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ЗАДАЧИ И УПРАЖНЕНИЯ ПО МАТЕМАТИЧЕСКОМУ АНАЛИЗУ

Под редакцией
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PROBLEMS IN MATHEMATICAL ANALYSIS

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TO THE READER

MIR Publishers would be glad to have your opinion of the translation and the design of this book.

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PREFACE

This collection of problems and exercises in mathematical analysis covers the maximum requirements of general courses in higher mathematics for higher technical schools. It contains over 3,000 problems sequentially arranged in Chapters I to X covering all branches of higher mathematics (with the exception of analytical geometry) given in college courses. Particular attention is given to the most important sections of the course that require established skills (the finding of limits, differentiation techniques, the graphing of functions, integration techniques, the applications of definite integrals, series, the solution of differential equations).

Since some institutes have extended courses of mathematics, the authors have included problems on field theory, the Fourier method, and approximate calculations. Experience shows that the number of problems given in this book not only fully satisfies the requirements of the student, as far as practical mastering of the various sections of the course goes, but also enables the instructor to supply a varied choice of problems in each section and to select problems for tests and examinations.

Each chapter begins with a brief theoretical introduction that covers the basic definitions and formulas of that section of the course. Here the most important typical problems are worked out in full. We believe that this will greatly simplify the work of the student. Answers are given to all computational problems; one asterisk indicates that hints to the solution are given in the answers, two asterisks, that the solution is given. The problems are frequently illustrated by drawings.

This collection of problems is the result of many years of teaching higher mathematics in the technical schools of the Soviet Union. It includes, in addition to original problems and examples, a large number of commonly used problems.

Chapter I

INTRODUCTION TO ANALYSIS

Sec. 1. Functions

1°. Real numbers. Rational and irrational numbers are collectively known as *real numbers*. The *absolute value* of a real number a is understood to be the nonnegative number $|a|$ defined by the conditions: $|a| = a$ if $a \geqslant 0$, and $|a| = -a$ if $a < 0$. The following inequality holds for all real numbers a and b:

$$|a+b| \leqslant |a|+|b|.$$

2°. Definition of a function. If to every value*) of a variable x, which belongs to some collection (set) E, there corresponds one and only one finite value of the quantity y, then y is said to be a *function* (single-valued) of x or a *dependent variable* defined on the set E, x is the *argument* or *independent variable*. The fact that y is a function of x is expressed in brief form by the notation $y=f(x)$ or $y=F(x)$, and the like.

If to every value of x belonging to some set E there corresponds one or several values of the variable y, then y is called a *multiple-valued function* of x defined on E. From now on we shall use the word "function" only in the meaning of a *single-valued* function, if not otherwise stated.

3° The domain of definition of a function. The collection of values of x for which the given function is defined is called the *domain of definition* (or the *domain*) of this function. In the simplest cases, the domain of a function is either a *closed interval* $[a, b]$, which is the set of real numbers x that satisfy the inequalities $a \leqslant x \leqslant b$, or an *open interval* (a, b), which is the set of real numbers that satisfy the inequalities $a < x < b$. Also possible is a more complex structure of the domain of definition of a function (see, for instance, Problem 21).

Example 1. Determine the domain of definition of the function

$$y=\frac{1}{\sqrt{x^2-1}}.$$

Solution. The function is defined if

$$x^2-1>0,$$

that is, if $|x|>1$. Thus, the domain of the function is a set of two intervals: $-\infty < x < -1$ and $1 < x < +\infty$.

4°. Inverse functions. If the equation $y=f(x)$ may be solved uniquely for the variable x, that is, if there is a function $x=g(y)$ such that $y \equiv f[g(y)]$,

---

*) Henceforth all values will be considered as real, if not otherwise stated.

then the function $x=g(y)$, or, in standard notation, $y=g(x)$, is the *inverse* of $y=f(x)$. Obviously, $g[f(x)]\equiv x$, that is, the function $f(x)$ is the *inverse* of $g(x)$ (and vice versa).

In the general case, the equation $y=f(x)$ defines a multiple-valued inverse function $x=f^{-1}(y)$ such that $y\equiv f[f^{-1}(y)]$ for all y that are values of the function $f(x)$

Example 2. Determine the inverse of the function

$$y=1-2^{-x}. \tag{1}$$

Solution. Solving equation (1) for x, we have

$$2^{-x}=1-y$$

and

$$x=-\frac{\log(1-y)}{\log 2}\,^{*)}. \tag{2}$$

Obviously, the domain of definition of the function (2) is $-\infty<y<1$.

5°. **Composite and implicit functions.** A function y of x defined by a series of equalities $y=f(u)$, where $u=\varphi(x)$, etc., is called a *composite function*, or a *function of a function*.

A function defined by an equation not solved for the dependent variable is called an *implicit function*. For example, the equation $x^3+y^3=1$ defines y as an implicit function of x.

6°. **The graph of a function.** A set of points (x, y) in an xy-plane, whose coordinates are connected by the equation $y=f(x)$, is called the *graph* of the given function.

1**. Prove that if a and b are real numbers then

$$\big||a|-|b|\big|\leqslant|a-b|\leqslant|a|+|b|.$$

2. Prove the following equalities:

a) $|ab|=|a|\cdot|b|$; c) $\left|\frac{a}{b}\right|=\frac{|a|}{|b|}\ (b\neq 0)$;

b) $|a|^2=a^2$; d) $\sqrt{a^2}=|a|$.

3. Solve the inequalities:

a) $|x-1|<3$; c) $|2x+1|<1$;

b) $|x+1|>2$; d) $|x-1|<|x+1|$.

4. Find $f(-1)$, $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(4)$, if $f(x)=x^3-6x^2+11x-6$.

5. Find $f(0)$, $f\left(-\frac{3}{4}\right)$, $f(-x)$, $f\left(\frac{1}{x}\right)$, $\frac{1}{f(x)}$, if $f(x)=\sqrt{1+x^2}$.

6. $f(x)=\arccos(\log x)$. Find $f\left(\frac{1}{10}\right)$, $f(1)$, $f(10)$.

7. The function $f(x)$ is linear. Find this function, if $f(-1)=2$ and $f(2)=-3$.

---

*) Log x is the logarithm of the number x to the base 10.

8. Find the rational integral function $f(x)$ of degree two, if $f(0)=1$, $f(1)=0$ and $f(3)=5$.

9. Given that $f(4)=-2$, $f(5)=6$. Approximate the value of $f(4,3)$ if we consider the function $f(x)$ on the interval $4\leqslant x\leqslant 5$ linear *(linear interpolation of a function)*.

10. Write the function

$$f(x)=\begin{cases} 0, & \text{if } x\leqslant 0, \\ x, & \text{if } x>0 \end{cases}$$

as a single formula using the absolute-value sign.

Determine the domains of definition of the following functions:

11. a) $y=\sqrt{x+1}$;
 b) $y=\sqrt[3]{x+1}$.

12. $y=\dfrac{1}{4-x^2}$.

13. a) $y=\sqrt{x^2-2}$;
 b) $y=x\sqrt{x^2-2}$.

14**. $y=\sqrt{2+x-x^2}$.

15. $y=\sqrt{-x}+\dfrac{1}{\sqrt{2+x}}$.

16. $y=\sqrt{x-x^3}$.

17. $y=\log\dfrac{2+x}{2-x}$.

18. $y=\log\dfrac{x^2-3x+2}{x+1}$.

19. $y=\arccos\dfrac{2x}{1+x}$.

20. $y=\arcsin\left(\log\dfrac{x}{10}\right)$.

21. Determine the domain of definition of the function

$$y=\sqrt{\sin 2x}.$$

22. $f(x)=2x^4-3x^3-5x^2+6x-10$. Find

$$\varphi(x)=\frac{1}{2}[f(x)+f(-x)] \quad\text{and}\quad \psi(x)=\frac{1}{2}[f(x)-f(-x)].$$

23. A function $f(x)$ defined in a symmetric region $-l<x<l$ is called *even* if $f(-x)=f(x)$ and *odd* if $f(-x)=-f(x)$.

Determine which of the following functions are even and which are odd:

a) $f(x)=\frac{1}{2}(a^x+a^{-x})$;

b) $f(x)=\sqrt{1+x+x^2}-\sqrt{1-x+x^2}$;

c) $f(x)=\sqrt[3]{(x+1)^2}+\sqrt[3]{(x-1)^2}$;

d) $f(x)=\log\dfrac{1+x}{1-x}$;

e) $f(x)=\log(x+\sqrt{1+x^2})$.

24. Prove that any function $f(x)$ defined in the interval $-l<x<l$ may be represented in the form of the sum of an even function and an odd function.

25. Prove that the product of two even functions or of two odd functions is an even function, and that the product of an even function by an odd function is an odd function.

26. A function $f(x)$ is called *periodic* if there exists a positive number T (the *period of the function*) such that $f(x+T)\equiv f(x)$ for all values of x within the domain of definition of $f(x)$.

Determine which of the following functions are periodic, and for the periodic functions find their least period T:

a) $f(x)=10\sin 3x$, d) $f(x)=\sin^2 x$;

b) $f(x)=a\sin\lambda x+b\cos\lambda x$; e) $f(x)=\sin(\sqrt{x})$.

c) $f(x)=\sqrt{\tan x}$;

27. Express the length of the segment $y=MN$ and the area S of the figure AMN as a function of $x=AM$ (Fig. 1). Construct the graphs of these functions.

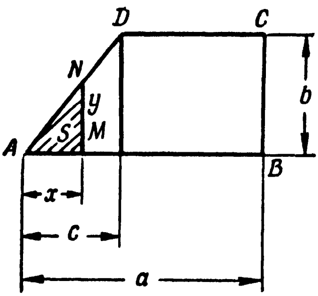

Fig. 1

28. The linear density (that is, mass per unit length) of a rod $AB=l$ (Fig. 2) on the segments $AC=l_1$, $CD=l_2$ and $DB=l_3$ ($l_1+l_2+l_3=l$) is equal to q_1, q_2 and q_3, respectively. Express the mass m of a variable segment $AM=x$ of this rod as a function of x. Construct the graph of this function.

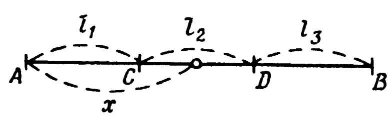

Fig. 2

29. Find $\varphi[\psi(x)]$ and $\psi[\varphi(x)]$, if $\varphi(x)=x^2$ and $\psi(x)=2^x$.

30. Find $f\{f[f(x)]\}$, if $f(x)=\frac{1}{1-x}$.

31. Find $f(x+1)$, if $f(x-1)=x^2$.

32. Let $f(n)$ be the sum of n terms of an arithmetic progression. Show that

$$f(n+3)-3f(n+2)+3f(n+1)-f(n)=0.$$

33. Show that if

$$f(x)=kx+b$$

and the numbers x_1, x_2, x_3 form an arithmetic progression, then the numbers $f(x_1)$, $f(x_2)$ and $f(x_3)$ likewise form such a progression.

34. Prove that if $f(x)$ is an exponential function, that is, $f(x)=a^x\ (a>0)$, and the numbers x_1, x_2, x_3 form an arithmetic progression, then the numbers $f(x_1)$, $f(x_2)$ and $f(x_3)$ form a geometric progression.

35. Let

$$f(x)=\log\frac{1+x}{1-x}.$$

Show that

$$f(x)+f(y)=f\left(\frac{x+y}{1+xy}\right).$$

36. Let $\varphi(x)=\frac{1}{2}(a^x+a^{-x})$ and $\psi(x)=\frac{1}{2}(a^x-a^{-x})$.
Show that

$$\varphi(x+y)=\varphi(x)\varphi(y)+\psi(x)\psi(y)$$

and

$$\psi(x+y)=\varphi(x)\psi(y)+\varphi(y)\psi(x).$$

37. Find $f(-1)$, $f(0)$, $f(1)$ if

$$f(x)=\begin{cases}\arcsin x \text{ for } -1\leqslant x\leqslant 0,\\ \arctan x \text{ for } 0<x\leqslant+\infty.\end{cases}$$

38. Determine the roots (zeros) of the region of positivity and of the region of negativity of the function y if:

a) $y=1+x$;
b) $y=2+x-x^2$;
c) $y=1-x+x^2$;
d) $y=x^3-3x$;
e) $y=\log\frac{2x}{1+x}$.

39. Find the inverse of the function y if:

a) $y=2x+3$;
b) $y=x^2-1$;
c) $y=\sqrt[3]{1-x^3}$;
d) $y=\log\frac{x}{2}$;
e) $y=\arctan 3x$.

In what regions will these inverse functions be defined?

40. Find the inverse of the function

$$y=\begin{cases}x, \text{ if } x\leqslant 0,\\ x^2, \text{ if } x>0.\end{cases}$$

41. Write the given functions as a series of equalities each member of which contains a simple elementary function (power, exponential, trigonometric, and the like):

a) $y=(2x-5)^{10}$;
b) $y=2^{\cos x}$;
c) $y=\log\tan\frac{x}{2}$;
d) $y=\arcsin(3^{-x^2})$.

42. Write as a single equation the composite functions represented as a series of equalities:

a) $y = u^2$, $u = \sin x$;

b) $y = \arctan u$, $u = \sqrt{v}$, $v = \log x$;

c) $y = \begin{cases} 2u, & \text{if } u \leqslant 0, \\ 0, & \text{if } u > 0; \end{cases}$

$u = x^2 - 1$.

43. Write, explicitly, functions of y defined by the equations:

a) $x^2 - \arccos y = \pi$;

b) $10^x + 10^y = 10$;

c) $x + |y| = 2y$.

Find the domains of definition of the given implicit functions.

Sec. 2. Graphs of Elementary Functions

Graphs of functions $y = f(x)$ are mainly constructed by marking a sufficiently dense net of points $M_i(x_i, y_i)$, where $y_i = f(x_i)$ $(i = 0, 1, 2, \ldots)$ and by connecting the points with a line that takes account of intermediate points. Calculations are best done by a slide rule.

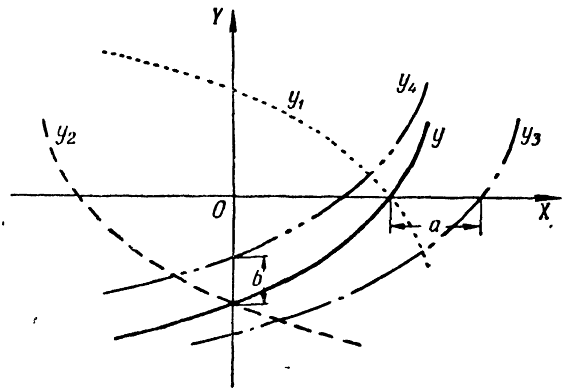

Fig. 3

Graphs of the basic elementary functions (see Appendix VI) are readily learned through their construction. Proceeding from the graph of

$$y = f(x), \tag{Γ}$$

we get the graphs of the following functions by means of simple geometric constructions:

1) $y_1 = -f(x)$ is the mirror image of the graph Γ about the x-axis;
2) $y_2 = f(-x)$ is the mirror image of the graph Γ about the y-axis;

3) $y_3 = f(x-a)$ is the Γ graph displaced along the x-axis by an amount a;
4) $y_4 = b + f(x)$ is the Γ graph displaced along the y-axis by an amount b (Fig. 3).

Example. Construct the graph of the function

$$y = \sin\left(x - \frac{\pi}{4}\right).$$

Solution. The desired line is a sine curve $y = \sin x$ displaced along the x-axis to the right by an amount $\frac{\pi}{4}$ (Fig. 4)

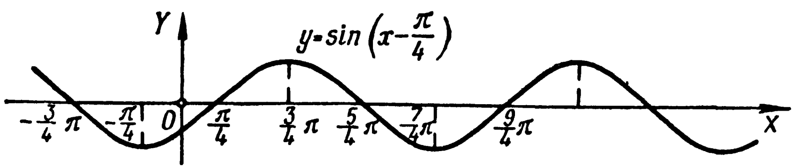

Fig. 4

Construct the graphs of the following linear functions (*straight lines*):

44. $y = kx$, if $k = 0, 1, 2, 1/2, -1, -2$.
45. $y = x + b$, if $b = 0, 1, 2, -1, -2$.
46. $y = 1.5x + 2$.

Construct the graphs of rational integral functions of degree two (*parabolas*).

47. $y = ax^2$, if $a = 1, 2, 1/2, -1, -2, 0$.
48. $y = x^2 + c$, if $c = 0, 1, 2, -1$.
49. $y = (x - x_0)^2$, if $x_0 = 0, 1, 2, -1$.
50. $y = y_0 + (x-1)^2$, if $y_0 = 0, 1, 2, -1$.
51*. $y = ax^2 + bx + c$, if: 1) $a = 1$, $b = -2$, $c = 3$; 2) $a = -2$, $b = 6$, $c = 0$.
52. $y = 2 + x - x^2$. Find the points of intersection of this parabola with the x-axis.

Construct the graphs of the following rational integral functions of degree above two:

53*. $y = x^3$ (*cubic parabola*).
54. $y = 2 + (x-1)^3$.
55. $y = x^3 - 3x + 2$.
56. $y = x^4$.
57. $y = 2x^2 - x^4$.

Construct the graphs of the following linear fractional functions (*hyperbolas*):

58*. $y = \frac{1}{x}$.

59. $y=\frac{1}{1-x}$.

60. $y=\frac{x-2}{x+2}$.

61*. $y=y_0+\frac{m}{x-x_0}$, if $x_0=1$, $y_0=-1$, $m=6$.

62*. $y=\frac{2x-3}{3x+2}$.

Construct the graphs of the fractional rational functions:

63. $y=x+\frac{1}{x}$.

64. $y=\frac{x^2}{x+1}$.

65*. $y=\frac{1}{x^2}$.

66. $y=\frac{1}{x^3}$.

67*. $y=\frac{10}{x^2+1}$ (*Witch of Agnesi*).

68. $y=\frac{2x}{x^2+1}$ (*Newton's serpentine*).

69. $y=x+\frac{1}{x^2}$.

70. $y=x^2+\frac{1}{x}$ (*trident of Newton*).

Construct the graphs of the irrational functions:

71*. $y=\sqrt{x}$.

72. $y=\sqrt[3]{x}$.

73*. $y=\sqrt[3]{x^2}$ (*Niele's parabola*).

74. $y=\pm x\sqrt{x}$ (*semicubical parabola*).

75*. $y=\pm\frac{3}{5}\sqrt{25-x^2}$ (*ellipse*).

76. $y=\pm\sqrt{x^2-1}$ (*hyperbola*).

77. $y=\frac{1}{\sqrt{1-x^2}}$.

78*. $y=\pm x\sqrt{\frac{x}{4-x}}$ (*cissoid of Diocles*).

79. $y=\pm x\sqrt{25-x^2}$.

Construct the graphs of the trigonometric functions:

80*. $y=\sin x$.

81*. $y=\cos x$.

82*. $y=\tan x$.

83*. $y=\cot x$.

84*. $y=\sec x$.

85*. $y=\operatorname{cosec} x$.

86. $y=A\sin x$, if $A=1,\ 10,\ 1/2,\ -2$.

87*. $y=\sin nx$, if $n=1,\ 2,\ 3,\ 1/2$.

88. $y=\sin(x-\varphi)$, if $\varphi=0,\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \pi,\ -\frac{\pi}{4}$.

89*. $y=5\sin(2x-3)$.

90*. $y=a\sin x+b\cos x$, if $a=6$, $b=-8$.

91. $y=\sin x+\cos x$.

92*. $y=\cos^2 x$.

93*. $y=x+\sin x$.

94*. $y=x\sin x$.

95. $y=\tan^2 x$.

96. $y=1-2\cos x$.

97. $y=\sin x-\frac{1}{3}\sin 3x$.

98. $y=\cos x+\frac{1}{2}\cos 2x$.

99*. $y=\cos\frac{\pi}{x}$.

100. $y=\pm\sqrt{\sin x}$.

Construct the graphs of the exponential and logarithmic functions:

101. $y=a^x$, if $a=2$, $\frac{1}{2}$, $e\,(e=2,\ 718\ \ldots)$ *).

102*. $y=\log_a x$, if $a=10,\ 2,\ \frac{1}{2},\ e$.

103*. $y=\sinh x$, where $\sinh x=1/2\,(e^x-e^{-x})$.

104*. $y=\cosh x$, where $\cosh x=1/2\,(e^x+e^{-x})$.

105*. $y=\tanh x$, where $\tanh x=\frac{\sinh x}{\cosh x}$.

106. $y=10^{\frac{1}{x}}$.

107*. $y=e^{-x^2}$ *(probability curve)*.

108. $y=2^{-\frac{1}{x^2}}$.

109. $y=\log x^2$.

110. $y=\log^2 x$.

111. $y=\log(\log x)$.

112. $y=\frac{1}{\log x}$.

113. $y=\log\frac{1}{x}$.

114. $y=\log(-x)$.

115. $y=\log_2(1+x)$.

116. $y=\log(\cos x)$.

117. $y=2^{-x}\sin x$.

Construct the graphs of the inverse trigonometric functions:

118*. $y=\arcsin x$.

119*. $y=\arccos x$.

120*. $y=\arctan x$.

121*. $y=\operatorname{arccot} x$.

122. $y=\arcsin\frac{1}{x}$.

123. $y=\arccos\frac{1}{x}$.

124. $y=x+\operatorname{arccot} x$.

Construct the graphs of the functions:

125. $y=|x|$.

126. $y=\frac{1}{2}(x+|x|)$.

127. a) $y=x|x|$; b) $y=\log_{\sqrt{2}}|x|$.

128. a) $y=\sin x+|\sin x|$; b) $y=\sin x-|\sin x|$.

129. $y=\begin{cases} 3-x^2 & \text{when } |x|\leqslant 1. \\ \frac{2}{|x|} & \text{when } |x|>1. \end{cases}$

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*) About the number e see p. 22 for more details.

130. a) $y=[x]$, b) $y=x-[x]$, where $[x]$ is the integral part of the number x, that is, the greatest integer less than or equal to x.

Construct the graphs of the following functions in the polar coordinate system (r, φ) $(r \geqslant 0)$:

131. $r=1$.

132*. $r=\frac{\varphi}{2}$ *(spiral of Archimedes).*

133*. $r=e^{\varphi}$ *(logarithmic spiral).*

134*. $r=\frac{\pi}{\varphi}$ *(hyperbolic spiral).*

135. $r=2\cos\varphi$ *(circle).*

136. $r=\frac{1}{\sin\varphi}$ *(straight line).*

137. $r=\sec^2\frac{\varphi}{2}$ *(parabola).*

138*. $r=10\sin 3\varphi$ *(three-leafed rose)*

139*. $r=a(1+\cos\varphi)$ $(a>0)$ *(cardioid).*

140*. $r^2=a^2\cos 2\varphi$ $(a>0)$ *(lemniscate).*

Construct the graphs of the functions represented parametrically:

141*. $x=t^3$, $y=t^2$ *(semicubical parabola).*

142*. $x=10\cos t$, $y=\sin t$ *(ellipse).*

143*. $x=10\cos^3 t$, $y=10\sin^3 t$ *(astroid).*

144*. $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$ *(involute of a circle).*

145*. $x=\frac{at}{1+t^3}$, $y=\frac{at^2}{1+t^3}$ *(folium of Descartes).*

146. $x=\frac{a}{\sqrt{1+t^2}}$, $y=\frac{at}{\sqrt{1+t^2}}$ *(semicircle).*

147. $x=2^t+2^{-t}$, $y=2^t-2^{-t}$ *(branch of a hyperbola).*

148. $x=2\cos^2 t$, $y=2\sin^2 t$ *(segment of a straight line).*

149. $x=t-t^2$, $y=t^2-t^3$.

150. $x=a(2\cos t-\cos 2t)$, $y=a(2\sin t-\sin 2t)$ *(cardioid).*

Construct the graphs of the following functions defined implicitly:

151*. $x^2+y^2=25$ *(circle).*

152. $xy=12$ *(hyperbola).*

153*. $y^2=2x$ *(parabola).*

154. $\frac{x^2}{100}+\frac{y^2}{64}=1$ *(ellipse).*

155. $y^2=x^2(100-x^2)$.

156*. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ *(astroid).*

157*. $x+y=10\log y$.

158. $x^2=\cos y$.

159*. $\sqrt{x^2+y^2}=e^{\arctan\frac{y}{x}}$ *(logarithmic spiral).*

160*. $x^3+y^3-3xy=0$ *(folium of Descartes).*

161. Derive the conversion formula from the Celsius scale (C) to the Fahrenheit scale (F) if it is known that 0°C corresponds to 32°F and 100°C corresponds to 212°F.

Construct the graph of the function obtained.

162. Inscribed in a triangle (base $b=10$, altitude $h=6$) is a rectangle (Fig. 5). Express the area of the rectangle y as a function of the base x.

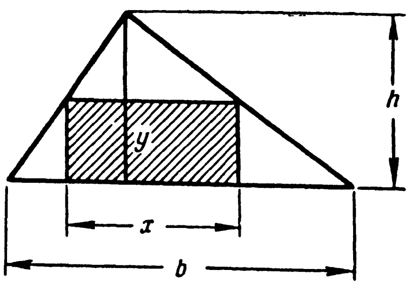

Fig. 5

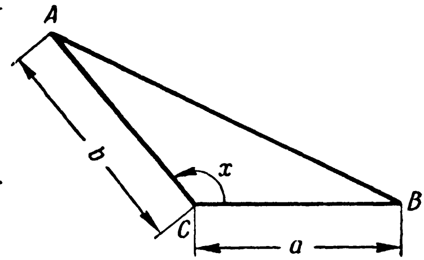

Fig 6

Construct the graph of this function and find its greatest value.

163. Given a triangle ACB with $BC=a$, $AC=b$ and a variable angle $\angle ACB=x$ (Fig. 6).

Express $y=$ area $\triangle ABC$ as a function of x. Plot the graph of this function and find its greatest value.

164. Give a graphic solution of the equations:

a) $2x^2-5x+2=0$; d) $10^{-x}=x$;

b) $x^3+x-1=0$; e) $x=1+0.5\sin x$;

c) $\log x=0.1x$; f) $\cot x=x$ $(0<x<\pi)$.

165. Solve the systems of equations graphically:

a) $xy=10$, $x+y=7$;

b) $xy=6$, $x^2+y^2=13$;

c) $x^2-x+y=4$, $y^2-2x=0$;

d) $x^2+y=10$, $x+y^2=6$;

e) $y=\sin x$, $y=\cos x$ $(0<x<2\pi)$.

Sec. 3. Limits

1°. The limit of a sequence. The number a is the ***limit of a sequence*** $x_1, x_2, \ldots, x_n, \ldots$, or

$$\lim_{n\to\infty} x_n = a,$$

if for any $\varepsilon > 0$ there is a number $N = N(\varepsilon)$ such that $|x_n - a| < \varepsilon$ when $n > N$.

Example 1. Show that

$$\lim_{n\to\infty} \frac{2n+1}{n+1} = 2. \tag{1}$$

Solution. Form the difference

$$\frac{2n+1}{n+1} - 2 = -\frac{1}{n+1}.$$

Evaluating the absolute value of this difference, we have:

$$\left|\frac{2n+1}{n+1} - 2\right| = \frac{1}{n+1} < \varepsilon, \tag{2}$$

if

$$n > \frac{1}{\varepsilon} - 1 = N(\varepsilon).$$

Thus, for every positive number ε there will be a number $N = \frac{1}{\varepsilon} - 1$ such that for $n > N$ we will have inequality (2) Consequently, the number 2 is the limit of the sequence $x_n = (2n+1)/(n+1)$, hence, formula (1) is true.

2°. The limit of a function. We say that a function $f(x) \to A$ as $x \to a$ (A and a are numbers), or

$$\lim_{x\to a} f(x) = A,$$

if for every $\varepsilon > 0$ we have $\delta = \delta(\varepsilon) > 0$ such that $|f(x) - A| < \varepsilon$ for $0 < |x - a| < \delta$.

Similarly,

$$\lim_{x\to\infty} f(x) = A,$$

if $|f(x) - A| < \varepsilon$ for $|x| > N(\varepsilon)$.

The following conventional notation is also used:

$$\lim_{x\to a} f(x) = \infty,$$

which means that $|f(x)| > E$ for $0 < |x - a| < \delta(E)$, where E is an arbitrary positive number

3°. One-sided limits. If $x < a$ and $x \to a$, then we write conventionally $x \to a - 0$; similarly, if $x > a$ and $x \to a$, then we write $x \to a + 0$. The numbers

$$f(a-0) = \lim_{x\to a-0} f(x) \text{ and } f(a+0) = \lim_{x\to a+0} f(x)$$

are called, respectively, the *limit on the left* of the function $f(x)$ at the point a and the *limit on the right* of the function $f(x)$ at the point a (if these numbers exist).

For the existence of the limit of a function $f(x)$ as $x \to a$, it is necessary and sufficient to have the following equality:

$$f(a-0) = f(a+0).$$

If the limits $\lim\limits_{x \to a} f_1(x)$ and $\lim\limits_{x \to a} f_2(x)$ exist, then the following theorems hold:

1) $\lim\limits_{x \to a} [f_1(x) + f_2(x)] = \lim\limits_{x \to a} f_1(x) + \lim\limits_{x \to a} f_2(x)$;

2) $\lim\limits_{x \to a} [f_1(x) f_2(x)] = \lim\limits_{x \to a} f_1(x) \cdot \lim\limits_{x \to a} f_2(x)$;

3) $\lim\limits_{x \to a} [f_1(x)/f_2(x)] = \lim\limits_{x \to a} f_1(x) / \lim\limits_{x \to a} f_2(x) \qquad (\lim\limits_{x \to a} f_2(x) \neq 0)$.

The following two limits are frequently used:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

and

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{\alpha \to 0} (1+\alpha)^{\frac{1}{\alpha}} = e = 2.71828 \ldots$$

Example 2. Find the limits on the right and left of the function

$$f(x) = \arctan \frac{1}{x}$$

as $x \to 0$.

Solution. We have

$$f(+0) = \lim_{x \to +0} \left(\arctan \frac{1}{x}\right) = \frac{\pi}{2}$$

and

$$f(-0) = \lim_{x \to -0} \left(\arctan \frac{1}{x}\right) = -\frac{\pi}{2}.$$

Obviously, the function $f(x)$ in this case has no limit as $x \to 0$.

166. Prove that as $n \to \infty$ the limit of the sequence

$$1, \ \frac{1}{4}, \ \ldots, \ \frac{1}{n^2}, \ \ldots$$

is equal to zero. For which values of n will we have the inequality

$$\frac{1}{n^2} < \varepsilon$$

(ε is an arbitrary positive number)?

Calculate numerically for a) $\varepsilon = 0.1$; b) $\varepsilon = 0.01$; c) $\varepsilon = 0.001$.

167. Prove that the limit of the sequence

$$x_n = \frac{n}{n+1} \quad (n = 1, 2, \ldots)$$

as $n \to \infty$ is unity. For which values of $n > N$ will we have the inequality

$$|x_n - 1| < \varepsilon$$

(ε is an arbitrary positive number)?

Find N for a) $\varepsilon = 0.1$; b) $\varepsilon = 0.01$; c) $\varepsilon = 0.001$.

168. Prove that

$$\lim_{x \to 2} x^2 = 4.$$

How should one choose, for a given positive number ε, some positive number δ so that the inequality

$$|x^2 - 4| < \varepsilon$$

should follow from

$$|x - 2| < \delta?$$

Compute δ for a) $\varepsilon = 0.1$; b) $\varepsilon = 0.01$; c) $\varepsilon = 0.001$.

169. Give the exact meaning of the following notations: a) $\lim\limits_{x \to +0} \log x = -\infty$; b) $\lim\limits_{x \to +\infty} 2^x = +\infty$; c) $\lim\limits_{x \to \infty} f(x) = \infty$.

170. Find the limits of the sequences:

a) $1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \ldots,\ \frac{(-1)^{n-1}}{n},\ \ldots;$

b) $\frac{2}{1},\ \frac{4}{3},\ \frac{6}{5},\ \ldots,\ \frac{2n}{2n-1},\ \ldots;$

c) $\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \ldots;$

d) $0.2,\ 0.23,\ 0.233,\ 0.2333,\ \ldots$

Find the limits:

171. $\lim\limits_{n \to \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \ldots + \frac{n-1}{n^2} \right).$

172. $\lim\limits_{n \to \infty} \frac{(n+1)(n+2)(n+3)}{n^3}.$

173. $\lim\limits_{n \to \infty} \left[\frac{1+3+5+7+\ldots+(2n-1)}{n+1} - \frac{2n+1}{2} \right].$

174. $\lim\limits_{n \to \infty} \frac{n+(-1)^n}{n-(-1)^n}.$

175. $\lim\limits_{n \to \infty} \frac{2^{n+1}+3^{n+1}}{2^n+3^n}.$

176. $\lim\limits_{n \to \infty} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} \right).$

177. $\lim\limits_{n \to \infty} \left[1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots + \frac{(-1)^{n-1}}{3^{n-1}} \right].$

178*. $\lim\limits_{n \to \infty} \frac{1^2+2^2+3^2+\ldots+n^2}{n^3}.$

179. $\lim\limits_{n\to\infty}(\sqrt{n+1}-\sqrt{n})$.

180. $\lim\limits_{n\to\infty}\dfrac{n\sin n!}{n^2+1}$.

When seeking the limit of a ratio of two integral polynomials in x as $x\to\infty$, it is useful first to divide both terms of the ratio by x^n, where n is the highest degree of these polynomials.

A similar procedure is also possible in many cases for fractions containing irrational terms.

Example 1.

$$\lim_{x\to\infty}\frac{(2x-3)(3x+5)(4x-6)}{3x^3+x-1}=$$

$$=\lim_{x\to\infty}\frac{\left(2-\frac{3}{x}\right)\left(3+\frac{5}{x}\right)\left(4-\frac{6}{x}\right)}{3+\frac{1}{x^2}-\frac{1}{x^3}}=\frac{2\cdot3\cdot4}{3}=8.$$

Example 2.

$$\lim_{x\to\infty}\frac{x}{\sqrt[3]{x^3+10}}=\lim_{x\to\infty}\frac{1}{\sqrt[3]{1+\frac{10}{x^3}}}=1.$$

181. $\lim\limits_{x\to\infty}\dfrac{(x+1)^2}{x^2+1}$.

182. $\lim\limits_{x\to\infty}\dfrac{1000x}{x^2-1}$.

183. $\lim\limits_{x\to\infty}\dfrac{x^2-5x+1}{3x+7}$.

184. $\lim\limits_{x\to\infty}\dfrac{2x^2-x+3}{x^3-8x+5}$.

185. $\lim\limits_{x\to\infty}\dfrac{(2x+3)^3(3x-2)^2}{x^5+5}$.

186. $\lim\limits_{x\to\infty}\dfrac{2x^2-3x-4}{\sqrt{x^4+1}}$.

187. $\lim\limits_{x\to\infty}\dfrac{2x+3}{x+\sqrt[3]{x}}$.

188. $\lim\limits_{x\to\infty}\dfrac{x^2}{10+x\sqrt{x}}$.

189. $\lim\limits_{x\to\infty}\dfrac{\sqrt[3]{x^2+1}}{x+1}$.

190. $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$.

If $P(x)$ and $Q(x)$ are integral polynomials and $P(a)\neq0$ or $Q(a)\neq0$, then the limit of the rational fraction

$$\lim_{x\to a}\frac{P(x)}{Q(x)}$$

is obtained directly.

But if $P(a)=Q(a)=0$, then it is advisable to cancel the binomial $x-a$ out of the fraction $\dfrac{P(x)}{Q(x)}$ once or several times.

Example 3.

$$\lim_{x\to2}\frac{x^2-4}{x^2-3x+2}=\lim_{x\to2}\frac{(x-2)(x+2)}{(x-2)(x-1)}=\lim_{x\to2}\frac{x+2}{x-1}=4.$$

191. $\lim\limits_{x \to -1} \frac{x^3+1}{x^2+1}$.

192. $\lim\limits_{x \to 5} \frac{x^2-5x+10}{x^2-25}$.

193. $\lim\limits_{x \to -1} \frac{x^2-1}{x^2+3x+2}$.

194. $\lim\limits_{x \to 2} \frac{x^2-2x}{x^2-4x+4}$.

195. $\lim\limits_{x \to 1} \frac{x^3-3x+2}{x^4-4x+3}$.

196. $\lim\limits_{x \to a} \frac{x^2-(a+1)x+a}{x^3-a^3}$.

197. $\lim\limits_{h \to 0} \frac{(x+h)^3-x^3}{h}$.

198. $\lim\limits_{x \to 1} \left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$.

The expressions containing irrational terms are in many cases rationalized by introducing a new variable.

Example 4. Find

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}.$$

Solution. Putting

$$1+x=y^6,$$

we have

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}=\lim_{y \to 1} \frac{y^3-1}{y^2-1}=\lim_{y \to 1} \frac{y^2+y+1}{y+1}=\frac{3}{2}.$$

199. $\lim\limits_{x \to 1} \frac{\sqrt{x}-1}{x-1}$.

200. $\lim\limits_{x \to 64} \frac{\sqrt{x}-8}{\sqrt[3]{x}-4}$.

201. $\lim\limits_{x \to 1} \frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$.

202. $\lim\limits_{x \to 1} \frac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}$.

Another way of finding the limit of an irrational expression is to transfer the irrational term from the numerator to the denominator, or vice versa, from the denominator to the numerator.

Example 5.

$$\lim_{x \to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}=\lim_{x \to a} \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}=$$

$$=\lim_{x \to a} \frac{1}{\sqrt{x}+\sqrt{a}}=\frac{1}{2\sqrt{a}} \quad (a>0).$$

203. $\lim\limits_{x \to 7} \frac{2-\sqrt{x-3}}{x^2-49}$.

204. $\lim\limits_{x \to 8} \frac{x-8}{\sqrt[3]{x}-2}$.

205. $\lim\limits_{x \to 1} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$.

206. $\lim\limits_{x \to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$.

207. $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

208. $\lim\limits_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$.

209. $\lim\limits_{h \to 0} \frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$.

210. $\lim\limits_{x \to 3} \frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}$.

211. $\lim\limits_{x \to +\infty} (\sqrt{x+a}-\sqrt{x})$.

212. $\lim\limits_{x \to +\infty} [\sqrt{x(x+a)}-x]$.

213. $\lim\limits_{x \to +\infty} (\sqrt{x^2-5x+6}-x)$.

214. $\lim\limits_{x \to +\infty} x(\sqrt{x^2+1}-x)$.

215. $\lim\limits_{x \to \infty} (x+\sqrt[3]{1-x^3})$.

The formula

$$\lim_{x \to 0} \frac{\sin x}{x}=1$$

is frequently used when solving the following examples. It is taken for granted that $\lim\limits_{x \to a} \sin x=\sin a$ and $\lim\limits_{x \to a} \cos x=\cos a$.

Example 6.

$$\lim_{x \to 0} \frac{\sin 5x}{x}=\lim_{x \to 0}\left(\frac{\sin 5x}{5x}\cdot 5\right)=1\cdot 5=5.$$

216. a) $\lim\limits_{x \to 2} \frac{\sin x}{x}$;

b) $\lim\limits_{x \to \infty} \frac{\sin x}{x}$.

217. $\lim\limits_{x \to 0} \frac{\sin 3x}{x}$.

218. $\lim\limits_{x \to 0} \frac{\sin 5x}{\sin 2x}$.

219. $\lim\limits_{x \to 1} \frac{\sin \pi x}{\sin 3\pi x}$.

220. $\lim\limits_{n \to \infty} \left(n \sin \frac{\pi}{n}\right)$.

221. $\lim\limits_{x \to 0} \frac{1-\cos x}{x^2}$.

222. $\lim\limits_{x \to a} \frac{\sin x-\sin a}{x-a}$.

223. $\lim\limits_{x \to a} \frac{\cos x-\cos a}{x-a}$.

224. $\lim\limits_{x \to -2} \frac{\tan \pi x}{x+2}$.

225. $\lim\limits_{h \to 0} \frac{\sin (x+h)-\sin x}{h}$.

226. $\lim\limits_{x \to \frac{\pi}{4}} \frac{\sin x-\cos x}{1-\tan x}$.

227. a) $\lim\limits_{x \to 0} x \sin \frac{1}{x}$;

b) $\lim\limits_{x \to \infty} x \sin \frac{1}{x}$.

228. $\lim\limits_{x \to 1} (1-x) \tan \frac{\pi x}{2}$.

229. $\lim\limits_{x \to 0} \cot 2x \cot \left(\frac{\pi}{2}-x\right)$.

230. $\lim\limits_{x \to \pi} \frac{1-\sin \frac{x}{2}}{\pi-x}$.

231. $\lim\limits_{x \to \frac{\pi}{3}} \frac{1-2\cos x}{\pi-3x}$.

232. $\lim\limits_{x \to 0} \frac{\cos mx-\cos nx}{x^2}$.

233. $\lim\limits_{x \to 0} \frac{\tan x-\sin x}{x^3}$.

234. $\lim\limits_{x \to 0} \frac{\arcsin x}{x}$.

235. $\lim\limits_{x \to 0} \frac{\arctan 2x}{\sin 3x}$.

236. $\lim\limits_{x \to 1} \frac{1-x^2}{\sin \pi x}$.

237. $\lim\limits_{x\to 0} \dfrac{x-\sin 2x}{x+\sin 3x}$.

238. $\lim\limits_{x\to 1} \dfrac{\cos \frac{\pi x}{2}}{1-\sqrt{x}}$.

239. $\lim\limits_{x\to 0} \dfrac{1-\sqrt{\cos x}}{x^2}$.

240. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$.

When taking limits of the form

$$\lim_{x\to a} [\varphi(x)]^{\psi(x)} = C \tag{3}$$

one should bear in mind that:

1) if there are final limits

$$\lim_{x\to a} \varphi(x) = A \text{ and } \lim_{x\to a} \psi(x) = B,$$

then $C = A^B$;

2) if $\lim\limits_{x\to a} \varphi(x) = A \neq 1$ and $\lim\limits_{x\to a} \psi(x) = \pm\infty$, then the problem of finding the limit of (3) is solved in straightforward fashion;

3) if $\lim\limits_{x\to a} \varphi(x) = 1$ and $\lim\limits_{x\to a} \psi(x) = \infty$, then we put $\varphi(x) = 1+\alpha(x)$, where $\alpha(x) \to 0$ as $x \to a$ and, hence,

$$C = \lim_{x\to a} \left\{[1+\alpha(x)]^{\frac{1}{\alpha(x)}}\right\}^{\alpha(x)\psi(x)} = e^{\lim\limits_{x\to a} \alpha(x)\psi(x)} = e^{\lim\limits_{x\to a} [\varphi(x)-1]\psi(x)},$$

where $e = 2.718\ldots$ is Napier's number.

Example 7. Find

$$\lim_{x\to 0} \left(\frac{\sin 2x}{x}\right)^{1+x}.$$

Solution. Here,

$$\lim_{x\to 0} \left(\frac{\sin 2x}{x}\right) = 2 \text{ and } \lim_{x\to 0} (1+x) = 1;$$

hence,

$$\lim_{x\to 0} \left(\frac{\sin 2x}{x}\right)^{1+x} = 2^1 = 2.$$

Example 8. Find

$$\lim_{x\to\infty} \left(\frac{x+1}{2x+1}\right)^{x^2}.$$

Solution. We have

$$\lim_{x\to\infty} \frac{x+1}{2x+1} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{2+\frac{1}{x}} = \frac{1}{2}$$

and

$$\lim_{x\to\infty} x^2 = +\infty.$$

Therefore,

$$\lim_{x\to\infty}\left(\frac{x+1}{2x+1}\right)^{x^2}=0.$$

Example 9. Find

$$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^{x}.$$

Solution. We have

$$\lim_{x\to\infty}\frac{x-1}{x+1}=\lim_{x\to\infty}\frac{1-\frac{1}{x}}{1+\frac{1}{x}}=1.$$

Transforming, as indicated above, we have

$$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^{x}=\lim_{x\to\infty}\left[1+\left(\frac{x-1}{x+1}-1\right)\right]^{x}=$$

$$=\lim_{x\to\infty}\left\{\left[1+\left(\frac{-2}{x+1}\right)\right]^{\frac{x+1}{-2}}\right\}^{-\frac{2x}{1+x}}=e^{\lim\limits_{x\to\infty}\frac{-2x}{x+1}}=e^{-2}.$$

In this case it is easier to find the limit without resorting to the general procedure:

$$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^{x}=\lim_{x\to\infty}\frac{\left(1-\frac{1}{x}\right)^{x}}{\left(1+\frac{1}{x}\right)^{x}}=\frac{\lim\limits_{x\to\infty}\left[\left(1-\frac{1}{x}\right)^{-x}\right]^{-1}}{\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}}=\frac{e^{-1}}{e}=e^{-2}.$$

Generally, it is useful to remember that

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{x}=e^{k}$$

241. $\lim\limits_{x\to 0}\left(\frac{2+x}{3-x}\right)^{x}$.

242. $\lim\limits_{x\to 1}\left(\frac{x-1}{x^2-1}\right)^{x+1}$.

243. $\lim\limits_{x\to\infty}\left(\frac{1}{x^2}\right)^{\frac{2x}{x+1}}$.

244. $\lim\limits_{x\to 0}\left(\frac{x^2-2x+3}{x^2-3x+2}\right)^{\frac{\sin x}{x}}$.

245. $\lim\limits_{x\to\infty}\left(\frac{x^2+2}{2x^2+1}\right)^{x^2}$.

246. $\lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}$.

247 $\lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^{x}$.

248. $\lim\limits_{x\to\infty}\left(\frac{x}{x+1}\right)^{x}$.

249. $\lim\limits_{x\to\infty}\left(\frac{x-1}{x+3}\right)^{x+2}$.

250. $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}$.

251. $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$.

252.** a) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x}}$; b) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$.

When solving the problems that follow, it is useful to know that if the limit $\lim\limits_{x\to a} f(x)$ exists and is positive, then

$$\lim_{x\to a}\,[\ln f(x)] = \ln\,[\lim_{x\to a} f(x)].$$

Example 10. Prove that

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1. \qquad (*)$$

Solution. We have

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\,[\ln(1+x)^{\frac{1}{x}}] = \ln\,[\lim_{x\to 0}(1+x)^{\frac{1}{x}}] = \ln e = 1.$$

Formula (*) is frequently used in the solution of problems.

253. $\lim\limits_{x\to\infty}[\ln(2x+1) - \ln(x+2)]$.

254. $\lim\limits_{x\to 0}\dfrac{\log(1+10x)}{x}$.

255. $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\ln\sqrt{\dfrac{1+x}{1-x}}\right)$.

256. $\lim\limits_{x\to+\infty} x\,[\ln(x+1) - \ln x]$.

257. $\lim\limits_{x\to 0}\dfrac{\ln(\cos x)}{x^2}$.

258*. $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}$.

259*. $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}$ $(a>0)$.

260*. $\lim\limits_{n\to\infty} n\,(\sqrt[n]{a}-1)$ $(a>0)$.

261. $\lim\limits_{x\to 0}\dfrac{e^{ax}-e^{bx}}{x}$.

262. $\lim\limits_{x\to 0}\dfrac{1-e^{-x}}{\sin x}$.

263. a) $\lim\limits_{x\to 0}\dfrac{\sinh x}{x}$;

b) $\lim\limits_{x\to 0}\dfrac{\cosh x-1}{x^2}$

(see Problems 103 and 104).

Find the following limits that occur on one side:

264. a) $\lim\limits_{x\to-\infty}\dfrac{x}{\sqrt{x^2+1}}$;

b) $\lim\limits_{x\to+\infty}\dfrac{x}{\sqrt{x^2+1}}$.

265. a) $\lim\limits_{x\to-\infty}\tanh x$;

b) $\lim\limits_{x\to+\infty}\tanh x$,

where $\tanh x = \dfrac{e^x-e^{-x}}{e^x+e^{-x}}$.

266. a) $\lim\limits_{x\to-0}\dfrac{1}{1+e^{\frac{1}{x}}}$;

b) $\lim\limits_{x\to+0}\dfrac{1}{1+e^{\frac{1}{x}}}$.

267. a) $\lim\limits_{x\to-\infty}\dfrac{\ln(1+e^x)}{x}$;

b) $\lim\limits_{x\to+\infty}\dfrac{\ln(1+e^x)}{x}$.

268. a) $\lim\limits_{x\to-0}\dfrac{|\sin x|}{x}$;

b) $\lim\limits_{x\to+0}\dfrac{|\sin x|}{x}$.

269. a) $\lim\limits_{x\to 1-0}\frac{x-1}{|x-1|}$;

b) $\lim\limits_{x\to 1+0}\frac{x-1}{|x-1|}$.

270. a) $\lim\limits_{x\to 2-0}\frac{x}{x-2}$;

b) $\lim\limits_{x\to 2+0}\frac{x}{x-2}$.

Construct the graphs of the following functions:

271.** $y=\lim\limits_{n\to\infty}(\cos^{2n}x)$.

272*. $y=\lim\limits_{n\to\infty}\frac{x}{1+x^n}$ $(x\geqslant 0)$.

273. $y=\lim\limits_{\alpha\to 0}\sqrt{x^2+\alpha^2}$.

274. $y=\lim\limits_{n\to\infty}(\arctan nx)$.

275. $y=\lim\limits_{n\to\infty}\sqrt[n]{1+x^n}$ $(x\geqslant 0)$.

276. Transform the following mixed periodic fraction into a common fraction:

$$\alpha=0.13555...$$

Regard it as the limit of the corresponding finite fraction.

277. What will happen to the roots of the quadratic equation

$$ax^2+bx+c=0,$$

if the coefficient a approaches zero while the coefficients b and c are constant, and $b\neq 0$?

278. Find the limit of the interior angle of a regular n-gon as $n\to\infty$.

279. Find the limit of the perimeters of regular n-gons inscribed in a circle of radius R and circumscribed about it as $n\to\infty$.

280. Find the limit of the sum of the lengths of the ordinates of the curve

$$y=e^{-x}\cos\pi x,$$

drawn at the points $x=0, 1, 2, \ldots, n$, as $n\to\infty$.

281. Find the limit of the sum of the areas of the squares constructed on the ordinates of the curve

$$y=2^{1-x}$$

as on bases, where $x=1, 2, 3, \ldots, n$, provided that $n\to\infty$.

282. Find the limit of the perimeter of a broken line $M_0M_1\ldots M_n$ inscribed in a logarithmic spiral

$$r=e^{-\varphi}$$

(as $n \to \infty$), if the vertices of this broken line have, respectively, the polar angles

$$\varphi_0 = 0,\ \varphi_1 = \frac{\pi}{2},\ \ldots,\ \varphi_n = \frac{n\pi}{2}.$$

283. A segment $AB = a$ (Fig. 7) is divided into n equal parts, each part serving as the base of an isosceles triangle with base angles $\alpha = 45°$. Show that the limit of the perimeter of the broken line thus formed differs from the length of AB despite the fact that in the limit the broken line "geometrically merges with the segment AB".

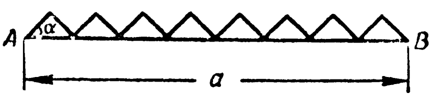

Fig. 7

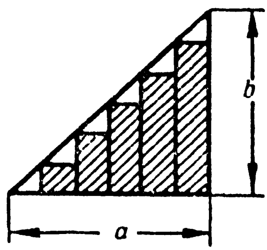

Fig 8

284. The point C_1 divides a segment $AB = l$ in half; the point C_2 divides a segment AC_1 in half; the point C_3 divides a segment C_2C_1 in half; the point C_4 divides C_2C_3 in half, and so on. Determine the limiting position of the point C_n when $n \to \infty$.

285. The side a of a right triangle is divided into n equal parts, on each of which is constructed an inscribed rectangle (Fig. 8). Determine the limit of the area of the step-like figure thus formed if $n \to \infty$.

286. Find the constants k and b from the equation

$$\lim_{x \to \infty} \left(kx + b - \frac{x^3 + 1}{x^2 + 1} \right) = 0. \qquad (1)$$

What is the geometric meaning of (1)?

287*. A certain chemical process proceeds in such fashion that the increase in quantity of a substance during each interval of time τ out of the infinite sequence of intervals $(i\tau, (i+1)\tau)$ $(i = 0, 1, 2, \ldots)$ is proportional to the quantity of the substance available at the commencement of each interval and to the length of the interval. Assuming that the quantity of substance at the initial time is Q_0, determine the quantity of substance $Q_t^{(n)}$ after the elapse of time t if the increase takes place each nth part of the time interval $\tau = \frac{t}{n}$.

Find $Q_t = \lim_{n \to \infty} Q_t^{(n)}$.

Sec. 4. Infinitely Small and Large Quantities

1°. Infinitely small quantities (infinitesimals). If

$$\lim_{x \to a} \alpha(x) = 0,$$

i.e., if $|\alpha(x)| < \varepsilon$ when $0 < |x-a| < \delta(\varepsilon)$, then the function $\alpha(x)$ is an *infinitesimal* as $x \longrightarrow a$. In similar fashion we define the infinitesimal $\alpha(x)$ as $x \longrightarrow \infty$.

The sum and product of a limited number of infinitesimals as $x \longrightarrow a$ are also infinitesimals as $x \to a$.

If $\alpha(x)$ and $\beta(x)$ are infinitesimals as $x \longrightarrow a$ and

$$\lim_{x \to a} \frac{\alpha(x)}{\beta(x)} = C,$$

where C is some number different from zero, then the functions $\alpha(x)$ and $\beta(x)$ are called *infinitesimals of the same order;* but if $C=0$, then we say that the function $\alpha(x)$ is an *infinitesimal of higher order* than $\beta(x)$. The function $\alpha(x)$ is called an *infinitesimal of order n* compared with the function $\beta(x)$ if

$$\lim_{x \to a} \frac{\alpha(x)}{[\beta(x)]^n} = C,$$

where $0 < |C| < +\infty$.

If

$$\lim_{x \to a} \frac{\alpha(x)}{\beta(x)} = 1,$$

then the functions $\alpha(x)$ and $\beta(x)$ are called *equivalent functions* as $x \longrightarrow a$:

$$\alpha(x) \sim \beta(x).$$

For example, for $x \longrightarrow 0$ we have

$$\sin x \sim x; \quad \tan x \sim x; \quad \ln(1+x) \sim x$$

and so forth.

The sum of two infinitesimals of different orders is equivalent to the term whose order is lower.

The limit of a ratio of two infinitesimals remains unchanged if the terms of the ratio are replaced by equivalent quantities. By virtue of this theorem, when taking the limit of a fraction

$$\lim_{x \to a} \frac{\alpha(x)}{\beta(x)},$$

where $\alpha(x) \longrightarrow 0$ and $\beta(x) \longrightarrow 0$ as $x \longrightarrow a$, we can subtract from (or add to) the numerator or denominator infinitesimals of higher orders chosen so that the resultant quantities should be equivalent to the original quantities.

Example 1.

$$\lim_{x \to 0} \frac{\sqrt[3]{x^3+2x^4}}{\ln(1+2x)} = \lim_{x \to 0} \frac{\sqrt[3]{x^3}}{2x} = \frac{1}{2}.$$

2°. Infinitely large quantities (infinites). If for an arbitrarily large number N there exists a $\delta(N)$ such that when $0 < |x-a| < \delta(N)$ we have the inequality

$$|f(x)| > N,$$

then the function $f(x)$ is called an *infinite* as $x \longrightarrow a$.

The definition of an infinite $f(x)$ as $x \to \infty$ is analogous. As in the case of infinitesimals, we introduce the concept of infinites of different orders.

288. Prove that the function

$$f(x)=\frac{\sin x}{x}$$

is an infinitesimal as $x \to \infty$. For what values of x is the inequality

$$|f(x)|<\varepsilon$$

fulfilled if ε is an arbitrary number?

Calculate for: a) $\varepsilon=0.1$; b) $\varepsilon=0.01$; c) $\varepsilon=0.001$.

289. Prove that the function

$$f(x)=1-x^2$$

is an infinitesimal for $x \to 1$. For what values of x is the inequality

$$|f(x)|<\varepsilon$$

fulfilled if ε is an arbitrary positive number? Calculate numerically for: a) $\varepsilon=0.1$; b) $\varepsilon=0.01$; c) $\varepsilon=0.001$.

290. Prove that the function

$$f(x)=\frac{1}{x-2}$$

is an infinite for $x \to 2$. In what neighbourhoods of $|x-2|<\delta$ is the inequality

$$|f(x)|>N$$

fulfilled if N is an arbitrary positive number?

Find δ if a) $N=10$; b) $N=100$; c) $N=1000$.

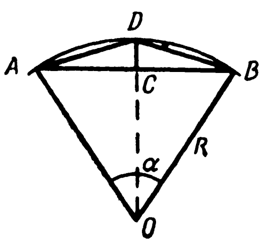

Fig. 9

291. Determine the order of smallness of: a) the surface of a sphere, b) the volume of a sphere if the radius of the sphere r is an infinitesimal of order one. What will the orders be of the radius of the sphere and the volume of the sphere with respect to its surface?

292. Let the central angle α of a circular sector ABO (Fig. 9) with radius R tend to zero. Determine the orders of the infinitesimals relative to the infinitesimal α: a) of the chord AB; b) of the line CD; c) of the area of ΔABD.

293. For $x \to 0$ determine the orders of smallness relative to x of the functions:

a) $\frac{2x}{1+x}$;

b) $\sqrt{x+\sqrt{x}}$;

c) $\sqrt[3]{x^2}-\sqrt{x^3}$;

d) $1-\cos x$;

e) $\tan x-\sin x$.

294. Prove that the length of an infinitesimal arc of a circle of constant radius is equivalent to the length of its chord.

295. Can we say that an infinitesimally small segment and an infinitesimally small semicircle constructed on this segment as a diameter are equivalent?

Using the theorem of the ratio of two infinitesimals, find

296. $\lim\limits_{x \to 0} \frac{\sin 3x \cdot \sin 5x}{(x-x^3)^2}$.

297. $\lim\limits_{x \to 0} \frac{\arcsin \frac{x}{\sqrt{1-x^2}}}{\ln(1-x)}$.

298. $\lim\limits_{x \to 1} \frac{\ln x}{1-x}$.

299. $\lim\limits_{x \to 0} \frac{\cos x-\cos 2x}{1-\cos x}$.

300. Prove that when $x \to 0$ the quantities $\frac{x}{2}$ and $\sqrt{1+x}-1$ are equivalent. Using this result, demonstrate that when $|x|$ is small we have the approximate equality

$$\sqrt{1+x} \approx 1+\frac{x}{2}. \tag{1}$$

Applying formula (1), approximate the following:

a) $\sqrt{1.06}$; b) $\sqrt{0.97}$; c) $\sqrt{10}$; d) $\sqrt{120}$

and compare the values obtained with tabular data.

301. Prove that when $x \to 0$ we have the following approximate equalities accurate to terms of order x^2:

a) $\frac{1}{1+x} \approx 1-x$;

b) $\sqrt{a^2+x} \approx a+\frac{x}{2a}$ $(a>0)$;

c) $(1+x)^n \approx 1+nx$ (n is a positive integer);

d) $\log(1+x)=Mx$,

where $M=\log e=0.43429\ldots$

Using these formulas, approximate:

1) $\frac{1}{1.02}$; 2) $\frac{1}{0.97}$; 3) $\frac{1}{105}$; 4) $\sqrt{15}$; 5) 1.04^3; 6) 0.93^4; 7) $\log 1.1$.

Compare the values obtained with tabular data.

302. Show that for $x \to \infty$ the rational integral function

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n \quad (a_0 \neq 0)$$

is an infinitely large quantity equivalent to the term of highest degree a_0x^n.

303. Let $x \to \infty$. Taking x to be an infinite of the first order, determine the order of growth of the functions:

a) $x^2 - 100x - 1{,}000$;

b) $\dfrac{x^5}{x+2}$;

c) $\sqrt{x+\sqrt{x}}$;

d) $\sqrt[3]{x-2x^2}$.

Sec. 5. Continuity of Functions

1°. Definition of continuity. A function $f(x)$ is *continuous* when $x=\xi$ (or "at the point ξ"), if: 1) this function is defined at the point ξ, that is, there exists a number $f(\xi)$; 2) there exists a finite limit $\lim\limits_{x\to\xi} f(x)$; 3) this limit is equal to the value of the function at the point ξ, i.e.,

$$\lim_{x\to\xi} f(x) = f(\xi). \tag{1}$$

Putting

$$x = \xi + \Delta\xi,$$

where $\Delta\xi \to 0$, condition (1) may be rewritten as

$$\lim_{\Delta\xi\to 0} \Delta f(\xi) = \lim_{\Delta\xi\to 0} [f(\xi+\Delta\xi) - f(\xi)] = 0, \tag{2}$$

or the function $f(x)$ is continuous at the point ξ if (and only if) at this point to an infinitesimal increment in the argument there corresponds an infinitesimal increment in the function.

If a function is continuous at every point of some region (interval, etc.), then it is said to be *continuous in this region*.

Example 1. Prove that the function

$$y = \sin x$$

is continuous for every value of the argument x.

Solution. We have

$$\Delta y = \sin(x+\Delta x) - \sin x = 2\sin\frac{\Delta x}{2}\cos\left(x+\frac{\Delta x}{2}\right) = \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cdot\cos\left(x+\frac{\Delta x}{2}\right)\cdot\Delta x.$$

Since

$$\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1 \quad\text{and}\quad \left|\cos\left(x+\frac{\Delta x}{2}\right)\right| \leqslant 1,$$

it follows that for any x we have

$$\lim_{\Delta x\to 0}\Delta y = 0.$$

Hence, the function $\sin x$ is continuous when $-\infty < x < +\infty$.

2°. **Points of discontinuity of a function.** We say that a function $f(x)$ has a *discontinuity* at $x=x_0$ (or at the point x_0) within the domain of definition of the function or on the boundary of this domain if there is a break in the continuity of the function at this point.

Example 2. The function $f(x)=\frac{1}{(1-x)^2}$ (Fig. 10 *a*) is discontinuous when $x=1$. This function is not defined at the point $x=1$, and no matter

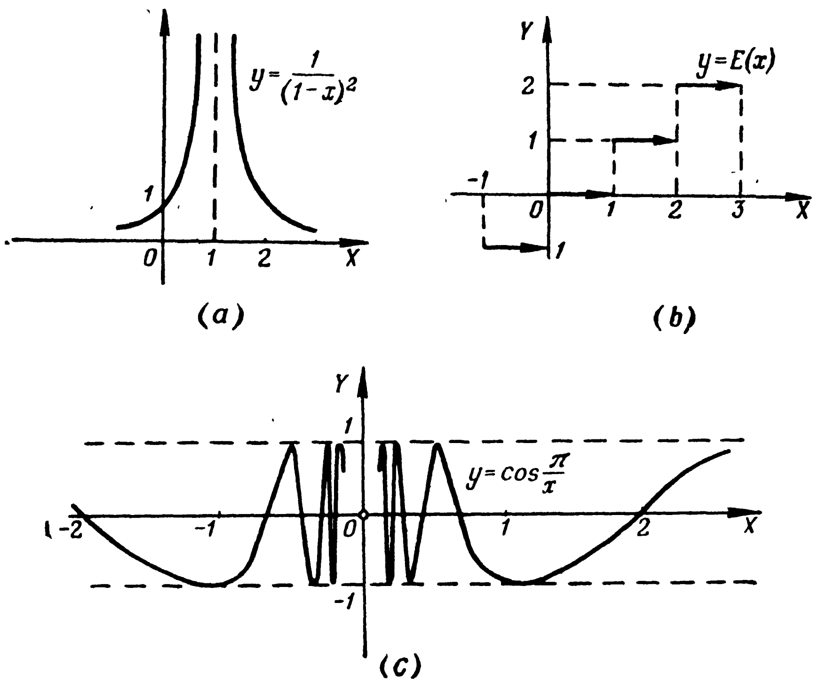

Fig. 10

how we choose the number $f(1)$, the redefined function $f(x)$ will not be continuous for $x=1$.

If the function $f(x)$ has *finite* limits:

$$\lim_{x\to x_0-0} f(x)=f(x_0-0) \quad \text{and} \quad \lim_{x\to x_0+0} f(x)=f(x_0+0),$$

and not all three numbers $f(x_0)$, $f(x_0-0)$, $f(x_0+0)$ are equal, then x_0 is called a *discontinuity of the first kind*. In particular, if

$$f(x_0-0)=f(x_0+0),$$

then x_0 is called a *removable discontinuity*.

For continuity of a function $f(x)$ at a point x_0, it is necessary and sufficient that

$$f(x_0)=f(x_0-0)=f(x_0+0).$$

Example 3. The function $f(x)=\frac{\sin x}{|x|}$ has a discontinuity of the first kind at $x=0$. Indeed, here,

$$f(+0)=\lim_{x\to+0}\frac{\sin x}{x}=+1$$

and

$$f(-0)=\lim_{x\to-0}\frac{\sin x}{-x}=-1.$$

Example 4. The function $y=E(x)$, where $E(x)$ denotes the integral part of the number x [i.e., $E(x)$ is an integer that satisfies the equality $x=E(x)+q$, where $0\leqslant q<1$], is discontinuous (Fig. 10*b*) at every integral point: $x=0$, ± 1, ± 2, ..., and all the discontinuities are of the first kind.

Indeed, if n is an integer, then $E(n-0)=n-1$ and $E(n+0)=n$. At all other points this function is, obviously, continuous.

Discontinuities of a function that are not of the first kind are called *discontinuities of the second kind*.

Infinite discontinuities also belong to discontinuities of the second kind. These are points x_0 such that at least one of the one-sided limits, $f(x_0-0)$ or $f(x_0+0)$, is equal to ∞ (see Example 2).

Example 5. The function $y=\cos\frac{\pi}{x}$ (Fig. 10*c*) at the point $x=0$ has a discontinuity of the second kind, since both one-sided limits are nonexistent here:

$$\lim_{x\to-0}\cos\frac{\pi}{x} \quad \text{and} \quad \lim_{x\to+0}\cos\frac{\pi}{x}.$$

3°. Properties of continuous functions. When testing functions for continuity, bear in mind the following theorems:

1) the sum and product of a limited number of functions continuous in some region is a function that is continuous in this region;

2) the quotient of two functions continuous in some region is a continuous function for all values of the argument of this region that do not make the divisor zero;

3) if a function $f(x)$ is continuous in an interval (a, b), and a set of its values is contained in the interval (A, B), and a function $\varphi(x)$ is continuous in (A, B), then the composite function $\varphi[f(x)]$ is continuous in (a, b).

A function $f(x)$ continuous in an interval $[a, b]$ has the following properties:

1) $f(x)$ is bounded on $[a, b]$, i.e., there is some number M such that $|f(x)|\leqslant M$ when $a\leqslant x\leqslant b$;

2) $f(x)$ has a minimum and a maximum value on $[a, b]$;

3) $f(x)$ takes on all intermediate values between the two given values; that is, if $f(\alpha)=A$ and $f(\beta)=B$ $(a\leqslant\alpha<\beta\leqslant b)$, then no matter what the number C between A and B, there will be at least one value $x=\gamma$ $(\alpha<\gamma<\beta)$ such that $f(\gamma)=C$.

In particular, if $f(\alpha)f(\beta)<0$, then the equation

$$f(x)=0$$

has at least one real root in the interval (α, β).

304. Show that the function $y=x^2$ is continuous for any value of the argument x.

305. Prove that the rational integral function

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$$

is continuous for any value of x.

306. Prove that the rational fractional function

$$R(x) = \frac{a_0x^n + a_1x^{n-1} + \ldots + a_n}{b_0x^m + b_1x^{m-1} + \ldots + b_m}$$

is continuous for all values of x except those that make the denominator zero.

307*. Prove that the function $y = \sqrt{x}$ is continuous for $x \geqslant 0$.

308. Prove that if the function $f(x)$ is continuous and non-negative in the interval (a, b), then the function

$$F(x) = \sqrt{f(x)}$$

is likewise continuous in this interval.

309*. Prove that the function $y = \cos x$ is continuous for any x.

310. For what values of x are the functions a) $\tan x$ and b) $\cot x$ continuous?

311*. Show that the function $y = |x|$ is continuous. Plot the graph of this function.

312. Prove that the absolute value of a continuous function is a continuous function.

313. A function is defined by the formulas

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{for } x \neq 2, \\ A & \text{for } x = 2. \end{cases}$$

How should one choose the value of the function $A = f(2)$ so that the thus redefined function $f(x)$ is continuous for $x = 2$? Plot the graph of the function $y = f(x)$.

314. The right side of the equation

$$f(x) = 1 - x \sin \frac{1}{x}$$

is meaningless for $x = 0$. How should one choose the value $f(0)$ so that $f(x)$ is continuous for $x = 0$?

315. The function

$$f(x) = \arctan \frac{1}{x-2}$$

is meaningless for $x = 2$. Is it possible to define the value of $f(2)$ in such a way that the redefined function should be continuous for $x = 2$?

316. The function $f(x)$ is not defined for $x=0$. Define $f(0)$ so that $f(x)$ is continuous for $x=0$, if:

a) $f(x)=\frac{(1+x)^n-1}{x}$ (n is a positive integer);

b) $f(x)=\frac{1-\cos x}{x^2}$;

c) $f(x)=\frac{\ln(1+x)-\ln(1-x)}{x}$;

d) $f(x)=\frac{e^x-e^{-x}}{x}$;

e) $f(x)=x^2\sin\frac{1}{x}$;

f) $f(x)=x\cot x$.

Investigate the following functions for continuity:

317. $y=\frac{x^2}{x-2}$.

318. $y=\frac{1+x^3}{1+x}$.

319. $y=\frac{\sqrt{7+x}-3}{x^2-4}$.

320. $y=\frac{x}{|x|}$.

321. a) $y=\sin\frac{\pi}{x}$;

b) $y=x\sin\frac{\pi}{x}$.

322. $y=\frac{x}{\sin x}$.

323. $y=\ln(\cos x)$.

324. $y=\ln\left|\tan\frac{x}{2}\right|$.

325. $y=\arctan\frac{1}{x}$.

326. $y=(1+x)\arctan\frac{1}{1-x^2}$.

327. $y=e^{\frac{1}{x+1}}$.

328. $y=e^{-\frac{1}{x^2}}$.

329. $y=\frac{1}{1+e^{\frac{1}{1-x}}}$.

330. $y=\begin{cases} x^2, & \text{for } x\leqslant 3, \\ 2x+1 & \text{for } x>3. \end{cases}$ Plot the graph of this function.

331. Prove that the Dirichlet function $\chi(x)$, which is zero for irrational x and unity for rational x, is discontinuous for every value of x.

Investigate the following functions for continuity and construct their graphs:

332. $y=\lim\limits_{n\to\infty}\frac{1}{1+x^n}$ $(x\geqslant 0)$.

333. $y=\lim\limits_{n\to\infty}(x\arctan nx)$.

334. a) $y=\operatorname{sgn} x$, b) $y=x \operatorname{sgn} x$, c) $y=\operatorname{sgn}(\sin x)$, where the function sgn x is defined by the formulas:

$$\operatorname{sgn} x=\begin{cases} +1, & \text{if } x>0, \\ 0, & \text{if } x=0, \\ -1, & \text{if } x<0. \end{cases}$$

335. a) $y=x-E(x)$, b) $y=xE(x)$, where $E(x)$ is the integral part of the number x.

336. Give an example to show that the sum of two discontinuous functions may be a continuous function.

337*. Let α be a regular positive fraction tending to zero $(0<\alpha<1)$. Can we put the limit of α into the equality

$$E(1+\alpha)=E(1-\alpha)+1,$$

which is true for all values of α?

338. Show that the equation

$$x^3-3x+1=0$$

has a real root in the interval (1,2). Approximate this root.

339. Prove that any polynomial $P(x)$ of odd power has at least one real root.

340. Prove that the equation

$$\tan x=x$$

has an infinite number of real roots.

Chapter II

DIFFERENTIATION OF FUNCTIONS

Sec. 1. Calculating Derivatives Directly

1°. Increment of the argument and increment of the function. If x and x_1 are values of the argument x, and $y=f(x)$ and $y_1=f(x_1)$ are corresponding values of the function $y=f(x)$, then

$$\Delta x = x_1 - x$$

is called the *increment of the argument* x in the interval (x, x_1), and

$$\Delta y = y_1 - y$$

or

$$\Delta y = f(x_1) - f(x) = f(x+\Delta x) - f(x) \qquad (1)$$

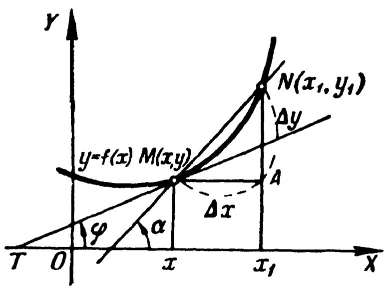

Fig. 11

is called the *increment of the function* y in the same interval (x, x_1) (Fig. 11, where $\Delta x = MA$ and $\Delta y = AN$). The ratio

$$\frac{\Delta y}{\Delta x} = \tan\alpha$$

is the slope of the secant MN of the graph of the function $y=f(x)$ (Fig. 11) and is called the *mean rate of change* of the function y over the interval $(x, x+\Delta x)$.

Example 1. For the function

$$y = x^2 - 5x + 6$$

calculate Δx and Δy, corresponding to a change in the argument:

a) from $x=1$ to $x=1.1$;
b) from $x=3$ to $x=2$.

Solution. We have

a) $\Delta x=1.1-1=0.1$,
$\Delta y=(1.1^2-5\cdot 1.1+6)-(1^2-5\cdot 1+6)=-0.29$;
b) $\Delta x=2-3=-1$,
$\Delta y=(2^2-5\cdot 2+6)-(3^2-5\cdot 3+6)=0$.

Example 2. In the case of the hyperbola $y=\frac{1}{x}$, find the slope of the secant passing through the points $M\left(3, \frac{1}{3}\right)$ and $N\left(10, \frac{1}{10}\right)$.

Solution. Here, $\Delta x=10-3=7$ and $\Delta y=\frac{1}{10}-\frac{1}{3}=-\frac{7}{30}$. Hence, $k=\frac{\Delta y}{\Delta x}=-\frac{1}{30}$.

2°. **The derivative.** *The derivative* $y'=\frac{dy}{dx}$ of a function $y=f(x)$ with respect to the argument x is the limit of the ratio $\frac{\Delta y}{\Delta x}$ when Δx approaches zero; that is,

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}.$$

The magnitude of the derivative yields the *slope* of the tangent MT to the graph of the function $y=f(x)$ at the point x (Fig. 11):

$$y'=\tan\varphi.$$

Finding the derivative y' is usually called *differentiation of the function*. The derivative $y'=f'(x)$ is the *rate of change of the function* at the point x.

Example 3. Find the derivative of the function

$$y=x^2.$$

Solution. From formula (1) we have

$$\Delta y=(x+\Delta x)^2-x^2=2x\Delta x+(\Delta x)^2$$

and

$$\frac{\Delta y}{\Delta x}=2x+\Delta x.$$

Hence,

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x)=2x.$$

3°. **One-sided derivatives.** The expressions

$$f'_-(x)=\lim_{\Delta x\to -0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

and

$$f'_+(x)=\lim_{\Delta x\to +0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

are called, respectively, the *left-hand* or *right-hand derivative* of the function $f(x)$ at the point x. For $f'(x)$ to exist, it is necessary and sufficient that

$$f'_-(x)=f'_+(x).$$

Example 4 Find $f'_-(0)$ and $f'_+(0)$ of the function

$$f(x)=|x|.$$

Solution. By the definition we have

$$f'_-(0)=\lim_{\Delta x\to -0}\frac{|\Delta x|}{\Delta x}=-1,$$

$$f'_+(0)=\lim_{\Delta x\to +0}\frac{|\Delta x|}{\Delta x}=1.$$

4°. Infinite derivative. If at some point we have

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\infty,$$

then we say that the continuous function $f(x)$ has an infinite derivative at x. In this case, the tangent to the graph of the function $y=f(x)$ is perpendicular to the x-axis.

Example 5. Find $f'(0)$ of the function

$$y=\sqrt[3]{x}.$$

Solution. We have

$$f'(0)=\lim_{\Delta x\to 0}\frac{\sqrt[3]{\Delta x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\sqrt[3]{\Delta x^2}}=\infty.$$

341. Find the increment of the function $y=x^2$ that corresponds to a change in argument:

a) from $x=1$ to $x_1=2$;
b) from $x=1$ to $x_1=1.1$;
c) from $x=1$ to $x_1=1+h$.

342. Find Δy of the function $y=\sqrt[3]{x}$ if:

a) $x=0$, $\Delta x=0.001$;
b) $x=8$, $\Delta x=-9$;
c) $x=a$, $\Delta x=h$.

343. Why can we, for the function $y=2x+3$, determine the increment Δy if all we know is the corresponding increment $\Delta x=5$, while for the function $y=x^2$ this cannot be done?

344. Find the increment Δy and the ratio $\frac{\Delta y}{\Delta x}$ for the functions:

a) $y=\frac{1}{(x^2-2)^2}$ for $x=1$ and $\Delta x=0.4$;
b) $y=\sqrt{x}$ for $x=0$ and $\Delta x=0.0001$;
c) $y=\log x$ for $x=100{,}000$ and $\Delta x=-90{,}000$.

345. Find Δy and $\frac{\Delta y}{\Delta x}$ which correspond to a change in argument from x to $x+\Delta x$ for the functions:

a) $y=ax+b$; d) $y=\sqrt{x}$;
b) $y=x^3$; e) $y=2^x$;
c) $y=\frac{1}{x^2}$; f) $y=\ln x$.

346. Find the slope of the secant to the parabola

$$y=2x-x^2,$$

if the abscissas of the points of intersection are equal:

a) $x_1=1,\ x_2=2$;
b) $x_1=1,\ x_2=0.9$;
c) $x_1=1,\ x_2=1+h$.

To what limit does the slope of the secant tend in the latter case if $h \to 0$?

347. What is the mean rate of change of the function $y=x^3$ in the interval $1 \leqslant x \leqslant 4$?

348. The law of motion of a point is $s=2t^2+3t+5$, where the distance s is given in centimetres and the time t is in seconds. What is the average velocity of the point over the interval of time from $t=1$ to $t=5$?

349. Find the mean rise of the curve $y=2^x$ in the interval $1 \leqslant x \leqslant 5$.

350. Find the mean rise of the curve $y=f(x)$ in the interval $[x,\ x+\Delta x]$.

351. What is to be understood by the rise of the curve $y=f(x)$ at a given point x?

352. Define: a) the mean rate of rotation; b) the instantaneous rate of rotation.

353. A hot body placed in a medium of lower temperature cools off. What is to be understood by: a) the mean rate of cooling; b) the rate of cooling at a given instant?

354. What is to be understood by the rate of reaction of a substance in a chemical reaction?

355. Let $m=f(x)$ be the mass of a non-homogeneous rod over the interval $[0,\ x]$. What is to be understood by: a) the mean linear density of the rod on the interval $[x,\ x+\Delta x]$; b) the linear density of the rod at a point x?

356. Find the ratio $\frac{\Delta y}{\Delta x}$ of the function $y=\frac{1}{x}$ at the point $x=2$, if: a) $\Delta x=1$; b) $\Delta x=0.1$; c) $\Delta x=0.01$. What is the derivative y' when $x=2$?

357**. Find the derivative of the function $y = \tan x$.

358. Find $y' = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ of the functions:

a) $y = x^3$; c) $y = \sqrt{x}$;
b) $y = \frac{1}{x^2}$; d) $y = \cot x$.

359. Calculate $f'(8)$, if $f(x) = \sqrt[3]{x}$.

360. Find $f'(0)$, $f'(1)$, $f'(2)$, if $f(x) = x(x-1)^2(x-2)^3$.

361. At what points does the derivative of the function $f(x) = x^2$ coincide numerically with the value of the function itself, that is, $f(x) = f'(x)$?

362. The law of motion of a point is $s = 5t^2$, where the distance s is in metres and the time t is in seconds. Find the speed at $t = 3$.

363. Find the slope of the tangent to the curve $y = 0.1x^3$ drawn at a point with abscissa $x = 2$.

364. Find the slope of the tangent to the curve $y = \sin x$ at the point $(\pi, 0)$.

365. Find the value of the derivative of the function $f(x) = \frac{1}{x}$ at the point $x = x_0$ $(x_0 \neq 0)$.

366*. What are the slopes of the tangents to the curves $y = \frac{1}{x}$ and $y = x^2$ at the point of their intersection? Find the angle between these tangents.

367**. Show that the following functions do not have finite derivatives at the indicated points:

a) $y = \sqrt[3]{x^2}$ at $x = 0$;
b) $y = \sqrt[5]{x-1}$ at $x = 1$;
c) $y = |\cos x|$ at $x = \frac{2k+1}{2}\pi$, $k = 0, \pm 1, \pm 2, \ldots$

Sec. 2. Tabular Differentiation

1°. **Basic rules for finding a derivative.** If c is a constant and $u = \varphi(x)$, $v = \psi(x)$ are functions that have derivatives, then

1) $(c)' = 0$;
2) $(x)' = 1$;
3) $(u \pm v)' = u' \pm v'$;
4) $(cu)' = cu'$;
5) $(uv)' = u'v + v'u$;
6) $\left(\frac{u}{v}\right)' = \frac{vu' - v'u}{v^2}$ $(v \neq 0)$;
7) $\left(\frac{c}{v}\right)' = \frac{-cv'}{v^2}$ $(v \neq 0)$.

2°. Table of derivatives of basic functions

I. $(x^n)' = nx^{n-1}$.

II. $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$ $(x > 0)$.

III. $(\sin x)' = \cos x$.

IV. $(\cos x)' = -\sin x$.

V. $(\tan x)' = \frac{1}{\cos^2 x}$.

VI. $(\cot x)' = \frac{-1}{\sin^2 x}$.

VII. $(\text{arc}\sin x)' = \frac{1}{\sqrt{1-x^2}}$ $(|x| < 1)$.

VIII. $(\text{arc}\cos x)' = \frac{-1}{\sqrt{1-x^2}}$ $(|x| < 1)$.

IX. $(\text{arc}\tan x)' = \frac{1}{1+x^2}$.

X. $(\text{arc}\cot x)' = \frac{-1}{x^2+1}$.

XI. $(a^x)' = a^x \ln a$.

XII. $(e^x)' = e^x$.

XIII. $(\ln x)' = \frac{1}{x}$ $(x > 0)$.

XIV. $(\log_a x)' = \frac{1}{x \ln a} = \frac{\log_a e}{x}$ $(x > 0,\ a > 0)$.

XV. $(\sinh x)' = \cosh x$.

XVI. $(\cosh x)' = \sinh x$.

XVII. $(\tanh x)' = \frac{1}{\cosh^2 x}$.

XVIII. $(\coth x)' = \frac{-1}{\sinh^2 x}$.

XIX. $(\text{arc}\sinh x)' = \frac{1}{\sqrt{1+x^2}}$.

XX. $(\text{arc}\cosh x)' = \frac{1}{\sqrt{x^2-1}}$ $(|x| > 1)$.

XXI. $(\text{arc}\tanh x)' = \frac{1}{1-x^2}$ $(|x| < 1)$.

XXII. $(\text{arc}\coth x)' = \frac{-1}{x^2-1}$ $(|x| > 1)$.

3°. Rule for differentiating a composite function. If $y = f(u)$ and $u = \varphi(x)$, that is, $y = f[\varphi(x)]$, where the functions y and u have derivatives, then

$$y'_x = y'_u u'_x \tag{1}$$

or in other notations

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

This rule extends to a series of any finite number of differentiable functions.

Example 1. Find the derivative of the function

$$y=(x^2-2x+3)^5.$$

Solution. Putting $y=u^5$, where $u=(x^2-2x+3)$, by formula (1) we will have

$$y'=(u^5)'_u\,(x^2-2x+3)'_x=5u^4\,(2x-2)=10\,(x-1)\,(x^2-2x+3)^4.$$

Example 2. Find the derivative of the function

$$y=\sin^3 4x.$$

Solution. Putting

$$y=u^3;\quad u=\sin v;\quad v=4x,$$

we find

$$y'=3u^2\cdot\cos v\cdot 4=12\sin^2 4x\cos 4x.$$

Find the derivatives of the following functions (the rule for differentiating a composite function is not used in problems 368-408).

A. Algebraic Functions

368. $y=x^5-4x^3+2x-3.$

369. $y=\frac{1}{4}-\frac{1}{3}x+x^2-0.5x^4.$

370. $y=ax^2+bx+c.$

371. $y=\frac{-5x^3}{a}.$

372. $y=at^m+bt^{m+n}.$

373. $y=\frac{ax^6+b}{\sqrt{a^2+b^2}}.$

374. $y=\frac{\pi}{x}+\ln 2.$

375. $y=3x^{\frac{2}{3}}-2x^{\frac{5}{2}}+x^{-3}.$

376*. $y=x^2\sqrt[3]{x^2}.$

377. $y=\frac{a}{\sqrt[3]{x^2}}-\frac{b}{x\sqrt[3]{x}}.$

378. $y=\frac{a+bx}{c+dx}.$

379. $y=\frac{2x+3}{x^2-5x+5}.$

380. $y=\frac{2}{2x-1}-\frac{1}{x}.$

381. $y=\frac{1+\sqrt{z}}{1-\sqrt{z}}.$

B. Inverse Circular and Trigonometric Functions

382. $y=5\sin x+3\cos x.$

383. $y=\tan x-\cot x.$

384. $y=\frac{\sin x+\cos x}{\sin x-\cos x}.$

385. $y=2t\sin t-(t^2-2)\cos t.$

386. $y=\operatorname{arc}\tan x+\operatorname{arc}\cot x.$

387. $y=x\cot x.$

388. $y=x\operatorname{arc}\sin x.$

389. $y=\frac{(1+x^2)\operatorname{arc}\tan x-x}{2}.$

C. Exponential and Logarithmic Functions

390. $y = x^7 \cdot e^x$.

391. $y = (x-1)e^x$.

392. $y = \frac{e^x}{x^2}$.

393. $y = \frac{x^5}{e^x}$.

394. $f(x) = e^x \cos x$.

395. $y = (x^2 - 2x + 2)e^x$.

396. $y = e^x \arcsin x$.

397. $y = \frac{x^2}{\ln x}$.

398. $y = x^3 \ln x - \frac{x^3}{3}$.

399. $y = \frac{1}{x} + 2\ln x - \frac{\ln x}{x}$.

400. $y = \ln x \log x - \ln a \log_a x$.

D. Hyperbolic and Inverse Hyperbolic Functions

401. $y = x \sinh x$.

402. $y = \frac{x^2}{\cosh x}$.

403. $y = \tanh x - x$.

404 $y = \frac{3 \coth x}{\ln x}$

405. $y = \arctan x - \operatorname{arc\,tanh} x$.

406. $y = \arcsin x \operatorname{arc\,sinh} x$.

407. $y = \frac{\operatorname{arc\,cosh} x}{x}$.

408. $y = \frac{\operatorname{arc\,coth} x}{1-x^2}$.

E. Composite Functions

In problems 409 to 466, use the rule for differentiating a composite function with one intermediate argument.

Find the derivatives of the following functions:

409.** $y = (1 + 3x - 5x^2)^{30}$.

Solution. Denote $1 + 3x - 5x^2 = u$; then $y = u^{30}$. We have:

$$y'_u = 30u^{29}; \quad u'_x = 3 - 10x;$$

$$y'_x = 30u^{29} \cdot (3 - 10x) = 30(1 + 3x - 5x^2)^{29} \cdot (3 - 10x).$$

410. $y = \left(\frac{ax+b}{c}\right)^3$.

411. $f(y) = (2a + 3by)^2$.

412. $y = (3 + 2x^2)^4$.

413. $y = \frac{3}{56(2x-1)^7} - \frac{1}{24(2x-1)^6} - \frac{1}{40(2x-1)^5}$.

414. $y = \sqrt{1 - x^2}$.

415. $y = \sqrt[3]{a + bx^3}$.

416. $y = (a^{2/3} - x^{2/3})^{3/2}$.

417. $y=(3-2\sin x)^5$.

Solution. $y'=5(3-2\sin x)^4\cdot(3-2\sin x)'=5(3-2\sin x)^4(-2\cos x)=-10\cos x(3-2\sin x)^4$.

418. $y=\tan x-\frac{1}{3}\tan^3 x+\frac{1}{5}\tan^5 x$.

419. $y=\sqrt{\cot x}-\sqrt{\cot \alpha}$.

420. $y=2x+5\cos^3 x$.

421*. $x=\operatorname{cosec}^2 t+\sec^2 t$.

422. $f(x)=-\frac{1}{6(1-3\cos x)^2}$.

423. $y=\frac{1}{3\cos^3 x}-\frac{1}{\cos x}$.

424. $y=\sqrt{\frac{3\sin x-2\cos x}{5}}$.

425. $y=\sqrt[3]{\sin^2 x}+\frac{1}{\cos^3 x}$.

426. $y=\sqrt{1+\arcsin x}$.

427. $y=\sqrt{\arctan x}-(\arcsin x)^3$.

428. $y=\frac{1}{\arctan x}$.

429. $y=\sqrt{xe^x+x}$.

430. $y=\sqrt[3]{2e^x-2^x+1}+\ln^5 x$.

431. $y=\sin 3x+\cos\frac{x}{5}+\tan\sqrt{x}$.

Solution. $y'=\cos 3x\cdot(3x)'-\sin\frac{x}{5}\left(\frac{x}{5}\right)'+\frac{1}{\cos^2\sqrt{x}}(\sqrt{x})'=3\cos 3x-\frac{1}{5}\sin\frac{x}{5}+\frac{1}{2\sqrt{x}\cos^2\sqrt{x}}$.

432. $y=\sin(x^2-5x+1)+\tan\frac{a}{x}$.

433. $f(x)=\cos(\alpha x+\beta)$.

434. $f(t)=\sin t\sin(t+\varphi)$.

435. $y=\frac{1+\cos 2x}{1-\cos 2x}$.

436. $f(x)=a\cot\frac{x}{a}$.

437. $y=-\frac{1}{20}\cos(5x^2)-\frac{1}{4}\cos x^2$.

438. $y=\arcsin 2x$.

Solution. $y'=\frac{1}{\sqrt{1-(2x)^2}}\cdot(2x)'=\frac{2}{\sqrt{1-4x^2}}$.

439. $y=\arcsin\frac{1}{x^2}$.

440. $f(x)=\arccos\sqrt{x}$.

441. $y=\arctan\frac{1}{x}$.

442. $y=\operatorname{arccot}\frac{1+x}{1-x}$.

443. $y=5e^{-x^2}$.

444. $y=\frac{1}{5^{x^2}}$.

445. $y=x^2\,10^{2x}$.

446. $f(t)=t\sin 2^t$.

447. $y=\arccos e^x$.

448. $y=\ln(2x+7)$.

449. $y=\log\sin x$.

450. $y=\ln(1-x^2)$.

451. $y=\ln^2 x-\ln(\ln x)$.

452. $y=\ln(e^x+5\sin x-4\arcsin x)$.

453. $y=\arctan(\ln x)+\ln(\arctan x)$.

454. $y=\sqrt{\ln x+1}+\ln(\sqrt{x}+1)$.

F. Miscellaneous Functions

455**. $y=\sin^2 5x\cos^2\frac{x}{3}$.

456. $y=-\frac{11}{2(x-2)^2}-\frac{4}{x-2}$.

457. $y=-\frac{15}{4(x-3)^4}-\frac{10}{3(x-3)^3}-\frac{1}{2(x-3)^2}$.

458. $y=\frac{x^8}{8(1-x^2)^4}$.

459. $y=\frac{\sqrt{2x^2-2x+1}}{x}$.

460. $y=\frac{x}{a^2\sqrt{a^2+x^2}}$.

461. $y=\frac{x^3}{3\sqrt{(1+x^2)^3}}$.

462. $y=\frac{3}{2}\sqrt[3]{x^2}+\frac{18}{7}x\sqrt[6]{x}+\frac{9}{5}x\sqrt[3]{x^2}+\frac{6}{13}x^2\sqrt[6]{x}$.

463. $y=\frac{1}{8}\sqrt[3]{(1+x^3)^8}-\frac{1}{5}\sqrt[3]{(1+x^3)^5}$.

464. $y=\frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}}$.

465. $y=x^4(a-2x^3)^2$.

466. $y=\left(\frac{a+bx^n}{a-bx^n}\right)^m$.

467. $y=\frac{9}{5(x+2)^5}-\frac{3}{(x+2)^4}+\frac{2}{(x+2)^3}-\frac{1}{2(x+2)^2}$.

468. $y=(a+x)\sqrt{a-x}$.

469. $y=\sqrt{(x+a)(x+b)(x+c)}$.

470. $z=\sqrt[3]{y+\sqrt{y}}$.

471. $f(t)=(2t+1)(3t+2)\sqrt[3]{3t+2}$.

472. $x=\frac{1}{\sqrt{2ay-y^2}}$.

473. $y=\ln(\sqrt{1+e^x}-1)-\ln(\sqrt{1+e^x}+1)$.

474. $y=\frac{1}{15}\cos^3 x(3\cos^2 x-5)$.

475. $y=\frac{(\tan^2 x-1)(\tan^4 x+10\tan^2 x+1)}{3\tan^3 x}$.

476. $y=\tan^5 5x$.

477. $y=\frac{1}{2}\sin(x^2)$.

478. $y=\sin^2(t^3)$.

479. $y=3\sin x\cos^2 x+\sin^3 x$.

480. $y=\frac{1}{3}\tan^3 x-\tan x+x$.

481. $y=-\frac{\cos x}{3\sin^3 x}+\frac{4}{3}\cot x$.

482. $y=\sqrt{\alpha\sin^2 x+\beta\cos^2 x}$.

483. $y=\arcsin x^2+\arccos x^2$.

484. $y=\frac{1}{2}(\arcsin x)^2\arccos x$.

485. $y=\arcsin\frac{x^2-1}{x^2}$.

486. $y=\arcsin\frac{x}{\sqrt{1+x^2}}$.

487. $y=\frac{\arccos x}{\sqrt{1-x^2}}$.

488. $y=\frac{1}{\sqrt{b}}\arcsin\left(x\sqrt{\frac{b}{a}}\right)$.

489. $y=\sqrt{a^2-x^2}+a\arcsin\frac{x}{a}$.

490. $y=x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}$.

491. $y=\arcsin(1-x)+\sqrt{2x-x^2}$.

492. $y=\left(x-\frac{1}{2}\right)\arcsin\sqrt{x}+\frac{1}{2}\sqrt{x-x^2}$.

493. $y=\ln(\arcsin 5x)$.

494. $y=\arcsin(\ln x)$.

495. $y=\arctan\frac{x\sin\alpha}{1-x\cos\alpha}$.

496. $y=\frac{2}{3}\arctan\frac{5\tan\frac{x}{2}+4}{3}$.

497. $y=3b^2\arctan\sqrt{\frac{x}{b-x}}-(3b+2x)\sqrt{bx-x^2}$.

498. $y=-\sqrt{2}\,\operatorname{arccot}\frac{\tan x}{\sqrt{2}}-x$.

499. $y=\sqrt{e^{ax}}$.

500. $y=e^{\sin^2 x}$.

501. $F(x)=(2ma^{mx}+b)^p$.

502. $F(t)=e^{\alpha t}\cos\beta t$.

503. $y=\frac{(\alpha\sin\beta x-\beta\cos\beta x)e^{\alpha x}}{\alpha^2+\beta^2}$.

504. $y=\frac{1}{10}e^{-x}(3\sin 3x-\cos 3x)$.

505. $y=x^n a^{-x^2}$.

506. $y=\sqrt{\cos x}\,a^{\sqrt{\cos x}}$.

507. $y=3^{\cot\frac{1}{x}}$.

508. $y=\ln(ax^2+bx+c)$.

509. $y=\ln(x+\sqrt{a^2+x^2})$.

510. $y=x-2\sqrt{x}+2\ln(1+\sqrt{x})$.

511. $y=\ln(a+x+\sqrt{2ax+x^2})$.

512. $y=\frac{1}{\ln^2 x}$.

513. $y=\ln\cos\frac{x-1}{x}$.

514*. $y=\ln\frac{(x-2)^5}{(x+1)^3}$.

515. $y=\ln\frac{(x-1)^3(x-2)}{x-3}$.

516. $y=-\frac{1}{2\sin^2 x}+\ln\tan x$.

517. $y=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln(x+\sqrt{x^2-a^2})$.

518. $y=\ln\ln(3-2x^3)$.

519. $y=5\ln^3(ax+b)$.

520. $y=\ln\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}$.

521. $y=\frac{m}{2}\ln(x^2-a^2)+\frac{n}{2a}\ln\frac{x-a}{x+a}$.

522. $y=x\cdot\sin\left(\ln x-\frac{\pi}{4}\right)$.

523. $y=\frac{1}{2}\ln\tan\frac{x}{2}-\frac{1}{2}\frac{\cos x}{\sin^2 x}$.

524. $f(x)=\sqrt{x^2+1}-\ln\frac{1+\sqrt{x^2+1}}{x}$.

525. $y=\frac{1}{3}\ln\frac{x^2-2x+1}{x^2+x+1}$.

526. $y=2^{\arcsin 3x}+(1-\arccos 3x)^2$.

527. $y=3^{\frac{\sin ax}{\cos bx}}+\frac{1}{3}\frac{\sin^3 ax}{\cos^3 bx}$.

528. $y=\frac{1}{\sqrt{3}}\ln\frac{\tan\frac{x}{2}+2-\sqrt{3}}{\tan\frac{x}{2}+2+\sqrt{3}}$.

529. $y=\arctan\ln x$.

530. $y=\ln\arcsin x+\frac{1}{2}\ln^2 x+\arcsin\ln x$.

531. $y=\arctan\ln\frac{1}{x}$.

532. $y=\frac{\sqrt{2}}{3}\arctan\frac{x}{\sqrt{2}}+\frac{1}{6}\ln\frac{x-1}{x+1}$.

533. $y=\ln\frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}}+2\arctan\sqrt{\sin x}$.

534. $y=\frac{3}{4}\ln\frac{x^2+1}{x^2-1}+\frac{1}{4}\ln\frac{x-1}{x+1}+\frac{1}{2}\arctan x$.

535. $f(x)=\frac{1}{2}\ln(1+x)-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$.

536. $f(x)=\frac{x\arcsin x}{\sqrt{1-x^2}}+\ln\sqrt{1-x^2}$.

537. $y=\sinh^3 2x$.

538. $y=e^{\alpha x}\cosh\beta x$.

539. $y=\tanh^3 2x$.

540. $y=\ln\sinh 2x$.

541. $y=\operatorname{arc\,sinh}\frac{x^2}{a^2}$.

542. $y=\operatorname{arc\,cosh}\ln x$.

543. $y=\operatorname{arc\,tanh}(\tan x)$.

544. $y=\operatorname{arc\,coth}(\sec x)$.

545. $y=\operatorname{arc\,tanh}\frac{2x}{1+x^2}$.

546. $y=\frac{1}{2}(x^2-1)\operatorname{arc\,tanh} x+\frac{1}{2}x$.

547. $y=\left(\frac{1}{2}x^2+\frac{1}{4}\right)\operatorname{arc\,sinh} x-\frac{1}{4}x\sqrt{1+x^2}$.

548. Find y', if:

a) $y=|x|$;

b) $y=x|x|$.

Construct the graphs of the functions y and y'.

549. Find y' if

$$y=\ln|x| \quad (x\neq 0).$$

550. Find $f'(x)$ if

$$f(x)=\begin{cases} 1-x & \text{for } x\leqslant 0,\\ e^{-x} & \text{for } x>0.\end{cases}$$

551. Calculate $f'(0)$ if

$$f(x)=e^{-x}\cos 3x.$$

Solution. $f'(x)=e^{-x}(-3\sin 3x)-e^{-x}\cos 3x$;

$f'(0)=e^0(-3\sin 0)-e^0\cos 0=-1$.

552. $f(x)=\ln(1+x)+\arcsin\frac{x}{2}$. Find $f'(1)$.

553. $y=\tan^3\frac{\pi x}{6}$. Find $\left(\frac{dy}{dx}\right)_{x=2}$.

554. Find $f'_+(0)$ and $f'_-(0)$ of the functions:

a) $f(x)=\sqrt{\sin(x^2)}$;

b) $f(x)=\arcsin\frac{a^2-x^2}{a^2+x^2}$;

c) $f(x)=\frac{x}{1+e^{\frac{1}{x}}}$, $x\neq 0$; $f(0)=0$;

d) $f(x)=x^2\sin\frac{1}{x}$, $x\neq 0$; $f(0)=0$;

e) $f(x)=x\sin\frac{1}{x}$ $x\neq 0$; $f(0)=0$

555. Find $f(0)+xf'(0)$ of the function $f(x)=e^{-x}$.

556. Find $f(3)+(x-3)f'(3)$ of the function $f(x)=\sqrt{1+x}$.

557. Given the functions $f(x)=\tan x$ and $\varphi(x)=\ln(1-x)$, find $\frac{f'(0)}{\varphi'(0)}$.

558. Given the functions $f(x)=1-x$ and $\varphi(x)=1-\sin\frac{\pi x}{2}$, find $\frac{\varphi'(1)}{f'(1)}$.

559. Prove that the derivative of an even function is an odd function, and the derivative of an odd function is an even function.

560. Prove that the derivative of a periodic function is also a periodic function.

561. Show that the function $y=xe^{-x}$ satisfies the equation $xy'=(1-x)y$.

562. Show that the function $y=xe^{-\frac{x^2}{2}}$ satisfies the equation $xy'=(1-x^2)y$.

563. Show that the function $y=\frac{1}{1+x+\ln x}$ satisfies the equation $xy'=y(y\ln x-1)$.

G. Logarithmic Derivative

A *logarithmic derivative* of a function $y=f(x)$ is the derivative of the logarithm of this function; that is,

$$(\ln y)'=\frac{y'}{y}=\frac{f'(x)}{f(x)}$$

Finding the derivative is sometimes simplified by first taking logs of the function.

Example. Find the derivative of the exponential function

$$y=u^v,$$

where $u=\varphi(x)$ and $v=\psi(x)$.

Solution. Taking logarithms we get

$$\ln y=v\ln u.$$

Differentiate both sides of this equation with respect *to* x:

$$(\ln y)'=v'\ln u+v(\ln u)',$$

or

$$\frac{1}{y}y'=v'\ln u+v\frac{1}{u}u',$$

whence

$$y'=y\left(v'\ln u+\frac{v}{u}u'\right),$$

or

$$y' = u^v \left(v' \ln u + \frac{v}{u} u' \right)$$

564. Find y', if

$$y = \sqrt[3]{x^2} \frac{1-x}{1+x^2} \sin^3 x \cos^2 x.$$

Solution. $\ln y = \frac{2}{3} \ln x + \ln (1-x) - \ln (1+x^2) + 3 \ln \sin x + 2 \ln \cos x;$

$$\frac{1}{y} y' = \frac{2}{3} \frac{1}{x} + \frac{(-1)}{1-x} - \frac{2x}{1+x^2} + 3 \frac{1}{\sin x} \cos x - \frac{2 \sin x}{\cos x},$$

whence $y' = y \left(\frac{2}{3x} - \frac{1}{1-x} - \frac{2x}{1+x^2} + 3 \cot x - 2 \tan x \right).$

565. Find y', if $y = (\sin x)^x$.

Solution. $\ln y = x \ln \sin x$; $\frac{1}{y} y' = \ln \sin x + x \cot x$;

$$y' = (\sin x)^x (\ln \sin x + x \cot x).$$

In the following problems find y' after first taking logs of the function $y = f(x)$:

566. $y = (x+1)(2x+1)(3x+1)$.

567. $y = \frac{(x+2)^2}{(x+1)^3 (x+3)^4}$.

568. $y = \sqrt{\frac{x(x-1)}{x-2}}$.

569. $y = x \sqrt[3]{\frac{x^2}{x^2+1}}$.

570. $y = \frac{(x-2)^9}{\sqrt{(x-1)^5 (x-3)^{11}}}$.

571. $y = \frac{\sqrt{x-1}}{\sqrt[3]{(x+2)^2} \sqrt{(x+3)^3}}$.

572. $y = x^x$.

573. $y = x^{x^2}$.

574. $y = \sqrt[x]{x}$.

575. $y = x^{\sqrt{x}}$.

576. $y = x^{x^x}$.

577. $y = x^{\sin x}$.

578. $y = (\cos x)^{\sin x}$.

579. $y = \left(1 + \frac{1}{x}\right)^x$.

580. $y = (\arctan x)^x$.

Sec. 3. The Derivatives of Functions Not Represented Explicitly

1°. The derivative of an inverse function. If a function $y = f(x)$ has a derivative $y'_x \neq 0$, then the derivative of the inverse function $x = f^{-1}(y)$ is

$$x'_y = \frac{1}{y'_x}$$

or

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}.$$

Example 1. Find the derivative x'_y, if

$$y=x+\ln x.$$

Solution. We have $y'_x=1+\frac{1}{x}=\frac{x+1}{x}$; hence, $x'_y=\frac{x}{x+1}$.

2°. **The derivatives of functions represented parametrically.** If a function y is related to an argument x by means of a parameter t,

$$\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases}$$

then

$$y'_x=\frac{y'_t}{x'_t},$$

or, in other notation,

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Example 2. Find $\frac{dy}{dx}$, if

$$\left.\begin{array}{l} x=a\cos t, \\ y=a\sin t \end{array}\right\}$$

Solution. We find $\frac{dx}{dt}=-a\sin t$ and $\frac{dy}{dt}=a\cos t$. Whence

$$\frac{dy}{dx}=-\frac{a\cos t}{a\sin t}=-\cot t.$$

3°. **The derivative of an implicit function.** If the relationship between x and y is given in implicit form,

$$F(x, y)=0, \tag{1}$$

then to find the derivative $y'_x=y'$ in the simplest cases it is sufficient: 1) to calculate the derivative, with respect to x, of the left side of equation (1), taking y as a function of x; 2) to equate this derivative to zero, that is, to put

$$\frac{d}{dx}F(x, y)=0, \tag{2}$$

and 3) to solve the resulting equation for y'.

Example 3. Find the derivative y'_x if

$$x^3+y^3-3axy=0. \tag{3}$$

Solution. Forming the derivative of the left side of (3) and equating it to zero, we get

$$3x^2+3y^2y'-3a(y+xy')=0,$$

whence

$$y' = \frac{x^2 - ay}{ax - y^2}.$$

581. Find the derivative x'_y if

a) $y = 3x + x^3$;

b) $y = x - \frac{1}{2}\sin x$;

c) $y = 0.1x + e^{\frac{x}{2}}$.

In the following problems, find the derivative $y' = \frac{dy}{dx}$ of the functions y represented parametrically:

582. $\begin{cases} x = 2t - 1, \\ y = t^3. \end{cases}$

583. $\begin{cases} x = \dfrac{1}{t+1}, \\ y = \left(\dfrac{t}{t+1}\right)^2. \end{cases}$

584. $\begin{cases} x = \dfrac{2at}{1+t^2}, \\ y = \dfrac{a(1-t^2)}{1+t^2}. \end{cases}$

585. $\begin{cases} x = \dfrac{3at}{1+t^3}, \\ y = \dfrac{3at^2}{1+t^3}. \end{cases}$

586. $\begin{cases} x = \sqrt{t}, \\ y = \sqrt[3]{t}. \end{cases}$

587. $\begin{cases} x = \sqrt{t^2+1}, \\ y = \dfrac{t-1}{\sqrt{t^2+1}}. \end{cases}$

588. $\begin{cases} x = a(\cos t + t\sin t), \\ y = a(\sin t - t\cos t). \end{cases}$

589. $\begin{cases} x = a\cos^2 t, \\ y = b\sin^2 t. \end{cases}$

590. $\begin{cases} x = a\cos^3 t, \\ y = b\sin^3 t. \end{cases}$

591. $\begin{cases} x = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}, \\ y = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}. \end{cases}$

592. $\begin{cases} x = \arccos \dfrac{1}{\sqrt{1+t^2}}, \\ y = \arcsin \dfrac{t}{\sqrt{1+t^2}}. \end{cases}$

593. $\begin{cases} x = e^{-t}, \\ y = e^{2t}. \end{cases}$

594. $\begin{cases} x = a\left(\ln \tan \dfrac{t}{2} + \cos t - \sin t\right), \\ y = a(\sin t + \cos t). \end{cases}$

595. Calculate $\frac{dy}{dx}$ when $t = \frac{\pi}{2}$ if

$$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t). \end{cases}$$

Solution. $\frac{dy}{dx} = \frac{a\sin t}{a(1-\cos t)} = \frac{\sin t}{1-\cos t}$

and

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{2}}=\frac{\sin\frac{\pi}{2}}{1-\cos\frac{\pi}{2}}=1.$$

596. Find $\frac{dy}{dx}$ when $t=1$ if $\begin{cases} x=t\ln t, \\ y=\frac{\ln t}{t}. \end{cases}$

597. Find $\frac{dy}{dx}$ when $t=\frac{\pi}{4}$ if $\begin{cases} x=e^t\cos t, \\ y=e^t\sin t. \end{cases}$

598. Prove that a function y represented parametrically by the equations

$$\begin{cases} x=2t+3t^2, \\ y=t^2+2t^3, \end{cases}$$

satisfies the equation

$$y=\left(\frac{dy}{dx}\right)^2+2\left(\frac{dy}{dx}\right)^3.$$

599. When $x=2$ the following equation is true:

$$x^2=2x.$$

Does it follow from this that

$$(x^2)'=(2x)'$$

when $x=2$?

600. Let $y=\sqrt{a^2-x^2}$. Is it possible to perform term-by-term differentiation of

$$x^2+y^2=a^2?$$

In the examples that follow it is required to find the derivative $y'=\frac{dy}{dx}$ of the implicit functions y.

601. $2x-5y+10=0$.

602. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

603. $x^3+y^3=a^3$.

604. $x^3+x^2y+y^2=0$.

605. $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

606. $\sqrt[3]{x^2}+\sqrt[3]{y^2}=\sqrt[3]{a^2}$.

607. $y^3=\frac{x-y}{x+y}$.

608. $y-0.3\sin y=x$.

609. $a\cos^2(x+y)=b$.

610. $\tan y=xy$.

611. $xy=\arctan\frac{x}{y}$.

612. $\arctan(x+y)=x$.

613. $e^y=x+y$.

614. $\ln x+e^{-\frac{y}{x}}=c$.

615. $\ln y+\frac{x}{y}=c$.

616. $\arctan\frac{y}{x}=\frac{1}{2}\ln(x^2+y^2)$.

617. $\sqrt{x^2+y^2}=c\arctan\frac{y}{x}$. **618.** $x^y=y^x$.

619. Find y' at the point $M(1,1)$, if

$$2y=1+xy^3.$$

Solution. Differentiating, we get $2y'=y^3+3xy^2y'$. Putting $x=1$ and $y=1$, we obtain $2y'=1+3y'$, whence $y'=-1$.

620. Find the derivatives y' of specified functions y at the indicated points:

a) $(x+y)^3=27(x-y)$ for $x=2$ and $y=1$;
b) $ye^y=e^{x+1}$ for $x=0$ and $y=1$;
c) $y^2=x+\ln\frac{y}{x}$ for $x=1$ and $y=1$.

Sec. 4. Geometrical and Mechanical Applications of the Derivative

1°. Equations of the tangent and the normal. From the geometric significance of a derivative it follows that the *equation of the tangent* to a curve $y=f(x)$ or $F(x, y)=0$ at a point $M(x_0, y_0)$ will be

$$y-y_0=y_0'(x-x_0),$$

where y_0' is the value of the derivative y' at the point $M(x_0, y_0)$. The straight line passing through the point of tangency perpendicularly to the tangent is called the *normal to the curve*. For the normal we have the equation

$$x-x_0+y_0'(y-y_0)=0.$$

2°. The angle between curves. The angle between the curves

$$y=f_1(x)$$

and

$$y=f_2(x)$$

at their common point $M_0(x_0, y_0)$ (Fig. 12) is the angle ω between the tangents M_0A and M_0B to these curves at the point M_0.

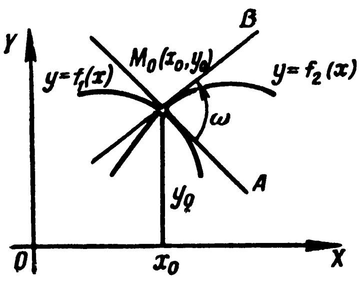

Fig. 12

Using a familiar formula of analytic geometry, we get

$$\tan\omega=\frac{f_2'(x_0)-f_1'(x_0)}{1+f_1'(x_0)\cdot f_2'(x_0)}.$$

3°. Segments associated with the tangent and the normal in a rectangular coordinate system. The tangent and the normal determine the following four segments (Fig. 13):

$t = TM$ is the so-called *segment of the tangent,*
$S_t = TK$ is the *subtangent,*
$n = NM$ is the *segment of the normal,*
$S_n = KN$ is the *subnormal.*

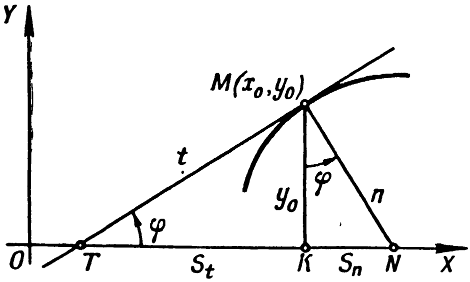

Fig. 13

Since $KM = |y_0|$ and $\tan \varphi = y_0'$, it follows that

$$t = TM = \left| \frac{y_0}{y_0'} \sqrt{1 + (y_0')^2} \right|; \quad n = NM = \left| y_0 \sqrt{1 + (y_0')^2} \right|;$$

$$S_t = TK = \left| \frac{y_0}{y_0'} \right|; \quad S_n = \left| y_0 y_0' \right|.$$

4°. **Segments associated with the tangent and the normal in a polar system of coordinates.** If a curve is given in polar coordinates by the equation $r = f(\varphi)$, then the angle μ formed by the tangent MT and the radius vector $r = OM$ (Fig. 14), is defined by the following formula:

$$\tan \mu = r \frac{d\varphi}{dr} = \frac{r}{r'}.$$

The tangent MT and the normal MN at the point M together with the radius vector of the point of tangency and with the perpendicular to the radius vector drawn through the pole O determine the following four segments (see Fig. 14):

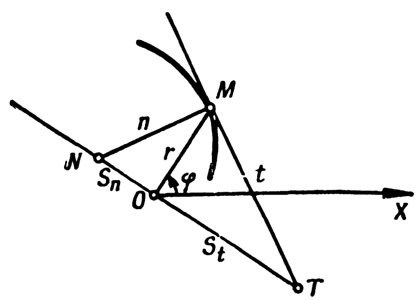

Fig. 14

$t = MT$ is the *segment of the polar tangent,*
$n = MN$ is the *segment of the polar normal,*
$S_t = OT$ is the *polar subtangent,*
$S_n = ON$ is the *polar subnormal.*

These segments are expressed by the following formulas:

$$t = MT = \frac{r}{|r'|}\sqrt{r^2+(r')^2}; \qquad S_t = OT = \frac{r^2}{|r'|};$$

$$n = MN = \sqrt{r^2+(r')^2}; \qquad S_n = ON = |r'|.$$

621. What angles φ are formed with the x-axis by the tangents to the curve $y = x - x^2$ at points with abscissas:

a) $x=0$; b) $x=1/2$; c) $x=1$?

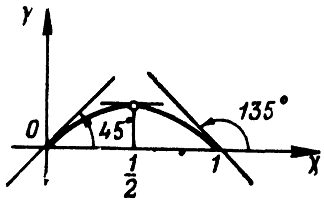

Fig. 15

Solution. We have $y' = 1-2x$. Whence a) $\tan\varphi = 1$, $\varphi = 45°$; b) $\tan\varphi = 0$, $\varphi = 0°$; c) $\tan\varphi = -1$, $\varphi = 135°$ (Fig. 15).

622. At what angles do the sine curves $y = \sin x$ and $y = \sin 2x$ intersect the axis of abscissas at the origin?

623. At what angle does the tangent curve $y = \tan x$ intersect the axis of abscissas at the origin?

624. At what angle does the curve $y = e^{0.5x}$ intersect the straight line $x = 2$?

625. Find the points at which the tangents to the curve $y = 3x^4 + 4x^3 - 12x^2 + 20$ are parallel to the x-axis.

626. At what point is the tangent to the parabola

$$y = x^2 - 7x + 3$$

parallel to the straight line $5x + y - 3 = 0$?

627. Find the equation of the parabola $y = x^2 + bx + c$ that is tangent to the straight line $x = y$ at the point (1,1).

628. Determine the slope of the tangent to the curve $x^3 + y^3 - xy - 7 = 0$ at the point (1,2).

629. At what point of the curve $y^2 = 2x^3$ is the tangent perpendicular to the straight line $4x - 3y + 2 = 0$?

630. Write the equation of the tangent and the normal to the parabola

$$y = \sqrt{x}$$

at the point with abscissa $x = 4$.

Solution. We have $y' = \frac{1}{2\sqrt{x}}$; whence the slope of the tangent is $k = [y']_{x=4} = \frac{1}{4}$. Since the point of tangency has coordinates $x = 4$, $y = 2$, it follows that the equation of the tangent is $y - 2 = 1/4\,(x-4)$ or $x - 4y + 4 = 0$.

Since the slope of the normal must be perpendicular,

$$k_1 = -4;$$

whence the equation of the normal: $y - 2 = -4\,(x-4)$ or $4x + y - 18 = 0$.

631. Write the equations of the tangent and the normal to the curve $y=x^3+2x^2-4x-3$ at the point $(-2,5)$.

632. Find the equations of the tangent and the normal to the curve

$$y=\sqrt[3]{x-1}$$

at the point (1,0).

633. Form the equations of the tangent and the normal to the curves at the indicated points:

a) $y=\tan 2x$ at the origin;

b) $y=\arcsin\frac{x-1}{2}$ at the point of intersection with the x-axis;

c) $y=\arccos 3x$ at the point of intersection with the y-axis;

d) $y=\ln x$ at the point of intersection with the x-axis;

e) $y=e^{1-x^2}$ at the points of intersection with the straight line $y=1$.

634. Write the equations of the tangent and the normal at the point (2,2) to the curve

$$x=\frac{1+t}{t^3},$$

$$y=\frac{3}{2t^2}+\frac{1}{2t}.$$

635. Write the equations of the tangent to the curve

$$x=t\cos t,\quad y=t\sin t$$

at the origin and at the point $t=\frac{\pi}{4}$.

636. Write the equations of the tangent and the normal to the curve $x^3+y^2+2x-6=0$ at the point with ordinate $y=3$.

637. Write the equation of the tangent to the curve $x^5+y^5-2xy=0$ at the point (1,1).

638. Write the equations of the tangents and the normals to the curve $y=(x-1)(x-2)(x-3)$ at the points of its intersection with the x-axis.

639. Write the equations of the tangent and the normal to the curve $y^4=4x^4+6xy$ at the point (1,2).

640*. Show that the segment of the tangent to the hyperbola $xy=a^2$ (the segment lies between the coordinate axes) is divided in two at the point of tangency.

641. Show that in the case of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ the segment of the tangent between the coordinate axes has a constant value equal to a.

642. Show that the normals to the involute of the circle

$$x = a(\cos t + t\sin t), \quad y = a(\sin t - t\cos t)$$

are tangents to the circle $x^2 + y^2 = a^2$.

643. Find the angle at which the parabolas $y = (x-2)^2$ and $y = -4 + 6x - x^2$ intersect.

644. At what angle do the parabolas $y = x^2$ and $y = x^3$ intersect?

645. Show that the curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 10$ are tangent to each other at the point (3,34). Will we have the same thing at (—2,4)?

646. Show that the hyperbolas

$$xy = a^2; \quad x^2 - y = b^2$$

intersect at a right angle.

647. Given a parabola $y^2 = 4x$. At the point (1,2) evaluate the lengths of the segments of the subtangent, subnormal, tangent, and normal.

648. Find the length of the segment of the subtangent of the curve $y = 2^x$ at any point of it.

649. Show that in the equilateral hyperbola $x^2 - y^2 = a^2$ the length of the normal at any point is equal to the radius vector of this point.

650. Show that the length of the segment of the subnormal in the hyperbola $x^2 - y^2 = a^2$ at any point is equal to the abscissa of this point.

651. Show that the segments of the subtangents of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = a^2$ at points with the same abscissas are equal. What procedure of construction of the tangent to the ellipse follows from this?

652. Find the length of the segment of the tangent, the normal, the subtangent, and the subnormal of the cycloid

$$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t) \end{cases}$$

at an arbitrary point $t = t_0$.

653. Find the angle between the tangent and the radius vector of the point of tangency in the case of the logarithmic spiral

$$r = ae^{k\varphi}.$$

654. Find the angle between the tangent and the radius vector of the point of tangency in the case of the lemniscate $r^2 = a^2\cos 2\varphi$.

655. Find the lengths of the segments of the polar subtangent, subnormal, tangent and normal, and also the angle between the tangent and the radius vector of the point of tangency in the case of the spiral of Archimedes

$$r = a\varphi$$

at a point with polar angle $\varphi = 2\pi$.

656. Find the lengths of the segments of the polar subtangent, subnormal, tangent, and normal, and also the angle between the tangent and the radius vector in the hyperbolic spiral $r = \frac{a}{\varphi}$ at an arbitrary point $\varphi = \varphi_0$; $r = r_0$.

657. The law of motion of a point on the x-axis is

$$x = 3t - t^3.$$

Find the velocity of the point at $t_0 = 0$, $t_1 = 1$, and $t_2 = 2$ (x is in centimetres and t is in seconds).

658. Moving along the x-axis are two points that have the following laws of motion: $x = 100 + 5t$ and $x = 1/2t^2$, where $t \geqslant 0$. With what speed are these points receding from each other at the time of encounter (x is in centimetres and t is in seconds)?

659. The end-points of a segment $AB = 5$ m are sliding along the coordinate axes OX and OY (Fig. 16). A is moving at 2 m/sec.

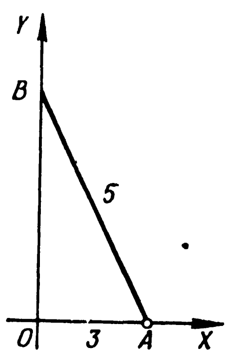

Fig. 16

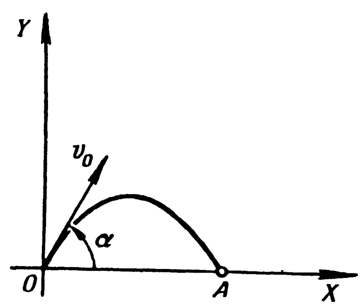

Fig. 17

What is the rate of motion of B when A is at a distance $OA = 3$ m from the origin?

660*. The law of motion of a material point thrown up at an angle α to the horizon with initial velocity v_0 (in the vertical plane OXY in Fig. 17) is given by the formulas (air resistance is disregarded):

$$x=v_0 t\cos\alpha,\qquad y=v_0 t\sin\alpha-\frac{gt^2}{2},$$

where t is the time and g is the acceleration of gravity. Find the trajectory of motion and the distance covered. Also determine the speed of motion and its direction.

661. A point is in motion along a hyperbola $y=\frac{10}{x}$ so that its abscissa x increases uniformly at a rate of 1 unit per second. What is the rate of change of its ordinate when the point passes through (5,2)?

662. At what point of the parabola $y^2=18x$ does the ordinate increase at twice the rate of the abscissa?

663. One side of a rectangle, $a=10$ cm, is of constant length, while the other side, b, increases at a constant rate of 4 cm/sec. At what rate are the diagonal of the rectangle and its area increasing when $b=30$ cm?

664. The radius of a sphere is increasing at a uniform rate of 5 cm/sec. At what rate are the area of the surface of the sphere and the volume of the sphere increasing when the radius becomes 50 cm?

665. A point is in motion along the spiral of Archimedes

$$r=a\varphi$$

($a=10$ cm) so that the angular velocity of rotation of its radius vector is constant and equal to 6° per second. Determine the rate of elongation of the radius vector r when $r=25$ cm.

666. A nonhomogeneous rod AB is 12 cm long. The mass of a part of it, AM, increases with the square of the distance of the moving point M from the end A and is 10 gm when $AM=2$ cm. Find the mass of the entire rod AB and the linear density at any point M. What is the linear density of the rod at A and B?

Sec. 5. Derivatives of Higher Orders

1°. Definition of higher derivatives. A *derivative of the second order*, or the *second derivative*, of the function $y=f(x)$ is the derivative of its derivative; that is,

$$y''=(y')'.$$

The second derivative may be denoted as

$$y'',\quad\text{or}\quad\frac{d^2y}{dx^2},\quad\text{or}\quad f''(x).$$

If $x=f(t)$ is the law of rectilinear motion of a point, then $\frac{d^2x}{dt^2}$ is the acceleration of this motion.

Generally, the nth *derivative* of a function $y=f(x)$ is the derivative of a derivative of order $(n-1)$. For the nth derivative we use the notation

$$y^{(n)}, \quad \text{or} \quad \frac{d^n y}{dx^n}, \quad \text{or} \quad f^{(n)}(x).$$

Example 1. Find the second derivative of the function

$$y=\ln(1-x).$$

Solution. $y'=\frac{-1}{1-x}$; $y''=\left(\frac{-1}{1-x}\right)'=\frac{1}{(1-x)^2}$.

2°. **Leibniz rule.** If the functions $u=\varphi(x)$ and $v=\psi(x)$ have derivatives up to the nth order inclusive, then to evaluate the nth derivative of a product of these functions we can use the *Leibniz rule* (or formula):

$$(uv)^{(n)}=u^{(n)}v+n\cdot u^{(n-1)}v'+\frac{n(n-1)}{1\cdot 2}u^{(-2)}v''+\ldots+uv^{(n)}.$$

3°. **Higher-order derivatives of functions represented parametrically.** If

$$\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases}$$

then the derivatives $y'_x=\frac{dy}{dx}$, $y''_{xx}=\frac{d^2y}{dx^2}$, ... can successively be calculated by the formulas

$$y'_x=\frac{y'_t}{x'_t}, \quad y''_{xx}=\left(y'_x\right)'_x=\frac{\left(y'_x\right)'_t}{x'_t}, \quad y'''_{xxx}=\frac{\left(y''_{xx}\right)'_t}{x'_t} \text{ and so forth.}$$

For a second derivative we have the formula

$$y''_{xx}=\frac{x'_t y''_{tt}-x''_{tt}y'_t}{(x'_t)^3}.$$

Example 2. Find y'', if

$$\begin{cases} x=a\cos t, \\ y=b\sin t. \end{cases}$$

Solution. We have

$$y'=\frac{(b\sin t)'_t}{(a\cos t)'_t}=\frac{b\cdot\cos t}{-a\sin t}=-\frac{b}{a}\cot t.$$

and

$$y''=\frac{\left(-\frac{b}{a}\cot t\right)'_t}{(a\cos t)'_t}=\frac{-\frac{b}{a}\cdot\frac{-1}{\sin^2 t}}{-a\sin t}=-\frac{b}{a^2\sin^3 t}.$$

A. Higher-Order Derivatives of Explicit Functions

In the examples that follow, find the second derivative of th given function.

667. $y = x^8 + 7x^6 - 5x + 4$.
668. $y = e^{x^2}$.
669. $y = \sin^2 x$.
670. $y = \ln \sqrt[3]{1 + x^2}$.
671. $y = \ln(x + \sqrt{a^2 + x^2})$.
672. $f(x) = (1 + x^2) \cdot \arctan x$.
673. $y = (\arcsin x)^2$.
674. $y = a \cosh \frac{x}{a}$.

675. Show that the function $y = \frac{x^2 + 2x + 2}{2}$ satisfies the differential equation $1 + y'^2 = 2yy''$.

676. Show that the function $y = \frac{1}{2} x^2 e^x$ satisfies the differential equation $y'' - 2y' + y = e^x$.

677. Show that the function $y = C_1 e^{-x} + C_2 e^{-2x}$ satisfies th equation $y'' + 3y' + 2y = 0$ for all constants C_1 and C_2.

678. Show that the function $y = e^{2x} \sin 5x$ satisfies the equation $y'' - 4y' + 29y = 0$.

679. Find y''', if $y = x^3 - 5x^2 + 7x - 2$.

680. Find $f'''(3)$, if $f(x) = (2x - 3)^5$.

681. Find y^{V} of the function $y = \ln(1 + x)$.

682. Find y^{VI} of the function $y = \sin 2x$.

683. Show that the function $y = e^{-x} \cos x$ satisfies the differential equation $y^{\mathrm{IV}} + 4y = 0$.

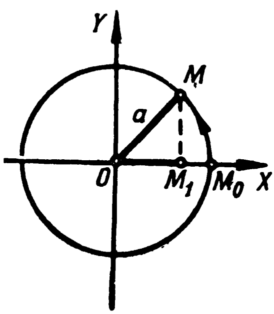

Fig. 18

684. Find $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$ if $f(x) = e^x \sin x$.

685. The equation of motion of a poin along the x-axis is

$$x = 100 + 5t - 0.001t^3.$$

Find the velocity and the acceleration c the point for times $t_0 = 0$, $t_1 = 1$, an $t_2 = 10$.

686. A point M is in motion around circle $x^2 + y^2 = a^2$ with constant angula velocity ω. Find the law of motion of it projection M_1 on the x-axis if at time $t =$ the point is at $M_0(a, 0)$ (Fig. 18). Find the velocity and the ac celeration of motion of M_1.

What is the velocity and the acceleration of M_1 at the in tial time and when it passes through the origin?

What are the maximum values of the absolute velocity and th absolute acceleration of M_1?

687. Find the nth derivative of the function $y=(ax+b)^n$, where n is a natural number.

688. Find the nth derivatives of the functions:

$$\text{a) } y=\frac{1}{1-x}; \quad \text{and} \quad \text{b) } y=\sqrt{x}.$$

689. Find the nth derivative of the functions:

a) $y=\sin x$; e) $y=\dfrac{1}{1+x}$;

b) $y=\cos 2x$; f) $y=\dfrac{1+x}{1-x}$;

c) $y=e^{-3x}$; g) $y=\sin^2 x$;

d) $y=\ln(1+x)$; h) $y=\ln(ax+b)$.

690. Using the Leibniz rule, find $y^{(n)}$, if:

a) $y=x\cdot e^x$; d) $y=\dfrac{1+x}{\sqrt{x}}$;

b) $y=x^2\cdot e^{-2x}$; e) $y=x^3\ln x$.

c) $y=(1-x^2)\cos x$;

691. Find $f^{(n)}(0)$, if $f(x)=\ln\dfrac{1}{1-x}$

B. Higher-Order Derivatives of Functions Represented Parametrically and of Implicit Functions

In the following problems find $\dfrac{d^2y}{dx^2}$.

692. a) $\begin{cases} x=\ln t, \\ y=t^3; \end{cases}$ b) $\begin{cases} x=\arctan t, \\ y=\ln(1+t^2); \end{cases}$ c) $\begin{cases} x=\arcsin t \\ y=\sqrt{1-t^2}. \end{cases}$

693. a) $\begin{cases} x=a\cos t, \\ y=a\sin t; \end{cases}$ c) $\begin{cases} x=a(t-\sin t), \\ y=a(1-\cos t); \end{cases}$

b) $\begin{cases} x=a\cos^3 t, \\ y=a\sin^3 t; \end{cases}$ d) $\begin{cases} x=a(\sin t-t\cos t), \\ y=a(\cos t+t\sin t). \end{cases}$

694. a) $\begin{cases} x=\cos 2t, \\ y=\sin^2 t; \end{cases}$ **695.** a) $\begin{cases} x=\arctan t, \\ y=\frac{1}{2}t^2; \end{cases}$

b) $\begin{cases} x=e^{-at}, \\ y=e^{at}. \end{cases}$ b) $\begin{cases} x=\ln t, \\ y=\frac{1}{1-t}. \end{cases}$

696. Find $\dfrac{d^2x}{dy^2}$, if $\begin{cases} x=e^t\cos t, \\ y=e^t\sin t. \end{cases}$

697. Find $\frac{d^2y}{dx^2}$ for $t=0$, if $\begin{cases} x=\ln(1+t^2), \\ y=t^2. \end{cases}$

698. Show that y (as a function of x) defined by the equations $x=\sin t$, $y=ae^{t\sqrt{2}}+be^{-t\sqrt{2}}$ for any constants a and b satisfies the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2y.$$

In the following examples find $y'''=\frac{d^3y}{dx^3}$.

699. $\begin{cases} x=\sec t, \\ y=\tan t. \end{cases}$

700. $\begin{cases} x=e^{-t}\cos t, \\ y=e^{-t}\sin t. \end{cases}$

701. $\begin{cases} x=e^{-t}, \\ y=t^3. \end{cases}$

702. Find $\frac{d^ny}{dx^n}$, if $\begin{cases} x=\ln t, \\ y=t^m. \end{cases}$

703. Knowing the function $y=f(x)$, find the derivatives x'', x''' of the inverse function $x=f^{-1}(y)$.

704. Find y'', if $x^2+y^2=1$.

Solution. By the rule for differentiating a composite function we have $2x+2yy'=0$; whence $y'=-\frac{x}{y}$ and $y''=-\left(\frac{x}{y}\right)'_x=-\frac{y-xy'}{y^2}$. Substituting the value of y', we finally get:

$$y''=-\frac{y^2+x^2}{y^3}=-\frac{1}{y^3}.$$

In the following examples it is required to determine the derivative y'' of the function $y=f(x)$ represented implicitly.

705. $y^2=2px$.

706. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

707. $y=x+\operatorname{arc}\tan y$.

708. Having the equation $y=x+\ln y$, find $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$.

709. Find y'' at the point (1,1) if

$$x^2+5xy+y^2-2x+y-6=0.$$

710. Find y'' at (0,1) if

$$x^4-xy+y^4=1.$$

711. a) The function y is defined implicitly by the equation

$$x^2+2xy+y^2-4x+2y-2=0.$$

Find $\frac{d^3y}{dx^3}$ at the point (1,1).

b) Find $\frac{d^3y}{dx^3}$, if $x^2+y^2=a^2$.

Sec. 6. Differentials of First and Higher Orders

1°. **First-order differential.** The *differential (first-order) of a function* $y=f(x)$ is the principal part of its increment, which part is linear relative to the increment $\Delta x = dx$ of the independent variable x. The differential of a function is equal to the product of its derivative by the differential of the independent variable

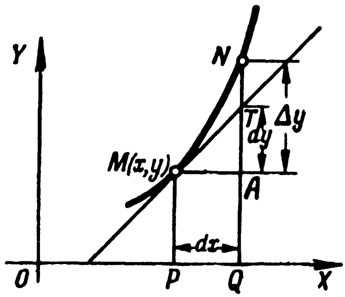

Fig. 19

$$dy = y'dx,$$

whence

$$y' = \frac{dy}{dx}.$$

If MN is an arc of the graph of the function $y=f(x)$ (Fig. 19), MT is the tangent at $M(x, y)$ and

$$PQ = \Delta x = dx,$$

then the increment in the ordinate of the tangent

$$AT = dy$$

and the segment $AN = \Delta y$.

Example 1. Find the increment and the differential of the function $y = 3x^2 - x$.

Solution. First method:

$$\Delta y = 3(x+\Delta x)^2 - (x+\Delta x) - 3x^2 + x$$

or

$$\Delta y = (6x-1)\,\Delta x + 3(\Delta x)^2.$$

Hence,

$$dy = (6x-1)\,\Delta x = (6x-1)\,dx.$$

Second method:

$$y' = 6x-1;\ dy = y'\,dx = (6x-1)\,dx.$$

Example 2. Calculate Δy and dy of the function $y = 3x^2 - x$ for $x = 1$ and $\Delta x = 0.01$.

Solution. $\Delta y = (6x-1)\cdot\Delta x + 3(\Delta x)^2 = 5\cdot 0.01 + 3\cdot(0.01)^2 = 0.0503$

and

$$dy = (6x-1)\,\Delta x = 5\cdot 0.01 = 0.0500.$$

2°. Principal properties of differentials.

1) $dc = 0$, where $c = \text{const}$.
2) $dx = \Delta x$, where x is an independent variable.
3) $d(cu) = c\,du$.
4) $d(u \pm v) = du \pm dv$.
5) $d(uv) = u\,dv + v\,du$.
6) $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \qquad (v \neq 0)$.
7) $df(u) = f'(u)\,du$.

3°. Applying the differential to approximate calculations. If the increment Δx of the argument x is small in absolute value, then the differential dy of the function $y = f(x)$ and the increment Δy of the function are approximately equal:

$$\Delta y \approx dy,$$

that is,

$$f(x + \Delta x) - f(x) \approx f'(x)\,\Delta x,$$

whence

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx.$$

Example 3. By how much (approximately) does the side of a square change if its area increases from 9 m² to 9.1 m²?

Solution. If x is the area of the square and y is its side, then

$$y = \sqrt{x}.$$

It is given that $x = 9$ and $\Delta x = 0.1$.

The increment Δy in the side of the square may be calculated approximately as follows:

$$\Delta y \approx dy = y'\,\Delta x = \frac{1}{2\sqrt{9}} \cdot 0.1 = 0.016 \text{ m}.$$

4°. Higher-order differentials. A *second-order differential* is the differential of a first-order differential:

$$d^2y = d(dy).$$

We similarly define the *differentials of the third* and higher orders.

If $y = f(x)$ and x is an independent variable, then

$$\begin{aligned} d^2y &= y''(dx)^2, \\ d^3y &= y'''(dx)^3, \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ d^ny &= y^{(n)}(dx)^n. \end{aligned}$$

But if $y = f(u)$, where $u = \varphi(x)$, then

$$\begin{aligned} d^2y &= y''(du)^2 + y'\,d^2u, \\ d^3y &= y'''(du)^3 + 3y''\,du \cdot d^2u + y'\,d^3u \end{aligned}$$

and so forth. (Here the primes denote derivatives with respect to u).

712. Find the increment Δy and the differential dy of the function $y = 5x + x^2$ for $x = 2$ and $\Delta x = 0.001$.

713. Without calculating the derivative, find

$$d(1-x^3)$$

for $x=1$ and $\Delta x=-\frac{1}{3}$.

714. The area of a square S with side x is given by $S=x^2$. Find the increment and the differential of this function and explain the geometric significance of the latter.

715. Give a geometric interpretation of the increment and differential of the following functions:

a) the area of a circle, $S=\pi x^2$;

b) the volume of a cube, $v=x^3$.

716. Show that when $\Delta x \to 0$, the increment in the function $y=2^x$, corresponding to an increment Δx in x, is, for any x, equivalent to the expression $2^x \ln 2\,\Delta x$.

717. For what value of x is the differential of the function $y=x^2$ not equivalent to the increment in this function as $\Delta x \to 0$?

718. Has the function $y=|x|$ a differential for $x=0$?

719. Using the derivative, find the differential of the function $y=\cos x$ for $x=\frac{\pi}{6}$ and $\Delta x=\frac{\pi}{36}$.

720. Find the differential of the function

$$y=\frac{2}{\sqrt{x}}$$

for $x=9$ and $\Delta x=-0.01$.

721. Calculate the differential of the function

$$y=\tan x$$

for $x=\frac{\pi}{3}$ and $\Delta x=\frac{\pi}{180}$.

In the following problems find the differentials of the given functions for arbitrary values of the argument and its increment.

722. $y=\frac{1}{x^m}$.

723. $y=\frac{x}{1-x}$.

724. $y=\arcsin\frac{x}{a}$.

725. $y=\arctan\frac{x}{a}$.

726. $y=e^{-x^2}$.

727. $y=x\ln x-x$.

728. $y=\ln\frac{1-x}{1+x}$.

729. $r=\cot\varphi+\operatorname{cosec}\varphi$.

730. $s=\arctan e^t$.

731 Find dy if $x^2+2xy-y^2=a^2$.

Solution. Taking advantage of the invariancy of the form of a differential, we obtain $2x\,dx+2(y\,dx+x\,dy)-2y\,dy=0$

Whence

$$dy=-\frac{x+y}{x-y}\,dx.$$

In the following examples find the differentials of the functions defined implicitly.

732. $(x+y)^2 \cdot (2x+y)^3 = 1$.

733. $y = e^{-\frac{x}{y}}$.

734. $\ln \sqrt{x^2+y^2} = \arctan \frac{y}{x}$.

735. Find dy at the point (1,2), if $y^3 - y = 6x^2$.

736. Find the approximate value of $\sin 31°$.

Solution. Putting $x = \text{arc}\, 30° = \frac{\pi}{6}$ and $\Delta x = \text{arc}\, 1° = \frac{\pi}{180}$, from formula (1) (see 3°) we have $\sin 31° \approx \sin 30° + \frac{\pi}{180} \cos 30° = 0.500 + 0.017 \cdot \frac{\sqrt{3}}{2} = 0.515$.

737. Replacing the increment of the function by the differential, calculate approximately:

a) $\cos 61°$; d) $\ln 0.9$;
b) $\tan 44°$; e) $\arctan 1.05$.
c) $e^{0.2}$;

738. What will be the approximate increase in the volume of a sphere if its radius $R = 15$ cm increases by 2 mm?

739. Derive the approximate formula (for $|\Delta x|$ that are small compared to x)

$$\sqrt{x+\Delta x} \approx \sqrt{x} + \frac{\Delta x}{2\sqrt{x}}.$$

Using it, approximate $\sqrt{5}$, $\sqrt{17}$, $\sqrt{70}$, $\sqrt{640}$.

740. Derive the approximate formula

$$\sqrt[3]{x+\Delta x} \approx \sqrt[3]{x} + \frac{\Delta x}{3\sqrt[3]{x^2}}$$

and find approximate values for $\sqrt[3]{10}$, $\sqrt[3]{70}$, $\sqrt[3]{200}$.

741. Approximate the functions:

a) $y = x^3 - 4x^2 + 5x + 3$ for $x = 1.03$;
b) $f(x) = \sqrt{1+x}$ for $x = 0.2$;
c) $f(x) = \sqrt[3]{\frac{1-x}{1+x}}$ for $x = 0.1$;
d) $y = e^{1-x^2}$ for $x = 1.05$.

742. Approximate $\tan 45°3'20''$.

743. Find the approximate value of $\arcsin 0.54$.

744. Approximate $\sqrt[4]{17}$.

745. Using Ohm's law, $I=\frac{E}{R}$, show that a small change in the current, due to a small change in the resistance, may be found approximately by the formula

$$\Delta I=-\frac{I}{R}\Delta R.$$

746. Show that, in determining the length of the radius, a relative error of 1% results in a relative error of approximately 2% in calculating the area of a circle and the surface of a sphere.

747. Compute d^2y, if $y=\cos 5x$.

Solution. $d^2y=y''(dx^2)=-25\cos 5x\,(dx)^2$.

748. $u=\sqrt{1-x^2}$, find d^2u.

749. $y=\arccos x$, find d^2y.

750. $y=\sin x\ln x$, find d^2y.

751. $z=\frac{\ln x}{x}$, find d^2z.

752. $z=x^2e^{-x}$, find d^3z.

753. $z=\frac{x^4}{2-x}$, find d^4z.

754. $u=3\sin(2x+5)$, find d^nu.

755. $y=e^{x\cos\alpha}\sin(x\sin\alpha)$, find d^ny.

Sec. 7. Mean-Value Theorems

1°. **Rolle's theorem.** If a function $f(x)$ is continuous on the interval $a\leqslant x\leqslant b$, has a derivative $f'(x)$ at every interior point of this interval, and

$$f(a)=f(b),$$

then the argument x has at least one value ξ, where $a<\xi<b$, such that

$$f'(\xi)=0.$$

2°. **Lagrange's theorem.** If a function $f(x)$ is continuous on the interval $a\leqslant x\leqslant b$ and has a derivative at every interior point of this interval, then

$$f(b)-f(a)=(b-a)f'(\xi),$$

where $a<\xi<b$.

3°. **Cauchy's theorem.** If the functions $f(x)$ and $F(x)$ are continuous on the interval $a\leqslant x\leqslant b$ and for $a<x<b$ have derivatives that do not vanish simultaneously, and $F(b)\neq F(a)$, then

$$\frac{f(b)-f(a)}{F(b)-F(a)}=\frac{f'(\xi)}{F'(\xi)},\quad \text{where } a<\xi<b.$$

756. Show that the function $f(x)=x-x^3$ on the intervals $-1\leqslant x\leqslant 0$ and $0\leqslant x\leqslant 1$ satisfies the Rolle theorem. Find the appropriate values of ξ.

Solution. The function $f(x)$ is continuous and differentiable for all values of x, and $f(-1)=f(0)=f(1)=0$. Hence, the Rolle theorem is applicable on the intervals $-1 \leqslant x \leqslant 0$ and $0 \leqslant x < 1$. To find ξ we form the equation $f'(x)=1-3x^2=0$. Whence $\xi_1=-\sqrt{\frac{1}{3}}$; $\xi_2=\sqrt{\frac{1}{3}}$, where $-1<\xi_1<0$ and $0<\xi_2<1$.

757. The function $f(x)=\sqrt[3]{(x-2)^2}$ takes on equal values $f(0)=f(4)=\sqrt[3]{4}$ at the end-points of the interval [0.4]. Does the Rolle theorem hold for this function on [0.4]?

758. Does the Rolle theorem hold for the function

$$f(x)=\tan x$$

on the interval $[0, \pi]$?

759. Let

$$f(x)=x(x+1)(x+2)(x+3).$$

Show that the equation

$$f'(x)=0$$

has three real roots.

760. The equation

$$e^x=1+x$$

obviously has a root $x=0$. Show that this equation cannot have any other real root.

761. Test whether the Lagrange theorem holds for the function

$$f(x)=x-x^3$$

on the interval $[-2,1]$ and find the appropriate intermediate value of ξ.

Solution. The function $f(x)=x-x^3$ is continuous and differentiable for all values of x, and $f'(x)=1-3x^2$ Whence, by the Lagrange formula, we have $f(1)-f(-2)=0-6=[1-(-2)]f'(\xi)$, that is, $f'(\xi)=-2$ Hence, $1-3\xi^2=-2$ and $\xi=\pm 1$; the only suitable value is $\xi=-1$, for which the inequality $-2<\xi<1$ holds

762. Test the validity of the Lagrange theorem and find the appropriate intermediate point ξ for the function $f(x)=x^{4/3}$ on the interval $[-1,1]$.

763. Given a segment of the parabola $y=x^2$ lying between two points $A(1,1)$ and $B(3,9)$, find a point the tangent to which is parallel to the chord AB.

764. Using the Lagrange theorem, prove the formula

$$\sin(x+h)-\sin x=h\cos\xi,$$

where $x<\xi<x+h$.

765. a) For the functions $f(x)=x^2+2$ and $F(x)=x^3-1$ test whether the Cauchy theorem holds on the interval [1,2] and find ξ;

b) do the same with respect to $f(x)=\sin x$ and $F(x)=\cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Sec. 8. Taylor's Formula

If a function $f(x)$ is continuous and has continuous derivatives up to the $(n-1)$th order inclusive on the interval $a \leqslant x \leqslant b$ (or $b \leqslant x \leqslant a$), and there is a finite derivative $f^{(n)}(x)$ at each interior point of the interval, then *Taylor's formula*

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots$$
$$\ldots+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{(x-a)^n}{n!}f^{(n)}(\xi),$$

where $\xi=a+\theta(x-a)$ and $0<\theta<1$, holds true on the interval.

In particular, when $a=0$ we have (*Maclaurin's formula*)

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+\frac{x^n}{n!}f^{(n)}(\xi),$$

where $\xi=\theta x$, $0<\theta<1$.

766. Expand the polynomial $f(x)=x^3-2x^2+3x+5$ in positive integral powers of the binomial $x-2$.

Solution. $f'(x)=3x^2-4x+3$; $f''(x)=6x-4$; $f'''(x)=6$; $f^{(n)}(x)=0$ for $n \geqslant 4$. Whence

$$f(2)=11;\ f'(2)=7;\ f''(2)=8;\ f'''(2)=6.$$

Therefore,

$$x^3-2x^2+3x+5=11+(x-2)\cdot 7+\frac{(x-2)^2}{2!}\cdot 8+\frac{(x-2)^3}{3!}\cdot 6$$

or

$$x^3-2x^2+3x+5=11+7(x-2)+4(x-2)^2+(x-2)^3.$$

767. Expand the function $f(x)=e^x$ in powers of $x+1$ to the term containing $(x+1)^3$.

Solution. $f^{(n)}(x)=e^x$ for all n, $f^{(n)}(-1)=\frac{1}{e}$. Hence,

$$e^x=\frac{1}{e}+(x+1)\frac{1}{e}+\frac{(x+1)^2}{2!}\frac{1}{e}+\frac{(x+1)}{3!}\frac{1}{e}+\frac{(x+1)^4}{4!}e^{\xi},$$

where $\xi=-1+\theta(x+1)$; $0<\theta<1$.

768. Expand the function $f(x)=\ln x$ in powers of $x-1$ up to the term with $(x-1)^2$.

769. Expand $f(x)=\sin x$ in powers of x up to the term containing x^3 and to the term containing x^5.

770. Expand $f(x)=e^x$ in powers of x up to the term containing x^{n-1}.

771. Show that $\sin(a+h)$ differs from

$$\sin a+h\cos a$$

by not more than $1/2\,h^2$.

772. Determine the origin of the approximate formulas:

a) $\sqrt{1+x}\approx 1+\frac{1}{2}x-\frac{1}{8}x^2, \quad |x|<1,$

b) $\sqrt[3]{1+x}\approx 1+\frac{1}{3}x-\frac{1}{9}x^2, \quad |x|<1$

and evaluate their errors.

773. Evaluate the error in the formula

$$e\approx 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}.$$

774. Due to its own weight, a heavy suspended thread lies in a catenary line $y=a\cosh\frac{x}{a}$. Show that for small $|x|$ the shape of the thread is approximately expressed by the parabola

$$y=a+\frac{x^2}{2a}.$$

775*. Show that for $|x|\ll a$, to within $\left(\frac{x}{a}\right)^2$, we have the approximate equality

$$e^{\frac{x}{a}}\approx\sqrt{\frac{a+x}{a-x}}.$$

Sec. 9. The L'Hospital-Bernoulli Rule for Evaluating Indeterminate Forms

1°. Evaluating the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Let the single-valued functions $f(x)$ and $\varphi(x)$ be differentiable for $0<|x-a|<h$; the derivative of one of them does not vanish.

If $f(x)$ and $\varphi(x)$ are both infinitesimals or both infinites as $x\to a$; that is, if the quotient $\frac{f(x)}{\varphi(x)}$, at $x=a$, is one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}$$

provided that the limit of the ratio of derivatives exists.

The rule is also applicable when $a=\infty$.

If the quotient $\frac{f'(x)}{\varphi'(x)}$ again yields an indeterminate form, at the point $x=a$, of one of the two above-mentioned types and $f'(x)$ and $\varphi'(x)$ satisfy all the requirements that have been stated for $f(x)$ and $\varphi(x)$, we can then pass to the ratio of second derivatives, etc.

However, it should be borne in mind that the limit of the ratio $\frac{f(x)}{\varphi(x)}$ may exist, whereas the ratios of the derivatives do not tend to any limit (see Example 809).

2°. **Other indeterminate forms.** To evaluate an indeterminate form like $0\cdot\infty$, transform the appropriate product $f_1(x)\cdot f_2(x)$, where $\lim\limits_{x\to a} f_1(x)=0$ and $\lim\limits_{x\to a} f_2(x)=\infty$, into the quetient $\frac{f_1(x)}{\frac{1}{f_2(x)}}$ (the form $\frac{0}{0}$ (or $\frac{f_2(x)}{\frac{1}{f_1(x)}}$ (the form $\frac{\infty}{\infty}$).

In the case of the indeterminate form $\infty-\infty$, one should transform the appropriate difference $f_1(x)-f_2(x)$ into the product $f_1(x)\left[1-\frac{f_2(x)}{f_1(x)}\right]$ and first evaluate the indeterminate form $\frac{f_2(x)}{f_1(x)}$; if $\lim\limits_{x\to a}\frac{f_2(x)}{f_1(x)}=1$, then we reduce the expression to the form

$$\frac{1-\frac{f_2(x)}{f_1(x)}}{\frac{1}{f_1(x)}} \text{ (the form } \frac{0}{0}\text{)}.$$

The indeterminate forms 1^∞, 0^0, ∞^0 are evaluated by first taking logarithms and then finding the limit of the logarithm of the power $[f_1(x)]^{f_2(x)}$ (which requires evaluating a form like $0\cdot\infty$).

In certain cases it is useful to combine the L'Hospital rule with the finding of limits by elementary techniques.

Example 1. Compute

$$\lim_{x\to 0}\frac{\ln x}{\cot x} \text{ (form } \frac{\infty}{\infty}\text{)}.$$

Solution. Applying the L'Hospital rule we have

$$\lim_{x\to 0}\frac{\ln x}{\cot x}=\lim_{x\to 0}\frac{(\ln x)'}{(\cot x)'}=-\lim_{x\to 0}\frac{\sin^2 x}{x}.$$

We get the indeterminate form $\frac{0}{0}$; however, we do not need to use the L'Hospital rule, since

$$\lim_{x\to 0}\frac{\sin^2 x}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\sin x=1\cdot 0=0.$$

We thus finally get

$$\lim_{x\to 0}\frac{\ln x}{\cot x}=0.$$

Example 2. Compute

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right) \text{ (form } \infty-\infty).$$

Reducing to a common denominator, we get

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x} \text{ (form } \frac{0}{0}).$$

Before applying the L'Hospital rule, we replace the denominator of the latter fraction by an equivalent infinitesimal (Ch. 1, Sec. 4) $x^2\sin^2 x \sim x^4$. We obtain

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4} \text{ (form } \frac{0}{0}).$$

The L'Hospital rule gives

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{2x-\sin 2x}{4x^3}=\lim_{x\to 0}\frac{2-2\cos 2x}{12x^2}.$$

Then, in elementary fashion, we find

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{1-\cos 2x}{6x^2}=\lim_{x\to 0}\frac{2\sin^2 x}{6x^2}=\frac{1}{3}.$$

Example 3. Compute

$$\lim_{x\to 0}(\cos 2x)^{\frac{3}{x^2}} \text{ (form } 1^\infty)$$

Taking logarithms and applying the L'Hospital rule, we get

$$\lim_{x\to 0}\ln(\cos 2x)^{\frac{3}{x^2}}=\lim_{x\to 0}\frac{3\ln\cos 2x}{x^2}=-6\lim_{x\to 0}\frac{\tan 2x}{2x}=-6.$$

Hence, $\lim\limits_{x\to 0}(\cos 2x)^{\frac{3}{x^2}}=e^{-6}$.

Find the indicated limits of functions in the following examples.

776. $\lim\limits_{x\to 1}\frac{x^3-2x^2-x+2}{x^3-7x+6}$.

Solution. $\lim\limits_{x\to 1}\frac{x^3-2x^2-x+2}{x^3-7x+6}=\lim\limits_{x\to 1}\frac{3x^2-4x-1}{3x^2-7}=\frac{1}{2}$.

777. $\lim\limits_{x\to 0}\frac{x\cos x-\sin x}{x^3}$.

778. $\lim\limits_{x\to 1}\frac{1-x}{1-\sin\frac{\pi x}{2}}$.

779. $\lim\limits_{x\to 0}\frac{\cosh x-1}{1-\cos x}$.

780. $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{x-\sin x}$.

781. $\lim\limits_{x \to \frac{\pi}{4}} \frac{\sec^2 x - 2\tan x}{1+\cos 4x}$.

782. $\lim\limits_{x \to \frac{\pi}{2}} \frac{\tan x}{\tan 5x}$.

783. $\lim\limits_{x \to \infty} \frac{e^x}{x^5}$.

784. $\lim\limits_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}}$.

785. $\lim\limits_{x \to 0} \frac{\frac{\pi}{x}}{\cot \frac{\pi x}{2}}$.

786. $\lim\limits_{x \to 0} \frac{\ln(\sin mx)}{\ln \sin x}$.

787. $\lim\limits_{x \to 0} (1-\cos x)\cot x$.

Solution. $\lim\limits_{x \to 0} (1-\cos x)\cot x = \lim\limits_{x \to 0} \frac{(1-\cos x)\cos x}{\sin x} = \lim\limits_{x \to 0} \frac{(1-\cos x)\cdot 1}{\sin x} =$

$= \lim\limits_{x \to 0} \frac{\sin x}{\cos x} = 0$

788. $\lim\limits_{x \to 1} (1-x)\tan \frac{\pi x}{2}$.

789. $\lim\limits_{x \to 0} \operatorname{arc\,sin} x \cot x$.

790. $\lim\limits_{x \to 0} (x^n e^{-x})$, $n>0$.

791. $\lim\limits_{x \to \infty} x \sin \frac{a}{x}$.

792. $\lim\limits_{x \to \infty} x^n \sin \frac{a}{x}$, $n>0$.

793. $\lim\limits_{x \to 1} \ln x \ln(x-1)$.

794. $\lim\limits_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$.

Solution. $\lim\limits_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim\limits_{x \to 1} \frac{x\ln x - x + 1}{(x-1)\ln x} =$

$$= \lim_{x \to 1} \frac{x \cdot \frac{1}{x} + \ln x - 1}{\ln x + \frac{1}{x}(x-1)} = \lim_{x \to 1} \frac{\ln x}{\ln x - \frac{1}{x} + 1} = \lim_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}.$$

795. $\lim\limits_{x \to 3} \left(\frac{1}{x-3} - \frac{5}{x^2-x-6}\right)$.

796. $\lim\limits_{x \to 1} \left[\frac{1}{2(1-\sqrt{x})} - \frac{1}{3(1-\sqrt[3]{x})}\right]$.

797. $\lim\limits_{x \to \frac{\pi}{2}} \left(\frac{x}{\cot x} - \frac{\pi}{2\cos x}\right)$.

798. $\lim\limits_{x \to 0} x^x$.

Solution. We have $x^x = y$; $\ln y = x\ln x$: $\lim\limits_{x \to 0} \ln y = \lim\limits_{x \to 0} x \ln x =$

$= \lim\limits_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \lim\limits_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$, whence $\lim\limits_{x \to 0} y = 1$, that is, $\lim\limits_{x \to 0} x^x = 1$.

799. $\lim\limits_{x\to+\infty} x^{\frac{1}{x}}$.

800. $\lim\limits_{x\to 0} x^{\frac{3}{4+\ln x}}$.

801. $\lim\limits_{x\to 0} x^{\sin x}$.

802. $\lim\limits_{x\to 1} (1-x)^{\cos\frac{\pi x}{2}}$.

803. $\lim\limits_{x\to 0} (1+x^2)^{\frac{1}{x}}$.

804. $\lim\limits_{x\to 1} x^{\frac{1}{1-x}}$.

805. $\lim\limits_{x\to 1}\left(\tan\frac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}}$.

806. $\lim\limits_{x\to 0} (\cot x)^{\frac{1}{\ln x}}$.

807. $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$.

808. $\lim\limits_{x\to 0} (\cot x)^{\sin x}$.

809. Prove that the limits of

a) $\lim\limits_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x}=0$;

b) $\lim\limits_{x\to\infty}\frac{x-\sin x}{x+\sin x}=1$

cannot be found by the L'Hospital-Bernoulli rule. Find these limits directly.

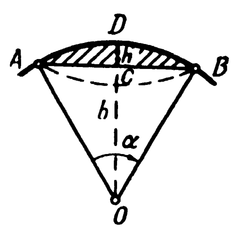

Fig. 20

810*. Show that the area of a circular segment with minor central angle α, which has a chord $AB=b$ and $CD=h$ (Fig. 20), is approximately

$$S\approx\frac{2}{3}bh$$

with an arbitrarily small relative error when $\alpha\to 0$.

Chapter III

THE EXTREMA OF A FUNCTION AND THE GEOMETRIC APPLICATIONS OF A DERIVATIVE

Sec. 1. The Extrema of a Function of One Argument

1°. **Increase and decrease of functions.** The function $y=f(x)$ is called *increasing (decreasing)* on some interval if, for any points x_1 and x_2 which belong to this interval, from the inequality $x_1<x_2$ we get the inequality $f(x_1)<f(x_2)$ (Fig. 21*a*) [$f(x_1)>f(x_2)$ (Fig. 21*b*)]. If $f(x)$ is continuous on the interval $[a, b]$ and $f'(x)>0$ [$f'(x)<0$] for $a<x<b$, then $f(x)$ increases (decreases) on the interval $[a, b]$.

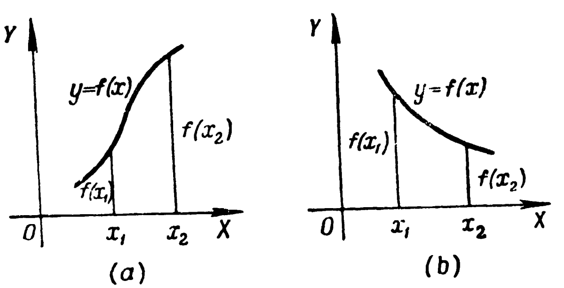

Fig. 21

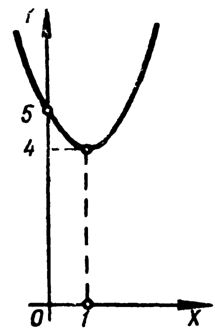

Fig. 22

In the simplest cases, the domain of definition of $f(x)$ may be subdivided into a finite number of intervals of increase and decrease of the function (*intervals of monotonicity*). These intervals are bounded by critical points x [where $f'(x)=0$ or $f'(x)$ does not exist].

Example 1. Test the following function for increase and decrease:

$$y=x^2-2x+5.$$

Solution. We find the derivative

$$y'=2x-2=2(x-1).$$

Whence $y'=0$ for $x=1$. On a number scale we get two intervals of monotonicity: $(-\infty, 1)$ and $(1, +\infty)$. From (1) we have: 1) if $-\infty<x<1$, then $y'<0$, and, hence, the function $f(x)$ decreases in the interval $(-\infty, 1)$; 2) if $1<x<+\infty$, then $y'>0$, and, hence, the function $f(x)$ increases in the interval $(1, +\infty)$ (Fig. 22).

Example 2. Determine the intervals of increase and decrease of the function

$$y=\frac{1}{x+2}.$$

Solution. Here, $x=-2$ is a discontinuity of the function and $y'=-\frac{1}{(x+2)^2}<0$ for $x\neq-2$. Hence, the function y decreases in the intervals $-\infty<x<-2$ and $-2<x<+\infty$.

Example 3. Test the following function for increase or decrease:

$$y=\frac{1}{5}x^5-\frac{1}{3}x^3.$$

Solution Here,

$$y'=x^4-x^2. \tag{2}$$

Solving the equation $x^4-x^2=0$, we find the points $x_1=-1$, $x_2=0$, $x_3=1$ at which the derivative y' vanishes. Since y' can change sign only when passing through points at which it vanishes or becomes discontinuous (in the given case, y' has no discontinuities), the derivative in each of the intervals $(-\infty, -1)$, $(-1, 0)$, $(0,1)$ and $(1, +\infty)$ retains its sign; for this reason, the function under investigation is monotonic in each of these intervals. To determine in which of the indicated intervals the function increases and in which it decreases, one has to determine the sign of the derivative in each of the intervals. To determine what the sign of y' is in the interval $(-\infty, -1)$, it is sufficient to determine the sign of y' at some point of the interval; for example, taking $x=-2$, we get from (2) $y'=12>0$, hence, $y'>0$ in the interval $(-\infty, -1)$ and the function in this interval increases Similarly, we find that $y'<0$ in the interval $(-1, 0)$ (as a check, we can take $x=-\frac{1}{2}$), $y'<0$ in the interval $(0,1)$ (here, we can use $x=1/2$) and $y'>0$ in the interval $(1, +\infty)$.

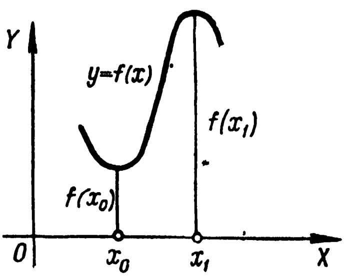

Fig 23

Thus, the function being tested increases in the interval $(-\infty, -1)$, decreases in the interval $(-1, 1)$ and again increases in the interval $(1, +\infty)$.

2°. **Extremum of a function.** If there exists a two-sided neighbourhood of a point x_0 such that for every point $x\neq x_0$ of this neighbourhood we have the inequality $f(x)>f(x_0)$, then the point x_0 is called the *minimum point* of the function $y=f(x)$, while the number $f(x_0)$ is called the *minimum* of the function $y=f(x)$. Similarly, if for any point $x\neq x_1$ of some neighbourhood of the point x_1, the inequality $f(x)<f(x_1)$ is fulfilled, then x_1 is called the *maximum point* of the function $f(x)$, and $f(x_1)$ is the *maximum* of the function (Fig. 23). The minimum point or maximum point of a function is its *extremal point* (bending point), and the minimum or maximum of a function is called the *extremum* of the function. If x_0 is an extremal point of the function $f(x)$, then $f'(x_0)=0$, or $f'(x_0)$ does not exist (necessary condition for the existence of an extremum). The converse is not true: points at which $f'(x)=0$, or $f'(x)$, does not exist (*critical points*) are not necessarily extremal points of the function $f(x)$.

The sufficient conditions for the existence and absence of an extremum of a continuous function $f(x)$ are given by the following rules:

1. If there exists a neighbourhood $(x_0-\delta, x_0+\delta)$ of a critical point x_0 such that $f'(x)>0$ for $x_0-\delta<x<x_0$ and $f'(x)<0$ for $x_0<x<x_0+\delta$, then x_0 is the maximum point of the function $f(x)$; and if $f'(x)<0$ for $x_0-\delta<x<x_0$ and $f'(x)>0$ for $x_0<x<x_0+\delta$, then x_0 is the minimum point of the function $f(x)$.

Finally, if there is some positive number δ such that $f'(x)$ retains its sign unchanged for $0<|x-x_0|<\delta$, then x_0 is not an extremal point of the function $f(x)$.

2. If $f'(x_0)=0$ and $f''(x_0)<0$, then x_0 is the maximum point; if $f'(x_0)=0$ and $f''(x_0)>0$, then x_0 is the minimum point; but if $f'(x_0)=0$, $f''(x_0)=0$, and $f'''(x_0)\neq 0$, then the point x_0 is not an extremal point.

More generally: let the first of the derivatives (not equal to zero at the point x_0) of the function $f(x)$ be of the order k. Then, if k is even, the point x_0 is an extremal point, namely, the maximum point, if $f^{(k)}(x_0)<0$; and it is the minimum point, if $f^{(k)}(x_0)>0$ But if k is odd, then x_0 is not an extremal point.

Example 4. Find the extrema of the function

$$y=2x+3\sqrt[3]{x^2}.$$

Solution. Find the derivative

$$y'=2+\frac{2}{\sqrt[3]{x}}=\frac{2}{\sqrt[3]{x}}(\sqrt[3]{x}+1). \tag{3}$$

Equating the derivative y' to zero, we get:

$$\sqrt[3]{x}+1=0.$$

Whence, we find the critical point $x_1=-1$. From formula (3) we have: if $x=-1-h$, where h is a sufficiently small positive number, then $y'>0$; but if $x=-1+h$, then $y'<0$*). Hence, $x_1=-1$ is the maximum point of the function y, and $y_{\max}=1$.

Equating the denominator of the expression of y' in (3) to zero, we get

$$\sqrt[3]{x}=0;$$

whence we find the second critical point of the function $x_2=0$, where there is no derivative y' For $x=-h$, we obviously have $y'<0$; for $x=h$ we have $y'>0$. Consequently, $x_2=0$ is the minimum point of the function y, and $y_{\min}=0$ (Fig. 24). It is also possible to test the behaviour of the function at the point $x=-1$ by means of the second derivative

$$y''=-\frac{2}{3x\sqrt[3]{x}}.$$

Here, $y''<0$ for $x_1=-1$ and, hence, $x_1=-1$ is the maximum point of the function.

3°. **Greatest and least values.** The least (greatest) value of a continuous function $f(x)$ on a given interval $[a, b]$ is attained either at the critical points of the function or at the end-points of the interval $[a, b]$.

---

*) If it is difficult to determine the sign of the derivative y', one can calculate arithmetically by taking for h a sufficiently small positive number.

Example 5. Find the greatest and least values of the function

$$y=x^3-3x+3$$

on the interval $-1^1/_2\leqslant x\leqslant 2^1/_2$.

Solution. Since

$$y'=3x^2-3,$$

it follows that the critical points of the function y are $x_1=-1$ and $x_2=1$.

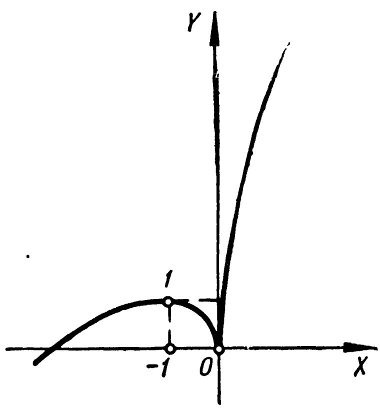

Fig. 24

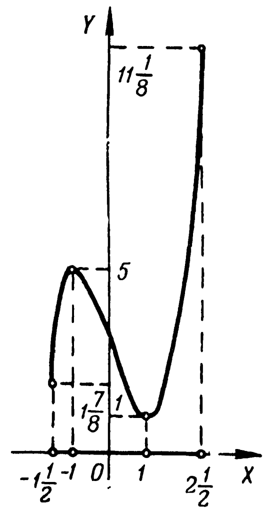

Fig. 25

Comparing the values of the function at these points and the values of the function at the end-points of the given interval

$$y(-1)=5;\ y(1)=1;\ y\left(-1\frac{1}{2}\right)=4\frac{1}{8};\qquad y\left(2\frac{1}{2}\right)=11\frac{1}{8},$$

we conclude (Fig. 25) that the function attains its least value, $m=1$, at the point $x=1$ (at the minimum point), and the greatest value $M=11\frac{1}{8}$ at the point $x=2^1/_2$ (at the right-hand end-point of the interval).

Determine the intervals of decrease and increase of the functions:

811. $y=1-4x-x^2$.

812. $y=(x-2)^2$.

813. $y=(x+4)^3$.

814. $y=x^2(x-3)$.

815. $y=\dfrac{x}{x-2}$.

816. $y=\dfrac{1}{(x-1)^2}$.

817. $y=\dfrac{x}{x^2-6x-16}$.

818. $y=(x-3)\sqrt{x}$.

819. $y=\frac{x}{3}-\sqrt[3]{x}$.

820. $y=x+\sin x$.

821. $y=x\ln x$.

822. $y=\arcsin(1+x)$.

823. $y=2e^{x^2-4x}$.

824. $y=2^{\frac{1}{x-a}}$.

825. $y=\frac{e^x}{x}$.

Test the following functions for extrema:

826. $y=x^2+4x+6$.

Solution. We find the derivative of the given function, $y'=2x+4$. Equating y' to zero, we get the critical value of the argument $x=-2$. Since $y'<0$ when $x<-2$, and $y'>0$ when $x>-2$, it follows that $x=-2$ is the minimum point of the function, and $y_{\min}=2$. We get the same result by utilizing the sign of the second derivative at the critical point $y''=2>0$.

827. $y=2+x-x^2$.

828. $y=x^3-3x^2+3x+2$.

829. $y=2x^3+3x^2-12x+5$.

Solution. We find the derivative

$$y'=6x^2+6x-12=6(x^2+x-2).$$

Equating the derivative y' to zero, we get the critical points $x_1=-2$ and $x_2=1$. To determine the nature of the extremum, we calculate the second derivative $y''=6(2x+1)$. Since $y''(-2)<0$, it follows that $x_1=-2$ is the maximum point of the function y, and $y_{\max}=25$. Similarly, we have $y''(1)>0$; therefore, $x_2=1$ is the minimum point of the function y and $y_{\min}=-2$.

830. $y=x^2(x-12)^2$.

831. $y=x(x-1)^2(x-2)^3$.

832. $y=\frac{x^3}{x^2+3}$.

833. $y=\frac{x^2-2x+2}{x-1}$.

834. $y=\frac{(x-2)(8-x)}{x^2}$.

835. $y=\frac{16}{x(4-x^2)}$.

836. $y=\frac{4}{\sqrt{x^2+8}}$.

837. $y=\frac{x}{\sqrt[3]{x^2-4}}$.

838. $y=\sqrt[3]{(x^2-1)^2}$.

839. $y=2\sin 2x+\sin 4x$.

840. $y=2\cos\frac{x}{2}+3\cos\frac{x}{3}$.

841. $y=x-\ln(1+x)$.

842. $y=x\ln x$.

843. $y=x\ln^2 x$.

844. $y=\cosh x$.

845. $y=xe^x$.

846. $y=x^2e^{-x}$.

847. $y=\frac{e^x}{x}$.

848. $y=x-\arctan x$.

Determine the least and greatest values of the functions on the indicated intervals (if the interval is not given, determine the greatest and least values of the function throughout the domain of definition).

849. $y=\frac{x}{1+x^2}$.

850. $y=\sqrt{x(10-x)}$.

851. $y=\sin^4 x+\cos^4 x$.

852. $y=\arccos x$.

853. $y=x^3$ on the interval $[-1,3]$.

854. $y=2x^3+3x^2-12x+1$
a) on the interval $[-1,5]$;
b) on the interval $[-10,12]$.

855. Show that for positive values of x we have the inequality

$$x+\frac{1}{x}\geqslant 2.$$

856. Determine the coefficients p and q of the quadratic trinomial $y=x^2+px+q$ so that this trinomial should have a minimum $y=3$ when $x=1$. Explain the result in geometrical terms.

857. Prove the inequality

$$e^x>1+x \quad \text{when } x\neq 0.$$

Solution. Consider the function

$$f(x)=e^x-(1+x).$$

In the usual way we find that this function has a single minimum $f(0)=0$. Hence,

$$f(x)>f(0) \quad \text{when } x\neq 0,$$
$$\text{and so } e^x>1+x \quad \text{when } x\neq 0,$$

as we set out to prove.

Prove the inequalities:

858. $x-\frac{x^3}{6}<\sin x<x$ when $x>0$.

859. $\cos x>1-\frac{x^2}{2}$ when $x\neq 0$.

860. $x-\frac{x^2}{2}<\ln(1+x)<x$ when $x>0$.

861. Separate a given positive number a into two summands such that their product is the greatest possible.

862. Bend a piece of wire of length l into a rectangle so that the area of the latter is greatest.

863. What right triangle of given perimeter $2p$ has the greatest area?

864. It is required to build a rectangular playground so that it should have a wire net on three sides and a long stone wall on the fourth. What is the optimum (in the sense of area) shape of the playground if l metres of wire netting are available?

865. It is required to make an open rectangular box of greatest capacity out of a square sheet of cardboard with side a by cutting squares at each of the angles and bending up the ends of the resulting cross-like figure.

866. An open tank with a square base must have a capacity of v litres. What size will it be if the least amount of tin is used?

867. Which cylinder of a given volume has the least overall surface?

868. In a given sphere inscribe a cylinder with the greatest volume.

869. In a given sphere inscribe a cylinder having the greatest lateral surface.

870. In a given sphere inscribe a cone with the greatest volume.

871. Inscribe in a given sphere a right circular cone with the greatest lateral surface.

872. About a given cylinder circumscribe a right cone of least volume (the planes and centres of their circular bases coincide).

873. Which of the cones circumscribed about a given sphere has the least volume?

874. A sheet of tin of width a has to be bent into an open cylindrical channel (Fig. 26). What should the central angle φ be so that the channel will have maximum capacity?

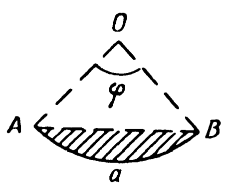

Fig. 26

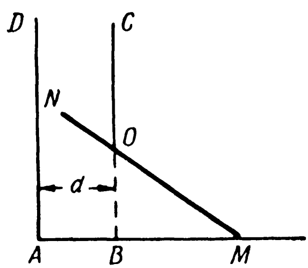

Fig. 27

875. Out of a circular sheet cut a sector such that when made into a funnel it will have the greatest possible capacity.

876. An open vessel consists of a cylinder with a hemisphere at the bottom; the walls are of constant thickness. What will the dimensions of the vessel be if a minimum of material is used for a given capacity?

877. Determine the least height $h = OB$ of the door of a vertical tower $ABCD$ so that this door can pass a rigid rod MN of length l, the end of which, M, slides along a horizontal straight line AB. The width of the tower is $d < l$ (Fig. 27).

878. A point $M_0(x_0, y_0)$ lies in the first quadrant of a coordinate plane. Draw a straight line through this point so that the triangle which it forms with the positive semi-axes is of least area.

879. Inscribe in a given ellipse a rectangle of largest area with sides parallel to the axes of the ellipse.

880. Inscribe a rectangle of maximum area in a segment of the parabola $y^2=2px$ cut off by the straight line $x=2a$.

881. On the curve $y=\frac{1}{1+x^2}$ find a point at which the tangent forms with the x-axis the greatest (in absolute value) angle.

882. A messenger leaving A on one side of a river has to get to B on the other side. Knowing that the velocity along the bank is k times that on the water, determine the angle at which the messenger has to cross the river so as to reach B in the shortest possible time. The width of the river is h and the distance between A and B along the bank is d.

883. On a straight line $AB=a$ connecting two sources of light A (of intensity p) and B (of intensity q), find the point M that receives least light (the intensity of illumination is inversely proportional to the square of the distance from the light source).

884. A lamp is suspended above the centre of a round table of radius r. At what distance should the lamp be above the table so that an object on the edge of the table will get the greatest illumination? (The intensity of illumination is directly proportional to the cosine of the angle of incidence of the light rays and is inversely proportional to the square of the distance from the light source.)

885. It is required to cut a beam of rectangular cross-section out of a round log of diameter d. What should the width x and the height y be of this cross-section so that the beam will offer maximum resistance a) to compression and b) to bending?

Note. The resistance of a beam to compression is proportional to the area of its cross-section, to bending—to the product of the width of the cross-section by the square of its height.

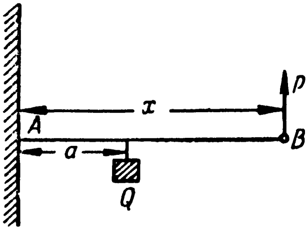

Fig. 2

886. A homogeneous rod AB, which can rotate about a point A (Fig. 28), is carrying a load Q kilograms at a distance of a cm from A and is held in equilibrium by a vertical force P applied to the free end B of the rod. A linear centimetre of the rod weighs q kilograms. Determine the length of the rod x so that the force P should be least, and find $P_{\min}$.

887*. The centres of three elastic spheres A, B, C are situated on a single straight line. Sphere A of mass M moving with velocity v strikes B, which, having acquired a certain velocity, strikes C of mass m. What mass should B have so that C will have the greatest possible velocity?

888. N identical electric cells can be formed into a battery in different ways by combining n cells in series and then combining the resulting groups $\left(\text{the number of groups is } \frac{N}{n}\right)$ in parallel. The current supplied by this battery is given by the formula

$$I=\frac{Nn\mathcal{E}}{NR+n^2r},$$

where $\mathcal{E}$ is the electromotive force of one cell, r is its internal resistance, and R is its external resistance.

For what value of n will the battery produce the greatest current?

889. Determine the diameter y of a circular opening in the body of a dam for which the discharge of water per second Q will be greatest, if $Q=cy\sqrt{h-y}$, where h is the depth of the lowest point of the opening (h and the empirical coefficient c are constant).

890. If $x_1, x_2, \ldots, x_n$ are the results of measurements of equal precision of a quantity x, then its most probable value will be that for which the sum of the squares of the errors

$$\sigma=\sum_{i=1}^{n}(x-x_i)^2$$

is of least value (the principle of least squares).

Prove that the most probable value of x is the arithmetic mean of the measurements.

Sec. 2. The Direction of Concavity. Points of Inflection

1°. The concavity of the graph of a function. We say that the graph of a differentiable function $y=f(x)$ is *concave down* in the interval (a,b) [*concave up* in the interval (a_1,b_1)] if for $a<x<b$ the arc of the curve is below (or for $a_1<x<b_1$, above) the tangent drawn at any point of the interval (a,b) or of the interval (a_1,b_1)] (Fig. 29). A sufficient condition for the concavity downwards (upwards) of a graph $y=f(x)$ is that the following inequality be fulfilled in the appropriate interval:

$$f''(x)<0 \quad [f''(x)>0].$$

2°. Points of inflection. A point $[x_0, f(x_0)]$ at which the direction of concavity of the graph of some function changes is called a *point of inflection* (Fig. 29).

For the abscissa of the point of inflection x_0 of the graph of a function $y=f(x)$ there is no second derivative $f''(x_0)=0$ or $f''(x_0)$. Points at which $f''(x)=0$ or $f''(x)$ does not exist are called *critical points of the second kind.* The critical point of the second kind x_0 is the abscissa of the point of inflection if $f''(x)$ retains constant signs in the intervals $x_0-\delta<x<x_0$ and $x_0<x<x_0+\delta$, where δ is some positive number; provided these signs are opposite. And it is not a point of inflection if the signs of $f''(x)$ are the same in the above-indicated intervals.

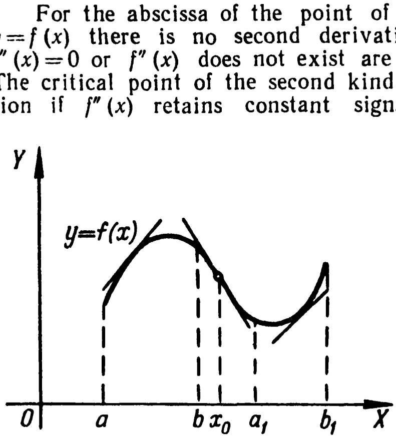

Fig. 29

Example 1. Determine the intervals of concavity and convexity and also the points of inflection of the Gaussian curve

$$y=e^{-x^2}.$$

Solution. We have

$$y'=-2xe^{-x}$$

and

$$y''=(4x^2-2)e^{-x^2}.$$

Equating the second derivative y'' to zero, we find the critical points of the second kind

$$x_1=-\frac{1}{\sqrt{2}} \quad \text{and} \quad x_2=\frac{1}{\sqrt{2}}.$$

These points divide the number scale $-\infty<x<+\infty$ into three intervals: I $(-\infty, x_1)$, II (x_1, x_2), and III $(x_2, +\infty)$. The signs of y'' will be, respec-

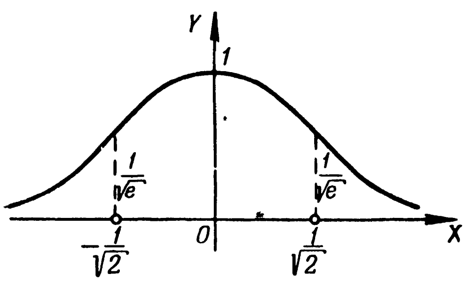

Fig. 30

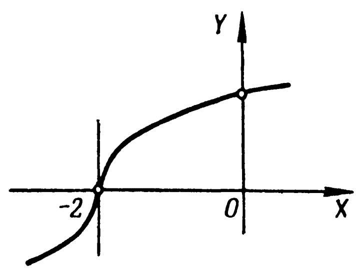

Fig. 31

tively, $+$, $-$, $+$ (this is obvious if, for example, we take one point in each of the intervals and substitute the corresponding values of x into y'') Therefore: 1) the curve is concave up when $-\infty<x<-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}<x<+\infty$; 2) the curve is concave down when $-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$. The points $\left(\frac{\pm 1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$ are points of inflection (Fig. 30).

It will be noted that due to the symmetry of the Gaussian curve about the y-axis, it would be sufficient to investigate the sign of the concavity of this curve on the semiaxis $0<x<+\infty$ alone.

Example 2. Find the points of inflection of the graph of the function

$$y=\sqrt[3]{x+2}.$$

Solution. We have:

$$y''=-\frac{2}{9}(x+2)^{-\frac{5}{3}}=\frac{-2}{9\sqrt[3]{(x+2)^5}}. \qquad (1)$$

It is obvious that y'' does not vanish anywhere.

Equating to zero the denominator of the fraction on the right of (1), we find that y'' does not exist for $x=-2$. Since $y''>0$ for $x<-2$ and $y''<0$ for $x>-2$, it follows that $(-2,0)$ is the point of inflection (Fig. 31). The tangent at this point is parallel to the axis of ordinates, since the first derivative y' is infinite at $x=-2$.

Find the intervals of concavity and the points of inflection of the graphs of the following functions:

891. $y=x^3-6x^2+12x+4$.

892. $y=(x+1)^4$.

893. $y=\frac{1}{x+3}$.

894. $y=\frac{x^3}{x^2+12}$.

895. $y=\sqrt[3]{4x^3-12x}$.

896. $y=\cos x$.

897. $y=x-\sin x$.

898. $y=x^2 \ln x$.

899. $y=\arctan x-x$.

900. $y=(1+x^2)e^x$.

Sec. 3. Asymptotes

1°. **Definition.** If a point (x,y) is in continuous motion along a curve $y=f(x)$ in such a way that at least one of its coordinates approaches infinity (and at the same time the distance of the point from some straight line tends to zero), then this straight line is called an *asymptote* of the curve.

2°. **Vertical asymptotes.** If there is a number a such that

$$\lim_{x\to a} f(x)=\pm\infty,$$

then the straight line $x=a$ is an asymptote *(vertical asymptote).*

3° **Inclined asymptotes.** If there are limits

$$\lim_{x\to+\infty}\frac{f(x)}{x}=k_1$$

and

$$\lim_{x\to+\infty}[f(x)-k_1x]=b_1,$$

then the straight line $y=k_1x+b_1$ will be an asymptote (a *right inclined asymptote* or, when $k_1=0$, a *right horizontal asymptote*).

If there are limits

$$\lim_{x\to-\infty}\frac{f(x)}{x}=k_2$$

and

$$\lim_{x\to-\infty} [f(x)-k_2x]=b_2,$$

then the straight line $y=k_2x+b_2$ is an asymptote (a *left inclined asymptote* or, when $k_2=0$, a *left horizontal asymptote*). The graph of the function $y=f(x)$ (we assume the function is single-valued) cannot have more than one right (inclined or horizontal) and more than one left (inclined or horizontal) asymptote.

Example 1. Find the asymptotes of the curve

$$y=\frac{x^2}{\sqrt{x^2-1}}.$$

Solution. Equating the denominator to zero, we get two vertical asymptotes:

$$x=-1 \quad \text{and} \quad x=1.$$

We seek the inclined asymptotes. For $x\to+\infty$ we obtain

$$k_1=\lim_{x\to+\infty}\frac{y}{x}=\lim_{x\to+\infty}\frac{x^2}{x\sqrt{x^2-1}}=1,$$

$$b_1=\lim_{x\to+\infty}(y-x)=\lim_{x\to+\infty}\frac{x^2-x\sqrt{x^2-1}}{\sqrt{x^2-1}}=0,$$

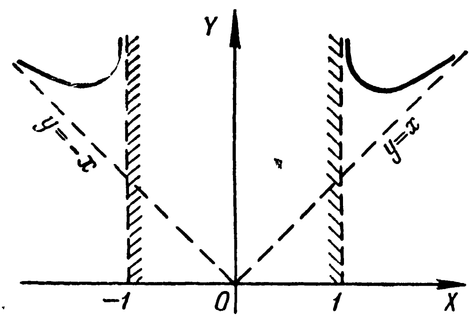

Fig. 32

hence, the straight line $y=x$ is the right asymptote. Similarly, when $x\to-\infty$, we have

$$k_2=\lim_{x\to-\infty}\frac{y}{x}=-1;$$

$$b_2=\lim_{x\to-\infty}(y+x)=0.$$

Thus, the left asymptote is $y=-x$ (Fig. 32). Testing a curve for asymptotes is simplified if we take into consideration the symmetry of the curve.

Example 2. Find the asymptotes of the curve

$$y=x+\ln x.$$

Solution. Since

$$\lim_{x\to+0} y=-\infty,$$

the straight line $x=0$ is a vertical asymptote (lower). Let us now test the curve only for the inclined right asymptote (since $x>0$).

We have:

$$k=\lim_{x\to+\infty}\frac{y}{x}=1,$$

$$b=\lim_{x\to+\infty}(y-x)=\lim_{x\to+\infty}\ln x=\infty.$$

Hence, there is no inclined asymptote.

If a curve is represented by the parametric equations $x=\varphi(t)$, $y=\psi(t)$, then we first test to find out whether there are any values of the parameter t for which one of the functions $\varphi(t)$ or $\psi(t)$ becomes infinite, while the other remains finite. When $\varphi(t_0)=\infty$ and $\psi(t_0)=c$, the curve has a horizontal asymptote $y=c$. When $\psi(t_0)=\infty$ and $\varphi(t_0)=c$, the curve has a vertical asymptote $x=c$.

If $\varphi(t_0)=\psi(t_0)=\infty$ and

$$\lim_{t\to t_0}\frac{\psi(t)}{\varphi(t)}=k;\quad \lim_{t\to t_0}[\psi(t)-k\varphi(t)]=b,$$

then the curve has an inclined asymptote $y=kx+b$.

If the curve is represented by a polar equation $r=f(\varphi)$, then we can find its asymptotes by the preceding rule after transforming the equation of the curve to the parametric form by the formulas $x=r\cos\varphi=f(\varphi)\cos\varphi$; $y=r\sin\varphi=f(\varphi)\sin\varphi$.

Find the asymptotes of the following curves:

901. $y=\dfrac{1}{(x-2)^2}$.

902. $y=\dfrac{x}{x^2-4x+3}$.

903. $y=\dfrac{x^2}{x^2-4}$.

904. $y=\dfrac{x^3}{x^2+9}$.

905. $y=\sqrt{x^2-1}$.

906. $y=\dfrac{x}{\sqrt{x^2+3}}$.

907. $y=\dfrac{x^2+1}{\sqrt{x^2-1}}$.

908. $y=x-2+\dfrac{x^2}{\sqrt{x^2+9}}$.

909. $y=e^{-x^2}+2$.

910. $y=\dfrac{1}{1-e^x}$.

911. $y=e^{\frac{1}{x}}$.

912. $y=\dfrac{\sin x}{x}$.

913. $y=\ln(1+x)$.

914. $x=t$; $y=t+2\arctan t$.

915. Find the asymptote of the hyperbolic spiral $r=\dfrac{a}{\varphi}$.

Sec. 4. Graphing Functions by Characteristic Points

In constructing the graph of a function, first find its domain of definition and then determine the behaviour of the function on the boundary of this domain. It is also useful to note any peculiarities of the function (if there are any), such as symmetry, periodicity, constancy of sign, monotonicity, etc.

Then find any points of discontinuity, bending points, points of inflection, asymptotes, etc. These elements help to determine the general nature of the graph of the function and to obtain a mathematically correct outline of it.

Example 1. Construct the graph of the function

$$y=\frac{x}{\sqrt[3]{x^2-1}}.$$

Solution. a) The function exists everywhere except at the points $x=\pm 1$. The function is odd, and therefore the graph is symmetric about the point $O(0, 0)$. This simplifies construction of the graph

b) The discontinuities are $x=-1$ and $x=1$; and $\lim\limits_{x\to 1\pm 0} y=\pm\infty$ and $\lim\limits_{x\to -1\pm 0} y=\pm\infty$; hence, the straight lines $x=\pm 1$ are vertical asymptotes of the graph.

c) We seek inclined asymptotes, and find

$$k_1=\lim_{x\to+\infty}\frac{y}{x}=0,$$

$$b_1=\lim_{x\to+\infty} y=\infty,$$

thus, there is no right asymptote. From the symmetry of the curve it follows that there is no left-hand asymptote either.

d) We find the critical points of the first and second kinds, that is, points at which the first (or, respectively, the second) derivative of the given function vanishes or does not exist.

We have:

$$y'=\frac{x^2-3}{3\sqrt[3]{(x^2-1)^4}}, \tag{1}$$

$$y''=\frac{2x(9-x^2)}{9\sqrt[3]{(x^2-1)^7}}. \tag{2}$$

The derivatives y' and y'' are nonexistent only at $x=\pm 1$, that is, only at points where the function y itself does not exist; and so the critical points are only those at which y' and y'' vanish.

From (1) and (2) it follows that

$$y'=0 \quad \text{when } x=\pm\sqrt{3};$$
$$y''=0 \quad \text{when } x=0 \text{ and } x=\pm 3.$$

Thus, y' retains a constant sign in each of the intervals $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, -1)$, $(-1, 1)$, $(1, \sqrt{3})$ and $(\sqrt{3}, +\infty)$, and y''—in each of the intervals $(-\infty, -3)$, $(-3, -1)$, $(-1, 0)$, $(0, 1)$, $(1, 3)$ and $(3, +\infty)$.

To determine the signs of y' (or, respectively, y'') in each of the indicated intervals, it is sufficient to determine the sign of y' (or y'') at some one point of each of these intervals.

It is convenient to tabulate the results of such an investigation (Table I), calculating also the ordinates of the characteristic points of the graph of the function. It will be noted that due to the oddness of the function y, it is enough to calculate only for $x \geqslant 0$; the left-hand half of the graph is constructed by the principle of odd symmetry.

Table I

x	0	(0, 1)	1	$(1, \sqrt{3})$	$\sqrt{3} \approx 1.73$	$(\sqrt{3}, 3)$	3	$(3, +\infty)$
y	0	$-$	$\pm\infty$	$+$	$\frac{\sqrt{3}}{\sqrt[3]{2}} \approx 1.37$	$+$	1.5	$+$
y'	$-$	$-$	non-exist	$-$	0	$+$	$+$	$+$
y''	0	$-$	non-exist	$+$	$+$	$+$	0	$-$
Conclusions	Point of inflection	Function decreases, graph is concave down	Discontinuit,	Function decreases, graph is concave up	Min. point	Function increases; graph is concave up	Point of inflection	Function increases; graph is concave down

e) Using the results of the investigation, we construct the graph of the function (Fig 33).

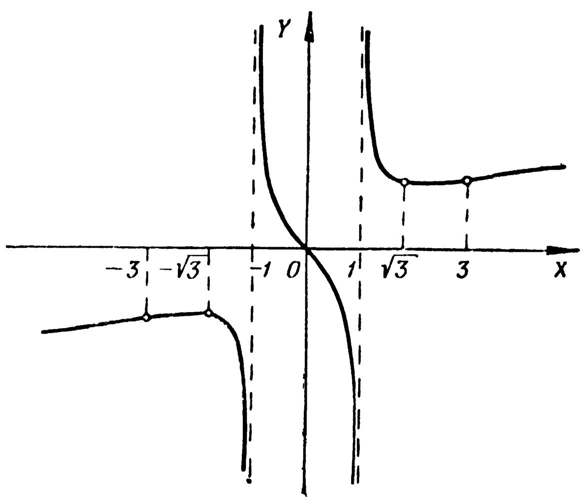

Fig. 33

Example 2. Graph the function

$$y=\frac{\ln x}{x}.$$

Solution. a) The domain of definition of the function is $0<x<+\infty$.

b) There are no discontinuities in the domain of definition, but as we approach the boundary point ($x=0$) of the domain of definition we have

$$\lim_{x\to 0} y=\lim_{x\to 0}\frac{\ln x}{x}=-\infty.$$

Hence, the straight line $x=0$ (ordinate axis) is a vertical asymptote.

c) We seek the right asymptote (there is no left asymptote, since x cannot tend to $-\infty$):

$$k=\lim_{x\to+\infty}\frac{y}{x}=0;$$

$$b=\lim_{x\to+\infty} y=0.$$

The right asymptote is the axis of abscissas: $y=0$.

d) We find the critical points; and have

$$y'=\frac{1-\ln x}{x^2},$$

$$y''=\frac{2\ln x-3}{x^3};$$

y' and y'' exist at all points of the domain of definition of the function and

$$y'=0 \text{ when } \ln x=1, \text{ that is, when } x=e;$$

$$y''=0 \text{ when } \ln x=\frac{3}{2}, \text{ that is, when } x=e^{3/2}.$$

We form a table, including the characteristic points (Table II). In addition to the characteristic points it is useful to find the points of intersection of

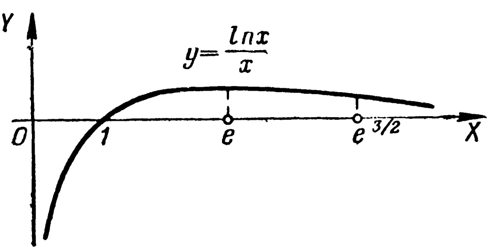

Fig. 34

the curve with the coordinate axes. Putting $y=0$, we find $x=1$ (the point of intersection of the curve with the axis of abscissas); the curve does not intersect the axis of ordinates

e) Utilizing the results of investigation, we construct the graph of the function (Fig. 34).

Table II

x	0	(0, 1)	1	(1, e)	$e \approx 2.72$	$\left(e, e^{\frac{3}{2}}\right)$	$e^{\frac{3}{2}} \approx 4.49$	$\left(e^{\frac{3}{2}}, +\infty\right)$
y	$-\infty$	$-$	0	+	$\frac{1}{e} \approx 0.37$	+	$\frac{3}{2\sqrt{e^3}} \approx 0.33$	+
y'	nonexist.	+	+	+	0	$-$	$-$	$-$
y''	nonexist	$-$	$-$	$-$	$-$	$-$	0	+
Conclusions	Boundary point of domain of def. of function Vertical asymptote	Funct increases graph is concave down	Funct incr. graph is concave down	Funct incr., graph is concave down	Max point of funct.	Funct decr., graph is concave down	Point of inflection	Function decreases; graph is concave up

Graph the following functions and determine for each function its domain of definition, discontinuities, extremal points, intervals of increase and decrease, points of inflection of its graph, the direction of concavity, and also the asymptotes.

916. $y=x^3-3x^2$.

917. $y=\frac{6x^2-x^4}{9}$.

918. $y=(x-1)^2(x+2)$.

919. $y=\frac{(x-2)^2(x+4)}{4}$.

920. $y=\frac{(x^2-5)^3}{125}$.

921. $y=\frac{x^2-2x+2}{x-1}$.

922. $y=\frac{x^4-3}{x}$.

923. $y=\frac{x^4+3}{x}$.

924. $y=x^2+\frac{2}{x}$.

925. $y=\frac{1}{x^2+3}$.

926. $y=\frac{8}{x^2-4}$.

927. $y=\frac{4x}{4+x^2}$.

928. $y=\frac{4x-12}{(x-2)^2}$.

929. $y=\frac{x}{x^2-4}$.

930. $y=\frac{16}{x^2(x-4)}$.

931. $y=\frac{3x^4+1}{x^3}$.

932. $y=\sqrt{x}+\sqrt{4-x}$.

933. $y=\sqrt{8+x}-\sqrt{8-x}$.

934. $y=x\sqrt{x+3}$.

935. $y=\sqrt{x^3-3x}$.

936. $y=\sqrt[3]{1-x^2}$.

937. $y=\sqrt[3]{1-x^3}$.

938. $y=2x+2-3\sqrt[3]{(x+1)^2}$.

939. $y=\sqrt[3]{x+1}-\sqrt[3]{x-1}$.

940. $y=\sqrt[3]{(x+4)^2}-\sqrt[3]{(x-4)^2}$.

941. $y=\sqrt[3]{(x-2)^2}+\sqrt[3]{(x-4)^2}$.

942. $y=\frac{4}{\sqrt{4-x^2}}$.

943. $y=\frac{8}{x\sqrt{x^2-4}}$.

944. $y=\frac{x}{\sqrt[3]{x^2-1}}$.

945. $y=\frac{x}{\sqrt[3]{(x-2)^2}}$.

946. $y=xe^{-x}$.

947. $y=\left(a+\frac{x^2}{a}\right)e^{\frac{x}{a}}$.

948. $y=e^{8x-x^2-14}$.

949. $y=(2+x^2)e^{-x^2}$.

950. $y=2|x|-x^2$.

951. $y=\frac{\ln x}{\sqrt{x}}$.

952. $y=\frac{x^2}{2}\ln\frac{x}{a}$.

953. $y=\frac{x}{\ln x}$.

954. $y=(x+1)\ln^2(x+1)$.

955. $y=\ln(x^2-1)+\frac{1}{x^2-1}$.

956. $y=\ln\frac{\sqrt{x^2+1}-1}{x}$.

957. $y=\ln(1+e^{-x})$.

958. $y=\ln\left(e+\frac{1}{x}\right)$.

959. $y=\sin x+\cos x$.

960. $y=\sin x+\frac{\sin 2x}{2}$.

961. $y=\cos x-\cos^2 x$.

962. $y=\sin^3 x+\cos^3 x$.

963. $y=\frac{1}{\sin x+\cos x}$.

964. $y=\frac{\sin x}{\sin\left(x+\frac{\pi}{4}\right)}$.

965. $y=\sin x\cdot\sin 2x$.

966. $y=\cos x\cdot\cos 2x$.

967. $y=x+\sin x$.

968. $y=\arcsin(1-\sqrt[3]{x^2})$.

969. $y=\frac{\arcsin x}{\sqrt{1-x^2}}$.

970. $y=2x-\tan x$.

971. $y=x\arctan x$.

972. $y=x\arctan\frac{1}{x}$ when $x\neq 0$ and $y=0$ when $x=0$.

973. $y=x-2\,\mathrm{arc\,cot}\,x$.

974. $y=\frac{x}{2}+\arctan x$.

975. $y=\ln\sin x$.

976. $y=\mathrm{arc\,cosh}\left(x+\frac{1}{x}\right)$.

977. $y=e^{\sin x}$.

978. $y=e^{\arcsin\sqrt{x}}$.

979. $y=e^{\arctan x}$.

980. $y=\ln\sin x$.

981. $y=\ln\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$.

982. $y=\ln x-\arctan x$.

983. $y=\cos x-\ln\cos x$.

984. $y=\arctan(\ln x)$.

985. $y=\arcsin\ln(x^2+1)$.

986. $y=x^x$.

987. $y=x^{\frac{1}{x}}$.

A good exercise is to graph the functions indicated in Examples 826-848.

Construct the graphs of the following functions represented parametrically.

988. $x=t^2-2t,\quad y=t^2+2t$.

989. $x=a\cos^3 t,\quad y=a\sin t\quad (a>0)$.

990. $x=te^t,\quad y=te^{-t}$.

991. $x=t+e^{-t},\quad y=2t+e^{-2t}$.

992. $x=a(\sinh t-t),\quad y=a(\cosh t-1)\quad (a>0)$.

Sec. 5. Differential of an Arc. Curvature

1°. Differential of an arc. The differential of an arc s of a plane curve represented by an equation in Cartesian coordinates x and y is expressed by the formula

$$ds=\sqrt{(dx)^2+(dy)^2};$$

here, if the equation of the curve is of the form

a) $y=f(x)$, then $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$;

b) $x=f_1(y)$, then $ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$;

c) $x=\varphi(t)$, $y=\psi(t)$, then $ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$;

d) $F(x, y)=0$, then $ds=\frac{\sqrt{F_x'^2+F_y'^2}}{|F_y'|}\,dx=\frac{\sqrt{F_x'^2+F_y'^2}}{|F_x'|}\,dy$.