numbers, and $b_i = \frac{1}{n-1}\sum_{j \ne i} a_j$ for all i, then*

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \ge \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}. \tag{4}$$

Proof. Let $a = \frac{a_1 + a_2 + \cdots + a_n}{n}$ Using the relations

$$\frac{(n-1)b_i}{a_i} = \frac{na}{a_i} - 1 \quad \text{and} \quad \frac{a_i}{b_i} = \frac{na}{b_i} - n + 1$$

for $i = 1, 2, \ldots, n$, the inequality becomes

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{n(n-2)}{a} \ge (n-1)\left(\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n}\right).$$

This inequality easily follows from generalized Popoviciu's Inequality, if we consider the convex function $f(x) = \frac{1}{x}$ for $x > 0$. For $n \ge 3$, one has equality if and only if $a_1 = a_2 = \cdots = 1$ □

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5. *If $x_1, x_2, \ldots, x_n$ are positive numbers such that*

$$x_1 + x_2 + \cdots + x_n = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}, \tag{5}$$

then

a) $$\frac{1}{1+(n-1)x_1} + \frac{1}{1+(n-1)x_2} + \cdots + \frac{1}{1+(n-1)x_n} \ge 1; \tag{6}$$

b) $$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \le 1. \tag{7}$$

Proof. a) This inequality may be derived from (4) using the following way. Suppose that inequality (6) is false; that is

$$\frac{1}{1+(n-1)x_1}+\frac{1}{1+(n-1)x_2}+\cdots+\frac{1}{1+(n-1)x_n}<1.$$

Then we will show that (5) also does not hold In order to show this, let $x_i=\dfrac{1-a_i}{(n-1)a_i}$ for all $i=1,2,\ldots,n$. Then, the above inequality yields

$$a_1+a_2+\cdots+a_n<1$$

and hence

$$1-a_i>\sum_{j\neq i}a_j=(n-1)b_i$$

for all $i=1,2,\ldots,n$ Consequently,

$$x_1+x_2+\cdots+x_n=\sum_{i=1}^{n}\frac{1-a_i}{(n-1)a_i}>\sum_{i=1}^{n}\frac{b_i}{a_i}$$

Taking account of (4), we get

$$x_1+x_2+\cdots+x_n>\sum_{i=1}^{n}\frac{b_i}{a_i}\geq\sum_{i=1}^{n}\frac{a_i}{b_i}>\sum_{i=1}^{n}\frac{(n-1)a_i}{1-a_i}=$$
$$=\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n},$$

which shows us (5) does not hold. For $n\geq 3$, one has equality if and only if $x_1=x_2=\cdots=x_n=1$

b) Substituting $1/x_i$ for x_i in (6) and noting that (5) is still satisfied gives us

$$\frac{x_1}{n-1+x_1}+\frac{x_2}{n-1+x_2}+\cdots+\frac{x_n}{n-1+x_n}\geq 1,$$

which is equivalent to (7). □

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6. *If* $a_1,a_2,\ldots,a_n$ $(n\geq 3)$ *are positive numbers such that* $a_1+a_2+\cdots+a_n=1$, *then*

$$\left(a_1+\frac{1}{a_1}-2\right)\left(a_2+\frac{1}{a_2}-2\right)\cdots\left(a_n+\frac{1}{a_n}-2\right)\geq\left(n+\frac{1}{n}-2\right)^n$$

Proof. Applying generalized Popoviciu's Inequality to the convex function $f(x) = -\ln x$ for $x > 0$, we get

$$(b_1 b_2 \dots b_n)^{n-1} \geq (a_1 a_2 \dots a_n) \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)^{n(n-2)},$$

where $b_i = \dfrac{1}{n-1} \sum_{j \neq i} a_j$ for all i. Under the condition $a_1 + a_2 + \cdots + a_n = 1$, this inequality becomes as follows

$$(1-a_1)^{n-1}(1-a_2)^{n-1} \dots (1-a_n)^{n-1} \geq n^n \left(1 - \frac{1}{n}\right)^{n^2-n} a_1 a_2 \dots a_n. \tag{8}$$

On the other hand, by the AM-GM Inequality, we have

$$(1 - a_1) + (1 - a_2) + \cdots + (1 - a_n) \geq n \sqrt[n]{(1 - a_1)(1 - a_2) \dots (1 - a_n)},$$

that is

$$\left(1 - \frac{1}{n}\right)^n \geq (1 - a_1)(1 - a_2) \dots (1 - a_n).$$

From this inequality, for $n \geq 3$ we obtain

$$\left(1 - \frac{1}{n}\right)^{n(n-3)} (1 - a_1)^2 (1 - a_2)^2 \dots (1 - a_n)^2 \geq$$
$$\geq (1 - a_1)^{n-1}(1 - a_2)^{n-1} \dots (1 - a_n)^{n-1}.$$

Multiplying this inequality and (8) yields the desired inequality. Equality occurs if and only if $a_1 = a_2 = \cdots = a_n = \dfrac{1}{n}$. □

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7. *If $x_1, x_2, \dots, x_n$ are positive numbers such that*

$$x_1 + x_2 + \cdots + x_n = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = ns,$$

then

$$\frac{1}{x_1 + n - 1} + \frac{1}{x_2 + n - 1} + \cdots + \frac{1}{x_n + n - 1} \geq$$
$$\geq \frac{1}{ns - x_1 + 1} + \frac{1}{ns - x_2 + 1} + \cdots + \frac{1}{ns - x_n + 1}.$$

Proof. By the Cauchy-Schwarz Inequality, we have

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \ge n^2,$$

whence $s \ge 1$ follows. Applying generalized Popoviciu's Inequality to the convex function $f(x) = \dfrac{1}{1+(n-1)x}$ for $x > 0$, we get

$$\sum_{i=1}^{n} \frac{1}{1+(n-1)x_i} + \frac{n(n-2)}{1+(n-1)s} \ge (n-1)\sum_{i=1}^{n} \frac{1}{ns - x_i + 1}$$

Thus, we still have to show that

$$(n-1)\sum_{i=1}^{n} \frac{1}{x_i + n - 1} \ge \sum_{i=1}^{n} \frac{1}{1+(n-1)x_i} + \frac{n(n-2)}{1+(n-1)s}$$

For $n \ge 3$, this inequality is equivalent to

$$\sum_{i=1}^{n} \frac{1}{(x_i + n - 1)\left(\dfrac{1}{x_i} + n - 1\right)} \ge \frac{1}{1+(n-1)s},$$

or

$$\frac{1}{A_1} + \frac{1}{A_2} + \cdots + \frac{1}{A_n} \ge \frac{1}{1+(n-1)s},$$

where $A_i = (n-1)\left(x_i + \dfrac{1}{x_i}\right) + n^2 - 2n + 2$. By the AM-GM Inequality, we have

$$\frac{1}{A_1} + \frac{1}{A_2} + \cdots + \frac{1}{A_n} \ge \frac{n^2}{A_1 + A_2 + \cdots + A_n} = \frac{n}{2(n-1)s + n^2 - 2n + 2}.$$

Consequently, it is enough to show that

$$\frac{n}{2(n-1)s + n^2 - 2n + 2} \ge \frac{1}{1+(n-1)s}.$$

It is easy to check that this inequality is true for $s \ge 1$. For $n \ge 3$, equality holds if and only if $x_1 = x_2 = \cdots = x_n = 1$ □

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8. *Let* $x_1, x_2, \ldots, x_n$ $(n \geq 3)$ *be positive numbers satisfying* $x_1x_2 \ldots x_n = 1$. *If* $0 < p \leq \frac{2n-1}{(n-1)^2}$, *then*

$$\frac{1}{\sqrt{1+px_1}} + \frac{1}{\sqrt{1+px_2}} + \cdots + \frac{1}{\sqrt{1+px_n}} \leq \frac{n}{\sqrt{1+p}}.$$

Proof. We suppose that the reverse inequality holds

$$\frac{1}{\sqrt{1+px_1}} + \frac{1}{\sqrt{1+px_2}} + \cdots + \frac{1}{\sqrt{1+px_n}} > \frac{n}{\sqrt{1+p}},$$

and show that this inequality implies $x_1x_2 \ldots x_n < 1$, which contradicts the hypothesis $x_1x_2 \ldots x_n = 1$. Using the substitution $1 + px_i = \frac{1+p}{a_i^2}$ $(a_i > 0)$ for all $i = 1, 2, \ldots, n$, we have to prove that $a_1 + a_2 + \cdots + a_n > n$ yields

$$\left(1+p-a_1^2\right)\left(1+p-a_2^2\right)\cdots\left(1+p-a_n^2\right) < p^n(a_1a_2\ldots a_n)^2.$$

Since the ratio $\left(1+p-a_1^2\right)/a_1^2$ is increasing when a_1 is decreasing, it suffices to consider the case $a_1+a_2+\cdots+a_n = n$. Denoting $1+p = q^2$, $1 < q \leq \frac{n}{n-1}$, the inequality becomes as

$$\left(q^2-a_1^2\right)\left(q^2-a_2^2\right)\cdots\left(q^2-a_n^2\right) \leq (q^2-1)^n(a_1a_2\ldots a_n)^2 \qquad (9)$$

Applying the generalized Popoviciu's Inequality to the convex function

$$f(x) = -\ln\left(\frac{n}{n-1} - x\right) \quad \text{for } 0 < x < 1,$$

gives us

$$(a_1a_2\ldots a_n)^{n-1} \geq [n-(n-1)a_1]\,[n-(n-1)a_2]\ldots[n-(n-1)a_n]. \qquad (10)$$

On the other hand, Jensen's Inequality applied to the convex function

$$f(x) = \ln\frac{n-(n-1)x}{q-x},$$

yields

$$\frac{[n-(n-1)a_1]\,[n-(n-1)a_2]\ldots[n-(n-1)a_n]}{(q-a_1)(q-a_2)\ldots(q-a_n)} \geq \frac{1}{(q-1)^n}. \qquad (11)$$

Multiplying (10) and (11) yields

$$(a_1a_2 \cdots a_n)^{n-1} \geq \frac{(q-a_1)(q-a_2)\cdots(q-a_n)}{(q-1)^n}$$

Thus, in order to prove (9), we will still have to show that

$$(a_1a_2\dots a_n)^{n-3}(q+a_1)(q+a_2)\cdots(q+a_n) \leq (q+1)^n.$$

By the AM-GM Inequality, we have

$$a_1a_2\dots a_n \leq \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^n = 1$$

and

$$(q+a_1)(q+a_2)\cdots(q+a_n) \leq \left(q+\frac{a_1+a_2+\cdots+a_n}{n}\right)^n = (q+1)^n,$$

from which the conclusion follows.

Equality holds if and only if $x_1 = x_2 = \cdots = x_n = 1$ □

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9. *If $x_1, x_2, \dots, x_n$ are positive numbers, then*

$$(n-1)\left(x_1^2+x_2^2+\cdots+x_n^2\right)+n\sqrt[n]{x_1^2x_2^2\cdots x_n^2} \geq (x_1+x_2+\cdots+x_n)^2.$$

Proof. This inequality follows by Theorem 2, using the convex function $f(x) = e^x$ and replacing $a_1, a_2, \dots, a_n$ with $2\ln x_1, 2\ln x_2, \dots, 2\ln x_n$, respectively. Finally, one uses the identity

$$2\sum_{1\leq i<j\leq n} x_ix_j = (x_1+x_2+\cdots+x_n)^2 - \left(x_1^2+x_2^2+\cdots+x_n^2\right)$$

For $n \geq 3$, equality holds if and only if $x_1 = x_2 = \cdots = x_n$ □

10. *If a, b, c, d are positive numbers such that $ab+bc+cd+da = 4$, then*

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{d}\right)\left(1+\frac{d}{a}\right) \geq (a+b+c+d)^2.$$

Proof. Applying Theorem 2 to the convex function $f(x) = -\ln x$, we get

$$(a+b)(b+c)(c+d)(d+a)(a+c)(b+d) \geq 4abcd(a+b+c+d)^2$$

Since $(a+c)(b+d) = ab+bc+cd+da = 4$, the inequality becomes

$$(a+b)(b+c)(c+d)(d+a) \geq abcd(a+b+c+d)^2,$$

or

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{d}\right)\left(1+\frac{d}{a}\right) \geq (a+b+c+d)^2$$

Equality occurs if and only if $a = b = c = d = 1$ □

Chapter 5

Inequalities involving EV-Theorem

The Equal Variable Theorem (called also $n-1$ Equal Variable Theorem on the Mathlinks Site - Inequalities Forum) is a powerful instrument to solving some difficult symmetric inequalities. First we will present the theoretical base of the method, then we will solve some inequalities, hardly attackable by other ways

5.1 Statement of results

In order to state and prove the Equal Variable Theorem (EV-Theorem) we will use the below Lemma and Proposition

Lemma *Let a, b, c be given non-negative real numbers, not all equal and at most one equal to zero, and let $x \le y \le z$ be non-negative real numbers such that*

$$x + y + z = a + b + c, \quad x^p + y^p + z^p = a^p + b^p + c^p,$$

where $p \in (-\infty, 0] \cup (1, \infty)$. For $p = 0$, the second equation is $xyz = abc > 0$ Then, there exist two non-negative real numbers x_1 and x_2, $x_1 < x_2$, such that $x \in [x_1, x_2]$

Moreover,

1) *if $x = x_1$ and $p \le 0$, then $0 < x < y = z$,*
2) *if $x = x_1$ and $p > 1$, then either $0 = x < y \le z$ or $0 < x < y = z$,*
3) *if $x \in (x_1, x_2)$, then $x < y < z$;*
4) *if $x = x_2$, then $x = y < z$*

A proof of Lemma is given in [8, 9]

Proposition *Let a, b, c be given non-negative real numbers, not all equal and at most one equal to zero, and let $0 \le x \le y \le z$ such that*

$$x + y + z = a + b + c, \quad x^p + y^p + z^p = a^p + b^p + c^p,$$

where $p \in (-\infty, 0] \cup (1, \infty)$. For $p = 0$, the second equation is $xyz = abc > 0$. Let $f(u)$ be a differentiable function on $(0, \infty)$, such that $g(x) = f'\left(x^{\frac{1}{p-1}}\right)$ is strictly convex on $(0, \infty)$, and let

$$F_3(x, y, z) = f(x) + f(y) + f(z)$$

1) *If $p \le 0$, then F_3 is maximal only for $0 < x = y < z$, and is minimal only for $0 < x < y = z$;*

2) *If $p > 1$ and either $f(u)$ is continuous at $u = 0$ or $\lim_{u \to 0} f(u) = -\infty$, then F_3 is maximal only for $0 < x = y < z$, and is minimal only for either $x = 0$ or $0 < x < y = z$.*

Proof On the assumption $x \le y \le z$, from the relations $y + z = a + b + c - x$ and $y^p + z^p = a^p + b^p + c^p - x^p$ we may express y and z in terms of x for $x \in [x_1, x_2]$. We claim that the function

$$F(x) = f(x) + f(y(x)) + f(z(x))$$

is minimal for $x = x_1$ and is maximal for $x = x_2$. If this assertion is true, then by Lemma it follows that

a) $F(x)$ is minimal for $0 < x = y < z$ – when $p \le 0$, or for either $x = 0$ or $0 < x < y = z$ – when $p > 1$;

b) $F(x)$ is maximal for $0 < x = y < z$.

In order to prove the claim above, assume that $x \in (x_1, x_2)$ By Lemma, we have $0 < x < y < z$ From

$$x + y(x) + z(x) = a + b + c \quad \text{and} \quad x^p + y^p(x) + z^p(x) = a^p + b^p + c^p,$$

we get

$$y' + z' = -1, \quad y^{p-1}y' + z^{p-1}z' = -x^{p-1},$$

hence

$$y' = \frac{x^{p-1} - z^{p-1}}{z^{p-1} - y^{p-1}}, \quad z' = \frac{x^{p-1} - y^{p-1}}{y^{p-1} - z^{p-1}}.$$

It is easy to check that this result is also valid for $p = 0$. We have

$$F'(x) = f'(x) + y'f'(y) + z'f'(z)$$

and

$$\frac{F'(x)}{(x^{p-1} - y^{p-1})(x^{p-1} - z^{p-1}} = \frac{g(x^{p-1})}{(x^{p-1} - y^{p-1})(x^{p-1} - z^{p-1})} + \\ + \frac{g(y^{p-1})}{(y^{p-1} - z^{p-1})(y^{p-1} - x^{p-1})} + \frac{g(z^{p-1})}{(z^{p-1} - x^{p-1})(z^{p-1} - y^{p-1})}$$

Since g is strictly convex, the right hand side is positive. On the other hand,

$$(x^{p-1} - y^{p-1})(x^{p-1} - z^{p-1}) > 0$$

Consequently, $F'(x) > 0$ and $F(x)$ is strictly increasing for $x \in (x_1, x_2)$ Excepting the trivial case when $p > 1$, $x_1 = 0$ and $\lim_{u \to 0} f(u) = -\infty$, the function $F(x)$ is continuous on $[x_1, x_2]$, and hence is minimal only for $x = x_1$, and is maximal only for $x = x_2$. □

Equal Variable Theorem (EV-Theorem). *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be given non-negative real numbers, and let $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \quad x_1^p + x_2^p + \cdots + x_n^p = a_1^p + a_2^p + \cdots + a_n^p,$$

where p is a real number, $p \neq 1$ For $p = 0$, the second equation is $x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n > 0$. Let $f(u)$ be a differentiable function on $(0, \infty)$, such that

$$g(x) = f'\left(x^{\frac{1}{p-1}}\right)$$

is strictly convex on $(0, \infty)$, and let

$$F_n(x_1, x_2, \ldots, x_n) = f(x_1) + f(x_2) + \cdots + f(x_n)$$

1) *If $p \leq 0$, then F_n is maximal for $0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n$, and is minimal for $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$,*

2) *If $p > 0$ and either $f(u)$ is continuous at $u = 0$ or $\lim_{u \to 0} f(u) = -\infty$, then F_n is maximal for $0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n$, and is minimal for $x_1 = \cdots = x_k = 0$ and $x_{k+2} = \cdots = x_n$, where $k \in \{0, 1, \ldots, n-1\}$.*

Proof We will consider two cases

Case $p \in (-\infty, 0] \cup (1, \infty)$. Excepting the trivial case when $p > 1$, $x_1 = 0$ and $\lim_{u \to 0} f(u) = -\infty$, the function $F_n(x_1, x_2, \dots, x_n)$ attains its minimum and maximum values, and the conclusion follows from Proposition above, using the contradiction way For example, let us consider the case $p \le 0$. In order to prove that F_n is maximal for $0 < x_1 = x_2 = \dots = x_{n-1} \le x_n$, we assume, for the sake of contradiction, that F_n attains its maximum at $(b_1, b_2, \dots, b_n)$ with $b_1 \le b_2 \le \dots \le b_n$ and $b_1 < b_{n-1}$ Let x_1, x_{n-1}, x_n be positive numbers such that $x_1 + x_{n-1} + x_n = b_1 + b_{n-1} + b_n$ and $x_1^p + x_{n-1}^p + x_n^p = b_1^p + b_{n-1}^p + b_n^p$. According to Proposition, the expression

$$F_3(x_1, x_{n-1}, x_n) = f(x_1) + f(x_{n-1}) + f(x_n)$$

is maximal only for $x_1 = x_{n-1} < x_n$, which contradicts our assumption that F_n attains its maximum at $(b_1, b_2, \dots, b_n)$ with $b_1 < b_{n-1}$.

Case $p \in (0, 1)$ This case reduces to the case $p > 1$, replacing each of a_i by $a_i^{\frac{1}{p}}$, each of x_i by $x_i^{\frac{1}{p}}$, then p by $\frac{1}{p}$ Thus, we get the sufficient condition that $h(x) = xf'\left(x^{\frac{1}{1-p}}\right)$ to be strictly convex on $(0, \infty)$ We claim that this condition is equivalent to the condition that $g(x) = f'\left(x^{\frac{1}{p-1}}\right)$ to be strictly convex on $(0, \infty)$ Actually, for our proof, it suffices to show that if $g(x)$ is strictly convex on $(0, \infty)$, then $h(x)$ is strictly convex on $(0, \infty)$. To show this, we see that $g\left(\frac{1}{x}\right) = \frac{1}{x} h(x)$ Since $g(x)$ is strictly convex on $(0, \infty)$, by Jensen's Inequality we have

$$ug\left(\frac{1}{x}\right) + vg\left(\frac{1}{y}\right) > (u + v)g\left(\frac{\frac{u}{x} + \frac{v}{y}}{u + v}\right),$$

for any positive x, y, u, v with $x \ne y$ This inequality is equivalent to

$$\frac{u}{x} h(x) + \frac{v}{y} h(y) > \left(\frac{u}{x} + \frac{v}{y}\right) h\left(\frac{u + v}{\frac{u}{x} + \frac{v}{y}}\right).$$

For $u = tx$ and $v = (1-t)y$, where $t \in (0,1)$, this inequality reduces to

$$th(x) + (1-t)h(y) > h(tx + (1-t)y),$$

which show us that $h(x)$ is strictly convex on $(0,\infty)$. □

Remark Let $0 < \alpha < \beta$ The EV-Theorem holds true when $x_1, x_2, \ldots, x_n \in (\alpha,\beta)$, the function f is differentiable on (α,β) and the function $g(x) = f'\left(x^{\frac{1}{p-1}}\right)$ is strictly convex on $(\alpha^{p-1}, \beta^{p-1})$ - for $p > 1$, or $(\beta^{p-1}, \alpha^{p-1})$ - for $p < 1$

By EV-Theorem, we easily obtain some particular results, which are useful in applications

Corollary 1. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be given non-negative numbers, and let $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,\ x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2.$$

Let f be a differentiable function on $(0,\infty)$, such that $g(x) = f'(x)$ is strictly convex on $(0,\infty)$. Moreover, either $f(x)$ is continuous at $x = 0$ or $\lim\limits_{x\to 0} f(x) = -\infty$. Then,

$$F_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is maximal for $0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n$, and is minimal for $x_1 = \cdots = x_k = 0$ and $x_{k+2} = \cdots = x_n$, where $k \in \{0, 1, \ldots, n-1\}$

Corollary 2. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be given positive numbers, and let $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,\ \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

Let f be a differentiable function on $(0,\infty)$ such that $g(x) = f'\left(\frac{1}{\sqrt{x}}\right)$ is strictly convex on $(0,\infty)$. Then, $F_n = f(x_1) + f(x_2) + \cdots + f(x_n)$ is maximal for $0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n$, and is minimal for $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$.

Corollary 3. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be given positive numbers, and let $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ such that*

$$x_1 + x_2 + \cdots + a_n = a_1 + a_2 + \cdots + a_n,\ x_1 x_2 \ldots x_n = a_1 a_2 \ldots a_n.$$

Let f be a differentiable function on $(0,\infty)$ such that $g(x) = f'\left(\frac{1}{x}\right)$ is strictly convex on $(0,\infty)$. Then, $F_n = f(x_1)+f(x_2)+\cdots+f(x_n)$ is maximal for $0<x_1=x_2=\cdots=x_{n-1}\le x_n$, and is minimal for $0<x_1\le x_2=x_3=\cdots=x_n$.

Corollary 4. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be given non-negative numbers, and let $0 \le x_1 \le x_2 \le \cdots \le x_n$ such that*

$$x_1+x_2+\cdots+x_n = a_1+a_2+\cdots+a_n, \; x_1^p+x_2^p+\cdots+x_n^p = a_1^p+a_2^p+\cdots+a_n^p,$$

where p is a real number, $p \ne 0$, $p \ne 1$.

a) For $p<0$, $P = x_1x_2\ldots x_n$ is minimal when

$$0 < x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is maximal when

$$0 < x_1 \le x_2 = x_3 = \cdots = x_n$$

b) For $p>0$, $P = x_1x_2\ldots x_n$ is maximal when

$$0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimal when either

$$x_1 = 0 \text{ or } 0 < x_1 \le x_2 = x_3 = \cdots = x_n$$

Proof. Apply EV-Theorem to the function $f(u) = p\ln u$. We see that $\lim_{u\to 0} f(u) = -\infty$ for $p>0$, and

$$f'(u) = \frac{p}{u}, \;\; g(x) = f'\left(x^{\frac{1}{p-1}}\right) = px^{\frac{1}{1-p}}, \;\; g''(x) = \frac{p^2}{(1-p)^2}\, x^{\frac{2p-1}{1-p}}.$$

Since $g''(x)>0$ for $x>0$, the function $g(x)$ is strictly convex on $(0,\infty)$, and the conclusion follows by EV-Theorem. □

Corollary 5. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be given non-negative numbers, and let $0 \le x_1 \le x_2 \le \cdots \le x_n$ such that*

$$x_1+x_2+\cdots+x_n = a_1+a_2+\cdots+a_n, \; x_1^p+x_2^p+\cdots+x_n^p = a_1^p+a_2^p+\cdots+a_n^p.$$

1. *Case $p \le 0$ ($p=0$ yields $x_1x_2\cdots x_n = a_1a_2\cdots a_n > 0$)*

a) For $q \in (p,0) \cup (1,\infty)$, $E = x_1^q + x_2^q + \cdots + x_n^q$ *is maximal when* $0<x_1=x_2=\cdots=x_{n-1}\le x_n$, *and is minimal when* $0<x_1\le x_2=x_3=\cdots=x_n$.

b) For $q \in (-\infty,p) \cup (0,1)$, $E = x_1^q + x_2^q + \cdots + x_n^q$ *is minimal when* $0<x_1=x_2=\cdots=x_{n-1}\le x_n$, *and is maximal when* $0<x_1\le x_2=x_3=\cdots=x_n$.

2. *Case* $0 < p < 1$.

a) For $q \in (0,p) \cup (1,\infty)$, $E = x_1^q + x_2^q + \cdots + x_n^q$ *is maximal when* $0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n$, *and is minimal when either* $x_1 = 0$ *or* $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

b) For $q \in (-\infty,0) \cup (p,1)$, $E = x_1^q + x_2^q + \cdots + x_n^q$ *is minimal when* $0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n$, *and is maximal when* $x_1 = \cdots = x_k = 0$ *and* $x_{k+2} = \cdots = x_n$, *where* $k \in \{0,1,\ldots,n-1\}$.

3 *Case* $p > 1$

a) For $q \in (0,1) \cup (p,\infty)$, $E = x_1^q + x_2^q + \cdots + x_n^q$ *is maximal when* $0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n$, *and is minimal when* $x_1 = \cdots = x_k = 0$ *and* $x_{k+2} = \cdots = x_n$, *where* $k \in \{0,1,\ldots,n-1\}$.

b) For $q \in (-\infty,0) \cup (1,p)$, $E = x_1^q + x_2^q + \cdots + x_n^q$ *is minimal when* $0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n$, *and is maximal when* $x_1 = \cdots = x_k = 0$ *and* $x_{k+2} = \cdots = x_n$, *where* $k \in \{0,1,\ldots,n-1\}$.

Proof. We will apply EV-Theorem to the function

$$f(u) = q(q-1)(q-p)u^q.$$

For $p > 0$, it is easy to check that either $f(u)$ is continuous at $u = 0$ (in the case $q > 0$) or $\lim_{u\to 0} f(u) = -\infty$ (in the case $q < 0$). We have

$$f'(u) = q^2(q-1)(q-p)u^{q-1}$$

and

$$g(x) = f'\left(x^{\frac{1}{p-1}}\right) = q^2(q-1)(q-p)x^{\frac{q-1}{p-1}},$$

$$g''(x) = \frac{q^2(q-1)^2(q-p)^2}{(p-1)^2}\, x^{\frac{2p-1}{1-p}}$$

Since $g''(x) > 0$ for $x > 0$, the function $g(x)$ is strictly convex on $(0,\infty)$, and the conclusion follows by EV-Theorem. □

Corollary 6. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be given non-negative numbers, let $p \in \{1, 2\}$ and let $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1^p + x_2^p + \cdots + x_n^p = a_1^p + a_2^p + \cdots + a_n^p$$

The expression $E = \sum x_1x_2x_3$ is maximal when $0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n$, and is minimal when $x_1 = \cdots = x_k = 0$ and $x_{k+2} = \cdots = x_n$, where $k \in \{0, 1, \ldots, n-1\}$.

Proof Taking into account the known relation

$$6\sum x_1x_2x_3 = \left(\sum x_1\right)^3 - 3\left(\sum x_1\right)\left(\sum x_1^2\right) + 2\sum x_1^3,$$

the statement follows by Corollary 5 (case $p = 2$ and $q = 3$, or $p = 3$ and $q = 2$) □

Corollary 7. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ by given non-negative numbers, and let $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ such that*

$$x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2,$$
$$x_1^3 + x_2^3 + \cdots + x_n^3 = a_1^3 + a_2^3 + \cdots + a_n^3$$

The expression $E = \sum x_1x_2x_3$ is maximal when $0 \leq x_1 = x_2 = \ldots = x_{n-1} \leq x_n$, and is minimal when $x_1 = \cdots = x_k = 0$ and $x_{k+2} = \cdots = x_n$, where $k \in \{0, 1, \ldots, n-1\}$.

Proof. According to the relation

$$6\sum x_1x_2x_3 = \left(\sum x_1\right)^3 - 3\left(\sum x_1\right)\left(\sum x_1^2\right) + 2\sum x_1^3,$$

the sum $\sum x_1x_2x_3$ is maximal (minimal) when $\sum x_1$ is maximal (minimal) Consequently, the statement follows by Corollary 5 (case $p = \frac{3}{2}$ and $q = \frac{1}{2}$), replacing $x_1, x_2, \ldots, x_n$ with $x_1^2, x_2^2, \ldots, x_n^2$, respectively □

5.2 Applications

1. If x, y, z are non-negative real numbers, then

$$x^4(y+z) + y^4(z+x) + z^4(x+y) \leq \frac{1}{12}(x+y+z)^5.$$

(*Vasile Cîrtoaje*, MS, 2005)

2. If x, y, z are non-negative real numbers such that $xy + yz + zx = 1$, then

$$x + y + z + 3\left(2\sqrt{3} - 3\right) xyz \geq 2.$$

3. If x, y, z are non-negative real numbers such that $ab + bc + ca = 1$, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} - \frac{1}{a+b+c} \geq 2.$$

(*Vasile Cîrtoaje*, MS, 2005)

4. Let x, y, z, t be non-negative real numbers such that $x + y + z + t = 3$. Prove that

$$x^2y^2z^2 + y^2z^2t^2 + z^2t^2x^2 + t^2x^2y^2 \leq 1.$$

5. Let x, y, z, t be non-negative real numbers such that $x + y + z + t = 4$ Prove that

$$xyz + yzt + ztx + txy + x^2y^2z^2 + y^2z^2t^2 + z^2t^2x^2 + t^2x^2y^2 \leq 8$$

(*Phan Thanh Nam*, Diendantoanhoc Forum, Vietnam)

6. Let x, y, z be non-negative real numbers such that $xy + yz + zx = 3$ Then

$$\sqrt{\frac{1+2x}{3}} + \sqrt{\frac{1+2y}{3}} + \sqrt{\frac{1+2z}{3}} \geq 3.$$

(*Vasile Cîrtoaje*, MS, 2006)

7. Let x, y, z be non-negative real numbers, no two of which are zero Then

$$\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} \geq \frac{9}{4(xy+yz+zx)}$$

(Iran, 1996)

8. Let x, y, z be non-negative real numbers, no two of which are zero. If $0 \leq r \leq \frac{5}{2}$, then

$$\sum \frac{1}{y^2 + yz + z^2} \geq \frac{3(1+r)}{x^2 + y^2 + z^2 + r(xy + yz + zx)}.$$

9. Let x, y, z be non-negative real numbers such that $x + y + z = 3$. If $r \geq \frac{8}{5}$, then

$$\frac{1}{r + x^2 + y^2} + \frac{1}{r + y^2 + z^2} + \frac{1}{r + z^2 + x^2} \leq \frac{3}{r+2}$$

(*Vasile Cîrtoaje*, MS, 2006)

10. Let x, y, z be non-negative numbers such that $x^2+y^2+z^2=3$. If $r \geq 10$, then

$$\frac{1}{r-(x+y)^2}+\frac{1}{r-(y+z)^2}+\frac{1}{r-(z+x)^2} \leq \frac{3}{r-4}.$$

(*Vasile Cîrtoaje*, MS, 2006)

11. If x, y, z are non-negative real numbers, then

$$\frac{yz}{3x^2+y^2+z^2}+\frac{zx}{3y^2+z^2+x^2}+\frac{xy}{3z^2+x^2+y^2} \leq \frac{3}{5}.$$

(*Vasile Cîrtoaje* and *Pham Kim Hung*, MS, 2005)

12. Let x, y, z be non-negative real numbers such that $x+y+z=2$. Prove that

$$\frac{yz}{x^2+1}+\frac{zx}{y^2+1}+\frac{xy}{z^2+1} \leq 1$$

(*Pham Kim Hung*, MS, 2005)

13. Let x, y, z be non-negative real numbers such that $x+y+z=2$. If $r_0 \leq r \leq 3$, where $r_0 = \dfrac{\ln 2}{\ln 3 - \ln 2} \approx 1.71$, then

$$x^r(y+z)+y^r(z+x)+z^r(x+y) \leq 2.$$

14. Let x, y, z be non-negative real numbers such that $xy+yz+zx=3$ If $1 < r \leq 2$, then

$$x^r(y+z)+y^r(z+x)+z^r(x+y) \geq 6$$

(*Walther Janous* and *Vasile Cîrtoaje*, CM, 5, 2003)

15. If $x_1, x_2, \ldots, x_n$ are positive numbers such that

$$x_1+x_2+\cdots+x_n=\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n},$$

then

$$\frac{1}{1+(n-1)x_1}+\frac{1}{1+(n-1)x_2}+\cdots+\frac{1}{1+(n-1)x_n} \geq 1.$$

(*Vasile Cîrtoaje*, A M M , 1996)

16. If a, b, c are positive real numbers such that $abc = 1$, then

$$a^3 + b^3 + c^3 + 15 \geq 6\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

(*Michael Rozenberg*, MS, 2006)

17. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. If m is a positive integer satisfying $m \geq n - 1$, then

$$a_1^m + a_2^m + \cdots + a_n^m + (m-1)n \geq m\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

(*Vasile Cîrtoaje*, MS, 2006)

18. Let $x_1, x_2, \ldots, x_n$ be non-negative numbers such that $x_1 + x_2 + \cdots x_n = n$ If k is a positive integer satisfying $2 \leq k \leq n + 2$, and $r = \left(\frac{n}{n-1}\right)^{k-1} - 1$, then

$$x_1^k + x_2^k + \cdots + x_n^k - n \geq nr(1 - x_1 x_2 \ldots x_n)$$

(*Vasile Cîrtoaje*, MS, 2005)

19. Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = n$ Then

$$x_1 + x_2 + \cdots + x_n - n \leq e_{n-1}(x_1 x_2 \ldots x_n - 1),$$

where $e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1} < e$

(*Gabriel Dospinescu* and *Călin Popa*, MS, 2004)

20. Let $x_1, x_2, \ldots, x_n$ be non-negative numbers such that $x_1 + x_2 + \cdots + x_n = n$. If $k \geq 3$ is a positive integer and $r = \frac{n^{k-1} - 1}{n - 1}$, then

$$x_1^k + x_2^k + \cdots + x_n^k - n \leq r\left(x_1^2 + x_2^2 + \cdots + x_n^2 - n\right).$$

(*Vasile Cîrtoaje*, MS, 2006)

21. If $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$x_1^n + x_2^n + \cdots + x_n^n + n(n-1)x_1 x_2 \ldots x_n \geq$$
$$\geq x_1 x_2 \ldots x_n (x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)$$

(*Vasile Cîrtoaje*, MS, 2004

22. If $x_1, x_2, \ldots, x_n$ are non-negative numbers, then

$$(n-1)\left(x_1^n + x_2^n + \cdots + x_n^n\right) + n x_1 x_2 \ldots x_n \ge$$
$$\ge \left(x_1 + x_2 + \cdots + x_n\right)\left(x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1}\right).$$

(*Janos Suranyi*, MSC-Hungary)

23. If $x_1, x_2, \ldots, x_n$ are non-negative numbers, then

$$(n-1)\left(x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1}\right) \ge$$
$$\ge \left(x_1 + x_2 + \cdots + x_n\right)\left(x_1^n + x_2^n + \cdots + x_n^n - x_1 x_2 \ldots x_n\right).$$

(*Vasile Cîrtoaje*, MS, 2006)

24. If $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$(x_1 + x_2 + \cdots + x_n - n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} - n\right) + x_1 x_2 \ldots x_n + \frac{1}{x_1 x_2 \ldots x_n} \ge 2.$$

(*Vasile Cîrtoaje*, MS, 2004)

25. If $x_1, x_2, \ldots, x_n$ are positive numbers such that $x_1 x_2 \ldots x_n = 1$, then

$$\left| \frac{1}{\sqrt{x_1 + x_2 + \cdots + x_n - n}} - \frac{1}{\sqrt{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} - n}} \right| < 1$$

(*Vasile Cîrtoaje*, GM-A, 3, 2004)

26. If $x_1, x_2, \ldots, x_n$ are non-negative numbers such that $x_1 + x_2 + \cdots + x_n = n$, then

$$(x_1 x_2 \ldots x_n)^{\frac{1}{\sqrt{n-1}}}\left(x_1^2 + x_2^2 + \cdots + x_n^2\right) \le n$$

(*Vasile Cîrtoaje*, MS, 2006)

27. Let x, y, z be non-negative numbers such that $xy + yz + zx = 3$, and let $p \ge \dfrac{\ln 9 - \ln 4}{\ln 3} \approx 0.738$. Then,

$$x^p + y^p + z^p \ge 3.$$

(*Vasile Cîrtoaje*, CM, No. 1, 2004)

28. Let x, y, z be non-negative numbers such that $x+y+z=3$, and let $p \geq \dfrac{\ln 9 - \ln 8}{\ln 3 - \ln 2} \approx 0\ 29$ Then,

$$x^p + y^p + z^p \geq xy + yz + zx$$

(*Vasile Cîrtoaje*, MS, 2005)

29. If $x_1, x_2, \ldots, x_n$ $(n \geq 4)$ are non-negative numbers such that

$$x_1 + x_2 + \cdots + x_n = n,$$

then

$$\frac{1}{n+1-x_2x_3\cdots x_n} + \frac{1}{n+1-x_3x_4\cdots x_1} + \cdots + \frac{1}{n+1-x_1x_2\cdots x_{n-1}} \leq 1$$

(*Vasile Cîrtoaje*, MS, 2004)

30. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b^2)} + \frac{1}{(1+c)^2} + \frac{2}{(1+a)(1+b)(1+c)} \geq 1.$$

31. Let a, b, c be non-negative numbers such that $a+b+c \geq 2$ and $ab+bc+ca \geq 1$ If $0 < r < 1$, then

$$a^r + b^r + c^r \geq 2.$$

(*Vasile Cîrtoaje*, MS, 2006)

32. Let a, b, c be positive numbers such that $(a+b+c)^3 = 32abc$. Find the minimum and the maximum of

$$E = \frac{a^4 + b^4 + c^4}{(a+b+c)^4}$$

(*Tran Nam Dung*, Vietnam, 2004)

33. Let $x_1, x_2, \ldots, x_n$ $(n \geq 3)$ be non-negative real numbers such that

$$\sum x_1 = 1.$$

If $m \in \{3, 4, \ldots, n\}$, then

$$1 + \frac{3m}{m-2}\sum x_1x_2x_3 \geq \frac{3m-1}{m-1}\sum x_1x_2.$$

(*Vasile Cîrtoaje*, MS, 2006)

34. Let x, y, z, t be non-negative real numbers such that

$$x^2 + y^2 + z^2 + t^2 = 1$$

Prove that

$$x^3 + y^3 + z^3 + t^3 + xyz + yzt + ztx + txy \leq 1.$$

(*Vasile Cîrtoaje* and *Pham Kim Hung*, MS, 2006)

5.3 Solutions

1. *If x, y, z are non-negative real numbers, then*

$$x^4(y+z) + y^4(z+x) + z^4(x+y) \leq \frac{1}{12}(x+y+z)^5.$$

Proof. Rewrite the inequality as

$$x^5 + y^5 + z^5 + \frac{1}{12}(x+y+z)^5 \geq (x+y+z)(x^4+y^4+z^4),$$

and apply Corollary 5 (case $p = 4$ and $q = 5$):

- *If $0 \leq x \leq y \leq z$ such that*

$$x + y + z = \text{constant} \ \ \textit{and} \ \ x^4 + y^4 + z^4 = \text{constant},$$

then the sum $x^5 + y^5 + z^5$ is minimal when either $x = 0$ or $0 < x \leq y = z$

Case $x = 0$. The inequality becomes

$$(y+z)(y^2 - 4yz + z^2)^2 \geq 0$$

Case $0 < x \leq y = z$ The inequality reduces to

$$(x+2y)^5 - 24x^4y - 24y^4(x+y) \geq 0$$

Since $(x+2y)^5 > (2y)^3(x+2y)^2$, it is enough to show that

$$y^2(x+2y)^2 - 3x^4 - 3y^3(x+y) \geq 0.$$

Indeed, we have

$$y^2(x+2y)^2 - 3x^4 - 3y^3(x+y) = y^4 - x^4 + x(y^3 - x^3) + x^2(y^2 - x^2) \geq 0$$

For $x \leq y \leq z$, one has equality when $(x, y, z) \sim \left(0, 3-\sqrt{3}, 3+\sqrt{3}\right)$. □

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2. *If x, y, z are non-negative real numbers such that $xy + yz + zx = 1$, then*

$$x + y + z + 3\left(2\sqrt{3} - 3\right)xyz \geq 2.$$

Proof We write the hypothesis in the form

$$(x + y + z)^2 = 2 + x^2 + y^2 + z^2,$$

the apply Corollary 4 (case $p = 2$)

- *If $0 \leq x \leq y \leq z$ such that*

$$x + y + z = \text{constant} \quad \textit{and} \quad x^2 + y^2 + z^2 = \text{constant},$$

then the product xyz is minimal when either $x = 0$ or $0 < x \leq y = z$.

Case $x = 0$. We must show that $yz = 1$ implies $y + z \geq 2$; this immediately follows from $y + z \geq 2\sqrt{yz}$.

Case $0 < x \leq y = z$ The hypothesis $xy + yz + zx = 1$ reduces to $2xy = (1 - y)(1 + y)$, and the inequality becomes successively:

$$x + 2y + 3\left(2\sqrt{3} - 3\right)xy^2 \geq 2,$$

$$x + 3\left(2\sqrt{3} - 3\right)xy^2 \geq \frac{4xy}{1 + y},$$

$$1 + 3\left(2\sqrt{3} - 3\right)y^2 \geq \frac{4y}{1 + y},$$

$$1 - 3y + 3\left(2\sqrt{3} - 3\right)y^2 + 3\left(2\sqrt{3} - 3\right)y^3 \geq 0,$$

$$\left(1 - \sqrt{3}y\right)^2\left[1 + \left(2\sqrt{3} - 3\right)y\right] \geq 0.$$

The last inequality is clearly true. For $x \leq y \leq z$, we have equality when either $(x, y, z) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ or $(x, y, z) = (0, 1, 1)$. □

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3. *If x, y, z are non-negative real numbers such that $ab + bc + ca = 1$, then*

$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} - \frac{1}{a + b + c} \geq 2$$

Proof. Let $b+c=2x$, $c+a=2y$ and $a+b=2z$. We have to prove that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{2}{x+y+z}\geq 4,$$

for $2(x^2+y^2+z^2)+1=(x+y+z)^2$ To do it, we will apply Corollary 5 (case $p=2$ and $q=-1$).

- *If* $0<x\leq y\leq z$ *such that*

$$x+y+z=\text{constant } \textit{and } x^2+y^2+z^2=\text{constant},$$

then the expression $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ *is minimal when* $0<x=y\leq z$

The case $0<x=y\leq z$ is equivalent to $a=b\geq c$ The hypothesis condition $ab+bc+ca=1$ reduces to $c=\frac{1-a^2}{2a}$, $0<a\leq 1$ We have

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}-\frac{1}{a+b+c}-2=\frac{1}{2a}+\frac{2}{a+c}-\frac{1}{2a+c}-2=$$
$$=\left(\frac{1}{2a}-\frac{1}{2a+c}\right)-2\left(1-\frac{1}{a+c}\right)=\frac{1-a^2}{2a(1+3a^2)}-\frac{2(1-a)^2}{1+a^2}=$$
$$=\frac{(1-a)(1-3a+5a^2-11a^3+12a^4}{2a(1+3a^2)(1+a^2)}.$$

Since $1-a\geq 0$, we need to show that

$$1-3a+5a^2-11a^3+12a^4\geq 0.$$

Indeed, we get

$$1-3a+5a^2-11a^3+12a^4=\left(1-\frac{3a}{2}\right)^2+11a^2\left(\frac{1}{2}-a\right)^2+a^4>0$$

For $a\geq b\geq c$, one has equality if and only if $(a,b,c)=(1,1,0)$. □

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4. *Let* x,y,z,t *be non-negative real numbers such that* $x+y+z+t=3$. *Prove that*

$$x^2y^2z^2+y^2z^2t^2+z^2t^2x^2+t^2x^2y^2\leq 1.$$

Proof Without loss of generality, we may assume that $x \le y \le z \le t$ For $x = 0$, the inequality becomes $y^2z^2t^2 \le 1$, with $y + z + t = 3$ From AM-GM Inequality

$$yzt \le \left(\frac{y+z+t}{3}\right)^3,$$

we get $yzt \le 1$, and hence $y^2z^2t^2 \le 1$.

For $x > 0$, rewrite the inequality in the form

$$(xyzt)^2\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{t^2}\right) \le 1,$$

and apply Corollary 5 (case $p = 0$ and $q = -2$):

- *If* $0 < x \le y \le z \le t$ *such that* $x + y + z + t = 3$ *and* $xyzt =$ constant, *then the expression* $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{t^2}$ *is maximal when* $0 < x \le y = z = t$

For $0 < x \le y = z = t$, from $x + y + z + t = 3$, we get $0 < y = z = t < 1$ and $x = 3(1 - y)$
The inequality reduce to

$$3x^2y^4 + y^6 \le 1.$$

By AM-GM Inequality, we get

$$x\frac{3y}{2}\cdot\frac{3y}{2} < \left(\frac{x+\frac{3y}{2}+\frac{3y}{2}}{3}\right)^3 = 1,$$

hence $xy^2 < \frac{4}{9}$ Thus, it suffices to show that $\frac{4}{3}xy^2 + y^6 \le 1$. Indeed, we have

$$\begin{aligned}1 - y^6 - \frac{4}{3}xy^2 &= 1 - y^6 - 4(1-y)y^2 = \\ &= (1-y)(1 + y - 3y^2 + y^3 + y^4 + y^5) > \\ &> (1-y)(1 + y - 3y^2 + y^4) = \\ &= (1-y)\left[(1-y^2)^2 + y(1-y)\right] > 0\end{aligned}$$

Equality occurs when $(x, y, z, t) = (0, 1, 1, 1)$ □

Remark This application solves the problem posted by *Gabriel Dospinescu* on Mathlinks Site-Inequalities Forum, in June 2005:

• *If x, y, z, t are non-negative numbers such that $x+y+z+t=4$, what is the maximum value of $x^2y^2z^2+y^2z^2t^2+z^2t^2x^2+t^2x^2y^2$?*

The maximum value is $\left(\frac{4}{3}\right)^6$, and is attained for $(x,y,z,t)=\left(0,\frac{4}{3},\frac{4}{3},\frac{4}{3}\right)$. To obtain this result we have to replace x, y, z, t in the above inequality by $\frac{3x}{4}, \frac{3y}{4}, \frac{3z}{4}, \frac{3t}{4}$, respectively

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5. *Let x, y, z, t be non-negative real numbers such that $x+y+z+t=4$ Prove that*

$$xyz+yzt+ztx+txy+x^2y^2z^2+y^2z^2t^2+z^2t^2x^2+t^2x^2y^2 \le 8$$

Proof Assume that $x \le y \le z \le t$ For $x=0$, the inequality reduces to $yzt+y^2z^2t^2 \le 8$, with $y+z+t=4$ From AM-GM Inequality

$$27yzt \le (y+z+t)^3,$$

we get $yzt \le \frac{64}{27}$, then

$$\frac{yzt+y^2z^2t^2}{8}=\frac{yzt}{8}(1+yzt) \le \frac{8}{27}\left(1+\frac{64}{27}\right)=\frac{728}{729}<1$$

For $x>0$, rewrite the inequality in the form

$$xyzt\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)+x^2y^2z^2t^2\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{t^2}\right) \le 8,$$

and apply Corollary 5 (case $p=0$ and $q<0$):

• *If $0<x\le y\le z\le t$ such that $x+y+z+t=4$ and $xyzt=$ constant, then the sums $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}$ and $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{t^2}$ are maximal when $0<x\le y=z=t$*

For $0<x\le y=z=t$, from $x+y+z+t=4$, we get $1\le y=z=t<\frac{4}{3}$ and $x=4-3t$. The inequality reduces to

$$3xt^2+t^3+3x^2t^4+t^6 \le 8,$$

$$4(7t^6-18t^5+12t^4-2t^3+3t^2-2) \le 0,$$

$$4t^2(t-1)^2E(t) \le 0,$$

where $E(t) = 7 - \frac{4}{t} - \frac{3}{t^2} - \frac{4}{t^3} - \frac{2}{t^4}$ Since $E(t) < E\left(\frac{4}{3}\right) = \frac{-1}{128}$, the last inequality is clearly true Equality occurs when $(x, y, z, t) = (1, 1, 1, 1)$. □

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6. *Let x, y, z be non-negative real numbers such that $xy + yz + zx = 3$. Then*

$$\sqrt{\frac{1+2x}{3}} + \sqrt{\frac{1+2y}{3}} + \sqrt{\frac{1+2z}{3}} \geq 3$$

Proof. We write the condition $xy + yz + zx = 3$ in the form

$$(x + y + z)^2 = 6 + x^2 + y^2 + z^2,$$

and then apply Corollary 1 to the function $f(u) = \sqrt{\frac{1+2u}{3}}$, $u \geq 0$ We have $g(x) = f'(x) = \frac{1}{\sqrt{3(1+2x)}}$ and from

$$g''(x) = \sqrt{3}(1 + 2x)^{\frac{-5}{2}} > 0,$$

it follows that $g(x)$ is strictly convex for $x \geq 0$. According to Corollary 1, *if* $0 \leq x \leq y \leq z$ *such that* $x + y + z =$ constant *and* $x^2 + y^2 + z^2 =$ constant, *then the sum*

$$\sqrt{\frac{1+2x}{3}} + \sqrt{\frac{1+2y}{3}} + \sqrt{\frac{1+2z}{3}}$$

is minimal when either $x = 0$ *or* $0 < x \leq y = z$.

Case $x = 0$. We have to show that

$$\sqrt{1+2y} + \sqrt{1+2z} \geq 3\sqrt{3} - 1 \quad \text{for} \quad yz = 3$$

By squaring, the inequality becomes

$$y + z + \sqrt{13 + 2(y+z)} \geq 13 - 3\sqrt{3}.$$

Indeed, we have $y + z \geq 2\sqrt{yz} = 2\sqrt{3}$, and therefore

$$y + z + \sqrt{13 + 2(y+z)} \geq 2\sqrt{3} + \sqrt{13 + 4\sqrt{3}} > 13 - 3\sqrt{3}.$$

Case $0 < x \le y = z$ From $xy + yz + zx = 3$, we get $x = \dfrac{3 - y^2}{2y}$, $0 < y \le \sqrt{3}$ The inequality becomes

$$\sqrt{1 + \frac{3 - y^2}{y}} + 2\sqrt{1 + 2y} \ge 3\sqrt{3}$$

Let us denote $t = \sqrt{\dfrac{1 + 2y}{3}}$, $\dfrac{1}{\sqrt{3}} < t \le \sqrt{\dfrac{1 + 2\sqrt{3}}{3}} < \dfrac{5}{4}$. The inequality transforms into

$$\sqrt{\frac{3 + 4t^2 - 3t^4}{2(3t^2 - 1)}} \ge 3 - 2t.$$

By squaring and dividing by 3, the inequality becomes

$$7 - 8t - 14t^2 + 24t^3 - 9t^4 \ge 0,$$

or, equivalently,

$$(1 - t)^2(7 + 6t - 9t^2) \ge 0$$

This inequality is true, because

$$7 + 6t - 9t^2 = 8 - (3t - 1)^2 > 8 - \left(\frac{15}{4} - 1\right)^2 = \frac{7}{16} > 0.$$

Equality occurs if and only if $(x, y, z) = (1, 1, 1)$. □

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7. *Let x, y, z be non-negative real numbers, no two of which are zero Then*

$$\frac{1}{(x + y)^2} + \frac{1}{(y + z)^2} + \frac{1}{(z + x)^2} \ge \frac{9}{4(xy + yz + zx)}.$$

Proof. Due to homogeneity, we may consider that $x + y + z = 1$. On this assumption, the inequality becomes

$$\frac{1}{(1 - x)^2} + \frac{1}{(1 - y)^2} + \frac{1}{(1 - z)^2} \ge \frac{9}{2 - 2(x^2 + y^2 + z^2)}.$$

To prove it, we will apply Corollary 1 to the function $f(u) = \dfrac{1}{(1 - u)^2}$, $0 \le u < 1$ We have $g(x) = f'(x) = \dfrac{2}{(1 - x)^3}$ and from

$$g''(x) = \frac{24}{1 - x)^5} > 0,$$

it follows that the function $g(x)$ is strictly convex for $0 \le x < 1$. According to Corollary 1 and Remark from section 5.1, *if* $0 \le x \le y \le z$ *such that* $x+y+z=1$ *and* $x^2+y^2+z^2 =$ constant, *then the sum*

$$\frac{1}{(1-x)^2}+\frac{1}{(1-y)^2}+\frac{1}{(1-z)^2}$$

is minimal when either $x=0$ *or* $0<x\le y=z$

Case $x=0$ The original inequality becomes

$$\frac{1}{y^2}+\frac{1}{(y+z)^2}+\frac{1}{z^2}\ge\frac{9}{4yz},$$

or

$$\frac{(y-z)^2(4y^2+7yz+4z^2)}{4y^2z^2(y+z)^2}\ge 0$$

Case $0<x\le y=z$. The original inequality becomes

$$\frac{2}{(x+y)^2}+\frac{1}{4y^2}\ge\frac{9}{4(2xy+y^2)},$$

or

$$\frac{x(x-y)^2}{2y^2(x+y)^2(2x+y)}\ge 0.$$

Equality occurs for $(x,y,z)\sim(1,1,1)$, as well as for $(x,y,z)\sim(0,1,1)$ or any cyclic permutation □

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8. *Let* x,y,z *be non-negative real numbers, no two of which are zero. If* $0\le r\le\frac{5}{2}$, *then*

$$\sum\frac{1}{y^2+yz+z^2}\ge\frac{3(1+r)}{x^2+y^2+z^2+r(xy+yz+zx)}$$

Proof. Due to homogeneity, we may consider $x+y+z=3$ Let

$$p=\frac{9+x^2+y^2+z^2}{6}.$$

Since

$$\frac{1}{2(y^2+yz+z^2)}=\frac{1}{(x+y+z)^2+x^2+y^2+z^2-2x(x+y+z)}=$$
$$=\frac{1}{6(p-x)},$$

the inequality becomes

$$\frac{1}{p-x}+\frac{1}{p-y}+\frac{1}{p-z}\geq\frac{3(1+r)}{2p-3+r(3-p)}.$$

To prove the inequality, we will apply Corollary 1 to the function

$$f(u)=\frac{1}{p-u},\ 0\leq u<p.$$

We have $g(x)=f'(x)=\dfrac{1}{(p-x)^2}$ and $g''(x)=\dfrac{6}{(p-x)^4}>0$
Therefore, $g(x)$ is strictly convex for $0\leq x<p$ According to Corollary 1 and Remark from the section 5 1, *if* $0\leq x\leq y\leq z$ *such that* $x+y+z=3$ *and* $x^2+y^2+z^2=$ constant, *then the sum*

$$\frac{1}{p-x}+\frac{1}{p-y}+\frac{1}{p-z}$$

is minimal when either $x=0$ *or* $0<x\leq y=z$

Case $x=0$ The original inequality becomes

$$\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{y^2+yz+z^2}\geq\frac{3(1+r)}{y^2+z^2+ryz},$$

or

$$s+\frac{1}{s+1}\geq\frac{3(1+r)}{s+r},$$

where $s=\dfrac{y}{z}+\dfrac{z}{y}$, $s\geq 2$. Write this inequality as

$$s^3+s^2-2s-3+r(s^2-2s-2)\geq 0.$$

Since $s^2-2s-2=(s-2)^2+2(s-1)>0$, it suffices to consider $r=\dfrac{5}{2}$
In this case, the inequality becomes $(s-2)(2s^2+11s+8)\geq 0$.

Case $0<x\leq y=z$ The original inequality becomes

$$\frac{2}{x^2+xy+y^2}+\frac{1}{3y^2}\geq\frac{3(1+r)}{x^2+2y^2+r(2xy+y^2)}.$$

Since the inequality is homogeneous, we may consider $x\leq y=1$. On this assumption, the inequality is equivalent to

$$x^4+x^3-7x+5+2r(x-1)^3\geq 0,$$

or

$$(x-1)^2\left[x^2+3x+5+2r(x-1)\right]\geq 0.$$

Since

$$x^2+3x+5+2r(x-1)=x^2+8x+(5-2r)(1-x)>0,$$

the proof is completed.

Equality occurs for $(x,y,z)\sim(1,1,1)$. In the particular case $r=\frac{5}{2}$, equality holds again for $(x,y,z)\sim(0,1,1)$ or any cyclic permutation.

Remark. For $r=2$, we get the known inequality

$$\sum\frac{1}{y^2+yz+z^2}\geq\frac{9}{(x+y+z)^2}.$$

□

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9. *Let x,y,z be non-negative real numbers such that $x+y+z=3$. If $r\geq\frac{8}{5}$, then*

$$\frac{1}{r+x^2+y^2}+\frac{1}{r+y^2+z^2}+\frac{1}{r+z^2+x^2}\leq\frac{3}{r+2}$$

Proof. Let $p=r+x^2+y^2+z^2$. We have show that

$$\frac{1}{p-x^2}+\frac{1}{p-y^2}+\frac{1}{p-z^2}\leq\frac{3}{r+2}$$

for $x+y+z=3$ and $x^2+y^2+z^2=p-r$ To prove this, we will apply Corollary 1 to the function $f(u)=\frac{1}{p-u^2}$, $0\leq u<\sqrt{p}$. We have

$$g(x)=f'(x)=\frac{2x}{(p-x^2)^2}$$

and

$$g''(x)=\frac{24x(p+x^2)}{(p-x^2)^4}.$$

Since $g''(x)>0$ for $x>0$, the function $g(x)$ is strictly convex for $0\leq x<\sqrt{p}$. According to Corollary 1, *if* $0\leq x\leq y\leq z$ *such that* $x+y+z=3$ *and* $x^2+y^2+z^2=$ constant, *then the sum*

$$\frac{1}{p-x^2}+\frac{1}{p-y^2}+\frac{1}{p-z^2}$$

is maximal when $0 \le x = y \le z$ Therefore, it suffices to consider only the case $x = y$ We have to show that for $2x + z = 3$ the inequality holds

$$\frac{1}{r + 2x^2} + \frac{2}{r + x^2 + z^2} \le \frac{3}{r + 2}$$

Write the inequality as follows

$$\frac{1}{r + 2x^2} + \frac{2}{r + 9 - 12x + 5x^2} \le \frac{3}{r + 2},$$

$$5x^4 - 12x^3 + (2r + 6)x^2 - 4(r - 1)x + 2r - 3 \ge 0,$$

$$(x - 1)^2(5x^2 - 2x + 2r - 3) \ge 0$$

Since

$$5x^2 - 2x + 2r - 3 = 5\left(x - \frac{1}{5}\right)^2 + 2\left(r - \frac{8}{5}\right) \ge 0,$$

the last inequality is clearly true. Equality occurs for $(x, y, z) = (1, 1, 1)$ In the case $r = \frac{8}{5}$, equality occurs again for $(x, y, z) = \left(\frac{1}{5}, \frac{1}{5}, \frac{13}{5}\right)$ or any cyclic permutation □

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10. *Let x, y, z be non-negative numbers such that $x^2 + y^2 + z^2 = 3$. If $r \ge 10$, then*

$$\frac{1}{r - (x + y)^2} + \frac{1}{r - (y + z)^2} + \frac{1}{r - (z + x)^2} \le \frac{3}{r - 4}.$$

Proof Let $s = x + y + z$ We have to show that

$$\frac{1}{r - (s - x)^2} + \frac{1}{r - (s - y)^2} + \frac{1}{r - (s - z)^2} \le \frac{3}{r - 4}$$

for $x + y + z = s$ and $x^2 + y^2 + z^2 = 3$. Apply Corollary 1 to the function $f(u) = \frac{-1}{r - (s - u)^2}$ for $0 \le u \le s$. We have $g(x) = f'(x) = \frac{2(s - x)}{[r - (s - x)^2]^2}$ and

$$g''(x) = \frac{24(s - x)\left[r + (s - x)^2\right]}{[r - (s - x)^2]^4}$$

Since $g''(x) > 0$ for $0 \le x < s$, the function $g(x)$ is strictly convex for $0 \le x \le s$ According to Corollary 1, *if* $0 \le x \le y \le z$ *such that*

$$x + y + z = \text{constant} \ \ \textit{and} \ \ x^2 + y^2 + z^2 = 3,$$

then the sum

$$\frac{-1}{r-(s-x)^2}+\frac{-1}{r-(s-y)^2}+\frac{-1}{r-(s-z)^2}$$

is minimal for either $x=0$ or $0<x\le y=z$. Therefore, it suffices to consider only the cases $x=0$ and $0<x\le y=z$

Case $x=0$. We have to show that $y^2+z^2=3$ implies

$$\frac{1}{r-y^2}+\frac{1}{r-z^2}+\frac{1}{r-(y+z)^2}\le\frac{3}{r-4}$$

Since

$$\frac{1}{r-y^2}+\frac{1}{r-z^2}=\frac{2r-3}{r^2-3r+y^2z^2}\le\frac{2r-3}{r(r-3)}$$

and $(y+z)^2\le 2(y^2+z^2)=6$, it suffices to prove that

$$\frac{2r-3}{r(r-3)}+\frac{1}{r-6}\le\frac{3}{r-4}.$$

This inequality reduces to

$$\frac{3(r^2-12r+24)}{r(r-3)(r-4)(r-6)}\ge 0,$$

and it is true because $r^2-12r+24=(r-2)(r-10)+4>0$.

Case $0<x\le y=z$. Write the inequality in the homogeneous form

$$\sum\frac{1}{r(x^2+y^2+z^2)-3(y+z)^2}\le\frac{3}{(r-4)(x^2+y^2+z^2)}$$

Since $y=z>0$, we may consider $y=z=1$ Setting $t=r(x^2+2)$, $t>2r\ge 20$, the inequality becomes

$$\frac{1}{t-12}+\frac{2}{t-3x^2-6x-3}\le\frac{3}{t-4x^2-8},$$

or

$$\frac{6(x-1)^2(t-2x^2-8x-18)}{(t-12)(t-3x^2-6x-3)(t-4x^2-8)}\ge 0.$$

The last inequality is true because

$$t-2x^2-8x-18=r(x^2+2)-2x^2-8x-18\ge$$
$$\ge 10(x^2+2)-2x^2-8x-18=2(2x-1)^2\ge 0$$

Equality occurs for $(x,y,z)=(1,1,1)$. In the case $r=10$, equality occurs again for $(x,y,z)=\left(\frac{1}{\sqrt3},\frac{2}{\sqrt3},\frac{2}{\sqrt3}\right)$ or any cyclic permutation. □

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11. *If x, y, z are non-negative real numbers, then*

$$\frac{yz}{3x^2+y^2+z^2}+\frac{zx}{3y^2+z^2+x^2}+\frac{xy}{3z^2+x^2+y^2}\le\frac{3}{5}$$

Proof Replacing x, y, z by $\sqrt{x}, \sqrt{y}, \sqrt{z}$ respectively, the inequality transforms into

$$\frac{\sqrt{yz}}{3x+y+z}+\frac{\sqrt{zx}}{3y+z+x}+\frac{\sqrt{xy}}{3z+x+y}\le\frac{3}{5}.$$

Without loss of generality, we may assume that $x \le y \le z$. For $x = 0$, the inequality reduces to $3\left(\sqrt{y}-\sqrt{z}\right)^2+\sqrt{yz}\ge 0$, which is clearly true For $x > 0$, since the inequality is homogeneous, we may assume that $x+y+z = 2$, and then rewrite the inequality in the form

$$\frac{1}{\sqrt{x}(x+1)}+\frac{1}{\sqrt{y}(y+1)}+\frac{1}{\sqrt{z}(z+1)}\le\frac{3}{2\sqrt{xyz}}$$

We will apply now Corollary 3 to the function $f(u)=\dfrac{-1}{\sqrt{u}(u+1)}$, $u > 0$.

We have $f'(u)=\dfrac{3u+1}{2u\sqrt{u}(u+1)^2}$ and

$$g(x)=f'\left(\frac{1}{x}\right)=\frac{x^2\sqrt{x}(x+3)}{2(x+1)^2},$$

$$g''(x)=\frac{\sqrt{x}(3x^2+11x^2+5x+45)}{8(x+1)^4}$$

Since $g''(x) > 0$ for $x > 0$, $g(x)$ is strictly convex on $(0, \infty)$. According to Corollary 3, *if* $0 < x \le y \le z$ *such that* $x + y + z = 2$ *and* $xyz =$ constant, *then the sum*

$$\frac{-1}{\sqrt{x}(x+1)}+\frac{-1}{\sqrt{y}(y+1)}+\frac{-1}{\sqrt{z}(z+1)}$$

is maximal when $0 < x \le y = z$

Therefore, it suffices to prove the original inequality for $y = z > 0$. Moreover, due to the homogeneity, we may consider $y = z = 1$ The inequality reduces to $9x^4 - 30x^3 + 37x^2 - 20x + 4 \ge 0$, which is equivalent to

$$(x-1)^2(3x-2)^2\ge 0.$$

Equality occurs for $(x, y, z) \sim (1, 1, 1)$, and also for $(x, y, z) \sim \left(\frac{2}{3}, 1, 1\right)$ or any cyclic permutation ☐

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12. *Let x, y, z be non-negative real numbers such that $x + y + z = 2$. Prove that*

$$\frac{yz}{x^2+1} + \frac{zx}{y^2+1} + \frac{xy}{z^2+1} \le 1.$$

Proof. We assume that $x \le y \le z$ For $x = 0$, the inequality reduces to $yz \le 1$, which is clearly true for $y + z = 2$ Otherwise, we rewrite the inequality in the form

$$\frac{1}{x(x^2+1)} + \frac{1}{y(y^2+1)} + \frac{1}{z(z^2+1)} \le \frac{1}{xyz}$$

and apply Corollary 3 to the function $f(u) = \dfrac{-1}{u(u^2+1)}$, $u > 0$ We have $f'(u) = \dfrac{3u^2+1}{u^2(u^2+1)^2}$ and

$$g(x) = f'\left(\frac{1}{x}\right) = \frac{x^4(x^2+3)}{(x^2+1)^2},$$

$$g''(x) = \frac{2x^2(x^6+5x^4-7x^2+12)}{(x^2+1)^4}.$$

Since $g''(x) > 0$ for $x > 0$, $g(x)$ is strictly convex on $(0, \infty)$. According to Corollary 3, *if* $0 < x \le y \le z$ *such that* $x + y + z = 2$ *and* $xyz =$ constant, *then the sum*

$$\frac{-1}{x(x^2+1)} + \frac{-1}{y(y^2+1)} + \frac{-1}{z(z^2+1)}$$

is minimal when $0 < x \le y = z$.

For $0 < x \le y = z$, from $x + y + z = 2$ we get $0 < y = z \le 1$ and $x = 2(1 - y)$ The inequality becomes

$$(y-1)^2(19y^2 - 18y + 5) \ge 0,$$

which is clearly true. For $x \le y \le z$, equality occurs $(x, y, z) = (0, 1, 1)$. □

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13. *Let x, y, z be non-negative real numbers such that $x + y + z = 2$. If $r_0 \le r \le 3$, where $r_0 = \dfrac{\ln 2}{\ln 3 - \ln 2} \approx 1\ 71$, then*

$$x^r(y+z) + y^r(z+x) + z^r(x+y) \le 2.$$

Proof Rewrite the inequality in the homogeneous form

$$x^{r+1}+y^{r+1}+z^{r+1}+2\left(\frac{x+y+z}{2}\right)^{r+1}\ge (x+y+z)(x^r+y^r+z^r),$$

and apply Corollary 5 (case $p=r$ and $q=r+1$)

• *For* $0\le x\le y\le z$ *such that*

$$x+y+z=\text{constant } \textit{and } x^r+y^r+z^r=\text{constant},$$

the sum $x^{r+1}+y^{r+1}+z^{r+1}$ *is minimal when either* $x=0$ *or* $0<x\le y=z$.

Case $x=0$. The initial inequality becomes

$$yz(y^{r-1}+z^{r-1})\le 2,$$

where $y+z=2$. Since $0<r-1\le 2$, by the Power Mean Inequality we have

$$\frac{y^{r-1}+z^{r-1}}{2}\le\left(\frac{y^2+z^2}{2}\right)^{\frac{r-1}{2}}.$$

Thus, it suffices to show that

$$yz\left(\frac{y^2+z^2}{2}\right)^{\frac{r-1}{2}}\le 1.$$

Taking account of $\dfrac{y^2+z^2}{2}=\dfrac{2(y^2+z^2)}{(y+z)^2}\ge 1$ and $\dfrac{r-1}{2}\le 1$, we have

$$1-yz\left(\frac{y^2+z^2}{2}\right)^{\frac{r-1}{2}}\ge 1-yz\left(\frac{y^2+z^2}{2}\right)=$$

$$=\frac{(y+z)^4}{16}-\frac{yz(y^2+z^2)}{2}=\frac{(y-z)^4}{16}\ge 0.$$

Case $0<x\le y=z$ In the homogeneous inequality we may leave aside the constraint $x+y+z=2$, and consider $y=z=1$, $0<x\le 1$. The inequality reduces to

$$\left(1+\frac{x}{2}\right)^{r+1}-x^r-x-1\ge 0.$$

Since $\left(1+\dfrac{x}{2}\right)^{r+1}$ is increasing and x^r is decreasing when r is increasing, it suffices to consider the case $r=r_0$ Let

$$f(x)=\left(1+\frac{x}{2}\right)^{r_0+1}-x^{r_0}-x-1$$

We have

$$f'(x) = \frac{r_0+1}{2}\left(1+\frac{x}{2}\right)^{r_0} - r_0 x^{r_0-1} - 1,$$
$$\frac{1}{r_0} f''(x) = \frac{r_0+1}{4}\left(1+\frac{x}{2}\right)^{r_0} - \frac{r_0-1}{x^{2-r_0}}$$

Since $f''(x)$ is strictly increasing on $(0,1]$, $f''(0_+) = -\infty$ and

$$\frac{1}{r_0} f''(1) = \frac{r_0+1}{4}\left(\frac{3}{2}\right)^{r_0} - r_0 + 1 = \frac{r_0+1}{2} - r_0 + 1 = \frac{3-r_0}{2} > 0,$$

there exists $x_1 \in (0,1)$ such that $f''(x_1) = 0$, $f''(x) < 0$ for $x \in (0, x_1)$, and $f''(x) > 0$ for $x \in (x_1, 1]$ Therefore, the function $f'(x)$ is strictly decreasing for $x \in [0, x_1]$, and strictly increasing for $x \in [x_1, 1]$ Since $f'(0) = \dfrac{r_0-1}{2} > 0$ and $f'(1) = \dfrac{r_0+1}{2}\left[\left(\dfrac{3}{2}\right)^{r_0} - 2\right] = 0$, there exists $x_2 \in (0, x_1)$ such that $f'(x_2) = 0$, $f'(x) > 0$ for $x \in [0, x_2)$, and $f'(x) < 0$ for $x \in (x_2, 1)$ Thus, the function $f(x)$ is strictly increasing for $x \in [0, x_2]$, and strictly decreasing for $x \in [x_2, 1]$. Since $f(0) = f(1) = 0$, it follows that $f(x) \geq 0$ for $0 < x \leq 1$, establishing the desired result

For $x \leq y \leq z$, equality occurs when $(x, y, z) = (0, 1, 1)$ Moreover, for $r = r_0$, equality holds again when $(x, y, z) = \left(\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right)$. □

Remark Using the above way, we can show that for $r > 3$ and $x + y + z = 2$, the expression

$$E(x,y,z) = x^r(y+z) + y^r(z+x) + z^r(x+y)$$

attains its maximal value when one of the numbers x, y, z is equal to zero. To prove this claim, it suffices to show that the inequality $E(x, y, z) \leq 2$ holds for $0 < x \leq y = z$, while is doesn't hold for $x = 0$ and any non-negative numbers y and z satisfying $y + z = 2$ Indeed, for $y = z = 1$ and $0 < x \leq 1$, by Bernoulli's Inequality, we get

$$\left(1+\frac{x}{2}\right)^{r+1} - x^r - x - 1 > 1 + \frac{(r+1)x}{2} - x^r - x - 1 = \frac{(r-1)x}{2} - x^r > x - x^r \geq 0.$$

In the special case $r = 4$, $E(x, y, z)$ is maximal when

$$(x,y,z) = \left(0, \frac{3-\sqrt{3}}{3}, \frac{3+\sqrt{3}}{3}\right),$$

as we have shown in the above application 1

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14. *Let x, y, z be non-negative real numbers such that $xy + yz + zx = 3$. If $1 < r \le 2$, then*

$$x^r(y+z) + y^r(z+x) + z^r(x+y) \ge 6$$

Proof. Rewrite the inequality in the homogeneous form

$$x^r(y+z) + y^r(z+x) + z^r(x+y) \ge 6\left(\frac{xy+yz+zx}{3}\right)^{\frac{r+1}{2}}.$$

For convenience, we may leave aside the condition $xy + yz + zx = 3$ Using now the condition $x + y + z = 1$, the inequality becomes

$$x^r(1-x) + y^r(1-y) + z^r(1-z) \ge 6\left(\frac{1-x^2-y^2-z^2}{6}\right)^{\frac{r+1}{2}}$$

Towards proving it, we will apply Corollary 1 to the function $f(u) = -u^r(1-u)$ for $0 \le u \le 1$ We have $f'(u) = -ru^{r-1} + (r+1)u^r$ and

$$g(x) = f'(x) = -rx^{r-1} + (r+1)x^r,$$
$$g''(x) = r(r-1)x^{r-3}[(r+1)x + 2 - r]$$

Since $g''(x) > 0$ for $x > 0$, $g(x)$ is strictly convex on $[0, \infty)$. According to Corollary 1, *if $0 \le x \le y \le z$ such that*

$$x + y + z = 1 \text{ and } x^2 + y^2 + z^2 = \text{constant},$$

the sum $f(x) + f(y) + f(z)$ is minimal for either $x = 0$ or $0 < x \le y = z$

Case $x = 0$. The original inequality becomes

$$yz\left(y^{r-1} + z^{r-1}\right) \ge 6,$$

where $yz = 3$ By the AM-GM Inequality, we have

$$yz\left(y^{r-1} + z^{r-1}\right) \ge 2(yz)^{\frac{r+1}{2}} = 2\ 3^{\frac{r+1}{2}} > 6.$$

Case $0 < x \le y = z$. The original inequality becomes

$$x^r y + y^r(x+y) \ge 3,$$

where $0 < x \leq y$ and $2xy + y^2 = 3$. From $0 < x \leq y$ and $2xy + y^2 = 3$ we obtain $0 < x \leq 1$ Let

$$f(x) = x^r y + y^r(x+y) - 3, \quad \text{with} \quad y = -x + \sqrt{x^2+3}.$$

We have to prove that $f(x) \geq 0$ for $0 < x \leq 1$ For $x=1$, we get $y=1$ and $f(1)=0$. Differentiating the equation $2xy + y^2 = 3$ yields $y' = \frac{-y}{x+y}$ Then,

$$f'(x) = rx^{r-1}y + y^r + \left[x^r + rxy^{r-1} + (r+1)y^r\right]y' =$$
$$= \frac{y\left[(r-1)x + ry\right]\left(x^{r-1} - y^{r-1}\right)}{x+y} \leq 0$$

The function $f(x)$ is strictly decreasing on $[0,1]$, and hence $f(x) \geq f(1) = 0$ for $0 < x \leq 1$ Equality occurs if and only if $(x,y,z) = (1,1,1)$. □

Remark. *Marian Tetiva* found a nice solution for the particular case $r = 2$. Write first the inequality in the form

$$(xy + yz + zx)(x + y + z) \geq 3(xyz + 2),$$

that is

$$x + y + z \geq xyz + 2$$

Assuming that $x \leq y \leq z$, the hypothesis $xy + yz + zx = 3$ implies $xy \leq 1$ and $yz \geq 1$ Hence

$$(1 - xy)(yz - 1) + (1 - y)^2 \geq 0,$$

or

$$y(x + y + z - xyz - 2) \geq 0,$$

from which the conclusion follows

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15. *If $x_1, x_2, \ldots, x_n$ are positive numbers such that*

$$x_1 + x_2 + \cdots + x_n = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n},$$

then

$$\frac{1}{1+(n-1)x_1} + \frac{1}{1+(n-1)x_2} + \cdots + \frac{1}{1+(n-1)x_n} \geq 1.$$

Proof. We have to consider two cases.

Case $n = 2$. The inequality is verified as equality.

Case $n \geq 3$. Assume that $0 < x_1 \leq x_2 \leq \cdots \leq x_n$, and then apply Corollary 2 to the function $f(u) = \dfrac{1}{1+(n-1)u}$ for $u > 0$

We have $f'(u) = \dfrac{-(n-1)}{[1+(n-1)u]^2}$ and

$$g(x) = f'\left(\frac{1}{\sqrt{x}}\right) = \frac{-(n-1)x}{(\sqrt{x}+n-1)^2},$$

$$g''(x) = \frac{3(n-1)^2}{2\sqrt{x}\,(\sqrt{x}+n-1)^4}.$$

Since $g''(x) > 0$, $g(x)$ is strictly convex on $(0, \infty)$. According to Corollary 2, *if* $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ *such that* $x_1 + x_2 + \cdots + x_n =$ constant *and* $\dfrac{1}{x_1} + \dfrac{1}{x_2} + \cdots + \dfrac{1}{x_n} =$ constant, *then the sum* $f(x_1) + f(x_2) + \cdots + f(x_n)$ *is minimal when* $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$

So, we have to prove the inequality

$$\frac{1}{1+(n-1)x} + \frac{n-1}{1+(n-1)y} \geq 1,$$

under the constraints $0 < x \leq 1 \leq y$ and

$$x + (n-1)y = \frac{1}{x} + \frac{n-1}{y}$$

The last relation is equivalent to

$$(n-1)(y-1) = \frac{y(1-x^2)}{x(1+y)}$$

Since

$$\frac{1}{1+(n-1)x} + \frac{n-1}{1+(n-1)y} - 1 =$$

$$= \frac{1}{1+(n-1)x} - \frac{1}{n} + \frac{n-1}{1+(n-1)y} - \frac{n-1}{n} =$$

$$= \frac{(n-1)(1-x)}{n\,[1+(n-1)x]} - \frac{(n-1)^2(y-1)}{n\,[1+(n-1)y]} =$$

$$= \frac{(n-1)(1-x)}{n\,[1+(n-1)x]} - \frac{(n-1)y(1-x^2)}{nx(1+y)\,[1+(n-1)y]},$$

we must show that

$$x(1+y)\left[1+(n-1)y\right] \geq y(1+x)\left[1+(n-1)x\right],$$

which reduces to

$$(y-x)\left[(n-1)xy-1\right] \geq 0.$$

Since $y - x \geq 0$, we still have to prove that

$$(n-1)xy \geq 1.$$

Indeed, from $x+(n-1)y = \frac{1}{x}+\frac{n-1}{y}$ we get $xy = \frac{y+(n-1)x}{x+(n-1)y}$, and hence

$$(n-1)xy - 1 = \frac{n(n-2)x}{x+(n-1)y} > 0.$$

For $n \geq 3$, one has equality if and only if $x_1 = x_2 = \cdots = x_n = 1$. □

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16. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$a^3 + b^3 + c^3 + 15 \geq 6\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Proof. Replacing a, b, c by $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, respectively, we have to show that

$$\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}+15 \geq 6(x+y+z)$$

for $xyz = 1$. Assume that $0 < x \leq y \leq z$ and apply Corollary 5 (case $p = 0$ and $q = -3$):

- *If $0 < x \leq y \leq z$ such that $x+y+z =$ constant and $xyz = 1$, then the sum $\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}$ is minimal when $0 < x = y \leq z$.*

Thus, it suffices to prove the inequality for $0 < x = y \leq 1 \leq z$ and $x^2 z = 1$, when it reduces to.

$$\frac{2}{x^3}+\frac{1}{z^3}+15 \geq 6(2x+z),$$

$$\frac{2}{x^3}+x^6+15 \geq 6\left(2x+\frac{1}{x^2}\right),$$

$$x^9 - 12x^4 + 15x^3 - 6x + 2 \geq 0,$$

$$(1-x)^2\left(2-2x-6x^2+5x^3+4x^4+3x^5+2x^6+x^7\right) \geq 0.$$

The last inequality is true if $2-2x-6x^2+5x^3+3x^4 \geq 0$ for $0 < x \leq 1$. Indeed, we have

$$2\left(2-2x-6x^2+5x^3+3x^4\right) =$$
$$(2-3x)^2\left(1+2x+\frac{3}{4}x^2\right)+x^3\left(1-\frac{3}{4}x\right) > 0.$$

Equality occurs if and only if $a=b=c=1$. □

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17. *Let $a_1, a_2, \dots, a_n$ be positive real numbers such that $a_1a_2\dots a_n = 1$. If m is a positive integer satisfying $m \geq n-1$, then*

$$a_1^m + a_2^m + \cdots + a_n^m + (m-1)n \geq m\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)$$

Proof. For $n=2$ (hence $m \geq 1$), the inequality reduces to

$$a_1^m + a_2^m + 2m - 2 \geq m(a_1+a_2)$$

We can prove it by summing the inequalities $a_1^m \geq 1 + m(a_1-1)$ and $a_2^m \geq 1+m(a_2-1)$, which are straightforward consequences of Bernoulli's Inequality. For $n \geq 3$, replacing $a_1, a_2, \dots, a_n$ by $\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}$, respectively, we have to show that

$$\frac{1}{x_1^m}+\frac{1}{x_2^m}+\cdots+\frac{1}{x_n^m}+(m-1)n \geq m(x_1+x_2+\cdots+x_n)$$

for $x_1x_2\cdots x_n = 1$. Assume that $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ and apply Corollary 5 (case $p=0$ and $q=-m$).

• *If $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ such that $x_1+x_2+\cdots+x_n =$ constant and $x_1x_2\dots x_n = 1$, then the sum $\frac{1}{x_1^m}+\frac{1}{x_2^m}+\cdots+\frac{1}{x_n^m}$ is minimal when $0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n$.*

Thus, it suffices to prove the inequality for $x_1 = x_2 = \cdots = x_{n-1} = x \leq 1$, $x_n = y$ and $x^{n-1}y = 1$, when it reduces to

$$\frac{n-1}{x^m}+\frac{1}{y^m}+(m-1)n \geq m(n-1)x+my$$

By the AM-GM Inequality, we have

$$\frac{n-1}{x^m}+(m-n+1) \geq \frac{m}{x^{n-1}} = my$$

Then, we still have to show that

$$\frac{1}{y^m} - 1 \geq m(n-1)(x-1)$$

This inequality is equivalent to

$$x^{mn-m} - 1 - m(n-1)(x-1) \geq 0.$$

Writing the inequality as

$$(x-1)\left[\left(x^{mn-m-1} - 1\right) + \left(x^{mn-m-2} - 1\right) + \cdots + (x-1)\right] \geq 0,$$

it is clearly true. For $n = 2$ and $m = 1$, the inequality becomes equality. Otherwise, equality occurs if and only if $a_1 = a_2 = \cdots = a_n = 1$. □

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18. *Let $x_1, x_2, \ldots, x_n$ be non-negative numbers such that $x_1 + x_2 + \cdots x_n = n$. If k is a positive integer satisfying $2 \leq k \leq n+2$, and $r = \left(\frac{n}{n-1}\right)^{k-1} - 1$, then*

$$x_1^k + x_2^k + \cdots + x_n^k - n \geq nr(1 - x_1 x_2 \ldots x_n).$$

Proof If $n = 2$, then the inequality reduces to $x_1^k + x_2^k - 2 \geq (2^k - 2)(1 - x_1 x_2)$ For $k = 2$ and $k = 3$, this inequality becomes equality, while for $k = 4$ it reduces to $6x_1x_2(1 - x_1x_2) \geq 0$, which is clearly true.

Consider now $n \geq 3$ and $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$. We will apply Corollary 4 (case $p = k > 0$).

• *If $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ such that $x_1 + x_2 + \cdots + x_n = n$ and $x_1^k + x_2^k + \cdots + x_n^k =$ constant, then the product $x_1 x_2 \ldots x_n$ is minimal when either $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$.*

Case $x_1 = 0$. The inequality reduces to

$$x_2^k + \cdots + x_n^k \geq \frac{n^k}{(n-1)^{k-1}},$$

with $x_2 + \cdots + x_n = n$. This inequality follows by applying Jensen's Inequality to the convex function $f(u) = u^k$:

$$x_2^k + \cdots + x_n^k \geq (n-1)\left(\frac{x_2 + \cdots + x_n}{n-1}\right)^k.$$

Case $0 < x_1 \le x_2 = x_3 = \cdots = x_n$ Denoting

$$x_1 = x \text{ and } x_2 = x_3 = \cdots = x_n = y,$$

we have to prove that for $0 < x \le 1 \le y$ and $x + (n-1)y = n$, the inequality holds:

$$x^k + (n-1)y^k + nrxy^{n-1} - n(r+1) \ge 0$$

We write the inequality as $f(x) \ge 0$, where

$$f(x) = x^k + (n-1)y^k + nrxy^{n-1} - n(r+1), \text{ with } y = \frac{n-x}{n-1}.$$

We see that $f(0) = f(1) = 0$. Since $y' = \dfrac{-1}{n-1}$, we have

$$\begin{aligned} f'(x) &= k\left(x^{k-1} - y^{k-1}\right) + nry^{n-2}(y-x) = \\ &= (y-x)\left[nry^{n-2} - k\left(y^{k-2} + y^{k-3}x + \cdots + x^{k-2}\right)\right] = \\ &= (y-x)y^{n-2}\left[nr - kg(x)\right], \end{aligned}$$

where

$$g(x) = \frac{1}{y^{n-k}} + \frac{x}{y^{n-k+1}} + \cdots + \frac{x^{k-2}}{y^{n-2}}$$

Since the function $y(x) = \dfrac{n-x}{n-1}$ is strictly decreasing, the function $g(x)$ is strictly increasing for $2 \le k \le n$ For $k = n+1$, we have

$$g(x) = y + x + \frac{x^2}{y} + \cdots + \frac{x^{n-1}}{y^{n-2}} = \frac{(n-2)x + n}{n-1} + \frac{x^2}{y} + \cdots + \frac{x^{n-1}}{y^{n-2}},$$

and for $k = n+2$, we have

$$\begin{aligned} g(x) &= y^2 + yx + x^2 + \frac{x^3}{y} + \cdots + \frac{x^n}{y^{n-2}} = \\ &= \frac{(n^2-3n+3)x^2 + n(n-3)x + n^2}{(n-1)^2} + \frac{x^3}{y} + \cdots + \frac{x^n}{y^{n-2}}. \end{aligned}$$

Therefore, the function $g(x)$ is strictly increasing for $2 \le k \le n+2$, and the function

$$h(x) = nr - kg(x)$$

is strictly decreasing. Note that

$$f'(x) = (y-x)y^{n-2}h(x)$$

We assert that $h(0) > 0$ and $h(1) < 0$ If our claim is true, then there exists $x_1 \in (0,1)$ such that $h(x_1) = 0$, $h(x) > 0$ for $x \in [0, x_1)$, and $h(x) < 0$ for $x \in (x_1, 1]$ Consequently, $f(x)$ is strictly increasing for $x \in [0, x_1]$, and strictly decreasing for $x \in [x_1, 1]$ Since $f(0) = f(1) = 0$, it follows that $f(x) \geq 0$ for $0 < x \leq 1$, and the proof is completed.

In order to prove that $h(0) > 0$, we assume that $h(0) \leq 0$ Then, $h(x) < 0$ for $x \in (0,1)$, $f'(x) < 0$ for $x \in (0,1)$, and $f(x)$ is strictly decreasing for $x \in [0,1]$, which contradicts $f(0) = f(1)$. Also, if $h(1) \geq 0$, then $h(x) > 0$ for $x \in (0,1)$, $f'(x) > 0$ for $x \in (0,1)$, and $f(x)$ is strictly increasing for $x \in [0,1]$, which also contradicts $f(0) = f(1)$

For $n \geq 3$ and $x_1 \leq x_2 \leq \cdots \leq x_n$, equality occurs when

$$x_1 = x_2 = \cdots = x_n = 1,$$

and also when $x_1 = 0$ and $x_2 = \cdots = x_n = \dfrac{n}{n-1}$. □

Remark 1. For $k = 2$, $k = 3$ and $k = 4$, we get the following nice inequalities

$$(n-1)\left(x_1^2 + x_2^2 + \cdots + x_n^2\right) + nx_1x_2\ldots x_n \geq n^2,$$

$$(n-1)^2\left(x_1^3 + x_2^3 + \cdots + x_n^3\right) + n(2n-1)x_1x_2\ldots x_n \geq n^3,$$

$$(n-1)^3\left(x_1^4 + x_2^4 + \cdots + x_n^4\right) + n(3n^2 - 3n + 1)x_1x_2\ldots x_n \geq n^4$$

for $x_1, x_2, \ldots, x_n$ non-negative numbers such that $x_1 + x_2 + \cdots + x_n = n$

Remark 2. For $k = n$, the inequality was posted in 2004 on Mathlinks Inequalities Forum by *Gabriel Dospinescu* and *Călin Popa.*

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19. *Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $\dfrac{1}{x_1} + \dfrac{1}{x_2} + \cdots + \dfrac{1}{x_n} = n$ Then*

$$x_1 + x_2 + \cdots + x_n - n \leq e_{n-1}(x_1x_2\ldots x_n - 1),$$

where $e_{n-1} = \left(1 + \dfrac{1}{n-1}\right)^{n-1} < e.$

Proof Replacing each of x_i by $\frac{1}{a_i}$, the statement becomes as follows

- *If $a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$a_1 a_2 \cdots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n + e_{n-1}\right) \le e_{n-1}$$

It is easy to check that the inequality holds for $n = 2$.

Consider now $n \ge 3$, assume that $0 < a_1 \le a_2 \le \cdots \le a_n$ and apply Corollary 4 (case $p = -1$) *If $0 < a_1 \le a_2 \le \cdots \le a_n$ such that*

$$a_1 + a_2 + \cdots + a_n = n \ \textit{and} \ \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = \text{constant},$$

then the product $a_1 a_2 \ldots a_n$ is maximal when $0 < a_1 \le a_2 = a_3 = \cdots = a_n$

Denoting $a_1 = x$ and $a_2 = a_3 = \cdots = a_n = y$, we have to prove that for $0 < x \le 1 \le y < \frac{n}{n-1}$ and $x + (n-1)y = n$, the inequality holds

$$y^{n-1} + (n-1)xy^{n-2} - (n - e_{n-1})xy^{n-1} \le e_{n-1}$$

Setting

$$f(x) = y^{n-1} + (n-1)xy^{n-2} - (n - e_{n-1})xy^{n-1} - e_{n-1},$$

with $y = \frac{n-x}{n-1}$, we must show that $f(x) \le 0$ for $0 < x \le 1$. We see that $f(0) = f(1) = 0$ Since $y' = \frac{-1}{n-1}$, we have

$$\frac{f'(x)}{y^{n-3}} = (y - x)\left[n - 2 - (n - e_{n-1})y\right] = (y - x)h(x),$$

where $h(x) = n - 2 - (n - e_{n-1})\frac{n-x}{n-1}$ is a linear increasing function

Let us show that $h(0) < 0$ and $h(1) > 0$ If $h(0) \ge 0$, then $h(x) > 0$ for $x \in (0,1)$, hence $f'(x) > 0$ for $x \in (0,1)$, and $f(x)$ is strictly increasing for $x \in [0,1]$, which contradicts $f(0) = f(1)$ Also, $h(1) = e_{n-1} - 2 > 0$.

From $h(0) < 0$ and $h(1) > 0$, it follows that there exists $x_1 \in (0,1)$ such that $h(x_1) = 0$, $h(x) < 0$ for $x \in [0, x_1)$, and $h(x) > 0$ for $x \in (x_1, 1]$ Consequently, $f(x)$ is strictly decreasing for $x \in [0, x_1]$, and strictly increasing for $x \in [x_1, 1]$ Since $f(0) = f(1) = 0$, it follows that $f(x) \le 0$ for $0 \le x \le 1$

For $n \ge 3$, equality occurs when $x_1 = x_2 = \cdots = x_n = 1$ □

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20. *Let $x_1, x_2, \ldots, x_n$ be non-negative numbers such that $x_1+x_2+\cdots+x_n=n$. If $k \geq 3$ is a positive integer and $r = \dfrac{n^{k-1}-1}{n-1}$, then*

$$x_1^k + x_2^k + \cdots + x_n^k - n \leq r\left(x_1^2 + x_2^2 + \cdots + x_n^2 - n\right)$$

Proof. There are two cases to consider.

Case $n = 2$. The inequality reduces to

$$x_1^k + x_2^k + (2^k - 2)x_1x_2 \leq 2^k$$

For $k = 3$, the inequality becomes equality. Consider now $k \geq 4$. We must show that $f(t) \leq 0$ for $t \in [0,1]$, where

$$f(t) = (1+t)^k + (1-t)^k + (2^k-2)(1-t^2) - 2^k.$$

We have

$$f'(t) = k\left[(1+t)^{k-1} - (1-t)^{k-1}\right] - (2^{k+1}-4)t,$$
$$f''(t) = k(k-1)\left[(1+t)^{k-2} + (1-t)^{k-2}\right] - 2^{k+1} + 4,$$
$$f'''(t) = k(k-1)(k-2)\left[(1+t)^{k-3} - (1-t)^{k-3}\right].$$

Since $f''' > 0$ for $t \in (0,1]$, the second derivative f'' is strictly increasing. Since $f''(0) = 2k(k-1) - 2^{k+1} + 4 < 0$ and $f''(1) = (k^2-k-8)2^{k-2} + 4 > 0$, there exists $t_1 \in (0,1)$ such that $f''(t_1) = 0$, $f''(t) < 0$ for $t \in [0, t_1)$, and $f''(t) > 0$ for $t \in (t_1, 1]$. Thus, the first derivative f' is strictly decreasing on $[0, t_1]$ and strictly increasing on $[t_1, 1]$. Since

$$f'(0) = 0 \text{ and } f'(1) = (k-4)2^{k-1} + 4 > 0,$$

there exists $t_2 \in (0,1)$ such that $f'(t_2) = 0$, $f'(t) < 0$ for $t \in (0, t_2)$, and $f'(t) > 0$ for $t \in (t_2, 1]$. Therefore, the function f is strictly decreasing on $[0, t_2]$ and strictly increasing on $[t_2, 1]$. Taking into account that $f(0) = f(1) = 0$, it follows that $f(t) \leq 0$ for $t \in [0,1]$.

Case $n \geq 3$. Assume that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ and apply Corollary 5 (case $p = 2$ and $q = k > p$).

• *If $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ such that $x_1 + x_2 + \cdots + x_n = n$ and $x_1^2 + x_2^2 + \cdots + x_n^2 =$ constant, then $x_1^k + x_2^k + \cdots + x_n^k$ is maximal when $0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n$.*

So, we have to prove the inequality

$$(n-1)x^k + y^k - n \le r\left[(n-1)x^2 + y^2 - n\right],$$

where $0 \le x \le 1 \le y$ and $(n-1)x + y = n$. Let

$$f(x) = (n-1)x^k + y^k - n - r\left[(n-1)x^2 + y^2 - n\right], \quad (n-1)x + y = n$$

We have to show that $f(x) \le 0$ for $x \in [0,1]$ Since $y' = -(n-1)$, we have

$$\frac{1}{k(n-1)} f'(x) = x^{k-1} - y^{k-1} - \frac{2r}{k}(x-y),$$

$$\frac{1}{k(k-1)(n-1)} f''(x) = x^{k-2} + (n-1)y^{k-2} - \frac{2nr}{k(k-1)},$$

$$\frac{1}{k(k-1)(k-2)(n-1)} f'''(x) = x^{k-3} - (n-1)^2 y^{k-3}.$$

Since $f''' < 0$ for $x \in [0,1]$, the second derivative f'' is strictly decreasing. Taking into account that $(n-1)r < n^{k-1}$ and

$$r = n^{k-2} + n^{k-3} + \dots + n + 1 > 2^{k-2} + 2^{k-3} + \dots + 2 + 1 = 2^{k-1} - 1,$$

we have

$$f''(0) = k(k-1)(n-1)^2 n^{k-2} - 2n(n-1)r > k(k-1)(n-1)^2 n^{k-2} - 2n^k \ge$$
$$\ge 6(n-1)^2 n^{k-2} - 2n^k = 2n^{k-2}(2n^2 - 6n + 3) > 0$$

and

$$\frac{f''(1)}{n(n-1)} = k(k-1) - 2r < k(k-1) - 2(2^{k-1} - 1) = k^2 - k + 2 - 2^k < 0.$$

Then, there exists $x_1 \in (0,1)$ such that $f''(x_1) = 0$, $f''(x) > 0$ for $x \in [0, x_1)$, and $f''(x) < 0$ for $x \in (x_1, 1]$ Thus, the first derivative f' is strictly increasing on $[0, x_1]$ and strictly decreasing on $[x_1, 1]$. Since

$$\frac{f'(0)}{n} = 2(n-1)r - k(n-1)n^{k-2} < 2n^{k-1} - k(n-1)n^{k-2} =$$
$$= -n^{k-2}[k(n-1) - 2n] \le -n^{k-2}[3(n-1) - 2n] = -n^{k-2}(n-3) \le 0$$

and $f'(1) = 0$, there exists $x_2 \in (0,1)$ such that $f'(x_2) = 0$, $f'(x) < 0$ for $x \in (0, x_2)$, and $f'(x) > 0$ for $x \in (x_2, 1]$. Therefore, the function f is strictly decreasing on $[0, x_2]$ and strictly increasing on $[x_2, 1]$. Since $f(0) = f(1) = 0$, it follows that $f(x) \le 0$ for $x \in [0,1]$ The proof is complete Equality occurs when $x_1 = x_2 = \dots = x_n$, as well as when $n-1$ of the numbers x_i are 0 □

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21. *If* $x_1, x_2, \ldots, x_n$ *are positive numbers, then*

$$x_1^n + x_2^n + \cdots + x_n^n + n(n-1)x_1x_2 \ldots x_n \geq$$
$$\geq x_1x_2 \ldots x_n(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)$$

Proof For $n = 2$, one has equality. For $n \geq 3$, assume that

$$0 < x_1 \leq x_2 \leq \cdots \leq x_n$$

and apply Corollary 5 (case $p = 0$):

• *If* $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ *such that* $x_1 + x_2 + \cdots + x_n =$ constant *and* $x_1x_2 \ldots x_n =$ constant, *then the sum* $x_1^n + x_2^n + \cdots + x_n^n$ *is minimal and the sum* $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$ *is maximal when* $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$.

Thus, it suffices to prove the homogeneous inequality for $0 < x_1 \leq 1$ and $x_2 = x_3 = \cdots = x_n = 1$. The inequality becomes

$$x_1^n + (n-2)x_1 \geq (n-1)x_1^2,$$

and is equivalent to $x_1(x_1 - 1)\left[\left(x_1^{n-2} - 1\right) + \left(x_1^{n-3} - 1\right) + \cdots + (x_1 - 1)\right] \geq 0$, which is clearly true.

Equality occurs if and only if $x_1 = x_2 = \cdots = x_n$ □

Remark. For $n = 3$, we get the third degree Schur's Inequality,

$$x_1^3 + x_2^3 + x_3^3 + 6x_1x_2x_3 \geq (x_1 + x_2 + x_3)(x_1x_2 + x_2x_3 + x_3x_1).$$

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22. *If* $x_1, x_2, \ldots, x_n$ *are non-negative numbers, then*

$$(n-1)\left(x_1^n + x_2^n + \cdots + x_n^n\right) + nx_1x_2 \ldots x_n \geq$$
$$\geq (x_1 + x_2 + \cdots + x_n)\left(x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1}\right).$$

Proof. For $n = 2$, one has equality For $n \geq 3$, assume that

$$0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$$

and apply Corollary 5 (case $p = n$ and $q = n-1$) and Corollary 4 (case $= n$

• *If* $0 \le x_1 \le x_2 \le \cdots \le x_n$ *such that* $x_1 + x_2 + \cdots + x_n =$ constant *and* $x_1^n + x_2^n + \cdots + x_n^n =$ constant, *then the sum* $x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1}$ *is maximal and the product* $x_1 x_2 \ldots x_n$ *is minimal when either* $x_1 = 0$ *or* $0 < x_1 \le x_2 = x_3 = \cdots = x_n$

Thus, it suffices to consider the cases $x_1 = 0$ and $0 < x_1 \le x_2 = x_3 = \ldots = x_n$

Case $x_1 = 0$ The inequality reduces to

$$(n-1)\left(x_2^n + \cdots + x_n^n\right) \ge (x_2 + \cdots + x_n)\left(x_2^{n-1} + \cdots + x_n^{n-1}\right),$$

which immediately follows by Chebyshev's Inequality

Case $0 < x_1 \le x_2 = x_3 = \cdots = x_n$. Setting $x_2 = x_3 = \cdots = x_n = 1$, the homogeneous inequality reduces to

$$(n-2)x_1^n + x_1 \ge (n-1)x_1^{n-1}$$

Rewriting this inequality as

$$x_1(x_1-1)\left[x_1^{n-3}(x_1-1) + x_1^{n-4}\left(x_1^2-1\right) + \cdots + \left(x_1^{n-2}-1\right)\right] \ge 0,$$

we see that it is clearly true For $n \ge 3$ and $x_1 \le x_2 \le \cdots \le x_n$, equality occurs when $x_1 = x_2 = \cdots = x_n$, and also when $x_2 = \cdots = x_n$ □

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23. *If* $x_1, x_2, \ldots, x_n$ *are non-negative numbers, then*

$$(n-1)\left(x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1}\right) \ge$$
$$\ge (x_1 + x_2 + \cdots + x_n)\left(x_1^n + x_2^n + \cdots + x_n^n - x_1 x_2 \cdots x_n\right)$$

Proof. For $n = 2$, one has equality For $n \ge 3$, assume that

$$0 \le x_1 \le x_2 \le \cdots \le x_n$$

and apply Corollary 5 (case $p = n+1$ and $q = n$) and Corollary 4 (case $p = n+1$)

• *If* $0 \le x_1 \le x_2 \le \cdots \le x_n$ *such that* $x_1 + x_2 + \cdots + x_n =$ constant *and* $x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1} =$ constant, *then the sum* $x_1^n + x_2^n + \cdots + x_n^n$ *is maximal an the product* $x_1 x_2 \ldots x_n$ *is minimal when either* $x_1 = 0$ *or* $0 < x_1 \le x_2 = x_3 = \cdots = x_n$

Thus, it suffices to consider the cases $x_1 = 0$ and $0 < x_1 \le x_2 = x_3 = \cdots = x_n$

Case $x_1 = 0$. The inequality reduces to

$$(n-1)\left(x_1^{n+1} + \cdots + x_n^{n+1}\right) \geq (x_2 + \cdots + x_n)\left(x_2^n + \cdots + x_n^n\right),$$

which immediately follows by Chebyshev's Inequality.

Case $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$. Setting $x_2 = x_3 = \cdots = x_n = 1$, the homogeneous inequality reduces to

$$(n-2)x_1^{n+1} + x_1^2 \geq (n-1)x_1^n.$$

Rewriting this inequality as

$$x_1^2(x_1 - 1)\left[x_1^{n-3}(x_1 - 1) + x_1^{n-4}\left(x_1^2 - 1\right) + \cdots + \left(x_1^{n-2} - 1\right)\right] \geq 0,$$

we see that it is clearly true. For $n \geq 3$ and $x_1 \leq x_2 \leq \cdots \leq x_n$, equality occurs when $x_1 = x_2 = \cdots = x_n$, and also when $x_1 = 0$ and $x_2 = \cdots = x_n$. □

Remark 1 We may reformulate the inequality above as follows

- *If* $x_1, x_2, \ldots, x_n$ *are non-negative numbers such that*

$$x_1 + x_2 + \cdots + x_n = n - 1,$$

then

$$x_1^n(1 - x_1) + x_2^n(1 - x_2) + \cdots + x_n^n(1 - x_n) \leq x_1 x_2 \ldots x_n$$

Remark 2. *Gjergji Zaimi* and *Keler Marku* generalized the above inequalities for any real k in the following form (problem 69 from chapter 8)

$$(n-1)\left(x_1^{n+k} + x_2^{n+k} + \cdots + x_n^{n+k}\right) + x_1 x_2 \ldots x_n\left(x_1^k + x_2^k + \cdots + x_n^k\right) \geq$$
$$\geq (x_1 + x_2 + \cdots + x_n)\left(x_1^{n+k-1} + x_2^{n+k-1} + \cdots + x_n^{n+k-1}\right).$$

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24. *If* $x_1, x_2, \ldots, x_n$ *are positive numbers, then*

$$(x_1 + x_2 + \cdots + x_n - n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} - n\right) + x_1 x_2 \ldots x_n + \frac{1}{x_1 x_2 \ldots x_n} \geq 2$$

Proof. For $n = 2$, the inequality reduces to

$$\frac{(1 - x_1)^2(1 - x_2)^2}{x_1 x_2} \geq 0.$$

For $n \geq 3$, assume that $0 < x_1 \leq x_2 \leq \cdots \leq x_n$. Since the inequality preserves its form by replacing each number x_i with $\frac{1}{x_i}$, we may consider $x_1 x_2 \ldots x_n \geq 1$ By the AM-GM Inequality we get

$$x_1 + x_2 + \cdots + x_n - n \geq n \sqrt[n]{x_1 x_2 \ldots x_n} - n \geq 0,$$

and thus we may apply Corollary 5 (case $p = 0$ and $q = -1$).

• *If* $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ *such that* $x_1 + x_2 + \cdots + x_n =$ constant *and* $x_1 x_2 \ldots x_n =$ constant, *then the sum* $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$ *is minimal when* $0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n$.

According to this statement, it suffices to consider

$$x_1 = x_2 = \cdots = x_{n-1} = x \text{ and } x_n = y,$$

when the inequality reduces to

$$[(n-1)x + y - n]\left(\frac{n-1}{x} + \frac{1}{y} - n\right) + x^{n-1}y + \frac{1}{x^{n-1}y} \geq 2,$$

or

$$\left(x^{n-1} + \frac{n-1}{x} - n\right)y + \left[\frac{1}{x^{n-1}} + (n-1)x - n\right]\frac{1}{y} \geq \frac{n(n-1)(x-1)^2}{x}.$$

Since

$$x^{n-1} + \frac{n-1}{x} - n = \frac{x-1}{x}\left[\left(x^{n-1} - 1\right) + \left(x^{n-2} - 1\right) + \cdots + (x-1)\right] =$$
$$= \frac{(x-1)^2}{x}\left[x^{n-2} + 2x^{n-3} + \cdots + (n-1)\right]$$

and

$$\frac{1}{x^{n-1}} + (n-1)x - n = \frac{(x-1)^2}{x}\left[\frac{1}{x^{n-2}} + \frac{2}{x^{n-3}} + \cdots + (n-1)\right],$$

it is enough to show that

$$\left[x^{n-2} + 2x^{n-3} + \cdots + (n-1)\right]y +$$
$$+ \left[\frac{1}{x^{n-2}} + \frac{2}{x^{n-3}} + \cdots + (n-1)\right]\frac{1}{y} \geq n(n-1).$$

This inequality is equivalent to

$$\left(x^{n-2}y + \frac{1}{x^{n-2}y} - 2\right) + \\ + 2\left(x^{n-3}y + \frac{1}{x^{n-3}y} - 2\right) + \cdots + (n-1)\left(y + \frac{1}{y} - 2\right) \geq 0,$$

or

$$\frac{\left(x^{n-2}y - 1\right)^2}{x^{n-2}y} + \frac{2\left(x^{n-3}y - 1\right)^2}{x^{n-3}y} + \cdots + \frac{(n-1)(y-1)^2}{y} \geq 0,$$

which is clearly true. Equality occurs in the given inequality if and only if $n-1$ of the numbers x_i are equal to 1 □

★

25. *If $x_1, x_2, \ldots, x_n$ are positive numbers such that $x_1x_2 \ldots x_n = 1$, then*

$$\left|\frac{1}{\sqrt{x_1 + x_2 + \cdots + x_n - n}} - \frac{1}{\sqrt{\dfrac{1}{x_1} + \dfrac{1}{x_2} + \cdots + \dfrac{1}{x_n} - n}}\right| < 1.$$

Proof. Let $A = x_1 + x_2 + \cdots + x_n - n$ and $B = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} - n$ From the AM-GM Inequality, it follows that A and B are positive. According to the preceding problem, the following inequality holds for any positive numbers $x_1, x_2, \ldots, x_{n+1}$:

$$(x_1 + x_2 + \cdots + x_{n+1} - n - 1)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n+1}} - n - 1\right) + \\ + x_1x_2 \ldots x_{n+1} + \frac{1}{x_1x_2 \ldots x_{n+1}} \geq 2$$

This inequality is equivalent to

$$(A - 1 + x_{n+1})\left(B - 1 + \frac{1}{x_{n+1}}\right) + x_{n+1} + \frac{1}{x_{n+1}} \geq 2$$

or

$$\frac{A}{x_{n+1}} + Bx_{n+1} + AB - A - B \geq 0.$$

Replacing x_{n+1} by $\sqrt{\frac{A}{B}}$, yields

$$2\sqrt{AB} + AB - A - B \geq 0,$$
$$AB \geq \left(\sqrt{A} - \sqrt{B}\right)^2,$$
$$1 \geq \left(\frac{1}{\sqrt{A}} - \frac{1}{\sqrt{B}}\right)^2,$$
$$1 \geq \left|\frac{1}{\sqrt{A}} - \frac{1}{\sqrt{B}}\right|$$

The last inequality is just the desired inequality. □

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26. *If $x_1, x_2, \ldots, x_n$ are non-negative numbers such that $x_1+x_2+\cdots+x_n = n$, then*

$$(x_1x_2\cdots x_n)^{\frac{1}{\sqrt{n-1}}}\left(x_1^2 + x_2^2 + \cdots + x_n^2\right) \leq n$$

Proof. For $n = 2$, the inequality reduces to $2(x_1x_2 - 1)^2 \geq 0$. For $n \geq 3$, assume that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ and apply Corollary 4 (case $p = 2$):

- *For* $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$,

$$x_1 + x_2 + \cdots + x_n = n \ \textit{and}\ x_1^2 + x_2^2 + \cdots + x_n^2 = \text{constant},$$

the product $x_1x_2\ldots x_n$ is maximal when $0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n$.

Consequently, it suffices to show the inequality for $x_1 = x_2 = \cdots = x_{n-1} = x$ and $x_n = y$, where $0 \leq x \leq 1 \leq y$ and $(n-1)x + y = n$. Under the circumstances, the inequality reduces to

$$x^{\sqrt{n-1}}y^{\frac{1}{\sqrt{n-1}}}\left[(n-1)x^2 + y^2\right] \leq n.$$

For $x = 0$, the inequality is trivial. For $x > 0$, it is equivalent to $f(x) \leq 0$, where

$$f(x) = \sqrt{n-1}\ln x + \frac{1}{\sqrt{n-1}}\ln y + \ln\left[(n-1)x^2 + y^2\right] - \ln n,$$

with $y = n - (n-1)x$.
We have $y' = -(n-1)$ and

$$\frac{f'(x)}{\sqrt{n-1}} = \frac{1}{x} - \frac{1}{y} + \frac{2\sqrt{n-1}(x-y)}{(n-1)x^2 + y^2} = \frac{(y-x)\left(\sqrt{n-1}x - y\right)^2}{xy\left[(n-1)x^2 + y^2\right]} \geq 0$$

Therefore, the function $f(x)$ is strictly increasing on $(0,1]$ and hence $f(x) \le f(1) = 0$.
Equality occurs if and only if $x_1 = x_2 = \cdots = x_n = 1$. □

Remark. For $n = 5$, we get the following nice statement:

- *If a, b, c, d are positive numbers such that $a^2 + b^2 + c^2 + d^2 + e^2 = 5$, then*

$$abcde\left(a^4 + b^4 + c^4 + d^4 + e^4\right) \le 5.$$

★

27. *Let x, y, z be non-negative numbers such that $xy + yz + zx = 3$, and let $p \ge \dfrac{\ln 9 - \ln 4}{\ln 3} \approx 0.738$. Then,*

$$x^p + y^p + z^p \ge 3$$

Proof. Let $r = \dfrac{\ln 9 - \ln 4}{\ln 3}$ By the Power-Mean Inequality, we have

$$\frac{x^p + y^p + z^p}{3} \ge \left(\frac{x^r + y^r + z^r}{3}\right)^{\frac{p}{r}}.$$

Thus, it suffices to show that

$$x^r + y^r + z^r \ge 3.$$

Let $x \le y \le z$. We consider two cases

Case $x = 0$. We have to show that $y^r + z^r \ge 3$ for $yz = 3$ Indeed, by the AM-GM Inequality, we get

$$y^r + z^r \ge 2(yz)^{\frac{r}{2}} = 2 \ 3^{\frac{r}{2}} = 3.$$

Case $x > 0$. The inequality $x^r + y^r + z^r \ge 3$ is equivalent to the homogeneous inequality

$$x^r + y^r + z^r \ge 3\left(\frac{xyz}{3}\right)^{\frac{r}{2}}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)^{\frac{r}{2}}.$$

Setting $x = a^{\frac{1}{r}}$, $y = b^{\frac{1}{r}}$, $z = c^{\frac{1}{r}}$ $(0 < a \le b \le c)$, the inequality becomes

$$a + b + c \ge 3\left(\frac{abc}{3}\right)^{\frac{1}{2}}\left(a^{\frac{-1}{r}} + b^{\frac{-1}{r}} + c^{\frac{-1}{r}}\right)^{\frac{r}{2}}.$$

To prove this inequality, we apply Corollary 5 (case $p = 0$ and $q = \frac{-1}{r}$)

- *If* $0 < a \le b \le c$ *such that* $a + b + c =$ constant *and* $abc =$ constant, *then the sum* $a^{\frac{-1}{r}} + b^{\frac{-1}{r}} + c^{\frac{-1}{r}}$ *is maximal when* $0 < a \le b = c$

So, it suffices to prove the inequality for $0 < a \le b = c$, that is, to prove the homogeneous inequality in x, y, z for $0 < x \le y = z = 1$ So the inequality reduces to

$$x^r + 2 \ge 3\left(\frac{2x+1}{3}\right)^{\frac{r}{2}}$$

Denoting

$$f(x) = \ln\frac{x^r+2}{3} - \frac{r}{2}\ln\frac{2x+1}{3},$$

we have to show that $f(x) \ge 0$ for $0 < x \le 1$ The derivative

$$f'(x) = \frac{rx^{r-1}}{x^r+2} - \frac{r}{2x+1} = \frac{r(x - 2x^{1-r} + 1)}{x^{1-r}(x^r+2)(2x+1)}$$

has the same sign as $g(x) = x - 2x^{1-r} + 1$ Since $g'(x) = 1 - \frac{2(1-r)}{x^r}$, we see that $g'(x) < 0$ for $x \in (0, x_1)$, and $g'(x) > 0$ for $x \in (x_1, 1]$, where $x_1 = (2-2r)^{\frac{1}{r}} \approx 0.416$ The function $g(x)$ is strictly decreasing on $[0, x_1]$, and strictly increasing on $[x_1, 1]$ Since $g(0) = 1$ and $g(1) = 0$, there exists $x_2 \in (0,1)$ such that $g(x_2) = 0$, $g(x) > 0$ for $x \in [0, x_2)$ and $g(x) < 0$ for $x \in (x_2, 1)$ Consequently, the function $f(x)$ is strictly increasing on $[0, x_2]$ and strictly decreasing on $[x_2, 1]$ Since $f(0) = f(1) = 0$, we have $f(x) \ge 0$ for $0 < x \le 1$, establishing the desired result.

Equality occurs for $x = y = z = 1$ Additionally, for $p = \frac{\ln 9 - \ln 4}{\ln 3}$ and $x \le y \le z$, equality holds again for $x = 0$ and $y = z = \sqrt{3}$ □

★

28. *Let* x, y, z *be non-negative numbers such that* $x + y + z = 3$, *and let* $p \ge \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2} \approx 0\,29$. *Then,*

$$x^p + y^p + z^p \ge xy + yz + zx$$

Proof. For $p \ge 1$, by Jensen's Inequality we have

$$x^p + y^p + z^p \ge 3\left(\frac{x+y+z}{3}\right)^p = 3 = \frac{1}{3}(x+y+z)^2 \ge xy + yz + zx$$

Assume now $p < 1$ Let $r = \dfrac{\ln 9 - \ln 8}{\ln 3 - \ln 2}$ and $x \le y \le z$ The inequality is equivalent to the homogeneous inequality

$$2(x^p + y^p + z^p)\left(\frac{x+y+z}{3}\right)^{2-p} + x^2 + y^2 + z^2 \ge (x+y+z)^2.$$

By Corollary 5 (case $0 < p < 1$ and $q = 2$), *for* $x \le y \le z$ *such that* $x + y + z =$ constant *and* $x^p + y^p + z^p =$ constant, *the sum* $x^2 + y^2 + z^2$ *is minimal when either* $x = 0$ *or* $0 < x \le y = z$.

Case $x = 0$. Returning to our original inequality, we have to show that $y^p + z^p \ge yz$ for $y + z = 3$ Indeed, by the AM-GM Inequality, we get

$$\begin{aligned} y^p + z^p - yz &\ge 2(yz)^{\frac{p}{2}} - yz = (yz)^{\frac{p}{2}}\left[2 - (yz)^{\frac{2-p}{2}}\right] \ge \\ &\ge (yz)^{\frac{p}{2}}\left[2 - \left(\frac{y+z}{2}\right)^{2-p}\right] = \\ &= (yz)^{\frac{p}{2}}\left[2 - \left(\frac{3}{2}\right)^{2-p}\right] \ge (yz)^{\frac{p}{2}}\left[2 - \left(\frac{3}{2}\right)^{2-r}\right] = 0 \end{aligned}$$

Case $0 < x \le y = z$. In the homogeneous inequality, we may leave aside the constraint $x + y + z = 3$, and consider $y = z = 1$ and $0 < x \le 1$ Thus, the inequality reduces to

$$(x^p + 2)\left(\frac{x+2}{3}\right)^{2-p} \ge 2x + 1.$$

To prove this inequality, we consider the function

$$f(x) = \ln(x^p + 1) + (2-p)\ln\frac{x+2}{3} - \ln(2x+1)$$

We must to show that $f(x) \ge 0$ for $0 < x \le 1$ We have

$$f'(x) = \frac{px^{p-1}}{x^p + 2} + \frac{2-p}{x} - \frac{2}{2x+1} = \frac{2g(x)}{x^{1-p}(x^p+1)(2x+1)},$$

where

$$g(x) = x^2 + (2p-1)x + p + 2(1-p)x^{2-p} - (p+2)x^{1-p},$$

and

$$\begin{aligned} g'(x) &= 2x + 2p - 1 + 2(1-p)(2-p)x^{1-p} - (p+2)(1-p)x^{-p}, \\ g''(x) &= 2 + 2(1-p)^2(2-p)x^{-p} + p(p+2)(1-p)x^{-p-1}. \end{aligned}$$

Since $g''(x) > 0$, the first derivative $g'(x)$ is strictly increasing on $(0,1]$. Taking into account that $g'(0_+) = -\infty$ and $g'(1) = 3(1-p)+3p^2 > 0$, there is $x_1 \in (0,1)$ such that $g'(x_1) = 0$, $g'(x) < 0$ for $x \in (0,x_1)$ and $g'(x) > 0$ for $x \in (x_1,1]$ Therefore, the function $g(x)$ is strictly decreasing on $[0,x_1]$ and strictly increasing on $[x_1,1]$. Since $g(0) = p > 0$ and $g(1) = 0$, there is $x_2 \in (0,x_1)$ such that $g(x_2) = 0$, $g(x) > 0$ for $x \in [0,x_2)$ and $g(x) < 0$ for $x \in (x_2,1]$. We have also $f'(x_2) = 0$, $f'(x) > 0$ for $x \in (0,x_2)$ and $f'(x) < 0$ for $x \in (x_2,1]$ According to this result, the function $f(x)$ is strictly increasing on $[0,x_2]$ and strictly decreasing on $[x_2,1]$. Since

$$f(0) = \ln 2 + (2-p)\ln\frac{2}{3} \geq \ln 2 + (2-r)\ln\frac{2}{3} = 0$$

and $f(1) = 0$, we get $f(x) \geq \min\{f(0), f(1)\} \geq 0$

Equality occurs for $x = y = z = 1$. Additionally, for $p = \dfrac{\ln 9 - \ln 8}{\ln 3 - \ln 2}$ and $x \leq y \leq z$, equality holds again when $x = 0$ and $y = z = \dfrac{3}{2}$. □

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29. *If $x_1, x_2, \ldots, x_n$ $(n \geq 4)$ are non-negative numbers such that*

$$x_1 + x_2 + \cdots + x_n = n,$$

then

$$\frac{1}{n+1-x_2x_3\cdots x_n} + \frac{1}{n+1-x_3x_4\cdots x_1} + \cdots + \frac{1}{n+1-x_1x_2\cdots x_{n-1}} \leq 1.$$

Proof Let $x_1 \leq x_2 \leq \cdots \leq x_n$ and $e_{n-1} = \left(1+\dfrac{1}{n-1}\right)^{n-1}$ By the AM-GM Inequality, we have

$$x_2 \cdots x_n \leq \left(\frac{x_2+\cdots+x_n}{n-1}\right)^{n-1} \leq \left(\frac{x_1+x_2+\cdots+x_n}{n-1}\right)^{n-1} = e_{n-1}$$

Hence

$$n+1-x_2x_3\cdots x_n \geq n+1-e_{n-1} > 0,$$

and all denominators of the inequality are positive

Case $x_1 = 0$ It is easy to show that the inequality holds

Case $x_1 > 0$ Suppose that $x_1x_2 \ldots x_n = (n+1)r =$ constant, $r > 0$ The inequality becomes

$$\frac{x_1}{x_1 - r} + \frac{x_2}{x_2 - r} + \cdots + \frac{x_n}{x_n - r} \le n + 1,$$

or

$$\frac{1}{x_1 - r} + \frac{1}{x_2 - r} + \cdots + \frac{1}{x_n - r} \le \frac{1}{r}.$$

By AM-GM Inequality, we have

$$(n+1)r = x_1x_2 \ldots x_n \le \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n = 1,$$

whence $r \le \frac{1}{n+1}$. From $x_n < x_1 + x_2 + \cdots + x_n = n < n + 1 \le \frac{1}{r}$, we get $x_n < \frac{1}{r}$. Therefore, we have $r < x_i < \frac{1}{r}$ for all numbers x_i.

We will apply now Corollary 3 to the function $f(u) = \frac{-1}{u - r}$, $u > r$.

We have $f'(u) = \frac{1}{(u-r)^2}$ and

$$g(x) = f'\left(\frac{1}{x}\right) = \frac{x^2}{(1 - rx)^2}, \quad g''(x) = \frac{4rx + 2}{(1 - rx)^4}.$$

Since $g''(x) > 0$, $g(x)$ is strictly convex on $\left[0, \frac{1}{r}\right)$. According to Corollary 3 and Remark from section 5 1, *if* $0 \le x_1 \le x_2 \le \cdots \le x_n$ *such that* $x_1 + x_2 + \cdots + x_n =$ constant *and* $x_1x_2 \ldots x_n =$ constant, *then the sum* $f(x_1) + f(x_2) + \cdots + f(x_n)$ *is minimal when* $x_1 \le x_2 = x_3 = \cdots = x_n$ Thus, to prove the original inequality, it suffices to consider the case $x_1 = x$ and $x_2 = x_3 = \cdots = x_n = y$, where $0 < x \le 1 \le y$ and $x + (n-1)y = n$.
We leave to the interested reader to end the proof. □

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30. *Let* a, b, c *be positive numbers such that* $abc = 1$. *Prove that*

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b^2)} + \frac{1}{(1+c)^2} + \frac{2}{(1+a)(1+b)(1+c)} \ge 1.$$

Proof. Denote $x = \dfrac{1}{1+a}$, $y = \dfrac{1}{1+b}$, $z = \dfrac{1}{1+c}$, $S = x + y + z$ and $Q = x^2 + y^2 + z^2$, where $0 < x, y, z < 1$ The hypothesis $abc = 1$ becomes $xyz = (1-x)(1-y)(1-z)$, that is

$$4xyz + x^2 + y^2 + z^2 = 1 + (x + y + z - 1)^2,$$

while the required inequality transforms into $x^2 + y^2 + z^2 + 2xyz \geq 1$, that is

$$(x + y + z - 1)^2 + x^2 + y^2 + z^2 \geq 1.$$

For the sake of contradiction, assume that $(x+y+z-1)^2+x^2+y^2+z^2 < 1$ It suffices to show that $4xyz+x^2+y^2+z^2 < 1+(x+y+z-1)^2$ According to Corollary 4 (case $p = 2$), *if* $0 \leq x \leq y \leq z$ *such that* $x+y+z =$ constant *and* $x^2+y^2+z^2 =$ constant, *then the product* xyz *is maximal when* $0 < x = y \leq z$ Therefore, it suffices to consider the case $x = y$ So, we have to show that $(2x+z-1)^2+2x^2+z^2 < 1$ implies $4x^2z+2x^2+z^2 < 1+(2x+z-1)^2$. Assuming the contrary, that $4x^2z + 2x^2 + z^2 \geq 1 + (2x + z - 1)^2$, which is equivalent to $z \geq \dfrac{(x-1)^2}{x^2 + (x-1)^2}$, it suffices to show that $(2x + z - 1)^2 + 2x^2 + z^2 \geq 1$. Since

$$2x + z - 1 \geq 2x + \frac{(x-1)^2}{x^2 + (x-1)^2} - 1 = \frac{x(4x^2 - 5x + 2)}{2x^2 - 2x + 1} > 0,$$

it is enough to prove the inequality for $z = \dfrac{(x-1)^2}{x^2 + (x-1)^2}$ We have

$$z^2 - 1 = \frac{-x^2(3x^2 - 4x + 2)}{(2x^2 - 2x + 1)^2},$$

and hence

$$(2x+z-1)^2+2x^2+z^2-1 = \frac{x^2(4x^2-5x+2)^2}{(2x^2 - 2x + 1)^2} + 2x^2 - \frac{x^2(3x^2-4x + 2)}{(2x^2 - 2x + 1)^2} =$$

$$= \frac{2x^2(12x^4 - 28x^3 + 27x^2 - 12x + 2)}{(2x^2 - 2x + 1)^2} =$$

$$= \frac{2x^2(2x - 1)^2(3x^2 - 4x + 2)}{(2x^2 - 2x + 1)^2} \geq 0$$

Equality in the given inequality occurs if and only if $a = b = c = 1$ □

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31. *Let a, b, c be non-negative numbers such that $a + b + c \geq 2$ and $ab + bc + ca \geq 1$. If $0 < r < 1$, then*

$$a^r + b^r + c^r \geq 2.$$

Proof. We may write the second condition as

$$(a + b + c)^2 - (a^2 + b^2 + c^2) \geq 2$$

This suggests us to apply Corollary 1 to the convex function $f(u) = -u^r$ *If $0 \leq a \leq b \leq c$ such that $a + b + c =$* constant *and $a^2 + b^2 + c^2 =$* constant, *then the sum $f(a) + f(b) + f(c)$ is maximal for either $a = 0$ or $0 < a \leq b = c$.*

Case $a = 0$. From $ab + bc + ca \geq 1$ we get $bc \geq 1$. Consequently,

$$a^r + b^r + c^r = b^r + c^r \geq 2\sqrt{b^r c^r} \geq 2$$

Case $0 < a \leq b = c$ If $c \geq 1$, then

$$a^r + b^r + c^r = a^r + 2c^r \geq 2c^r \geq 2$$

If $c < 1$, then $0 < a \leq b = c < 1$ and hence

$$a^r + b^r + c^r > a + b + c \geq 2.$$

For $a \leq b \leq c$, equality in the original inequality occurs if and only if $a = 0$ and $b = c = 1$. □

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32. *Let a, b, c be positive numbers such that $(a + b + c)^3 = 32abc$. Find the minimum and the maximum of*

$$E = \frac{a^4 + b^4 + c^4}{(a + b + c)^4}.$$

Proof. We will apply Corollary 5 (case $p = 0$, $q = 4$)

- *If $0 < a \leq b \leq c$ such that $a + b + c =$* constant *and $abc =$* constant, *then the sum $a^4 + b^4 + c^4$ is minimal when $0 < a \leq b = c$ and is maximal when $0 < a = b \leq c$.*

Due to homogeneity, E is minimal for $0<a\le b=c=1$ and $(a+2)^3=32a$, and is maximal for $a=b=1\le c$ and $(c+2)^3=32c$. Since the equation $(x+2)^3=32x$ has the roots 2 and $-4\pm 2\sqrt{5}$, it follows that E is minimal for $(a,b,c)\sim\left(2\sqrt{5}-4,1,1\right)$ or any cyclic permutation, and is maximal for $(a,b,c)\sim(1,1,2)$ or any cyclic permutation. The extremal values of the expression E are $\dfrac{383-165\sqrt{5}}{256}$ and $\dfrac{9}{128}$, respectively □

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33. *Let $x_1,x_2,\ldots,x_n$ $(n\ge 3)$ be non-negative real numbers such that*

$$\sum x_1=1$$

If $m\in\{3,4,\ldots,n\}$, then

$$1+\frac{3m}{m-2}\sum x_1x_2x_3\ge\frac{3m-1}{m-1}\sum x_1x_2.$$

Proof (after an idea of *Yuan Shyong Ooi*). Since

$$2\sum x_1x_2=\left(\sum x_1\right)^2-\sum x_1^2,$$

we may apply Corollary 6 (case $p=2$)·

- *For $0\le x_1\le x_2\le\cdots\le x_n$ such that*

$$\sum x_1=1 \text{ and } \sum x_1^2=\text{constant},$$

the sum $\sum x_1x_2x_3$ is minimal when

$$x_1=\cdots=x_k=0 \text{ and } x_{k+2}=\cdots=x_n,$$

where $k\in\{0,1,\ldots,n-1\}$.

Thus, it suffices to consider the case

$$x_1=\cdots=x_k=0 \text{ and } x=x_{k+1}\le x_{k+2}=\cdots=x_n=y$$

On the other hand, taking into account that

$$\left(\sum x_1\right)^2=\sum x_1^2+2\sum x_1x_2$$

and

$$\left(\sum x_1\right)^3=\sum x_1^3+3\left(\sum x_1\right)\left(\sum x_1x_2\right)-3\sum x_1x_2x_3,$$

we get

$$2\sum x_1x_2 = 1 - \sum x_1^2$$

and

$$6\sum x_1x_2x_3 = 1 - 3\sum x_1^2 + 2\sum x_1^3$$

Therefore, the inequality becomes

$$1 + m(m-1)\sum x_1^3 \geq (2m-1)\sum x_1^2,$$

or

$$\sum_{i=1}^{n} f(x_i) \geq 0,$$

where

$$f(t) = t(1-mt)\left[1-(m-1)t\right].$$

We have to prove that

$$f(x) + (n-k-1)f(y) \geq 0$$

for $x + (n-k-1)y = 1$, $0 \leq x \leq y$, $0 \leq x \leq \dfrac{1}{n-k}$. From

$$f''(t) = 6m(m-1)t - 2(2m-1),$$

it follows that f is convex for $t \geq \dfrac{2m-1}{3m(m-1)}$.

a) *Case* $x \geq \dfrac{2m-1}{3m(m-1)}$. By Jensen's Inequality we have

$$f(x) + (n-k-1)f(y) \geq (n-k)f\left(\frac{x+(n-k-1)y}{n-k}\right) =$$

$$= (n-k)f\left(\frac{1}{n-k}\right) = \frac{(k-n+m)(k-n+m-1)}{(n-k)^2} \geq 0,$$

because $(k-n+m)$ is an integer number.

b) *Case* $x < \dfrac{2m-1}{3m(m-1)}$. Since

$$1 - mx > 1 - \frac{2m-1}{3(m-1)} = \frac{m-2}{3(m-1)} > 0,$$

we have

$$f(x) = x(1-mx)\left[1-(m-1)x\right] \geq 0.$$

Consider now three cases in terms of k.

Case $k \le n-m-1$ Since

$$1-my = 1-\frac{m(1-x)}{n-k-1} \ge 1-(1-x) = x \ge 0,$$

we have $f(y) = y(1-my)\,[1-(m-1)y] \ge 0$, and hence

$$f(x)+(n-k-1)f(y) \ge 0$$

Case $k \ge n-m+1$. Since

$$\begin{aligned}(m-1)y-1 &= \frac{(m-1)(1-x)}{n-k-1}-1 \ge \frac{(m-1)(1-x)}{m-2}-1 = \\ &= \frac{1}{m-2}-\frac{(m-1)x}{m-2} > \frac{1}{m-2}-\frac{m-1}{m-2}\,\frac{2m-1}{3m(m-1)} = \\ &= \frac{m+1}{3m(m-2)} > 0,\end{aligned}$$

we have $f(y) = y(my-1)\,[(m-1)y-1] > 0$, and hence

$$f(x)+(n-k-1)f(y) \ge 0.$$

Case $k = n-m$ We have $x+(m-1)y = 1$,

$$f(y) = y(my-1)[(m-1)y-1] = \frac{x(x-1)(1-mx)}{(m-1)^2}$$

and

$$\begin{aligned}f(x)+(n-k-1)f(y) &= f(x)+(m-1)f(y) = \\ &= x(1-mx)\,[1-(m-1)x]+\frac{x(x-1)(1-mx)}{m-1} = \\ &= \frac{(m-2)x(1-mx)^2}{m-1} \ge 0\end{aligned}$$

This completes the proof Equality occurs if m or $m-1$ of the numbers $x_1, x_2, \ldots, x_n$ are equal and the others are zero □

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34. *Let x, y, z, t be non-negative real numbers such that*

$$x^2+y^2+z^2+t^2 = 1$$

Prove that

$$x^3+y^3+z^3+t^3+xyz+yzt+ztx+txy \le 1.$$

Proof. Assume that $x \le y \le z \le t$ and apply Corollary 7:

- *For* $0 \le x \le y \le z \le t$ *such that*

$$x^2 + y^2 + z^2 + t^2 = 1 \text{ and } x^3 + y^3 + z^3 + t^3 = \text{constant},$$

the expression $xyz + yzt + ztx + txy$ *is maximal when* $0 \le x = y = z \le t$.

Consequently, we have to show that

$$4x^3 + t^3 + 3x^2 t \le 1$$

for $3x^2 + t^2 = 1$, $0 \le x \le \frac{1}{2} \le t$ Let

$$f(x) = 4x^3 + t^3 + 3x^2 t.$$

Taking into account that $t' = \frac{-3x}{t}$, we have

$$f'(x) = 12x^2 + 3(t^2 + x^2)t' + 6xt =$$
$$= \frac{3x(t-x)(3x-t)}{t} = \frac{3x(t-x)(12x^2-1)}{t(3x+t)}$$

Since $f'(x) < 0$ for $x \in \left(0, \frac{1}{2\sqrt{3}}\right)$, $f'\left(\frac{1}{2\sqrt{3}}\right) = 0$ and $f'(x) > 0$ for $x \in \left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right)$, the function $f(x)$ is strictly decreasing on $\left[0, \frac{1}{2\sqrt{3}}\right]$ and strictly increasing on $\left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$ Therefore, $f(x) \le \max\left\{f(0), f\left(\frac{1}{2}\right)\right\}$ Since $f(0) = f\left(\frac{1}{2}\right) = 1$, we get $f(x) \le 1$, as desired. Equality occurs for $(x, y, z, t) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, and also for $x, y, z, t) = (1, 0, 0, 0)$ or any permutation thereof □

Remark. Similarly, we can prove the below more general statement:

If $x_1, x_2, \ldots, x_n$ *are non-negative real numbers such that* $\sum x_1^2 = 1$, *then*

$$\sum x_1^3 + \frac{6}{(n-2)(\sqrt{n}+1)} \sum x_1 x_2 x_3 \le 1.$$

(*Vasile Cîrtoaje*, MS, 2006)

Chapter 6

Arithmetic/Geometric Compensation Method

The Arithmetic Compensation Method and the Geometric Compensation Method can be used to prove some difficult symmetric inequalities [10]

6.1 Arithmetic Compensation Method

Arithmetic Compensation Theorem (AC-Theorem). *Let $s > 0$ and let $F(x_1, x_2, \ldots, x_n)$ be a symmetrical continuous function on the compact set in $\mathbb{R}^n$*

$$S = \{(x_1, x_2, \ldots, x_n) \quad x_1 + x_2 + \cdots + x_n = s,\ x_1 \geq 0, \quad , x_n \geq 0\}.$$

If

$$F(x_1, x_2, x_3, \ldots, x_n) \leq$$
$$\leq \max\left\{F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n\right), F(0, x_1+x_2, x_3, \ldots, x_n)\right\} \quad (1)$$

for all $(x_1, x_2, \ldots, x_n) \in S$ with $x_1 > x_2 > 0$, then

$$F(x_1, x_2, x_3, \ldots, x_n) \leq \max_{1 \leq k \leq n} F\left(\frac{s}{k}, \ldots \frac{s}{k}, 0, \ldots, 0\right) \quad (2)$$

for all $(x_1, x_2, \ldots, x_n) \in S$.

Proof. Since the function f is continuous on the compact set S, F attains a maximum value at one or more points of the set Let $(x_1, x_2, \ldots, x_n)$ be

such a maximum point For the sake of contradiction, assume that there exist two numbers x_i and x_j such that $x_i > x_j > 0$; for convenience, let us consider $i = 1$ and $j = 2$ (hence $x_1 > x_2 > 0$)
According to the hypothesis, there are two cases to consider

a) *Case* $F(x_1, x_2, x_3, \ldots, x_n) <$
$$< \max\left\{F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n\right), F(0, x_1+x_2, x_3, \ldots, x_n)\right\}$$

But is false because F is maximal at $(x_1, x_2, \ldots, x_n)$, and the theorem is proved

b) *Case* $F(x_1, x_2, x_3, \ldots, x_n) =$
$$= \max\left\{F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n\right), F(0, x_1+x_2, x_3, \ldots, x_n)\right\}.$$

The function F attains again its maximum value at $(y_1, y_2, \ldots, y_n)$ with $y_i = x_i$ for $i \geq 3$ and either $y_1 = y_2 = \dfrac{x_1+x_2}{2}$ or $y_1 = 0$ and $y_2 = x_1 + x_2$. If there are not two numbers y_i and y_j such that $y_i > y_j > 0$, then the proof is finished. Otherwise, we iterate the preceding process, eventually finding a maximum point $(z_1, z_2, \ldots, z_n)$ such that all $z_i \in \left\{0, \dfrac{s}{k}\right\}$, where $1 \leq k \leq n$ □

Remark 1. In order to prove the condition (1), it suffices to show that $x_1 > x_2 > 0$ and
$$F(x_1, x_2, x_3, \ldots, x_n) > F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n\right)$$
involve
$$F(x_1, x_2, x_3, \ldots, x_4) \leq F(0, x_1+x_2, x_3, \ldots, x_n).$$

Remark 2 The AC-Theorem holds by replacing (1) and (2) with
$$F(x_1, x_2, x_3, \ldots, x_n) \geq$$
$$\geq \max\left\{F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n\right), F(0, x_1+x_2, x_3, \ldots, x_n)\right\} \quad (1')$$
and
$$F(x_1, x_2, x_3, \ldots, x_n) \geq \min_{1 \leq k \leq n} F\left(\frac{s}{k}, \ldots, \frac{s}{k}, 0, \ldots, 0\right), \quad (2')$$

respectively In order to prove the condition (1'), it suffices to show that $x_1 > x_2 > 0$ and

$$F(x_1, x_2, x_3, \ldots, x_n) < F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n\right)$$

involve

$$F(x_1, x_2, x_3, \ldots, x_n) \geq F(0, x_1+x_2, x_3, \ldots, x_n)$$

6.2 Geometric Compensation Method

Geometric Compensation Theorem (GC-Theorem) *Let $f(t)$ be a continuous function defined on $[0,\infty)$ such that for any couple (x,y) with $x > y > 0$, the inequality holds*

$$f(x) + f(y) \leq \max\left\{2f\left(\sqrt{xy}\right), a+b\right\},$$

where $a = f(0)$ and $b = \lim_{t\to\infty} f(t)$

Let $p > 0$, let $x_1, x_2, \ldots, x_n$ be positive numbers such that

$$x_1 x_2 \ldots x_n = p^n,$$

let $k_1, k_2 \in \{1, 2, \ldots, n-1\}$ and let

$$\delta = \max_{\substack{k_1+k_2\leq n \\ c\geq 0}} \left[k_1 a + k_2 b + (n-k_1-k_2) f(c)\right].$$

Then,

$$f(x_1) + f(x_2) + \cdots + f(x_n) \leq \max\{\delta, n f(p)\}.$$

Proof. Here we will prove this theorem only for the case in which the inequality in the hypothesis is strict, that is

$$f(x) + f(y) < \max\left\{2f\left(\sqrt{xy}\right), a+b\right\}$$

When the inequality is nonstrict, the proof is similar to one from the AC-Theorem

Denote by D the supremum of the function

$$F(x_1, x_2, \ldots, x_n) = f(x_1) + f(x_2) + \cdots + f(x_n)$$

on the set $S=\{(x_1,x_2,\ldots,x_n) \quad x_1x_2\ldots x_n=p^n, x_1>0,\ldots,x_n>0\}$ in $\mathbb{R}^n$

Suppose first that the supremum is attained at $(x_1,x_2,\ldots,x_n)\in S$. We infer that $x_1=x_2=\cdots=x_n=p$. For the sake of contradiction, we assume that there exist two indices i and j such that $x_i>x_j>0$ From the hypothesis it follows that the function F increases when the numbers x_i and x_j are replaced either by $x_i'=\sqrt{x_ix_j}$ and $x_j'=\sqrt{x_ix_j}$, or by $x_i'\to 0$ and $x_j'\to\infty$ (such that $x_i'x_j'=x_ix_j$) Consequently, F is not maximal at $(x_1,x_2,\ldots,x_n)$, which is a contradiction

Suppose now that the supremum D is not attained at a point of S. Thus, we may write D in the form

$$D=k_1a+k_2b+f(x_1)+\cdots+f(x_{n-k_1-k_2}),$$

where $k_1,k_2\in\{1,2,\ldots,n-1\}$ such that $k_1+k_2\le n$, and $x_1,\ldots,x_{n-k_1-k_2}>0$ We have to show that $x_1=\cdots=x_{n-k_1-k_2}$. Indeed, if there exist two indices $i,j\in\{1,\ldots,n-k_1-k_2\}$ such that $x_i>x_j>0$, then the sum $f(x_i)+f(x_j)$ increases when the numbers x_i and x_j are replaced by either $x_i'=\sqrt{x_ix_j}$ and $x_j'=\sqrt{x_ix_j}$, or $x_i'\to 0$ and $x_j'\to\infty$ (such that $x_i'x_j'=x_ix_j$). Consequently, D is not the supremum of F, contradiction. □

6.3 Applications

1. If $a,b,c,d\ge 0$ such that $a+b+c+d=4$, then

$$a)\qquad \frac{1}{5-abc}+\frac{1}{5-bcd}+\frac{1}{5-cda}+\frac{1}{5-dab}\le 1,$$

$$b)\qquad \frac{1}{4-abc}+\frac{1}{4-bcd}+\frac{1}{4-cda}+\frac{1}{4-dab}\le \frac{15}{11}$$

(*Vasile Cîrtoaje*, MS, 2005)

2. Let m and n be integer numbers such that $n\ge 3$ and $1<m<n$, and let $x_1,x_2,\ldots,x_n$ be non-negative numbers such that $x_1+x_2+\cdots+x_n=n$ If $p>\left(\frac{n}{m}\right)^m$, then the function

$$F(x_1,x_2,\ldots,x_n)=\sum_{1\le i_1<\cdots<i_m\le n}\frac{1}{p-x_{i_1}x_{i_2}\ldots x_{i_m}}$$

is maximal for $x_1 = \cdots = x_k = \dfrac{n}{k}$ and $x_{k+1} = \cdots = x_n = 0$, where $k \in \{m, m+1, \ldots, n\}$.

(*Vasile Cîrtoaje*, MS, 2005)

3. Let a, b, c, d be non-negative real numbers such that $a + b + c + d = 1$ Prove that

$$a) \quad 4(a^3 + b^3 + c^3 + d^3) + 15(abc + bcd + cda + dab) \geq 1;$$

$$b) \quad 11(a^3 + b^3 + c^3 + d^3) + 21(abc + bcd + cda + dab) \geq 2.$$

(*Vasile Cîrtoaje*, MS, 2006)

4. If $x_1, x_2, \ldots, x_n$ $(n \geq 3)$ are non-negative real numbers, then

$$a) \quad \sum x_1^3 + 3\sum x_1x_2x_3 \geq \sum x_1x_2(x_1 + x_2);$$

$$b) \quad \frac{n-1}{2}\sum x_1^3 + \frac{3}{n-2}\sum x_1x_2x_3 \geq \sum x_1x_2(x_1 + x_2).$$

(*Vasile Cîrtoaje*, MS, 2006)

5. Let a, b, c, d be non-negative real numbers.

a) If $a^2 + b^2 + c^2 + d^2 = 2$, then

$$a^3 + b^3 + c^3 + d^3 + abc + bcd + cda + dab \geq 2;$$

b) If $a^2 + b^2 + c^2 + d^2 = 3$, then

$$3(a^3 + b^3 + c^3 + d^3) + 2(abc + bcd + cda + dab) \geq 11$$

(*Vasile Cîrtoaje*, MS, 2006)

6. If $a, b, c, d \geq 0$ such that $a + b + c + d = 2$, then

$$\frac{1}{1+3a^2} + \frac{1}{1+3b^2} + \frac{1}{1+3c^2} + \frac{1}{1+3d^2} \geq \frac{16}{7}.$$

7. If $x_1, x_2, \ldots, x_n$ are non-negative real numbers such that

$$x_1 + x_2 + \cdots + x_n = s,$$

then

$$\frac{1}{1+x_1^2} + \frac{1}{1+x_2^2} + \cdots + \frac{1}{1+x_n^2} \geq n - \max_{1 \leq k \leq n} \frac{ks^2}{k^2 + s^2}.$$

(*Vasile Cîrtoaje*, MS, 2006)

8. Let $s > 0$, and let $x_1, x_2, \ldots, x_n$ be non-negative real numbers such that $x_1 + x_2 + \cdots + x_n = s$. Then,

$$\left(1+x_1^2\right)\left(1+x_2^2\right)\ldots\left(1+x_n^2\right) \le \max_{1\le k\le n}\left(1+\frac{s^2}{k^2}\right)^k .$$

(*Vasile Cîrtoaje*, CM, 8, 2005)

9. If $a, b, c, d \ge 0$ such that $a+b+c+d=1$, then

$$\frac{(1+2a)(1+2b)(1+2c)(1+2d)}{(1-a)(1-b)(1-c)(1-d)} \ge \frac{125}{8}.$$

10. Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers such that

$$x_1 + x_2 + \cdots + x_n = 1.$$

If $m > -1$, then

$$\prod_{i=1}^{n} \frac{1+mx_i}{1-x_i} \ge \min_{2\le k\le n}\left(\frac{k+m}{k-1}\right)^k .$$

(*Vasile Cîrtoaje*, CM, 7, 2004)

11. Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers such that

$$x_1 + x_2 + \cdots + x_n = \frac{2}{3}.$$

Then

$$\sum_{1\le i<j\le n} \frac{x_i x_j}{(1-x_i)(1-x_j)} \le \frac{1}{4}$$

(*Vasile Cîrtoaje*, MS, 2005)

12. Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers such that

$$x_1 + x_2 + \cdots + x_n = 1$$

and no $n-1$ of which are zero. Then

$$\sum_{1\le i<j\le n} \frac{x_i x_j}{(1-x_i)(1-x_j)} \ge \frac{n}{2(n-1)}.$$

(*Gabriel Dospinescu*, MS, 2005)

13. If $a,b,c,d \geq 0$ such that $a+b+c+d=4$, then

$$(1+3a)(1+3b)(1+3c)(1+3d) \leq 125+131abcd.$$

(*Pham Kim Hung*, MS, 2006)

14. If $a,b,c,d \geq 0$ such that $a+b+c+d=4$, then

$$(1+3a^2)(1+3b^2)(1+3c^2)(1+3d^2) \leq 255+a^2b^2c^2d^2.$$

(*Vasile Cîrtoaje*, MS, 2006)

15. Let $x_1, x_2, \ldots, x_n$ be positive numbers satisfying

$$\sqrt[n]{x_1 x_2 \ldots x_n} = p \leq \frac{1}{n-1}$$

Prove that

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n} \leq \frac{n}{1+p}.$$

16. If $a_1, a_2, \ldots, a_n$ are positive numbers such that

$$\sqrt[n]{a_1 a_2 \ldots a_n} = p \leq \sqrt{\frac{n}{n-1}} - 1,$$

then

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} \leq \frac{n}{(1+p)^2}.$$

6.4 Solutions

1. *If $a,b,c,d \geq 0$ such that $a+b+c+d=4$, then*

a) $$\frac{1}{5-abc} + \frac{1}{5-bcd} + \frac{1}{5-cda} + \frac{1}{5-dab} \leq 1;$$

b) $$\frac{1}{4-abc} + \frac{1}{4-bcd} + \frac{1}{4-cda} + \frac{1}{4-dab} \leq \frac{15}{11}.$$

Proof. If at least two of the numbers a,b,c,d are equal to zero, then the inequalities are clearly true. Assume now that at most one of a,b,c,d is equal to zero.

a) Denote the left hand side of the inequality by $F(a,b,c,d)$ We will show that $F(a,b,c,d) > F(t,t,c,d)$ involves $F(a,b,c,d) \le F(0,2t,c,d)$ for $a > b > 0$ and $t = \frac{a+b}{2}$. Then, by AC-Theorem it follows that

$$F(a,b,c,d) \le \max\left\{F(4,0,0,0), F(2,2,0,0), F\left(\frac{4}{3},\frac{4}{3},\frac{4}{3},0\right), F(1,1,1,1)\right\}$$

Since $F(4,0,0,0) = F(2,2,0,0) = \frac{4}{5}$, $F\left(\frac{4}{3},\frac{4}{3},\frac{4}{3},0\right) = \frac{348}{355}$ and $F(1,1,1,1) = 1$, we get $F(a,b,c,d) \le 1$

Let us show now that $F(a,b,c,d) > F(t,t,c,d)$ involves $F(a,b,c,d) \le F(0,2t,c,d)$ for $a > b > 0$ and $t = \frac{a+b}{2}$ Write the given inequality $F(a,b,c,d) > F(t,t,c,d)$ as

$$\frac{2(5-tcd)}{(5-acd)(5-bcd)} - \frac{2}{5-tcd} >$$
$$> \left(\frac{1}{5-t^2c} - \frac{1}{5-abc}\right) + \left(\frac{1}{5-t^2d} - \frac{1}{5-abd}\right).$$

Dividing by the positive factor $t^2 - ab$, the inequality becomes

$$\frac{2c^2d^2}{(5-acd)(5-bcd)(5-tcd)} > \frac{c}{(5-abc)(5-t^2c)} + \frac{d}{(5-abd)(5-t^2d)}.$$

Since

$$\frac{c}{(5-abc)(5-t^2c)} + \frac{d}{(5-abd)(5-t^2d)} > \frac{c}{5(5-t^2c)} + \frac{d}{5(5-t^2d)},$$

we get

$$\frac{2c^2d^2}{(5-acd)(5-bcd)(5-tcd)} > \frac{c}{5(5-t^2c)} + \frac{d}{5(5-t^2d)} \tag{1}$$

Similarly, write the required inequality $F(a,b,c,d) \le F(0,2t,c,d)$ as follows

$$\left(\frac{1}{5-abc} - \frac{1}{5}\right) + \left(\frac{1}{5-abd} - \frac{1}{5}\right) + \left(\frac{1}{5-acd} + \frac{1}{5-bcd}\right) \le \frac{1}{5} + \frac{1}{5-2tcd},$$

$$\frac{abc}{5(5-abc)} + \frac{abd}{5(5-abd)} + \frac{2(5-tcd)}{(5-acd)(5-bcd)} \le \frac{2(5-tcd)}{5(5-2tcd)},$$

$$\frac{c}{5(5-abc)} + \frac{d}{5(5-abd)} \le \frac{2c^2d^2(5-tcd)}{5(5-acd)(5-bcd)(5-2tcd)}$$

Since

$$\frac{5-tcd}{5-2tcd} \geq \frac{5}{5-tcd},$$

it suffices to show that

$$\frac{c}{5(5-abc)} + \frac{d}{5(5-abd)} \leq \frac{2c^2d^2}{(5-acd)(5-bcd)(5-tcd)}$$

This inequality immediately follows from (1). Equality occurs if and only if $a=b=c=d=1$.

b) Let

$$F(a,b,c,d) = \frac{1}{4-abc} + \frac{1}{4-bcd} + \frac{1}{4-cda} + \frac{1}{4-dab}$$

As in the preceding case, we can show that $F(a,b,c,d) > F\left(\frac{a+b}{2}, \frac{a+b}{2}, c, d\right)$ involves $F(a,b,c,d) \leq F(0, a+b, c, d)$ for $a > b > 0$. Then, by AC-Theorem, we have

$$F(a,b,c,d) \leq \max\left\{F(4,0,0,0), F(2,2,0,0), F\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 0\right), F(1,1,1,1)\right\}.$$

Since $F(4,0,0,0) = F(2,2,0,0) = 1$, $F\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 0\right) = \frac{15}{11}$ and $F(1,1,1,1) = \frac{4}{3}$, the desired inequality follows. Equality occurs when one of a, b, c, d equals 0 and the others equal $\frac{4}{3}$. □

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2. *Let m and n be integer numbers such that $n \geq 3$ and $1 < m < n$, and let $x_1, x_2, \ldots, x_n$ be non-negative numbers such that $x_1 + x_2 + \cdots + x_n = n$. If $p > \left(\frac{n}{m}\right)^m$, then the function*

$$F(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} \frac{1}{p - x_{i_1} x_{i_2} \ldots x_{i_m}}$$

is maximal for $x_1 = \cdots = x_k = \frac{n}{k}$ and $x_{k+1} = \cdots = x_n = 0$, where $k \in \{m, m+1, \ldots, n\}$.

Proof. For $p > \left(\frac{n}{m}\right)^m$, we have

$$x_{i_1}x_{i_2} \cdots x_{i_m} \le \left(\frac{x_{i_1}+x_{i_2}+\cdots+x_{i_m}}{m}\right)^m \le \left(\frac{n}{m}\right)^m < p$$

If at least $n-m+1$ of the numbers x_i are equal to zero, then the function F is minimal. Therefore, we will assume now that at most $n-m$ of the numbers x_i are equal to zero; that is, at least m of the numbers x_i are strict positive. According to AC-Theorem, it suffices to show that for $x > y > 0$ and $t = \frac{x+y}{2}$, the inequality $F(x,y,x_3,\ldots,x_n) > F(t,t,x_3,\ldots,x_n)$ involves $F(x,y,x_3,\ldots,x_n) \le F(0,2t,x_3,\ldots,x_n)$.

For convenience, let us denote

$$A_i = x_{i_1} \cdots x_{i_{m-2}},\quad B_i = x_{i_1} \cdots x_{i_{m-1}},\quad C_i = x_{i_1} \cdots x_{i_m}$$

and

$$\sum f(A_i) = \sum_{3\le i_1<\cdots<i_{m-2}\le n} f\left(x_{i_1}\cdots x_{i_{m-2}}\right),$$

$$\sum f(B_i) = \sum_{3\le i_1<\cdots<i_{m-1}\le n} f\left(x_{i_1}\cdots x_{i_{m-1}}\right),$$

$$\sum f(C_i) = \sum_{3\le i_1<\cdots<i_m\le n} f\left(x_{i_1}\cdots x_{i_m}\right),$$

where f is an arbitrary function.

We have

$$F(x,y,x_3,\ldots,x_n) = \sum\frac{1}{p-xyA_i}+\sum\frac{1}{p-xB_i}+\sum\frac{1}{p-yB_i}+\sum\frac{1}{p-C_i} =$$

$$= \sum\frac{1}{p-xyA_i}+\sum\frac{2(p-tB_i)}{(p-xB_i)(p-yB_i)}+\sum\frac{1}{p-C_i},$$

$$F(t,t,x_3,\ldots,x_n) = \sum\frac{1}{p-t^2A_i}+\sum\frac{2}{p-tB_i}+\sum\frac{1}{p-C_i}$$

and

$$F(0,2t,x_3,\ldots,x_n) = \binom{n-2}{m-2}\frac{1}{p}+\sum\frac{2(p-tB_i)}{p(p-2tB_i)}+\sum\frac{1}{p-C_i}$$

Thus, we may write the inequality $F(x,y,x_3,\ldots,x_n) > F(t,t,x_3,\ldots,x_n)$ in the form

$$\sum\frac{2(p-tB_i)}{(p-xB_i)(p-yB_i)}-\sum\frac{2}{p-tB_i} > \sum\left(\frac{1}{p-t^2A_i}-\frac{1}{p-xyA_i}\right)$$

After combining and dividing by the positive factor $t^2 - xy$, we obtain

$$\sum \frac{2B_i^2}{(p-xB_i)(p-yB_i)(p-tB_i)} > \sum \frac{A_i}{(p-xyA_i)(p-t^2A_i)}.$$

Since at least $m-2$ of the numbers $x_3, \ldots, x_n$ are non-zero, we have

$$\sum \frac{A_i}{(p-xyA_i)(p-t^2A_i)} > \sum \frac{A_i}{p(p-xyA_i)}$$

Consequently,

$$\sum \frac{2B_i^2}{(p-xB_i)(p-yB_i)(p-tB_i)} > \sum \frac{A_i}{p(p-xyA_i)}. \qquad (2)$$

Similarly, we may write the required inequality

$$F(x, y, x_3, \ldots, x_n) < F(0, 2t, x_3, \ldots, x_n)$$

as follows:

$$\sum \left(\frac{1}{p-xyA_i} - \frac{1}{p}\right) + \sum \frac{2(p-tB_i)}{(p-xB_i)(p-yB_i)} \le \sum \frac{2(p-tB_i)}{p(p-2tB_i)},$$

$$\sum \frac{xyA_i}{p(p-xyA_i)} \le \sum \frac{2xyB_i^2(p-tB_i)}{p(p-xB_i)(p-yB_i)(p-2tB_i)},$$

$$\sum \frac{A_i}{p-xyA_i} \le \sum \frac{2B_i^2(p-tB_i)}{(p-xB_i)(p-yB_i)(p-2tB_i)}$$

Since

$$\frac{p-tB_i}{p-2tB_i} \ge \frac{p}{p-tB_i},$$

it suffices to show that

$$\sum \frac{A_i}{p-xyA_i} \le \sum \frac{2pB_i^2}{(p-xB_i)(p-yB_i)(p-tB_i)}$$

But this inequality immediately follows from (2) □

From the above statement, we can deduce the following results.

Proposition. *Let m and n be arbitrary integers such that*

$$n \ge 3 \text{ and } 1 < m < n,$$

and let

$$q = \frac{\binom{n}{m} - 1}{\binom{n}{m}\left(\frac{m}{n}\right)^m - 1}$$

Let $x_1, x_2, \ldots, x_n$ *be non-negative numbers such that* $x_1 + x_2 + \cdots + x_n = n$, *and let*

$$F(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < \cdots < i_m \le n} \frac{1}{p - x_{i_1} x_{i_2} \ldots x_{i_m}}.$$

a) *For* $p \ge q$, *we have*

$$F(x_1, x_2, \ldots, x_n) \le F(1, 1, \ldots, 1);$$

b) *For* $\left(\frac{n}{m}\right)^m < p \le q$, *we have*

$$F(x_1, x_2, \ldots, x_n) \le F\left(\frac{n}{m}, \ldots, \frac{n}{m}, 0, \ldots, 0\right).$$

Corollary 8. *Let* $a_1, a_2, \ldots, a_n$ *be non-negative real numbers such that* $a_1 + a_2 + \cdots + a_n = n$, *and let*

$$F(a_1, a_2, \ldots, a_n) = \frac{1}{p - a_2 a_3 \ldots a_n} + \frac{1}{p - a_3 a_4 \ldots a_1} + \cdots + \frac{1}{p - a_1 a_2 \ldots a_{n-1}}$$

If

$$e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1} \quad \textit{and} \quad q = \frac{(n-1)e_{n-1}}{n - e_{n-1}},$$

then

a) $F(a_1, a_2, \ldots, a_n) \le \frac{n}{p-1}$, *for* $p \ge q$;

b) $F(a_1, a_2, \ldots, a_n) \le \frac{n-1}{p} + \frac{1}{p - e_{n-1}}$, *for* $e_{n-1} < p \le q$.

Corollary 9. *Let* $a_1, a_2, \ldots, a_n$ *be non-negative real numbers such that* $a_1 + a_2 + \cdots + a_n = n$, *and let*

$$F(a_1, a_2, \ldots, a_n) = \sum_{1 \le i < j \le n} \frac{1}{p - a_i a_j}.$$

Then

a) $F(a_1, a_2, \ldots, a_n) \le \dfrac{n(n-1)}{2(p-1)}$, *for* $p \ge \dfrac{n(n+1)}{2}$;

b) $F(a_1, a_2, \ldots, a_n) \le \dfrac{(n-2)(n+1)}{2p} + \dfrac{4}{4p-n^2}$, *for* $\dfrac{n^2}{4} < p \le \dfrac{n(n+1)}{2}$.

For $p = n+1$, from Corollary 1 we get the following nice statement:

- *Let* $a_1, a_2, \ldots, a_n \ge 0$ $(n \ge 4)$ *such that* $a_1 + a_2 + \cdots + a_n = n$. *Then*

$$\frac{1}{n+1-a_2a_3\cdots a_n} + \frac{1}{n+1-a_3a_4\cdots a_1} + \cdots + \frac{1}{n+1-a_1a_2\cdots a_{n-1}} \le 1$$

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3. *Let* a, b, c, d *be non-negative real numbers such that* $a+b+c+d=1$. *Prove that*

a) $4(a^3+b^3+c^3+d^3) + 15(abc+bcd+cda+dab) \ge 1$;

b) $11(a^3+b^3+c^3+d^3) + 21(abc+bcd+cda+dab) \ge 2$

Proof. Let p and q be real numbers, and let

$$F(a,b,c,d) = p(a^3+b^3+c^3+d^3) + q(abc+bcd+cda+dab)$$

We claim that

$$F(a,b,c,d) \ge$$
$$\ge \min\left\{F(1,0,0,0), F\left(\frac{1}{2},\frac{1}{2},0,0,\right), F\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right), F\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right\} = \quad (3)$$
$$= \min\left\{p, \frac{p}{4}, \frac{3p+q}{27}, \frac{p+q}{16}\right\}.$$

On this assumption, in the case a) with $p=4$ and $q=15$, we get

$$F(a,b,c,d) \ge \min\left\{4,1,1,\frac{19}{16}\right\} = 1,$$

which is in fact the desired inequality. Equality holds if one of a, b, c, d is zero and the others equal $\frac{1}{3}$, and also if two of a, b, c, d are zero and the others equal $\frac{1}{2}$

Similarly, in the case b) with $p = 11$ and $q = 21$, we get

$$F(a,b,c,d) \ge \min\left\{11, \frac{11}{4}, 2, 2\right\} = 2.$$

Equality holds if all numbers a, b, c, d equal $\frac{1}{4}$, and also if one of a, b, c, d is zero and the others equal $\frac{1}{3}$

In order to prove (3), we will use AC-Theorem, showing that the inequality $F(a,b,c,d) < F(t,t,c,d)$ involves $F(a,b,c,d) \geq F(0,a+b,c,d)$ for $a > b > 0$ and $t = \frac{a+b}{2}$. The inequality $F(a,b,c,d) < F(t,t,c,d)$ is equivalent to

$$p(a^3 + b^3 - 2t^3) < q(c+d)(t^2 - ab),$$

or

$$3p(a+b) < q(c+d). \tag{4}$$

On the other hand, the required inequality $F(a,b,c,d) \geq F(0,a+b,c,d)$ is equivalent to

$$p\left[a^3 + b^3 - (a+b)^3\right] + qab(c+d) \geq 0,$$

or

$$q(c+d) \geq 3p(a+b)$$

Clearly, (4) yields the last inequality. □

Remark The inequality a) has the homogeneous form

$$4(a^3 + b^3 + c^3 + d^3) + 15(abc + bcd + cda + dab) \geq (a+b+c+d)^3.$$

Since

$$(a+b+c+d)^3 = \sum a^3 + 3\sum ab(a+b) + 6\sum abc,$$

we get the inequality

$$\sum a^3 + 3\sum abc \geq \sum ab(a+b).$$

For $d = 0$, this inequality transforms into the third degree Schur's Inequality

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a)$$

Similarly, we may write the inequality b) in the homogeneous form

$$\sum a^3 + \sum abc \geq \frac{2}{3}\sum ab(a+b).$$

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4. *If $x_1, x_2, \ldots, x_n$ $(n \geq 3)$ are non-negative real numbers, then*

a) $$\sum x_1^3 + 3\sum x_1x_2x_3 \geq \sum x_1x_2(x_1+x_2);$$

b) $$\frac{n-1}{2}\sum x_1^3 + \frac{3}{n-2}\sum x_1x_2x_3 \geq \sum x_1x_2(x_1+x_2).$$

Proof. For convenience, let us denote the sum

$$\sum_{i+1\leq i_1<i_2<\cdots<i_j\leq n} x_{i_1}x_{i_2}\cdots x_{i_j}$$

by $\sum x_{i+1}x_{i+2}\ldots x_{i+j}$. Let p and q be real numbers, and let

$$F(x_1, x_2, \ldots, x_n) = p\sum x_1^3 + q\sum x_1x_2x_3 - \sum x_1x_2(x_1+x_2)$$

Since

$$\sum x_1x_2(x_1+x_2) = \left(\sum x_1\right)\left(\sum x_1^2\right) - \sum x_1^3,$$

we get

$$F(x_1, x_2, \ldots, x_n) = (p+1)\sum x_1^3 + q\sum x_1x_2x_3 - \left(\sum x_1\right)\left(\sum x_1^2\right).$$

If $x_1, x_2, \ldots, x_n$ are zero, the inequality is trivial Otherwise, due to homogeneity, we may consider that $\sum x_1 = 1$. We claim that

$$F(x_1, x_2, x_3, \ldots, x_n) < F(t, t, x_3, \ldots, x_n)$$

involves

$$F(x_1, x_2, x_3, \ldots, x_n) \geq F(0, x_1+x_2, x_3, \ldots, x_n)$$

for $x_1 > x_2 > 0$ and $t = \dfrac{x_1+x_2}{2}$. Then, by the AC-Theorem it follows that

$$F(x_1, x_2, x_3, \ldots, x_n) \geq \min_{1\leq k\leq n} f(k),$$

where

$$f(k) = \frac{6(p+1) + q(k-1)(k-2) - 6k}{6k^2}$$

is the value of F for $x_1 = \cdots = x_k = \dfrac{1}{k}$ and $x_{k+1} = \cdots = x_n = 0$.

In the case a), with $p = 1$ and $q = 3$, we get

$$f(k) = \frac{(k-2)(k-3)}{2k^2} \geq 0$$

for $k \in \{1, 2, \ldots, n\}$, and hence $F(x_1, x_2, x_3, \ldots, x_n) \geq 0$. Equality holds if two or three of the numbers $x_1, x_2, \ldots, x_n$ are equal and the others are zero

In the case b), with $p = \dfrac{n-1}{2}$ and $q = \dfrac{3}{n-2}$, we also have $F(x_1, x_2, x_3, \ldots, x_n) \geq 0$, because

$$f(k) = \frac{(k-n)(k-n+1)}{2k^2(n-2)} \geq 0.$$

Equality holds if either all numbers $x_1, x_2, \ldots, x_n$ are equal or $n-1$ of them are equal and the other is zero.

Taking into account that

$$\sum x_1x_2x_3 = x_1x_2\sum x_3 + (x_1+x_2)\sum x_3x_4 + \sum x_3x_4x_5,$$

the inequality $F(x_1, x_2, x_3, \ldots, x_n) < F(t, t, x_3, \ldots, x_n)$ is equivalent to

$$(p+1)\left(x_1^3 + x_2^3 - 2t^3\right) + q(x_1x_2 - t^2)\sum x_3 -$$
$$-\left(x_1 + x_2 + \sum x_3\right)\left(x_1^2 + x_2^2 - 2t^2\right) < 0.$$

Dividing by $(x_1 - x_2)^2$, it becomes

$$(3p+1)(x_1+x_2) < (q+2)\sum x_3 \tag{5}$$

On the other hand, the required inequality

$$F(x_1, x_2, x_3, \ldots, x_n) \geq F(0, 2t, x_3, \ldots, x_n)$$

is equivalent to

$$(p+1)\left(x_1^3 + x_2^3 - 8t^3\right) + qx_1x_2\sum x_3 - \left(x_1 + x_2 + \sum x_3\right)\left(x_1^2 + x_2^2 - 4t^2\right) \geq 0$$

Dividing by x_1x_2, we get the inequality

$$(q+2)\sum x_3 \geq (3p+1)(x_1+x_2),$$

which immediately follows from (5)

Remark For $m \in \{3, 4, \ldots, n\}$, $p = \dfrac{m-1}{2}$ and $q = \dfrac{3}{m-2}$, we find

$$f(k) = \frac{(k-m)(k-m+1)}{2k^2(m-2)}.$$

Since $f(k) \geq 0$ for $k \in \{1, 2, \ldots, n\}$, it follows that the following inequality holds for any $m \in \{3, 4, \ldots, n\}$

$$\frac{m-1}{2}\sum x_1^3 + \frac{3}{m-2}\sum x_1x_2x_3 \geq \sum x_1x_2(x_1+x_2). \tag{6}$$

Equality occurs if m or $m-1$ of the numbers $x_1, x_2, \ldots, x_n$ are equal and the others are zero.

Since

$$\sum x_1x_2(x_1+x_2) = \left(\sum x_1\right)\left(\sum x_1^2\right) - \sum x_1^3,$$

the inequality is equivalent to

$$\frac{m+1}{2}\sum x_1^3 + \frac{3}{m-2}\sum x_1x_2x_3 \geq \left(\sum x_1\right)\left(\sum x_1^2\right). \tag{7}$$

Since $\sum x_1^2 = \left(\sum x_1\right)^2 - 2\sum x_1x_2$ and

$$\sum x_1^3 = 3\sum x_1x_2x_3 + \left(\sum x_1\right)^3 - 3\left(\sum x_1\right)\left(\sum x_1x_2\right),$$

we may write the inequality in the form

$$\left(\sum x_1\right)^3 + \frac{3m}{m-2}\sum x_1x_2x_3 \geq \frac{3m-1}{m-1}\left(\sum x_1\right)\left(\sum x_1x_2\right). \tag{8}$$

Notice that the equivalent inequalities (6), (7) and (8) are valid for $m \in \{3, 4, \ldots, n\}$, but are not valid if $m \in (3, n)$ is not integer □

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5. *Let a, b, c, d be non-negative real numbers.*

a) If $a^2 + b^2 + c^2 + d^2 = 2$, then

$$a^3 + b^3 + c^3 + d^3 + abc + bcd + cda + dab \geq 2,$$

b) If $a^2 + b^2 + c^2 + d^2 = 3$, then

$$3(a^3 + b^3 + c^3 + d^3) + 2(abc + bcd + cda + dab) \geq 11$$

Proof. a) Let

$$F(a, b, c, d) = a^3 + b^3 + c^3 + d^3 + abc + bcd + cda + dab.$$

We claim that for $a > b > 0$ and $t = \sqrt{\dfrac{a^2+b^2}{2}}$, the inequality $F(a,b,c,d) < F(t,t,c,d)$ involves $F(a,b,c,d) \geq F\left(0, \sqrt{2}t, c, d\right)$. We see that

$$a^2+b^2+c^2+d^2 = t^2+t^2+c^2+d^2$$

and

$$a^2+b^2+c^2+d^2 = 0^2+\left(\sqrt{2}t\right)^2+c^2+d^2.$$

Then, by AC-Theorem we have

$$F(a,b,c,d) \geq$$
$$\geq \min\left\{F\left(\sqrt{2},0,0,0\right), F(1,1,0,0), F\left(\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}},0\right), F\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\right\} =$$
$$= \min\left\{2\sqrt{2}, 2, \frac{8\sqrt{6}}{9}, 4\sqrt{2}\right\} = 2,$$

from which the conclusion follows.

The inequality $F(a,b,c,d) < F(t,t,c,d)$ is equivalent to

$$a^3+b^3-2t^3 < cd(2t-a-b)+(c+d)(t^2-ab),$$

or

$$\frac{a^4+b^4+4abt^2}{2(a^3+b^3+2t^2)} < \frac{cd}{a+b+2t}+\frac{c+d}{2} \tag{9}$$

Similarly, the required inequality $F(a,b,c,d) \geq F\left(0, \sqrt{2}t, c, d\right)$ is equivalent to

$$cd\left(a+b-\sqrt{2}t\right)+ab(c+d) \geq 2\sqrt{2}t^3-a^3-b^3,$$

or

$$\frac{cd}{a+b+\sqrt{2}t}+\frac{c+d}{2} \geq \frac{3abt^2}{a^3+b^3+2\sqrt{2}t^3}$$

To prove this inequality, it suffices to show that

$$\frac{cd}{a+b+2t}+\frac{c+d}{2} \geq \frac{3abt^2}{a^3+b^3+2t^3}.$$

Taking account of (9), we have to show that

$$a^4+b^4+4abt^2 \geq 6abt^2$$

This inequality is equivalent to

$$(a-b)^2(a^2+ab+b^2) \geq 0,$$

which is clearly true Equality occurs when two of a, b, c, d are zero and the others equal 1

b) Let

$$F(a,b,c,d) = 3(a^3+b^3+c^3+d^3) + 2(abc+bcd+cda+dab).$$

As in the preceding case, we can show that $F(a,b,c,d) < F(t,t,c,d)$ involves $F(a,b,c,d) \geq F\left(0, \sqrt{2}t, c, d\right)$ for $a > b > 0$ and $t = \sqrt{\dfrac{a^2+b^2}{2}}$ Then, by AC-Theorem we have

$$F(a,b,c,d) \geq$$

$$\geq \min\left\{F\left(\sqrt{3},0,0,0\right), F\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 0, 0\right), F(1,1,1,0), F\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)\right\} =$$

$$= \min\left\{9\sqrt{3}, 9\sqrt{\frac{3}{2}}, 11, \frac{15\sqrt{3}}{2}\right\} = 11,$$

from which the conclusion follows.

The given inequality $F(a,b,c,d) < F(t,t,c,d)$ is equivalent to

$$\frac{3(a^4+b^4+4abt^2)}{2(a^3+b^3+2t^3)} < \frac{2cd}{a+b+2t} + c + d, \tag{10}$$

while the required inequality $F(a,b,c,d) \geq F\left(0, \sqrt{2}t, c, d\right)$ is equivalent to

$$\frac{2cd}{a+b+\sqrt{2}t} + c + d \geq \frac{9abt^2}{a^3+b^3+2\sqrt{2}t^3}$$

In order to prove this inequality, it suffices to show that

$$\frac{2cd}{a+b+2t} + c + d \geq \frac{9abt^2}{a^3+b^3+2t^3}.$$

This inequality follows from (10) and

$$a^4+b^4+4abt^2 \geq 6abt^2,$$

which is equivalent to

$$(a-b)^2(a^2+ab+b^2) \geq 0$$

Equality occurs when one of a, b, c, d is zero and the others equal 1. □

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6. *If $a, b, c, d \geq 0$ such that $a + b + c + d = 2$, then*

$$\frac{1}{1+3a^2} + \frac{1}{1+3b^2} + \frac{1}{1+3c^2} + \frac{1}{1+3d^2} \geq \frac{16}{7}.$$

Proof. Let

$$F(a,b,c,d) = \frac{1}{1+3a^2} + \frac{1}{1+3b^2} + \frac{1}{1+3c^2} + \frac{1}{1+3d^2}$$

We claim that for $a > b > c$ and $t = \dfrac{a+b}{2}$, the inequality $F(a,b,c,d) < F(t,t,c,d)$ involves $F(a,b,c,d) \geq F(0,2t,c,d)$. Then, by AC-Theorem we have

$$F(a,b,c,d) \geq$$
$$\leq \min\left\{F(2,0,0,0), F(1,1,0,0), F\left(\frac{2}{3},\frac{2}{3},\frac{2}{3},0\right), F\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right\} =$$
$$= \min\left\{\frac{40}{13},\frac{5}{2},\frac{16}{7},\frac{16}{7}\right\} = \frac{16}{7}.$$

The inequality $F(a,b,c,d) < F(t,t,c,d)$ is equivalent to

$$\frac{1}{1+3a^2} + \frac{1}{1+3b^2} < \frac{2}{1+3t^2},$$

or

$$\frac{(6t^2-1+3ab)(t^2-ab)}{(1+3a^2)(1+3b^2)(1+3t^2)} < 0.$$

Since $t^2 - ab > 0$, we get

$$6t^2 - 1 + 3ab < 0 \tag{11}$$

Similarly, the required inequality $F(a,b,c,d) \geq F(0,2t,c,d)$ is equivalent to each of the following

$$\frac{1}{1+3a^2} + \frac{1}{1+3b^2} \geq 1 + \frac{1}{1+3(a+b)^2},$$

$$\frac{ab(1-3ab-18abt^2)}{(1+3a^2)(1+3b^2)(1+12t^2)} \geq 0,$$

$$\frac{1}{3ab} - 1 - 6t^2 \geq 0$$

Using (11), we have

$$\frac{1}{3ab} - 1 - 6t^2 \geq \frac{1}{3ab} + 3ab - 2 = \frac{(1-3ab)^2}{3ab} > 0$$

The last inequality is strict because (11) yields $1 - 3ab > 6t^2 > 0$ Equality occurs when $a = b = c = d = \frac{1}{2}$, and also when one of a, b, c, d is zero and the others equal $\frac{2}{3}$. □

★

7. *If* $x_1, x_2, \ldots, x_n$ *are non-negative real numbers such that*

$$x_1 + x_2 + \cdots + x_n = s,$$

then

$$\frac{1}{1+x_1^2} + \frac{1}{1+x_2^2} + \cdots + \frac{1}{1+x_n^2} \geq n - \max_{1 \leq k \leq n} \frac{ks^2}{k^2+s^2}.$$

Proof. Let

$$F(x_1, x_2, \ldots, x_n) = \frac{1}{1+x_1^2} + \frac{1}{1+x_2^2} + \cdots + \frac{1}{1+x_n^2}.$$

We have to show that F is minimal for

$$x_1 = \cdots = x_k = \frac{1}{k} \text{ and } x_{k+1} = \cdots = x_n = 0,$$

where $k \in \{1, 2, \ldots, n\}$. By AC-Theorem, it suffices to show that for $x > y > 0$ and $t = \frac{x+y}{2}$, the inequality

$$F(x, y, x_3, \ldots, x_n) < F(t, t, x_3, \ldots, x_n)$$

involves

$$F(x, y, x_3, \ldots, x_n) \geq F(0, 2t, x_3, \ldots, x_n).$$

To this effect we can be use the same way as above. □

Remark. From this application, we can deduce the following result.

- *Let* $x_1, x_2, \ldots, x_n$ *be non-negative real numbers such that*

$$x_1 + x_2 + \cdots + x_n = s,$$

and let

$$F(x_1, x_2, \ldots, x_n) = \frac{1}{1+x_1^2} + \frac{1}{1+x_2^2} + \cdots + \frac{1}{1+x_n^2}.$$

a) *If* $k \in \{1, 2, \ldots, n-1\}$ *and* $\sqrt{k(k-1)} \le s \le \sqrt{(k(k+1)}$, *then*

$$F(x_1, x_2, \ldots, x_n) \ge \frac{k^2 n + (n-k)s^2}{k^2 + s^2};$$

b) *If* $s \ge \sqrt{n(n-1)}$, *then*

$$F(x_1, x_2, \ldots, x_n) \ge \frac{n^3}{n^2 + s^2}.$$

★

8. *Let* $s > 0$, *and let* $x_1, x_2, \ldots, x_n$ *be non-negative real numbers such that* $x_1 + x_2 + \cdots + x_n = s$. *Then,*

$$\left(1 + x_1^2\right)\left(1 + x_2^2\right) \ldots \left(1 + x_n^2\right) \le \max_{1 \le k \le n} \left(1 + \frac{s^2}{k^2}\right)^k.$$

Proof. Let

$$F(x_1, x_2, \ldots, x_n) = \left(1 + x_1^2\right)\left(1 + x_2^2\right) \ldots \left(1 + x_n^2\right).$$

By AC-Theorem, it suffices to show that for $x > y > 0$ and $t = \dfrac{x+y}{2}$,

$$F(x, y, x_3, \ldots, x_n) > F(t, t, x_3, \ldots, x_n)$$

involves

$$F(x, y, x_3, \ldots, x_n) \le F(0, 2t, x_3, \ldots, x_n)$$

Since

$$F(x, y, x_3, \ldots, x_n) - F(t, t, x_3, \ldots, x_n) =$$
$$= (t^2 - xy)(2 - xy - t^2)\left(1 + x_3^2\right) \ldots \left(1 + x_n^2\right)$$

and

$$F(x, y, x_3, \ldots, x_n) - F(0, 2t, x_3, \ldots, x_n) = xy(xy - 2)\left(1 + x_3^2\right) \ldots \left(1 + x_n^2\right),$$

we have to show that $2 - xy - t^2 > 0$ implies $xy - 2 \le 0$. Indeed, we have $2 - xy > t^2 > 0$. □

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9. *If* $a, b, c, d \geq 0$ *such that* $a + b + c + d = 1$, *then*

$$\frac{(1+2a)(1+2b)(1+2c)(1+2d)}{(1-a)(1-b)(1-c)(1-d)} \geq \frac{125}{8}.$$

Proof. Let us denote the left hand side of the inequality by $F(a, b, c, d)$ We claim that for $a > b > 0$ and $t = \frac{a+b}{2}$, the inequality $F(a,b,c,d) < F(t,t,c,d)$ involves $F(a,b,c,d) \geq F(0,2t,c,d)$. Then, by AC-Theorem we have

$$\begin{aligned} &F(a,b,c,d) \geq \\ &\leq \min\left\{F\left(\frac{1}{2},\frac{1}{2},0,0\right), F\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right), F\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right\} = \\ &= \min\left\{16, \frac{125}{8}, 16\right\} = \frac{125}{8}. \end{aligned}$$

The inequality $F(a, b, c, d) < F(t, t, c, d)$ is equivalent to

$$\frac{(1+2a)(1+2b)}{(1-a)(1-b)} < \left(\frac{1+2t}{1-t}\right)^2,$$

or

$$\frac{3(4t-1)(t^2-ab)}{(1-t)(1-a)(1-b)} < 0.$$

Since $t^2 - ab > 0$, it follows that $t < \frac{1}{4}$ On the other hand, the desired inequality $F(a,b,c,d) \geq F(0,2t,c,d)$ is equivalent to

$$\frac{(1+2a)(1+2b)}{(1-a)(1-b)} \geq \frac{1+4t}{1-2t},$$

or

$$\frac{3(-4t+1)ab}{(1-2t)(1-a)(1-b)} \geq 0$$

Since $t < \frac{1}{4}$, the inequality is clearly true. Equality occurs when one of the numbers a, b, c, d equals 0, and the others equal $\frac{1}{3}$. □

★

10. *Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers such that*

$$x_1 + x_2 + \cdots + x_n = 1$$

If $m > -1$, then

$$\prod_{i=1}^{n} \frac{1+mx_i}{1-x_i} \ge \min_{2\le k\le n} \left(\frac{k+m}{k-1}\right)^k .$$

Proof. Let

$$F(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \frac{1+mx_i}{1-x_i} .$$

We have to show that F is minimal for

$$x_1 = \cdots = x_k = \frac{1}{k} \text{ and } x_{k+1} = \cdots = x_n = 0,$$

where $k \in \{2, 3, \ldots, n\}$. By AC-Theorem, it suffices to show that for $x > y > 0$ and $t = \dfrac{x+y}{2}$, the inequality

$$F(x, y, x_3, \ldots, x_n) < F(t, t, x_3, \ldots, x_n)$$

involves

$$F(x, y, x_3, \ldots, x_n) \ge F(0, 2t, x_3, \ldots, x_n).$$

We may write the inequality $F(x, y, x_3, \ldots, x_n) < F(t, t, x_3, \ldots, x_n)$ as follows

$$\frac{(1+mx)(1+my)}{(1-x)(1-y)} < \left(\frac{1+mt}{1-t}\right)^2 ,$$

$$\frac{(m+1)(2mt-m+1)(t^2-xy)}{(1-t)(1-x)(1-y)} < 0.$$

Since $t^2 - xy > 0$, we get $2mt - m + 1 < 0$ Similarly, the desired inequality

$$F(x, y, x_3, \ldots, x_n) \ge F(0, 2t, x_3, \ldots, x_n)$$

is equivalent to

$$\frac{(1+mx)(1+my)}{(1-x)(1-y)} \ge \frac{1+2mt}{1-2t}$$

or

$$\frac{(m+1)(-2mt+m-1)xy}{(1-2t)(1-x)(1-y)} \ge 0.$$

The last inequality is true because $2mt - m + 1 < 0$. □

Remark. From this application, we can deduce the following result

- *Let m_k be the positive root of the equation in m,*

$$\left(1+\frac{1+m}{k}\right)^{k+1}=\left(1+\frac{1+m}{k-1}\right)^{k}.$$

Then,

a) $\sqrt{5}=m_2>m_3>\cdots>m_{n-1}>1$;

b) $\prod_{i=1}^{n}\left(\frac{1+mx_i}{1-x_i}\right)\geq(m+2)^2$, for $m\geq m_2$;

c) $\prod_{i=1}^{n}\left(\frac{1+mx_i}{1-x_i}\right)\geq\left(\frac{k+m}{k-1}\right)^{k}$, for $m_k\leq m\leq m_{k-1}$ and $k\in\{3,\ldots,n-1\}$;

d) $\prod_{i=1}^{n}\left(\frac{1+mx_i}{1-x_i}\right)\geq\left(\frac{m+n}{n-1}\right)^{n}$, for $-1<m\leq m_{n-1}$.

★

11. *Let $x_1,x_2,\ldots,x_n$ be non-negative real numbers such that*

$$x_1+x_2+\cdots+x_n=\frac{2}{3}$$

Then

$$\sum_{1\leq i<j\leq n}\frac{x_ix_j}{(1-x_i)(1-x_j)}\leq\frac{1}{4}.$$

Proof. Let

$$F(x_1,x_2,\ldots,x_n)=\sum_{1\leq i<j\leq n}\frac{x_ix_j}{(1-x_i)(1-x_j)}.$$

We claim that for $x>y>0$ and $t=\frac{x+y}{2}$, the inequality

$$F(x,y,x_3,\ldots,x_n)>F(t,t,x_3,\ldots,x_n)$$

involves

$$F(x,y,x_3,\ldots,x_n)\leq F(0,2t,x_3,\ldots,x_n).$$

Then, by AC-Theorem, we have

$$F(x_1,x_2,\ldots,x_n)\leq\max_{1\leq k\leq n}F\left(\frac{2}{3k},\ldots,\frac{2}{3k},0,\ldots,0\right)=\max_{1\leq k\leq n}\frac{2k(k-1)}{3k-2^{\ 2}}.$$

Since

$$\frac{2k(k-1)}{(3k-2)^2} = \frac{1}{4} - \frac{(k-2)^2}{4(3k-2)^2} \le \frac{1}{4},$$

the desired inequality follows

The inequality $F(x, y, x_3, \ldots, x_n) > F(t, t, x_3, \ldots, x_n)$ is equivalent to

$$\frac{xy}{(1-x)(1-y)} - \frac{t^2}{(1-t)^2} + \left(\frac{x}{1-x} + \frac{y}{1-y} - \frac{2t}{1-t}\right)\sum_{j=3}^{n}\frac{x_j}{1-x_j} > 0,$$

or

$$\frac{t^2 - xy}{(1-x)(1-y)(1-t)^2}\left[2t - 1 + 2(1-t)\sum_{j=3}^{n}\frac{x_j}{1-x_j}\right] > 0$$

Since $t^2 - xy > 0$, we get

$$2t - 1 + 2(1-t)\sum_{j=3}^{n}\frac{x_j}{1-x_j} > 0. \tag{12}$$

The required inequality $F(x, y, x_3, \ldots, x_n) \le F(0, 2t, x_3, \ldots, x_n)$ is equivalent to

$$\frac{xy}{(1-x)(1-y)} + \left(\frac{x}{1-x} + \frac{y}{1-y} - \frac{2t}{1-2t}\right)\sum_{j=3}^{n}\frac{x_j}{1-x_j} \le 0,$$

or

$$\frac{xy}{(1-x)(1-y)(1-2t)}\left[1 - 2t + 2(t-1)\sum_{j=3}^{n}\frac{x_j}{1-x_j}\right] \le 0.$$

Taking account of (12), this inequality is clearly true

Equality occurs if and only if two of x_i are equal to $\frac{1}{3}$, and the others are zero □

Remark 1. From the above proof, we can formulate a more general statement.

• *Let* $0 < s < 1$, *let* $x_1, x_2, \ldots, x_n$ *be non-negative real numbers such that* $x_1 + x_2 + \cdots + x_n = s$ *and let*

$$F(x_1, x_2, \ldots, x_n) = \sum_{1 \le i < j \le n}\frac{x_i x_j}{(1-x_i)(1-x_j)}$$

Then,

$$F(x_1, x_2, \ldots, x_n) \le \max_{1<k<n} F\left(\frac{s}{k}, \ldots, \frac{s}{k}, 0, \ldots, 0\right) = \frac{s^2}{2}\max_{2\le k\le n}\frac{k(k-1)}{(k-s)^2}.$$

Remark 2 For $x_1 + x_2 + \cdots + x_n = 1$, the inequality holds

$$\sum_{1 \le i < j \le n} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} \le 1$$

Indeed, assuming that $x_1 \ge x_2 \ge \cdots \ge x_n$, we have

$$\sum_{1 \le i < j \le n} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} \le \frac{1}{(1 - x_1)(1 - x_2)} \sum_{1 \le i < j \le n} x_i x_j =$$
$$= \frac{1 - \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)}{2(1 - x_1)(1 - x_2)} \le$$
$$\le \frac{1 - \left(x_1^2 + x_2^2\right)}{2(1 - x_1)(1 - x_2)} = 1 - \frac{(x_1 + x_2 - 1)^2}{2(1 - x_1)(1 - x_2)} \le 1.$$

Under the assumption $x_1 \ge x_2 \ge \cdots \ge x_n$, equality occurs if and only if $x_3 = \cdots = x_n = 0$.

★

12. *Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers such that*

$$x_1 + x_2 + \cdots + x_n = 1$$

and no $n - 1$ of which are zero. Then

$$\sum_{1 \le i < j \le n} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} \ge \frac{n}{2(n - 1)}$$

Proof. For $n = 2$, the inequality becomes equality Consider now that $n \ge 3$. We will show that the inequality holds if one of x_i is larger than $\frac{3}{4}$. Indeed, if $x_1 > \frac{3}{4}$, than

$$\sum_{1 \le i < j \le n} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} \ge \sum_{j=2}^{n} \frac{x_1 x_j}{(1 - x_1)(1 - x_j)} > \frac{x_1}{1 - x_1} \sum_{j=2}^{n} x_j =$$
$$= x_1 > \frac{3}{4} \ge \frac{n}{2(n - 1)}.$$

Consider now that $0 \le x_i \le \frac{3}{4}$ for $i = 1, 2, \ldots, n$. Let

$$F(x_1, x_2, \ldots, x_n) = \sum_{1 \le i < j \le n} \frac{x_i x_j}{(1 - x_i)(1 - x_j)}.$$

We claim that for $x>y>0$ and $t=\dfrac{x+y}{2}$, the inequality

$$F(x,y,x_3,\dots,x_n) < F(t,t,x_3,\dots,x_n)$$

involves

$$F(x,y,x_3,\dots,x_n) \ge F(0,2t,x_3,\dots,x_n).$$

Since the symmetric function $F(x_1,x_2,\dots,x_n)$ is continuous for $0 \le x_i \le \dfrac{3}{4}$, by AC-Theorem we have

$$F(x_1,x_2,\dots,x_n) \ge \min_{2\le k\le n} F\left(\frac{1}{k},\dots,\frac{1}{k},0,\dots,0\right) = \\ = \min_{2\le k\le n} \frac{k}{2(k-1)} = \frac{n}{2(n-1)},$$

which is just the required inequality.

From the preceding proof, we may assert that the given condition

$$F(x,y,x_3,\dots,x_n) < F(t,t,x_3,\dots,x_n)$$

yields

$$2t-1+2(1-t)\sum_{j=3}^{n}\frac{x_j}{1-x_j} < 0,$$

whereas the inequality $F(x,y,x_3,\dots,x_n) \ge F(0,2t,x_3,\dots,x_n)$ is true if we show that

$$1-2t+2(t-1)\sum_{j=3}^{n}\frac{x_j}{1-x_j} \ge 0.$$

The conclusion follows. For $n \ge 3$ equality occurs if and only if $x_1=x_2=\cdots=x_n=\dfrac{1}{n}$. □

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13. *If $a,b,c,d \ge 0$ such that $a+b+c+d=4$, then*

$$(1+3a)(1+3b)(1+3c)(1+3d) \le 125+131abcd.$$

Proof. Let

$$F(a,b,c,d) = (1+3a)(1+3b)(1+3c)(1+3d) - 131abcd.$$

We claim that for $a > b > 0$, the inequality $F(a,b,c,d) > F\left(\frac{a+b}{2}, \frac{a+b}{2}, c, d\right)$ involves $F(a,b,c,d) \le F(0, a+b, c, d)$. Then, by AC-Theorem we have

$$F(a,b,c,d) \le \max\left\{F(4,0,0,0), F(2,2,0,0), F\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 0\right), F(1,1,1,1)\right\}$$

From

$$F(4,0,0,0) = 13, \ F(2,2,0,0) = 49,$$
$$F\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 0\right) = 125, \ F(1,1,1,1) = 125,$$

we get $F(a,b,c,d) \le 125$, which is the desired inequality

Since the inequality $F(a,b,c,d) > F\left(\frac{a+b}{2}, \frac{a+b}{2}, c, d\right)$ is equivalent to

$$(a-b)^2 [131cd - 9(1+3c)(1+3d)] > 0,$$

whereas the inequality $F(a,b,c,d) \le F(0, a+b, c, d)$ is equivalent to

$$ab [9(1+3c)(1+3d) - 131cd] < 0,$$

the conclusion follows. Equality occurs when $a = b = c = d = 1$, and again when one of the numbers a, b, c, d is 0 and the others are equal to $\frac{4}{3}$. □

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14. *If $a, b, c, d \ge 0$ such that $a + b + c + d = 4$, then*

$$(1+3a^2)(1+3b^2)(1+3c^2)(1+3d^2) \le 255 + a^2b^2c^2d^2.$$

Proof. Let

$$F(a,b,c,d) = (1+3a^2)(1+3b^2)(1+3c^2)(1+3d^2) - a^2b^2c^2d^2$$

We claim that for $a > b > 0$ and $t = \frac{a+b}{2}$, the inequality $F(a,b,c,d) > F(t,t,c,d)$ involves $F(a,b,c,d) \le F(0, 2t, c, d)$ Then, by AC-Theorem we have

$$F(a,b,c,d) \le \max\left\{F(4,0,0,0), F(2,2,0,0), F\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 0\right), F(1,1,1,1)\right\}.$$

Since $F(4,0,0,0) = 49$, $F(2,2,0,0) = 169$, $F\left(\frac{4}{3},\frac{4}{3},\frac{4}{3},0\right) = \frac{6859}{27} < 255$ and $F(1,1,1,1) = 255$, we get the desired result $F(a,b,c,d) \le 255$

The inequality $F(a,b,c,d) > F(t,t,c,d)$ is equivalent to

$$(t^2 - ab)\left[c^2d^2(t^2+ab) - 3(3t^2+3ab-2)(1+3c^2)(1+3d^2)\right] > 0$$

This inequality implies

$$\frac{c^2d^2}{3(1+3c^2)(1+3d^2)} > \frac{3t^2+3ab-2}{t^2+ab}.$$

On the other hand, the required inequality $F(a,b,c,d) \le F(0,2t,c,d)$ is equivalent to

$$ab\left[3(3ab-2)(1+3c^2)(1+3d^2) - abc^2d^2\right] \le 0,$$

and it is true if

$$\frac{c^2d^2}{3(1+3c^2)(1+3d^2)} \ge \frac{3ab-2}{ab}.$$

To prove this, it suffices to show that

$$\frac{3t^2+3ab-2}{t^2+ab} \ge \frac{3ab-2}{ab}.$$

Indeed, we have

$$\frac{3t^2+3ab-2}{t^2+ab} = 3 - \frac{2}{t^2+ab} > 3 - \frac{2}{ab} = \frac{3ab-2}{ab}$$

Equality occurs when $a = b = c = d = 1$, and again when one of the numbers a, b, c, d is 0 and the others are equal to $\frac{4}{3}$. □

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15. *Let $x_1, x_2, \ldots, x_n$ be positive numbers satisfying*

$$\sqrt[n]{x_1x_2\ldots x_n} = p \le \frac{1}{n-1}.$$

Prove that

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n} \le \frac{n}{1+p}$$

Proof. We apply GC-Theorem to the continuous function $f(t) = \frac{1}{1+t}$, $t \geq 0$. Let us show that for $x > y > 0$ the inequality holds

$$f(x) + f(y) \leq \max\{2f(\sqrt{xy}), a + b\},$$

where $a = f(0) = 1$ and $b = \lim_{t\to\infty} f(t) = 0$. Rewrite the inequality as

$$\max\left\{\frac{2}{1+\sqrt{xy}}, 1\right\} \geq \frac{1}{1+x} + \frac{1}{1+y}.$$

For $xy < 1$, we have

$$\max\left\{\frac{2}{1+\sqrt{xy}}, 1\right\} - \frac{1}{1+x} - \frac{1}{1+y} = \frac{2}{1+\sqrt{xy}} - \frac{1}{1+x} - \frac{1}{1+y} =$$
$$= \frac{\left(\sqrt{x}-\sqrt{y}\right)^2\left(1-\sqrt{xy}\right)}{(1+x)(1+y)\left(1+\sqrt{xy}\right)} > 0$$

For $xy \geq 1$, we have

$$\max\left\{\frac{2}{1+\sqrt{xy}}, 1\right\} - \frac{1}{1+x} - \frac{1}{1+y} = 1 - \frac{1}{1+x} - \frac{1}{1+y} =$$
$$= \frac{xy-1}{(1+x)(1+y)} \geq 0.$$

On the other hand, we have

$$\delta = \max_{\substack{k_1+k_2\leq n\\ c\geq 0}} [k_1 a + k_2 b + (n-k_1-k_2)f(c)] =$$
$$= \max_{\substack{k_1+k_2\leq n\\ c\geq 0}} \left(k_1 + \frac{n-k_1-k_2}{1+c}\right) = \max_{k_1+k_2\leq n}(k_1 + n - k_1 - k_2) =$$
$$= \max_{1\leq k_2\leq n-1}(n-k_2) = n-1$$

and

$$\max\{\delta, nf(p)\} = \max\left\{n-1, \frac{n}{1+p}\right\} = \frac{n}{1+p}.$$

By GC-Theorem it follows that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \leq \max\{\delta, nf(p)\} = \frac{n}{1+p},$$

which is the desired result.

For $n \geq 3$, equality occurs if and only if $x_1 = x_2 = \cdots = x_n = p$. □

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16. *If $a_1, a_2, \ldots, a_n$ are positive numbers such that*

$$\sqrt[n]{a_1 a_2 \ldots a_n} = p \le \sqrt{\frac{n}{n-1}} - 1,$$

then

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} \le \frac{n}{(1+p)^2}.$$

Proof. We apply GC-Theorem to the continuous function $f(t) = \dfrac{1}{(1+t)^2}$, $t \ge 0$. Let us show that for $x > y > 0$, the inequality holds

$$f(x) + f(y) < \max\left\{2f\left(\sqrt{xy}\right), a + b\right\},$$

where $a = f(0) = 1$ and $b = \lim_{t\to\infty} f(t) = 0$

Setting $t = \sqrt{xy}$ and $s = 1 + x + y$ $(s > 1 + 2t)$, we have

$$f(x) + f(y) = \frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} = \frac{s^2 + 1 - 2t^2}{(s+t^2)^2},$$

and the above inequality may be written as

$$\max\left\{\frac{2}{(1+t)^2}, 1\right\} > \frac{s^2 + 1 - 2t^2}{(s+t^2)^2}.$$

For $t \ge \sqrt{2} - 1$, we get

$$\max\left\{\frac{2}{(1+t)^2}, 1\right\} - \frac{s^2 + 1 - 2t^2}{(s+t^2)^2} = 1 - \frac{s^2 + 1 - 2t^2}{(s+t^2)^2} =$$
$$= \frac{2st^2 + t^4 + 2t^2 - 1}{(s+t^2)^2} > \frac{2(1+2t)t^2 + t^4 + 2t^2 - 1}{(s+t^2)^2} =$$
$$= \frac{t^2(t^2+2)^2 - 1}{(s+t^2)^2} = \frac{(t+1)^2(t^2+2t-1)}{(s+t^2)^2} \ge 0.$$

For $t < \sqrt{2} - 1$, we get

$$\max\left\{\frac{2}{(1+t)^2}, 1\right\} - \frac{s^2+1-2t^2}{(s+t^2)^2} = \frac{2}{(1+t)^2} - \frac{s^2+1-2t^2}{(s+t^2)^2} =$$
$$= \frac{(s-1-2t)\left[(1-2t-t^2)s+1-t^2-2t^3\right]}{(1+t)^2(s+t^2)^2} >$$
$$> \frac{(s-1-2t)\left[(1-2t-t^2)(1+2t)+1-t^2-2t^3\right]}{(1+t)^2(s+t^2)^2} =$$
$$= \frac{2(s-1-2t)[1-3t^2-2t^3]}{(1+t)^2(s+t^2)^2} = \frac{2(s-1-2t)[(1+t)(1-t-2t^2)]}{(1+t)^2(s+t^2)^2} > 0.$$

On the other hand,

$$\delta = \max_{\substack{k_1+k_2\le n \\ c\ge 0}} [k_1 a + k_2 b + (n-k_1-k_2)f(c)] = \max_{\substack{k_1+k_2\le n \\ c\ge 0}} \left[k_1 + \frac{n-k_1-k_2}{(1+c)^2}\right] =$$
$$= \max_{k_1+k_2\le n} (k_1+n-k_1-k_2) = \max_{1\le k_2\le n-1} (n-k_2) = n-1$$

and

$$\max\{\delta, nf(p)\} = \max\left\{n-1, \frac{n}{(1+p)^2}\right\} = \frac{n}{(1+p)^2}.$$

By GC-Theorem it follows that

$$f(x_1)+f(x_2)+\cdots+f(x_n) \le \max\{\delta, nf(p)\} = \frac{n}{(1+p)^2}.$$

Equality occurs if and only if $x_1 = x_2 = \cdots = x_n = p$. □

Chapter 7

Symmetric inequalities with three variables involving fractions

In this chapter we are mainly concerned with some inequalities involving symmetric expressions as ones below, where a, b, c are non-negative real numbers, and $r > -2$, p and q are given real numbers

$$E_1 = \frac{a(b+c)+pbc}{b^2+rbc+c^2} + \frac{b(c+a)+pca}{c^2+rca+a^2} + \frac{c(a+b)+pab}{a^2+rab+b^2},$$

$$E_2 = \frac{a^2+qbc}{b^2+rbc+c^2} + \frac{b^2+qca}{c^2+rca+a^2} + \frac{c^2+qab}{a^2+rab+b^2}$$

7.1 Inequalities involving E_1

1. Let a, b, c be non-negative real numbers, no two of which are zero Then,

$$\frac{a(b+c)}{b^2+bc+c^2} + \frac{b(c+a)}{c^2+ca+a^2} + \frac{c(a+b)}{a^2+ab+b} \geq 2.$$

2. Let a, b, c be non-negative real numbers, no two of which are zero. Then,

$$\frac{ab-bc+ca}{b^2+c^2} + \frac{bc-ca+ab}{c^2+a^2} + \frac{ca-ab+bc}{a^2+b^2} \geq \frac{3}{2}.$$

3. Let a, b, c be non-negative real numbers, no two of which are zero Then,

$$\frac{ab-2bc+ca}{b^2-bc+c^2} + \frac{bc-2ca+ab}{c^2-ca+a^2} + \frac{ca-2ab+bc}{a^2-ab+b^2} \geq 0$$

(*Vasile Cîrtoaje*, MS, 2005)

4. Let a, b, c be non-negative real numbers, no two of which are zero. Then,

$$\frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{1}{(a+b)^2} \geq \frac{9}{4(ab+bc+ca)}$$

(*Iran*, 1996)

5. Let a, b, c be non-negative real numbers, no two of which are zero.
If $r > -2$, then

$$\sum \frac{ab+(r-1)bc+ca}{b^2+rbc+c^2} \geq \frac{3(r+1)}{r+2}.$$

(*Vasile Cîrtoaje*, MS, 2005)

6. Let a, b, c be non-negative real numbers, no two of which are zero. Then,

$$\sum \frac{ab+4bc+ca}{b^2+c^2} \geq 4$$

7. Let a, b, c be non-negative real numbers, no two of which are zero. If $r > -2$, then

$$\sum \frac{ab+(r+2)^2bc+ca}{b^2+rbc+c^2} \geq r+4.$$

8. Let a, b, c be non-negative real numbers, no two of which are zero, let p, r be real numbers $(r > -2)$ and let

$$E(a,b,c) = \sum \frac{ab+pbc+ca}{b^2+rbc+c^2}$$

Then,

a) $E(a,b,c) \geq \dfrac{3(p+2)}{r+2}$, for $p \leq r-1$;

b) $E(a,b,c) \geq \dfrac{p}{r+2} + 2$, for $r-1 \leq p \leq (r+2)^2$;

c) $E(a,b,c) \geq 2\sqrt{p} - r$, for $p \geq (r+2)^2$.

(*Vasile Cîrtoaje*, MS, 2006)

7.2 Solutions

1. *Let a, b, c be non-negative real numbers, no two of which are zero. Then,*

$$\frac{a(b+c)}{b^2+bc+c^2} + \frac{b(c+a)}{c^2+ca+a^2} + \frac{c(a+b)}{a^2+ab+b} \geq 2$$

First Solution By the Cauchy-Schwarz Inequality we have

$$\sum \frac{a(b+c)}{b^2+bc+c^2} \geq \frac{(a+b+c)^2}{\sum \frac{a(b^2+bc+c^2)}{bc}}.$$

Thus it is enough to show that

$$(a+b+c)^2 \geq 2\sum \frac{a(b^2+bc+c^2)}{b+c}.$$

Since

$$\frac{a(b^2+bc+c^2)}{b+c} = a\left(b+c-\frac{bc}{b+c}\right) = ab+ca-\frac{abc}{b+c},$$

the inequality becomes

$$2abc\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \geq 2(ab+bc+ca)-a^2-b^2-c^2.$$

Taking into account that the AM-HM Inequality yields

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \geq \frac{9}{(b+c)+(c+a)+(a+b)},$$

it suffices to show that

$$\frac{9abc}{a+b+c} \geq 2(ab+bc+ca)-a^2-b^2-c^2.$$

This inequality is equivalent to the well-know Schur's Inequality of third degree

$$a^3+b^3+c^3+3abc \geq ab(a+b)+bc(b+c)+ca(c+a)$$

Equality occurs for the following four cases: $a=b=c$, $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$.

Second Solution By direct calculation, we may reduce the inequality to

$$\sum bc(b^4+c^4) \geq \sum b^2c^2(b^2+c^2),$$

which is equivalent to the evident inequality

$$\sum bc(b-c)(b^3-c^3) \geq 0.$$

Third solution (by *Darij Grinberg*) The hint is to multiply the both sides of the inequality by $a+b+c$ We have

$$(a+b+c)\left[\sum \frac{a(b+c)}{b^2+bc+c^2}-2\right]=\sum\left[\frac{a(b+c)(a+b+c)}{b^2+bc+c^2}-2a\right]=$$

$$=\sum \frac{a(ab+ac-b^2-c^2)}{b^2+bc+c^2}=\sum \frac{ab(a-b)-ca(c-a)}{b^2+bc+c^2}=$$

$$=\sum \frac{ab(a-b)}{b^2+bc+c^2}-\sum \frac{ab(a-b)}{c^2+ca+a^2}=$$

$$=\sum ab(a-b)\left(\frac{1}{b^2+bc+c^2}-\frac{1}{c^2+ca+a^2}\right)=$$

$$=(a+b+c)\sum \frac{ab(a-b)^2}{(b^2+bc+c^2)(c^2+ca+a^2)}\geq 0$$

From this solution, the following interesting identity follows.

$$\sum \frac{a(b+c)}{b^2+bc+c^2}=2+\sum \frac{bc(b-c)^2}{(a^2+ab+b^2)(a^2+ac+c^2)}.$$

★

2. *Let a,b,c be non-negative real numbers, no two of which are zero. Then,*

$$\frac{ab-bc+ca}{b^2+c^2}+\frac{bc-ca+ab}{c^2+a^2}+\frac{ca-ab+bc}{a^2+b^2}\geq \frac{3}{2}$$

Solution. We have

$$\sum\left(\frac{ab-bc+ca}{b^2+c^2}-\frac{1}{2}\right)=\sum \frac{(b+c)(2a-b-c)}{2(b^2+c^2)}=$$

$$=\sum \frac{(b+c)(a-b)}{2(b^2+c^2)}+\sum \frac{(b+c)(a-c)}{2(b^2+c^2)}=$$

$$=\sum \frac{(b+c)(a-b)}{2(b^2+c^2)}+\sum \frac{(c+a)(b-a)}{2(c^2+a^2)}=$$

$$=\sum \frac{(a-b)^2(ab+bc+ca-c^2)}{2(b^2+c^2)(c^2+a^2)}.$$

Thus, the inequality is equivalent to

$$(b-c)^2S_a+(c-a)^2S_b+(a-b)^2S_c\geq 0,$$

where

$$S_a= b^2+c^2)(ab+bc+ca-a^2).$$

Without loss of generality, assume that $a \geq b \geq c$. It is easy to check that $S_b \geq 0$ and $S_c > 0$. For nontrivial case $S_a < 0$, it suffices to show that

$$(b-c)^2 S_a + (c-a)^2 S_b \geq 0,$$

that is

$$(a^2+c^2)(ab+bc+ca-b^2)(a-c)^2 \geq (b^2+c^2)(a^2-ab-bc-ca)(b-c)^2.$$

This inequality follows by multiplying up the inequalities

$$a^2+c^2 \geq b^2+c^2,$$
$$a-c \geq b-c,$$
$$(ab+bc+ca-b^2)(a-c) \geq (a^2-ab-bc-ca)(b-c)$$

The last inequality reduces to

$$2a(a-c)+2b(b-c) \geq 0,$$

which is clearly true. Equality occurs for $a=b=c$, $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$

★

3. *Let a, b, c be non-negative real numbers, no two of which are zero. Then,*

$$\frac{ab-2bc+ca}{b^2-bc+c^2} + \frac{bc-2ca+ab}{c^2-ca+a^2} + \frac{ca-2ab+bc}{a^2-ab+b^2} \geq 0$$

Solution. For $a=0$, the inequality reduces to

$$\frac{-2bc}{b^2-bc+c^2} + \frac{b}{c} + \frac{c}{b} \geq 0,$$

which is equivalent to

$$(b-c)^2(b^2+bc+c^2) \geq 0.$$

For $a, b, c > 0$, the inequality follows immediately applying Lemma below to the function $f(x) = \dfrac{-1}{x}$. Equality occurs for $a=b=c$, $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$.

Lemma. *Let $f(x)$ be an increasing function on $(0,\infty)$. If a,b,c are positive real numbers, then*

$$\frac{2f(a)-f(b)-f(c)}{b^2-bc+c^2}+\frac{2f(b)-f(c)-f(a)}{c^2-ca+a^2}+\frac{2f(c)-f(a)-f(b)}{a^2-ab+b^2}\geq 0$$

In order to prove Lemma, assume that $a\geq b\geq c$, denote

$$X=f(a)-f(b),\quad Y=f(b)-f(c),$$
$$A=b^2-bc+c^2,\quad B=c^2-ca+a^2,\quad C=a^2-ab+b^2,$$

and write the inequality as

$$X\left(\frac{2}{A}-\frac{1}{B}-\frac{1}{C}\right)+Y\left(\frac{1}{A}+\frac{1}{B}-\frac{2}{C}\right)\geq 0.$$

Since $X\geq 0$ and $Y\geq 0$, it suffices to show that $\frac{2}{A}-\frac{1}{B}-\frac{1}{C}\geq 0$ and $\frac{1}{A}+\frac{1}{B}-\frac{2}{C}\geq 0$. Taking into account that $B-A=(a-b)(a+b-c)\geq 0$ and $C-A=(a-c)(a+c-b)\geq 0$, we get

$$\frac{2}{A}-\frac{1}{B}-\frac{1}{C}=\frac{B-A}{AB}+\frac{C-A}{AC}\geq 0.$$

On the other hand

$$\frac{1}{A}+\frac{1}{B}-\frac{2}{C}=\frac{B(C-A)-A(B-C)}{ABC}=$$
$$=\frac{B(a-c)(a+c-b)-A(b-c)(a-b-c)}{ABC}.$$

The inequality $\frac{1}{A}+\frac{1}{B}-\frac{2}{C}\geq 0$ is true since $B\geq A$, $a-c\geq b-c$ and $a+c-b>a-b-c$

★

4. *Let a,b,c be non-negative real numbers, no two of which are zero. Then,*

$$\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2}\geq\frac{9}{4(ab+bc+ca)}$$

First Solution. Assume that $a \le b \le c$ and denote $x = b + c$, $y = c + a$, $z = a + b$ Then, we have to show that

$$\left(2\sum yz - \sum x^2\right)\left(\sum \frac{1}{x^2}\right) \ge 9$$

for $x \ge y \ge z > 0$ and $x \le y + z$ We have

$$\left(2\sum yz - \sum x^2\right)\left(\sum \frac{1}{x^2}\right) - 9 =$$
$$= \left(\sum x^2\right)\left(\sum \frac{1}{x^2}\right) - 9 - 2\left(\sum x^2 - \sum yz\right)\left(\sum \frac{1}{x^2}\right) =$$
$$= \sum \left(\frac{y}{z} - \frac{z}{y}\right)^2 - \left[\sum (y-z)^2\right]\left(\sum \frac{1}{x^2}\right) = \sum (y-z)^2 \left(\frac{2}{yz} - \frac{1}{x^2}\right)$$

Therefore, we may write the inequality as

$$\sum (y-z)^2 S_x \ge 0,$$

where

$$S_x = \frac{2}{yz} - \frac{1}{x^2}$$

Since $S_x > 0$, $S_y = \dfrac{2}{xz} - \dfrac{1}{y^2} \ge \dfrac{2}{(y+z)z} - \dfrac{1}{y^2} = \dfrac{(y-z)(2y+z)}{(y+z)y^2 z} \ge 0$ and

$$y^2 S_y + z^2 S_z = 2\left(\frac{y^3 + z^3}{xyz} - 1\right) \ge 2\left(\frac{y+z}{x} - 1\right) \ge 0,$$

we get

$$\sum (y-z)^2 S_x \ge (x-z)^2 S_y + (x-y)^2 S_z =$$
$$= \left(\frac{x-z}{y}\right)^2 y^2 S_y + \left(\frac{x-y}{z}\right)^2 z^2 S_z \ge \left[\left(\frac{x-z}{y}\right)^2 - \left(\frac{x-y}{z}\right)^2\right] y^2 S_y =$$
$$= \frac{(y-z)(y+z-x)\left[z(x-z) + y(x-y)\right]}{z^2} S_y \ge 0$$

Equality occurs for $a = b = c$, $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$

Second Solution Since the inequality is homogeneous, we may assume that $ab + bc + ca = 1$ In this case, the inequality becomes

$$\frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{1}{(a+b)^2} \ge \frac{9}{4}.$$

We will show that the following sharper inequality holds for $ab+bc+ca=1$:

$$\sum \frac{1}{(b+c)^2} \geq \frac{9}{4} + \frac{1}{4}\sum \frac{bc(b-c)^2}{(b+c)^2}.$$

Since

$$\frac{1}{4}\sum \frac{bc(b-c)^2}{(b+c)^2} = \frac{1}{4}\frac{bc(b+c)^2 - 4b^2c^2}{(b+c)^2} = \frac{1}{4}\sum bc - \sum \frac{b^2c^2}{(b+c)^2} =$$
$$= \frac{1}{4} - \sum \frac{b^2c^2}{(b+c)^2},$$

we may write the inequality in the form

$$\frac{1+b^2c^2}{(b+c)^2} + \frac{1+c^2a^2}{(c+a)^2} + \frac{1+a^2b^2}{(a+b)^2} \geq \frac{5}{2}. \tag{1}$$

This inequality was given at Mathlinks Contest in 2005. Taking into account that

$$\frac{1+b^2c^2}{(b+c)^2} = \left(\frac{ab+bc+ca}{b+c}\right)^2 + \frac{b^2c^2}{(b+c)^2} =$$
$$= \left(a + \frac{bc}{b+c}\right)^2 + \frac{b^2c^2}{(b+c)^2} = a^2 + \frac{2abc}{b+c} + \frac{2b^2c^2}{(b+c)^2},$$

we may write (1) in the homogeneous form

$$\sum a^2 - \sum bc + a\sum\left(\frac{2bc}{b+c} - \frac{b+c}{2}\right) + \left[\frac{2b^2c^2}{(b+c)^2} - \frac{bc}{2}\right] \geq 0,$$

which is equivalent to

$$\sum (b-c)^2 \left[\frac{1}{2} - \frac{a}{2(b+c)} - \frac{bc}{2(b+c)^2}\right] \geq 0,$$

or

$$(b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c \geq 0,$$

where

$$S_a = \frac{b^2+bc+c^2-ab-ac}{(b+c)^2}.$$

Without loss of generality, assume that $a \geq b \geq c$ We see that

$$S_b = \frac{c^2+ca+a^2-bc-ba}{(c+a)^2} = \frac{(a-b)(a+c)+c^2}{(c+a)^2} \geq 0$$

and

$$S_c = \frac{a^2+ab+b^2-ac-bc}{(a+b)^2} \geq \frac{a^2+ab+b^2-b(a+b)}{(a+b)^2} > 0.$$

For nontrivial case $S_a < 0$, it suffices to show that

$$(b-c)^2 S_a + (c-a)^2 S_b \geq 0,$$

that is

$$\frac{(c^2+ca+a^2-bc-ba)(a-c)^2}{(a+c)^2} \geq \frac{(ab+ac-b^2-bc-c^2)(b-c)^2}{(b+c)^2}.$$

This inequality follows by multiplying the inequalities

$$c^2+ca+a^2-bc-ba \geq ab+ac-b^2-bc-c^2$$

and

$$\frac{(a-c)^2}{(a+c)^2} \geq \frac{(b-c)^2}{(b+c)^2}.$$

The first inequality is equivalent to $(a-b)^2+2c^2 \geq 0$, while the second inequality is equivalent to $\frac{a-c}{a+c} \geq \frac{b-c}{b+c}$, that is $c(a-b) \geq 0$.

Equality in the original inequality and also in (1) occurs for $a=b=c$, as well as for $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$.

Remark. Michael Rozenberg noticed that

$$\begin{aligned}
\sum (b-c)^2 S_a &= \sum (b-c)\frac{b^3-c^3-a(b^2-c^2)}{(b+c)^2} = \\
&= \sum (b-c)\frac{b^2(b-a)+c^2(a-c)}{(b+c)^2} = \\
&= \sum (b-c)(b-a)\frac{b^2}{(b+c)^2} + \sum (b-c)(a-c)\frac{c^2}{(b+c)^2} = \\
&= \sum (a-b)(a-c)\frac{a^2}{(a+b)^2} + \sum (c-a)(b-a)\frac{a^2}{(c+a)^2} = \\
&= \sum (a-b)(a-c)S_a,
\end{aligned}$$

where

$$S_a = \left(\frac{a}{a+b}\right)^2 + \left(\frac{a}{a+c}\right)^2.$$

Assume that $a \geq b \geq c$. Since $(c-a)(c-b) \geq 0$, it suffices to show that

$$(a-b)(a-c)S_a + (b-c)(b-a)S_b \geq 0$$

But since $a-b \ge 0$ and $a-c \ge b-c \ge 0$, it suffices to prove that $S_a - S_b \ge 0$. We have

$$S_a - S_b = \frac{a^2-b^2}{(a+b)^2} + \left(\frac{a}{a+c}\right)^2 - \left(\frac{b}{b+c}\right)^2 =$$
$$= \frac{a-b}{a+b} + \frac{c(a-b)}{(a+c)(b+c)}\left(\frac{a}{a+c} + \frac{b}{b+c}\right) \ge 0.$$

Third Solution (after an idea of *Marian Tetiva*) Let

$$E(a,b,c) = \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{1}{(a+b)^2}.$$

Because of the symmetry, we may assume without loss of generality that $a = \min\{a,b,c\}$ Under this assumption, we will prove the desired inequality by using the following chain of inequalities

$$E(a,b,c) \ge E(a,t,t) \ge \frac{9}{4t(2a+t)},$$

where $t = \sqrt{(a+b)(a+c)} - a$. It is easy to check that $t(2a+t) = ab+bc+ca$ This relation emphasizes the trick of the solution, to intercalate between the two sides of the inequality a new expression for $E(a,b,c)$ obtained by equating two of the three variables (b and c) such that the expression $ab+bc+ca$ holds unchanged

In order to prove the inequality $E(a,b,c) \ge E(a,t,t)$, we write it in the form

$$\frac{(a+b)^2+(a+c)^2}{(a+b)^2(a+c)^2} - \frac{2}{(a+t)^2} \ge \frac{1}{4t^2} - \frac{1}{(b+c)^2}.$$

Taking into account that $a+t = \sqrt{(a+b)(a+c)}$ and

$$b+c-2t = 2a+b+c-2\sqrt{(a+b)(a+c)} =$$
$$= \left(\sqrt{a+b} - \sqrt{a+c}\right)^2 = \frac{(b-c)^2}{\left(\sqrt{a+b}+\sqrt{a+c}\right)^2},$$

the inequality is equivalent to

$$\frac{(b-c)^2}{(a+b)^2(a+c)^2} \ge \frac{(b-c)^2(b+c+2t)}{4t^2(b+c)^2\left(\sqrt{a+b}+\sqrt{a+c}\right)^2}.$$

Since $(b-c)^2 \ge 0$, it is enough to show that

$$4t^2(b+c)^2\left(\sqrt{a+b}+\sqrt{a+c}\right)^2 \ge (a+b)^2(a+c)^2(b+c+2t)$$

This inequality follows by multiplying the inequalities

$$\left(\sqrt{a+b}+\sqrt{a+c}\right)^2 \ge b+c+2t$$

and

$$4t^2(b+c)^2 \ge (a+b)^2(a+c)^2.$$

The first inequality is true because

$$\left(\sqrt{a+b}+\sqrt{a+c}\right)^2 = 2a+b+c+2\sqrt{(a+b)(a+c)} =$$
$$= 2a+b+c+2a+2t \ge b+c+2t$$

With regard to the second inequality, since $t \ge \sqrt{bc}$ (easy to check) and $a \le \sqrt{bc}$ (from $a = \min\{a,b,c\}$), we have

$$2t(b+c)-(a+b)(a+c) \ge 2\sqrt{bc}(b+c)-\left(\sqrt{bc}+b\right)\left(\sqrt{bc}+c\right) =$$
$$= \sqrt{bc}\left(\sqrt{b}-\sqrt{c}\right)^2 \ge 0.$$

Finally, the inequality $E(a,t,t) \ge \dfrac{9}{4t(2a+t)}$ is equivalent to

$$\frac{1}{4t^2}+\frac{2}{(a+t)^2} \ge \frac{9}{4t(2a+t)}.$$

We have

$$\frac{1}{4t^2}+\frac{2}{(a+t)^2}-\frac{9}{4t(2a+t)} = \frac{a(a-t)^2}{2t^2(2a+t)(a+t)^2} \ge 0.$$

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5. *Let a,b,c be non-negative real numbers, no two of which are zero. If $r > -2$, then*

$$\sum \frac{ab+(r-1)bc+ca}{b^2+rbc+c^2} \ge \frac{3(r+1)}{r+2}.$$

Solution. In order to prove this inequality we will apply the expanding way and will use then the following strong inequalities:

$$\sum a^3 + 3abc \ge \sum bc(b+c),$$
$$(a-b)^2(b-c)^2(c-a)^2 \ge 0,$$
$$\sum bc(b-c)^4 \ge 0.$$

By expanding, we may write the inequality as $E \geq 0$, where

$$E = (r+2)\left[A + (r-1)B\right] - 3(r+1)C$$

and

$$\begin{aligned}
A &= \sum a(b+c)(a^2+rab+b^2)(c^2+rca+a^2) = \\
&= (r^2+r+2)abc\sum bc(b+c) + r\sum b^2c^2(b^2+c^2) + \\
&+ 2rabc\sum a^3 + 6ra^2b^2c^2 + \sum bc(b^4+c^4) + 2\sum b^3c^3, \\
B &= \sum bc(a^2+rab+b^2)(c^2+rca+a^2) = \\
&= 3r^2a^2b^2c^2 + (2r+1)abc\sum bc(b+c) + \sum b^3c^3 + abc\sum a^3, \\
C &= (a^2+rab+b^2)(b^2+rbc+c^2)(c^2+rca+a^2) = \\
&= (r^3+2)a^2b^2c^2 + r(r+1)abc\sum bc(b+c) + \\
&+ r\sum b^3c^3 + rabc\sum a^3 + \sum b^2c^2(b^2+c^2).
\end{aligned}$$

After some manipulations, we get

$$E = (r+2)X + (r^2-1)Y + 2(r-1)abcZ,$$

where

$$\begin{aligned}
X &= \sum bc(b^4+c^4) - \sum b^2c^2(b^2+c^2) = \sum bc(b^2+bc+c^2)(b-c)^2 \geq 0, \\
Y &= \sum b^2c^2(b^2+c^2) - 2\sum b^3c^3 = \sum b^2c^2(b-c)^2 \geq 0, \\
Z &= \sum a^3 - \sum bc(b+c) + 3abc \geq 0
\end{aligned}$$

The inequality $Z \geq 0$ is well-known Schur's Inequality. We have two cases to consider.

Case $r \geq 1$ Since $X, Y, Z \geq 0$, it is clear that

$$E = (r+2)X + (r^2-1)Y + 2(r-1)abcZ \geq 0$$

Case $-2 < r < 1$ Setting $r = -2$ in

$$\begin{aligned}
E &= (r+2)X + (r^2-1)Y + 2(r-1)abcZ = \\
&= (r+2)[A + (r-1)B] - 3(r+1)C,
\end{aligned}$$

we get

$$Y - 2abcZ = C = (a-b)^2(b-c)^2(c-a)^2 \geq 0.$$

Thus, it follows that $2(r-1)abcZ \geq (r-1)Y$, and hence

$$E \geq (r+2)X + (r^2-1)Y + (r-1)Y = (r+2)[X+(r-1)Y] \geq$$
$$\geq (r+2)(X-3Y) = (r+2)\sum bc(b-c)^4 \geq 0$$

Equality in the given inequality occurs in the following four cases $a=b=c$, $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$.

Remark 1. Actually, we found for E the following non-negative forms

$$E = (r+2)\sum bc(b^2+bc+c^2)(b-c)^2 + (r^2-1)\sum b^2c^2(b-c)^2 +$$
$$+ 2(r-1)abc\sum a(a-b)(a-c)$$

for $r \geq 1$, and

$$E = (1-r)\prod(b-c)^2 + (r+2)\sum bc(b-c)^4 + (r+2)^2\sum b^2c^2(b-c)^2$$

for $-2 < r \leq 1$

Remark 2. In the particular cases $r=1$, $r=0$, $r=-1$ and $r=2$ we obtain the inequalities from the previous applications 1, 2, 3 and 4, respectively.

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6. *Let a,b,c be non-negative real numbers, no two of which are zero. Then,*

$$\sum \frac{ab+4bc+ca}{b^2+c^2} \geq 4.$$

Solution. First notice us that equality occurs when one of a,b,c is zero and the others are equal. Let $a \leq b \leq c$ and

$$E(a,b,c) = \sum \frac{ab+4bc+ca}{b^2+c^2}.$$

We will show that

$$E(a,b,c) \geq E(0,b,c) \geq 4.$$

For $a=0$ we have $E(a,b,c)-E(0,b,c)=0$, and for $a>0$ we get

$$\frac{E(a,b,c)-E(0,b,c)}{a} = \frac{b+c}{b^2+c^2} + \frac{4c^2+b(c-a)}{c(c^2+a^2)} + \frac{4b^2+c(b-a)}{b(a^2+b^2)} > 0$$

Letting now $x = \frac{b}{c} + \frac{c}{b}$, we find

$$E(0,b,c) - 4 = \frac{4bc}{b^2+c^2} + \frac{b}{c} + \frac{c}{b} - 4 = \frac{4}{x} + x - 4 = \frac{(x-2)^2}{x} \geq 0$$

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7. *Let a, b, c be non-negative real numbers, no two of which are zero. If $r > -2$, then*

$$\sum \frac{ab + (r+2)^2 bc + ca}{b^2 + rbc + c^2} \geq r + 4.$$

Solution. Let $a \leq b \leq c$ and

$$E(a,b,c) = \sum \frac{ab + (r+2)^2 bc + ca}{b^2 + rbc + c^2}.$$

In order to prove the desired inequality we consider two cases

I. *Case* $(r+2)^2 b^2 \geq (r-1)bc + ca$.

We will show that

$$E(a,b,c) \geq E(0,b,c) \geq r + 4.$$

For $a = 0$ we have $E(a,b,c) - E(0,b,c) = 0$, and for $a > 0$ we get

$$\frac{E(a,b,c) - E(0,b,c)}{a} = \frac{b+c}{b^2 + rbc + c^2} + \frac{(r+2)^2 c^2 - (r-1)bc - ab}{c(c^2 + rca + a^2)} +$$
$$+ \frac{(r+2)^2 b^2 - (r-1)bc - ca}{b(a^2 + rab + b^2)} >$$
$$> \frac{(r+2)^2 c^2 - (r-1)bc - ab}{c(c^2 + rca + a^2)} \geq$$
$$\geq \frac{(r+2)^2 bc - (r-1)bc - bc}{c(c^2 + rca + a^2)} = \frac{(r^2 + 3r + 4)bc}{c(c^2 + rca + a^2)} > 0.$$

Letting now $x = \frac{b}{c} + \frac{c}{b}$, we find

$$E(0,b,c) - r - 4 = \frac{(r+2)^2 bc}{b^2 + rbc + c^2} + \frac{b}{c} + \frac{c}{b} - r - 4 =$$
$$= \frac{(r+2)^2}{x + r} + x - r - 4 = \frac{(x-2)^2}{x + r} \geq 0.$$

II *Case* $(r-1)bc + ca > (r+2)^2 b^2$

This condition yields $(r-1)b + a > 0$, $(r-1)b + b > 0$, and hence $r > 0$. Towards proving the desired inequality, it suffices to show that

$$\frac{ca + (r+2)^2 ab + bc}{a^2 + rab + b^2} \geq r + 4.$$

Indeed, using the condition

$$c > \frac{(r+2)^2b^2}{(r-1)b+a}$$

yields

$$\frac{c(a+b)+(r+2)^2ab}{a^2+rab+b^2} - r - 4 > (r+2)^2 b\frac{\dfrac{b(a+b)}{(r-1)b+a}+a}{a^2+rab+b^2} - r - 4 =$$
$$= (r+2)^2\frac{b}{(r-1)b+a} - r - 4 \geq \frac{(r+2)^2}{r} - r - 4 = \frac{4}{r} > 0,$$

and the inequality is proved

Equality occurs in the original inequality when one of a, b, c is 0 and the others are equal

Remark For $r = -1$, we get the inequality from the application 6 Moreover, the inequality is also valid for $r = -2$; that is

$$\frac{a(b+c)}{(b-c)^2} + \frac{b(c+a)}{(c-a)^2} + \frac{c(a+b)}{(a-b)^2} > 2$$

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8. *Let a, b, c be non-negative real numbers, no two of which are zero, let p, r be real numbers ($r > -2$) and let*

$$E(a,b,c) = \sum \frac{ab+pbc+ca}{b^2+rbc+c^2}.$$

Then,

a) $E(a,b,c) \geq \dfrac{3(p+2)}{r+2}$, *for* $p \leq r-1$;

b) $E(a,b,c) \geq \dfrac{p}{r+2} + 2$, *for* $r - 1 \leq p \leq (r+2)^2$;

c) $E(a,b,c) \geq 2\sqrt{p} - r$, *for* $p \geq (r+2)^2$.

Solution. a) For fixed a, b, c and r, consider the linear function

$$f_1(p) = \sum \frac{ab+pbc+ca}{b^2+rbc+c^2} - \frac{3(p+2)}{r+2}$$

Since

$$\sum \frac{bc}{b^2+rbc+c^2} - \frac{3}{r+2} = \sum \frac{-(b-c)^2}{(r+2)(b^2+rbc+c^2)} \leq 0,$$

the function $f_1(p)$ is decreasing Therefore, it suffices to prove that $f_1(r-1) \geq 0$. Taking into account application 5 from this section, the conclusion follows Equality occurs for $a = b = c$. In the case $p = r - 1$, equality again for $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$

b) For fixed a, b, c and r, consider the linear function

$$f_2(p) = \sum \frac{ab + pbc + ca}{b^2 + rbc + c^2} - \frac{p}{r+2} - 2.$$

Since $r - 1 \leq p \leq (r+2)^2$, it suffices to prove that $f_2(r-1) \geq 0$ and $f_2((r+2)^2) \geq 0$ Taking into account applications 5 and 7, the conclusion follows For $r - 1 < p \leq (r+2)^2$, equality occurs if and only if $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$

c) The condition $p \geq (r+2)^2$ involves $p > 0$ Let $a \leq b \leq c$ and

$$E(a, b, c) = \sum \frac{ab + pbc + ca}{b^2 + rbc + c^2}.$$

We have two cases to consider.

I *Case* $pb^2 \geq (r-1)bc + ca$

We will show that

$$E(a, b, c) \geq E(0, b, c) \geq 2\sqrt{p} - r$$

For $a = 0$ we have $E(a, b, c) - E(0, b, c) = 0$, and for $a > 0$ we get

$$\begin{aligned} \frac{E(a,b,c) - E(0,b,c)}{a} &= \frac{b+c}{b^2 + rbc + c^2} + \frac{pc^2 - (r-1)bc - ab}{c(c^2 + rca + a^2)} + \\ &+ \frac{pb^2 - (r-1)bc - ca}{b(a^2 + rab + b^2)} > \frac{pc^2 - (r-1)bc - ab}{c(c^2 + rca + a^2)} \geq \\ &\geq \frac{(r+2)^2 bc - (r-1)bc - bc}{c(c^2 + rca + a^2)} = \frac{(r^2 + 3r + 4)bc}{c(c^2 + rca + a^2)} > 0 \end{aligned}$$

Letting now $x = \frac{b}{c} + \frac{c}{b}$, we get

$$\begin{aligned} E(0, b, c) - 2\sqrt{p} + r &= \frac{pbc}{b^2 + rbc + c^2} + \frac{b}{c} + \frac{c}{b} - 2\sqrt{p} + r = \\ &= \frac{p}{x + r} + x - 2\sqrt{p} + r = \frac{\left(x + r - \sqrt{p}\right)^2}{x + r} \geq 0 \end{aligned}$$

II *Case* $(r-1)bc+ca>pb^2$.
Since $p>0$, this condition yields $(r-1)b+a>0$, $(r-1)b+b>0$, hence $r>0$ In order to prove the desired inequality, it suffices to show that

$$\frac{ca+pab+bc}{a^2+rab+b^2}\geq 2\sqrt{p}-r.$$

Indeed,

$$\frac{c(a+b)+pab}{a^2+rab+b^2}-2\sqrt{p}+r>pb\frac{\frac{b(a+b)}{(r-1)b+a}+a}{a^2+rab+b^2}-2\sqrt{p}+r=$$

$$=\frac{pb}{(r-1)b+a}-2\sqrt{p}+r\geq\frac{p}{r}-2\sqrt{p}+r=\frac{1}{r}(r-\sqrt{p})^2\geq 0,$$

and the inequality is proved.

For $a\leq b\leq c$, equality occurs if and only if $a=0$ and $\frac{b}{c}+\frac{c}{b}=\sqrt{p}-r$

Remark 1. This application generalizes the preceding applications 1 - 7 Moreover, the inequality c) is also valid for $r=-2$ and $p\geq 0$, that is,

$$\sum\frac{ab+pbc+ca}{(b-c)^2}\geq 2(\sqrt{p}+1).$$

On the assumption $a=\min\{a,b,c\}$ and $p>0$, equality occurs if and only if $a=0$ and $\frac{b}{c}+\frac{c}{b}=\sqrt{p}+2$

Remark 2. For $p=1$, we get the following inequalities

$$\sum\frac{1}{b^2+rbc+c^2}\geq\frac{9}{(r+2)(ab+bc+ca)},\quad \text{for}\quad r\geq 2,$$

$$\sum\frac{1}{b^2+rbc+c^2}\geq\frac{2r+5}{(r+2)(ab+bc+ca)},\quad \text{for}\quad -1\leq r\leq 2;$$

$$\sum\frac{1}{b^2+rbc+c^2}\geq\frac{2-r}{ab+bc+ca},\quad \text{for}\quad -2\leq r\leq -1.$$

Remark 3. For $p+r=2$, we get the inequalities

$$\sum\frac{b+c}{b^2+rbc+c^2}\geq\frac{18}{(r+2)(a+b+c)},\quad \text{for}\quad r\geq\frac{2}{3},$$

$$\sum\frac{b+c}{b^2+rbc+c^2}\geq\frac{2(r+6)}{(r+2)(a+b+c)},\quad \text{for}\quad \frac{\sqrt{17}-5}{2}\leq r\leq\frac{2}{3},$$

$$\sum\frac{b+c}{b^2+rbc+c^2}\geq\frac{3-r+2\sqrt{2-r}}{a+b+c},\quad \text{for}\quad -2\leq r\leq\frac{\sqrt{17}-5}{2}.$$

7.3 Inequalities involving E_2

1. Let a, b, c be non-negative real numbers, no two of which are zero Then,

$$\frac{2a^2+bc}{b^2+c^2}+\frac{2b^2+ca}{c^2+a^2}+\frac{2c^2+ab}{a^2+b^2}\geq\frac{9}{2}$$

2. Let a, b, c be non-negative real numbers, no two of which are zero. Then

$$\frac{a^2+bc}{b^2+bc+c^2}+\frac{b^2+ca}{c^2+ca+a^2}+\frac{c^2+ab}{a^2+ab+b^2}\geq 2.$$

(*Vasile Cîrtoaje*, MS, 2005)

3. Let a, b, c be non-negative real numbers, no two of which are zero. Then,

a) $$\frac{a^2+2bc}{b+c}+\frac{b^2+2ca}{c+a}+\frac{c^2+2ab}{a+b}\geq\frac{3}{2}(a+b+c);$$

b) $$\frac{a^2+2bc}{(b+c)^2}+\frac{b^2+2ca}{(c+a)^2}+\frac{c^2+2ab}{(a+b)^2}\geq\frac{9}{4};$$

c) $$\frac{2a^2+5bc}{(b+c)^2}+\frac{2b^2+5ca}{(c+a)^2}+\frac{2c^2+5ab}{(a+b)^2}\geq\frac{21}{4}.$$

(*Vasile Cîrtoaje*, MS, 2005)

4. Let a, b, c be non-negative real numbers, no two of which are zero. Then,

$$\frac{a^2-bc}{2b^2-3bc+2c^2}+\frac{b^2-ca}{2c^2-3ca+2a^2}+\frac{c^2-ab}{2a^2-3ab+2b^2}\geq 0$$

(*Vasile Cîrtoaje*, MS, 2005)

5. Let a, b, c be non-negative real numbers, no two of which are zero Then,

$$\frac{a^2}{2b^2-bc+2c^2}+\frac{b^2}{2c^2-ca+2a^2}+\frac{c^2}{2a^2-ab+2b^2}\geq 1.$$

(*Vasile Cîrtoaje*, MS, 2005)

6. Let a, b, c be non-negative real numbers, no two of which are zero Then,

$$\frac{2a^2-bc}{b^2-bc+c^2}+\frac{2b^2-ca}{c^2-ca+a^2}+\frac{2c^2-ab}{a^2-ab+b^2}\geq 3$$

(*Vasile Cîrtoaje*, MS, 2005)

7. Let a, b, c be non-negative real numbers, no two of which are zero. If $r > -2$, then

$$\sum \frac{2a^2 + (2r+1)bc}{b^2 + rbc + c^2} \geq \frac{3(2r+3)}{r+2}$$

(*Vasile Cîrtoaje*, MS, 2005)

8. Let a, b, c be non-negative real numbers, no two of which are zero. Then

$$\frac{a^2 + 16bc}{b^2 + c^2} + \frac{b^2 + 16ca}{c^2 + a^2} + \frac{c^2 + 16ab}{a^2 + b^2} \geq 10.$$

(*Vasile Cîrtoaje*, MS, 2005)

9. Let a, b, c be non-negative real numbers, no two of which are zero. If $r > -2$, then

$$\sum \frac{a^2 + 4(r+2)^2 bc}{b^2 + rbc + c^2} \geq 4r + 10.$$

(*Vasile Cîrtoaje*, MS, 2005)

10. Let a, b, c be non-negative real numbers, no two of which are zero, let q, r be real numbers $(r > -2)$ and let

$$E(a, b, c) = \sum \frac{a^2 + qbc}{b^2 + rbc + c^2}.$$

Then,

a) $E(a, b, c) \geq \dfrac{3(q+1)}{r+2}$, for $q \leq \dfrac{2r+1}{2}$;

b) $E(a, b, c) \geq \dfrac{q}{r+2} + 2$, for $\dfrac{2r+1}{2} \leq q \leq 4(r+2)^2$,

c) $E(a, b, c) \geq 4kr + 12k^2 - 2$, for $q = 4k(r+2k)^2$, $k \geq 1$

(*Vasile Cîrtoaje*, MS, 2005)

7.4 Solutions

1. *Let a, b, c be non-negative real numbers, no two of which are zero. Then,*

$$\frac{2a^2 + bc}{b^2 + c^2} + \frac{2b^2 + ca}{c^2 + a^2} + \frac{2c^2 + ab}{a^2 + b^2} \geq \frac{9}{2}.$$

First Solution. Since

$$\frac{2(2a^2+bc)}{b^2+c^2}-3=\frac{2(2a^2-b^2-c^2)}{b^2+c^2}-\frac{(b-c)^2}{b^2+c^2},$$

we may write the inequality as

$$2\sum\frac{2a^2-b^2-c^2}{b^2+c^2}\geq\sum\frac{(b-c)^2}{b^2+c^2}.$$

But

$$\sum\frac{2a^2-b^2-c^2}{b^2+c^2}=\sum\frac{a^2-b^2}{b^2+c^2}+\sum\frac{a^2-c^2}{b^2+c^2}=\sum\frac{a^2-b^2}{b^2+c^2}+\sum\frac{b^2-a^2}{c^2+a^2}=$$
$$=\sum(a^2-b^2)\left(\frac{1}{b^2+c^2}-\frac{1}{c^2+a^2}\right)=$$
$$=\sum\frac{(a^2-b^2)^2}{(b^2+c^2)(c^2+a^2)}.$$

Consequently, the inequality is equivalent to

$$2\sum\frac{(b^2-c^2)^2}{(c^2+a^2)(a^2+b^2)}\geq\sum\frac{(b-c)^2}{b^2+c^2}.$$

Since $(b^2-c^2)^2\geq(b-c)^2(b^2+c^2)$, it is enough to show that

$$(b-c)^2S_a+(c-a)^2S_b+(a-b)^2S_c\geq 0,$$

where

$$S_a=2(b^2+c^2)^2-(c^2+a^2)(a^2+b^2).$$

Without loss of generality, we may assume that $a\geq b\geq c$. We have

$$S_b=2(c^2+a^2)-(a^2+b^2)(b^2+c^2)\geq$$
$$\geq 2(c^2+a^2)(c^2+b^2)-(a^2+b^2)(b^2+c^2)=$$
$$=(b^2+c^2)(a^2-b^2+2c^2)\geq 0,$$
$$S_c=2(a^2+b^2)^2-(b^2+c^2)(c^2+a^2)>0$$

and

$$S_a+S_b=(a^2-b^2)^2+2c^2(a^2+b^2+2c^2)\geq 0$$

Therefore,

$$(b-c)^2S_a+(c-a)^2S_b+(a-b)^2S_c\geq$$
$$\geq(b-c)^2S_a+(a-c)^2S_b\geq(b-c)^2(S_a+S_b)\geq 0$$

Equality occurs when $a = b = c$, and also when one of a, b, c is 0 and the others are equal.

Second Solution (by *Darij Grinberg*). Since $bc \geq \dfrac{2b^2c^2}{b^2+c^2}$ and

$$\frac{2a^2+bc}{b^2+c^2} \geq \frac{2(a^2b^2+b^2c^2+c^2a^2)}{(b^2+c^2)^2},$$

we have

$$\sum \frac{2a^2+bc}{b^2+c^2} \geq 2(a^2b^2+b^2c^2+c^2a^2)\sum \frac{1}{(b^2+c^2)^2}.$$

Therefore, is it enough to show that

$$\sum \frac{1}{(b^2+c^2)^2} \geq \frac{9}{4(a^2b^2+b^2c^2+c^2a^2)}.$$

This inequality is just Iran Inequality (see application 7 1.4)

★

2. *Let a, b, c be non-negative real numbers, no two of which are zero. Then*

$$\frac{a^2+bc}{b^2+bc+c^2} + \frac{b^2+ca}{c^2+ca+a^2} + \frac{c^2+ab}{a^2+ab+b^2} \geq 2$$

Solution. By Cauchy-Schwarz Inequality, we have

$$\sum \frac{a^2}{b^2+bc+c^2} \geq \frac{\left(\sum a^2\right)^2}{\sum a^2(b^2+bc+c^2)} = 1 + \frac{\sum a^4 - abc\sum a}{2\sum b^2c^2 + abc\sum a}$$

and

$$\sum \frac{bc}{b^2+bc+c^2} \geq \frac{\left(\sum bc\right)^2}{\sum bc(b^2+bc+c^2)} = 1 - \frac{\sum bc(b^2+c^2) - 2abc\sum a}{\sum b^2c^2 + \sum bc(b^2+c^2)}$$

Thus, it suffices to show that

$$\frac{X}{A} \geq \frac{Y}{B},$$

where

$$X = \sum a^4 - abc\sum a, \qquad Y = \sum bc(b^2+c^2) - 2abc\sum a,$$
$$A = 2\sum b^2c^2 + abc\sum a, \qquad B = \sum b^2c^2 + \sum bc(b^2+c^2).$$

Since

$$X \geq \sum b^2c^2 - abc\sum a = \frac{1}{2}\sum a^2(b-c)^2 \geq 0,$$

it is enough to show that $B \geq A$ and $X \geq Y$ We have

$$\begin{aligned} B - A &= \sum bc(b^2+c^2) - \sum b^2c^2 - abc\sum a = \\ &= \sum bc(b-c)^2 + \sum b^2c^2 - abc\sum a = \\ &= \sum bc(b-c)^2 + \frac{1}{2}\sum a^2(b-c)^2 \geq 0 \end{aligned}$$

and

$$X - Y = \sum a^4 + abc\sum a - \sum bc(b^2+c^2) \geq 0.$$

The last inequality $X \geq Y$ is just Schur's Inequality of fourth degree. This completes the proof. Equality holds if and only if $a = b = c$.

Remark. Actually, the following sharper inequality holds:

$$\frac{2a^2+3bc}{b^2+bc+c^2} + \frac{2b^2+3ca}{c^2+ca+a^2} + \frac{2c^2+3ab}{a^2+ab+b^2} \geq 5.$$

We will prove it at application 7 2.7.

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3. *Let a, b, c be non-negative real numbers, no two of which are zero. Then,*

$$a) \qquad \frac{a^2+2bc}{b+c} + \frac{b^2+2ca}{c+a} + \frac{c^2+2ab}{a+b} \geq \frac{3}{2}(a+b+c);$$

$$b) \qquad \frac{a^2+2bc}{(b+c)^2} + \frac{b^2+2ca}{(c+a)^2} + \frac{c^2+2ab}{(a+b)^2} \geq \frac{9}{4},$$

$$c) \qquad \frac{2a^2+5bc}{(b+c)^2} + \frac{2b^2+5ca}{(c+a)^2} + \frac{2c^2+5ab}{(a+b)^2} \geq \frac{21}{4}.$$

Solution. a) Since

$$\frac{a^2+2bc}{b+c} - \frac{3(b+c)}{4} = \frac{2(2a^2-b^2-c^2)-(b-c)^2}{4(b+c)},$$

the inequality is equivalent to

$$2\sum \frac{2a^2-b^2-c^2}{b+c} \geq \sum \frac{(b-c)^2}{b+c}.$$

Taking into account that

$$\sum \frac{2a^2-b^2-c^2}{b+c} = \sum \frac{a^2-b^2}{b+c} + \sum \frac{a^2-c^2}{b+c} = \sum \frac{b^2-c^2}{c+a} + \sum \frac{c^2-b^2}{a+b} =$$
$$= \sum (b^2-c^2)\left(\frac{1}{c+a} - \frac{1}{a+b}\right) = \sum \frac{(b-c)^2(b+c)}{(c+a)(a+b)},$$

the inequality transforms into one of type

$$(b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c \geq 0,$$

where

$$S_a = 2(b+c)^2 - (c+a)(a+b).$$

Without loss of generality, we may assume that $a \geq b \geq c$. We have

$$S_b = 2(c+a)^2 - (a+b)(b+c) \geq 2(c+a)(c+b) - (a+b)(b+c) =$$
$$= (b+c)(a-b+2c) \geq 0,$$
$$S_c = 2(a+b)^2 - (b+c)(c+a) > 0$$

and

$$S_a + S_b = (a-b)^2 + 2c(a+b+2c) \geq 0.$$

Therefore,

$$(b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c \geq$$
$$\geq (b-c)^2 S_a + (a-c)^2 S_b \geq (b-c)^2 (S_a + S_b) \geq 0.$$

Equality occurs when $a = b = c$, and also when one of a, b, c is 0 and the others are equal.

b) Applying Cauchy-Schwarz Inequality and then the inequality a), we have

$$\sum \frac{a^2+2bc}{(b+c)^2} \geq \frac{\left(\sum \frac{a^2+2bc}{b+c}\right)^2}{\sum (a^2+2bc)} \geq \frac{\frac{9}{4}(a+b+c)^2}{(a+b+c)^2} = \frac{9}{4}.$$

Equality occurs if and only if $a = b = c$.

c) Write the inequality as follows

$$\sum \left[\frac{2a^2+5bc}{(b+c)^2} - \frac{7}{4} \right] \ge 0,$$

$$\sum \frac{4(a^2-b^2)+4(a^2-c^2)-3(b-c)^2}{(b+c)^2} \ge 0,$$

$$4\sum \frac{a^2-b^2}{(b+c)^2} + 4\sum \frac{b^2-a^2}{(c+a)^2} - 3\sum \frac{(a-b)^2}{(a+b)^2} \ge 0,$$

$$4\sum \frac{(a-b)^2(a+b)(a+b+2c)}{(b+c)^2(c+a)^2} - 3\sum \frac{(a-b)^2}{(a+b)^2} \ge 0$$

Setting $b+c=x$, $c+a=y$ and $a+b=z$, we may write the inequality in the form

$$(y-z)^2 S_x + (z-x)^2 S_y + (x-y)^2 S_z \ge 0,$$

where

$$S_x = 4x^3(y+z) - 3y^2z^2,\ S_y = 4y^3(z+x) - 3z^2x^2,\ S_z = 4z^3(x+y) - 3x^2y^2.$$

Without loss of generality, assume that $0 < x \le y \le z$. Taking into account that

$$x+y-z = 2c \ge 0,$$

we have

$$S_z > 3y(z^3 - x^2y) \ge 0$$

and

$$S_y \ge 4x^2y(z+x) - 3x^2z(x+y) = x^2[4xy + z(y-3x)].$$

If $y-3x \ge 0$ then $S_y > 0$, and if $y-3x<0$ then

$$S_y \ge x^2[4xy + (x+y)(y-3x)] = x^2(3x+y)(y-x) \ge 0.$$

Since $S_y \ge 0$ and $S_z > 0$, it suffices to show that $S_x + S_y \ge 0$ We have

$$\begin{aligned} S_x + S_y &= 4xy(x^2+y^2) + 4(x^3+y^3)z - 3(x^2+y^2)z^2 \ge \\ &\ge 4xy(x^2+y^2) + 4(x^3+y^3)z - 3(x^2+y^2)(x+y)z = \\ &= 4xy(x^2+y^2) + (x^2-4xy+y^2)(x+y)z \end{aligned}$$

If $x^2-4xy+y^2 \ge 0$ then $S_x+S_y>0$, and if $x^2-4xy+y^2<0$ then

$$\begin{aligned} S_x + S_y &= 4xy(x^2+y^2) + (x^2-4xy+y^2)(x+y)^2 \ge \\ &\ge 2xy(x+y)^2 + (x^2-4xy+y^2)(x+y)^2 = \\ &= (x-y)^2(x+y)^2 \ge 0 \end{aligned}$$

Equality occurs for $a=b=c$, $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$

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4. *Let a,b,c be non-negative real numbers, no two of which are zero Then,*

$$\frac{a^2-bc}{2b^2-3bc+2c^2}+\frac{b^2-ca}{2c^2-3ca+2a^2}+\frac{c^2-ab}{2a^2-3ab+2b^2}\geq 0.$$

First Solution The main idea is to apply the Cauchy-Schawarz Inequality after we made the numerators of the fractions to be non-negative and as small as possible To do this, we write the inequality as

$$\frac{a^2+2(b-c)^2}{2b^2-3bc+2c^2}+\frac{b^2+2(c-a)^2}{2c^2-3ca+2a^2}+\frac{c^2+2(a-b)^2}{2a^2-3ab+2b^2}\geq 3.$$

According to Cauchy-Schwarz Inequality, we have

$$\sum\frac{a^2+2(b-c)^2}{2b^2-3bc+2c^2}\geq\frac{\left(5\sum a^2-4\sum bc\right)^2}{\sum(2b^2-3bc+2c^2)\left[a^2+2(b-c)^2\right]},$$

and it remains to show that

$$\left(5\sum a^2-4\sum bc\right)^2\geq 3\sum(2b^2-3bc+2c^2)\left[a^2+2(b-c)^2\right].$$

This inequality is equivalent to

$$\sum a^4+abc\sum a+2\sum bc(b^2+c^2)\geq 6\sum b^2c^2.$$

We can get it by summing up the inequality

$$\sum a^4+abc\sum a\geq\sum bc(b^2+c^2)$$

to

$$3\sum bc(b^2+c^2)\geq 6\sum b^2c^2.$$

The first inequality is well-known Schur's Inequality of fourth degree, while the second inequality is equivalent to

$$3\sum bc(b-c)^2\geq 0.$$

Equality occurs for $a=b=c$, $a=0$ and $b=c$ and $c=a$, $c=0$ and $a=b$

Second Solution (by *Ho Chung Siu*). Since

$$\sum \frac{2(a^2-bc)}{2b^2-3bc+2c^2} = \sum \frac{(a-b)(a+c)+(a-c)(a+b)}{2b^2-3bc+2c^2} =$$
$$= \sum \frac{(a-b)(a+c)}{2b^2-3bc+2c^2} + \sum \frac{(b-a)(b+c)}{2c^2-3ca+2a^2} =$$
$$= \sum (a-b)\left(\frac{a+c}{2b^2-3bc+2c^2} - \frac{b+c}{2c^2-3ca+2a^2}\right) =$$
$$= \sum (a-b)^2 \frac{2(a^2+ab+b^2)-c(a+b+c)}{(2b^2-3bc+2c^2)(2c^2-3ca+2a^2)},$$

the inequality is equivalent to

$$(b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c \geq 0,$$

where

$$S_a = (2b^2-3bc+2c^2)\left[2(b^2+bc+c^2)-a(a+b+c)\right].$$

Without loss of generality, we assume that $a \geq b \geq c$. Since $S_b \geq 0$ and $S_c > 0$, it suffices to show that $S_a + S_b \geq 0$; that is

$$(2c^2-3ca+2a^2)\left[2(c^2+ca+a^2)-b(a+b+c)\right] \geq$$
$$\geq (2b^2-3bc+2c^2)\left[a(a+b+c)-2(b^2+bc+c^2)\right].$$

We may get it by multiplying the inequalities

$$2c^2-3ca+2a^2 \geq 2b^2-3bc+2c^2$$

and

$$2(c^2+ca+a^2)-b(a+b+c) \geq a(a+b+c)-2(b^2+bc+c^2)$$

The first inequality is equivalent to $(a-b)(2a+2b-3c) \geq 0$, while the second inequality is equivalent to $(a-b)^2 + c(a+b+c) \geq 0$

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5. *Let a, b, c be non-negative real numbers, no two of which are zero. Then,*

$$\frac{a^2}{2b^2-bc \quad 2c^2} + \frac{b^2}{2c^2-ca+2a^2} + \frac{c^2}{2a^2-ab+2b^2} \geq 1$$

Solution. By Cauchy-Schwarz Inequality, we have

$$\sum a^2(2b^2 - bc + 2c^2)\sum \frac{a^2}{2b^2 - bc + 2c^2} \ge \left(\sum a^2\right)^2.$$

Thus, it suffices to show that

$$\left(\sum a^2\right)^2 \ge \sum a^2(2b^2 - bc + 2c^2).$$

The inequality is equivalent to

$$\sum a^4 + abc\sum a \ge 2\sum b^2c^2$$

We may obtain it by adding the fourth degree Schur's Inequality

$$\sum a^4 + abc\sum a \ge \sum bc(b^2 + c^2)$$

to

$$\sum bc(b^2 + c^2) \ge 2\sum b^2c^2$$

The last inequality reduces to $\sum bc(b-c)^2 \ge 0$ Equality occurs when $a = b = c$, and also when one of a, b, c is 0 and the others are equal

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6. *Let a, b, c be non-negative real numbers, no two of which are zero. Then,*

$$\frac{2a^2 - bc}{b^2 - bc + c^2} + \frac{2b^2 - ca}{c^2 - ca + a^2} + \frac{2c^2 - ab}{a^2 - ab + b^2} \ge 3$$

First Solution Write the inequality such that the numerators of the fractions to be non-negative and as smaller possible, that is

$$\sum \frac{2a^2 + (b-c)^2}{b^2 - bc + c^2} \ge 6.$$

Applying now the Cauchy-Schwarz Inequality, we get

$$\sum \frac{2a^2 + (b-c)^2}{b^2 - bc + c^2} \ge \frac{4\left(2\sum a^2 - \sum bc\right)^2}{\sum(b^2 - bc + c^2)\left(2a^2 + (b-c)^2\right)}$$

We still have to show that

$$2\left(2\sum a^2 - \sum bc\right)^2 \ge 3\sum(b^2 - bc + c^2)\left(2a^2 + (b-c)^2\right)$$

This inequality reduces to

$$2\sum a^4 + 2abc\sum a + \sum bc(b^2+c^2) \geq 6\sum b^2c^2$$

We can get it by summing up the inequalities

$$\sum a^4 + abc\sum a \geq \sum bc(b^2+c^2)$$

and

$$\sum bc(b^2+c^2) \geq 2\sum b^2c^2,$$

multiplying by 2 and 3, respectively The first inequality is well-known Schur's Inequality of fourth degree, while the second inequality is equivalent to

$$\sum bc(b-c)^2 \geq 0.$$

Equality occurs for $a=b=c$, $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$

Second Solution. The inequality follows by applying Lemma from application 7.1.3 to the increasing function $f(x)=x^2$ We get

$$\frac{2a^2-b^2-c^2}{b^2-bc+c^2}+\frac{2b^2-c^2-a^2}{c^2-ca+a^2}+\frac{2c^2-a^2-b^2}{a^2-ab+b^2)} \geq 0,$$

which is equivalent to the desired inequality.

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7. *Let a,b,c be non-negative real numbers, no two of which are zero. If $r>-2$, then*

$$\sum\frac{2a^2+(2r+1)bc}{b^2+rbc+c^2} \geq \frac{3(2r+3)}{r+2}.$$

Solution. There two cases to consider.

I. *Case* $r\geq -1$. Since

$$\frac{2a^2+(2r+1)bc}{b^2+rbc+c^2}-\frac{2r+3}{r+2}=\frac{2a^2-b^2-c^2}{b^2+rbc+c^2}+\frac{b^2+c^2+(2r+1)bc}{b^2+rbc+c^2}-\frac{2r+3}{r+2}=$$

$$=\frac{2a^2-b^2-c^2}{b^2+rbc+c^2}-\frac{(r+1)(b-c)^2}{(r+2)(b^2+rbc+c^2)},$$

we may write the inequality in the form

$$\sum\frac{2a^2-b^2-c^2}{b^2+rbc+c^2} \geq \frac{r+1}{r+2}\sum\frac{(b-c)^2}{b^2+rbc+c^2}.$$

Since

$$\sum \frac{2a^2-b^2-c^2}{b^2+rbc+c^2} = \sum \frac{a^2-b^2}{b^2+rbc+c^2} + \sum \frac{a^2-c^2}{b^2+rbc+c^2} =$$
$$= \sum \frac{a^2-b^2}{b^2+rbc+c^2} + \sum \frac{b^2-a^2}{c^2+rca+a^2} =$$
$$= \sum (a^2-b^2)\left(\frac{1}{b^2+rbc+c^2} - \frac{1}{c^2+rca+a^2}\right) =$$
$$= \sum \frac{(a^2-b^2)(a-b)(a+b+rc)}{(b^2+rbc+c^2)(c^2+rca+a^2)},$$

the inequality is equivalent to

$$(b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c \geq 0,$$

where

$$S_a = (r+2)(b+c)(ra+b+c)(b^2+rbc+c^2)-$$
$$-(r+1)(c^2+rca+a^2)(a^2+rab+b^2).$$

Due to symmetry, we may assume that $a \geq b \geq c$ To prove the inequality, it suffices to show that $S_b \geq 0$, $S_c \geq 0$ and $S_a + S_b \geq 0$.

We can prove that $S_b \geq 0$ by multiplying the inequalities

$$(r+2)(c+a)(a+rb+c) \geq (r+1)(a^2+rab+b^2)$$

and

$$c^2+rca+a^2 \geq b^2+rbc+c^2.$$

The first inequality is equivalent to

$$(2+r)c^2 + (2+r)(2a+rb)c + (a-b)[a+(1+r)b] \geq 0,$$

and is true because

$$2a+rb = 2(a-b)+(2+r)b > 0$$

and

$$a+(1+r)b = a-b+(2+r)b > 0.$$

The second inequality is also true because

$$c^2+rca+a^2-(b^2+rbc+c^2) = (a-b)(a+b+rc) =$$
$$= (a-b)[(a-c)+(b-c)+(2+r)c] \geq 0.$$

We can prove that $S_c \geq 0$ by multiplying the inequalities

$$r+2 > r+1,$$
$$(a+b)(a+b+rc) > c^2+rca+a^2,$$
$$a^2+rab+b^2 \geq b^2+rbc+c^2.$$

Indeed,

$$(a+b)(a+b+rc)-c^2-rca-a^2 = b^2-c^2+b[2(a-c)+(2+r)c] > 0$$

and

$$a^2+rab+b^2-(b^2+rbc+c^2) = (a-c)[a-b+(1+r)b+c] \geq 0.$$

In order to prove the inequality $S_a+S_b \geq 0$, we write it in the form

$$c_4c^4+c_3c^3+c_2c^2+c_1c+c_0 \geq 0,$$

where

$$\begin{aligned}
c_4 &= 2(2+r),\\
c_3 &= 2(1+r)(2+r)(a+b),\\
c_2 &= 2(1+r)^2(a^2+b^2+rab),\\
c_1 &= (4+3r)(a^3+b^3)+r(1+r)ab(a+b) \geq (2+r)^2ab(a+b),\\
c_0 &= (a-b)^2\left[a^2+b^2+(2+r)ab\right].
\end{aligned}$$

Since $c_4 > 0$, $c_3 \geq 0$, $c_2 \geq 0$, $c_1 > 0$ and $c_0 \geq 0$, the conclusion follows.

II. *Case* $-2 < r \leq -\frac{1}{2}$. The hint is to apply Cauchy-Schwarz Inequality after we made the numerators of the fractions to be non-negative and as small as possible. To do this, we write the inequality in the form

$$\sum \left[\frac{2a^2+(1+2r)bc}{b^2+rbc+c^2}-\frac{1+2r}{2+r}\right] \geq \frac{6}{2+r},$$

or

$$\sum \frac{E_a}{b^2+rbc+c^2} \geq 6,$$

where

$$E_a = 4+2r)a^2-(1+2r)(b-c)^2 \geq 0.$$

We will show that

$$\sum \frac{E_a}{b^2+rbc+c^2} \geq \frac{\left(\sum E_a\right)^2}{\sum(b^2+rbc+c^2)E_a} \geq 6$$

The left inequality follows by Cauchy-Schwarz Inequality. In order to prove the right inequality, we see that

$$\sum E_a = 2(1-r)\sum a^2 + 2(1+2r)\sum bc,$$

$$\left(\sum E_a\right)^2 = 4(1-r)^2\sum a^4 + 12(1+2r^2)\sum b^2c^2 + \\ + 8(1+r-2r^2)\sum bc(b^2+c^2) + 8(2+r)(1+2r)abc\sum a,$$

and

$$\sum(b^2+rbc+c^2)E_a = -2(1+2r)\sum a^4 + 2(3+r+2r^2)\sum b^2c^2 + \\ + (2-r)(1+2r)\sum bc(b^2+c^2) + 2r(2+r)abc\sum a$$

Thus, the inequality becomes as follows

$$2(2+r)^2\left(\sum a^4 + abc\sum a\right) - (2+r)(1+2r)\sum bc(b^2+c^2) - \\ - 6(2+r)\sum b^2c^2 \geq 0,$$

$$2(2+r)\left[\sum a^4 + abc\sum a - \sum bc(b^2+c^2)\right] + \\ + 3\left[\sum bc(b^2+c^2) - 2\sum b^2c^2\right] \geq 0$$

Since

$$\sum bc(b^2+c^2) - 2\sum b^2c^2 = \sum bc(b-c)^2 \geq 0,$$

and

$$\sum a^4 + abc\sum a - \sum bc(b^2+c^2) \geq 0$$

is well-known Schur's Inequality of fourth degree, the proof is complete Equality occurs for $a=b=c$, $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$

Remark For $r=2$, $r=1$, $r=0$, $r=\frac{-1}{4}$, $r=\frac{-1}{2}$, $r=-1$ and $r=\frac{-3}{2}$, we get the following particular inequalities.

$$\frac{2a^2+5bc}{(b+c)^2}+\frac{2b^2+5ca}{(c+a)^2}+\frac{2c^2+5ab}{(a+b)^2}\geq\frac{21}{4},$$

$$\frac{2a^2+3bc}{b^2+bc+c^2}+\frac{2b^2+3ca}{c^2+ca+a^2}+\frac{2c^2+3ab}{a^2+ab+b^2}\geq 5,$$

$$\frac{2a^2+bc}{b^2+c^2}+\frac{2b^2+ca}{c^2+a^2}+\frac{2c^2+ab}{a^2+b^2}\geq\frac{9}{2},$$

$$\frac{1}{4b^2-bc+4c^2}+\frac{1}{4c^2-ca+4a^2}+\frac{1}{4a^2-ab+4b^2}\geq\frac{9}{7(a^2+b^2+c^2)},$$

$$\frac{a^2}{2b^2-bc+2c^2}+\frac{b^2}{2c^2-ca+2a^2}+\frac{c^2}{2a^2-ab+2b^2}\geq 1,$$

$$\frac{2a^2-bc}{b^2-bc+c^2}+\frac{2b^2-ca}{c^2-ca+a^2}+\frac{2c^2-ab}{a^2-ab+b^2}\geq 3,$$

$$\frac{a^2-bc}{2b^2-3bc+2c^2}+\frac{b^2-ca}{2c^2-3ca+2a^2}+\frac{c^2-ab}{2a^2-3ab+2b^2}\geq 0.$$

In all these inequalities, equality occurs for $a=b=c$, and also for $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$

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8. *Let a,b,c be non-negative real numbers, no two of which are zero. Then*

$$\frac{a^2+16bc}{b^2+c^2}+\frac{b^2+16ca}{c^2+a^2}+\frac{c^2+16ab}{a^2+b^2}\geq 10.$$

Solution. Let $a\leq b\leq c$ and

$$E(a,b,c)=\frac{a^2+16bc}{b^2+c^2}+\frac{b^2+16ca}{c^2+a^2}+\frac{c^2+16ab}{a^2+b^2}.$$

In order to prove the inequality, we consider two cases

I *Case* $16b^3\geq ac^2$. We will show that

$$E(a,b,c)\geq E(0,b,c)\geq 10$$

We have

$$E(a,b,c)-E(0,b,c)=\frac{a^2}{b^2+c^2}+\frac{a(16c^3-ab^2)}{c^2\ c^2+a^2}+\frac{a(16b^3-ac^2)}{b^2(a^2+b^2)}\geq 0$$

since $c^3 - ab^2 \geq 0$ and $16b^3 - ac^2 \geq 0$ Letting now $x = \frac{b}{c} + \frac{c}{b}$, we find

$$E(0,b,c) - 10 = \frac{16bc}{b^2+c^2} + \frac{b^2}{c^2} + \frac{c^2}{b^2} - 10 = \frac{16}{x} + x^2 - 12 =$$
$$= \frac{(x-2)^2(x+4)}{x} \geq 0.$$

II. *Case* $ac^2 > 16b^3$ It suffices to show that

$$\frac{c^2+16ab}{a^2+b^2} \geq 10.$$

Indeed, we have

$$\frac{c^2+16ab}{a^2+b^2} - 10 > \frac{\frac{16b^3}{a} + 16ab}{a^2+b^2} - 10 = \frac{16b}{a} - 10 > 16 - 10 > 0$$

Equality occurs when one of a, b, c is 0 and the others are equal.

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9. *Let a, b, c be non-negative real numbers, no two of which are zero. If $r > -2$, then*

$$\sum \frac{a^2 + 4(r+2)^2bc}{b^2+rbc+c^2} \geq 4r + 10$$

Solution. Let $a \leq b \leq c$ and

$$E(a,b,c) = \sum \frac{a^2 + 4(r+2)^2bc}{b^2+rbc+c^2}$$

I. *Case* $4(r+2)^2b^3 \geq c^2(a+rb)$. We will show that

$$E(a,b,c) \geq E(0,b,c) \geq 10.$$

For the nontrivial case $a > 0$, we have

$$\frac{E(a,b,c) - E(0,b,c)}{a} = \frac{a}{b^2+rbc+c^2} + \frac{4(r+2)^2c^3 - b^2(a+rc)}{c^2(c^2+rca+a^2)} +$$
$$+ \frac{4(r+2)^2b^3 - c^2(a+rb)}{b^2(a^2+rab+b^2)} > \frac{4(r+2)^2c^3 - b^2(a+rc)}{c^2(c^2+rca+a^2)} \geq$$
$$\geq \frac{4(r+2)^2b^2c - b^2(c+rc)}{c^2(c^2+rca+a^2)} = \frac{(4r^2+15r+15)b^2c}{c^2(c^2+rca+a^2)} > 0$$

Letting now $x = \frac{b}{c} + \frac{c}{b}$, we find

$$E(0,b,c) - 4r - 10 = \frac{4(r+2)^2bc}{b^2+rbc+c^2} + \frac{b^2}{c^2} + \frac{c^2}{b^2} - 4r - 10 =$$
$$= \frac{4(r+2)^2}{x+r} + x^2 - 4r - 12 = \frac{(x-2)^2(x+r+4)}{x+r} \geq 0.$$

II *Case* $c^2(a+rb) > 4(r+2)^2b^3$ This case implies $a+rb>0$, $b+rb>0$, and hence $1+r>0$. In order to prove the desired inequality, it suffices to show that

$$\frac{c^2+4(r+2)^2ab}{a^2+rab+b^2} \geq 4r+10.$$

Indeed, we have

$$\frac{c^2+4(r+2)^2ab}{a^2+rab+b^2} - 4r - 10 > 4(r+2)^2b\frac{\frac{b^2}{a+rb}+a}{a^2+rab+b^2} - 4r - 10 =$$
$$= 4(r+2)^2\frac{b}{a+rb} - 4r - 10 \geq \frac{4(r+2)^2}{r+1} - 4r - 10 = \frac{2(r+3)}{r+1} > 0.$$

Equality occurs in the given inequality when one of a,b,c is 0 and the others are equal.

Remark. For $r=2$, we obtain the inequality

$$\frac{a^2+64bc}{(b+c)^2} + \frac{b^2+64ca}{(c+a)^2} + \frac{c^2+64ab}{(a+b)^2} \geq 18$$

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10. *Let a,b,c be non-negative real numbers, no two of which are zero, let q,r be real numbers $(r>-2)$ and let*

$$E(a,b,c) = \sum \frac{a^2+qbc}{b^2+rbc+c^2}$$

Then,

a) $E(a,b,c) \geq \frac{3(q+1)}{r+2}$, *for* $q \leq \frac{2r+1}{2}$,

b) $E(a,b,c) \geq \frac{q}{r+2} + 2$, *for* $\frac{2r+1}{2} \leq q \leq 4(r+2)^2$;

c) $E(a,b,c \geq 4kr + 12k^2 - 2$, *for* $q = 4k(r+2k)^2$, $k \geq 1$.

Solution. a) For a, b, c and r fixed, consider the linear function

$$f_1(q) = \sum \frac{a^2 + qbc}{b^2 + rbc + c^2} - \frac{3(q+1)}{r+2}$$

Since

$$\sum \frac{bc}{b^2 + rbc + c^2} - \frac{3}{r+2} = -3\sum \frac{(b-c)^2}{(r+2)(b^2 + rbc + c^2)} \le 0,$$

$f_1(q)$ is decreasing Therefore, it suffices to prove that $f_1\left(\frac{2r+1}{2}\right) \ge 0$ Taking into account the preceding application 7 from this section, the conclusion follows Equality occurs for $a = b = c$. In the case $q = \frac{2r+1}{2}$, equality occurs again for $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$

b) For fixed a, b, c and r, consider the linear function

$$f_2(q) = \sum \frac{a^2 + qbc}{b^2 + rbc + c^2} - \frac{q}{r+2} - 2.$$

Since $\frac{2r+1}{2} \le q \le 4(r+2)^2$, it suffices to prove that $f_2\left(\frac{2r+1}{2}\right) \ge 0$ and $f_2\left(4(r+2)^2\right) \ge 0$ According to the preceding applications 7 and 9 from this section, the conclusion follows For $\frac{2r+1}{2} < q \le 4(r+2)^2$, equality occurs if and only if $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$

c) Let $a \le b \le c$ and

$$E(a,b,c) = \sum \frac{a^2 + qbc}{b^2 + rbc + c^2}$$

In order to prove the required inequality, we consider two cases.

I *Case* $qb^3 \ge c^2(a + rb)$ We will show that

$$E(a,b,c) \ge E(0,b,c) \ge 4kr + 12k^2 - 2$$

For nontrivial case $a > 0$, we have

$$\frac{E(a,b,c) - E(0,b,c)}{a} = \frac{a}{b^2 + rbc + c^2} + \frac{qc^3 - b^2(a+rc)}{c^2(c^2 + rca + a^2)} +$$

$$+ \frac{qb^3 - c^2(a+rb)}{b^2(a^2 + rab + b^2)} > \frac{qc^3 - b^2(a+rc)}{c^2(c^2 + rca + a^2)} \ge$$

$$\ge \frac{qb^2c - b^2(c+rc)}{c^2(c^2 + rca + a^2)} = \frac{(q-1-r)b^2c}{c^2(c^2 + rca + a^2)}$$

Since

$$q-1-r=4k(r+2k)^2-1-r\geq 4(r+2)^2-1-r=$$
$$=(2r+3)^2+3(r+2)>0,$$

it follows that $E(a,b,c)-E(0,b,c)>0$.

Letting now $x=\frac{b}{c}+\frac{c}{b}$, we find

$$E(0,b,c)=\frac{qbc}{b^2+rbc+c^2}+\frac{b^2}{c^2}+\frac{c^2}{b^2}=\frac{4k(r+2k)^2}{x+r}+x^2-2$$

and

$$E(0,b,c)-4kr-12k^2+2=\frac{4k(r+2k)^2}{x+r}+x^2-4kr-12k^2=$$
$$=\frac{(x-2k)^2(x+r+4k)}{x+r}\geq 0.$$

II. *Case* $c^2(a+rb)>qb^3$. This case implies $a+rb>0$, $b+rb>0$, and hence $1+r>0$. In order to prove the required inequality, it suffices to show that

$$\frac{c^2+qab}{a^2+rab+b^2}\geq 4kr+12k^2-2.$$

Since

$$\frac{c^2+qab}{a^2+rab+b^2}>qb\frac{\frac{b^2}{a+rb}+a}{a^2+rab+b^2}=\frac{qb}{a+rb}\geq\frac{q}{1+r}=\frac{4k(r+2k)^2}{1+r},$$

it is enough to show that

$$\frac{4k(r+2k)^2}{1+r}\geq 4kr+12k^2-2$$

This inequality is equivalent to

$$\frac{4k(k-1)(1+r)+4k(2k-1)^2}{1+r}+2\geq 0,$$

which is clearly true for $k\geq 1$

On the assumption that $a\leq b\leq c$, equality occurs if and only if $a=0$ and $\frac{b}{c}+\frac{c}{b}=2k$

Remark 1. The application generalizes the preceding applications 1 - 9 from this section In addition, the inequality c) is also valid for $r = -2$, that is

$$\sum \frac{a^2 + 16k(k-1)^2bc}{(b-c)^2} \geq 12k^2 - 8k - 2, \quad \text{for } k \geq 1.$$

For $a = \min\{a, b, c\}$ and $k > 1$, equality occurs if and only if $a = 0$ and $\frac{b}{c} + \frac{c}{b} = 2k$.

Remark 2. For $r = 0$, we get the following inequalities·

$$\sum \frac{a^2 + qbc}{b^2 + c^2} \geq \frac{3(q+1)}{2}, \quad \text{for} \quad q \leq \frac{1}{2},$$

$$\sum \frac{a^2 + qbc}{b^2 + c^2} \geq \frac{q}{2} + 2, \quad \text{for} \quad \frac{1}{2} \leq q \leq 16;$$

$$\sum \frac{a^2 + qbc}{b^2 + c^2} \geq 3\sqrt[3]{\frac{q^2}{4}} - 2, \quad \text{for} \quad q \geq 16.$$

Remark 3. For $r = -1$ and $q = 1$, from b) we get the inequality

$$\sum \frac{1}{b^2 - bc + c^2} \geq \frac{6}{a^2 + b^2 + c^2}.$$

Similarly, for $r = -\frac{1}{2}$ and $q = \frac{1}{2}$, from b) we get the inequality

$$\sum \frac{1}{2b^2 - bc + 2c^2} \geq \frac{8}{3(a^2 + b^2 + c^2)}.$$

Equality occurs in both inequalities when $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$.

Remark 4. For $r = 2$ and $q = 150$, from c) we get the inequality

$$\frac{a^2 + 150bc}{(b+c)^2} + \frac{b^2 + 150ca}{(c+a)^2} + \frac{c^2 + 150ab}{(a+b)^2} \geq 37,$$

with equality for $(a, b, c) \sim \left(0, 1, \frac{3+\sqrt{5}}{2}\right)$ or any permutation thereof

7.5 Inequalities involving E_1 and E_2

1. Let a, b, c be non-negative real numbers, no two of which are zero If

$$r > -2, \quad \alpha \ge 0, \quad \alpha(1-r) + \beta = \frac{2r+1}{2},$$

then

$$\sum \frac{a^2 + \alpha a(b+c) + \beta bc}{b^2 + rbc + c^2} \ge \frac{3(1+2\alpha+\beta)}{r+2}$$

2. Let a, b, c be non-negative real numbers, no two of which are zero. If

$$r > -2, \quad \alpha \ge 0, \quad \frac{2r+1}{2} + \alpha(r-1) \le \beta \le 4(r+2)^2 + \alpha(r-1),$$

then

$$\sum \frac{a^2 + \alpha a(b+c) + \beta bc}{b^2 + rbc + c^2} \ge 2 + 2\alpha + \frac{\beta}{r+2}.$$

7.6 Solutions

1. *Let a, b, c be non-negative real numbers, no two of which are zero. If*

$$r > -2, \quad \alpha \ge 0, \quad \alpha(1-r) + \beta = \frac{2r+1}{2},$$

then

$$\sum \frac{a^2 + \alpha a(b+c) + \beta bc}{b^2 + rbc + c^2} \ge \frac{3(1+2\alpha+\beta)}{r+2}$$

Solution. The inequality follows by the inequality from application 7 1.5,

$$\sum \frac{a(b+c) + (r-1)bc}{b^2 + rbc + c^2} \ge \frac{3(r+1)}{r+2},$$

and the inequality from application 7.2.7,

$$\sum \frac{2a^2 + (2r+1)bc}{b^2 + rbc + c^2} \ge \frac{3(2r+3)}{r+2}.$$

Adding the first inequality multiplied by α to the second inequality divided by 2 yields the desired inequality.

Equality occurs if and only if $a = b = c$, $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$.

Remark 1. The particular case $\beta = 0$ yields the following statement

• *Let a, b, c be non-negative real numbers, no two of which are zero. If $\gamma \geq 0$ and $r = \frac{2-\gamma}{2(1+\gamma)}$, then*

$$\sum \frac{a(\gamma a + b + c)}{b^2 + rbc + c^2} \geq \frac{3(\gamma + 2)}{r + 2},$$

with equality if and only if $a = b = c$, $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$.

For $\gamma = 1$ and $\gamma = 2$, we get the inequalities

$$\sum \frac{a}{4(b^2 + c^2) + bc} \geq \frac{1}{a + b + c},$$

$$\sum \frac{a(2a + b + c)}{b^2 + c^2} \geq 6,$$

respectively Note that the first inequality yields

$$\sum \frac{a}{b^2 + c^2} \geq \frac{4}{a + b + c},$$

with equality if and only if $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$

Remark 2. The particular case $\beta = \alpha^2$ yields the following statement:

• *Let a, b, c be non-negative real numbers, no two of which are zero. If $\alpha \geq 0$ and $r = \alpha - \frac{1}{2(\alpha + 1)}$, then*

$$\sum \frac{(a + \alpha b)(a + \alpha c)}{b^2 + rbc + c^2} \geq \frac{3(1 + \alpha)^2}{r + 2},$$

with equality if and only if $a = b = c$, $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$

For $\alpha = 1$, we get the inequality

$$\sum \frac{(a + b)(a + c)}{4(b^2 + c^2) + 3bc} \geq \frac{12}{11}.$$

Remark 3. The particular case $\beta = \alpha - r$ yields the following statement:

• *Let a, b, c be non-negative real numbers, no two of which are zero. If $\frac{-1}{4} \leq r < 2$ and $\alpha = \frac{1 + 4r}{2(2 - r)}$, then*

$$\sum \frac{1}{b^2 + rbc + c^2} \geq \frac{9(1 + \alpha)}{r + 2} \cdot \frac{1}{a^2 + b^2 + c^2 + \alpha(ab + bc + ca)},$$

with equality if and only if $a = b = c$, $a = 0$ *and* $b = c$, $b = 0$ *and* $c = a$, $c = 0$ *and* $a = b$.

For $r = \frac{1}{2}$ and $r = \frac{7}{8}$, we get the inequalities

$$\sum \frac{1}{2(b^2 + c^2) + bc} \geq \frac{18}{5(a^2 + b^2 + c^2 + ab + bc + ca)},$$

$$\sum \frac{1}{8(b^2 + c^2) + 7bc} \geq \frac{27}{23(a + b + c)^2},$$

respectively Since

$$\frac{3}{b^2 + bc + c^2} \geq \frac{23}{8(b^2 + c^2) + 7bc},$$

from the last inequality we obtain the known inequality

$$\sum \frac{1}{b^2 + bc + c^2} \geq \frac{9}{(a + b + c)^2},$$

with equality if and only if $a = b = c$.

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2. *Let* a, b, c *be non-negative real numbers, no two of which are zero. If*

$$r > -2, \;\; \alpha \geq 0, \;\; \frac{2r + 1}{2} + \alpha(r - 1) \leq \beta \leq 4(r + 2)^2 + \alpha(r - 1),$$

then

$$\sum \frac{a^2 + \alpha a(b + c) + \beta bc}{b^2 + rbc + c^2} \geq 2 + 2\alpha + \frac{\beta}{r + 2}.$$

Solution. The inequality follows by the inequality from application 7.1.5,

$$\sum \frac{a(b + c) + (r - 1)bc}{b^2 + rbc + c^2} \geq \frac{3(r + 1)}{r + 2},$$

and the inequality b) from application 7.2.10,

$$\sum \frac{a^2 + qbc}{b^2 + rbc + c^2} \geq \frac{q}{r + 2} + 2,$$

where $\frac{2r + 1}{2} \leq q \leq 4(r + 2)^2$. Adding the first inequality multiplied by α to the second inequality and denoting $\alpha(r - 1) + q = \beta$ yields the desired ine uality.

Equality occurs for $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$ In the case

$$\beta = \frac{2r+1}{2} + \alpha(r-1),$$

equality occurs again for $a = b = c$.

Remark The particular case $\beta = \alpha - r$ yields the following statement

- *Let a, b, c be non-negative real numbers, no two of which are zero. If*

$$-2 < r < 2, \quad \frac{1+4r}{2(2-r)} \le \alpha \le \frac{r+4(2+r)^2}{2-r}, \quad \gamma = 4 + 2\alpha + \frac{\alpha+2}{r+2},$$

then

$$\sum \frac{1}{b^2+rbc+c^2} \ge \frac{\gamma}{a^2+b^2+c^2+\alpha(ab+bc+ca)},$$

with equality for $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$.

For $\alpha = 1$ and $\alpha = 2$, we get the statement:

- *Let a, b, c be non-negative numbers, no two of which are zero.*

a) *If* $-1 \le r \le \frac{1}{2}$, *then*

$$\sum \frac{1}{b^2+rbc+c^2} \ge \frac{3(2r+5)}{(r+2)(a^2+b^2+c^2+ab+bc+ca)},$$

b) *If* $\frac{-3}{4} \le r \le \frac{7}{4}$, *then*

$$\sum \frac{1}{b^2+rbc+c^2} \ge \frac{4(2r+5)}{(r+2)(a+b+c)^2}$$

Equality in a) and b) occurs for $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$. Moreover, the first inequality becomes equality for $r = \frac{1}{2}$ and $a = b = c$, while the second inequality becomes equality for $r = \frac{7}{4}$ and $a = b = c$

For $r = 0$, from a) and b) we obtain the inequalities

$$\sum \frac{1}{b^2+c^2} \ge \frac{15}{2(a^2+b^2+c^2+ab+bc+ca)};$$

$$\sum \frac{1}{b^2+c^2} \ge \frac{10}{(a+b+c)^2}$$

Actually, inequality b) holds for $-r_1 \le r \le \frac{7}{4}$, where

$$r_1 = \frac{25-\sqrt{97}}{12} \approx 1.2626$$

Marian Tetiva proved this inequality for $r=-1$, that is

$$\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}+\frac{1}{a^2-ab+b^2} \ge \frac{12}{(a+b+c)^2}.$$

Assuming that $a=\min\{a,b,c\}$, we have

$$\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}+\frac{1}{a^2-ab+b^2} \ge$$
$$\ge \frac{1}{b^2-bc+c^2}+\frac{1}{c^2}+\frac{1}{b^2} \ge \frac{12}{(b+c)^2} \ge \frac{12}{(a+b+c)^2}.$$

The middle inequality is true because

$$\frac{1}{b^2-bc+c^2}+\frac{1}{c^2}+\frac{1}{b^2}-\frac{12}{(b+c)^2} \ge \frac{1}{b^2-bc+c^2}+\frac{1}{c^2}+\frac{1}{b^2}-\frac{3}{bc}=$$
$$=\frac{(b-c)^4}{b^2c^2(b^2-bc+c^2)} \ge 0.$$

7.7 Other related inequalities

1. Let a,b,c be non-negative real numbers, no two of which are zero. Then,

$$\frac{a^2(b+c)^2}{b^2+c^2}+\frac{b^2(c+a)^2}{c^2+a^2}+\frac{c^2(a+b)^2}{a^2+b^2} \ge 2(ab+bc+ca).$$

2. Let a,b,c be non-negative real numbers such that $ab+bc+ca=1$ Then,

$$\frac{(1+ab)^2}{a^2+b^2+4ab}+\frac{(1+bc)^2}{b^2+c^2+4bc}+\frac{(1+ca)^2}{c^2+a^2+4ca} \ge \frac{8}{3}.$$

3. Let a,b,c be non-negative real numbers such that $ab+bc+ca=1$. If $r \ge 0$, then

$$\sum \frac{(1-bc)^2+rbc}{b^2+rbc+c^2} \ge \frac{3r+4}{r+2}$$

(*Vasile Cîrtoaje*, MS, 2006)

4. Let a, b, c be non-negative real numbers, no two of which are zero Prove that

$$\frac{\sqrt{bc + 4a(b+c)}}{b+c} + \frac{\sqrt{ca + 4b(c+a)}}{c+a} + \frac{\sqrt{ab + 4c(a+b)}}{a+b} \geq \frac{9}{2}$$

5. Let a, b, c be positive numbers. Prove that

$$\frac{\sqrt{a^2 + bc}}{b+c} + \frac{\sqrt{b^2 + ca}}{c+a} + \frac{\sqrt{c^2 + ab}}{a+b} \geq \frac{3\sqrt{2}}{2}.$$

(*Vasile Cîrtoaje*, MS, 2006)

6. Let a, b, c be non-negative real numbers, no two of which are zero Prove that

$$\sqrt{\frac{2a(b+c)}{(2b+c)(b+2c)}} + \sqrt{\frac{2b(c+a)}{(2c+a)(c+2a)}} + \sqrt{\frac{2c(a+b)}{(2a+b)(a+2b)}} \geq 2$$

7. Let a, b, c be non-negative real numbers, no two of which are zero Prove that

$$a) \qquad \frac{a^3 + 3abc}{b+c} + \frac{b^3 + 3abc}{c+a} + \frac{a^3 + 3abc}{a+b} \geq 2(ab + bc + ca),$$

$$b) \qquad \frac{a^3 + 3abc}{(b+c)^3} + \frac{b^3 + 3abc}{(c+a)^3} + \frac{c^3 + 3abc}{(a+b)^3} \geq \frac{3}{2}.$$

(*Vasile Cîrtoaje*, MS, 2005)

8. Let a, b, c be non-negative real numbers, no two of which are zero Prove that

$$a) \qquad \frac{a^2 + 2bc}{b+c} + \frac{b^2 + 2ca}{c+a} + \frac{c^2 + 2ab}{a+b} \geq \frac{3}{2}(a+b+c);$$

$$b) \qquad \frac{a^3 + 2abc}{b+c} + \frac{b^3 + 2abc}{c+a} + \frac{c^3 + 2abc}{a+b} \geq \frac{1}{2}(a+b+c)^2.$$

9. Let a, b, c be non-negative real numbers, no two of which are zero Prove that

$$\frac{a\sqrt{a^2 + 3bc}}{b+c} + \frac{b\sqrt{b^2 + 3ca}}{c+a} + \frac{c\sqrt{c^2 + 3ab}}{a+b} \geq a + b + c.$$

(*Cezar Lupu*, MS, 2006)

10. Let a, b, c be non-negative real numbers, no two of which are zero. If $r \geq 3+\sqrt{7}$, then

$$\frac{1}{ra^2+bc}+\frac{1}{rb^2+ca}+\frac{1}{rc^2+ab} \geq \frac{9}{(r+1)(ab+bc+ca)}.$$

(*Vasile Cîrtoaje*, MS, 2005)

11. Let a, b, c be non-negative real numbers, no two of which are zero. If $\frac{2}{3} \leq r \leq 3+\sqrt{7}$, then

$$\frac{1}{ra^2+bc}+\frac{1}{rb^2+ca}+\frac{1}{rc^2+ab} \geq \frac{r+2}{r(ab+bc+ca)}.$$

(*Vasile Cîrtoaje*, MS, 2005)

12. Let a, b, c be non-negative numbers, no two of which are zero Prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab} \geq \frac{6}{a^2+b^2+c^2+ab+bc+ca}$$

13. Let a, b, c be non-negative real numbers, no two of which are zero. Prove that

$$\frac{1}{22a^2+5bc}+\frac{1}{22b^2+5ca}+\frac{1}{22c^2+5ab} \geq \frac{1}{(a+b+c)^2}.$$

(*Vasile Cîrtoaje*, MS, 2005)

14. Let a, b, c be non-negative real numbers, no two of which are zero. Prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab} \geq \frac{8}{(a+b+c)^2}.$$

(*Vasile Cîrtoaje*, MS, 2005)

15. Let a, b, c be non-negative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab} \geq \frac{12}{(a+b+c)^2}.$$

(*Vasile Cîrtoaje*, MS, 2005)

16. Let a, b, c be non-negative numbers such that $a+b+c=2$. Prove that

$$(a^2+bc)(b^2+ca)(c^2+ab) \leq 1.$$

Vasile Cîrtoaje, MS, 2005)

17. If a, b, c are non-negative numbers, then

$$a)\qquad \frac{a^2-bc}{2a^2+b^2+c^2}+\frac{b^2-ca}{2b^2+c^2+a^2}+\frac{c^2-ab}{2c^2+a^2+b^2}\ge 0,$$

$$b)\qquad \frac{a^2-bc}{\sqrt{2a^2+b^2+c^2}}+\frac{b^2-ca}{\sqrt{2b^2+c^2+a^2}}+\frac{c^2-ab}{\sqrt{2c^2+a^2+b^2}}\ge 0$$

(*Nguyen Anh Tuan*, MS, 2005)

18. If a, b, c are the side lengths of an triangle, then

$$\frac{a^2-bc}{3a^2+b^2+c^2}+\frac{b^2-ca}{3b^2+c^2+a^2}+\frac{c^2-ab}{3b^2+c^2+a^2}\le 0$$

(*Nguyen Anh Tuan*, MS, 2006)

7.8 Solutions

1. *Let a, b, c be non-negative real numbers, no two of which are zero. Then,*

$$\frac{a^2(b+c)^2}{b^2+c^2}+\frac{b^2(c+a)^2}{c^2+a^2}+\frac{c^2(a+b)^2}{a^2+b^2}\ge 2(ab+bc+ca)$$

Solution. We have

$$\sum\frac{a^2(b+c)^2}{b^2+c^2}-2\sum bc=\sum a^2+2\sum\frac{a^2bc}{b^2+c^2}-2\sum bc=$$

$$=2\left(\sum a^2-\sum bc\right)-\sum a^2\left(1-\frac{2bc}{b^2+c^2}\right)=$$

$$=\sum(b-c)^2-\sum\frac{a^2(b-c)^2}{b^2+c^2}=$$

$$=\sum\left(1-\frac{a^2}{b^2+c^2}\right)(b-c)^2$$

Without loss of generality, assume that $a\ge b\ge c$. Since $1-\dfrac{c^2}{a^2+b^2}>0$, it suffices to show that

$$\left(1-\frac{a^2}{b^2+c^2}\right)(b-c)^2+\left(1-\frac{b^2}{c^2+a^2}\right)(c-a)^2\ge 0$$

Write this inequality as

$$\frac{(a^2-b^2+c^2)(a-c)^2}{a^2+c^2}\ge\frac{(a^2-b^2-c^2)(b-c)^2}{b^2+c^2}.$$

We may get it by multiplying the inequalities

$$a^2-b^2+c^2 \ge a^2-b^2-c^2, \quad \frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{b^2+c^2}$$

The latter inequality is true since

$$\frac{(a-c)^2}{a^2+c^2}-\frac{(b-c)^2}{b^2+c^2}=\frac{2bc}{b^2+c^2}-\frac{2ac}{a^2+c^2}=\frac{2c(a-b)(ab-c^2)}{(b^2+c^2)(a^2+c^2)} \ge 0.$$

Equality occurs for $a=b=c$, and also for $a=0$ and $b=c$ and $c=a$, $c=0$ and $a=b$.

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2. *Let a,b,c be non-negative real numbers such that $ab+bc+ca=1$ Then,*

$$\frac{(1+ab)^2}{a^2+b^2+4ab}+\frac{(1+bc)^2}{b^2+c^2+4bc}+\frac{(1+ca)^2}{c^2+a^2+4ca} \ge \frac{8}{3}.$$

Solution. Since

$$\frac{(1+bc)^2}{b^2+c^2+4bc}=\frac{[a(b+c)+2bc]^2}{b^2+c^2+4bc}=\frac{a^2(b+c)^2+4abc(b+c)+4b^2c^2}{b^2+c^2+4bc}=$$

$$=\frac{a^2(b^2+c^2+4bc)-2a^2bc+4abc(b+c)+4b^2c^2}{b^2+c^2+4bc}=$$

$$=a^2-\frac{2a^2bc}{b^2+c^2+4bc}+\frac{4abc(b+c)}{b^2+c^2+4bc}+\frac{4b^2c^2}{b^2+c^2+4bc},$$

we may write the inequality in the form

$$\sum(a^2-bc)-\frac{1}{3}\sum a^2\left(\frac{6bc}{b^2+c^2+4bc}-1\right)+$$
$$+\frac{1}{3}\sum(a^2-bc)+\frac{2}{3}\sum a(b+c)\left(\frac{6bc}{b^2+c^2+4bc}-1\right)+$$
$$+\frac{2}{3}\sum bc\left(\frac{6bc}{b^2+c^2+4bc}-1\right) \ge 0,$$

or

$$\sum(b-c)^2\left[\frac{1}{2}+\frac{1}{3}\,\frac{a^2}{b^2+c^2+4bc}-\frac{1}{6}-\frac{2}{3}\,\frac{a(b+c)}{b^2+c^2+4bc}-\frac{2}{3}\,\frac{bc}{b^2+c^2+4bc}\right] \ge 0$$

This inequality is equivalent to

$$\sum\frac{(b-c)^2(b+c-a)^2}{b^2+c^2+4bc} \ge 0,$$

that is clearly true Equality occurs for $a = b = c = \frac{1}{\sqrt{3}}$, and also for $a = 0$ and $b = c = 1$ or any permutation thereof.

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3. *Let a, b, c be non-negative real numbers such that $ab + bc + ca = 1$ If $r \geq 0$, then*

$$\sum \frac{(1-bc)^2 + rbc}{b^2 + rbc + c^2} \geq \frac{3r+4}{r+2} \tag{1}$$

Solution. Since

$$\frac{(1-bc)^2 + rbc}{b^2 + c^2 + rbc} = \frac{a^2(b+c)^2 + rbc(ab+bc+ca)}{b^2 + c^2 + rbc} =$$

$$= \frac{a^2\left[b^2 + c^2 + rbc + (2-r)bc\right] + rbc(ab+bc+ca)}{b^2 + c^2 + rbc} =$$

$$= a^2 + \frac{(2-r)a^2bc}{b^2 + c^2 + rbc} + \frac{rabc(b+c)}{b^2 + c^2 + rbc} + \frac{rb^2c^2}{b^2 + c^2 + rbc},$$

we may write successively the inequality as

$$\sum(a^2 - bc) + \frac{2-r}{2+r}\sum a^2\left[\frac{(2+r)bc}{b^2 + c^2 + rbc} - 1\right] + \frac{2-r}{2+r}\sum(a^2 - bc) +$$

$$+ \frac{r}{2+r}\sum a(b+c)\left[\frac{(2+r)bc}{b^2 + c^2 + rbc} - 1\right] +$$

$$+ \frac{r}{2+r}\sum bc\left[\frac{(2+r)bc}{b^2 + c^2 + rbc} - 1\right] \geq 0,$$

$$\sum(b-c)^2\left[\frac{1}{2} - \frac{2-r}{2+r}\,\frac{a^2}{b^2 + c^2 + rbc} + \frac{2-r}{2(2+r)} - \right.$$

$$\left. - \frac{r}{2+r}\,\frac{a(b+c)}{b^2 + c^2 + rbc} - \frac{r}{2+r}\,\frac{bc}{b^2 + c^2 + rbc}\right] \geq 0,$$

$$\sum \frac{(b-c)^2}{b^2 + c^2 + rbc} S_a \geq 0, \tag{2}$$

where

$$S_a = 2(b^2 + c^2 - a^2) + r(a-b)(a-c).$$

Assume that $a \geq b \geq c$ and consider two cases.

I *Case* $0 \le r \le 4$. Since

$$S_c = 2(a^2+b^2-c^2) + r(c-a)(c-b) \ge$$
$$\ge 2(a^2+b^2-c^2) - 2(c-a)(c-b) =$$
$$= 2(a^2-ab+b^2) + 2c(a+b-2c) > 0,$$

it suffices to prove that

$$\frac{(b-c)^2}{b^2+c^2+rbc} S_a + \frac{(c-a)^2}{c^2+a^2+rca} S_b \ge 0.$$

Since

$$S_b = 2(a^2+c^2-b^2) + r(b-a)(b-c) \ge$$
$$\ge 2(a^2+c^2-b^2) + 4(b-a)(b-c) = 2(a-b+c)^2 \ge 0,$$

we may prove the inequality by multiplying the inequalities

$$S_b \ge -S_a$$

and

$$\frac{(a-c)^2}{a^2+c^2+rac} \ge \frac{(b-c)^2}{b^2+c^2+rbc}.$$

Indeed, we have

$$S_b + S_a = r(a-b)^2 + 4c^2 \ge 0$$

and

$$\frac{(a-c)^2}{a^2+c^2+rac} - \frac{(b-c)^2}{b^2+c^2+rbc} = 1 - \frac{(2+r)ac}{a^2+c^2+rac} - 1 + \frac{(2+r)bc}{b^2+c^2+rbc} =$$
$$= (2+r)\left(\frac{bc}{b^2+c^2+rbc} - \frac{ac}{a^2+c^2+rac}\right) =$$
$$= \frac{(2+r)c(a-b)(ab-c^2)}{(b^2+c^2+rbc)(a^2+c^2+rac)} \ge 0.$$

II *Case* $r > 4$. Since

$$S_a = 2(b^2+c^2-a^2) + 4(a-b)(a-c) + (r-4)(a-b)(a-c) =$$
$$= 2(a-b-c)^2 + (r-4)(a-b)(a-c) \ge$$
$$\ge (r-4)(a-b)(a-c)$$

and, similarly,

$$S_b \geq (r-4)(b-c)(b-a), \quad S_c \geq (r-4)(c-a)(c-b),$$

to prove (2) it suffices to show that

$$\frac{b-c}{b^2+c^2+rbc} - \frac{a-c}{c^2+a^2+rca} + \frac{a-b}{a^2+b^2+rab} \geq 0$$

This inequality is equivalent to

$$(a-b)(b-c)(a-c)\left[a^2+b^2+c^2+(1+r)(ab+bc+ca\right] \geq 0,$$

which is clearly true This completes the proof

Equality occurs for $a=b=c=\dfrac{1}{\sqrt{3}}$, and also for $a=0$ and $b=c=1$ or any other permutation.

Remark 1. Since

$$(1-bc)^2+rbc = 1+(r-2)bc+b^2c^2 = ab+(r-1)bc+ca+b^2c^2,$$

we may write (1) as

$$\sum \frac{ab+(r-1)bc+ca}{b^2+rbc+c^2} + \sum \frac{b^2c^2}{b^2+c^2+rbc} \geq \frac{3r+4}{r+2}$$

On the other hand,

$$\sum \frac{b^2c^2}{b^2+rbc+c^2} \leq \sum \frac{bc}{r+2} = \frac{1}{2+r}.$$

Therefore, from (1) we get

$$\sum \frac{ab+(r-1)bc+ca}{b^2+rbc+c^2} \geq \frac{3(1+r)}{2+r} \qquad .$$

According to this result, we may say that the inequality (1) is sharper than the one from application 7.1 5 As a consequence, the inequality (1) for $r=2$, that is

$$\sum \frac{1+b^2c^2}{(b+c)^2} \geq \frac{5}{2},$$

is sharper than the well-known Iran Inequality

$$\sum \frac{1}{(b+c)^2} \geq \frac{9}{4}$$

Remark 2. For $r = 0$ and $r = 4$, we get the inequalities from the preceding applications 1 and 2 in this section. Besides, for $r = 1$, we obtain

$$\sum \frac{1 - bc + b^2c^2}{b^2 + bc + c^2} \geq \frac{7}{3}$$

Remark 3. We conjecture that the inequality (1) holds true for any $r > -2$.

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4. *Let a, b, c be non-negative real numbers, no two of which are zero Prove that*

$$\frac{\sqrt{bc + 4a(b+c)}}{b+c} + \frac{\sqrt{ca + 4b(c+a)}}{c+a} + \frac{\sqrt{ab + 4c(a+b)}}{a+b} \geq \frac{9}{2}$$

Solution. Squaring and setting $A = bc + 4a(b+c)$, $B = ca + 4b(c+a)$, $C = ab + 4c(a+b)$, the inequality becomes

$$\sum \frac{A}{(b+c)^2} + 2\sum \frac{\sqrt{BC}}{(c+a)(a+b)} \geq \frac{81}{4}.$$

In order to prove this inequality, we will use the ingenious identity (due to *Sung-Yoon Kim*)

$$(b+c)^2BC - 4\left[a(b^2+c^2) + 2bc(b+c) + 3abc\right]^2 = abc(b-c)^2(a+4b+4c),$$

which implies

$$\sqrt{BC} \geq \frac{2a(b^2+c^2) + 4bc(b+c) + 6abc}{b+c},$$

and hence

$$2\sum \frac{\sqrt{BC}}{(c+a)(a+b)} \geq \frac{4\sum a(b^2+c^2) + 8\sum bc(b+c) + 36abc}{(a+b)(b+c)(c+a)} =$$

$$= \frac{12\sum bc(b+c) + 36abc}{(a+b)(b+c)(c+a)}$$

On the other hand, taking into account Iran Inequality (see application 7 1.4)

$$\sum \frac{ab + bc + ca}{(b+c)^2} \geq \frac{9}{4},$$

we have

$$\sum \frac{A}{(b+c)^2} = \sum \frac{ab+bc+ca}{(b+c)^2} + 3\sum \frac{a}{b+c} \geq \frac{9}{4} + 3\sum \frac{a}{b+c}$$

Then, it suffices to show that

$$3\sum \frac{a}{b+c} + \frac{12\sum bc(b+c) + 36abc}{(a+b)(b+c)(c+a)} \geq 18$$

This inequality is equivalent to

$$\sum a(a+b)(a+c) + 4\sum bc(b+c) + 12abc \geq 6(a+b)(b+c)(c+a)$$

or

$$\sum a^3 + 3abc \geq \sum bc(b+c).$$

Since the last inequality is just the third degree Schur's Inequality, the proof is completed Equality occurs for $a = b = c$, as well as for $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$.

★

5. *Let a, b, c be positive numbers. Prove that*

$$\frac{\sqrt{a^2+bc}}{b+c} + \frac{\sqrt{b^2+ca}}{c+a} + \frac{\sqrt{c^2+ab}}{a+b} \geq \frac{3\sqrt{2}}{2}$$

First Solution. Since

$$\sum \frac{\sqrt{a^2+bc}}{b+c} - \frac{3\sqrt{2}}{2} = \sum \left(\frac{\sqrt{a^2+bc}}{b+c} - \frac{1}{\sqrt{2}} \right) =$$

$$= \frac{2a^2-b^2-c^2}{\sqrt{2}(b+c)\left[\sqrt{2(a^2+bc)}+b+c\right]},$$

we may write the inequality as

$$\sum \frac{2a^2-b^2-c^2}{E_a} \geq 0,$$

where

$$E_a = (b+c)\sqrt{2(a^2+bc)} + (b+c)^2.$$

Let us consider $a \le b \le c$ We have $E_a \ge E_b$, since $(b+c)^2 \ge (c+a)^2$ and

$$(b+c)\sqrt{a^2+bc} - (c+a)\sqrt{b^2+ca} =$$
$$= \frac{c(b-a)(a^2+b^2+c^2-ab+bc+ca)}{(b+c)\sqrt{a^2+bc}+(c+a)\sqrt{b^2+ca}} \ge 0$$

Analogously, we have $E_b \ge E_c$, because $(c+a)^2 \ge (a+b)^2$ and

$$(c+a)\sqrt{b^2+ca} - (a+b)\sqrt{c^2+ab} =$$
$$= \frac{a(c-b)(a^2+b^2+c^2+ab-bc+ca)}{(c+a)\sqrt{b^2+ca}+(a+b)\sqrt{c^2+ab}} \ge 0.$$

Since

$$2a^2-b^2-c^2 \le 2b^2-c^2-a^2 \le 2c^2-a^2-b^2$$

and

$$\frac{1}{E_a} \le \frac{1}{E_b} \le \frac{1}{E_c},$$

by Chebyshev's Inequality we get

$$\sum \frac{2a^2-b^2-c^2}{E_a} \ge \frac{1}{3}\left[\sum(2a^2-b^2-c^2)\right]\left(\sum \frac{1}{E_a}\right) = 0.$$

Equality occurs if and only if $a=b=c$.

Second Solution For x, y, z positive number, the well-known inequality holds

$$x+y+z \ge \sqrt{3(xy+yz+zx)}$$

Thus, it suffices to show that

$$\sum \frac{\sqrt{(b^2+ca)(c^2+ab)}}{(c+a)(a+b)} \ge \frac{3}{2}$$

Setting $a=x^2$, $b=y^2$, $c=z^2$, where $x, y, z > 0$, the inequality becomes

$$2\sum(y^2+z^2)\sqrt{(y^4+z^2x^2)(z^4+x^2y^2)} \ge 3(x^2+y^2)(y^2+z^2)(z^2+x^2).$$

The Cauchy-Schwarz Inequality gives us

$$\sqrt{(y^2+z^2)(y^4+z^2x^2)} \ge y^3+z^2x$$

and

$$\sqrt{(z^2+y^2)(z^4+x^2y^2)} \geq z^3+xy^2$$

Multiplying these inequalities yields

$$(y^2+z^2)\sqrt{(y^4+z^2x^2)(z^4+x^2y^2)} \geq (y^3+z^2x)(z^3+xy^2) =$$
$$= y^3z^3 + x(y^5+x^5) + x^2y^2z^2$$

Therefore, it suffices to show that

$$2\sum y^3z^3 + 2\sum x(y^5+z^5) + 6x^2y^2z^2 \geq 3(x^2+y^2)(y^2+z^2)(z^2+x^2).$$

This inequality is equivalent to

$$2\sum y^3z^3 + 2\sum yz(y^4+z^4) \geq 3\sum y^2z^2(y^2+z^2)$$

or

$$\sum yz(y-z)^2(2y^2+yz+2z^2) \geq 0$$

Since the last inequality is clearly true, the proof is completed

★

6. *Let a, b, c be non-negative real numbers, no two of which are zero. Prove that*

$$\sqrt{\frac{2a(b+c)}{(2b+c)(b+2c)}} + \sqrt{\frac{2b(c+a)}{(2c+a)(c+2a)}} + \sqrt{\frac{2c(a+b)}{(2a+b)(a+2b)}} \geq 2$$

Solution. Using the substitution $a = x^2$, $b = y^2$, $c = z^2$, where $x, y, z \geq 0$, the inequality becomes

$$\sum x\sqrt{\frac{2(y^2+z^2)}{(2y^2+z^2)(y^2+2z^2)}} \geq 2$$

We will show that

$$\sqrt{\frac{2(y^2+z^2)}{(2y^2+z^2)(y^2+2z^2)}} \geq \frac{y+z}{y^2+yz+z^2}.$$

Indeed, by squaring and direct calculation, the inequality reduces to $y^2z^2(y-z)^2 \geq 0$, which is clearly true Therefore, it suffices to prove that

$$\sum \frac{x(y+z)}{y^2+yz+z^2} \geq 2,$$

which is just the inequality from the application 7.1.1. Equality occurs for $a = b = c$, and also for $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$.

★

7. *Let a, b, c be non-negative real numbers, no two of which are zero. Prove that*

$$a)\qquad \frac{a^3+3abc}{b+c}+\frac{b^3+3abc}{c+a}+\frac{a^3+3abc}{a+b}\geq 2(ab+bc+ca);$$

$$b)\qquad \frac{a^3+3abc}{(b+c)^3}+\frac{b^3+3abc}{(c+a)^3}+\frac{c^3+3abc}{(a+b)^3}\geq \frac{3}{2}.$$

Solution. a) We have

$$\sum \frac{a^3+3abc}{b+c}-2\sum bc=\sum\left[\frac{a^3+3abc}{b+c}-a(b+c)\right]=$$
$$=\sum \frac{a}{b+c}(a^2+bc-b^2-c^2)=$$
$$=\sum \frac{a(a-b)(a-c)}{b+c}+\sum \frac{a(ab+ac-b^2-c^2)}{b+c}.$$

Since

$$\sum \frac{a(ab+ac-b^2-c^2)}{b+c}=\sum \frac{ab(a-b)}{b+c}+\sum \frac{ac(a-c)}{b+c}=$$
$$=\sum \frac{ab(a-b)}{b+c}+\sum \frac{ba(b-a)}{c+a}=\sum \frac{ab(a-b)^2}{(b+c)(c+a)}\geq 0,$$

it remains to show that

$$\sum \frac{a(a-b)(a-c)}{b+c}\geq 0$$

This inequality is a particular case of the following more general statement.

- *If $a\geq b\geq c$ are real numbers and $X\geq Y\geq Z\geq 0$, then*

$$X(a-b)(a-c)+Y(b-c)(b-a)+Z(c-a)(c-b)\geq 0.$$

Notice that the inequality follows by adding the evident inequality

$$Z(c-a)(c-b)\geq 0$$

to

$$X(a-b)(a-c)+Y(b-c)(b-a)\geq 0.$$

To prove the latter inequality it suffices to show that $X(a-c)\geq Y(b-c)$. This inequality is true because $X\geq Y$ and $a-c\geq b-c\geq 0$.

Returning to our problem, we set $X = \frac{a}{b+c}$, $Y = \frac{b}{c+a}$, $Z = \frac{c}{a+b}$, and to see that $X \geq Y \geq Z \geq 0$ Equality occurs for $a = b = c$, and also for $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$

Remark The above statement is also valid for $0 \leq X \leq Y \leq Z$ We can prove this claim by adding the evident inequality $X(a-b)(a-c) \geq 0$ to

$$Y(b-c)(b-a) + Z(c-a)(c-b) \geq 0.$$

To prove the latter it suffices to show that $Z(a-c) \geq Y(a-b)$ This inequality is true because $Z \geq Y$ and $a - c \geq a - b \geq 0$.

b) Let $a \geq b \geq c$. Since

$$\frac{a^3+3abc}{b+c} \geq \frac{b^3+3abc}{c+a} \geq \frac{c^3+3abc}{a+b}$$

and

$$\frac{1}{(b+c)^2} \geq \frac{1}{(c+a)^2} \geq \frac{1}{(a+b)^2},$$

by Cebyshev's Inequality we get

$$\sum \frac{a^3+3abc}{(b+c)^3} \geq \frac{1}{3}\left(\sum \frac{a^3+3abc}{b+c}\right)\sum \frac{1}{(b+c)^2}$$

Taking into account Iran Inequality (application 7.1.4)

$$\sum \frac{1}{(b+c)^2} \geq \frac{9}{4(ab+bc+ca)},$$

it is enough to show that

$$\sum \frac{a^3+3abc}{b+c} \geq 2\sum bc,$$

which is just the inequality a) Equality occurs if and only if $a = b = c$.

★

8. *Let a, b, c be non-negative real numbers, no two of which are zero. Prove that*

a) $$\frac{a^2+2bc}{b+c} + \frac{b^2+2ca}{c+a} + \frac{c^2+2ab}{a+b} \geq \frac{3}{2}(a+b+c);$$

b) $$\frac{a^3+2abc}{b+c} + \frac{b^3+2abc}{c+a} + \frac{c^3+2abc}{a+b} \geq \frac{1}{2}(a+b+c)^2$$

Solution. a) We have

$$2\sum \frac{a^2+2bc}{b+c} - 3(a+b+c) = \sum \left(\frac{2a^2}{b+c} - a\right) + \sum \left(\frac{4bc}{b+c} - b - c\right) =$$
$$= \sum \frac{a(2a-b-c)}{b+c} - \sum \frac{(b-c)^2}{b+c}$$

and

$$\sum \frac{a(2a-b-c)}{b+c} = \sum \frac{a(a-b)}{b+c} + \sum \frac{a(a-c)}{b+c} =$$
$$= \sum \frac{a(a-b)}{b+c} + \sum \frac{b(b-a)}{c+a} = (a+b+c)\sum \frac{(a-b)^2}{(b+c)(c+a)} =$$
$$= (a+b+c)\sum \frac{(b-c)^2}{(a+b)(a+c)}.$$

Therefore, we may rewrite the inequality as

$$\sum (b-c)^2\ S_a \geq 0,$$

where

$$S_a = (a+b+c)(b+c) - (a+b)(a+c).$$

Without loss of generality, assume that $a \geq b \geq c$. We have

$$S_b = (a+b+c)(c+a) - (b+c)(b+a) \geq$$
$$\geq (a+b)(c+a) - (b+c)(b+a) = a^2 - b^2 \geq 0,$$
$$S_c = (a+b+c)(a+b) - (c+a)(c+b) \geq$$
$$\geq (a+c)(a+b) - (c+a)(c+b) = a^2 - c^2 \geq 0$$

and

$$S_a + S_b = (a+b+c)(a+b+2c) - (a+b)(a+b+2c) =$$
$$= c(a+b+2c) \geq 0$$

Then

$$\sum (b-c)^2 S_a \geq (b-c)^2 S_a + (a-c)^2 S_b \geq$$
$$\geq (b-c)^2 S_a + (b-c)^2 S_b = (S_a+S_b)(b-c)^2 \geq 0.$$

b) Since

$$\sum \frac{a^3+2abc}{b+c} = \sum \left(\frac{a^3+2abc}{b+c} + a^2 + 2bc \right) - \sum (a^2+2bc) =$$
$$= (a+b+c) \sum \frac{a^2+2bc}{b+c} - (a+b+c)^2,$$

the inequality becomes

$$(a+b+c) \sum \frac{a^2+2bc}{b+c} \geq \frac{3}{2} (a+b+c)^2$$

Dividing by $a+b+c$, we get the inequality a).
Equality occurs in both inequalities for $a=b=c$, and also for $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$.

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9. *Let a,b,c be non-negative real numbers, no two of which are zero. Prove that*

$$\frac{a\sqrt{a^2+3bc}}{b+c} + \frac{b\sqrt{b^2+3ca}}{c+a} + \frac{c\sqrt{c^2+3ab}}{a+b} \geq a+b+c.$$

Solution. (by *Yuan Shyong Ooi*). By the AM-GM Inequality, we have

$$\frac{a\sqrt{a^2+3bc}}{b+c} = \frac{a(a^2+3bc)}{\sqrt{(b+c)^2(a^2+3bc)}} \geq$$
$$\geq \frac{2a(a^2+3bc)}{(b+c)^2+(a^2+3bc)} = \frac{2a^3+6abc}{S+5bc},$$

where $S = a^2+b^2+c^2$. Since

$$\frac{2a^3+6abc}{S+5bc} - a = \frac{a^3+abc-a(b^2+c^2)}{S+5bc},$$

it suffices to show that

$$AX+BY+CZ \geq 0,$$

where

$$A = \frac{1}{S+5bc}, \quad B = \frac{1}{S+5ca}, \quad C = \frac{1}{S+5ab},$$
$$X = a^3+abc-a(b^2+c^2),$$
$$Y = b^3+abc-b(c^2+a^2),$$
$$Z = c^3+abc-c(a^2+b^2).$$

Without loss of generality, assume that $a \geq b \geq c$ Since

$$A \geq B \geq C,$$
$$X = a(a^2 - b^2) + ac(b - c) \geq 0,$$
$$Z = c(c^2 - b^2) + ac(b - a) \leq 0$$

and

$$X + Y + X = \sum a^3 + 3abc - \sum a(b^2 + c^2) \geq 0$$

(Schur's Inequality), we have

$$AX + BY + CZ \geq BX + BY + BZ = B(X + Y + Z) \geq 0$$

Equality occurs for $a = b = c$, and also for $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$

★

10. *Let a, b, c be non-negative real numbers, no two of which are zero. If $r \geq 3 + \sqrt{7}$, then*

$$\frac{1}{ra^2 + bc} + \frac{1}{rb^2 + ca} + \frac{1}{rc^2 + ab} \geq \frac{9}{(r+1)(ab + bc + ca)}.$$

First Solution. We write the inequality as

$$(r+1)(ab + bc + ca) \sum (rb^2 + ca)(rc^2 + ab) \geq$$
$$\geq 9(ra^2 + bc)(rb^2 + ca)(rc^2 + ab).$$

Since

$$\sum (rb^2 + ca)(rc^2 + ab) = r^2 \sum b^2c^2 + abc \sum a + r \sum bc(b^2 + c^2),$$
$$(ab + bc + ca) \sum (rb^2 + ca)(rc^2 + ab) = r \sum b^2c^2(b^2 + c^2) +$$
$$+ r^2 \sum b^3c^3 + (r^2 + r + 1)abc \sum bc(b + c) + 2rabc \sum a^3 + 3a^2b^2c^2$$

and

$$(ra^2 + bc)(rb^2 + ca)(rc^2 + ab) = r^2 \sum b^3c^3 + rabc \sum a^3 + (r^3 + 1)a^2b^2c^2,$$

the inequality becomes

$$r(r+1) \sum b^2c^2(b^2 + c^2) + r^2(r - 8) \sum b^3c^3 +$$
$$+ (r+1)(r^2 + r + 1)abc \sum bc(b + c) \geq$$
$$\geq r(7 - 2r)abc \sum a^3 + 3(r+1)(3r^2 - 3r + 2)a^2b^2c^2.$$

On the other hand,

$$(a-b)^2(b-c)^2(c-a)^2 = \sum b^2c^2(b^2+c^2) - 2\sum b^3c^3 + \\ + 2abc\sum bc(b+c) - 2abc\sum a^3 - 6a^2b^2c^2$$

Then, the inequality is equivalent to

$$r(r+1)(a-b)^2(b-c)^2(c-a)^2 + r(r^2-6r+2)\sum b^3c^3 + \\ + (r^3+1)abc\sum bc(b+c) + r(4r-5)abc\sum a^3 \ge \\ \ge 3(r+1)(3r^2-5r+2)a^2b^2c^2$$

Since $r^2-6r+2 \ge 0$ for $r \ge 3+\sqrt{7}$, by AM-GM Inequality we get

$$r(r^2-6r+2)\sum b^3c^3 + (r^3+1)abc\sum bc(b+c) + r(4r-5)abc\sum a^3 \ge \\ \ge 3r(r^2-6r+2)a^2b^2c^2 + 6(r^3+1)a^2b^2c^2 + 3r(4r-5)a^2b^2c^2 = \\ = 3(r+1)(3r^2-5r+2)a^2b^2c^2,$$

from which the required inequality follows. Equality occurs when $a=b=c$. For $r=3+\sqrt{7}$, equality occurs again when $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$

Second Solution (by *Pham Kim Hung*). Write the inequality as

$$\sum f(a,b,c) \ge 0,$$

where

$$f(a,b,c) = \frac{(r+1)(ab+bc+ca)}{ra^2+bc} - 3$$

Since

$$f(a,b,c) = \frac{3ra(b+c-2a)-(r-2)(ab-2bc+ca)}{2(ra^2+bc)} = \\ = \frac{[3ra+(r-2)c](b-a)+[3ra+(r-2)b](c-a)}{2(ra^2+bc)},$$

we have

$$\sum f(a,b,c) = \sum \frac{[3ra+(r-2)c](b-a)}{2(ra^2+bc)} + \sum \frac{[3rb+(r-2)c](a-b)}{2(rb^2+ca)} = \\ = \frac{1}{2}\sum(a-b)\left[\frac{3rb+(r-2)c}{rb^2+ca} - \frac{3ra+(r-2)c}{ra^2+bc}\right] = \\ = \frac{1}{2(ra^2+bc)(rb^2+ca)(rc^2+ab)}\sum(a-b)^2E_c,$$

where

$$E_c = (rc^2 + ab)[3r^2ab + r(r-5)(a+b)c - (r-2)c^2].$$

Without loss of generality, assume that $a \ge b \ge c$. Since $r \ge 3 + \sqrt{7}$ implies $r(r-5) \ge r-2$, and hence

$$r(r-5)(a+b)c - (r-2)c^2 \ge (r-2)c(a+b-c),$$

it suffices to show that

$$3r^2 \sum ab(rc^2 + ab)(a-b)^2 + (r-2)\sum (a-b)^2 S_c \ge 0,$$

where

$$S_c = (rc^2 + ab)c(a+b-c).$$

Since

$$3r^2 \sum ab(rc^2 + ab)(a-b)^2 \ge 3r^2a^2b^2(a-b)^2 \ge r(r-2)a^2b^2(a-b)^2,$$

it is enough to prove that

$$ra^2b^2(a-b)^2 + \sum (a-b)^2 S_c \ge 0.$$

We have

$$S_a = (ra^2 + bc)a(b+c-a) \ge (ra^2 + bc)a(b-a),$$

$$S_b = (rb^2 + ca)b(c+a-b) \ge (rb^2 + ca)b(a-b) \ge 0,$$

$$S_a + \frac{a^2}{b^2}S_b \ge (a-b)\left[-a(ra^2 + bc) + \frac{a^2}{b}(rb^2 + ca)\right] =$$

$$= \frac{a(a-b)^2(ca - rab + bc)}{b} \ge \frac{a(a-b)^2(-rab)}{b} = -ra^2(a-b)^2$$

and $S_c \ge 0$ Therefore,

$$\sum (a-b)^2 S_c \ge (b-c)^2 S_a + (a-c)^2 S_b \ge$$

$$\ge (b-c)^2 S_a + \frac{a^2}{b^2}(b-c)^2 S_b = (b-c)^2\left(S_a + \frac{a^2}{b^2}S_b\right) \ge$$

$$\ge -ra^2(b-c)^2(a-b)^2,$$

and finally

$$ra^2b^2(a-b)^2 + \sum (a-b)^2 S_c \ge ra^2(a-b)^2\left[b^2 - (b-c)^2\right] \ge 0.$$

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11. *Let a, b, c be non-negative real numbers, no two of which are zero. If $\frac{2}{3} \le r \le 3+\sqrt{7}$, then*

$$\frac{1}{ra^2+bc}+\frac{1}{rb^2+ca}+\frac{1}{rc^2+ab} \ge \frac{r+2}{r(ab+bc+ca)}$$

Solution. We write the inequality as

$$r(ab+bc+ca)\sum(rb^2+ca)(rc^2+ab) \ge$$
$$\ge (r+2)(ra^2+bc)(rb^2+ca)(rc^2+ab)$$

As in preceding proof (first solution), we may rewrite the inequality as

$$r^2\sum b^2c^2(b^2+c^2)-2r^2\sum b^3c^3+r(r^2+r+1)abc\sum bc(b+c) \ge$$
$$\ge r(2-r)abc\sum a^3+(r^4+2r^3-2r+2)a^2b^2c^2,$$

or

$$r^2(a-b)^2(b-c)^2(c-a)^2+r(r^2-r+1)abc\sum bc(b+c)+$$
$$+r(3r-2)abc\sum a^3 \ge (r^4+2r^3-6r^2-2r+2)a^2b^2c^2$$

Since $3r-2 \ge 0$, by AM-GM Inequality we get

$$r(r^2-r+1)abc\sum bc(b+c)+r(3r-2)abc\sum a^3 \ge$$
$$\ge 3r(r^2-r+1)a^2b^2c^2+3r(3r-2)a^2b^2c^2=3r^2(2r+1)a^2b^2c^2$$

So, it suffices to show that

$$3r^2(2r+1) \ge r^4+2r^3-6r^2-2r+2.$$

This inequality is equivalent to $(r+1)^2(6r-2-r^2) \ge 0$, and is true for $\frac{2}{3} \le r \le 3+\sqrt{7}$ Equality occurs when $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$ For $r=3+\sqrt{7}$, equality occurs again when $a=b=c$.

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12. *Let a, b, c be non-negative numbers, no two of which are zero. Prove that*

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab} \ge \frac{6}{a^2+b^2+c^2+ab+bc+ca}$$

Solution. Applying Cauchy-Schwarz Inequality, we have

$$\sum \frac{1}{2a^2+bc} \geq \frac{4(a+b+c)^2}{\sum(b+c)^2(2a^2+bc)}.$$

So, it suffices to prove that

$$2(a+b+c)^2(a^2+b^2+c^2+ab+bc+ca) \geq 3\sum(b+c)^2(2a^2+bc).$$

Since

$$\begin{aligned}(a+b+c)^2&(a^2+b^2+c^2+ab+bc+ca) = \\ &= \left(\sum a^2 + 2\sum bc\right)\left(\sum a^2 + \sum bc\right) = \\ &= \left(\sum a^2\right)^2 + 3\left(\sum a^2\right)\left(\sum bc\right) + 2\left(\sum bc\right)^2 = \\ &= \sum a^4 + 3\sum bc(b^2+c^2) + 4\sum b^2c^2 + 7abc\sum a\end{aligned}$$

and

$$\begin{aligned}\sum(b+c)^2(2a^2+bc) &= \sum(b^2+c^2+2bc)(2a^2+bc) = \\ &= \sum bc(b^2+c^2) + 6\sum b^2c^2 + 4abc\sum a,\end{aligned}$$

the inequality transforms into

$$2\sum a^4 + 3\sum bc(b^2+c^2) + 2abc\sum a \geq 10\sum b^2c^2$$

We may obtain this inequality by adding Schur's Inequality of fourth degree

$$\sum a^4 + abc\sum a \geq \sum bc(b^2+c^2),$$

multiplied by 2, to

$$5\sum bc(b^2+c^2) \geq 10\sum b^2c^2.$$

The latter inequality is equivalent to

$$5\sum bc(b-c)^2 \geq 0$$

Equality occurs for $a=b=c$, and also for $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$

★

13. *Let a, b, c be non-negative real numbers, no two of which are zero. Prove that*

$$\frac{1}{22a^2+5bc}+\frac{1}{22b^2+5ca}+\frac{1}{22c^2+5ab}\geq\frac{1}{(a+b+c)^2}$$

Solution. By Cauchy-Schwarz Inequality, we have

$$\sum\frac{1}{22a^2+5bc}\geq\frac{4(a+b+c)^2}{\sum(b+c)^2(22a^2+5bc)}.$$

Therefore, it suffices to prove that

$$4(a+b+c)^4\geq\sum(b+c)^2(22a^2+5bc).$$

Since

$$\begin{aligned}(a+b+c)^4&=\left(\sum a^2+2\sum bc\right)^2=\\&=\left(\sum a^2\right)^2+4\left(\sum a^2\right)\left(\sum bc\right)+4\left(\sum bc\right)^2=\\&=\sum a^4+4\sum bc(b^2+c^2)+6\sum b^2c^2+12abc\sum a\end{aligned}$$

and

$$\begin{aligned}\sum(b+c)^2(22a^2+5bc)&=\sum(b^2+c^2+2bc)(22a^2+5bc)=\\&=5\sum bc(b^2+c^2)+54\sum b^2c^2+44abc\sum a,\end{aligned}$$

the inequality becomes

$$4\sum a^4+11\sum bc(b^2+c^2)+4abc\sum a\geq30\sum b^2c^2,$$

or, dividing by 4,

$$\sum a^4+abc\sum a-\sum bc(b^2+c^2)+\frac{15}{4}\sum bc(b-c)^2\geq0$$

Taking into account Schur's Inequality of fourth degree

$$\sum a^4+abc\sum a\geq\sum bc(b^2+c^2),$$

the conclusion follows Equality occurs if and only if $a=b=c$.

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14. *Let a, b, c be non-negative real numbers, no two of which are zero. Prove that*

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\geq\frac{8}{(a+b+c)^2}.$$

Solution. By Cauchy-Schwarz Inequality, we have

$$\sum\frac{1}{2a^2+bc}\geq\frac{4(a+b+c)^2}{\sum(b+c)^2(2a^2+bc)}.$$

Therefore, it suffices to prove that

$$(a+b+c)^4\geq 2\sum(b+c)^2(2a^2+bc).$$

Since

$$(a+b+c)^4=\sum a^4+4\sum bc(b^2+c^2)+6\sum b^2c^2+12abc\sum a$$

and

$$\sum(b+c)^2(2a^2+bc)=\sum(b^2+c^2+2bc)(2a^2+bc)=$$
$$=\sum bc(b^2+c^2)+6\sum b^2c^2+4abc\sum a,$$

the inequality becomes

$$\sum a^4+2\sum bc(b^2+c^2)+4abc\sum a\geq 6\sum b^2c^2.$$

We will prove that the stronger inequality

$$\sum a^4+2\sum bc(b^2+c^2)+abc\sum a\geq 6\sum b^2c^2.$$

This inequality follows by summing Schur's Inequality of fourth degree

$$\sum a^4+abc\sum a\geq\sum bc(b^2+c^2)$$

to the inequality

$$3\sum bc(b^2+c^2)\geq 6\sum b^2c^2,$$

which is equivalent to $3\sum bc(b-c)^2\geq 0$. Equality occurs if and only if $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$.

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15. *Let a, b, c be non-negative real numbers, no two of which are zero. Prove that*

$$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab} \geq \frac{12}{(a+b+c)^2}.$$

Solution. Due to homogeneity, we may assume that $a+b+c=1$ Under this assumption, we write the inequality in the form

$$\frac{1-a^2-bc}{a^2+bc}+\frac{1-b^2-ca}{b^2+ca}+\frac{1-c^2-ab}{c^2+ab} \geq 9$$

Since $1-a^2-bc=(a+b+c)^2-a^2-bc>0$ and, analogously, $1-b^2-ca>0$ and $1-c^2-ab>0$, by Cauchy-Schwarz Inequality we get

$$\sum \frac{1-a^2-bc}{a^2+bc} \geq \frac{\left[\sum(1-a^2-bc)\right]^2}{\sum(1-a^2-bc)(a^2+bc)}$$

Thus, it is enough to show that

$$\frac{\left[3-\sum(a^2+bc)\right]^2}{\sum(a^2+bc)-\sum(a^2+bc)^2} \geq 9$$

Let us denote $ab+bc+ca=x$ Since

$$\sum a^2 = 1-2x,\ \sum b^2c^2 = x^2-2abc,\ \sum(a^2+bc)=1-x,$$

$$\sum a^4 = \left(\sum a^2\right)^2 - 2\sum b^2c^2 = 1-4x+2x^2+4abc,$$

$$\sum(a^2+bc)^2 = 2abc+\sum a^4+\sum b^2c^2 = 1-4x+3x^2+4abc,$$

the inequality becomes

$$\frac{(2+x)^2}{3x-3x^2-4abc} \geq 9,$$

or

$$(1-4x)(4-7x)+36abc \geq 0.$$

The inequality is clearly true for $x \leq \frac{1}{4}$. Consider now that $x > \frac{1}{4}$ By Schur's Inequality of third degree

$$(a+b+c)^3+9abc \geq 4(a+b+c)(ab+bc+ca),$$

it follows that $1+9abc \geq 4x$. Therefore

$$36abc \geq 16x - 4$$

and

$$(1-4x)(4-7x) + 36abc \geq (1-4x)(4-7x) + 16x-4 = 7x(4x-1) > 0.$$

Equality occurs if and only if $a=0$ and $b=c$, $b=0$ and $c=a$, $c=0$ and $a=b$

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16. *Let a, b, c be non-negative numbers such that $a+b+c=2$. Prove that*

$$(a^2+bc)(b^2+ca)(c^2+ab) \leq 1.$$

Solution. Without loss of generality, assume that $a \geq b \geq c$. Since

$$a^2+bc \leq \left(a+\frac{c}{2}\right)^2$$

and

$$(b^2+ca)(c^2+ab) \leq \frac{1}{4}(b^2+ca+c^2+ab)^2,$$

it suffices to show that

$$(2a+c)^2(b^2+c^2+ab+ac)^2 \leq 16.$$

Let

$$E(a,b,c) = (2a+c)(b^2+c^2+ab+ac)$$

We will show that

$$E(a,b,c) \leq E(a,b+c,0) \leq 4.$$

Indeed, we have

$$E(a,b,c) - E(a,b+c,0) = c(b^2+c^2+ac-3ab) \leq 0$$

and

$$E(a,b+c,0) - 4 = 2a(b+c)(a+b+c) - 4 =$$
$$= 4a(2-a) - 4 = -4(a-1)^2 \leq 0.$$

Equality occurs if and only if $a=0$ and $b=c=1$, $b=0$ and $c=a=1$, $c=0$ and $a=b=1$

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17. *If a, b, c are non-negative numbers, then*

$$a)\qquad \frac{a^2-bc}{2a^2+b^2+c^2}+\frac{b^2-ca}{2b^2+c^2+a^2}+\frac{c^2-ab}{2c^2+a^2+b^2}\geq 0;$$

$$b)\qquad \frac{a^2-bc}{\sqrt{2a^2+b^2+c^2}}+\frac{b^2-ca}{\sqrt{2b^2+c^2+a^2}}+\frac{c^2-ab}{\sqrt{2c^2+a^2+b^2}}\geq 0$$

Solution. a) Since

$$\frac{1}{2}-\frac{a^2-bc}{2a^2+b^2+c^2}=\frac{(b+c)^2}{2a^2+b^2+c^2},$$

we may rewrite the inequality as

$$\sum \frac{(b+c)^2}{2a^2+b^2+c^2}\leq 3$$

Applying Cauchy-Schwarz Inequality, we have

$$\left[(a^2+b^2)+(a^2+c^2)\right]\left(\frac{b^2}{a^2+b^2}+\frac{c^2}{a^2+c^2}\right)\geq (b+c)^2;$$

that is

$$\frac{(b+c)^2}{2a^2+b^2+c^2}\leq \frac{b^2}{a^2+b^2}+\frac{c^2}{a^2+c^2}$$

Therefore,

$$\sum \frac{(b+c)^2}{2a^2+b^2+c^2}\leq \sum \frac{b^2}{a^2+b^2}+\sum \frac{c^2}{a^2+c^2}=$$

$$=\sum \frac{b^2}{a^2+b^2}+\sum \frac{a^2}{b^2+a^2}=3$$

Equality occurs if and only if $a=b=c$.

b) **First Solution** (by *Pham Huu Duc*). Since

$$\frac{2(a^2-bc)}{\sqrt{2a^2+b^2+c^2}}=\sqrt{2a^2+b^2+c^2}-\frac{(b+c)^2}{\sqrt{2a^2+b^2+c^2}},$$

we may write the inequality as

$$\sum \sqrt{2a^2+b^2+c^2}\geq \sum \frac{(b+c)^2}{\sqrt{2a^2+b^2+c^2}}.$$

We will show that

$$\sum\sqrt{\frac{2a^2+b^2+c^2}{2}} \ge \sum\sqrt{b^2+c^2} \ge \sum\frac{(b+c)^2}{\sqrt{2(2a^2+b^2+c^2)}}.$$

Using the inequality $\sqrt{2(x+y)} \ge \sqrt{x}+\sqrt{y}$ yields

$$\sum\sqrt{\frac{2a^2+b^2+c^2}{2}} \ge \frac{1}{2}\sum\left(\sqrt{a^2+b^2}+\sqrt{a^2+c^2}\right) = \sum\sqrt{b^2+c^2}$$

Using again the inequality $\sqrt{2(x+y)} \ge \sqrt{x}+\sqrt{y}$ and then the Cauchy-Schwarz Inequality, we have

$$\sum\frac{(b+c)^2}{\sqrt{2(2a^2+b^2+c^2)}} \le \sum\frac{(b+c)^2}{\sqrt{a^2+b^2}+\sqrt{a^2+c^2}} \le$$

$$\le \sum\left(\frac{b^2}{\sqrt{a^2+b^2}}+\frac{c^2}{\sqrt{a^2+c^2}}\right) = \sum\left(\frac{c^2}{\sqrt{b^2+c^2}}+\frac{b^2}{\sqrt{c^2+b^2}}\right) =$$

$$= \sum\sqrt{b^2+c^2},$$

which completes the proof. Equality occurs if and only if $a=b=c$.

Second Solution Write the inequality as

$$\sum\frac{a^2-bc}{A} \ge 0,$$

where $A=\sqrt{2a^2+b^2+c^2}$, $B=\sqrt{2b^2+c^2+a^2}$ and $C=\sqrt{2c^2+a^2+b^2}$. We have

$$2\sum\frac{a^2-bc}{A} = \sum\frac{(a-b)(a+c)+(a-c)(a+b)}{A} =$$

$$= \sum\frac{(a-b)(a+c)}{A} + \sum\frac{(b-a)(b+c)}{B} =$$

$$= \sum(a-b)\left(\frac{a+c}{A}-\frac{b+c}{B}\right) =$$

$$= \sum\frac{a-b}{AB}\;\frac{(a+c)^2B^2-(b+c)^2A^2}{(a+c)B+(b+c)A} =$$

$$= \sum\frac{(a-b)^2}{AB}\cdot\frac{C_1}{(a+c)B+(b+c)A},$$

where

$$C_1 = a^3+b^3+2c^3+ab(a+b)+c(a^2+b^2 \;\; + c(a-b^{\;2}$$

Since $C_1 > 0$, the inequality is clearly true.

Remark We can prove that for $0 \le p \le 1 + 2\sqrt{2}$, the inequality holds

$$\frac{a^2 - bc}{\sqrt{pa^2 + b^2 + c^2}} + \frac{b^2 - ca}{\sqrt{pb^2 + c^2 + a^2}} + \frac{c^2 - ab}{\sqrt{pc^2 + a^2 + b^2}} \ge 0$$

Using the same method as above one, we get

$$C_1 = (a^2 + b^2 + c^2)(a + b + 2c) - (p - 1)c(2ab + bc + ca) \ge$$
$$\ge (a^2 + b^2 + c^2)(a + b + 2c) - 2\sqrt{2}c(2ab + bc + ca)$$

Let $a + b = 2x$. Since $a^2 + b^2 \ge 2x^2$ and $ab \le x^2$, it follows that

$$C_1 \ge (2x^2 + c^2)(2x + 2c) - 2\sqrt{2}c(2x^2 + 2cx) = 2(x + c)\left(x\sqrt{2} - c\right)^2 \ge 0$$

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18. *If a, b, c are the side lengths of an triangle, then*

$$\frac{a^2 - bc}{3a^2 + b^2 + c^2} + \frac{b^2 - ca}{3b^2 + c^2 + a^2} + \frac{c^2 - ab}{3b^2 + c^2 + a^2} \le 0$$

Solution. We have

$$2\sum \frac{a^2 - bc}{3a^2 + b^2 + c^2} = \sum \frac{(a - b)(a + c) + (a - c)(a + b)}{3a^2 + b^2 + c^2} =$$
$$= \sum \frac{(a - b)(a + c)}{3a^2 + b^2 + c^2} + \sum \frac{(b - a)(b + c)}{3b^2 + c^2 + a^2} =$$
$$= \sum (a - b)\left(\frac{a + c}{3a^2 + b^2 + c^2} - \frac{b + c}{3b^2 + c^2 + a^2}\right) =$$
$$= (a^2 + b^2 + c^2 - 2ab - 2bc - 2ca)\sum \frac{(a - b)^2}{(3a^2 + b^2 + c^2)(3b^2 + c^2 + a^2)}.$$

Since

$$a^2 + b^2 + c^2 - 2ab - 2bc - 2ca = a(a - b - c) + b(b - c - a) + c(c - a - b) < 0,$$

the conclusion follows. Equality occurs if and only if $a = b = c$

Remark We can also prove that in any triangle the inequality holds

$$\frac{a^4 - b^2c^2}{3a^4 + b^4 + c^4} + \frac{b^4 - c^2a^2}{3b^4 + c^4 + a^4} + \frac{c^4 - a^2b^2}{3b^4 + c^4 + a^4} \le 0$$

Using the same method as above, we get

$$2\sum \frac{a^4 - b^2c^2}{3a^4 + b^4 + c^4} = P\sum \frac{(a^2 - b^2)^2}{(3a^4 + b^4 + c^4)(3b^4 + c^4 + a^4)}$$

where

$$P = (a + b + c)(a + b - c)(b + c - a)(c + a - b) > 0$$

Chapter 8

Final problem set

8.1 Applications

19. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\sqrt{\frac{a+b}{b+1}} + \sqrt{\frac{b+c}{c+1}} + \sqrt{\frac{c+a}{a+1}} \geq 3.$$

(*Vasile Cîrtoaje*, MC, 2005)

20. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\sqrt{\frac{a}{b+3}} + \sqrt{\frac{b}{c+3}} + \sqrt{\frac{c}{a+3}} \geq \frac{3}{2}$$

(*Vasile Cîrtoaje*, MS, 2005)

21. Let a, b, c be non-negative numbers such that $a + b + c = 3$. Prove that

$$\frac{5-3bc}{1+a} + \frac{5-3ca}{1+b} + \frac{5-3ab}{1+c} \geq ab + bc + ca.$$

(*Vasile Cîrtoaje*, MS, 2005)

22. Let a, b, c, d be non-negative numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$(abc)^3 + (bcd)^3 + (cda)^3 + (dab)^3 \leq 4.$$

(*Vasile Cîrtoaje*, MS, 2004)

23. Let a, b, c be non-negative numbers, no two of which are zero Then,

$$\sqrt{\frac{a}{4a+5b}}+\sqrt{\frac{b}{4b+5c}}+\sqrt{\frac{c}{4c+5a}} \leq 1.$$

(*Vasile Cîrtoaje*, GM-A, 1, 2004)

24. Let $a_1, a_2, \ldots, a_n$ be positive numbers Prove that

$$(a) \qquad \frac{(a_1+a_2+\cdots+a_n)^2}{(a_1^2+1)(a_2^2+1)\cdots(a_n^2+1)} \leq \frac{(n-1)^{n-1}}{n^{n-2}},$$

$$(b) \qquad \frac{a_1+a_2+\cdots+a_n}{(a_1^2+1)(a_2^2+1)\cdots(a_n^2+1)} \leq \frac{(2n-1)^{n-\frac{1}{2}}}{2^n n^{n-1}}$$

(*Vasile Cîrtoaje*, GM-B, 6, 1994)

25. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Prove that

$$a_1b_1+\cdots+a_nb_n+\sqrt{(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)} \geq \frac{2}{n}(a_1+\cdots+a_n)(b_1+\cdots+b_n)$$

(*Vasile Cîrtoaje*, Kvant, 11, 1989)

26. Let k and n be positive integers with $k < n$, and let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_n$. Prove that

$$(a_1+a_2+\cdots+a_n)^2 \geq n(a_1a_{k+1}+a_2a_{k+2}+\cdots+a_na_k)$$

in the following cases:
(a) for $n = 2k$;
(b) for $n = 4k$.

(*Vasile Cîrtoaje*, CM, 5, 2005)

27. Let a, b, c, d be positive numbers such that $abcd = 1$. Prove that

$$\frac{1}{1+a+a^2+a^3}+\frac{1}{1+b+b^2+b^3}+\frac{1}{1+c+c^2+c^3}+\frac{1}{1+d+d^2+d^3} \geq 1.$$

(*Vasile Cîrtoaje*, GM-B, 11, 1999)

28. If a, b, c are non-negative numbers, then

$$9(a^4+1)(b^4+1)(c^4+1) \geq 8(a^2b^2c^2+abc+1)^2.$$

(*Vasile Cîrtoaje*, GM-B, 3, 2004)

29. If a, b, c, d are non-negative numbers, then

$$\frac{(1+a^3)(1+b^3)(1+c^3)(1+d^3)}{(1+a^2)(1+b^2)(1+c^2)(1+d^2)} \geq \frac{1+abcd}{2}$$

(*Vasile Cîrtoaje*, GM-B, 10, 2002)

30. Let a, b, c be non-negative numbers, no two of which are zero. Then,

$$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} \geq \frac{9}{(a+b+c)^2}.$$

(*Vasile Cîrtoaje*, GM-B, 9, 2000)

31. Let a, b, c be positive numbers, and let

$$x = a + \frac{1}{b} - 1, \; y = b + \frac{1}{c} - 1, \; z = c + \frac{1}{a} - 1.$$

Prove that

$$xy + yz + zx \geq 3$$

(*Vasile Cîrtoaje*, GM-B, 1, 1991)

32. Let a, b, c be positive numbers, no two of which are zero. If n is a positive integer, then

$$\frac{2a^n - b^n - c^n}{b^2 - bc + c^2} + \frac{2b^n - c^n - a^n}{c^2 - ca + a^2} + \frac{2c^n - a^n - b^n}{a^2 - ab + b^2} \geq 0$$

(*Vasile Cîrtoaje*, GM-B, 1, 2004)

33. Let $0 \leq a < b$ and let $a_1, a_2, \ldots, a_n \in [a, b]$. Prove that

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \ldots a_n} \leq (n-1)\left(\sqrt{b} - \sqrt{a}\right)^2$$

(*Vasile Cîrtoaje* and *Gabriel Dospinescu*, MS, 2005)

34. Let a, b, c and x, y, z be positive numbers such that $x + y + z = a + b + c$. Prove that

$$ax^2 + by^2 + cz^2 + xyz \geq 4abc$$

(*Vasile Cîrtoaje*, GM-A, 4, 1987)

35. Let a, b, c and x, y, z be positive numbers such that $x + y + z = a + b + c$. Prove that

$$\frac{x(3x+a)}{bc} + \frac{y(3y+a)}{ca} + \frac{z(3z+a)}{ab} \geq 12.$$

36. Let a, b, c be positive numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{9}{a+b+c}$$

37. Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 a_2 \ldots a_n = 1$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{4n}{n + a_1 + a_2 + \cdots + a_n} \geq n + 2.$$

(*Vasile Cîrtoaje*, MS, 2005)

38. Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 a_2 \ldots a_n = 1$. Prove that

$$a_1 + a_2 + \cdots + a_n - n + 1 \geq \sqrt[n-1]{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n + 1}$$

(*Vasile Cîrtoaje*, MS, 2006)

39. Let $r > 1$ and let a, b, c be non-negative numbers such that $ab + bc + ca = 3$. Prove that

$$a^r(b+c) + b^r(c+a) + c^r(a+b) \geq 6.$$

40. Let a, b, c be positive real numbers such that $abc \geq 1$ Prove that

(a) $\qquad a^{\frac{a}{b}} b^{\frac{b}{c}} c^{\frac{c}{a}} \geq 1;$

(b) $\qquad a^{\frac{a}{b}} b^{\frac{b}{c}} c^{c} \geq 1.$

(*Vasile Cîrtoaje*, CM, 4, 2005)

41. Let a, b, c, d be non-negative numbers. Prove that

$$4(a^3 + b^3 + c^3 + d^3) + 15(abc + bcd + cda + dab) \geq (a + b + c + d)^3.$$

42. Let a, b, c be positive numbers such that

$$(a + b - c)\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) = 4.$$

Prove that

$$(a^4 + b^4 + c^4)\left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4}\right) \geq 2304.$$

(*Vasile Cîrtoaje*, MC, 2005)

43. Let a, b, c be positive numbers Prove that

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab}>\frac{2}{ab+bc+ca}.$$

(*Vasile Cîrtoaje*, MS, 2005)

44. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab}\geq 1+\frac{ab+bc+ca}{a^2+b^2+c^2}$$

(*Vasile Cîrtoaje*, MS, 2006)

45. Let a, b, c be non-negative numbers, no two of which are zero. Then

$$\frac{(b+c)^2}{a^2+bc}+\frac{(c+a)^2}{b^2+ca}+\frac{(a+b)^2}{c^2+ab}\geq 6.$$

(*Peter Scholze* and *Darij Grinberg*, MS, 2005)

46. Let a, b, c be non-negative numbers, no two of which are zero Then

$$\frac{b+c}{2a^2+bc}+\frac{c+a}{2b^2+ca}+\frac{a+b}{2c^2+ab}\geq\frac{6}{a+b+c}$$

(*Vasile Cîrtoaje*, MS, 2006)

47. If a, b, c are non-negative numbers, then

$$a\sqrt{a^2+3bc}+b\sqrt{b^2+3ca}+c\sqrt{c^2+3ab}\geq 2(ab+bc+ca).$$

(*Vasile Cîrtoaje*, MS, 2005)

48. Let a, b, c be non-negative numbers, no two of which are zero. Then

$$\frac{a^2-bc}{\sqrt{a^2+bc}}+\frac{b^2-ca}{\sqrt{b^2+ca}}+\frac{c^2-ab}{\sqrt{c^2+ab}}\geq 0$$

(*Vasile Cîrtoaje*, MS, 2005)

49. If a, b, c are non-negative numbers, then

$$(a^2-bc)\sqrt{a^2+4bc}+(b^2-ca)\sqrt{b^2+4ca}+(c^2-ab)\sqrt{c^2+4ab}\geq 0$$

(*Vasile Cîrtoaje*, MS, 2005)

50. If a, b, c are positive numbers, then

$$\frac{a^2-bc}{\sqrt{8a^2+(b+c)^2}}+\frac{b^2-ca}{\sqrt{8b^2+(c+a)^2}}+\frac{c^2-ab}{\sqrt{8c^2+(a+b)^2}}\geq 0.$$

(*Vasile Cîrtoaje*, MS, 2006)

51. If a, b, c are non-negative numbers, then

$$\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab}\leq \frac{3}{2}(a+b+c)$$

(*Pham Kim Hung*, MS, 2005)

52. Let a, b, c be non-negative numbers such that $a^2+b^2+c^2=3$. Then,

$$21+18abc\geq 13(ab+bc+ca)$$

(*Vasile Cîrtoaje*, MS, 2005)

53. Let a, b, c be non-negative numbers such that $a^2+b^2+c^2=3$. Then

$$\frac{1}{5-2ab}+\frac{1}{5-2bc}+\frac{1}{5-2ca}\leq 1$$

(*Vasile Cîrtoaje*, MS, 2005)

54. Let a, b, c be non-negative numbers such that $a^2+b^2+c^2=3$ Then,

$$(2-ab)(2-bc)(2-ca)\geq 1.$$

(*Vasile Cîrtoaje*, MS, 2005)

55. Let a, b, c be non-negative numbers such that $a+b+c=2$ Prove that

$$\frac{bc}{a^2+1}+\frac{ca}{b^2+1}+\frac{ab}{c^2+1}\leq 1.$$

(*Pham Kim Hung*, MS, 2005)

56. Let a, b, c be non-negative numbers, no two of which are zero. Then,

$$\frac{a^3+3abc}{(b+c)^2}+\frac{b^3+3abc}{(c+a)^2}+\frac{c^3+3abc}{(a+b)^2}\geq a+b+c$$

Vasile Cîrtoaje, MS, 2005)

57. Let a, b, c be positive numbers such that $a^4+b^4+c^4=3$. Then,

$$a)\qquad \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge 3;$$

$$b)\qquad \frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\ge \frac{3}{2}.$$

(*Alexey Gladkich*, MS, 2005)

58. If a, b, c are positive numbers, then

$$\frac{a^3-b^3}{a+b}+\frac{b^3-c^3}{b+c}+\frac{c^3-a^3}{c+a}\le \frac{(a-b)^2+(b-c)^2+(c-a)^2}{8}.$$

(*Marian Tetiva* and *Darij Grinberg*, MS, 2005)

59. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{a^2}{(2a+b)(2a+c)}+\frac{b^2}{(2b+c)(2b+a)}+\frac{c^2}{(2c+a)(2c+b)}\le \frac{1}{3}.$$

(*Tigran Sloyan*, MS, 2005)

60. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{1}{5(a^2+b^2)-ab}+\frac{1}{5(b^2+c^2)-bc}+\frac{1}{5(c^2+a^2)-ca}\ge \frac{1}{a^2+b^2+c^2}.$$

(*Vasile Cîrtoaje*, MS, 2006)

61. Let a, b, c be non-negative real numbers such that $a^2+b^2+c^2=1$. Prove that

$$\frac{bc}{a^2+1}+\frac{ca}{b^2+1}+\frac{ab}{c^2+1}\le \frac{3}{4}.$$

(*Pham Kim Hung*, MS, 2005)

62. Let a, b, c be non-negative numbers such that $a^2+b^2+c^2=1$. Prove that

$$\frac{1}{3+a^2-2bc}+\frac{1}{3+b^2-2ca}+\frac{1}{3+c^2-2ab}\le \frac{9}{8}.$$

(*Vasile Cîrtoaje* and *Wolfgang Berndt*, MS, 2006)

63. If a, b, c are positive numbers, then

$$\frac{4a^2-b^2-c^2}{a(b+c)}+\frac{4b^2-c^2-a^2}{b(c+a)}+\frac{4c^2-a^2-b^2}{c(a+b)}\le 3$$

(*Vasile Cîrtoaje*, MS, 2006)

64. If a, b, c are positive numbers such that $abc=1$, then

$$a^2+b^2+c^2+6\ge\frac{3}{2}\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

(*Vasile Cîrtoaje*, MS, 2006)

65. Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1+a_2+\cdots+a_n=n$ Prove that

$$a_1a_2\ldots a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n+3\right)\le 3.$$

(*Vasile Cîrtoaje*, MS, 2004)

66. Let a, b, c be the side lengths of a triangle If $a^2+b^2+c^2=3$, then

$$ab+bc+ca\ge 1+2abc.$$

(*Vasile Cîrtoaje*, MS, 2005)

67. Let a, b, c be the side lengths of a triangle. If $a^2+b^2+c^2=3$, then

$$a+b+c\ge 2+abc$$

(*Vasile Cîrtoaje*, MS, 2005)

68. If a, b, c are the side lengths of a non-isosceles triangle, then

$$a)\qquad \left|\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}\right|>5;$$

$$b)\qquad \left|\frac{a^2+b^2}{a^2-b^2}+\frac{b^2+c^2}{b^2-c^2}+\frac{c^2+a^2}{c^2-a^2}\right|>3.$$

(*Vasile Cîrtoaje*, GM-B, 3, 2003)

69. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$a^2\left(\frac{b}{c}-1\right)+b^2\left(\frac{c}{a}-1\right)+c^2\left(\frac{a}{b}-1\right)\ge 0.$$

(*Vasile Cîrtoaje*, Moldova TST, 2006)

70. Let a, b, c be the lengths of the sides of an triangle. Prove that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 6\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

(Vietnam TST, 2006)

71. If $a_1, a_2, a_3, a_4, a_5, a_6 \in \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$, then

$$\frac{a_1-a_2}{a_2+a_3}+\frac{a_2-a_3}{a_3+a_4}+\cdots+\frac{a_6-a_1}{a_1+a_2} \geq 0.$$

(*Vasile Cîrtoaje*, AJ, 7-8, 2002)

72. Let a, b, c be positive numbers such that $a^2+b^2+c^2 \geq 3$ Prove that

$$\frac{a^5-a^2}{a^5+b^2+c^2}+\frac{b^5-b^2}{a^2+b^5+c^2}+\frac{c^5-c^2}{a^2+b^2+c^5} \geq 0.$$

(*Vasile Cîrtoaje*, MS, 2005)

73. Let a, b, c be positive numbers such that $x+y+z \geq 3$ Then,

$$\frac{1}{x^3+y+z}+\frac{1}{x+y^3+z}+\frac{1}{x+y+z^3} \leq 1.$$

(*Vasile Cîrtoaje*, MS, 2005)

74. Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1 x_2 \ldots x_n \geq 1$. If $\alpha > 1$, then

$$\sum \frac{x_1^\alpha}{x_1^\alpha+x_2+\cdots+x_n} \geq 1$$

(*Vasile Cîrtoaje*, CM, 2, 2006)

75. Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1 x_2 \ldots x_n \geq 1$. If $n \geq 3$ and $\frac{-2}{n-2} \leq \alpha < 1$, then

$$\sum \frac{x_1^\alpha}{x_1^\alpha+x_2+\cdots+x_n} \leq 1.$$

(*Vasile Cîrtoaje*, CM, 2, 2006)

76. Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1 x_2 \ldots x_n \geq 1$ If $\alpha > 1$, then

$$\sum \frac{x_1}{x_1^\alpha + x_2 + \cdots + x_n} \leq 1.$$

(*Vasile Cîrtoaje*, CM, 2, 2006)

77. Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1 x_2 \ldots x_n \geq 1$. If $-1 - \dfrac{2}{n-2} \leq \alpha < 1$, then

$$\sum \frac{x_1}{x_1^\alpha + x_2 + \cdots + x_n} \geq 1$$

(*Vasile Cîrtoaje*, CM, 2, 2006)

78. Let $n \geq 3$ be an integer and let p be a real number such that $1 < p < n-1$ If $0 < x_1, x_2, \ldots, x_n \leq \dfrac{pn - p - 1}{p(n - p - 1)}$ such that $x_1 x_2 \ldots x_n = 1$, then

$$\frac{1}{1 + px_1} + \frac{1}{1 + px_2} + \cdots + \frac{1}{1 + px_n} \geq \frac{n}{1 + p}.$$

(*Vasile Cîrtoaje*, GM-A, 1, 2005)

79. Let a, b, c be positive numbers such that $abc = 1$ Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{2}{(1+a)(1+b)(1+c)} \geq 1.$$

(*Pham Van Thuan*, MS, 2006)

80. Let a, b, c be positive numbers such that $abc = 1$ Prove that

$$a^2 + b^2 + c^2 + 9(ab + bc + ca) \geq 10(a + b + c)$$

81. Let a, b, c be non-negative numbers such that $ab + bc + ca = 3$ Prove that

$$\frac{a(b^2 + c^2)}{a^2 + bc} + \frac{b(c^2 + a^2)}{b^2 + ca} + \frac{c(a^2 + b^2)}{c^2 + ab} \geq 3.$$

(*Pham Huu Duc*, MS, 2006)

82. If a, b, c are positive numbers, then

$$a + b + c + \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{6(a^2 + b^2 + c^2)}{a + b + c}$$

(*Pham Huu Duc*, MS, 2006

83. If a, b, c are positive numbers, then

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{3(a^3+b^3+c^3)}{2(a^2+b^2+c^2)}$$

(*Pham Huu Duc*, MS, 2006)

84. If a, b, c are given non-negative numbers, find the minimum value $E(a, b, c$ of the expression

$$E=\frac{ax}{y+z}+\frac{by}{z+x}+\frac{cz}{x+y}$$

for any positive numbers x, y, z.

(*Vasile Cîrtoaje*, MS, 2006)

85. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq a^2+b^2+c^2.$$

(*Vasile Cîrtoaje*, Romania TST, 2006)

86. Let a, b, c be non-negative real numbers such that $a+b+c=3$. Prove that

$$(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)\leq 12.$$

(*Pham Kim Hung*, MS, 2006)

87. Let a, b, c be non-negative real numbers such that $a+b+c=1$. Prove that

$$\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}\geq 2$$

(*Phan Thanh Nam*)

88. If a, b, c are non-negative real numbers, then

$$a^3+b^3+c^3+3abc\geq\sum bc\sqrt{2(b^2+c^2)}.$$

89. If a, b, c are non-negative real numbers, then

$$(1+a^2)(1+b^2)(1+c^2)\geq\frac{15}{16}(1+a+b+c)^2.$$

(*Vasile Cîrtoaje*, MS, 2006)

90. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2) \geq (a+b+c+d)^2.$$

(*Pham Kim Hung*, MS, 2006)

91. If $x_1, x_2, \ldots, x_n$ are non-negative numbers, then

$$x_1 + x_2 + \cdots + x_n \geq (n-1)\sqrt[n]{x_1 x_2 \cdots x_n} + \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}.$$

(*Vasile Cîrtoaje*, MS, 2006)

92. If k is a real number and $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$(n-1)\left(x_1^{n+k} + x_2^{n+k} + \cdots + x_n^{n+k}\right) + x_1 x_2 \ldots x_n \left(x_1^k + x_2^k + \cdots + x_n^k\right) \geq$$
$$\geq (x_1 + x_2 + \cdots + x_n)\left(x_1^{n+k-1} + x_2^{n+k-1} + \cdots + x_n^{n+k-1}\right).$$

(*Gjergji Zaimi* and *Keler Marku*, MS, 2006

93. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{a^4}{a^3+b^3} + \frac{b^4}{b^3+c^3} + \frac{c^4}{c^3+a^3} \geq \frac{a+b+c}{2}.$$

8.2 Solutions

1. *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\sqrt{\frac{a+b}{b+1}} + \sqrt{\frac{b+c}{c+1}} + \sqrt{\frac{c+a}{a+1}} \geq 3.$$

Solution. By AM-GM Inequality, it follows that

$$\sqrt{\frac{a+b}{b+1}} + \sqrt{\frac{b+c}{c+1}} + \sqrt{\frac{c+a}{a+1}} \geq 3\sqrt[6]{\frac{(a+b)(b+c)(c+a)}{(b+1)(c+1)(a+1)}}$$

Thus, we still have to show that

$$(a+b)(b+c)(c+a) \geq (a+1)(b+1)(c+1)$$

Let $A = a+b+c$ and $B = ab+bc+ca$. The AM-GM Inequality yields $A \geq 3$ and $B \geq 3$ Since

$$(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc = AB - 1$$

and

$$(a+1)(b+1)(c+1) = A + B + 2,$$

we have

$$(a+b)(b+c)(c+1) - (a+1)(b+1)(c+1) =$$
$$= AB - A - B - 3 = (A-1)(B-1) - 4 \geq 2 \cdot 2 - 4 = 0$$

Equality occurs for $a = b = c = 1$.

Remark The inequality holds for the extended condition

$$ab + bc + ca \geq 3$$

Letting $a = tx$, $b = ty$ and $c = tz$, where $t > 0$ and $x, y, z \geq 0$ such that $xy + yz + zx = 3$, the inequality

$$(a+b)(b+c)(c+a) \geq (a+1)(b+1)(c+1)$$

becomes

$$(x+y)(y+z)(z+x) \geq \left(x + \frac{1}{t}\right)\left(y + \frac{1}{t}\right)\left(z + \frac{1}{t}\right).$$

From $ab+bc+ca \geq 3$ we get $t \geq 1$. It is easy to see that it suffices to consider only the case $t = 1$, which is equivalent to the condition $ab + bc + ca = 3$ In this case, from

$$(a+b+c)^2 \geq 3(ab+bc+ca)$$

we get $a + b + c \geq 3$, and from $ab + bc + ca \geq 3\sqrt{a^2b^2c^2}$ we get $abc \leq 3$. Finally,

$$(a+b)(b+c)(c+a) - (a+1)(b+1)(c+1) =$$
$$= (ab+bc+ca-1)(a+b+c-1) - 2(1+abc) =$$
$$= 2(a+b+c-3) + 2(1-abc) \geq 0$$

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2. *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\sqrt{\frac{a}{b+3}} + \sqrt{\frac{b}{c+3}} + \sqrt{\frac{c}{a+3}} \geq \frac{3}{2}.$$

Solution. Setting $a = \frac{x}{y}$, $b = \frac{z}{x}$, $c = \frac{y}{z}$, the inequality becomes

$$\frac{x}{\sqrt{y(3x+z)}} + \frac{y}{\sqrt{z(3y+x)}} + \frac{z}{\sqrt{x(3z+y)}} \geq \frac{3}{2}$$

By Jensen's Inequality applied to the convex function $f(t) = \frac{1}{\sqrt{t}}$, we get

$$\frac{x}{\sqrt{y(3x+z)}} + \frac{y}{\sqrt{z(3y+x)}} + \frac{z}{\sqrt{x(3z+y)}} \geq$$
$$\geq (x+y+z)\sqrt{\frac{x+y+z}{xy(3x+z)+yz(3y+x)+zx(3z+y)}}$$

Using this result, it is enough to show that

$$4(x+y+z)^3 \geq 27(x^2y + y^2z + z^2x + xyz).$$

Let $x = \min\{x, y, z\}$ Denoting $y = x + p$, $z = x + q$ $(p, q \geq 0)$, the inequality transforms into

$$9(p^2 - pq + q^2) + (a-2b)^2(4a+b) \geq 0,$$

which is clearly true. Equality occurs only for $a = b = c = 1$

★

3. *Let a, b, c be non-negative numbers such that $a + b + c = 3$. Prove that*

$$\frac{5-3bc}{1+a} + \frac{5-3ca}{1+b} + \frac{5-3ab}{1+c} \geq ab + bc + ca.$$

Solution. Let $s = ab + bc + ca$. The well-known inequality

$$(a+b+c)^2 \geq 3(ab+bc+ca)$$

implies $s \leq 3$. We write now the inequality as follows:

$$\left(\frac{5-3bc}{1+a} - bc\right) + \left(\frac{5-3ca}{1+b} - ca\right) + \left(\frac{5-3ab}{1+c} - ab\right) \geq 0,$$

$$\frac{5-4bc-abc}{1+a}+\frac{5-4ca-abc}{1+b}+\frac{5-4ab-abc}{1+c}\geq 0,$$
$$\sum(1+b)(1+c)(5-4bc-abc)\geq 0,$$
$$\sum(4-a+bc)(5-4bc-abc)\geq 0,$$
$$45+3abc\geq 11\sum bc+4\sum b^2c^2+abc\sum bc,$$
$$45+27abc\geq 11s+4s^2+abcs.$$

Since $s\leq 3$, it suffices to show that

$$45+24abc\geq 11s+4s^2.$$

For $s<\frac{9}{4}$, we have

$$45+24abc-11s-4s^2\geq 45-11s-4s^2>45-\frac{99}{4}-\frac{81}{4}=0.$$

Consider now $\frac{9}{4}\leq s\leq 3$. By Schur's Inequality

$$(a+b+c)^3+9abc\geq 4(a+b+c)(ab+bc+ca),$$

it follows that $9+3abc\geq 4s$. Then,

$$45+24abc-11s-4s^2\geq 45+8(4s-9)-11s-4s^2=$$
$$=21s-27-4s^2=(3-s)(4s-9)\geq 0,$$

which completes the proof. Equality occurs for $(a,b,c)=(1,1,1)$ and also for $(a,b,c)=\left(0,\frac{3}{2},\frac{3}{2}\right)$ or any cyclic permutation

★

4. *Let a,b,c,d be non-negative numbers such that $a^2+b^2+c^2+d^2=4$. Prove that*

$$(abc)^3+(bcd)^3+(cda)^3+(dab)^3\leq 4.$$

Solution. Setting $x=a^2$, $y=b^2$, $z=c^2$ and $t=d^2$, the inequality becomes

$$(xyz)^{3/2}+(yzt)^{3/2}+(ztx)^{3/2}+(txy)^{3/2}\leq 4,$$

where x, y, z and t are positive numbers such that $x+y+z+t=4$. By AM-GM Inequality, we have

$$1+x+y+z \geq 4\sqrt[4]{xyz}.$$

Thus,

$$\sqrt{xyz} \leq \left(\frac{1+x+y+z}{4}\right)^2 = \left(\frac{5-t}{4}\right)^2, \ (xyz)^{3/2} \leq \left(\frac{5-t}{4}\right)^2 xyz.$$

Analogously,

$$(yzt)^{3/2} \leq \left(\frac{5-x}{4}\right)^2 yzt, \ (ztx)^{3/2} \leq \left(\frac{5-y}{4}\right)^2 ztx, \ (txy)^{3/2} \leq \left(\frac{5-z}{4}\right)^2 txy.$$

Taking account of these inequalities, it suffices to show that

$$\left(\frac{5-t}{4}\right)^2 xyz + \left(\frac{5-x}{4}\right)^2 yzt + \left(\frac{5-y}{4}\right)^2 ztx + \left(\frac{5-z}{4}\right)^2 txy \leq 4$$

This inequality is equivalent to $E(x,y,z,t) \leq 0$, where

$$E(x,y,z,t) = 25(xyz+yzt+ztx+txy) - 64 - 36xyzt.$$

Without loss of generality, we may assume that $x \geq y \geq z \geq t$. We will show that E is maximal for $x=z$, and hence for $x=y=z$ To prove this, it is enough to show that $x>z$ implies

$$E(x,y,z,t) < E\left(\frac{x+z}{2}, y, \frac{x+z}{2}, t\right).$$

Indeed,

$$E\left(\frac{x+z}{2}, y, \frac{x+z}{2}, t\right) - E(x,y,z,t) =$$
$$= \frac{25(y+t)-36yt}{4}(x-z)^2 \geq \frac{25(y+t)-9(y+t)^2}{4}(x-z)^2 =$$
$$= \frac{(y+t)[25-9(y+t)]}{4}(x-z)^2 > 0,$$

since

$$y+t = \frac{y+t}{2} + \frac{y+t}{2} \leq \frac{y+t}{2} + \frac{x+z}{2} = 2$$

We need now to show that $E(x,y,z,t) \le 0$ for $x = y = z \le \frac{4}{3}$. We have

$$E(x,x,x,4-3x) = 4(27x^4 - 86x^3 + 75x^2 - 16) =$$
$$= 4(x-1)^2(27x^2 - 32x - 16) \le 0,$$

since $27x^2 - 32x - 16 = 9(3x-4) + 4(x-4) < 0$. This completes the proof Equality occurs for $a = b = c = d = 1$

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5. *Let a, b, c be non-negative numbers, no two of which are zero. Then,*

$$\sqrt{\frac{a}{4a+5b}} + \sqrt{\frac{b}{4b+5c}} + \sqrt{\frac{c}{4c+5a}} \le 1.$$

Solution. If one of a, b, c is zero, the inequality is clearly true. Otherwise, setting $x = \frac{b}{a}$, $y = \frac{c}{b}$ and $z = \frac{a}{c}$ (such that $xyz = 1$), the inequality becomes

$$\frac{1}{\sqrt{4+5x}} + \frac{1}{\sqrt{4+5y}} + \frac{1}{\sqrt{4+5z}} \le 1.$$

Assume now that $x \ge y \ge z$. The condition $xyz = 1$ yields $x \ge 1$ and $yz \le 1$. We may obtain the inequality by adding up the inequalities

$$\frac{1}{\sqrt{4+5y}} + \frac{1}{\sqrt{4+5z}} \le \frac{2}{\sqrt{4+5\sqrt{yz}}},$$
$$\frac{1}{\sqrt{4+5x}} + \frac{2}{\sqrt{4+5\sqrt{yz}}} \le 1.$$

The former inequality is satisfied as equality for $y = z$. For $y > z$, let us denote $s = \frac{y+z}{2}$ and $p = \sqrt{yz} (s > p, p \le 1)$. Squaring and dividing then by $\frac{10(s-p)}{(4+5p)\sqrt{(4+5y)(4+5z)}}$, the inequality becomes successively

$$\frac{1}{4+5y} + \frac{1}{4+5z} - \frac{2}{4+5p} \le \frac{2}{4+5p} - \frac{2}{\sqrt{(4+5y)(4+5z)}},$$
$$\frac{5p-4}{\sqrt{(4+5y)(4+5z)}} \le \frac{8}{4+5p+\sqrt{(4+5y)(4+5z)}},$$
$$25p^2 - 16 \le (12-5p)\sqrt{25p^2 + 40s + 16}$$

The last inequality is true because

$$(12-5p)\sqrt{25p^2+40s+16}-25p^2+16>$$
$$>(12-5p)\sqrt{25p^2+40p+16}-25p^2+16=2(8-5p)(5p+4)>0$$

To prove the latter inequality,

$$\frac{1}{\sqrt{4+5x}}+\frac{2}{\sqrt{4+5\sqrt{yz}}}\le 1,$$

let $\sqrt{4+5\sqrt{yz}}=3t,\ \frac{2}{3}<t\le 1$ Since

$$x=\frac{1}{yz}=\frac{25}{(9t^2-4)^2},$$

the inequality becomes

$$\frac{9t^2-4}{3\sqrt{36t^4-32t^2+21}}+\frac{2}{3t}\le 1,$$
$$(2-3t)\left(\sqrt{36t^4-32t^2+21}-3t^2-2t\right)\le 0.$$

Since $2-3t<0$, we still have to show that $\sqrt{36t^4-32t^2+21}\ge 3t^2+2t$ By squaring, we get

$$9t^4-4t^3-12t^2+7\ge 0.$$

This inequality is equivalent to

$$(t-1)^2(9t^2+14t+7)\ge 0,$$

which is clearly true. Equality in the given inequality occurs if and only if $a=b=c$.

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6. *Let $a_1,a_2,\ldots,a_n$ be positive numbers. Prove that*

a) $$\frac{(a_1+a_2+\cdots+a_n)^2}{(a_1^2+1)(a_2^2+1)\cdots(a_n^2+1)}\le\frac{(n-1)^{n-1}}{n^{n-2}},$$

b) $$\frac{a_1+a_2+\cdots+a_n}{(a_1^2+1)(a_2^2+1)\cdots(a_n^2+1)}\le\frac{(2n-1)^{n-\frac{1}{2}}}{2^n n^{n-1}}$$

Solution. (by *Gabriel Dospinescu*) a) Let m be a positive integer $(m \geq n)$, and let $a_i = \dfrac{x_i}{\sqrt{m-1}}$ for all i Assume that $x_1 \leq \cdots \leq x_k \leq 1 \leq x_{k+1} \leq \cdots \leq x_n$. By Bernoulli's Inequality we have

$$\left(\frac{m-1}{m}\right)^n \prod_{i=1}^{n}\left(a_i^2+1\right) = \left(\frac{m-1}{m}\right)^n \prod_{i=1}^{n}\left(\frac{x_i^2}{m-1}+1\right) =$$

$$= \prod_{i=1}^{k}\left(1+\frac{x_i^2-1}{m}\right) = \prod_{i=1}^{k}\left(1+\frac{x_i^2-1}{m}\right)\prod_{i=k+1}^{n}\left(1+\frac{x_i^2-1}{m}\right) \geq$$

$$\geq \left(1+\sum_{i=1}^{k}\frac{x_i^2-1}{m}\right)\left(1+\sum_{i=k+1}^{n}\frac{x_i^2-1}{m}\right) =$$

$$= \frac{1}{m^2}\left(x_1^2+\cdots+x_k^2+m-k\right)\left(m+k-n+x_{k+1}^n+\cdots+x_n^2\right).$$

Applying now the Cauchy-Schwarz Inequality to the m-tuples

$$(x_1, \ldots, x_k, 1, \ldots, 1) \text{ and } (1, \ldots, 1, x_{k+1}, \ldots, x_n),$$

we get

$$\left(x_1^2+\cdots+x_k^2+m-k\right)\left(m+k-n+x_{k+1}^2+\cdots+x_n^2\right) \geq$$
$$\geq (x_1+\cdots+x_k+m-n+x_{k+1}+\cdots+x_n)^2 =$$
$$= (m-n+x_1+x_2+\cdots+x_n)^2,$$

and hence

$$\left(\frac{m-1}{m}\right)^n \prod_{i=1}^{n}\left(a_i^2+1\right) \geq \frac{1}{m^2}(m-n+x_1+x_2+\cdots+x_n)^2,$$

or

$$\prod_{i=1}^{n}\left(a_i^2+1\right) \geq \frac{m^{n-2}}{(m-1)^{n-1}}\left(\frac{m-n}{\sqrt{m-1}}+a_1+a_2+\cdots+a_n\right)^2.$$

Equality occurs when $a_1 = a_2 = \cdots = a_n = \dfrac{1}{\sqrt{m-1}}$.

a) Choosing $m = n$, we get the desired inequality. Equality occurs for

$$a_1 = a_2 = \cdots = a_n = \frac{1}{\sqrt{n-1}}.$$

b) Since

$$\left(\frac{m-n}{\sqrt{m-1}}+a_1+a_2+\cdots+a_n\right)^2 \geq \frac{4(m-n)}{\sqrt{m-1}}(a_1+a_2+\cdots+a_n),$$

we get

$$\prod_{i=1}^{n}\left(a_i^2+1\right) \geq \frac{4m^{n-2}(m-n)}{(m-1)^{n-\frac{1}{2}}}\left(a_1+a_2+\cdots+a_n\right).$$

Choosing $m = 2n$, we get the required inequality Equality occurs for

$$a_1 = a_2 = \cdots = a_n = \frac{1}{\sqrt{2n-1}}.$$

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7. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Prove that*

$$a_1b_1+\cdots+a_nb_n+\sqrt{(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)} \geq \frac{2}{n}(a_1+\cdots+a_n)(b_1+\cdots+b_n)$$

First Solution. Write the inequality as

$$\sqrt{\left(a_1^2+\cdots+a_n^2\right)\left(b_1^2+\cdots+b_n^2\right)} \geq a_1(2b-b_1)+\cdots+a_n(2b-b_n),$$

where $b = \dfrac{b_1+b_2+\cdots+b_n}{n}$ Setting now $x_i = 2b - b_i$ for all indices i, we have

$$\sum x_i^2 = \sum\left(4b^2-4bb_i+b_i^2\right) = 4nb^2 - 4b\sum b_i + \sum b_i^2 = \sum b_i^2,$$

and the inequality reduces to

$$\sqrt{\left(a_1^2+\cdots+a_n^2\right)\left(x_1^2+\cdots+x_n^2\right)} \geq a_1x_1+\cdots+a_nx_n,$$

which is just the Cauchy-Schwarz Inequality. This completes the proof In the case $a_1a_2\cdots a_n \neq 0$, equality occurs for

$$\frac{2b-b_1}{a_1} = \frac{2b-b_2}{a_2} = \cdots = \frac{2b-b_n}{a_n} \geq 0.$$

Second Solution Consider the nontrivial case $a_1^2+a_2^2+\cdots+a_n^2 > 0$, denote

$$x = \sqrt{\frac{b_1^2+b_2^2+\cdots+b_n^2}{a_1^2+a_2^2+\cdots+a_n^2}}$$

and use the substitution $b_i = xx_i$ for all i We see that

$$a_1^2+a_2^2+\cdots+a_n^2 = x_1^2+x_2^2+\cdots+x_n^2.$$

The inequality becomes

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + a_1^2 + a_2^2 + \cdots + a_n^2 \geq$$
$$\geq \frac{2}{n}(a_1 + a_2 + \cdots + a_n)(x_1 + x_2 + \cdots + x_n),$$

or

$$(a_1 + x_1)^2 + (a_2 + x_2)^2 + \cdots + (a_n + x_n)^2 \geq$$
$$\geq \frac{4}{n}(a_1 + a_2 + \cdots + a_n)(x_1 + x_2 + \cdots + x_n).$$

Since

$$[(a_1 + a_2 + \cdots + a_n) + (x_1 + x_2 + \cdots + x_n)]^2 \geq$$
$$\geq 4(a_1 + a_2 + \cdots + a_n)(x_1 + x_2 + \cdots + x_n),$$

it suffices to show that

$$(a_1 + x_1)^2 + (a_2 + x_2)^2 + \cdots + (a_n + x_n)^2 \geq$$
$$\geq \frac{1}{n}[(a_1 + a_2 + \cdots + a_n) + (x_1 + x_2 + \cdots + x_n)]^2.$$

This one reduces to the well-known inequality

$$n\left(y_1^2 + y_2^2 + \cdots + y_n^2\right) \geq (y_1 + y_2 + \cdots + y_n)^2,$$

where $y_i = a_i + x_i$ for all i.

Remark Setting $b_i = \frac{1}{a_i}$ for all i, we get the following inequality

$$n^2 + n\sqrt{\left(a_1^2 + +a_2^2 + \cdots + a_n^2\right)\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}\right)} \geq$$
$$\geq 2(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

For even n $(n = 2k)$ and $a_1 \leq a_2 \leq \cdots \leq a_n$, equality occurs when

$$a_1 = a_2 = \cdots = a_k \text{ and } a_{k+1} = a_{k+2} = \cdots = a_{2k}.$$

For odd n, equality occurs only when $a_1 = a_2 = \cdots = a_n$. We conjecture that for odd n, the following stronger inequality holds

$$n^2 + 1 + \sqrt{(n^2 - 1)\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \ldots \frac{1}{a_n^2}\right) - n^2 + 1} \geq$$
$$\geq 2(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

If $a_1 \leq a_2 \leq \cdots \leq a_n$ and $n = 2k+1$, equality occurs for either

$$a_1 = a_2 = \cdots = a_k \text{ and } a_{k+1} = a_{k+2} = \cdots = a_{2k+1},$$

or

$$a_1 = a_2 = \cdots = a_{k+1} \text{ and } a_{k+2} = a_{k+3} = \cdots = a_{2k+1}.$$

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8. *Let k and n be positive integers with $k < n$, and let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_n$. Prove that*

$$(a_1 + a_2 + \cdots + a_n)^2 \geq n(a_1 a_{k+1} + a_2 a_{k+2} + \cdots + a_n a_k)$$

in the following cases:

a) for $n = 2k$;

b) for $n = 4k$.

Solution. a) We have to prove that

$$(a_1 + a_2 + \cdots + a_{2k}) \geq 4k(a_1 a_{k+1} + a_2 a_{k+2} + \cdots + a_k a_{2k}).$$

Let x be a real number such that $a_k \leq x \leq a_{k+1}$. We have

$$(x - a_1)(a_{k+1} - x) + (x - a_2)(a_{k+2} - x) + \cdots + (x - a_k)(a_{2k} - x) \geq 0$$

This can be manipulated into

$$4kx(a_1 + a_2 + \cdots + a_{2k}) \geq 4k^2x^2 + 4k(a_1 a_{k+1} + a_2 a_{k+2} + \cdots + a_k a_{2k}).$$

Summing this inequality to

$$(a_1 + a_2 + \cdots + a_{2k})^2 + 4k^2x^2 \geq 4kx(a_1 + a_2 + \cdots + a_{2k})$$

yields the desired inequality. Equality occurs for

$$a_{j+1} = a_{j+2} = \cdots = a_{j+k} = \frac{a_1 + a_2 + \cdots + a_{2k}}{2k},$$

where $j \in \{1, 2, \ldots, k-1\}$.

(b) We have to prove that

$$(a_1 + a_2 + \cdots + a_{4k}{}^2 \geq 4k(a_1 a_{k+1} + a_2 a_{k+2} + \cdots + a_{4k} a_k$$

The inequality is equivalent to

$$(b_1 + b_2 + \cdots + b_{2k})^2 \geq 4k(b_1 b_{k+1} + b_2 b_{k+2} + \cdots + b_k b_{2k}),$$

where $b_i = a_i + a_{2k+i}$ for $1 \leq i \leq 2k$. Since $b_1 \leq b_2 \leq \cdots \leq b_{2k}$, this is just the preceding inequality. Equality occurs for

$$\begin{cases} a_{j+1} = a_{j+2} = \cdots = a_{j+k} = a \\ a_{j+2k+1} = a_{j+2k+2} = \cdots = a_{j+3k} = b, \\ a_1 + a_2 + \cdots + a_{4k} = 2k(a+b) \end{cases}$$

where $a \leq b$ are real numbers and $j \in \{1, 2, \ldots, k-1\}$

Remark Actually, the inequality is valid in the more general case $2 \leq \frac{n}{k} \leq 4$.

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9. *Let a, b, c, d be positive numbers such that $abcd = 1$. Prove that*

$$\frac{1}{1+a+a^2+a^3} + \frac{1}{1+b+b^2+b^3} + \frac{1}{1+c+c^2+c^3} + \frac{1}{1+d+d^2+d^3} \geq 1.$$

Solution. The inequality can be obtained by summing the inequalities

$$\frac{1}{1+a+a^2+a^3} + \frac{1}{1+b+b^2+b^3} \geq \frac{1}{1+(ab)^{3/2}},$$

$$\frac{1}{1+c+c^2+c^3} + \frac{1}{1+d+d^2+d^3} \geq \frac{1}{1+(cd)^{3/2}} = \frac{(ab)^{3/2}}{1+(ab)^{3/2}}$$

Each of these inequalities is of the type

$$\frac{1}{1+x^2+x^4+x^6} + \frac{1}{1+y^2+y^4+y^6} \geq \frac{1}{1+x^3y^3},$$

where x and y are positive numbers. Using the substitutions $p = xy$ and $s = x^2 + xy + y^2$ ($s \geq 3p$), the inequality becomes as follows:

$$p^3(x^6+y^6)+p^2(p-1)(x^4+y^4)-p^2(p^2-p+1)(x^2+y^2)-p^6-p^4+2p^3-p^2+1 \geq 0$$

$$p^3(x^3-y^3)^2 + p^2(p-1)(x^2-y^2)^2 - p^2(p^2-p+1)(x-y)^2 + p^6-p^4-p^2+1 \geq 0,$$

$$p^2(s+1)(ps-1)(x-y)^2 + (p^2-1)(p^4-1) \geq 0.$$

If $ps - 1 \ge 0$, then the inequality is clearly true Consider now that $ps < 1$. From $ps < 1$ and $s \ge 3p$, we get $p^2 < \frac{1}{3}$. Write the inequality in the form

$$(1-p^2)(1-p^4) \ge p^2(1+s)(1-ps)(x-y)^2.$$

Since

$$p(x-y)^2 = ps - 3p^2 < 1 - 3p^2 < 1 - p^2,$$

it suffices to show that

$$1 - p^4 \ge p(1+s)(1-ps)$$

Indeed, we have

$$p(1+s)(1-ps) \le \frac{1}{4}\left[p(1+s) + (1-ps)\right]^2 = \frac{(1+p)^2}{4} < \frac{1+p^2}{2} < 1 - p^4.$$

Equality occurs if and only if $a = b = c = d = 1$.

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10. *If a, b, c are non-negative numbers, then*

$$9(a^4+1)(b^4+1)(c^4+1) \ge 8(a^2b^2c^2 + abc + 1)^2.$$

Solution. If at least one of a, b, c is zero, then the inequality becomes trivial. Consider now that a, b, c are positive numbers For $a = b = c$ the inequality reduces to

$$9(a^4+1)^3 \ge 8(a^6 + a^3 + 1)^2,$$

or

$$9\left(a^2 + \frac{1}{a^2}\right)^3 \ge 8\left(a^3 + \frac{1}{a^3} + 1\right)^2$$

Setting $a + \frac{1}{a} = x$, the inequality can be written as follows

$$9(x^2-2)^3 \ge 8(x^3 - 3x + 1)^2,$$
$$x^6 - 6x^4 - 16x^3 + 36x^2 + 48x - 80 \ge 0,$$
$$(x-2)^2\left[x(x^3-8) + 4(x^3-5) + 6x^2\right] \ge 0.$$

Since $x \ge 2$, the last inequality is clearly true. Multiplying the inequalities

$$9(a^4+1)^3 \ge 8(a^6+a^3+1)^2,\ 9(b^4+1)^3 \ge 8(b^6+b^3+1)^2,\ 9(c^4+1)^3 \ge 8(c^6+c^3+1)^2,$$

yields

$$\left[9(a^4+1)(b^4+1)(c^4+1)\right]^3 \ge 8^3(a^6+a^3+1)^2(b^6+b^3+1)^2(c^6+c^3+1)^2$$

Using now Hölder's Inequality

$$(a^6+a^3+1)(b^6+b^3+1)(c^6+c^3+1) \ge (a^2b^2c^2+abc+1)^3,$$

the conclusion follows. Equality occurs only for $a=b=c=1$.

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11. *If a,b,c,d are non-negative numbers, then*

$$\frac{(1+a^3)(1+b^3)(1+c^3)(1+d^3)}{(1+a^2)(1+b^2)(1+c^2)(1+d^2)} \ge \frac{1+abcd}{2}$$

Solution. For $a=b=c=d$, the inequality becomes

$$\left(\frac{a^3+1}{a^2+1}\right)^4 \ge \frac{a^4+1}{2}.$$

We will show that

$$\left(\frac{a^3+1}{a^2+1}\right)^4 \ge \left(\frac{a^3+1}{a+1}\right)^2 \ge \frac{a^4+1}{2}$$

The left side inequality is equivalent to $(a^3+1)(a+1) \ge (a^2+1)^2$, which reduces to $a(a-1)^2 \ge 0$, while the right side inequality is equivalent to $2(a^2-a+1)^2 \ge a^4+1$, which reduces to $(a-1)^4 \ge 0$

Multiplying now the inequalities

$$\frac{a^3+1}{a^2+1} \ge \sqrt[4]{\frac{a^4+1}{2}}, \qquad \frac{b^3+1}{b^2+1} \ge \sqrt[4]{\frac{b^4+1}{2}},$$

$$\frac{c^3+1}{c^2+1} \ge \sqrt[4]{\frac{c^4+1}{2}}, \qquad \frac{d^3+1}{d^2+1} \ge \sqrt[4]{\frac{d^4+1}{2}}$$

yields

$$\frac{(a^3+1)(b^3+1)(c^3+1)(d^3+1)}{(a^2+1)(b^2+1)(c^2+1)(d^2+1)} \ge \frac{1}{2}\sqrt[4]{(a^4+1)(b^4+1)(c^4+1)(d^4+1)}.$$

Applying twice the Cauchy-Schwarz Inequality produces

$$(a^4+1)(b^4+1)(c^4+1)(d^4+1) \ge (a^2b^2+1)(c^2d^2+1)^2 \ge (abcd+1)^4,$$

from which the desired inequality follows. Equality holds for $a=b=c=d=1$.

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12. *Let a, b, c be non-negative numbers, no two of which are zero. Then,*

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\geq\frac{9}{(a+b+c)^2}$$

Solution. Let $s = ab + bc + ca$ Due to homogeneity, we may consider $a + b + c = 1$. Since

$$\frac{1}{a^2+ab+b^2}=\frac{1}{(a+b+c)^2-(ab+bc+ca)-(a+b+c)c}=\frac{1}{1-s-c},$$

the inequality successively becomes

$$\frac{1}{1-s-c}+\frac{1}{1-s-a}+\frac{1}{1-s-b}\geq 9,$$
$$9s^3-6s^2-3s+1+9abc\geq 0,$$
$$s(3s-1)^2+1-4s+9abc\geq 0.$$

The last inequality is true because $1 - 4s + 9abc \geq 0$ by Schur's Inequality

$$(a+b+c)^3+9abc\geq 4(a+b+c)(ab+bc+ca)$$

Equality occurs if and only if $a = b = c$.

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13. *Let a, b, c be positive numbers, and let*

$$x=a+\frac{1}{b}-1,\ y=b+\frac{1}{c}-1,\ z=c+\frac{1}{a}-1.$$

Prove that

$$xy+yz+zx\geq 3.$$

Solution. Without loss of generality, assume that $x = \max\{x, y, z\}$ Then,

$$x\geq\frac{1}{3}(x+y+z)=\frac{1}{3}\left(a+\frac{1}{a}+b+\frac{1}{b}+c+\frac{1}{c}-3\right)\geq\frac{1}{3}(2+2+2-3)=1$$

On the other hand,

$$(x+1)(y+1)(z+1)=abc+\frac{1}{abc}+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq$$
$$\geq 2+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=5+x+y+z,$$

and hence

$$xyz + xy + yz + zx \geq 4$$

Since $y + z = \frac{1}{a} + b + \frac{(c-1)^2}{c} > 0$, two cases are possible a) $yz \leq 0$, b) $y > 0$ and $z > 0$

a) *Case* $yz \leq 0$ We have $xyz \leq 0$, and from $xyz + xy + yz + zx \geq 4$ it follows that $xy + yz + zx \geq 4 > 3$

b) *Case* $y, z > 0$ Let $d = \sqrt{\frac{xy + yz + zx}{3}}$. We have to show that $d \geq 1$ By the AM-GM Inequality, we have $xyz \leq d^3$ Thus $xyz + xy + yz + zx \geq 4$ we get $d^3 + 3d^2 \geq 4$, $(d-1)(d+2)^2 \geq 0$, $d \geq 1$ Equality occurs for $a = b = c = 1$

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14. *Let a, b, c be positive numbers, no two of which are zero. If n is a positive integer, then*

$$\frac{2a^n - b^n - c^n}{b^2 - bc + c^2} + \frac{2b^n - c^n - a^n}{c^2 - ca + a^2} + \frac{2c^n - a^n - b^n}{a^2 - ab + b^2} \geq 0$$

First Solution Let E be the left hand side of the inequality, and let

$$X = 2a^n - b^n - c^n,\ Y = 2b^n - c^n - a^n,\ Z = 2c^n - a^n - b^n$$

Since $X + Y + Z = 0$, we have

$$E = \left(\frac{1}{b^2 - bc + c^2} - \frac{1}{c^2 - ca + a^2}\right) X + \left(\frac{1}{a^2 - ab + b^2} - \frac{1}{c^2 - ca + a^2}\right) Z =$$

$$= \frac{1}{c^2 - ca + a^2}\left[\frac{(a-b)(a+b-c)X}{b^2 - bc + c^2} + \frac{(c-b)(c+b-a)Z}{a^2 - ab + b^2}\right]$$

Thus, the inequality becomes

$$\frac{(a-b)(a+b-c)X}{b^2 - bc + c^2} + \frac{(c-b)(c+b-a)Z}{a^2 - ab + b^2} \geq 0$$

Since the inequality is symmetric, it suffices to consider the following two cases. 1) $a \geq b \geq c$, $b + c \geq a$, 2) $c \geq a \geq b$, $a + b < c$.

In the first case, as well as in second case with $X \leq 0$, the inequality is true since $(a-b)(a+b-c)X \geq 0$ and $(c-b)(c+b-a)Z \geq 0$

In the second case with $X > 0$, we rewrite the inequality as

$$\frac{(c-b)(c+b-a)Z}{a^2-ab+b^2} \geq \frac{(a-b)(c-a-b)X}{b^2-bc+c^2}$$

This inequality is true since

$$\begin{gathered} Z \geq X > 0, \\ c-b \geq a-b \geq 0, \\ c+b-a \geq c-a-b > 0, \\ b^2-bc+c^2 \geq a^2-ab+b^2 > 0. \end{gathered}$$

Equality occurs if and only if $a=b=c$.

Second Solution (after a *Ho Chung Siu*'s idea). Let E be the left hand side of the inequality, and let

$$\begin{aligned} &A = b^n - c^n, && B = c^n - a^n, && C = a^n - b^n, \\ &X = b^2-bc+c^2, && Y = c^2-ca+a^2, && Z = a^2-ab+b^2. \end{aligned}$$

Without loss of generality, assume that $a \geq b \geq c$. We have $A \geq 0$, $C \geq 0$, and

$$E = \frac{A+2C}{X} + \frac{A-C}{Y} - \frac{2A+C}{Z} = A\left(\frac{1}{X}+\frac{1}{Y}-\frac{2}{Z}\right) + C\left(\frac{2}{X}-\frac{1}{Y}-\frac{1}{Z}\right)$$

To prove the desired inequality it suffices to show that $\frac{1}{X}+\frac{1}{Y}-\frac{2}{Z} \geq 0$ and $\frac{2}{X}-\frac{1}{Y}-\frac{1}{Z} \geq 0$. Since $Y-X = (a-b)(a+b-c) \geq 0$ and $Z-X = (a-c)(a-b+c) \geq 0$, the second inequality is obviously true In order to prove the first inequality, we write it as

$$\frac{1}{X}-\frac{1}{Z} \geq \frac{1}{Z}-\frac{1}{Y},$$

or

$$(a-c)(a+c-b)(a^2+c^2-ac) \geq (b-c)(a-b-c)(b^2+c^2-bc).$$

The inequality is trivial for $a-b-c \leq 0$ For $a-b-c > 0$, the inequality follows from $a-c > b-c$, $a+c-b \geq a-b-c$, $a^2+c^2-ac > b^2+c^2-bc$.

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15. *Let* $0 \le a < b$ *and let* $a_1, a_2, \ldots, a_n \in [a, b]$. *Prove that*

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \ldots a_n} \le (n-1)\left(\sqrt{b} - \sqrt{a}\right)^2.$$

Solution. First we will show that the left hand side of the inequality is maximal when $a_1, a_2, \ldots, a_n \in \{a, b\}$. To prove this claim, consider $a_2, \ldots, a_n$ fixed and assume, for the sake of contradiction, that

$$f(a_1) = a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \ldots a_n}$$

is maximal for $a < a_1 < b$; that is $f(a_1) > f(a)$ and $f(a_1) > f(b)$. Let $x_i = \sqrt[n]{a_i}$ for all i, and let $c = \sqrt[n]{a}$, $d = \sqrt[n]{b}$ $(c < x_1 < d)$. From

$$\begin{aligned} f(a_1) - f(a) &= x_1^n - c^n - n(x_1 - c)x_2 \ldots x_n = \\ &= (x_1 - c)\left(x_1^{n-1} + x_1^{n-2}c + \cdots + c^{n-1} - nx_2 \ldots x_n\right) > 0, \end{aligned}$$

we get

$$x_1^{n-1} + x_1^{n-2}c + \cdots + c^{n-1} > nx_2 \ldots x_n.$$

Analogously, from

$$\begin{aligned} f(a_1) - f(b) &= x_1^n - d^n - n(x_1 - d)x_2 \ldots x_n = \\ &= (x_1 - d)\left(x_1^{n-1} + x_1^{n-2}d + \cdots + d^{n-1} - nx_2 \ldots x_n\right) > 0, \end{aligned}$$

we get

$$nx_2 \ldots x_n > x_1^{n-1} + x_1^{n-2}d + \cdots + d^{n-1}.$$

Adding up the obtained inequalities yields

$$x_1^{n-1} + x_1^{n-2}c + \cdots + c^{n-1} > x_1^{n-1} + x_1^{n-2}d + \cdots + d^{n-1},$$

which is clearly false.

Since the left hand side of the given inequality is maximal when

$$a_1, a_2, \ldots, a_n \in \{a, b\},$$

it suffices to consider that

$$a_1 = \cdots = a_k = a \text{ and } a_{k+1} = \cdots = a_n = b,$$

where $k \in \{1, 2, \ldots, n-1\}$. The inequality reduces to

$$(n-k-1)a + (k-1)b + na^{\frac{k}{n}}b^{\frac{n-k}{n}} \ge (2n-2)\sqrt{ab},$$

which immediately follows by the AM-GM Inequality

For $n \geq 3$, equality occurs if and only if $a = 0$, one of a_i is equal to 0 and all the other a_i are equal to b

Remark This inequality is an improved generalization of the following problem from USA TST 2000, proposed by *Titus Andreescu*:

If a, b, c are positive numbers, then

$$a + b + c - 3\sqrt[3]{abc} \leq 3 \max\left\{\left(\sqrt{a} - \sqrt{b}\right)^2, \left(\sqrt{b} - \sqrt{c}\right)^2, \left(\sqrt{c} - \sqrt{a}\right)^2\right\}.$$

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16. *Let a, b, c and x, y, z be positive numbers such that $x + y + z = a + b + c$. Prove that*

$$ax^2 + by^2 + cz^2 + xyz \geq 4abc$$

First Solution Let $p = b - \dfrac{z + x}{2}$, $q = c - \dfrac{x + y}{2}$ and $r = a - \dfrac{y + z}{2}$ Among the numbers p, q and r always there are two of them with the same sign let us say $pq \geq 0$ We have

$$b = p + \frac{x + z}{2},\ c = q + \frac{x + y}{2},\ a = x + y + z - b - c = \frac{y + z}{2} - p - q,$$

and so

$$ax^2 + by^2 + cz^2 + xyz - 4abc = \left(\frac{y + z}{2} - p - q\right)x^2 + \left(p + \frac{x + z}{2}\right)y^2 +$$
$$+ \left(q + \frac{x + y}{2}\right)z^2 + xyz - 4\left(\frac{y + z}{2} - p - q\right)\left(p + \frac{x + z}{2}\right)\left(q + \frac{x + y}{2}\right) =$$
$$= 4pq(p + q) + 2p^2(x + y) + 2q^2(x + z) + 4pqx =$$
$$= 4q^2\left(p + \frac{x + z}{2}\right) + 4p^2\left(q + \frac{x + y}{2}\right) + 4pqx = 4(q^2b + p^2c + pqx) \geq 0$$

Equality occurs if and only if $x = b + c - a$, $y = c + a - b$, $z = a + b - c$. We can also write these equality conditions as $a = \dfrac{y + z}{2}$, $b = \dfrac{z + x}{2}$, $c = \dfrac{x + y}{2}$

Second Solution We will consider two cases

Case $x^2 \geq 4bc$ We have $ax^2 \geq 4abc$, and hence $ax^2 + by^2 + cz^2 + xyz \geq 4abc$

Case $x^2 < 4bc$ Let $u = a + b + c = x + y + z$ Substituting $z = u - x - y$ and $a = u - b - c$, the inequality becomes

$$cu^2 + \left[(x^2 - 4bc) - 2c(x + y) + xy\right]u - (b + c)(x^2 - 4bc) +$$
$$+ by^2 + c(x + y)^2 - xy(x + y) \geq 0$$

The quadratic in u has the discriminant

$$\delta = (x^2 - 4bc)(2c - x - y)^2.$$

Since $\delta \le 0$, the inequality is clearly true

Third Solution The inequality is a direct consequence of the identity

$$2(yzu + zxv + xyw)(ax^2 + by^2 + cz^2 + xyz - 4abc) =$$
$$= xu(v-w)^2 + yv(w-u)^2 + zw(u-v)^2 + 2uvw(x+y+z-a-b-c),$$

where $u = 2ax + yz$, $v = 2by + zx$ and $w = 2cz + xy$.

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17. *Let a, b, c and x, y, z be positive numbers such that $x + y + z = a + b + c$. Prove that*

$$\frac{x(3x+a)}{bc} + \frac{y(3y+a)}{ca} + \frac{z(3z+a)}{ab} \ge 12.$$

Solution. Write the inequality in the form

$$ax^2 + by^2 + cz^2 + \frac{1}{3}(a^2x + b^2y + c^2z) \ge 4abc$$

Applying the Cauchy-Schwarz Inequality, we have

$$a^2x + b^2y + c^2z \ge \frac{(a+b+c)^2}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} = \frac{xyz(x+y+z)^2}{xy + yz + zx} \ge 3xyz.$$

Thus, it suffices to show that

$$ax^2 + by^2 + cz^2 + xyz \ge 4abc,$$

which is just the preceding inequality.

One has equality for $x = y = z = a = b = c$

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18. *Let a, b, c be positive numbers such that $a^2 + b^2 + c^2 = 3$. Prove that*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \frac{9}{a+b+c}.$$

Solution. By the Cauchy-Schwarz Inequality, we have

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{(a+b+c)^2}{ab+bc+ca}.$$

Thus, we still have to show that

$$(a+b+c)^3 \geq 9(ab+bc+ca).$$

By squaring and homogenizing, this inequality becomes

$$(a+b+c)^6 \geq 27(ab+bc+ca)^2(a^2+b^2+c^2).$$

Without loss of generality, we assume that $a+b+c=3$ Setting $t=ab+bc+ca$ reduces the inequality to

$$27 \geq t^2(9-2t).$$

Indeed,

$$27-t^2(9-2t)=2t^3-9t^2+27=(t-3)^2(2t+3)\geq 0$$

Equality occurs if and only if $a=b=c=1$.

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19. Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1a_2\ldots a_n=1$. Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}+\frac{4n}{n+a_1+a_2+\cdots+a_n} \geq n+2.$$

Solution. Let $a=\sqrt[n-1]{\dfrac{a_1+a_2+\cdots+a_n}{n}}$ By the AM-GM Inequality we get $a \geq 1$, and by Maclaurin's Inequality we have

$$\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n} \geq \sqrt[n-1]{\frac{\sum \frac{1}{a_1a_2\ldots a_{n-1}}}{n}}=\sqrt[n-1]{\frac{a_1+a_2+\cdots+a_n}{n}}=a.$$

Thus, it suffices to prove that

$$na+\frac{4}{1+a^{n-1}} \geq n+2.$$

Since $a \geq 1$, it is enough to show that

$$na + \frac{4}{1+a^n} \geq n+2.$$

This inequality is equivalent to

$$(a-1)\left[n(a^n+1) - 2(a^{n-1} + \cdots + a + 1)\right] \geq 0.$$

We have

$$n(a^{n-1}+1) - 2(a^{n-1} + \cdots + a + 1) = \sum_{i=0}^{n-1}(a^{n-1} + 1 - a^{n-i-1}a^i) =$$

$$= \sum_{i=0}^{n-1}(a^i - 1)(a^{n-i-1} - 1) \geq 0,$$

and the proof is completed. One has equality for $a_1 = a_2 = \cdots = a_n = 1$.

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20. *Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 a_2 \ldots a_n = 1$. Prove that*

$$a_1 + a_2 + \cdots + a_n - n + 1 \geq \sqrt[n-1]{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n + 1}$$

Solution. Let $a = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$ By the AM-GM Inequality we get $a \geq 1$, and by Maclaurin's Inequality we have

$$a = \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n-1]{\frac{\sum a_1 a_2 \ldots a_{n-1}}{n}} = \sqrt[n-1]{\frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{n}},$$

and hence

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \leq na^{n-1}.$$

Thus, it suffices to show that

$$na - n + 1 \geq \sqrt[n-1]{na^{n-1} - n + 1}.$$

We write this inequality in the form

$$\left[1 + (n-1)\left(1 - \frac{1}{a}\right)\right]^{n-1} \geq n - \frac{n-1}{a^{n-1}}$$

Using Bernoulli's Inequality yields

$$\left[1+(n-1)\left(1-\frac{1}{a}\right)\right]^{n-1}-n+\frac{n-1}{a^{n-1}}\geq 1+(n-1)^2\left(1-\frac{1}{a}\right)-n+\frac{n-1}{a^{n-1}}=$$
$$=(n-1)\left[(n-1)\left(1-\frac{1}{a}\right)-\left(1-\frac{1}{a^{n-1}}\right)\right]=$$
$$=(n-1)\left(1-\frac{1}{a}\right)\left[n-1-\left(1+\frac{1}{a}+\frac{1}{a^2}+\cdots+\frac{1}{a^{n-2}}\right)\right]=$$
$$=(n-1)\left(1-\frac{1}{a}\right)\left[\left(1-\frac{1}{a}\right)+\left(1-\frac{1}{a^2}\right)+\cdots+\left(1-\frac{1}{a^{n-2}}\right)\right]\geq 0,$$

from which the conclusion follows Equality occurs for $a_1=a_2=\cdots=a_n=1$

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21. *Let $r>1$ and let a,b,c be non-negative numbers such that $ab+bc+ca=3$. Prove that*

$$a^r(b+c)+b^r(c+a)+c^r(a+b)\geq 6.$$

Solution. Let $E=a^r(b+c)+b^r(c+a)+c^r(a+b)$ We will consider two cases, depending on r

Case $r\geq 2$. Applying Jensen's Inequality to the convex function

$$f(x)=x^{r-1},$$

gives us

$$E=(ab+ac)a^{r-1}+(bc+ba)b^{r-1}+(ca+cb)c^{r-1}\geq$$
$$\geq 2(ab+bc+ca)\left[\frac{a^2(b+c)+b^2(c+a)+c^2(a+b)}{2(ab+bc+ca)}\right]^{r-1}=$$
$$=6\left[\frac{a^2(b+c)+b^2(c+a)+c^2(a+b)}{6}\right]^{r-1}$$

Thus, it suffices to show that

$$a^2(b+c)+b^2(c+a)+c^2(a+b)\geq 6.$$

Write this inequality as

$$ab+bc+ca)(a+b+c)\geq 3abc+6$$

It is true because $a+b+c \geq 3$ and $abc \leq 3$ The former inequality follows by the well-known inequality $(a+b+c)^2 \geq 3(ab+bc+ca)$, while the latter by the AM-GM Inequality

$$ab+bc+ca \geq 3\sqrt[3]{a^2b^2c^2}.$$

Case $1 < r < 2$. According to the condition $ab+bc+ca=3$, we have

$$a(b+c) = 3-bc,\ b(c+a) = 3-ca,\ c(a+b) = 3-ab,$$

and

$$E = a^{r-1}(3-bc) + b^{r-1}(3-ca) + c^{r-1}(3-ab) =$$
$$= 3(a^{r-1}+b^{r-1}+c^{r-1}) - a^{r-1}b^{r-1}c^{r-1}\left[(ab)^{2-r}+(bc)^{2-r}+(ca)^{2-r}\right].$$

Since $0 < 2-r < 1$, the function $f(x) = x^{2-r}$ is concave for $x \geq 0$ Thus, by Jensen's Inequality we have

$$\frac{(ab)^{2-r}+(bc)^{2-r}+(ca)^{2-r}}{3} \leq \left(\frac{ab+bc+ca}{3}\right)^{2-r} = 1,$$

and hence

$$E \geq 3(a^{r-1}+b^{r-1}+c^{r-1}) - 3a^{r-1}b^{r-1}c^{r-1}.$$

Consequently, it suffices to show that

$$a^{r-1}+b^{r-1}+c^{r-1} \geq a^{r-1}b^{r-1}c^{r-1}+2.$$

Because the inequality is symmetric, we may assume that $a \geq b \geq c$. Let $x = \sqrt{ab}$ From $a \geq b \geq c$ and $ab+bc+ca=3$, we get $1 \leq x \leq \sqrt{3}$. Write now the inequality as

$$a^{r-1}+b^{r-1}-2 \geq \left(a^{r-1}b^{r-1}-1\right)\left(\frac{3-ab}{a+b}\right)^{r-1}.$$

The AM-GM Inequality yields $a+b \geq 2x$ and $a^{r-1}+b^{r-1} \geq 2x^{r-1}$. Thus, it suffices to show that

$$2(x^{r-1}-1) \geq (x^{2r-2}-1)\left(\frac{3-x^2}{2x}\right)^{r-1}$$

Since $x \geq 1$, we have to prove that

$$2 \geq (x^{r-1}+1)\left(\frac{3-x^2}{2x}\right)^{r-1}$$

Write this inequality as

$$2 \geq \left(\frac{3-x^2}{2}\right)^{r-1} + \left(\frac{3-x^2}{2x}\right)^{r-1}$$

Since $1 \geq \frac{3-x^2}{2} \geq \frac{3-x^2}{2x}$, the inequality is clearly true. Equality occurs if and only if $a = b = c = 1$.

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22. *Let a, b, c be positive real numbers such that $abc \geq 1$. Prove that*

$$(a) \qquad a^{\frac{a}{b}} b^{\frac{b}{c}} c^{\frac{c}{a}} \geq 1;$$

$$(b) \qquad a^{\frac{a}{b}} b^{\frac{b}{c}} c^{c} \geq 1$$

Solution. (a) Using the substitution $x = \frac{a}{r}$, $y = \frac{b}{r}$ and $z = \frac{c}{r}$, where $r = \sqrt[3]{abc} \geq 1$, we have $xyz = 1$ and

$$a^{\frac{a}{b}} b^{\frac{b}{c}} c^{\frac{c}{a}} = x^{\frac{x}{y}} y^{\frac{y}{z}} z^{\frac{z}{x}} r^{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}} \geq x^{\frac{x}{y}} y^{\frac{y}{z}} z^{\frac{z}{x}}.$$

Therefore, it suffices to show that

$$x^{\frac{x}{y}} y^{\frac{y}{z}} z^{\frac{z}{x}} \geq 1,$$

or, equivalently,

$$\frac{x}{y}\ln x + \frac{y}{z}\ln y + \frac{z}{x}\ln z \geq 0.$$

Since the function $f(x) = x\ln x$ is convex, by Jensen's Inequality we get

$$\frac{1}{y}\cdot x\ln x + \frac{1}{z}\ y\ln y + \frac{1}{x}\cdot z\ln z \geq \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)\ln\frac{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}{\frac{1}{y}+\frac{1}{z}+\frac{1}{x}},$$

and it remains to show that

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{1}{y} + \frac{1}{z} + \frac{1}{x}.$$

By the AM-GM Inequality we have

$$\frac{x}{y} + 2\frac{y}{z} \geq 3\sqrt[3]{\frac{x}{y}\left(\frac{y}{z}\right)^2} = \frac{3}{z},$$

and, analogously,

$$\frac{y}{z} + 2\frac{z}{x} \geq \frac{3}{x}, \ \frac{z}{x} + 2\frac{x}{y} \geq \frac{3}{y}$$

Adding these inequalities yields the required inequality.

(b) Write the inequality in the form

$$\frac{a}{b}\ln a + \frac{b}{c}\ln b + c\ln c \geq 0.$$

As above, by Jensen's Inequality we get

$$\frac{1}{b}\cdot a\ln a + \frac{1}{c}\cdot b\ln b + c\ln c \geq \left(\frac{a}{b} + \frac{b}{c} + c\right)\ln\frac{\frac{a}{b} + \frac{b}{c} + c}{\frac{1}{b} + \frac{1}{c} + 1}$$

Thus, it remains to show that

$$\frac{a}{b} + \frac{b}{c} + c \geq \frac{1}{b} + \frac{1}{c} + 1.$$

Since $a \geq \frac{1}{bc}$, it suffices to show that

$$\frac{1}{b^2c} + \frac{b}{c} + c \geq \frac{1}{b} + \frac{1}{c} + 1$$

This inequality is equivalent to

$$\frac{1}{b^2} + b + c^2 \geq \frac{c}{b} + 1 + c,$$

or

$$\left(2c - 1 - \frac{1}{b}\right)^2 + \left(1 - \frac{1}{b}\right)^2(4b+3) \geq 0.$$

Equality in both inequalities occurs for $a = b = c = 1$.

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23. *Let a, b, c, d be non-negative numbers. Prove that*

$$4(a^3 + b^3 + c^3 + d^3) + 15(abc + bcd + cda + dab) \geq (a + b + c + d)^3.$$

Solution. Let

$$E(a,b,c,d)=4(a^3+b^3+c^3+d^3)+15(abc+bcd+cda+dab)-(a+b+c+d)^3.$$

Assume $a \le b \le c \le d$, then show that

$$E(a,b,c,d) \ge E(0,a+b,c,d) \ge 0.$$

We have

$$E(a,b,c,d)-E(0,a+b,c,d) = 4\left[a^3+b^3-(a+b)^3\right]+15ab(c+d) =$$
$$= 3ab\left[5(c+d)-4(a+b)\right] \ge 0.$$

Setting $a+b=x$, we get

$$E(0,a+b,c,d)=E(0,x,c,d)=4(x^3+c^3+d^3)+15xcd-(x+c+d)^3.$$

It is easy to check that the inequality $E(0,x,c,d) \ge 0$ is equivalent to Schur's Inequality

$$x^3+c^3+d^3+3xcd \ge xc(x+c)+cd(c+d)+dx(d+x)$$

Under the assumption $a \le b \le c \le d$, equality occurs for

$$(a,b,c,d) \sim (0,1,1,1) \text{ and } (a,b,c,d) \sim (0,0,1,1).$$

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24. Let a,b,c be positive numbers such that

$$(a+b-c)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)=4.$$

Prove that

$$(a^4+b^4+c^4)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right) \ge 2304.$$

Solution. Without loss of generality, assume that $a \ge b$. Let $\frac{a}{b}=u$, $u \ge 1$ Since

$$(a+b-c)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)=(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)-\left[(a+b)\frac{1}{c}+\left(\frac{1}{a}+\frac{1}{b}\right)c\right]+1 \le$$
$$\le (a+b)\left(\frac{1}{a}+\frac{1}{b}\right)-2\sqrt{(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}+1=\left[\sqrt{(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}-1\right]^2=$$
$$=\left(\sqrt{u+\frac{1}{u}+2}-1\right)^2,$$

it follows that $\left(\sqrt{u+\frac{1}{u}+2}-1\right)^2 \geq 4$, and hence $u+\frac{1}{u} \geq 7$. On the other hand,

$$(a^4+b^4+c^4)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right) = (a^4+b^4)\left(\frac{1}{a^4}+\frac{1}{b^4}\right)+$$
$$+\left[(a^4+b^4)\frac{1}{c^4}+\left(\frac{1}{a^4}+\frac{1}{b^4}\right)c^4\right]+1 \geq$$
$$\geq (a^4+b^4)\left(\frac{1}{a^4}+\frac{1}{b^4}\right)+2\sqrt{(a^4+b^4)\left(\frac{1}{a^4}+\frac{1}{b^4}\right)}+1 =$$
$$= \left[\sqrt{(a^4+b^4)\left(\frac{1}{a^4}+\frac{1}{b^4}\right)}+1\right]^2 =$$
$$= \left(u^2+\frac{1}{u^2}+1\right)^2 = \left[\left(u+\frac{1}{u}\right)^2-1\right]^2 \geq (7^2-1)^2 = 2304$$

Equality occurs when $ab = c^2$ and $\frac{a}{b}+\frac{b}{a} = 7$. For $a \geq b$, the equality conditions are equivalent to $\frac{a}{c} = \frac{3+\sqrt{5}}{2}$ and $\frac{b}{c} = \frac{3-\sqrt{5}}{2}$.

★

25. *Let a, b, c be positive numbers. Prove that*

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab} > \frac{2}{ab+bc+ca}$$

First solution. Without loss of generality, assume that $a \geq b \geq c$. We have

$$\frac{ab+bc+ca}{b^2+2ca} = 1-\frac{(b-a)(b-c)}{b^2+2ca}, \quad \frac{ab+bc+ca}{c^2+2ab} = 1-\frac{(c-a)(c-b)}{c^2+2ab},$$

and hence

$$\frac{ab+bc+ca}{b^2+2ca}+\frac{ab+bc+ca}{c^2+2ab} = 2+(b-c)^2\frac{2a^2-3a(b+c)+bc}{(b^2+2ca)(c^2+2ab)}.$$

Thus, the inequality becomes

$$\frac{ab+bc+ca}{a^2+2bc} > (b-c)^2\frac{3a(b+c)-bc-2a^2}{(b^2+2ac)(c^2+2ab)}.$$

This inequality is clearly true if $2a^2+bc \geq 3a(b+c)$.

Since $ab+bc+ca-3a(b+c)+bc+2a^2 = 2(a-b)(a-c) \geq 0$, it suffices to show that

$$(b^2+2ac)(c^2+2ab) \geq (b-c)^2(a^2+2bc).$$

similarly, since $ab+bc+ca-3c(a+b)+ab+2c^2 = 2(c-a)(c-b) \geq 0$, it suffices to show that

$$(a^2+2bc)(b^2+2ac) \geq (a-b)^2(c^2+2ab).$$

By multiplying these two sufficient inequalities, we get

$$(b^2+2ac)^2 \geq (b-c)^2(a-b)^2,$$

which is equivalent to

$$2b^2+3ac \geq b(a+c).$$

If the last inequality is true, then the given inequality holds. On the other hand, as shown above, the given inequality holds if $2a^2+bc \geq 3a(b+c)$. Thus, it suffices to show that

$$(2b^2+3ac)+(2a^2+bc) \geq b(a+c)+3a(b+c).$$

This inequality reduces to $2(a-b)^2 \geq 0$, which clearly is true.

Second solution (by *Darij Grinberg*). We will show that the following sharper inequality holds

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab} \geq \frac{2}{ab+bc+ca}+\frac{1}{a^2+b^2+c^2},$$

with equality for $a=b$, or $b=c$, or $c=a$. Taking account of

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab} =$$
$$= \frac{(ab+bc+ca)(2a^2+2b^2+2c^2+ab+bc+ca)}{(a^2+2bc)(b^2+2ca)(c^2+2ab)},$$

we can show that the inequality is equivalent to

$$(a-b)^2(b-c)^2(c-a)^2(2a^2+2b^2+2c^2+ab+bc+ca) \geq 0$$

26. *Let a, b, c be non-negative numbers, no two of which are zero. Prove that*

$$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab}\geq 1+\frac{ab+bc+ca}{a^2+b^2+c^2}$$

Solution. The inequality follows by adding the above inequality

$$\frac{ab+bc+ca}{a^2+2bc}+\frac{ab+bc+ca}{b^2+2ca}+\frac{ab+bc+ca}{c^2+2ab}\geq 2+\frac{ab+bc+ca}{a^2+b^2+c^2}$$

to the inequality

$$1\geq\frac{bc}{a^2+2bc}+\frac{ca}{b^2+2ca}+\frac{ab}{c^2+2ab}.$$

The last inequality is equivalent to

$$\frac{a^2}{a^2+2bc}+\frac{b^2}{b^2+2ca}+\frac{c^2}{c^2+2ab}\geq 1$$

According to the Cauchy-Schwarz Inequality, we have

$$\sum\frac{a^2}{a^2+2bc}\geq\frac{\left(\sum a\right)^2}{\sum(a^2+2bc)}=1.$$

Equality occurs if and only if $a=b=c$.

★

27. *Let a, b, c be non-negative numbers, no two of which are zero. Then*

$$\frac{(b+c)^2}{a^2+bc}+\frac{(c+a)^2}{b^2+ca}+\frac{(a+b)^2}{c^2+ab}\geq 6.$$

First Solution. By the Cauchy-Schwarz Inequality we have

$$\sum\frac{(b+c)^2}{a^2+bc}\geq\frac{\left[\sum(b+c)^2\right]^2}{\sum(a^2+bc)(b+c)^2}$$

Thus, it suffices to show that

$$2\left(\sum a^2+\sum bc\right)^2\geq 3\sum(a^2+bc)(b^2+c^2+2bc)$$

Since

$$\left(\sum a^2+\sum bc\right)^2=\left(\sum a^2\right)^2+\left(\sum bc\right)^2+2\left(\sum a^2\right)\left(\sum bc\right)=$$
$$=\sum a^4+3\sum b^2c^2+4abc\sum a+2\sum bc(b^2+c^2)$$

and

$$\sum(a^2+bc)(b^2+c^2+2bc) = 4\sum b^2c^2 + 2abc\sum a + \sum bc(b^2+c^2),$$

the inequality becomes

$$2\sum a^4 + 2abc\sum a + \sum bc(b^2+c^2) \ge 6\sum b^2c^2.$$

We can obtain this inequality by summing the inequalities

$$\sum bc(b^2+c^2) \ge 2\sum b^2c^2$$

and

$$\sum a^4 + abc\sum a \ge \sum bc(b^2+c^2)$$

multiplied by 3 and 2, respectively. The first inequality is equivalent to

$$\sum bc(b-c)^2 \ge 0,$$

while the second inequality is just the fourth degree Schur's Inequality. Equality occurs for $(a,b,c) \sim (1,1,1)$, and also for $(a,b,c) \sim (0,1,1)$ or any cyclic permutation

Second Solution (by *Pham Kim Hung*) Since

$$\sum\left[\frac{(b+c)^2}{a^2+bc} - 2\right] = \sum\frac{b^2+c^2-2a^2}{a^2+bc} =$$
$$= \sum\frac{b^2-a^2}{a^2+bc} + \sum\frac{c^2-a^2}{a^2+bc} = \sum\frac{b^2-a^2}{a^2+bc} + \sum\frac{a^2-b^2}{b^2+ca} =$$
$$= \sum\frac{(a^2-b^2)(a-b)(a+b-c)}{(a^2+bc)(b^2+ca)},$$

we may write the inequality in the form

$$E = \sum(b-c)^2(a^2+bc)(b+c)(b+c-a) \ge 0$$

Due to symmetry, we may assume that $a \ge b \ge c$. Since

$$(a-b)^2(c^2+ab)(a+b)(a+b-c) \ge 0,$$

it suffices to show that

$$b-c\;^2\;a^2+bc)(b+c)(b+c-a) + (c-a)^2(b^2+ca)(c+a)(c+a-b) \ge 0.$$

Write the inequality as

$$(a-c)^2(b^2+ac)(a+c)(a+c-b) \geq (b-c)^2(a^2+bc)(b+c)(a-b-c)$$

Since $a+c \geq b+c$, $a+c-b \geq 0$ and $a+c-b \geq a-b-c$, it suffices to show that

$$(a-c)^2(b^2+ac) \geq (b-c)^2(a^2+bc).$$

We can obtain this inequality by multiplying the obvious inequalities

$$a^2(b^2+ac) \geq b^2(a^2+bc), \quad b^2(a-c)^2 \geq a^2(b-c)^2.$$

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28. *Let a, b, c be non-negative numbers, no two of which are zero. Then*

$$\frac{b+c}{2a^2+bc} + \frac{c+a}{2b^2+ca} + \frac{a+b}{2c^2+ab} \geq \frac{6}{a+b+c}$$

Solution (by *Bin Zhao*) Write the inequality as $E \geq 0$, where

$$E = \sum \left[\frac{(b+c)(a+b+c)}{2a^2+bc} - 2\right] = \sum \frac{b^2+c^2-4a^2+ab+ac}{a^2+bc} =$$

$$= \sum \frac{(b+2a)(b-a)+(c+2a)(c-a)}{2a^2+bc} =$$

$$= \sum \frac{(b+2a)(b-a)}{2a^2+bc} - \sum \frac{(a+2b)(a-b)}{2b^2+ca} =$$

$$= \frac{2}{(a^2+bc)(b^2+ca)(c^2+ab)} \sum (a-b)^2(2c^2+ab)(a^2+b^2+3ab-bc-ca).$$

Due to symmetry, we may assume that $a \geq b \geq c$. Since

$$a^2+b^2+3ab-bc-ca = a(a-c)+b(b-c)+3ab > 0$$

it suffices to show that

$$(b-c)^2(2a^2+bc)(b^2+c^2+3bc-ca-ab)+$$
$$+(c-a)^2(2b^2+ca)(c^2+a^2+3ca-ab-bc) \geq 0$$

Write this inequality as

$$(a-c)^2(2b^2+ac)(a^2+c^2+3ac-ab-bc) \geq$$
$$\geq (b-c)^2(2a^2+bc)(ab+ac-b^2-c^2-3bc).$$

Since

$$a^2 + c^2 + 3ac - ab - bc = (a+c)(a-b) + c(c+2a) \geq 0$$

and

$$(a^2+c^2+3ac-ab-bc)-(ab+ac-b^2-c^2-3bc)=(a-b)^2+2c(a+b+c)\geq 0$$

it suffices to show that

$$(a-c)^2(2b^2+ac) \geq (b-c)^2(2a^2+bc).$$

This inequality follows by multiplying the inequalities

$$a^2(2b^2+ac) \geq b^2(2a^2+bc), \;\; b^2(a-c)^2 \geq a^2(b-c)^2.$$

Equality occurs for $(a,b,c) \sim (1,1,1)$, and also for $(a,b,c) \sim (0,1,1)$ or any cyclic permutation

★

29. *If a, b, c are non-negative numbers, then*

$$a\sqrt{a^2+3bc} + b\sqrt{b^2+3ca} + c\sqrt{c^2+3ab} \geq 2(ab+bc+ca).$$

First Solution. Without loss of generality, assume that $a \geq b \geq c$. For $c = 0$, the inequality reduces to $(a-b)^2 \geq 0$ Consider now $a \geq b \geq c > 0$, and rewrite the inequality as follows

$$\sum a\left(\sqrt{a^2+3bc} - b - c\right) \geq 0,$$

$$\sum a\frac{a^2+bc-b^2-c^2}{\sqrt{a^2+3bc}+b+c} \geq 0,$$

$$\frac{X}{(b+c)A} + \frac{Y}{(c+a)B} + \frac{Z}{(a+b)C} \geq 0,$$

where

$$A = \sqrt{a^2+3bc} + b + c, \; B = \sqrt{b^2+3ca} + c + a, \; C = \sqrt{c^2+3ab} + a + b,$$

$$X = a^3(b+c) - a(b^3+c^3), \; Y = b^3(c+a) - b(c^3+a^3), \; Z = c^3(a+b) - c(a^3+b^3).$$

We see that $X + Y + Z = 0$. We have

$$X = a(b+c)\left[a^2 - b^2 + c(b-c)\right] \geq 0,$$

$$Z = c(a+b)\left[c^2 - a^2 + b(a-b)\right] \leq 0$$

and

$$\frac{X}{(b+c)A}+\frac{Y}{(c+a)B}+\frac{Z}{(a+b)C}=\frac{X}{(b+c)A}-\frac{X+Z}{(c+a)B}+\frac{Z}{(a+b)C}=$$
$$=X\left[\frac{1}{(b+c)A}-\frac{1}{(c+a)B}\right]+(-Z)\left[\frac{1}{(c+a)B}-\frac{1}{(a+b)C}\right]\geq$$
$$\geq\frac{X}{c+a}\left(\frac{1}{A}-\frac{1}{B}\right)+\frac{(-Z)}{a+b}\left(\frac{1}{B}-\frac{1}{C}\right)$$

To finish the proof, it is enough to show that $A\leq B\leq C$. The inequality $A\leq B$ is equivalent to each of the following inequalities

$$\sqrt{a^2+3bc}-a\leq\sqrt{b^2+3ca}-b,$$
$$\frac{3bc}{\sqrt{a^2+3bc}+a}\leq\frac{3ca}{\sqrt{b^2+3ca}+b},$$
$$b^2+\sqrt{b^4+3ab^2c}\leq a^2+\sqrt{a^4+3a^2bc}$$

Since $b\leq a$, the last inequality is clearly true Similarly, the inequality $B\leq C$ is equivalent to

$$c^2+\sqrt{c^4+3abc^2}\leq b^2+\sqrt{b^4+3ab^2c},$$

which is also true

For $a\geq b\geq c$, equality occurs when either $(a,b,c)\sim(1,1,1)$ or $(a,b,c)\sim(1,1,0)$

Second Solution (by *Ho Chung Siu*) Assume that $a\geq b\geq c>0$, and rewrite the inequality in the form

$$\frac{a(a^2+bc-b^2-c^2)}{A}+\frac{b(b^2+ca-c^2-a^2)}{B}+\frac{c(c^2+ab-a^2-b^2)}{C}\geq 0,$$

where

$$A=\sqrt{a^2+3bc}+b+c,\ B=\sqrt{b^2+3ca}+c+a,\ C=\sqrt{c^2+3ab}+a+b$$

As shown above, we have $A\leq B\leq C$ We will prove that

$$\sum\frac{a(a^2+bc-b^2-c^2)}{A}\geq\sum\frac{a(a-b)(a-c)}{A}\geq 0.$$

The left side inequality is equivalent to

$$\sum\frac{a(ab+ac-b^2-c^2)}{A}\geq 0.$$

It is true because

$$\sum \frac{a(ab+ac-b^2-c^2)}{A} = \sum \frac{ab(a-b)}{A} - \sum \frac{ca(c-a)}{A} =$$
$$= \sum \frac{ab(a-b)}{A} - \sum \frac{ab(a-b)}{B} = \sum ab(a-b)\left(\frac{1}{A} - \frac{1}{B}\right) \geq 0.$$

In order to prove the right side inequality, we write it as

$$A_1(a-b)(a-c) + B_1(b-c)(b-a) + C_1(c-a)(c-b) \geq 0,$$

or

$$B_1(a-b)^2 + (A_1 - B_1)(a-b)(a-c) + C_1(c-a)(c-b) \geq 0,$$

where $A_1 = \frac{a}{A}$, $B_1 = \frac{b}{B}$, $C_1 = \frac{c}{C}$. Since $a \geq b \geq c$ and $A \leq B \leq C$, we have $A_1 \geq B_1 \geq C_1 > 0$, from which the inequality follows.

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30. *Let a, b, c be non-negative numbers, no two of which are zero. Then*

$$\frac{a^2-bc}{\sqrt{a^2+bc}} + \frac{b^2-ca}{\sqrt{b^2+ca}} + \frac{c^2-ab}{\sqrt{c^2+ab}} \geq 0.$$

First Solution. We write the inequality in the form

$$\frac{X}{A} + \frac{Y}{B} + \frac{Z}{C} \geq 0,$$

where

$$X = (a^2-bc)(b+c), \;\; Y = (b^2-ca)(c+a), \;\; Z = (c^2-ab)(a+b),$$
$$A = (b+c)\sqrt{a^2+bc}, \;\; B = (c+a)\sqrt{b^2+ca}, \;\; C = (a+b)\sqrt{c^2+ab}.$$

Without loss of generality, consider that $a \geq b \geq c$. It is easy to check that $X + Y + Z = 0$ Moreover, we claim that $X \geq Y \geq Z$ and $A \leq B \leq c$. Indeed,

$$X - Y = ab(a-b) + 2(a^2-b^2)c + (a-b)c^2 \geq 0,$$
$$Y - Z = bc(b-c) + 2(b^2-c^2)a + (b-c)a^2 \geq 0$$

and

$$B^2 - A^2 = (a-b)c^3 + (a^2-b^2)c^2 + c(a-b)(a^2-ab+b^2) \geq 0,$$
$$C^2 - B^2 = b-c)a^3 + (b^2-c^2)a^2 + a(b-c)(b^2-bc+c^2) \geq 0.$$

Then, by Chebyshev's Inequality we get

$$3\left(\frac{X}{A}+\frac{Y}{B}+\frac{Z}{C}\right)\geq(X+Y+Z)\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)=0$$

Equality occurs if and only if $a=b=c$

Second Solution Write the inequality as

$$\sum\frac{a^2-bc}{A}\geq 0,$$

where $A=\sqrt{a^2+bc}$, $B=\sqrt{b^2+ca}$ and $C=\sqrt{c^2+ab}$ We have

$$2\sum\frac{a^2-bc}{A}=\sum\frac{(a-b)(a+c)+(a-c)(a+b)}{A}=$$
$$=\sum\frac{(a-b)(a+c)}{A}+\sum\frac{(b-a)(b+c)}{B}=\sum(a-b)\left(\frac{a+c}{A}-\frac{b+c}{B}\right)=$$
$$=\sum\frac{a-b}{AB}\cdot\frac{(a+c)^2B^2-(b+c)^2A^2}{(a+c)B+(b+c)A}=\sum\frac{c(a-b)^2}{AB}\quad\frac{C_1}{(a+c)B+(b+c)A},$$

where $C_1=a^2+b^2+c^2-ab+bc+ca$. Since $C_1>0$, the inequality is clearly true.

Remark. Similarly, we can prove that for $0\leq p\leq\frac{3}{2}$, the inequality holds

$$\frac{a^2-bc}{\sqrt{pa^2+bc}}+\frac{b^2-ca}{\sqrt{pb^2+ca}}+\frac{c^2-ab}{\sqrt{pc^2+ab}}\geq 0.$$

By the second method, we get

$$C_1=a^2+ab+b^2+2(a+b)c+c^2-p(2ab+bc+ca)\geq$$
$$\geq a^2+ab+b^2+2(a+b)c+c^2-\frac{3}{2}(2ab+bc+ca)=$$
$$=\frac{2(a-b)^2+c(a+b+2c)}{2}\geq 0.$$

★

31. *If a,b,c are non-negative numbers, then*

$$(a^2-bc)\sqrt{a^2+4bc}+(b^2-ca)\sqrt{b^2+4ca}+(c^2-ab)\sqrt{c^2+4ab}\geq 0.$$

Solution. It two of a, b, c are zero, then one has equality. Otherwise, we write the inequality in the form

$$AX + BY + CZ \geq 0,$$

where

$$X = (a^2 - bc)(b + c),\ Y = (b^2 - ca)(c + a),\ Z = (c^2 - ab)(a + b),$$
$$A = \frac{\sqrt{a^2 + 4bc}}{b + c}, \qquad B = \frac{\sqrt{b^2 + 4ca}}{c + a}, \qquad C = \frac{\sqrt{c^2 + 4ab}}{a + b}.$$

Consider now, without loss of generality, that $a \geq b \geq c$. We have $X + Y + Z = 0$, $X \geq 0$ and $Z \leq 0$. Moreover,

$$X - Y = ab(a - b) + 2(a^2 - b^2)c + (a - b)c^2 \geq 0$$

and

$$A^2 - B^2 = \frac{a^4 - b^4 + 2(a^3 - b^3)c + (a^2 - b^2)c^2 + 4abc(a - b) - 4(a - b)c^3}{(b + c)^2(c + a)^2} \geq$$
$$\geq \frac{4abc(a - b) - 4(a - b)c^3}{(b + c)^2(c + a)^2} = \frac{4c(a - b)(ab - c^2)}{(b + c)^2(c + a)^2} \geq 0.$$

Since

$$AX + BY + CZ = (A - B)(X - Y) - (A + B - 2C)Z,$$

it suffices to show that

$$A + B - 2C \geq 0.$$

Taking account of $A + B \geq 2\sqrt{AB}$, it is enough to prove that $AB \geq C^2$. Using the Cauchy-Schwarz Inequality, we get

$$AB \geq \frac{ab + 4c\sqrt{ab}}{(b + c)(c + a)}$$

Since $4c\sqrt{ab} \geq 2c\sqrt{ab} + 2c^2$, we will still have to show that

$$(a + b)^2 \left(2c\sqrt{ab} + 2c^2\right) \geq (b + c)(c + a)(c^2 + 4ab).$$

This inequality is equivalent to

$$ab(a - b)^2 + 2c\sqrt{ab}(a + b)\left(\sqrt{a} - \sqrt{b}\right)^2 + c^2\left[2(a + b)^2 - 5ab - c(a + b) - c^2\right] \geq 0,$$

which is true because

$$2(a+b)^2-5ab-c(a+b)-c^2=a(a-b)+a(a-c)+b(b-c)+b^2-c^2\geq 0$$

For $a\geq b\geq c$, equality occurs when either

$$(a,b,c)\sim(1,1,1)\ \text{ or }\ (a,b,c)\sim(1,1,0).$$

★

32. *If a,b,c are positive numbers, then*

$$\frac{a^2-bc}{\sqrt{8a^2+(b+c)^2}}+\frac{b^2-ca}{\sqrt{8b^2+(c+a)^2}}+\frac{c^2-ab}{\sqrt{8c^2+(a+b)^2}}\geq 0.$$

Solution. Write the inequality as

$$\sum\frac{a^2-bc}{A}\geq 0,$$

where $A=\sqrt{8a^2+(b+c)^2}$, $B=\sqrt{8b^2+(c+a)^2}$ and $C=\sqrt{8c^2+(a+b)^2}$. We have

$$\begin{aligned}2\sum\frac{a^2-bc}{A}&=\sum\frac{(a-b)(a+c)+(a-c)(a+b)}{A}=\\&=\sum\frac{(a-b)(a+c)}{A}+\sum\frac{(b-a)(b+c)}{B}=\\&=\sum\frac{a-b}{AB}\cdot\frac{(a+c)^2B^2-(b+c)^2A^2}{(a+c)B+(b+c)A}=\\&=\sum\frac{(a-b)^2}{AB}\cdot\frac{C_1}{(a+c)B+(b+c)A},\end{aligned}$$

where

$$C_1=[(a+c)+(b+c)]\left[(a+c)^2+(b+c)^2\right]-8ac(b+c)-8bc(a+c).$$

Let us denote $x=a+c$ and $y=b+c$. Since $4ac\leq x^2$ and $4bc\leq y^2$, we obtain

$$C_1=(x+y)(x^2+y^2)-2x^2y-2y^2x=(x+y)(x-y)^2\geq 0$$

Equality occurs if and only if $a=b=c$.

★

33. *If a, b, c are non-negative numbers, then*

$$\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab}\le \frac{3}{2}(a+b+c).$$

First Solution (by *Tsoi Yun Pui*). Assume that $a \ge b \ge c$. Since

$$\sqrt{a^2+bc}\le a+\frac{c}{2}$$

and

$$\sqrt{b^2+ca}+\sqrt{c^2+ab}\le\sqrt{2(b^2+ca)+2(c^2+ab)},$$

it suffices to show that

$$\sqrt{2(b^2+c^2+ab+ca)}\le\frac{a+3b+2c}{2}.$$

By squaring, the inequality becomes

$$a^2+b^2-4c^2-2ab+12bc-4ca\ge 0,$$

or

$$(a-b-2c)^2+8c(b-c)\ge 0,$$

which is clearly true.
For $a \ge b \ge c$, equality occurs if and only if $(a,b,c)\sim(1,1,0)$

Second Solution. For $a=b=c=0$, the inequality is trivial. Consider now $a \ge b \ge c$, $a>0$. Since

$$\sqrt{b^2+ca}+\sqrt{c^2+ab}\le\sqrt{2(b^2+c^2)+2a(b+c)},$$

it suffices to show that

$$\sqrt{a^2+bc}+\sqrt{2(b^2+c^2)+2a(b+c)}\le\frac{3}{2}(a+b+c).$$

Denoting $s=\dfrac{b+c}{2}$ $(s\le a)$ and $p=bc$, the inequality becomes

$$2\sqrt{a^2+p}+4\sqrt{2s^2-p+as}\le 3(a+2s),$$
$$4\sqrt{2s^2-p+as}\le 3(a+2s)-2\sqrt{a^2+p}.$$

By squaring, the last inequality transforms into

$$12\ a+2s)\sqrt{a^2+p}\le 13a^2+20as+4s^2+20p$$

or

$$12(a+2s)\left(\sqrt{a^2+p}-a\right) \le (a-2s)^2 + 20p.$$

Since $(a-2s)^2 \ge 0$ and

$$\sqrt{a^2+p}-a = \frac{p}{\sqrt{a^2+p}+a} \le \frac{p}{2a},$$

it suffices to show that

$$\frac{6(a+2s)p}{a} \le 20p.$$

This inequality is equivalent to $p(6s-7a) \le 0$, which is clearly true

★

34. *Let a, b, c be non-negative numbers such that $a^2+b^2+c^2=3$. Then,*

$$21 + 18abc \ge 13(ab+bc+ca).$$

Solution. We will use Schur's Inequality of fourth degree

$$a^4+b^4+c^4+2abc(a+b+c) \ge (a^2+b^2+c^2)(ab+bc+ca).$$

Let $s = a+b+c$. From $(a+b+c)^2 \ge a^2+b^2+c^2$, we get $s \ge \sqrt{3}$ Taking account of

$$a^4+b^4+c^4 = (a^2+b^2+c^2)^2 - 2(a^2b^2+b^2c^2+c^2a^2) =$$
$$= 9 - 2(ab+bc+ca)^2 + 4abcs$$

and

$$ab+bc+ca = \frac{s^2-3}{2},$$

from Schur's Inequality above, we obtain

$$abc \ge \frac{s^4-3s^2-18}{12s}.$$

Returning to the our inequality, we have

$$21+18abc-13(ab+bc+ca) \ge 21 + \frac{3(s^4-3s^2-18)}{2s} - \frac{13(s^2-3)}{2} =$$
$$= \frac{3s^4-13s^3-9s^2+81s-54}{2s} = \frac{(s-3)^2(3s^2+5s-6)}{2s} \ge 0.$$

Equality occurs if and only if $a=b=c=1$.

★

35. *Let a, b, c be non-negative numbers such that $a^2 + b^2 + c^2 = 3$. Then*

$$\frac{1}{5-2ab} + \frac{1}{5-2bc} + \frac{1}{5-2ca} \le 1.$$

First Solution. Let $s = a + b + c$. Then,

$$ab + bc + ca = \frac{s^2-3}{2},$$

and from

$$a^2 + b^2 + c^2 \le (a+b+c)^2 \le 3(a^2+b^2+c^2),$$

we get $\sqrt{3} \le s \le 3$. By expanding, our inequality becomes

$$4a^2b^2c^2 - 8abc(a+b+c) + 15(ab+bc+ca) - 25 \le 0,$$

or

$$8(s-abc)^2 + 7s^2 - 95 \le 0.$$

As shown in the preceding proof, fourth degree Schur's Inequality implies

$$abc \ge \frac{s^4 - 3s^2 - 18}{12s}.$$

Then, since $s - abc \ge s - \dfrac{s^3}{27} > 0$ and

$$0 < s - abc \le s - \frac{s^4 - 3s^2 - 18}{12s} = \frac{18 + 15s^2 - s^4}{12s},$$

it suffices to show that

$$\frac{(18 + 15s^2 - s^4)^2}{18s^2} + 7s^2 - 95 \le 0.$$

Substituting $s^2 = 9x$, $\dfrac{1}{3} \le x \le 1$, the inequality becomes successively

$$(2 + 15x - 9x^2)^2 + 2x(63x - 95) \le 0,$$
$$81x^4 - 270x^3 + 315x^2 - 130x + 4 \le 0,$$
$$(x-1)(81x^3 - 189x^2 + 126x - 4) \le 0.$$

Since

$$81x^3 - 189x^2 + 126x - 4 = 9(9x^3 - 21x^2 + 14x - 2) + 14 =$$
$$= 9(1-x)(-9x^2 + 12x - 2) + 14,$$

it suffices to show that $-9x^2 + 12x - 2 \geq 0$ Indeed, we have

$$-9x^2 + 12x - 2 = 1 + 3(3x-1)(1-x) > 0.$$

equality occurs if and only if $a = b = c = 1$

Second Solution. In the proof of the problem 62 from the first chapter we have shown that the following inequality holds for $p \geq 6$

$$\frac{1}{p - a^2b^2} + \frac{1}{p - b^2c^2} + \frac{1}{p - c^2a^2} \leq \frac{3}{p-1}.$$

Choosing $p = \frac{25}{4}$, the inequality becomes as follows

$$\frac{1}{(5-2ab)(5+2ab)} + \frac{1}{(5-2bc)(5+2bc)} + \frac{1}{(5-2ca)(5+2ca)} \leq \frac{1}{7}$$

or

$$\sum \frac{1}{5-2ab} + \sum \frac{1}{5+2ab} \leq \frac{10}{7}.$$

If we show that

$$\frac{3}{7} \leq \sum \frac{1}{5+2ab},$$

the proof is finished Indeed, this inequality follows by the Cauchy-Schwarz Inequality

$$\sum \frac{1}{5+2ab} \geq \frac{9}{\sum(5+2ab)} = \frac{9}{15 + 2(ab+bc+ca)}.$$

and the well-known inequality

$$ab + bc + ca \leq a^2 + b^2 + c^2 = 3.$$

★

36. *Let a, b, c be non-negative numbers such that $a^2 + b^2 + c^2 = 3$. Then,*

$$(2-ab)(2-bc)(2-ca) \geq 1.$$

First Solution We will use Schur's Inequality of third degree

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca).$$

Let $s = a+b+c$, $s \leq 3$ From

$$2(ab+bc+ca) = s^2 - (a^2+b^2+c^2) = s^2 - 3$$

and Schur's Inequality, we obtain

$$9abc \geq s^3 - 6s.$$

We have

$$(2-ab)(2-bc)(2-ca)-1 = 7-4(ab+bc+ca) + 2abc(a+b+c) - a^2b^2c^2 =$$
$$= 7 - 2(s^2-3) + 2abcs - a^2b^2c^2 = 13 - s^2 - (s-abc)^2.$$

Since

$$0 < s - abc \leq s - \frac{s^3-6s}{9} = \frac{15s - s^3}{9},$$

it suffices to show that

$$13 - s^2 - \frac{s^2(15-s^2)^2}{81} \geq 0.$$

Substituting $s = 3\sqrt{x}$, $x \leq 1$, the inequality becomes

$$13 - 34x + 30x^2 - 9x^3 \geq 0.$$

It is true because

$$13 - 34x + 30x^2 - 9x^3 = (1-x)(13 - 21x + 9x^2) =$$
$$= (1-x)\left[1 + 3(1-x)(4-3x)\right].$$

Equality occurs if and only if $a = b = c = 1$.

Second Solution (by *Marian Tetiva*) We will use the "mixing variables" method. Assume, without loss of generality, that $a \leq 1$ and then show that

$$(2-bc)(2-ca)(2-ab) \geq (2-x^2)(2-ax)^2 \geq 1$$

for $x = \sqrt{\dfrac{b^2+c^2}{2}} = \sqrt{\dfrac{3-a^2}{2}}$.

The left inequality follows by multiplying the inequalities

$$2 - bc \geq 2 - x^2$$

and

$$(2 - ca)(2 - ab) \geq (2 - ax)^2.$$

After some manipulations, the last inequality becomes

$$\frac{4a(b-c)^2}{b+c+2x} \geq a^2(b-c)^2.$$

So, it is enough to show that

$$4 \geq a(b+c) + 2ax.$$

We have

$$4 - a(b+c) - 2ax \geq 4(1 - ax) = 2\left(2 - a\sqrt{6 - 2a^2}\right) \geq 0,$$

because

$$2 - a\sqrt{6-2a^2} = \frac{2(1-a^2)(2-a^2)}{2 + a\sqrt{6-2a^2}} \geq 0.$$

The right inequality $(2 - x^2)(2 - ax)^2 \geq 1$ is equivalent to

$$(1 + a^2)(2 - ax)^2 \geq 2.$$

Since $2(1 + a^2) \geq (1 + a)^2$, it suffices to show that

$$(1 + a)^2(2 - ax)^2 \geq 4$$

or

$$(1 + a)(2 - ax) \geq 2.$$

We have

$$(1+a)(2-ax)-2 = a(2-x-ax) = \frac{a(a^4 + 2a^3 - 2a^2 - 6a + 5)}{2(2 + x + ax)} =$$

$$= \frac{a(a-1)^2(a^2 + 4a + 5)}{2(2 + x + ax)} \geq 0.$$

★

37. *Let a, b, c be non-negative numbers such that $a+b+c=2$. Prove that*

$$\frac{bc}{a^2+1}+\frac{ca}{b^2+1}+\frac{ab}{c^2+1}\le 1.$$

Solution. Write the inequality as

$$\sum bc(b^2+1)(c^2+1)\le (a^2+1)(b^2+1)(c^2+1),$$

or

$$\sum b^3c^3+\sum bc(b^2+c^2)+\sum bc\le a^2b^2c^2+\sum b^2c^2+\sum a^2+1$$

Let $x=ab+bc+ca$ and $p=abc$ From $(a+b+c)^2\ge 3(ab+bc+ca)$ we get $x\le\frac{4}{3}$, and from $a+b+c\ge 3\sqrt[3]{abc}$ we get $p\le\frac{8}{27}$. We have

$$\sum a^2=4-2x,\ \ \sum b^2c^2=x^2-4p,$$
$$\sum bc(b+c)=\left(\sum a\right)\left(\sum bc\right)-3abc=2x-3p,$$
$$\sum a^3=\left(\sum a\right)\left(\sum a^2\right)-\sum bc(b+c)=8-6x+3p,$$
$$\sum a^4=\left(\sum a^2\right)^2-2\sum b^2c^2=16-16x+2x^2+8p,$$
$$\sum bc(b^2+c^2)=\left(\sum bc\right)\left(\sum a^2\right)-abc\sum a=4x-2x^2-2p,$$
$$\sum b^3c^3=\left(\sum bc\right)\left(\sum b^2c^2\right)-abc\sum bc(b+c)=x^3-6px+3p^2$$

Thus, the inequality is equivalent to

$$E=(1-x)(5-2x+x^2)+(6x-2)p-2p^2\ge 0$$

First Solution. To kill the terms in p and p^2, we will use the non-negative expression

$$A=(a-b)^2(b-c)^2(c-a)^2\quad\text{and}\quad B=\sum a^2(a-b)(a-c)$$

From

$$A=\sum b^2c^2(b^2+c^2)-2\sum b^3c^3+2abc\sum bc(b+c)-2abc\sum a^3-6a^2b^2c^2=$$
$$=4x^2(1-x)+4(9x-8)p-27p^2$$

and

$$B=\sum a^4+abc\sum a-\sum bc(b^2+c^2)=4(1-x)(4-x)+12p,$$

we get

$$6A+\frac{5}{2}(1+9x)B=2(1-x)(20+175x-33x^2)+81\left[(6x-2)p-2p^2\right],$$

and hence

$$81E=6A+\frac{5}{2}(1+9x)B+(1-x)^2(365-147x)\geq 0$$

Equality holds for $a=0$ and $b=c=1$, $b=0$ and $c=a=1$ and $a=b=1$

Second Solution. We will consider three cases.

Case $x<\frac{2}{3}$. Since

$$(6x-2)p-2p^2=6x-4+2(1-p)(2-3x+p)>6x-4,$$

we have

$$E>(1-x)(5-2x+x^2)+6x-4=(1-x)(1+x^2)+2x^2>0.$$

Case $\frac{2}{3}\leq x\leq 1$. Since

$$(6x-2)p-2p^2=2p(3x-1-p)\geq 2p(3x-2)\geq 0,$$

we have

$$E\geq(1-x)(5-2x+x^2)\geq 0.$$

Case $1<x\leq\frac{4}{3}$ As shown at the first solution, Schur's Inequality

$$\sum a^2(a-b)(a-c)\geq 0$$

implies

$$p\geq\frac{(x-1)(4-x)}{3}.$$

Since

$$(6x-2)p-2p^2=\frac{1}{4}(3x-1)^2-\frac{1}{2}(3x-1-2p)^2$$

and $3x-1-2p>2(1-p)>0$, it suffices to prove the inequality $E\geq 0$ for $p=\frac{(x-1)(4-x)}{3}$. In this case we get

$$E=\frac{(x-1)^2(37-11x-2x^2)}{9},$$

and clearly $E\geq 0$, since $x\leq\frac{4}{3}$.

★

38. *Let a, b, c be non-negative numbers, no two of which are zero. Then,*

$$\frac{a^3+3abc}{(b+c)^2}+\frac{b^3+3abc}{(c+a)^2}+\frac{c^3+3abc}{(a+b)^2}\ge a+b+c.$$

First Solution. Since

$$\frac{a^3+3abc}{(b+c)^2}-a=\frac{a(a^2+bc-b^2-c^2)}{(b+c)^2}=\frac{a^3(b+c)-a(b^3+c^3)}{(b+c)^3},$$

we can write the inequality in the form

$$\frac{X}{(b+c)^3}+\frac{Y}{(c+a)^3}+\frac{Z}{(a+b)^3}\ge 0,$$

where

$$X=a^3(b+c)-a(b^3+c^3),\ Y=b^3(c+a)-b(c^3+a^3),\ Z=c^3(a+b)-c(a^3+b^3).$$

We see that $X+Y+Z=0$. Without loss of generality, assume that $a\ge b\ge c$. We have

$$X=ab(a^2-b^2)+ac(a^2-c^2)\ge 0,$$
$$Z=ac(c^2-a^2)+bc(c^2-b^2)\le 0,$$

and

$$\frac{X}{(b+c)^3}+\frac{Y}{(c+a)^3}+\frac{Z}{(a+b)^3}=\frac{X}{(b+c)^3}-\frac{X+Z}{(c+a)^3}+\frac{Z}{(a+b)^3}=$$
$$=X\left[\frac{1}{(b+c)^3}-\frac{1}{(c+a)^3}\right]+(-Z)\left[\frac{1}{(c+a)^3}-\frac{1}{(a+b)^3}\right]\ge 0.$$

For $a\ge b\ge c$, equality occurs when either $(a,b,c)\sim(1,1,1)$ or $(a,b,c)\sim(1,1,0)$.

Second Solution (by *Ho Chung Siu*). As above, write the inequality in the form

$$\sum\frac{a(a^2+bc-b^2-c^2)}{(b+c)^2}\ge 0$$

Since

$$a(a^2+bc-b^2-c^2)=a(a-b)(a-c)+ab(a-b)-ca(c-a)$$

and

$$\sum \frac{a(a-b)(a-c)}{(b+c)^2} \geq 0,$$

it suffices to show that

$$\sum \frac{ab(a-b)}{(b+c)^2} - \sum \frac{ca(c-a)}{(b+c)^2} \geq 0.$$

Taking into account that

$$\sum \frac{ca(c-a)}{(b+c)^2} = \sum \frac{ab(a-b)}{(c+a)^2},$$

the last inequality becomes

$$\sum ab(a-b)\left[\frac{1}{(b+c)^2} - \frac{1}{(c+a)^2}\right] \geq 0,$$

or

$$\sum \frac{ab(a+b+2c)(a-b)^2}{(b+c)^2(c+a)^2} \geq 0.$$

★

39. *Let a, b, c be positive numbers such that $a^4 + b^4 + c^4 = 3$. Then,*

$$a) \qquad \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq 3;$$

$$b) \qquad \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{3}{2}.$$

Solution (by *Pam Kim Hung*). a) By Hölder's Inequality, we have

$$\left(\sum \frac{a^2}{b}\right)\left(\sum \frac{a^2}{b}\right)\left(\sum a^2b^2\right) \geq \left(\sum a^2\right)^3.$$

Therefore, it suffices to show that

$$\left(\sum a^2\right)^3 \geq 9 \sum a^2b^2.$$

Write the inequality in the homogeneous form

$$\left(\sum a^2\right)^3 \geq 3\left(\sum a^2b^2\right)\sqrt{3\sum a^4},$$

or

$$x^3 \geq 3y\sqrt{3(x^2-2y)},$$

where $x = \sum a^2$ and $y = \sum a^2b^2$ By squaring, the inequality becomes

$$x^6 - 27x^2y^2 + 54y^3 \geq 0.$$

It is true because

$$x^6 - 27x^2y^2 + 54y^3 = (x^2 - 3y)^2(x^2 + 6y) \geq 0.$$

Equality holds if and only if $(a, b, c) = (1, 1, 1)$.

b) By Hölder's Inequality, we have

$$\left(\sum \frac{a^2}{b+c}\right)\left(\sum \frac{a^2}{b+c}\right)\left[\sum a^2(b+c)^2\right] \geq \left(\sum a^2\right)^3$$

Therefore, it suffices to show that

$$\left(\sum a^2\right)^3 \geq \frac{9}{4}\sum a^2(b+c)^2.$$

Using the above inequality $\left(\sum a^2\right)^3 \geq 9\sum a^2b^2$, we still have to prove that

$$\sum a^2b^2 \geq \frac{1}{4}\sum a^2(b+c)^2$$

This inequality is equivalent to

$$\sum a^2(b-c)^2 \geq 0,$$

which is clearly true. Equality occurs if and only if $(a, b, c) = (1, 1, 1)$

★

40. *If a, b, c are positive numbers, then*

$$\frac{a^3 - b^3}{a+b} + \frac{b^3 - c^3}{b+c} + \frac{c^3 - a^3}{c+a} \leq \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{8}.$$

Solution (by *Darij Grinberg*). Since

$$\begin{aligned}\sum(b^3-c^3)(a+b)(c+a) &= \sum(b^3-c^3)a^3 + \sum(b^3-c^3)(ab+bc+ca) = \\ &= a^2b^3 + b^2c^3 + c^2a^3 - a^3b^2 - b^3c^2 - c^3a^2 = \\ &= (a-b)(b-c)(c-a)(a+b+bc+ca),\end{aligned}$$

the inequality is equivalent to

$$\frac{(a-b)(b-c)(c-a)(ab+bc+ca)}{(a+b)(b+c)(c+a)} \le \frac{(a-b)^2+(b-c)^2+(c-a)^2}{8}.$$

Assume that $a=\min\{a,b,c\}$. For $a \le c \le b$, the inequality is trivial, because its left hand side is either negative or zero. Consider now that $a \le b < c$, and denote $b=a+x$ and $c=a+y$ $(0 \le x < y)$. Since

$$(a+b)(b+c)(c+a) > (b+c)(ab+bc+ca) > (b+c-2a)(ab+bc+ca),$$

it suffices to show that

$$\frac{(b-a)(c-b)(c-a)}{b+c-2a} \le \frac{(a-b)^2+(b-c)^2+(c-a)^2}{8},$$

that is

$$\frac{xy(y-x)}{x+y} \le \frac{x^2+(y-x)^2+y^2}{8}$$

This inequality is equivalent to

$$x^3+y(2x-y)^2 \ge 0,$$

which is clearly true. Equality occurs if and only if $(a,b,c) \sim (1,1,1)$

★

41. *Let a,b,c be non-negative numbers, no two of which are zero. Prove that*

$$\frac{a^2}{(2a+b)(2a+c)}+\frac{b^2}{(2b+c)(2b+a)}+\frac{c^2}{(2c+a)(2c+b)} \le \frac{1}{3}.$$

Solution. The inequality is equivalent to each of the inequalities

$$\sum\left[\frac{a}{3(a+b+c)}-\frac{a^2}{(2a+b)(2a+c)}\right] \ge 0,$$

$$\sum \frac{a(a-b)(a-c)}{(2a+b)(2a+c)} \ge 0.$$

Due to symmetry, we may consider that $a \ge b \ge c$. Since

$$\frac{c(c-a)(c-b)}{(2c+a)(2c+b)} \ge 0,$$

it suffices to show that

$$\frac{a(a-b)(a-c)}{(2a+b)(2a+c)}+\frac{b(b-c)(b-a)}{(2b+c)(2b+a)}\geq 0$$

Writing this inequality in the form

$$(a-b)^2\left[(a+b)(2ab-c^2)+c(a^2+b^2+5ab)\right]\geq 0,$$

we see that it is true. For $a\geq b\geq c$, equality occurs when either $(a,b,c)\sim(1,1,1)$ or $(a,b,c)\sim(1,1,0)$

★

42. *Let a,b,c be non-negative numbers, no two of which are zero. Prove that*

$$\frac{1}{5(a^2+b^2)-ab}+\frac{1}{5(b^2+c^2)-bc}+\frac{1}{5(c^2+a^2)-ca}\geq\frac{1}{a^2+b^2+c^2}.$$

Solution. The hint is to apply Cauchy-Schwarz Inequality after making the numerators of the homogeneous fractions to be non-negative and as small as possible To do this, we see that

$$\frac{a^2+b^2+c^2}{5(b^2+c^2)-bc}-\frac{1}{5}=\frac{5a^2+bc}{25(b^2+c^2)-5bc}>0$$

Thus, we may write the inequality as

$$\sum\frac{5a^2+bc}{5(b^2+c^2)-bc}\geq 2.$$

According to Cauchy-Schwarz Inequality, we have

$$\sum\frac{5a^2+bc}{5(b^2+c^2)-bc}\geq\frac{\left(5\sum a^2+\sum bc\right)^2}{\sum(5b^2+5c^2-bc)(5a^2+bc)},$$

and it remains to show that

$$\left(5\sum a^2+\sum bc\right)^2\geq 2\sum\left(5b^2+5c^2-bc\right)\left(5a^2+bc\right).$$

This inequality reduces to

$$25\sum a^4+22abc\sum a\geq 47\sum b^2c^2$$

We can get it by summing the inequalities

$$\sum a^4 + abc\sum a \geq 2\sum b^2c^2$$

and

$$\sum a^4 \geq \sum b^2c^2,$$

multiplied by 22 and 3, respectively. The former inequality follows by summing up the well-known fourth degree Schur's Inequality

$$\sum a^4 + abc\sum a \geq \sum bc(b^2 + c^2)$$

to

$$\sum bc(b^2 + c^2) \geq 2\sum b^2c^2.$$

The last inequality is equivalent to

$$\sum bc(b - c)^2 \geq 0.$$

Equality occurs for $a = b = c$

★

43. *Let a, b, c be non-negative real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that*

$$\frac{bc}{a^2+1} + \frac{ca}{b^2+1} + \frac{ab}{c^2+1} \leq \frac{3}{4}.$$

Solution. Since

$$\frac{1}{2} - \frac{bc}{a^2+1} = \frac{a^2+1-2bc}{2(a^2+1)} = \frac{2a^2+(b-c)^2}{2(a^2+1)} \geq 0,$$

write the inequality as

$$\sum \frac{a^2+1-2bc}{a^2+1} \geq \frac{3}{2},$$

and apply the Cauchy-Schwarz Inequality·

$$\sum \frac{a^2+1-2bc}{a^2+1} \geq \frac{\left[\sum(a^2-2bc+1)\right]^2}{\sum(a^2+1)(a^2+1-2bc)} =$$

$$= \frac{4\left(2-\sum bc\right)^2}{\sum(a^2+1)(a^2+1-2bc)}.$$

Thus, it suffices to show that

$$8\left(2-\sum bc\right)^2 \ge 3\sum(a^2+1)(a^2+1-2bc).$$

This inequality is equivalent to

$$17+8\sum b^2c^2+22abc\sum a \ge 3\sum a^4+26\sum bc.$$

Taking account of $\sum a^4 = 1-2\sum b^2c^2$, the inequality becomes successively as follows

$$7+7\sum b^2c^2+11abc\sum a \ge 13\sum bc,$$

$$7+7\left(\sum bc\right)^2 \ge 13\sum bc+3abc\sum a,$$

$$\left(1-\sum bc\right)\left(7-6\sum bc\right)+\left(\sum bc\right)^2-3abc\sum a \ge 0,$$

$$\left(1-\sum bc\right)\left(7-6\sum bc\right)+\frac{1}{2}\sum a^2(b-c)^2 \ge 0.$$

Since $\sum bc \le \sum a^2 = 1$, the last inequality is clearly true. Equality occurs for $a=b=c=\dfrac{1}{\sqrt{3}}$.

★

44. *Let a, b, c be non-negative numbers such that $a^2+b^2+c^2=1$ Prove that*

$$\frac{1}{3+a^2-2bc}+\frac{1}{3+b^2-2ca}+\frac{1}{3+c^2-2ab} \le \frac{9}{8}.$$

Solution. Since

$$\frac{1}{2}-\frac{1}{3+a^2-2bc}=\frac{1+a^2-2bc}{2(3+a^2-2bc)}=\frac{2a^2+(b-c)^2}{2(3+a^2-2bc)} \ge 0,$$

write the inequality as

$$\sum\frac{1+a^2-2bc}{3+a^2-2bc} \ge \frac{3}{4}.$$

By the Cauchy-Schwarz Inequality we have

$$\sum\frac{1+a^2-2bc}{3+a^2-2bc} \ge \frac{\left[\sum(1+a^2-2bc)\right]^2}{\sum(3+a^2-2bc)(1+a^2-2bc)} =$$

$$=\frac{4\left(2-\sum bc\right)^2}{8-4\sum bc+\sum(1+a^2-2bc)^2}.$$

Thus, it suffices to show that

$$16\left(2-\sum bc\right)^2 \geq 24-12\sum bc+3\sum(1+a^2-2bc)^2.$$

This inequality is equivalent to

$$25+4\sum b^2c^2+44abc\sum a \geq 3\sum a^4+40\sum bc.$$

Since $\sum a^4 = 1-2\sum b^2c^2$, the inequality becomes

$$11+5\sum b^2c^2+22abc\sum a \geq 20\sum bc.$$

Setting $x = ab+bc+ca$, we may write the inequality as

$$11-20x+5x^2+12abc\sum a \geq 0$$

The Schur's Inequality of fourth degree

$$\sum a^4+2abc\sum a \geq \left(\sum a^2\right)\left(\sum bc\right)$$

is equivalent to

$$6abc\sum a \geq 2x^2+x-1.$$

Therefore,

$$11-20x+5x^2+12abc\sum a \geq 11-20x+5x^2+2(2x^2+x-1) =$$
$$= 9(x-1)^2 \geq 0.$$

Equality occurs for $a=b=c=\dfrac{1}{\sqrt{3}}$.

★

45. *If a, b, c are positive numbers, then*

$$\frac{4a^2-b^2-c^2}{a(b+c)}+\frac{4b^2-c^2-a^2}{b(c+a)}+\frac{4c^2-a^2-b^2}{c(a+b)} \leq 3.$$

Solution. Write the inequality as $E \geq 0$, where

$$E=\sum\frac{b^2+c^2-4a^2+a(b+c)}{a(b+c)}.$$

We have

$$
\begin{aligned}
2E - \sum \frac{(b-c)^2}{a(b+c)} &= \sum \frac{(b+c)^2 - 8a^2 + 2a(b+c)}{a(b+c)} = \\
&= \sum \frac{(b+c)^2 - 4a^2 + 2a(b+c-2a)}{a(b+c)} = \\
&= \sum \frac{(b+c-2a)(b+c+4a)}{a(b+c)} = \\
&= \sum (b+c-2a)\left(\frac{1}{a} + \frac{4}{b+c}\right) = \\
&= \sum (c-a)\left(\frac{1}{a} + \frac{4}{b+c}\right) - \sum (a-b)\left(\frac{1}{a} + \frac{4}{b+c}\right) = \\
&= \sum (b-c)\left(\frac{1}{c} + \frac{4}{a+b}\right) - \sum (b-c)\left(\frac{1}{b} + \frac{4}{c+a}\right) = \\
&= \sum (b-c)^2 \left[\frac{1}{bc} - \frac{4}{(a+b)(c+a)}\right] \geq \\
&\geq \sum (b-c)^2 \left(\frac{1}{bc} - \frac{4}{ab+bc+ca}\right),
\end{aligned}
$$

and hence

$$
\begin{aligned}
2E &\geq \sum (b-c)^2 \left(\frac{1}{ab+ac} + \frac{1}{bc} - \frac{4}{ab+bc+ca}\right) = \\
&= \sum (b-c)^2 \left[\frac{ab+bc+ca}{abc(b+c)} - \frac{4}{ab+bc+ca}\right] = \\
&= \frac{1}{abc(ab+bc+ca)} \sum \frac{(b-c)^2(ab-bc+ca)^2}{b+c} \geq 0.
\end{aligned}
$$

Equality holds if and only if $a = b = c$.

★

46. *If a, b, c are positive numbers such that $abc = 1$, then*

$$
a^2 + b^2 + c^2 + 6 \geq \frac{3}{2}\left(a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)
$$

Solution (by *Michael Rozenberg*). Without loss of generality, assume that $a = \min\{a, b, c\}$. Let $x = \sqrt{bc}$ $(x \geq 1)$ and

$$
F(a, b, c) = a^2 + b^2 + c^2 + 6 - \frac{3}{2}\left(a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).
$$

We will show that

$$F(a,b,c) \geq F\left(a, \sqrt{bc}, \sqrt{bc}\right) \geq 0.$$

We have

$$F(a,b,c)-F\left(a,\sqrt{bc},\sqrt{bc}\right) = (b-c)^2 - \frac{3}{2}\left(b+c-2\sqrt{bc}+\frac{1}{b}+\frac{1}{c}-\frac{2}{\sqrt{bc}}\right) =$$
$$= \frac{1}{2}\left(\sqrt{b}-\sqrt{c}\right)^2\left[2\left(\sqrt{b}+\sqrt{c}\right)^2-3-\frac{3}{bc}\right] \geq$$
$$\geq \frac{1}{2}\left(\sqrt{b}-\sqrt{c}\right)^2\left(8\sqrt{bc}-3-\frac{3}{bc}\right) \geq$$
$$\geq \frac{1}{2}\left(\sqrt{b}-\sqrt{c}\right)^2(8-3-3) \geq 0$$

and

$$F\left(a,\sqrt{bc},\sqrt{bc}\right) = F\left(\frac{1}{x^2},x,x\right) = \frac{x^6-6x^5+12x^4-6x^3-3x^2+2}{2x^4} =$$
$$= \frac{(x-1)^2(x^4-4x^3+3x^2+4x+2)}{2x^4} =$$
$$= \frac{(x-1)^2\left[(x^2-2x-1)^2+x^2+1\right]}{2x^4} \geq 0.$$

Equality holds if and only if $a = b = c = 1$

★

47. *Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that*

$$a_1 a_2 \ldots a_n \left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n+3\right) \leq 3$$

Solution. We will use the induction way. For $n = 2$, the inequality is true because it reduces to $a_1 a_2 \leq 1$ with $a_1 + a_2 = 2$. Assume now that $a_1 \geq a_2 \geq \cdots \geq a_n$ and denote by $E_n(a_1, a_2, \ldots, a_n)$ the left hand side of the inequality We have

$$a_n \leq \frac{a_1+a_2+\cdots+a_n}{n} = 1 \leq a_1.$$

We will show that

$$E_n(a_1,a_2,\ldots,a_n) \leq E_n(b_1,a_2,\ldots,a_{n-1},1) \leq 3,$$

where $b_1 = a_1 + a_n - 1$, $b_1 > 0$

The right side inequality follows by the inductive hypothesis, because $b_1 + a_2 + \cdots + a_{n-1} = n - 1$ and

$$E_n(b_1, a_2, \ldots, a_{n-1}, 1) = E_{n-1}(b_1, a_2, \ldots, a_{n-1}) \leq 3.$$

The left side inequality is equivalent to

$$(1 - a_1)(1 - a_n)\left(\frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} - n + 3\right) \leq 0.$$

It is true, since $1 - a_1 \leq 0$, $1 - a_n \geq 0$ and

$$\frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} \geq \frac{(n-2)^2}{a_2 + \cdots + a_{n-1}} = \frac{(n-2)^2}{n - a_1 - a_n} > \frac{(n-2)^2}{n - a_1} \geq$$
$$\geq \frac{(n-2)^2}{n-1} > n - 3.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n = 1$.

★

48. *Let a, b, c be the side lengths of a triangle. If $a^2 + b^2 + c^2 = 3$, then*

$$ab + bc + ca \geq 1 + 2abc.$$

Solution. Write the inequality in the homogeneous form

$$\sqrt{\frac{a^2 + b^2 + c^2}{3}}\left[3(ab + bc + ca) - \left(a^2 + b^2 + c^2\right)\right] \geq 6abc.$$

Since

$$\sqrt{\frac{a^2 + b^2 + c^2}{3}} \geq \frac{a + b + c}{3},$$

it suffices to show that

$$(a + b + c)\left[3(ab + bc + ca) - (a^2 + b^2 + c^2)\right] \geq 18abc.$$

Using the classical substitution $a = y+z$, $b = z+x$ and $c = x+y$ $(x, y, z > 0)$, the inequality becomes

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x + y) + yz(y + z) + zx(z + x),$$

which is just Schur's Inequality. Equality occurs if and only if $a = b = c = 1$.

★

49. *Let a, b, c be the side lengths of a triangle. If $a^2+b^2+c^2=3$, then*

$$a+b+c \geq 2+abc.$$

Solution. Without loss of generality, assume that $a \geq b \geq c$ From

$$3=a^2+b^2+c^2 \geq a^2+\frac{1}{2}(b+c)^2 > \frac{3}{2}a^2,$$

it follows that $a<\sqrt{2}$ Let

$$E(a,b,c)=a+b+c-2-abc$$

and $t=\sqrt{\dfrac{b^2+c^2}{2}}$, $t \leq 1 \leq a$. We will show that

$$E(a,b,c) \geq E(a,t,t) \geq 0.$$

With regard to the left side inequality, we have

$$E(a,b,c)-E(a,t,t)=a(t^2-bc)-(2t-b-c)=\frac{a(b-c)^2}{2}-\frac{(b-c)^2}{2t+b+c}=$$

$$=\frac{(b-c)^2}{2}\left(\frac{3a}{a^2+2t^2}-\frac{2}{2t+b+c}\right)=$$

$$=\frac{(b-c)^2\left[2t(3a-2t)+a(3b+3c-2a)\right]}{2(a^2+2t^2)(2t+b+c)} \geq 0,$$

because $3a-2t>2(a-t) \geq 0$ and $3(b+c)-2a>2(b+c-a)>0$.

Since $t \leq 1$ and

$$E(a,t,t)=a+2t-2-at^2=(1-t)(a+at-2),$$

the right inequality $E(a,t,t) \geq 0$ is true if and only if $at \geq 2-a$; that is

$$a\sqrt{\frac{3-a^2}{2}} \geq 2-a$$

By squaring throughout, the inequality becomes

$$(a-1)(8-a^2-a^3) \geq 0.$$

Since $1 \leq a<\sqrt{2}$, we have $a-1 \geq 0$ and $8-a-a^3>8-2-2\sqrt{2}>0$. Equality occurs if and only if $a=b=c=1$.

★

50. *If a,b,c are the side lengths of a non-isosceles triangle, then*

$$a)\qquad \left|\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}\right|>5;$$

$$b)\qquad \left|\frac{a^2+b^2}{a^2-b^2}+\frac{b^2+c^2}{b^2-c^2}+\frac{c^2+a^2}{c^2-a^2}\right|>3.$$

Solution. Since the inequalities are symmetric, we will consider $a>b>c$.

a) Set $x=a-c$ and $y=b-c$. From $a>b>c$ and $a<b+c$, it follows that $x>y>0$ and $c>x-y$. So, we have

$$\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}=\frac{2c+x+y}{x-y}+\frac{2c+y}{y}-\frac{2c+x}{x}=$$
$$=2c\left(\frac{1}{x-y}+\frac{1}{y}-\frac{1}{x}\right)+\frac{x+y}{x-y}>$$
$$>\frac{2c}{y}+\frac{x+y}{x-y}>\frac{2(x-y)}{y}+\frac{x+y}{x-y}=$$
$$=2\left(\frac{x-y}{y}+\frac{y}{x-y}\right)+1\geq 5.$$

b) We will show that

$$\frac{a^2+b^2}{a^2-b^2}+\frac{b^2+c^2}{b^2-c^2}+\frac{c^2+a^2}{c^2-a^2}>3;$$

that is

$$\frac{b^2}{a^2-b^2}+\frac{c^2}{b^2-c^2}>\frac{a^2}{a^2-c^2}.$$

Since $\dfrac{a^2}{a^2-c}<\dfrac{(b+c)^2}{a^2-c^2}$, it suffices to show that

$$\frac{b^2}{a^2-b^2}+\frac{c^2}{b^2-c^2}>\frac{(b+c)^2}{a^2-c^2}.$$

This inequality is equivalent to each of the following inequalities:

$$b^2\left(\frac{1}{a^2-b^2}-\frac{1}{a^2-c^2}\right)+c^2\left(\frac{1}{b^2-c^2}-\frac{1}{a^2-c^2}\right)>\frac{2bc}{a^2-c^2},$$

$$\frac{b^2(b^2-c^2)}{a^2-b^2}+\frac{c^2(a^2-b^2)}{b^2-c^2}>2bc,$$

$$\left[b(b^2-c^2)-c(a^2-b^2)\right]^2>0.$$

Under the condition $a > b > c$, equality occurs for a degenerate triangle with $a = b + c$ and $\frac{b}{c} = x_1$, where $x_1 \cong 1.5321$ is the positive root of the equation $x^3 - 3x - 1 = 0$

★

51. *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$a^2\left(\frac{b}{c} - 1\right) + b^2\left(\frac{c}{a} - 1\right) + c^2\left(\frac{a}{b} - 1\right) \geq 0.$$

First Solution. Using the substitutions $a = \frac{1}{x}$, $b = \frac{1}{y}$ and $c = \frac{1}{z}$, the inequality becomes

$$\frac{1}{x^2}\left(\frac{z}{y} - 1\right) + \frac{1}{y^2}\left(\frac{x}{z} - 1\right) + \frac{1}{z^2}\left(\frac{y}{x} - 1\right) \geq 0$$

or

$$yz^2(z - y) + zx^2(x - z) + xy^2(y - x) \geq 0.$$

Without loss of generality, assume that $a = \min\{a, b, c\}$, and hence $x = \max\{x, y, z\}$. Denoting the left hand side of the last inequality by $E(x, y, z)$, we will show that

$$E(x, y, z) \geq E(y, y, z) \geq 0.$$

We have

$$E(x,y,z) - E(y,y,z) = z(x^3 - y^3) - z^2(x^2 - y^2) + y^3(x - y) - y^2(x^2 - y^2) =$$
$$= (x - y)(x - z)(xz + yz - y^2).$$

Since $(x - y)(x - z) \geq 0$ and

$$xz + yz - y^2 \geq y(2z - y) = \frac{2b - c}{b^2 c} = \frac{(b - a) + (a + b - c)}{b^2 c} > 0,$$

it follows that $E(x, y, z) - E(y, y, z) \geq 0$ On the other hand, we have

$$E(y, y, z) = yz(y - z)^2 \geq 0$$

Equality occurs for $a = b = c$.

Second Solution (by *Alexandru Zamorzaev*). Write the inequality as

$$E(a,b,c) \geq 0,$$

where

$$E(a,b,c) = a^3b^2 + b^3c^2 + c^3a^2 - abc(a^2 + b^2 + c^2).$$

Since

$$2E(a,b,c) = \sum a^3(b-c)^2 - \sum a^2(b^3 - c^3)$$

and

$$\sum a^2(b^3 - c^3) = \sum a^2(b-c)^3,$$

we have

$$2E(a,b,c) = \sum a^2(b-c)^2(a-b+c) \geq 0.$$

★

52. *Let a, b, c be the lengths of the sides of an triangle. Prove that*

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 6\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

Solution. Since

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 9 = \sum \frac{(b-c)^2}{bc}$$

and

$$2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) - 3 = \sum \frac{2a-b-c}{b+c} = \sum \frac{a-b}{b+c} + \sum \frac{a-c}{b+c} =$$

$$= \sum \frac{a-b}{b+c} + \sum \frac{b-a}{c+a} = \sum \frac{(a-b)^2}{(b+c)(c+a)} = \sum \frac{(b-c)^2}{(c+a)(a+b)},$$

we may write the inequality as

$$\sum (b-c)^2 S_a \geq 0,$$

where

$$S_a = \frac{1}{bc} - \frac{3}{c+a)\ a+b)}.$$

Without loss of generality, assume that $a \geq b \geq c$ Since $S_a > 0$,

$$S_b = \frac{1}{ca} - \frac{3}{(a+b)(b+c)} = \frac{a(b-c) + c(b-a) + b^2}{ac(a+b)(b+c)} =$$
$$= \frac{a(b-c) + c(b+c-a) + b^2 - c^2}{ac(a+b)(b+c)} =$$
$$= \frac{(b-c)(a+b+c) + c(b+c-a)}{ac(a+b)(b+c)} > 0$$

and

$$\sum (b-c)^2 S_a \geq (c-a)^2 S_b + (a-b)^2 S_c \geq (a-b)^2 (S_b + S_c),$$

it suffices to show that

$$S_b + S_c \geq 0$$

This inequality is equivalent to

$$(a+b)(a+c)(b+c)^2 \geq 3abc(2a+b+c).$$

Let $b + c = 2x$ We have $a^2 \geq x^2 \geq bc$, and hence

$$(a+b)(a+c)(b+c)^2 - 3abc(2a+b+c) = 4x^2(a^2+2ax+bc) - 6abc(a+x) =$$
$$= 4ax^2(a+2x) - 2bc(3a^2+3ax-2x^2) \geq 4ax^2(a+2x) - 2x^2(3a^2+3ax-2x^2) =$$
$$= 2x^2(2x^2 + ax - a^2) = 2x^2(x+a)(2x-a) = 2x^2(x+a)(b+c-a) > 0.$$

Equality occurs if and only if $a = b = c$.

★

53. *If* $a_1, a_2, a_3, a_4, a_5, a_6 \in \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$, *then*

$$\frac{a_1 - a_2}{a_2 + a_3} + \frac{a_2 - a_3}{a_3 + a_4} + \cdots + \frac{a_6 - a_1}{a_1 + a_2} \geq 0.$$

Solution. Write the inequality as

$$\sum \left(\frac{a_1 - a_2}{a_2 + a_3} + \frac{1}{2}\right) \geq 3, \quad \sum \frac{2a_1 - a_2 + a_3}{a_2 + a_3} \geq 6.$$

Since $2a_1 - a_2 + a_3 \geq \frac{2}{\sqrt{3}} - \sqrt{3} + \frac{1}{\sqrt{3}} = 0$, by the Cauchy-Schwarz Inequality we get

$$\sum \frac{2a_1 - a_2 + a_3}{a_2 + a_3} \geq \frac{\left[\sum (2a_1 - a_2 + a_3)\right]^2}{\sum (a_2 + a_3)(2a_1 - a_2 + a_3)} = \frac{2\left(\sum a_1\right)^2}{\sum a_1 a_2 + \sum a_1 a_3}.$$

Thus, we still have to show that

$$\left(\sum a_1\right)^2 \geq 3\left(\sum a_1 a_1 + \sum a_1 a_3\right).$$

Indeed, letting $x = a_1 + a_4$, $y = a_2 + a_5$, $z = a_3 + a_6$, we have

$$\left(\sum a_1\right)^2 - 3\left(\sum a_1 a_2 + \sum a_1 a_3\right) = (x+y+z)^2 - 3(xy+yz+zx) \geq 0.$$

Equality occurs if and only if $a_1 = a_3 = a_5$ and $a_2 = a_4 = a_6$

★

54. *Let a, b, c be positive numbers such that $a^2 + b^2 + c^2 \geq 3$. Prove that*

$$\frac{a^5 - a^2}{a^5 + b^2 + c^2} + \frac{b^5 - b^2}{a^2 + b^5 + c^2} + \frac{c^5 - c^2}{a^2 + b^2 + c^5} \geq 0$$

Solution. The inequality is equivalent to

$$\frac{1}{a^5 + b^2 + c^2} + \frac{1}{a^2 + b^5 + c^2} + \frac{1}{a^2 + b^2 + c^5} \leq \frac{3}{a^2 + b^2 + c^2}.$$

Letting $a = tx$, $b = ty$ and $c = tz$, where $t > 0$ and $x, y, z > 0$ such that $x^2 + y^2 + z^2 = 3$, the condition $a^2 + b^2 + c^2 \geq 3$ imply $t \geq 1$, and the inequality becomes

$$\frac{1}{t^3x^5 + y^2 + z^2} + \frac{1}{x^2 + t^3y^5 + z^2} + \frac{1}{x^2 + y^2 + t^3z^5} \leq 1.$$

We see that it suffices to prove this inequality for $t = 1$. In this case, we may write the inequality in the form

$$\frac{1}{x^5 + 3 - x^2} + \frac{1}{y^5 + 3 - y^2} + \frac{1}{z^5 + 3 - z^2} \leq 1.$$

Without loss of generality, assume that $x \geq y \geq z$. Two cases are to consider.

Case $x \leq \sqrt{2}$. We have also $y, z \leq \sqrt{2}$. The desired inequality follows by adding the inequalities

$$\frac{1}{x^5 + 3 - x^2} \leq \frac{3 - x^2}{6}, \quad \frac{1}{y^5 + 3 - y^2} \leq \frac{3 - y^2}{6}, \quad \frac{1}{z^5 + 3 - z^2} \leq \frac{3 - z^2}{6}$$

We have

$$\frac{1}{x^5 + 3 - x^2} - \frac{3 - x^2}{6} = \frac{(x-1)^2(x^5 + 2x^4 - 3x^2 - 6x - 3)}{6(x^5 + 3 - x^2)}$$

and

$$x^5+2x^4-3x^2-6x-3=x^2\left(x^3+2x^2-3-\frac{6}{x}-\frac{3}{x^2}\right)\le$$
$$\le x^2\left(2\sqrt{2}+4-3-3\sqrt{2}-\frac{3}{2}\right)=-x^2\left(\frac{1}{2}+\sqrt{2}\right)<0.$$

Case $x>\sqrt{2}$. From $x^2+y^2+z^2=3$, it follows that $y^2+z^2<1$ Since

$$\frac{1}{x^5+3-x^2}+\frac{1}{y^5+3-y^2}+\frac{1}{z^5+3-z^2}<\frac{1}{x^5+3-x^2}+\frac{1}{3-y^2}+\frac{1}{3-z^2}$$

and

$$\frac{1}{x^5+3-x^2}<\frac{1}{2\sqrt{2}x^2+3-x^2}=\frac{1}{\left(2\sqrt{2}-1\right)x^2+3}<\frac{1}{\left(2\sqrt{2}-1\right)2+3}<\frac{1}{6},$$

it suffices to show that

$$\frac{1}{3-y^2}+\frac{1}{3-z^2}\le\frac{5}{6}.$$

Indeed,

$$\frac{1}{3-y^2}+\frac{1}{3-z^2}-\frac{5}{6}=\frac{9(y^2+z^2-1)-5y^2z^2}{6(3-y^2)(3-z^2)}<0,$$

which completes the proof. Equality occurs if and only if $a=b=c=1$.

Remark Since $abc\ge 1$ yields $a^2+b^2+c^2\ge 3$ (by the AM-GM Inequality), we get the following statement.

- *If* a,b,c *are positive numbers such that* $abc\ge 1$, *then*

$$\frac{a^5-a^2}{a^5+b^2+c^2}+\frac{b^5-b^2}{a^2+b^5+c^2}+\frac{c^5-c^2}{a^2+b^2+c^5}\ge 0.$$

This is a problem from IMO-2005, proposed by *Hojoo Lee*. A special award was given to *Iurie Boreico* from Moldova, who noticed in his solution that

$$\frac{a^2(a^3-1)}{a^5+b^2+c^2}\ge\frac{a^3-1}{a(a^2+b^2+c^2)},$$

and hence,

$$\sum\frac{a^5-a^2}{a^5+b^2+c^2}\ge\frac{1}{a^2+b^2+c^2}\sum\left(a^2-\frac{1}{a}\right)\ge$$
$$\ge\frac{1}{a^2+b^2+c^2}\sum(a^2-bc)=\frac{1}{2(a^2+b^2+c^2)}\sum(a-b)^2\ge 0$$

★

55. *Let a, b, c be positive numbers such that $x+y+z \geq 3$. Then,*

$$\frac{1}{x^3+y+z}+\frac{1}{x+y^3+z}+\frac{1}{x+y+z^3} \leq 1.$$

Solution. It is easy to check that it suffices to consider $x+y+z=3$. In this case, we may write the inequality in the form

$$\frac{1}{x^3-x+3}+\frac{1}{y^3-y+3}+\frac{1}{z^3-z+3} \leq 1.$$

Without loss of generality, assume that $x \geq y \geq z$ Two cases are to consider.

Case $x \leq 2$. We have also $y, z \leq \sqrt{2}$ The desired inequality follows by adding the inequalities

$$\frac{1}{x^3-x+3} \leq \frac{5-2x}{9}, \quad \frac{1}{y^3-y+3} \leq \frac{5-2y}{9}, \quad \frac{1}{z^3-z+3} \leq \frac{5-2z}{9}$$

Indeed,

$$\frac{1}{x^3-x+3}-\frac{5-2x}{9}=\frac{(x-1)^2(2x+3)(x-2)}{9(x^2-x+3)} \leq 0.$$

Case $x > 2$ From $x+y+z=3$, it follows that $y+z<1$. We have

$$\frac{1}{x^3-x+3}+\frac{1}{y^3-y+3}+\frac{1}{z^3-z+3} < \frac{1}{x^3-x+3}+\frac{1}{3-y}+\frac{1}{3-z} <$$
$$< \frac{1}{9}+\frac{1}{3-y}+\frac{1}{3-z}.$$

Thus, it is enough to show that

$$\frac{1}{3-y}+\frac{1}{3-z} \leq \frac{8}{9}.$$

Since $y+z<1$, we get

$$\frac{1}{3-y}+\frac{1}{3-z}-\frac{8}{9}=\frac{-3-15(1-y-z)-8yz}{9(3-y)(3-z)} < 0.$$

Equality occurs if and only if $x=y=z=1$.

Conjecture. *If $x_1, x_2, \ldots, x_n$ are non-negative numbers such that*

$$x_1+x_2+\cdots+x_n \geq n,$$

then for any $p > 1$ the inequalities hold

(a) $$\frac{1}{x_1^p + x_2 + \cdots + x_n} + \frac{1}{x_1 + x_2^p + \cdots + x_n} + \frac{1}{x_1 + x_2 + \cdots + x_n^p} \le 1;$$

(b) $$\frac{x_1}{x_1^p + x_2 + \cdots + x_n} + \frac{x_2}{x_1 + x_2^p + \cdots + x_n} + \frac{x_n}{x_1 + x_2 + \cdots + x_n^p} \le 1.$$

★

56. *Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1 x_2 \ldots x_n \ge 1$. If $\alpha > 1$, then*

$$\sum \frac{x_1^\alpha}{x_1^\alpha + x_2 + \cdots + x_n} \ge 1.$$

Solution. First we observe that it suffices to consider only the case

$$x_1 x_2 \ldots x_n = 1.$$

In order to show this, let $r = \sqrt[n]{x_1 x_2 \ldots x_n}$ and $y_i = \dfrac{x_i}{r}$ for $i = 1, 2, \ldots, n$. Note that $r \ge 1$ and $y_1 y_2 \ldots y_n = 1$. Thus, the inequality becomes

$$\sum \frac{y_1^\alpha}{y_1^\alpha + \dfrac{y_2 + \cdots + y_n}{r^{\alpha-1}}} \ge 1,$$

and we see that it suffices to prove it only for $r = 1$, that is, for $x_1 x_2 \ldots x_n = 1$

Under the assumption $x_1 x_2 \ldots x_n = 1$, we will show that there exists a suitable real p such that

$$\frac{x_1^\alpha}{x_1^\alpha + x_2 + \cdots + x_n} \ge \frac{x_1^p}{x_1^p + x_2^p + \cdots + x_n^p}. \tag{1}$$

If this claim is valid, then adding (1) with the analogous inequalities for $x_2, \ldots, x_n$ will yield the required inequality. Inequality (1) is equivalent to

$$x_2^p + \cdots + x_n^p \ge (x_2 \ldots x_n)^{\alpha - p}(x_2 + \cdots + x_n).$$

Choosing

$$p = \frac{(n-1)\alpha + 1}{n}, \ p > 1$$

reduces the inequality to the homogeneous inequality

$$x_2^p + \cdots + x_n^p \ge (x_2 \ldots x_n)^{\frac{p-1}{n-1}}(x_2 + \cdots + x_n)$$

Since

$$(x_2 \ldots x_n)^{\frac{p-1}{n-1}} \le \left(\frac{x_2 + \cdots + x_n}{n-1}\right)^{p-1}$$

(by the AM-GM Inequality), it is enough to show that

$$\frac{x_2^p + \cdots + x_n^p}{n-1} \ge \left(\frac{x_2 + \quad + x_n}{n-1}\right)^p.$$

This inequality follows by applying Jensen's Inequality to the convex function $f(x) = x^p$. Equality in the given inequality occurs if and only if $x_1 = x_2 = \cdots = x_n = 1$.

★

57. *Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1 x_2 \ . \ x_n \ge 1$. If $n \ge 3$ and $\dfrac{-2}{n-2} \le \alpha < 1$, then*

$$\sum \frac{x_1^\alpha}{x_1^\alpha + x_2 + \cdots + x_n} \le 1.$$

Solution. The first part of the proof is similar to the proof of the preceding inequality. Finally, we have to prove the inequality

$$x_2 + \cdots + x_n \ge (x_2 \ldots x_n)^{\frac{1-p}{n-1}} (x_2^p + \cdot \ \cdot + x_n^p)$$

for

$$p = \frac{(n-1)\alpha + 1}{n}, \quad \frac{-1}{n-2} \le p < 1.$$

For $p = \dfrac{-1}{n-2}$, the inequality reduces to

$$x_2 + \cdots + x_n \ge \sqrt[n-2]{x_3 \ldots x_n} + \cdots + \sqrt[n-2]{x_2 \ldots x_{n-1}},$$

which can be proved adding the inequalities

$$\frac{x_3 + \cdots + x_n}{n-2} \ge \sqrt[n-2]{x_3 \ldots x_n}, \quad . \, , \quad \frac{x_2 + \cdots + x_{n-1}}{n-2} \ge \sqrt[n-2]{x_2 \ldots x_{n-1}}.$$

For $\dfrac{-1}{n-2} < p < 1$, by the Weighted AM-GM Inequality, we have

$$\frac{1 + (n-2)p}{1-p} x_2 + x_3 + \cdots + x_n \ge \frac{n-1}{1-p} x_2^p (x_2 x_3 \ldots \ x_n)^{\frac{1-p}{n-1}}.$$

Adding this inequality to the analogous ones for $x_3, . \ . , x_n$, we get the required inequality. Equality occurs in the given inequality if and only if $x_1 = x_2 = \cdots = x_n = 1$.

★

58. *Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1 x_2 \ldots x_n \geq 1$. If $\alpha > 1$, then*

$$\sum \frac{x_1}{x_1^\alpha + x_2 + \cdots + x_n} \leq 1.$$

Solution. We will consider two cases $1 < \alpha \leq n+1$ and $\alpha \geq n - \frac{1}{n-1}$

Case $1 < \alpha \leq n+1$. Since $x_1 x_2 \ldots x_n \geq 1$ implies $x_1 + x_2 + \cdots + x_n \geq n$ (by the AM-GM Inequality), it suffices to show that the required inequality holds for $x_1 + x_2 + \cdots + x_n \geq n$. We may consider only the case $x_1 + x_2 + \cdots + x_n = n$. Indeed, if we set $r = \frac{x_1 + x_2 + \cdots + x_n}{n}$ and $y_i = \frac{x_i}{r}$ for $i = 1, 2, \ldots, n$, then $r \geq 1$ and $y_1 + y_2 + \cdots + y_n = 1$. The inequality becomes

$$\sum \frac{y_1}{r^{\alpha-1} y_1^\alpha + y_2 + \cdots + y_n} \leq 1,$$

and we see that it suffices to prove it for $r = 1$; that is, for $x_1 + x_2 + \cdots + x_n = n$. Under this assumption, write the required inequality in the form

$$\frac{x_1}{x_1^\alpha - x_1 + n} + \frac{x_2}{x_2^\alpha - x_2 + n} + \cdots + \frac{x_n}{x_n^\alpha - x_n + n} \leq 1.$$

For any $x > 0$, by Bernoulli's Inequality, we have

$$x^\alpha = [1 + (x-1)]^\alpha \geq 1 + \alpha(x-1),$$

and hence,

$$x^\alpha - x + n \geq n - \alpha + 1 + (\alpha - 1)x > 0$$

Consequently, it is suffices to show that

$$\sum_{i=1}^{n} \frac{x_i}{n - \alpha + 1 + (\alpha - 1)x_i} \leq 1.$$

This inequality clearly holds for $\alpha = n+1$. For $\alpha < n+1$, using

$$\frac{(\alpha-1)x_i}{n - \alpha + 1 + (\alpha - 1)x_i} = 1 - \frac{n - \alpha + 1}{n - \alpha + 1 + (\alpha - 1)x_i},$$

it may be rewritten as

$$\sum_{i=1}^{n} \frac{1}{n - \alpha + 1 + (\alpha - 1)x_i} \geq 1.$$

Setting $y_i = n - \alpha + 1 + (\alpha - 1)x_i$ for $i = 1, 2, \ldots, n$, we have $y_i > 0$ and $y_1 + y_2 + \cdots + y_n = n^2$ The inequality reduces to

$$\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n} \geq 1,$$

which is an immediate consequence of the AM-HM Inequality

$$(y_1 + y_2 + \cdots + y_n)\left(\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}\right) \geq n^2.$$

Case $\alpha \geq n - \frac{1}{n-1}$. As above, we may consider that $x_1 x_2 \ldots x_n = 1$. Under this assumption it suffices to show that

$$\frac{(n-1)x_1}{x_1^\alpha + x_2 + \ldots x_n} + \frac{x_1^p}{x_1^p + x_2^p + \cdots + x_n^p} \leq 1 \tag{2}$$

for a suitable real p, and to add then this inequality to the analogously inequalities for $x_2, \ldots, x_n$. Set $t = \sqrt[n-1]{x_2 \ldots x_n}$. By the AM-GM Inequality, we have $x_2 + \cdots + x_n \geq (n-1)t$ and $x_2^p + \cdots + x_n^p \geq (n-1)t^p$. Thus, it suffices to show that

$$\frac{(n-1)x_1}{x_1^\alpha + (n-1)t} + \frac{x_1^p}{x_1^p + (n-1)t^p} \leq 1.$$

Since $x_1 = \frac{1}{t^{n-1}}$, this inequality is equivalent to

$$(n-1)t^{n+q} - (n-1)t^q - t^{q-np} + 1 \geq 0,$$

where $q = (n-1)(\alpha - 1)$. We will now show that the inequality holds for

$$p = \frac{(n-1)(\alpha - n - 1)}{n}.$$

Indeed, for this value of p, the inequality successively becomes the following:

$$(n-1)t^{n+q} - (n-1)t^q - t^{n(n-1)} + 1 \geq 0,$$

$$(n-1)t^q(t^n - 1) - (t^n - 1)\left(t^{n^2-2n} + t^{n^2-3n} + \cdots + 1\right) \geq 0,$$

$$(t^n - 1)\left[\left(t^q - t^{n^2-2n}\right) + \left(t^q - t^{n^2-3n}\right) + \ldots (t^q - 1)\right] \geq 0.$$

We see that the last inequality is true for $q \geq n^2 - 2n$; that is, for $\alpha \geq n - \frac{1}{n-1}$. Equality occurs in the given inequality if and only if $x_1 = x_2 = \cdots = x_n = 1$.

★

59. *Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1 x_2 \ldots x_n \geq 1$ If $-1-\dfrac{2}{n-2} \leq \alpha < 1$, then*

$$\sum \frac{x_1}{x_1^\alpha + x_2 + \cdots + x_n} \geq 1$$

Solution. It suffices to consider only the case where $x_1 x_2 \ldots x_n = 1$. By the Cauchy-Schwarz Inequality, we have

$$\sum \frac{x_1}{x_1^\alpha + x_2 + \cdots + x_n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{\sum x_1 (x_1^\alpha + x_2 + \cdots + x_n)} =$$

$$= \frac{(x_1 + x_2 + \cdots + x_n)^2}{(x_1 + x_2 + \cdots + x_n)^2 + \sum_{i=1}^n x_i^{1+\alpha} - \sum_{i=1}^n x_i^2}.$$

Thus, we still have to show that

$$\sum_{i=1}^n x_i^2 \geq \sum_{i=1}^n x_i^{1+\alpha}$$

Case $-1 \leq \alpha < 1$. We can prove the inequality using Chebyshev's Inequality and the AM-GM Inequality, as follows:

$$\sum_{i=1}^n x_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^n x_i^{1-\alpha}\right)\left(\sum_{i=1}^n x_i^{1+\alpha}\right) \geq$$

$$\geq (x_1 x_2 \ldots x_n)^{(1-\alpha)/n} \sum_{i=1}^n x_i^{1+\alpha} = \sum_{i=1}^n x_i^{1+\alpha}.$$

Case $-1-\dfrac{2}{n-1} \leq \alpha < -1$. It is convenient to replace the numbers $x_1, x_2, \ldots, x_n$ by $x_1^{(n-1)/2}, x_2^{(n-1)/2}, \ldots, x_n^{(n-1)/2}$, respectively. We also use the substitution $q = \dfrac{(n-1)(1+\alpha)}{2}$, and note that $-1 \leq q < 0$. Thus, we have to prove that

$$\sum_{i=1}^n x_i^{n-1} \geq \sum_{i=1}^n x_i^q,$$

when $x_1 x_2 \ldots x_n = 1$. Using the well-known Maclaurin Inequality

$$\sum_{i=1}^n x_i^{n-1} \geq \sum_{\text{ciclic}} x_2 \ldots x_n,$$

and Chebyshev's Inequality together with the AM-GM Inequality, we get the desired inequality

$$\sum_{i=1}^{n} x_i^{n-1} \geq \sum_{i=1}^{n} \frac{1}{x_i} \geq \frac{1}{n}\left(\sum_{i=1}^{n} x_i^{-1-q}\right)\left(\sum_{i=1}^{n} x_i^{q}\right) \geq$$
$$\geq \sqrt[n]{(x_1x_2\ldots x_n)^{-1-q}}\sum_{i=1}^{n} x_i^{q} = \sum_{i=1}^{n} x_i^{q}$$

Now the proof is complete. Equality in the given inequality occurs if and only if $x_1 = x_2 = \cdots = x_n = 1$.

★

60. *Let $n \geq 3$ be an integer and let p be a real number such that $1<p<n-1$. If $0 < x_1, x_2, \ldots, x_n \leq \dfrac{pn-p-1}{p(n-p-1)}$ such that $x_1x_2\ldots x_n = 1$, then*

$$\frac{1}{1+px_1} + \frac{1}{1+px_2} + \cdots + \frac{1}{1+px_n} \geq \frac{n}{1+p}$$

Solution. We will prove by induction that

$$\frac{1}{1+qx_1} + \frac{1}{1+qx_2} + \cdots + \frac{1}{1+qx_n} \geq \frac{n}{1+q} \tag{3}$$

for any $q \geq p$. For $n = 2$, the inequality reduces to

$$\frac{(q-1)(x_1-1)^2}{(1+qx_1)(1+qx_2)} \geq 0,$$

which is clearly true. Consider now that the inequality holds for $n-1$, $n \geq 3$. Without loss of generality, assume that

$$x_{n-1} = \min\{x_1, x_2, \ldots, x_n\} \text{ and } x_n = \max\{x_1, x_2, \ldots, x_n\}.$$

The condition $x_1x_2\ldots x_n = 1$ implies $x_{n-1} \leq 1$ and $x_n \geq 1$. We must show that the inequality (3) holds for $x_1x_2\ldots x_n = 1$ and $x_1, x_2, \ldots, x_n \leq p_n$, where

$$p_n = \frac{pn-p-1}{p(n-p-1)}.$$

Without loss of generality, assume that

$$x_{n-1} = \min\{x_1, x_2, \ldots, x_n\} \text{ and } x_n = \max\{x_1, x_2, \ldots, x_n\}.$$

The condition $x_1x_2 \quad x_n = 1$ implies $x_{n-1} \le 1$ and $x_n \ge 1$ Since $x_{n-1}x_n \le x_n$, we have

$$x_1, \ \ldots, x_{n-2}, x_{n-1}x_n \le p_n \le p_{n-1},$$

and, by the inductive hypothesis, the inequality holds

$$\frac{1}{1+qx_1} + \quad \cdot + \frac{1}{1+qx_{n-2}} + \frac{1}{1+qx_{n-1}x_n} \ge \frac{n-1}{1+q}$$

for any $q \ge p$ with $1 < p < n-2$, that is, for any $q \ge p$ with $1 < p < n-1$ So, it remains to show that

$$\frac{1}{1+qx_{n-1}} + \frac{1}{1+qx_n} \ge \frac{1}{1+qx_{n-1}x_n} + \frac{1}{1+q},$$

which is equivalent to

$$(1-x_{n-1})(x_n-1)\left(q^2x_{n-1}x_n - 1\right) \ge 0.$$

Since this inequality is true for $q^2x_{n-1}x_n \ge 1$, we still have to show that (3) holds for $q^2x_{n-1}x_n < 1$ On this assumption, we have

$$\frac{1}{1+qx_{n-1}} + \frac{1}{1+qx_n} = 1 - \frac{qx_{n-1}}{1+qx_{n-1}} + \frac{1}{1+qx_n} >$$

$$> 1 - \frac{1}{1+qx_n} + \frac{1}{1+qx_n} = 1$$

Thus, it suffices to prove that

$$\frac{1}{1+qx_1} + \frac{1}{1+qx_2} + \ \cdot \ + \frac{1}{1+qx_{n-2}} \ge \frac{n-q-1}{1+q}$$

Taking into account that $x_i \le p_n$ for $i = 1, 2, \ldots, n-2$, we get

$$\sum_{i=1}^{n-2} \frac{1}{1+qx_i} - \frac{n-q-1}{1+q} \ge \frac{n-2}{1+qp_n} - \frac{n-q-1}{1+q} =$$

$$= \frac{nq - q - 1 - q(n-q-1)p_n}{(1+qp_n)(1+q)} =$$

$$= \frac{(q-p)\left[(pn-p-1)q + n - p - 1\right]}{p(n-p-1)(1+qp_n)(1+q)} \ge 0.$$

Equality in the original inequality occurs if and only if $x_1 = x_2 = \quad \cdot = x_n = 1$.

Remark For $p \to n-1$, we obtain the well-known inequality

$$\frac{1}{1+(n-1)x_1} + \frac{1}{1+(n-1)x_2} + \quad \cdot + \frac{1}{1+(n-1)x_n} \ge 1,$$

which holds for any positive numbers $x_1, x_2, \ \ldots, x_n$ with $x_1x_2\ldots x_n = 1$

★

61. *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{2}{(1+a)(1+b)(1+c)} \geq 1.$$

First Solution There are two of the numbers a, b, c either greater than or equal to 1, or less than or equal to 1. Let a and b be the numbers having this property; that is $(1-a)(1-b) \geq 0$. Since

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} - \frac{1}{1+ab} = \frac{ab(a-b)^2 + (1-ab)^2}{(1+a)^2(1+b)^2(1+ab)} \geq 0,$$

it suffices to show that

$$\frac{1}{1+ab} + \frac{1}{(1+c)^2} + \frac{2}{(1+a)(1+b)(1+c)} \geq 1$$

Substituting c by $\frac{1}{ab}$, the inequality becomes

$$\frac{1}{1+ab} + \frac{a^2b^2}{(1+ab)^2} + \frac{2ab}{(1+a)(1+b)(1+ab)} \geq 1.$$

This inequality is equivalent to

$$\frac{ab(1-a)(1-b)}{(1+a)(1+b)(1+ab)^2} \geq 0,$$

which is clearly true. Equality occurs if and only if $a = b = c = 1$.

Second Solution (after an idea of *Pham Kim Hung*) Set $\frac{1}{1+a} = \frac{1+x}{2}$, $\frac{1}{1+b} = \frac{1+y}{2}$ and $\frac{1}{1+c} = \frac{1+z}{2}$. That is $a = \frac{1-x}{1+x}$, $b = \frac{1-y}{1+y}$ and $c = \frac{1-z}{1+z}$, where $-1 < x, y, z < 1$ We have to prove that $x+y+z+xyz = 0$ implies

$$(1+x)^2 + (1+y)^2 + (1+z)^2 + (1+x)(1+y)(1+z) \geq 4;$$

that is

$$x^2 + y^2 + z^2 + 2(x+y+z) + xy + yz + zx \geq 0$$

Since

$$xy + yz + zx = \frac{(x+y+z)^2 - x^2 - y^2 - z^2}{2},$$

the inequality transforms into

$$x^2 + y^2 + z^2 + 4(x+y+z) + (x+y+z)^2 \geq 0.$$

Substituting $x + y + z$ by $-xyz$, the inequality becomes

$$x^2 + y^2 + z^2 + x^2y^2z^2 \geq 4xyz.$$

By the AM-GM Inequality, we get

$$x^2 + y^2 + z^2 + x^2y^2z^2 \geq 4\sqrt[4]{x^4y^4z^4} = 4|xyz| \geq 4xyz$$

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62. *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$a^2 + b^2 + c^2 + 9(ab + bc + ca) \geq 10(a + b + c).$$

First Solution. Write the inequality as $f(a, b, c) \geq 0$, where

$$f(a,b,c) = a^2 + b^2 + c^2 + 9\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 10(a + b + c).$$

Without loss of generality, assume that $a \geq 1$ We will show that

$$f(a,b,c) \geq f\left(a, \sqrt{bc}, \sqrt{bc}\right) \geq 0$$

The left inequality is true, because

$$f(a,b,c) - f\left(a, \sqrt{bc}, \sqrt{bc}\right) = (b-c)^2 + \frac{9\left(\sqrt{b}-\sqrt{c}\right)^2}{bc} - 10\left(\sqrt{b}-\sqrt{c}\right)^2 =$$
$$= \left(\sqrt{b} - \sqrt{c}\right)^2 \left[\left(\sqrt{b} + \sqrt{c}\right)^2 + \frac{9}{bc} - 10\right]$$

and

$$\left(\sqrt{b} + \sqrt{c}\right)^2 + \frac{9}{bc} - 10 \geq 4\sqrt{bc} + \frac{9}{bc} - 10 = \frac{4}{\sqrt{a}} + 9a - 10 \geq$$
$$\geq \frac{4}{a} + 9a - 10 > \frac{1}{a} + 9a - 10 = \frac{(a-1)(9a-1)}{a} \geq 0.$$

With regard to the right inequality, substituting $a = x^2$, $x \geq 1$, we have

$$f\left(a, \sqrt{bc}, \sqrt{bc}\right) = f\left(x^2, \frac{1}{x}, \frac{1}{x}\right) = \frac{x^6 - 10x^4 + 18x^3 - 20x + 11}{x^2} =$$

$$= \frac{(x-1)^2(x^4 + 2x^3 - 7x^2 + 2x + 11)}{x^2} \geq$$

$$\geq \frac{(x-1)^2(x^4 + 2x^3 - 7x^2 + 2x + 10)}{x^2} =$$

$$= \frac{(x-1)^2(x+1)(x^3 + x^2 - 8x + 10)}{x^2}.$$

Since

$$x^3 + x^2 - 8x + 10 \geq 2x^2 - 8x + 10 = 2(x-2)^2 + 2 > 0,$$

it follows that $f\left(a, \sqrt{bc}, \sqrt{bc}\right) \geq 0$ Equality in the given inequality occurs if and only if $a = b = c = 1$.

Second Solution We write the inequality as

$$\sum \left(a^2 + \frac{9}{a} - 10a + 17\ln a\right) \geq 0.$$

So, it suffices to show that the function $f(x) = x^2 + \frac{9}{x} - 10x + 17\ln x$ is non-negative for $x > 0$. Since

$$f'(x) = 2x - \frac{9}{x^2} - 10 + \frac{17}{x} = \frac{2x^3 - 10x^2 + 17x - 9}{x^2} =$$

$$= \frac{(x-1)(2x^2 - 8x + 9)}{x^2}$$

and $2x^2 - 8x + 9 = 2(x-2)^2 + 1 > 0$, it follows that $f'(x)$ is negative for $0 < x < 1$ and positive for $x > 1$ Therefore, $f(x)$ is strictly decreasing for $0 < x \leq 1$ and strictly increasing for $x \geq 1$ This result implies $f(x) \geq f(1) = 0$.

Remark. Actually, the following stronger inequality holds

$$a^2 + b^2 + c^2 + 15(ab + bc + ca) \geq 16(a + b + c)$$

for any positive number a, b, c satisfying $abc = 1$. This inequality can be proved using the mixing method as in the first solution above Finally, we find that the inequality $f\left(a, \sqrt{bc}, \sqrt{bc}\right) \geq 0$ holds if and only if

$$x^4 + 2x^3 - 13x^2 + 2x + 17 \geq 0$$

Writing this inequality in the form

$$1 + (x+1)(x-2)(x^2+3x-8) \geq 0,$$

we see that it is true for $x \geq 2$ Also, for $1 \leq x < 2$, we have

$$1 + (x+1)(x-2)(x^2+3x-8) = 1-(2+x-x^2)(x^2+3x-8) \geq$$
$$\geq 1 - \frac{1}{4}\left[(2+x-x^2)+(x^2+3x-8)\right]^2 = 4(x-1)(2-x) \geq 0.$$

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63. *Let a, b, c be non-negative numbers such that $ab + bc + ca = 3$ Prove that*

$$\frac{a(b^2+c^2)}{a^2+bc} + \frac{b(c^2+a^2)}{b^2+ca} + \frac{c(a^2+b^2)}{c^2+ab} \geq 3.$$

Solution. Taking into account the known inequality

$$(x+y+z)^2 \geq 3(xy+yz+zx),$$

it suffices to prove the stronger inequality

$$\sum \frac{bc(a^2+b^2)(c^2+a^2)}{(b^2+ca)(c^2+ab)} \geq 3$$

In order to homogenize this inequality, we replace the right hand side by $\sum bc$ Since

$$\frac{bc(a^2+b^2)(c^2+a^2)}{(b^2+ca)(c^2+ab)} - bc = \frac{abc(a^3-b^3-c^3+ab^2+ac^2-abc)}{(b^2+ca)(c^2+ab)},$$

we have to show that

$$\sum(a^2+bc)(a^3-b^3-c^3+ab^2+ac^2-abc) \geq 0$$

This inequality is equivalent to

$$\sum a^5 + 2abc\sum a^2 \geq \sum bc(b^3+c^3) + abc\sum bc,$$

or

$$\sum a^3(a-b)(a-c) + \frac{abc}{2}\sum(b-c)^2 \geq 0.$$

Since $\sum a^3(a-b)(a-c) \geq 0$ by Schur's Inequality, the inequality is clearly true Equality occurs if and only if $a = b = c = 1$

★

64. *If a, b, c are positive numbers, then*

$$a+b+c+\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq \frac{6(a^2+b^2+c^2)}{a+b+c}.$$

Solution. Write the inequality as follows

$$\sum_{\text{cyc}}\left(\frac{a^2}{b}-2a+b\right) \geq 6\left(\frac{a^2+b^2+c^2}{a+b+c}-\frac{a+b+c}{3}\right),$$

$$\sum_{\text{cyc}}\frac{(a-b)^2}{b} \geq \frac{2}{a+b+c}\sum_{\text{cyc}}(a-b)^2,$$

$$(b-c)^2A+(c-a)^2B+(a-b)^2C \geq 0,$$

where

$$A=\frac{a+b}{c}-1, \quad B=\frac{b+c}{a}-1, \quad C=\frac{c+a}{b}-1.$$

Taking into account the identity

$$(b-c)^2A+(c-a)^2B+(a-b)^2C=$$
$$=\frac{[(a-c)B+(a-b)C]^2+(b-c)^2(AB+BC+CA)}{B+C},$$

it suffices to show that $B+C>0$ and $AB+BC+CA \geq 0$ We have

$$B+C=\frac{(a-b)^2}{ab}+c\left(\frac{1}{a}+\frac{1}{b}\right)>0$$

and

$$AB+BC+CA=3+\frac{a^3+b^3+c^3+3abc-ab(a+b)-bc(b+c)-ca(c+a)}{abc} \geq 3,$$

according to Schur's Inequality of three degree

$$a^3+b^3+c^3+3abc \geq ab(a+b)+bc(b+c)+ca(c+a).$$

Equality occurs if and only if $a=b=c$.

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65. *If a, b, c are positive numbers, then*

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \geq \frac{3(a^3+b^3+c^3)}{2(a^2+b^2+c^2)}.$$

First Solution (after an idea of *Kunihiko Chikaya*). Since

$$(a+b)(b+c)(c+a)\sum \frac{a^2}{b+c} = \sum a^2(a+b)(a+c) =$$
$$= abc\sum a + \sum a^4 + \sum a^3(b+c) = abc\sum a + \left(\sum a^3\right)\left(\sum a\right) =$$
$$= \left(\sum a\right)\left(abc + \sum a^3\right),$$

the inequality can be written in the form

$$2\left(\sum a\right)\left(\sum a^2\right)\left(abc + \sum a^3\right) \geq 3(a+b)(b+c)(c+a)\left(\sum a^3\right)$$

This inequality can be obtained by multiplying the inequalities

$$2\left(\sum a\right)\left(\sum a^2 + 3\sum bc\right) \geq 9(a+b)(b+c)(c+a)$$

and

$$3\left(\sum a^2\right)\left(abc + \sum a^3\right) \geq \left(\sum a^3\right)\left(\sum a^2 + 3\sum bc\right)$$

The first inequality is equivalent to

$$2\sum a^3 \geq \sum bc(b+c),$$

which is true because

$$2\sum a^3 - \sum bc(b+c) = \sum (b^3+c^3) - \sum bc(b+c) = \sum (b+c)(b-c)^2 \geq 0.$$

Setting $X = \sum a^3 - 3abc = \left(\sum a\right)^3 - 3\left(\sum a\right)\left(\sum bc\right)$, the second inequality is successively equivalent to

$$3\left(\sum a^2\right)(X+4abc) \geq (X+3abc)\left(\sum a^2 + 3\sum bc\right),$$
$$X\left(2\sum a^2 - 3\sum bc\right) + 9abc\left(\sum a^2 - \sum bc\right) \geq 0,$$
$$\left(\sum a^2 - \sum bc\right)\left[\left(\sum a\right)\left(2\sum a^2 - 3\sum bc\right) + 9abc\right] \geq 0,$$
$$\left(\sum a^2 - \sum bc\right)\left[2\sum a^3 - \sum bc(b+c)\right] \geq 0,$$
$$\left[\sum (b-c)^2\right]\left[\sum (b+c)(b-c)^2\right] \geq 0.$$

The last inequality is clearly true, and the proof is completed Equality occurs if and only if $a = b = c$

Second Solution. Write the inequality as $A \geq B$, where

$$A = 2\sum \frac{a^2}{b+c} - \sum a, \quad B = 3\frac{a^3+b^3+c^3}{a^2+b^2+c^2} - \sum a.$$

Since

$$A = \sum \frac{a(a-b)+a(a-c)}{b+c} = \sum \frac{a(a-b)}{b+c} + \sum \frac{a(a-c)}{b+c} =$$
$$= \sum \frac{b(b-c)}{c+a} + \sum \frac{c(c-b)}{a+b} = (a+b+c)\sum \frac{(b-c)^2}{(a+b)(a+c)}$$

and

$$B = \frac{1}{a^2+b^2+c^2}\sum(b^3+c^3-b^2c-bc^2) =$$
$$= \frac{1}{a^2+b^2+c^2}\sum(b+c)(b-c)^2,$$

we may write the inequality in the form

$$S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \geq 0,$$

where

$$S_a = \frac{a+b+c}{(a+b)(a+c)} - \frac{b+c}{a^2+b^2+c^2} = \frac{a^3+b^3+c^3-2abc}{(a+b)(a+c)(a^2+b^2+c^2)}.$$

By the AM-GM Inequality, we have $a^3+b^3+c^3 \geq 3abc$. Hence

$$S_a \geq \frac{abc}{(a+b)(a+c)(a^2+b^2+c^2)} \geq 0,$$

and, similarly, $S_b \geq 0$ and $S_c \geq 0$

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66. *If a, b, c are given positive numbers, find the minimum value $E(a, b, c)$ of the expression*

$$E = \frac{ax}{y+z} + \frac{by}{z+x} + \frac{cz}{x+y}$$

for any non-negative numbers x, y, z, no of which are zero.

Solution Consider that $a = \max\{a, b, c\}$ Since

$$E = \sum \frac{ax}{y+z} = \sum \frac{a(x+y+z) - a(y+z)}{y+z} =$$
$$= (x+y+z)\sum \frac{a}{y+z} - \sum a =$$
$$= \frac{1}{2}\left[\sum(y+z)\right]\left(\sum \frac{a}{y+z}\right) - \sum a,$$

by the Cauchy-Schwarz Inequality we get

$$E \geq \frac{1}{2}\left(\sum \sqrt{a}\right)^2 - \sum a = \sqrt{ab} + \sqrt{bc} + \sqrt{ca} - \frac{a+b+c}{2},$$

with equality if and only if $\frac{\sqrt{a}}{y+z} = \frac{\sqrt{b}}{z+x} = \frac{\sqrt{c}}{x+y}$. Consequently, for $\sqrt{a} \leq \sqrt{b} + \sqrt{c}$ the expression E has the minimum value

$$E(a, b, c) = \sqrt{ab} + \sqrt{bc} + \sqrt{ca} - \frac{a+b+c}{2}$$

for $x = \sqrt{b} + \sqrt{c} - \sqrt{a}$, $y = \sqrt{c} + \sqrt{a} - \sqrt{b}$, $z = \sqrt{a} + \sqrt{b} - \sqrt{c}$

We assert now that for $\sqrt{a} \geq \sqrt{b} + \sqrt{c}$, the expression E is minimal for $x = 0$ and $\frac{y}{z} = \sqrt{\frac{c}{b}}$, and its minimum value is

$$E(a, b, c) = 2\sqrt{bc}.$$

Since $\sqrt{a} \geq \sqrt{b} + \sqrt{c}$, it suffices to show that

$$\frac{Ax}{y+z} + \frac{by}{z+x} + \frac{cz}{x+y} \geq 2\sqrt{bc},$$

where $A = \left(\sqrt{b} + \sqrt{c}\right)^2$

Setting $y + z = 2X$, $z + x = 2Y$, $x + y = 2Z$, the inequality becomes

$$\frac{A(Y+Z-X)}{X} + \frac{b(Z+X-Y)}{Y} + \frac{c(X+Y-Z)}{Z} \geq 4\sqrt{bc},$$
$$\left(A\frac{Y}{X} + b\frac{X}{Y}\right) + \left(b\frac{Z}{Y} + c\frac{Y}{Z}\right) + \left(c\frac{X}{Z} + A\frac{Z}{X}\right) \geq 2A + 2\sqrt{bc}$$

The last inequality follows immediately from

$$A\frac{Y}{X} + b\frac{X}{Y} \geq 2\sqrt{Ab}, \quad b\frac{Z}{Y} + c\frac{Y}{Z} \geq 2\sqrt{bc}, \quad c\frac{X}{Z} + A\frac{Z}{X} \geq 2\sqrt{cA}$$

Finally, for $a = \max\{a, b, c\}$ we have

$$E(a, bc) = \begin{cases} \sqrt{ab} + \sqrt{bc} + \sqrt{ca} - \dfrac{a+b+c}{2}, & \text{if} \quad \sqrt{a} \le \sqrt{b} + \sqrt{c} \\ 2\sqrt{bc}, & \text{if} \quad \sqrt{a} \ge \sqrt{b} + \sqrt{c}. \end{cases}$$

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67. *Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that*

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \ge a^2 + b^2 + c^2.$$

First Solution. Write the inequality in the form

$$\sum \left(\frac{1}{a^2} - a^2 + 4a - 4 \right) \ge 0,$$

which is equivalent to

$$\sum \frac{(1-a)^2(1+2a-a^2)}{a^2} \ge 0.$$

Without loss of generality, we may assume that $a \ge b \ge c$. We have to consider two cases.

Case $a \le 1 + \sqrt{2}$. Since $c \le b \le a \le 1 + \sqrt{2}$, we have $1 + 2a - a^2 \ge 0$, $1 + 2b - b^2 \ge 0$ and $1 + 2c - c^2 \ge 0$.

Case $a > 1 + \sqrt{2}$. Since $b + c = 3 - a < 2 - \sqrt{2} < \frac{2}{3}$, we have

$$bc \le \frac{1}{4}(b+c)^2 < \frac{1}{9},$$

and hence

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} > \frac{1}{b^2} + \frac{1}{c^2} \ge \frac{2}{bc} > 18 > (a+b+c)^2 > a^2 + b^2 + c^2.$$

Equality occurs if and only if $a = b = c = 1$

Second Solution. Since

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \ge \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca},$$

it suffices to show that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \ge a^2 + b^2 + c^2,$$

or

$$abc(a^2 + b^2 + c^2) \leq 3.$$

Let $x = ab + bc + ca$. From the well-known inequality

$$(ab + bc + ca)^2 \geq 3abc(a + b + c)$$

we get $abc \leq \frac{x^2}{9}$. On the other hand, from

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca),$$

we have $a^2 + b^2 + c^2 = 9 - 2x$. Therefore,

$$abc(a^2 + b^2 + c^2) - 3 \leq \frac{x^2}{9}(9 - 2x) - 3 = \frac{-(x - 3)^2(2x + 3)}{9} \leq 0.$$

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68. *Let a, b, c be non-negative real numbers such that $a + b + c = 3$. Prove that*

$$(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \leq 12.$$

Solution. Assume that $a \leq b \leq c$. We see that equality occurs for

$$(a, b, c) = (0, 1, 2).$$

Since

$$a^2 - ab + b^2 \leq b^2$$

and

$$c^2 - ca + a^2 \leq (a + c)^2,$$

it suffices to show that

$$x^2y \leq 6^3,$$

where $x = 3b(a + c)$ and $y = 2(b^2 - bc + c^2)$ Note that $x = y = 6$ for the equality case $(a, b, c) = (0, 1, 2)$ From the AM-GM Inequality, we have

$$x^2y \leq \left(\frac{2x + y}{3}\right)^3.$$

Consequently, it suffices to show that

$$x + \frac{y}{2} \leq 9.$$

Indeed,

$$9 - x - \frac{y}{2} = (a+b+c)^2 - 3(ab+bc) - (b^2 - bc + c^2) =$$
$$= a(a - b + 2c) \geq 0.$$

On the assumption $a \leq b \leq c$, equality occurs if and only if $(a,b,c) = (0,1,2)$.

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69. *Let a, b, c be non-negative real numbers such that $a+b+c = 1$. Prove that*

$$\sqrt{a+b^2} + \sqrt{b+c^2} + \sqrt{c+a^2} \geq 2$$

Solution. We will use the inequality

$$\sqrt{x} + \sqrt{y} \geq \sqrt{z} + \sqrt{x+y-z},$$

which is valid for any non-negative numbers x, y, z with $z = \min\{x, y, z\}$ Indeed, twice squaring reduces the inequality to $(x-z)(y-z) \geq 0$ Assume now that $c = \min\{a, b, c\}$ and denote $x = a + b^2$, $y = b + c^2$ and $z = c + b^2$. Since

$$x - z = a - c \geq 0,$$
$$y - z = (b-c)(1-b-c) = (b-c)a \geq 0,$$

and

$$x + y - z = a + b - c + c^2 = 1 - 2c + c^2 = (1-c)^2,$$

by the above inequality we get

$$\sqrt{a+b^2} + \sqrt{b+c^2} \geq \sqrt{c+b^2} + 1 - c.$$

Therefore, we still have to prove that

$$\sqrt{c+a^2} + \sqrt{c+b^2} \geq 1 + c.$$

By squaring, the inequality becomes

$$2\sqrt{(c+a^2)(c+b^2)} \geq 1 + c^2 - a^2 - b^2,$$

or

$$\sqrt{(c+a^2)(c+b^2)} \geq c + ab,$$

which is clearly true. Equality occurs in the original inequality for $(a,b,c) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, as well as for $(a,b,c) = (1,0,0)$ and any cyclic permutation thereof.

★

70. *If a, b, c are non-negative real numbers, then*

$$a^3 + b^3 + c^3 + 3abc \geq \sum bc\sqrt{2(b^2 + c^2)}.$$

Solution. Write the inequality as

$$a^3 + b^3 + c^3 + 3abc - \sum bc(b + c) \geq \sum bc\left[\sqrt{2(b^2 + c^2)} - b - c\right],$$

or

$$\frac{1}{2}\sum(b - c)^2(b + c - a) \geq \sum bc\frac{(b - c)^2}{\sqrt{2(b^2 + c^2)} + b + c}.$$

Since $\sqrt{2(b^2 + c^2)} \geq b + c$, it suffices to show that

$$\sum(b - c)^2 S_a \geq 0,$$

where

$$S_a = b + c - a - \frac{bc}{b + c} = c - a + \frac{b^2}{b + c}.$$

Assuming $a \geq b \geq c$, we have

$$S_b = a - b + \frac{c^2}{c + a} \geq 0,$$

$$S_c = b - c + \frac{a^2}{a + b} \geq 0$$

and

$$\sum(b - c)^2 S_a \geq (b - c)^2 S_a + (a - c)^2 S_b \geq (b - c)^2(S_a + S_b).$$

Since

$$S_a + S_b = 2c - \frac{bc}{b + c} - \frac{ca}{c + a} = c\left(2 - \frac{b}{b + c} - \frac{a}{c + a}\right) =$$
$$= c^2\left(\frac{1}{b + c} + \frac{1}{c + a}\right) \geq 0,$$

the proof is completed. Equality occurs for $a = b = c$, $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$.

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71. *If a, b, c are non-negative real numbers, then*

$$(1+a^2)(1+b^2)(1+c^2) \geq \frac{15}{16}(1+a+b+c)^2$$

Solution. We can see that equality occurs for $a = b = c = \frac{1}{2}$. Substituting a, b, c by $\frac{x}{2}, \frac{y}{2}, \frac{z}{2}$, the inequality becomes

$$(x^2+4)(y^2+4)(z^2+4) \geq 5(x+y+z+2)^2.$$

Among x, y, z there are two numbers either less than or equal to 1, or greater than or equal to 1. Let y and z be these numbers By Bernoulli's Inequality, we have

$$\left(1+\frac{y^2-1}{5}\right)\left(1+\frac{z^2-1}{5}\right) \geq 1+\frac{y^2-1}{5}+\frac{z^2-1}{5},$$

$$(y^2+4)(z^2+4) \geq 5(y^2+z^2+3)$$

Hence, it suffices to show that

$$(x^2+4)(y^2+z^2+3) \geq (x+y+z+2)^2.$$

Writing this inequality as

$$(x^2+1+1+2)(1+y^2+z^2+2) \geq (x+y+z+2)^2,$$

we recognize the Cauchy-Schwarz Inequality

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72. *Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that*

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2) \geq (a+b+c+d)^2$$

Solution. Among a, b, c, d there are two numbers less than or equal to 1, or greater than or equal to 1 Let b and d be these numbers, that is $(b-1)(d-1) \geq 0$. By the Cauchy-Schwarz Inequality, we have

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2) = (1+a^2+b^2+a^2b^2)(c^2+1+d^2+c^2d^2) \geq$$
$$\geq \ c+a+bd+abcd)^2$$

Since $abcd = 1$, it suffices to show that

$$c + a + bd + 1 \geq a + b + c + d.$$

This inequality is equivalent to $(b-1)(d-1) \geq 0$, which is true. Equality occurs if and only if $a = b = c = d = 1$

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73. *If $x_1, x_2, \dots, x_n$ are non-negative numbers, then*

$$x_1 + x_2 + \cdots + x_n \geq (n-1)\sqrt[n]{x_1 x_2 \dots x_n} + \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}$$

First Solution (by *Michael Rozenberg*). Let us denote

$$x = \frac{x_1 + x_2 + \cdots + x_n}{n}, \quad y = \sqrt{\frac{2\sum\limits_{1 \leq i < j \leq n} x_i x_j}{n(n-1)}}, \quad z = \sqrt[n]{x_1 x_2 \dots x_n},$$

where $x \geq y \geq z$ (Maclaurin's Inequalities). The inequality becomes

$$nx - (n-1)z \geq \sqrt{\frac{n^2x^2 - n(n-1)y^2}{n}},$$

or

$$nx^2 - 2nxz + (n-1)z^2 + y^2 \geq 0.$$

Since $y \geq z$, we have

$$nx^2 - 2nxz + (n-1)z^2 + y^2 = n(x-z)^2 + (y^2 - z^2) \geq 0.$$

Equality in the original inequality occurs if and only if $x_1 = x_2 = \cdots = x_n$.

Second Solution. Let us denote

$$x = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}, \quad y = \frac{x_1 + x_2 + \cdots + x_n}{n}, \quad z = \sqrt[n]{x_1 x_2 \dots x_n},$$

where $x \geq y \geq z$ We may write the inequality as

$$n(y - z) \geq x - z.$$

Since

$$x - z = \frac{x^2 - z^2}{x + z} \geq \frac{x^2 - z^2}{y + z},$$

it suffices to show that

$$n(y-z) \geq \frac{x^2 - z^2}{y+z}$$

This inequality is equivalent to each of the following inequalities

$$ny^2 - x^2 \geq (n-1)z^2,$$
$$(x_1 + x_2 + \cdots + x_n)^2 - \left(x_1^2 + x_2^2 + \cdots + x_n^2\right) \geq n(n-1)z^2,$$
$$\sum_{1 \leq i < j \leq n} x_i x_j \geq \frac{n(n-1)}{2} z^2.$$

The last inequality immediately follows by the AM-GM Inequality.

Remark In a similar manner, we can prove the following inequality for any integer $k \geq 2$·

$$n^{k-2}(x_1 + x_2 + \cdots + x_n) \geq (n^{k-1} - 1)\sqrt[n]{x_1 x_2 \ldots x} + \sqrt[k]{\frac{x_1^k + x_2^k + \cdots + x_n^k}{n}}.$$

For $k = n = 3$ we get an inequality from the Austria National Olympiad 2006

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74. *If k is a real number and $x_1, x_2, \ldots, x_n$ are positive numbers, then*

$$(n-1)\left(x_1^{n+k} + x_2^{n+k} + \cdots + x_n^{n+k}\right) + x_1 x_2 \ldots x_n \left(x_1^k + x_2^k + \cdots + x_n^k\right) \geq$$
$$\geq (x_1 + x_2 + \cdots + x_n)\left(x_1^{n+k-1} + x_2^{n+k-1} + \cdots + x_n^{n+k-1}\right).$$

Solution. We will proceed by induction on n as *Gabriel Dospinescu* had proceed in [3] to prove Suranyi's Inequality (case $k = 0$). For $n = 2$ we have an identity, while for $n = 3$ we get Schur's Inequality

$$\sum x_1^{k+1}(x_1 - x_2)(x_1 - x_3) \geq 0$$

Suppose that the inequality is true for n numbers and let us prove it for $n + 1$ numbers. Since the inequality is homogeneous, we may consider that

$$x_1 + x_2 + \quad + x_n = n$$

In addition, let us denote x_{n+1} by x and

$$X = x_1^{n+k+1} + x_2^{n+k+1} + \cdots + x_n^{n+k+1},$$
$$Y = x_1^{n+k} + x_2^{n+k} + \cdots + x_n^{n+k},$$
$$Z = x_1^{n+k-1} + x_2^{n+k-1} + \cdots + x_n^{n+k-1},$$
$$W = x_1 x_2 \ldots x_n.$$

We have to show that

$$n(X + x^{n+k+1}) + Wx\left(x_1^k + x_2^k + \cdots + x_n^k + x^k\right) \geq (n+x)(Y + x^{n+k}),$$

under the inductive hypothesis

$$(n-1)Y + W\left(x_1^k + x_2^k + \cdots + x_n^k\right) \geq nZ.$$

Using this last inequality, it suffices to show that

$$n(X - Y) - nx(Y - Z) + x^{k+1}\left[W - nx^{n-1} + (n-1)x^n\right] \geq 0.$$

We will consider two cases depending on k.

Case $k \geq 1 - n$ According to Chebyshev's Inequality, we have

$$nY \geq (x_1 + x_2 + \cdots + x_n)\left(x_1^{n+k-1} + x_2^{n+k-1} + \cdots + x_n^{n+k-1}\right),$$

and hence $Y - Z \geq 0$. Since the inequality is symmetric, we may consider that

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} = x, \quad 0 < x \leq 1.$$

Thus,

$$n(X-Y) - n(Y-Z)x \geq n(X-Y) - n(Y-Z) = n(X - 2Y + Z) =$$
$$= \sum_{i=1}^{n} x_i^{n+k-1}(x_i - 1)^2 \geq 0,$$

and we still have to show that

$$W - nx^{n-1} + (n-1)x^n \geq 0$$

Indeed, since $x_i - x \geq 0$ for all x_i, by Bernoulli's Inequality we have

$$W = x^n \prod_{i=1}^{n}\left(1 + \frac{x_i - x}{x}\right) \geq x^n\left(1 + \sum_{i=1}^{n}\frac{x_i - x}{x}\right) = nx^{n-1} - (n-1)x^n.$$

Case $k \leq 1 - n$. According to Chebyshev's Inequality, we have

$$nY \leq (x_1 + x_2 + \cdots + x_n)\left(x_1^{n+k-1} + x_2^{n+k-1} + \cdots + x_n^{n+k-1}\right),$$

and hence $Y - Z \leq 0$. Since the inequality is symmetric, we may consider that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} = x, \; x \geq 1.$$

Then,

$$n(X-Y)-n(Y-Z)x \ge n(X-Y)-n(Y-Z) \ge 0$$

and

$$W = x^n \prod_{i=1}^{n}\left(1+\frac{x_i-x}{x}\right) \ge x^n\left(1+\sum_{i=1}^{n}\frac{x_i-x}{x}\right) = nx^{n-1}-(n-1)x^n,$$

since $-1 < \dfrac{x_i - x}{x} \le 0$ for all x_i. This completes the proof
Equality holds for $n \ge 3$ if and only if $x_1 = x_2 = \cdots = x_n$.

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75. *Let a, b, c be non-negative numbers, no two of which are zero. Prove that*

$$\frac{a^4}{a^3+b^3}+\frac{b^4}{b^3+c^3}+\frac{c^4}{c^3+a^3} \ge \frac{a+b+c}{2}.$$

Solution. Without loss of generality, assume that $c = \max\{a, b, c\}$. Write the inequality as

$$\left(\frac{a^4}{a^3+b^3}-\frac{a}{2}\right)+\left(\frac{b^4}{b^3+c^3}-\frac{b}{2}\right)+\left(\frac{c^4}{c^3+a^3}-\frac{c}{2}\right) \ge 0,$$

$$\frac{a(a^3-b^3)}{a^3+b^3}+\frac{b(b^3-c^3)}{b^3+c^3}+\frac{c(c^3-a^3)}{c^3+a^3} \ge 0$$

Since

$$\frac{a(a^3-b^3)}{a^3+b^3}-\frac{b(a^3-b^3)}{a^3+b^3}=\frac{(a-b)(a^3-b^3)}{a^3+b^3} \ge 0,$$

it suffices to show that

$$\frac{b(a^3-b^3)}{a^3+b^3}+\frac{b(b^3-c^3)}{b^3+c^3}+\frac{c(c^3-a^3)}{c^3+a^3} \ge 0.$$

Taking account of

$$\frac{b(a^3-b^3)}{a^3+b^3}+\frac{b(b^3-c^3)}{b^3+c^3}=\frac{2b^4(a^3-c^3)}{(a^3+b^3)(b^3+c^3)},$$

the last inequality is equivalent to

$$(c^3-a^3)(c-b)\left[a^3(2b^3+b^2c+bc^2+c^3)-b^3c(b^2+bc-c^2)\right] \ge 0, \qquad (4)$$

or

$$a^3(2b^3+b^2c+bc^2+c^3)-b^3c(b^2+bc-c^2) \ge 0. \qquad (5)$$

Case $a \geq b$ We have

$$a^3(2b^3 + b^2c + bc^2 + c^3) - b^3c(b^2 + bc - c^2) \geq$$
$$\geq b^3(2b^3 + b^2c + bc^2 + c^3) - b^3c(b^2 + bc - c^2) = 2b^3(b^3 + c^3) \geq 0.$$

Case $0 \leq a < b \leq c$ According to (4), the original cyclic inequality is true if

$$(a^3 - b^3)(a - c)\left[b^3(2c^3 + c^2a + ca^2 + a^3) - c^3a(c^2 + ca - a^2)\right] \geq 0.$$

Since $(a^3 - b^3)(a - c) > 0$, it suffices to show that

$$b^3(2c^3 + c^2a + ca^2 + a^3) - c^3a(c^2 + ca - a^2) \geq 0. \tag{6}$$

To finish the proof, we will show that (5) holds for $5b^3 \leq 5a^3 + c^3$, and (6) holds for $5b^3 \geq 5a^3 + c^3$. Due to homogeneity, we will consider $c = 1$ We must show that

$$a^3(2b^3 + b^2 + b + 1) - b^3(b^2 + b - 1) \geq 0 \tag{7}$$

for $5b^3 \leq 5a^3 + 1$, and

$$b^3(2 + a + a^2 + a^3) - a(1 + a - a^2) \geq 0 \tag{8}$$

for $5b^3 \geq 5a^3 + 1$ The inequality (7) is clearly true for $b^2 + b - 1 \leq 0$ For $b^2 + b - 1 > 0$ and $5b^3 \leq 5a^3 + 1$, we have

$$5a^3(2b^3 + b^2 + b + 1) - 5b^3(b^2 + b - 1) \geq$$
$$\geq (5b^3 - 1)(2b^3 + b^2 + b + 1) - 5b^3(b^2 + b - 1) =$$
$$= 10b^6 + 8b^3 - b^2 - b - 1 = 8b^6 + (b^2 + b - 1)(2b^4 - 2b^3 + 4b^2 + 2b + 1) >$$
$$> (b^2 + b - 1)(b^4 - 2b^3 + b^2) = b^2(b^2 + b - 1)(b - 1)^2 \geq 0.$$

The inequality (8) is true for $5b^3 \geq 5a^3 + 1$ because

$$5b^3(2 + a + a^2 + a^3) - 5a(1 + a - a^2) \geq$$
$$\geq (5a^3 + 1)(2 + a + a^2 + a^3) - 5a(1 + a - a^2) =$$
$$= 5a^6 + 5a^5 + 5a^4 + 16a^3 - 4a^2 - 4a + 2 >$$
$$> 12a^3 - 4a^2 - 5a + 2 = (2a - 1)^2(3a + 2) \geq 0$$

Equality occurs if and only if $a = b = c$

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List of Symbols

Symbol	Description
AJ	Archimedes Journal
BMO	Balkan Mathematical Olympiad
CM	Crux Mathematicorum
GM-A	Gazeta Matematică – A
GM-B	Gazeta Matematică – B
MC	Mathlinks Contest
MS	Mathlinks Site
ONI	Old and New Inequalities
RMT	Revista Matematică Timişoara
TST	Team Selection Test
AM-GM Inequality	Arithmetic Mean-Geometric Mean Inequality
AM-HM Inequality	Arithmetic Mean-Harmonic Mean Inequality
AC Method	Arithmetic Compensation Method
EV Method	Equal Variable Method
GC Method	Geometric Compensation Method
LCF Theorem	Left Concave Function Theorem
LCRCF Theorem	Left Concave-Right Convex Function Theorem
RCF Theorem	Right Convex Function Theorem
$\sum$	$\sum_{cyclic}$
$\sum f(x_{i+1}x_{i+2}\cdots x_{i+j})$	$\sum_{i+1\le i_1<i_2<\cdots<i_j\le n} f(x_{i_1}x_{i_2}\cdots x_{i_j})$

Glossary

(1) AM-GM (Arithmetic Mean-Geometric Mean) Inequality

If $a_1, a_2, \ldots, a_n$ are non-negative real numbers, then

$$a_1 + a_2 + \cdots + a_n \geq n \sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$

(2) Weighted AM-GM Inequality

Let $w_1, w_2, \ldots, w_n$ be positive real numbers with

$$w_1 + w_2 + \cdots + w_n = 1.$$

If $a_1, a_2, \ldots, a_n$ are non-negative real numbers, then

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

(3) AM-HM (Arithmetic Mean-Harmonic Mean) Inequality

If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

(4) Power Mean Inequality

For positive real numbers $a_1, a_2, \ldots, a_n$, the power mean of order r is defined by

$$M_r = \begin{cases} \left(\dfrac{a_1^r + a_2^r + \cdots + a_n^r}{n}\right)^{\frac{1}{r}} & \text{for } r \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n} & \text{for } r = 0 \end{cases}$$

If not all a_i's are equal, then M_r is strictly increasing for $r \in \mathbb{R}$. For instant, $M_2 \geq M_1 \geq M_0$ implies

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

(5) Weighted Power Mean Inequality

Let $p_1, p_2, \ldots, p_n$ be positive real numbers with

$$p_1 + p_2 + \cdots + p_n = 1$$

For positive real numbers $a_1, a_2, \ldots, a_n$, the weighted power mean of order r is defined by

$$M_r = \begin{cases} (p_1 a_1^r + p_2 a_2^r + \cdots + p_n a_n^r)^{\frac{1}{r}} & \text{for } r \neq 0 \\ a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} & \text{for } r = 0 \end{cases}$$

If not all a_i's are equal, then M_r is strictly increasing for $r \in \mathbb{R}$

(6) Bernoulli's Inequality

For any real numbers $x \geq -1$, we have

a) $(1+x)^r \geq 1 + rx$ for $r \geq 1$,

b) $(1+x)^r \leq 1 + rx$ for $0 \leq r \leq 1$

In addition, if $a_1, a_2, \ldots, a_n$ are real numbers such that either $a_1, a_2, \ldots, a_n \geq 0$ or $-1 \leq a_1, a_2, \ldots, a_n \leq 0$, then

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1 + a_1 + a_2 + \cdots + a_n.$$

(7) Cauchy-Schwarz Inequality

For any real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, we have

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1 + a_1 + a_2 + \cdots + a_n,$$

with equality if and only if a_i and b_i are proportional for all i

(8) Hölder's Inequality

Let x_{ij} ($i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$) be non-negative real numbers. Then

$$\prod_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} \right)^{\frac{1}{m}} \geq \sum_{j=1}^{n} \left(\prod_{i=1}^{m} x_{ij}^{\frac{1}{m}} \right).$$

More general, if $p_1, p_2, \ldots, p_m$ are positive real numbers with

$$p_1 + p_2 + \cdots + p_m = 1,$$

then

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\right)^{p_i} \geq \sum_{j=1}^{n}\left(\prod_{i=1}^{m} x_{ij}^{p_i}\right).$$

(9) Minkowski's Inequality

For any real number $r \geq 1$ and any positive real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, the inequality holds

$$\left(\sum_{i=1}^{n}(a_i + b_i)^r\right)^{\frac{1}{r}} \leq \left(\sum_{i=1}^{n} a_i^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^{n} b_i^r\right)^{\frac{1}{r}}$$

(10) Cyclic Sum

If f is a function of n variables, define the cyclic sum as

$$\sum_{\text{cyc}} f(a_1, a_2, \ldots, a_n) = f(a_1, a_2, \ldots, a_n) + f(a_2, a_3, \ldots, a_1) +$$

$$+ \cdots + f(a_n, a_1, \ldots, a_{n-1})$$

In our book, the symbols $\sum_{\text{cyc}}$ and $\sum$ are identical.

(11) Schur's Inequality

For any non-negative real numbers a, b, c and any positive number r, the inequality holds

$$\sum a^r(a-b)(a-c) \geq 0,$$

with equality if and only if $a = b = c$, $a = 0$ and $b = c$, $b = 0$ and $c = a$, $c = 0$ and $a = b$.

For $r = 1$, we get the third degree Schur's Inequality

$$a^3 + b^3 + c^3 + 3abc \geq \sum bc(b+c),$$

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

$$\sum (b-c)^2(b+c-a) \geq 0.$$

For $r = 2$, we get the fourth degree Schur's Inequality

$$a^4 + b^4 + c^4 + abc\sum a \geq \sum bc(b^2 + c^2).$$

(12) Maclaurin's Inequality

If $a_1, a_2, \ldots, a_n$ are non-negative real numbers, then

$$S_1 \geq S_2 \geq \cdots \geq S_n,$$

where

$$S_k = \sqrt[k]{\frac{\sum a_1 a_2 \ldots a_k}{\binom{n}{k}}}$$

(13) Chebyshev's Inequality

Let $a_1 \leq a_2 \leq \cdots \leq a_n$ be real numbers

a) If $b_1 \leq b_2 \leq \cdots \leq b_n$, then

$$n \sum_{i=1}^{n} a_i b_i \geq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right),$$

b) If $b_1 \geq b_2 \geq \cdots \geq b_n$, then

$$n \sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right)$$

(14) Convex functions

A function f defined on an interval $\mathbb{I}$ of real numbers is said to be convex if for all $x, y \in \mathbb{I}$ and any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$,

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y) \tag{1}$$

If (1) is strict for all $x \neq y$ and $\alpha, \beta > 0$, then f is said to be strictly convex. If (1) is reversed, then f is said to be (strictly) concave

The inequality (1) is equivalent to

$$\frac{f(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{f(x_2)}{(x_2 - x_3)(x_2 - x_1)} + \frac{f(x_3)}{(x_3 - x_1)(x_3 - x_2)} \geq 0,$$

where x_1, x_2, x_3 are distinct numbers in $\mathbb{I}$

If f is differentiable on $\mathbb{I}$, then f is (strictly) convex if and only if the derivative f' is (strictly) increasing

If f is continuous on $[a, b]$ and f'' exists on (a, b), then f is convex on $[a, b]$ if and only if $f'' \geq 0$ If $f'' > 0$, then f is strictly convex.

If f'' exists on (a,b), then f is convex on (a,b) if and only if $f'' \geq 0$. If $f'' > 0$, then f is strictly convex.

(15) Jensen's Inequality

Let $w_1, w_2, \ldots, w_n$ be positive real numbers. If f is convex on an interval $\mathbb{I}$, then for any $a_1, a_2, \ldots, a_n \in \mathbb{I}$, the inequality holds

$$\frac{w_1 f(a_1) + w_2 f(a_2) + \cdots + w_n f(a_n)}{w_1 + w_2 + \cdots + w_n} \geq f\left(\frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{w_1 + w_2 + \cdots + w_n}\right).$$

If f is strictly convex, then equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

(16) Karamata's Majorization Inequality

We say that a vector $\vec{A} = (a_1, a_2, \cdots, a_n)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ majorizes a vector $\vec{B} = (b_1, b_2, \cdots, b_n)$ with $b_1 \geq b_2 \geq \cdots \geq b_n$, and write it as $\vec{A} \succeq \vec{B}$, if

$$\begin{aligned}
a_1 &\geq b_1, \\
a_1 + a_2 &\geq b_1 + b_2, \\
&\cdots\cdots \\
a_1 + a_2 + \cdots + a_{n-1} &\geq b_1 + b_2 + \cdots + b_{n-1}, \\
a_1 + a_2 + \cdots + a_n &= b_1 + b_2 + \cdots + b_n.
\end{aligned}$$

If f is a convex function on an interval $\mathbb{I}$, and a vector $\vec{A} = (a_1, a_2, \ldots, a_n)$ with $a_i \in \mathbb{I}$ majorizes a vector $\vec{B} = (b_1, b_2, \ldots, b_n)$ with $b_i \in \mathbb{I}$, then

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(b_1) + f(b_2) + \cdots + f(b_n).$$

If f'' exists on (a, b), then f is convex on (a, b) if and only if $f'' \geq 0$. If $f'' > 0$, then f is strictly convex.

(15) Jensen's Inequality

Let $w_1, w_2, \ldots, w_n$ be positive real numbers. If f is convex on an interval $\mathbb{I}$, then for any $a_1, a_2, \ldots, a_n \in \mathbb{I}$, the inequality holds

$$\frac{w_1 f(a_1) + w_2 f(a_2) + \cdots + w_n f(a_n)}{w_1 + w_2 + \cdots + w_n} \geq f\left(\frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{w_1 + w_2 + \cdots + w_n}\right).$$

If f is strictly convex, then equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

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$$\begin{aligned} a_1 &\geq b_1, \\ a_1 + a_2 &\geq b_1 + b_2, \\ &\cdots\cdots \\ a_1 + a_2 + \cdots + a_{n-1} &\geq b_1 + b_2 + \cdots + b_{n-1}, \\ a_1 + a_2 + \cdots + a_n &= b_1 + b_2 + \cdots + b_n. \end{aligned}$$

If f is a convex function on an interval $\mathbb{I}$, and a vector $\vec{A} = (a_1, a_2, \ldots, a_n)$ with $a_i \in \mathbb{I}$ majorizes a vector $\vec{B} = (b_1, b_2, \ldots, b_n)$ with $b_i \in \mathbb{I}$, then

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(b_1) + f(b_2) + \cdots + f(b_n).$$