1920. Find d^2z if

$$z=f(u, v), \text{ where } u=ax, \ v=by.$$

1921. Find d^2z if

$$z=f(u, v), \text{ where } u=xe^y, \ v=ye^x.$$

1922. Find d^3z if

$$z=e^x \cos y.$$

1923. Find the third differential of the function

$$z=x\cos y+y\sin x.$$

Determine all third partial derivatives.

1924. Find $df(1, 2)$ and $d^2f(1, 2)$ if

$$f(x, y)=x^2+xy+y^2-4\ln x-10\ln y.$$

1925. Find $d^2f(0, 0, 0)$ if

$$f(x, y, z)=x^2+2y^2+3z^2-2xy+4xz+2yz.$$

Sec. 8. Integration of Total Differentials

1°. The condition for a total differential. For an expression $P(x, y)\,dx+Q(x, y)\,dy$, where the functions $P(x, y)$ and $Q(x, y)$ are continuous in a simply connected region D together with their first partial derivatives, to be (in D) the total differential of some function $u(x, y)$, it is necessary and sufficient that

$$\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}.$$

Example 1. Make sure that the expression

$$(2x+y)\,dx+(x+2y)\,dy$$

is a total differential of some function, and find that function.

Solution. In the given case, $P=2x+y$, $Q=x+2y$. Therefore, $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}=1$, and, hence,

$$(2x+y)\,dx+(x+2y)\,dy=du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy,$$

where u is the desired function.

It is given that $\frac{\partial u}{\partial x}=2x+y$; therefore,

$$u=\int(2x+y)\,dx=x^2+xy+\varphi(y).$$

But on the other hand $\frac{\partial u}{\partial y}=x+\varphi'(y)=x+2y$, whence $\varphi'(y)=2y$, $\varphi(y)=y^2+C$ and

$$u=x^2+xy+y^2+C.$$

Finally we have

$$(2x+y)\,dx+(x+2y)\,dy=d(x^2+xy+y^2+C).$$

2°. The case of three variables. Similarly, the expression

$$P(x, y, z)\,dx+Q(x, y, z)\,dy+R(x, y, z)\,dz,$$

where $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ are, together with their first partial derivatives, continuous functions of the variables x, y and z, is the total differential of some function $u(x, y, z)$ if and only if the following conditions are fulfilled:

$$\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y},\quad \frac{\partial R}{\partial y}\equiv\frac{\partial Q}{\partial z},\quad \frac{\partial P}{\partial z}\equiv\frac{\partial R}{\partial x}.$$

Example 2. Be sure that the expression

$$(3x^2+3y-1)\,dx+(z^2+3x)\,dy+(2yz+1)\,dz$$

is the total differential of some function, and find that function.

Solution. Here, $P=3x^2+3y-1$, $Q=z^2+3x$, $R=2yz+1$. We establish the fact that

$$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}=3,\quad \frac{\partial R}{\partial y}=\frac{\partial Q}{\partial z}=2z,\quad \frac{\partial P}{\partial z}=\frac{\partial R}{\partial x}=0$$

and, hence,

$$(3x^2+3y-1)dx+(z^2+3x)\,dy+(2yz+1)\,dz=du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz,$$

where u is the sought-for function.

We have

$$\frac{\partial u}{\partial x}=3x^2+3y-1,$$

hence,

$$u=\int(3x^2+3y-1)\,dx=x^3+3xy-x+\varphi(y, z).$$

On the other hand,

$$\frac{\partial u}{\partial y}=3x+\frac{\partial\varphi}{\partial y}=z^2+3x,$$
$$\frac{\partial u}{\partial z}=\frac{\partial\varphi}{\partial z}=2yz+1,$$

whence $\frac{\partial\varphi}{\partial y}=z^2$ and $\frac{\partial\varphi}{\partial z}=2yz+1$. The problem reduces to finding the function of two variables $\varphi(y, z)$ whose partial derivatives are known and the condition for total differential is fulfilled.

We find φ:

$$\varphi(y, z)=\int z^2dy=yz^2+\psi(z),$$
$$\frac{\partial\varphi}{\partial z}=2yz+\psi'(z)=2yz+1,$$
$$\psi'(z)=1,\quad \psi(z)=z+C,$$

that is, $\varphi(y, z)=yz^2+z+C$. And finally,

$$u=x^3+3xy-x+yz^2+z+C.$$

Having convinced yourself that the expressions given below are total differentials of certain functions, find these functions.

1926. $y\,dx+x\,dy$.

1927. $(\cos x+3x^2y)\,dx+(x^3-y^2)\,dy$.

1928. $\frac{(x+2y)\,dx+y\,dy}{(x+y)^2}$.

1929. $\frac{x+2y}{x^2+y^2}\,dx-\frac{2x-y}{x^2+y^2}\,dy$.

1930. $\frac{1}{y}\,dx-\frac{x}{y^2}\,dy$.

1931. $\frac{x}{\sqrt{x^2+y^2}}\,dx+\frac{y}{\sqrt{x^2+y^2}}\,dy$.

1932. Determine the constants a and b in such a manner that the expression

$$\frac{(ax^2+2xy+y^2)\,dx-(x^2+2xy+by^2)\,dy}{(x^2+y^2)^2}$$

should be a total differential of some function z, and find that function.

Convince yourself that the expressions given below are total differentials of some functions and find these functions.

1933. $(2x+y+z)\,dx+(x+2y+z)\,dy+(x+y+2z)\,dz$.

1934. $(3x^2+2y^2+3z)\,dx+(4xy+2y-z)\,dy+(3x-y-2)\,dz$.

1935. $(2xyz-3y^2z+8xy^2+2)\,dx+$
$$+(x^2z-6xyz+8x^2y+1)\,dy+(x^2y-3xy^2+3)\,dz.$$

1936. $\left(\frac{1}{y}-\frac{z}{x^2}\right)dx+\left(\frac{1}{z}-\frac{x}{y^2}\right)dy+\left(\frac{1}{x}-\frac{y}{z^2}\right)dz$.

1937. $\frac{x\,dx+y\,dy+z\,dz}{\sqrt{x^2+y^2+z^2}}$.

1938*. Given the projections of a force on the coordinate axes

$$X=\frac{y}{(x+y)^2}, \quad Y=\frac{\lambda x}{(x+y)^2},$$

where λ is a constant. What must the coefficient λ be for the force to have a potential?

1939. What condition must the function $f(x, y)$ satisfy for the expression

$$f(x, y)\,(dx+dy)$$

to be a total differential?

1940. Find the function u if

$$du=f(xy)\,(y\,dx+x\,dy).$$

Sec. 9. Differentiation of Implicit Functions

1°. The case of one independent variable. If the equation $f(x, y)=0$, where $f(x, y)$ is a differentiable function of the variables x and y, defines y as a function of x, then the derivative of this implicitly defined function, provided that $f'_y(x, y) \neq 0$, may be found from the formula

$$\frac{dy}{dx}=-\frac{f'_x(x, y)}{f'_y(x, y)}. \tag{1}$$

Higher-order derivatives are found by successive differentiation of formula (1)

Example 1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if

$$(x^2+y^2)^3-3(x^2+y^2)+1=0.$$

Solution. Denoting the left-hand side of this equation by $f(x, y)$, we find the partial derivatives

$$f'_x(x, y)=3(x^2+y^2)^2\cdot 2x-3\cdot 2x=6x[(x^2+y^2)^2-1],$$

$$f'_y(x, y)=3(x^2+y^2)^2\cdot 2y-3\cdot 2y=6y[(x^2+y^2)^2-1].$$

Whence, applying formula (1), we get

$$\frac{dy}{dx}=-\frac{f'_x(x, y)}{f'_y(x, y)}=-\frac{6x[(x^2+y^2)^2-1]}{6y[(x^2+y^2)^2-1]}=-\frac{x}{y}.$$

To find the second derivative, differentiate with respect to x the first derivative which we have found, taking into consideration the fact that y is a function of x·

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(-\frac{x}{y}\right)=-\frac{1\cdot y-x\frac{dy}{dx}}{y^2}=-\frac{y-x\left(-\frac{x}{y}\right)}{y^2}=-\frac{y^2+x^2}{y^3}.$$

2°. The case of several independent variables. Similarly, if the equation $F(x, y, z)=0$, where $F(x, y, z)$ is a differentiable function of the variables x, y and z, defines z as a function of the independent variables x and y and $F'_z(x, y, z) \neq 0$, then the partial derivatives of this implicitly represented function can, generally speaking, be found from the formulas

$$\frac{\partial z}{\partial x}=-\frac{F'_x(x, y, z)}{F'_z(x, y, z)}, \quad \frac{\partial z}{\partial y}=-\frac{F'_y(x, y, z)}{F'_z(x, y, z)}. \tag{2}$$

Here is another way of finding the derivatives of the function z: differentiating the equation $F(x, y, z)=0$, we find

$$\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz=0.$$

Whence it is possible to determine dz, and, therefore,

$$\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

Example 2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$x^2-2y^2+3z^2-yz+y=0.$$

Solution. First method. Denoting the left side of this equation by $F(x, y, z)$, we find the partial derivatives

$$F'_x(x, y, z)=2x, \quad F'_y(x, y, z)=-4y-z+1, \quad F'_z(x, y, z)=6z-y.$$

Applying formulas (2), we get

$$\frac{\partial z}{\partial x}=-\frac{F'_x(x, y, z)}{F'_z(x, y, z)}=-\frac{2x}{6z-y}; \quad \frac{\partial z}{\partial y}=-\frac{F'_y(x, y, z)}{F'_z(x, y, z)}=-\frac{1-4y-z}{6z-y}.$$

Second method. Differentiating the given equation, we obtain

$$2x\,dx-4y\,dy+6z\,dz-y\,dz-z\,dy+dy=0.$$

Whence we determine dz, that is, the total differential of the implicit function:

$$dz=\frac{2x\,dx+(1-4y-z)\,dy}{y-6z}.$$

Comparing with the formula $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$, we see that

$$\frac{\partial z}{\partial x}=\frac{2x}{y-6z}, \quad \frac{\partial z}{\partial y}=\frac{1-4y-z}{y-6z}.$$

3°. A system of implicit functions. If a system of two equations

$$\begin{cases} F(x, y, u, v)=0, \\ G(x, y, u, v)=0 \end{cases}$$

defines u and v as functions of the variables x and y and the Jacobian

$$\frac{D(F, G)}{D(u, v)}=\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} \neq 0,$$

then the differentials of these functions (and hence their partial derivatives as well) may be found from the following set of equations

$$\begin{cases} \frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial u}du+\frac{\partial F}{\partial v}dv=0, \\ \frac{\partial G}{\partial x}dx+\frac{\partial G}{\partial y}dy+\frac{\partial G}{\partial u}du+\frac{\partial G}{\partial v}dv=0. \end{cases} \tag{3}$$

Example 3. The equations

$$u+v=x+y, \quad xu+yv=1$$

define u and v as functions of x and y; find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$.

Solution. First method. Differentiating both equations with respect to x, we obtain

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=1,$$

$$u+x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial x}=0,$$

whence

$$\frac{\partial u}{\partial x}=-\frac{u+y}{x-y}, \quad \frac{\partial v}{\partial x}=\frac{u+x}{x-y}.$$

Similarly we find

$$\frac{\partial u}{\partial y}=-\frac{v+y}{x-y}, \quad \frac{\partial v}{\partial y}=\frac{v+x}{x-y}.$$

Second method. By differentiation we find two equations that connect the differentials of all four variables:

$$du+dv=dx+dy,$$
$$x\,du+u\,dx+y\,dv+v\,dy=0.$$

Solving this system for the differentials du and dv, we obtain

$$du=-\frac{(u+y)\,dx+(v+y)\,dy}{x-y}, \qquad dv=\frac{(u+x)\,dx+(v+x)\,dy}{x-y}.$$

Whence

$$\frac{\partial u}{\partial x}=-\frac{u+y}{x-y}, \quad \frac{\partial u}{\partial y}=-\frac{v+y}{x-y},$$
$$\frac{\partial v}{\partial x}=\frac{u+x}{x-y}, \quad \frac{\partial v}{\partial y}=\frac{v+x}{x-y}.$$

4°. Parametric representation of a function. If a function z of the variables x and y is represented parametrically by the equations

$$x=x(u,\ v),\ y=y(u,\ v),\ z=z(u,\ v)$$

and

$$\frac{D(x,\ y)}{D(u,\ v)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0,$$

then the differential of this function may be found from the following system of equations

$$\left\{\begin{aligned} dx&=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv, \\ dy&=\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv, \\ dz&=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv. \end{aligned}\right. \tag{4}$$

Knowing the differential $dz=p\,dx+q\,dy$, we find the partial derivatives $\frac{\partial z}{\partial x}=p$ and $\frac{\partial z}{\partial y}=q$.

Example 4. The function z of the arguments x and y is defined by the equations

$$x=u+v, \quad y=u^2+v^2, \quad z=u^3+v^3 \quad (u \neq v).$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution. First method. By differentiation we find three equations that connect the differentials of all five variables:

$$\begin{cases} dx=du+dv, \\ dy=2u\,du+2v\,dv, \\ dz=3u^2du+3v^2dv. \end{cases}$$

From the first two equations we determine du and dv:

$$du=\frac{2v\,dx-dy}{2(v-u)}, \quad dv=\frac{dy-2u\,dx}{2(v-u)}.$$

Substituting into the third equation the values of du and dv just found, we have:

$$dz=3u^2\frac{2v\,dx-dy}{2(v-u)}+3v^2\frac{dy-2u\,dx}{2(v-u)}=$$
$$=\frac{6uv(u-v)\,dx+3(v^2-u^2)\,dy}{2(v-u)}=-3uv\,dx+\frac{3}{2}(u+v)\,dy.$$

Whence

$$\frac{\partial z}{\partial x}=-3uv, \quad \frac{\partial z}{\partial y}=\frac{3}{2}(u+v).$$

Second method. From the third given equation we can find

$$\frac{\partial z}{\partial x}=3u^2\frac{\partial u}{\partial x}+3v^2\frac{\partial v}{\partial x}; \quad \frac{\partial z}{\partial y}=3u^2\frac{\partial u}{\partial y}+3v^2\frac{\partial v}{\partial y}. \tag{5}$$

Differentiate the first two equations first with respect to x and then with respect to y:

$$\begin{cases} 1=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}, & 0=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}, \\ 0=2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}, & 1=2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}. \end{cases}$$

From the first system we find

$$\frac{\partial u}{\partial x}=\frac{v}{v-u}, \quad \frac{\partial v}{\partial x}=\frac{u}{u-v}.$$

From the second system we find

$$\frac{\partial u}{\partial y}=\frac{1}{2(u-v)}, \quad \frac{\partial v}{\partial y}=\frac{1}{2(v-u)}.$$

Substituting the expressions $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ into formula (5), we obtain

$$\frac{\partial z}{\partial x}=3u^2\frac{v}{v-u}+3v^2\frac{u}{u-v}=-3uv,$$
$$\frac{\partial z}{\partial y}=3u^2\frac{1}{2(u-v)}+3v^2\cdot\frac{1}{2(v-u)}=\frac{3}{2}(u+v).$$

1941. Let y be a function of x defined by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

1942. y is a function defined by the equation

$$x^2+y^2+2axy=0 \quad (a>1).$$

Show that $\frac{d^2y}{dx^2}=0$ and explain the result obtained.

1943. Find $\frac{dy}{dx}$ if $y=1+y^x$.

1944. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $y=x+\ln y$.

1945. Find $\left(\frac{dy}{dx}\right)_{x=1}$ and $\left(\frac{d^2y}{dx^2}\right)_{x=1}$ if

$$x^2-2xy+y^2+x+y-2=0.$$

Taking advantage of the results obtained, show approximately the portions of the given curve in the neighbourhood of the point $x=1$.

1946. The function y is defined by the equation

$$\ln\sqrt{x^2+y^2}=a\arctan\frac{y}{x}\ (a\neq 0).$$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

1947. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if

$$1+xy-\ln(e^{xy}+e^{-xy})=0.$$

1948. The function z of the variables x and y is defined by the equation

$$x^3+2y^3+z^3-3xyz-2y+3=0.$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

1949. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$x\cos y+y\cos z+z\cos x=1.$$

1950. The function z is defined by the equation

$$x^2+y^2-z^2-xy=0.$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the system of values $x=-1$, $y=0$, $z=1$.

1951. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$ if

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

1952. $f(x, y, z)=0$. Show that $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x}=-1$.

1953. $z=\varphi(x, y)$, where y is a function of x defined by the equation $\psi(x, y)=0$. Find $\frac{dz}{dx}$.

1954. Find dz and d^2z, if

$$x^2+y^2+z^2=a^2.$$

1955. z is a function of the variables x and y defined by the equation

$$2x^2+2y^2+z^2-8xz-z+8=0.$$

Find dz and d^2z for the values $x=2$, $y=0$, $z=1$.

1956. Find dz and d^2z, if $\ln z=x+y+z-1$. What are the first- and second-order derivatives of the function z?

1957. Let the function z be defined by the equation

$$x^2+y^2+z^2=\varphi(ax+by+cz),$$

where φ is an arbitrary differentiable function and a, b, c are constants. Show that

$$(cy-bz)\frac{\partial z}{\partial x}+(az-cx)\frac{\partial z}{\partial y}=bx-ay.$$

1958. Show that the function z defined by the equation

$$F(x-az,\ y-bz)=0,$$

where F is an arbitrary differentiable function of two arguments, satisfies the equation

$$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=1.$$

1959. $F\left(\frac{x}{z}, \frac{y}{z}\right)=0$. Show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$.

1960. Show that the function z defined by the equation $y=x\varphi(z)+\psi(z)$ satisfies the equation

$$\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial z}{\partial y}\right)^2-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial^2 z}{\partial y^2}\left(\frac{\partial z}{\partial x}\right)^2=0.$$

1961. The functions y and z of the independent variable x are defined by a system of equations $x^2+y^2-z^2=0$, $x^2+2y^2+3z^2=4$. Find $\frac{dy}{dx}$, $\frac{dz}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^2z}{dx^2}$ for $x=1$, $y=0$, $z=1$.

1962. The functions y and z of the independent variable x are defined by the following system of equations:

$$xyz=a, \quad x+y+z=b.$$

Find dy, dz, d^2y, d^2z.

1963. The functions u and v of the independent variables x and y are defined implicitly by the system of equations

$$u=x+y, \quad uv=y.$$

Calculate

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x\,\partial y}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial x\,\partial y}, \frac{\partial^2 v}{\partial y^2}$$

for $x=0$, $y=1$.

1964. The functions u and v of the independent variables x and y are defined implicitly by the system of equations

$$u+v=x, \quad u-yv=0.$$

Find du, dv, d^2u, d^2v.

1965. The functions u and v of the variables x and y are defined implicitly by the system of equations

$$x=\varphi(u, v), \quad y=\psi(u, v).$$

Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$.

1966. a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if $x=u\cos v$, $y=u\sin v$, $z=cv$.

b) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if $x=u+v$, $y=u-v$, $z=uv$.

c) Find dz, if $x=e^{u+v}$, $y=e^{u-v}$, $z=uv$.

1967. $z=F(r, \varphi)$ where r and φ are functions of the variables x and y defined by the system of equations

$$x=r\cos\varphi, \quad y=r\sin\varphi.$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

1968. Regarding z as a function of x and y, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if

$$x=a\cos\varphi\cos\psi, \quad y=b\sin\varphi\cos\psi, \quad z=c\sin\psi.$$

Sec. 10. Change of Variables

When changing variables in differential expressions, the derivatives in them should be expressed in terms of other derivatives by the rules of differentiation of a composite function.

1°. Change of variables in expressions containing ordinary derivatives.

Example 1. Transform the equation

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\frac{a^2}{x^2}y=0$$

putting $x=\frac{1}{t}$.

Solution. Express the derivatives of y with respect to x in terms of the derivatives of y with respect to t. We have

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{dy}{dt}}{-\frac{1}{t^2}}=-t^2\frac{dy}{dt},$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=-\left(2t\frac{dy}{dt}+t^2\frac{d^2y}{dt^2}\right)(-t^2)=2t^3\frac{dy}{dt}+t^4\frac{d^2y}{dt^2}.$$

Substituting the expressions of the derivatives just found into the given equation and replacing x by $\frac{1}{t}$, we get

$$\frac{1}{t^2}\cdot t^3\left(2\frac{dy}{dt}+t\frac{d^2y}{dt^2}\right)+2\cdot\frac{1}{t}\left(-t^2\frac{dy}{dt}\right)+a^2t^2y=0$$

or

$$\frac{d^2y}{dt^2}+a^2y=0.$$

Example 2. Transform the equation

$$x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3-\frac{dy}{dx}=0,$$

taking y for the argument and x for the function.

Solution. Express the derivatives of y with respect to x in terms of the derivatives of x with respect to y.

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}};$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{\frac{dx}{dy}}\right)=\frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right)\frac{dy}{dx}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}\cdot\frac{1}{\frac{dx}{dy}}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}.$$

Substituting these expressions of the derivatives into the given equation, we will have

$$x\left[-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}\right]+\frac{1}{\left(\frac{dx}{dy}\right)^3}-\frac{1}{\frac{dx}{dy}}=0,$$

or, finally,

$$x\frac{d^2x}{dy^2}-1+\left(\frac{dx}{dy}\right)^2=0.$$

Example 3. Transform the equation

$$\frac{dy}{dx}=\frac{x+y}{x-y},$$

by passing to the polar coordinates

$$x=r\cos\varphi,\quad y=r\sin\varphi. \tag{1}$$

Solution. Considering r as a function of φ, from formula (1) we have

$$dx=\cos\varphi\,dr-r\sin\varphi\,d\varphi,\quad dy=\sin\varphi\,dr+r\cos\varphi\,d\varphi,$$

whence

$$\frac{dy}{dx}=\frac{\sin\varphi\,dr+r\cos\varphi\,d\varphi}{\cos\varphi\,dr-r\sin\varphi\,d\varphi}=\frac{\sin\varphi\frac{dr}{d\varphi}+r\cos\varphi}{\cos\varphi\frac{dr}{d\varphi}-r\sin\varphi}.$$

Putting into the given equation the expressions for x, y, and $\frac{dy}{dx}$, we will have

$$\frac{\sin\varphi\frac{dr}{d\varphi}+r\cos\varphi}{\cos\varphi\frac{dr}{d\varphi}-r\sin\varphi}=\frac{r\cos\varphi+r\sin\varphi}{r\cos\varphi-r\sin\varphi},$$

or, after simplifications,

$$\frac{dr}{d\varphi}=r.$$

2° Change of variables in expressions containing partial derivatives.
Example 4. Take the equation of oscillations of a string

$$\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2}\qquad(a\neq 0)$$

and change it to the new independent variables α and β, where $\alpha=x-at$, $\beta=x+at$.

Solution. Let us express the partial derivatives of u with respect to x and t in terms of the partial derivatives of u with respect to α and β. Applying the formulas for differentiating a composite function

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial\alpha}\frac{\partial\alpha}{\partial t}+\frac{\partial u}{\partial\beta}\frac{\partial\beta}{\partial t},\quad \frac{\partial u}{\partial x}=\frac{\partial u}{\partial\alpha}\frac{\partial\alpha}{\partial x}+\frac{\partial u}{\partial\beta}\frac{\partial\beta}{\partial x},$$

we get

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial\alpha}(-a)+\frac{\partial u}{\partial\beta}a=a\left(\frac{\partial u}{\partial\beta}-\frac{\partial u}{\partial\alpha}\right),$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial\alpha}\cdot 1+\frac{\partial u}{\partial\beta}\cdot 1=\frac{\partial u}{\partial\alpha}+\frac{\partial u}{\partial\beta}.$$

Differentiate again using the same formulas:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)=\frac{\partial}{\partial \alpha}\left(\frac{\partial u}{\partial t}\right)\frac{\partial \alpha}{\partial t}+\frac{\partial}{\partial \beta}\left(\frac{\partial u}{\partial t}\right)\frac{\partial \beta}{\partial t}=$$

$$=a\left(\frac{\partial^2 u}{\partial \alpha\,\partial \beta}-\frac{\partial^2 u}{\partial \alpha^2}\right)(-a)+a\left(\frac{\partial^2 u}{\partial \beta^2}-\frac{\partial^2 u}{\partial \alpha\,\partial \beta}\right)a=$$

$$=a^2\left(\frac{\partial^2 u}{\partial \alpha^2}-2\frac{\partial^2 u}{\partial \alpha\,\partial \beta}+\frac{\partial^2 u}{\partial \beta^2}\right);$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial \alpha}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \alpha}{\partial x}+\frac{\partial}{\partial \beta}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \beta}{\partial x}=$$

$$=\left(\frac{\partial^2 u}{\partial \alpha^2}+\frac{\partial^2 u}{\partial \alpha\,\partial \beta}\right)\cdot 1+\left(\frac{\partial^2 u}{\partial \alpha\,\partial \beta}+\frac{\partial^2 u}{\partial \beta^2}\right)\cdot 1=$$

$$=\frac{\partial^2 u}{\partial \alpha^2}+2\frac{\partial^2 u}{\partial \alpha\,\partial \beta}+\frac{\partial^2 u}{\partial \beta^2}.$$

Substituting into the given equation, we will have

$$a^2\left(\frac{\partial^2 u}{\partial \alpha^2}-2\frac{\partial^2 u}{\partial \alpha\,\partial \beta}+\frac{\partial^2 u}{\partial \beta^2}\right)=a^2\left(\frac{\partial^2 u}{\partial \alpha^2}+2\frac{\partial^2 u}{\partial \alpha\,\partial \beta}+\frac{\partial^2 u}{\partial \beta^2}\right)$$

or

$$\frac{\partial^2 u}{\partial \alpha\,\partial \beta}=0.$$

Example 5. Transform the equation $x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=z^2$, taking $u=x$, $v=\frac{1}{y}-\frac{1}{x}$ for the new independent variables, and $w=\frac{1}{z}-\frac{1}{x}$ for the new function.

Solution. Let us express the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of the partial derivatives $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$. To do this, differentiate the given relationships between the old and new variables:

$$du=dx,\quad dv=\frac{dx}{x^2}-\frac{dy}{y^2},\quad dw=\frac{dx}{x^2}-\frac{dz}{z^2}.$$

On the other hand,

$$dw=\frac{\partial w}{\partial u}du+\frac{\partial w}{\partial v}dv.$$

Therefore,

$$\frac{\partial w}{\partial u}du+\frac{\partial w}{\partial v}dv=\frac{dx}{x^2}-\frac{dz}{z^2}$$

or

$$\frac{\partial w}{\partial u}dx+\frac{\partial w}{\partial v}\left(\frac{dx}{x^2}-\frac{dy}{y^2}\right)=\frac{dx}{x^2}-\frac{dz}{z^2}.$$

Whence

$$dz=z^2\left(\frac{1}{x^2}-\frac{\partial w}{\partial u}-\frac{1}{x^2}\frac{\partial w}{\partial v}\right)dx+\frac{z^2}{y^2}\frac{\partial w}{\partial v}dy$$

and, consequently,

$$\frac{\partial z}{\partial x}=z^2\left(\frac{1}{x^2}-\frac{\partial w}{\partial u}-\frac{1}{x^2}\frac{\partial w}{\partial v}\right)$$

and

$$\frac{\partial z}{\partial y}=\frac{z^2}{y^2}\frac{\partial w}{\partial v}.$$

Substituting these expressions into the given equation, we get

$$x^2z^2\left(\frac{1}{x^2}-\frac{\partial w}{\partial u}-\frac{1}{x^2}\frac{\partial w}{\partial v}\right)+z^2\frac{\partial w}{\partial v}=z^2$$

or

$$\frac{\partial w}{\partial u}=0.$$

1969. Transform the equation

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+y=0,$$

putting $x=e^t$.

1970. Transform the equation

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0,$$

putting $x=\cos t$.

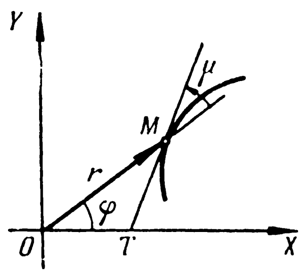

Fig 69

1971 Transform the following equations, taking y as the argument:

a) $\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)^2=0$,

b) $\frac{dy}{dx}\frac{d^3y}{dx^3}-3\left(\frac{d^2y}{dx^2}\right)^2=0$.

1972. The tangent of the angle μ formed by the tangent line MT and the radius vector OM of the point of tangency (Fig. 69) is expressed as follows:

$$\tan\mu=\frac{y'-\frac{y}{x}}{1+\frac{y}{x}y'}.$$

Transform this expression by passing to polar coordinates: $x=r\cos\varphi$, $y=r\sin\varphi$.

1973. Express, in the polar coordinates $x=r\cos\varphi$, $y=r\sin\varphi$, the formula of the curvature of the curve

$$K=\frac{y''}{[1+(y')^2]^{3/2}}.$$

1974. Transform the following equation to new independent variables u and v:

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0,$$

if $u=x$, $v=x^2+y^2$.

1975. Transform the following equation to new independent variables u and v:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}-z=0,$$

if $u=x$, $v=\frac{y}{x}$.

1976. Transform the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

to the polar coordinates

$$x=r\cos\varphi,\quad y=r\sin\varphi.$$

1977. Transform the equation

$$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=0,$$

putting $u=xy$ and $v=\frac{x}{y}$.

1978. Transform the equation

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=(y-x)\,z,$$

by introducing new independent variables

$$u=x^2+y^2,\quad v=\frac{1}{x}+\frac{1}{y}$$

and the new function $w=\ln z-(x+y)$.

1979. Transform the equation

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}=0,$$

taking $u=x+y$, $v=\frac{y}{x}$ for the new independent variables and $w=\frac{z}{x}$ for the new function.

1980. Transform the equation

$$\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}=0,$$

putting $u=x+y$, $v=x-y$, $w=xy-z$, where $w=w(u, v)$.

Sec. 11. The Tangent Plane and the Normal to a Surface

1°. The equations of a tangent plane and a normal for the case of explicit representation of a surface. The *tangent plane* to a surface at a point M (point of tangency) is a plane in which lie all the tangents at the point M to various curves drawn on the surface through this point.

The *normal* to the surface is the perpendicular to the tangent plane at the point of tangency

If the equation of a surface, in a rectangular coordinate system, is given in explicit form, $z=f(x, y)$, where $f(x, y)$ is a differentiable function, then the equation of the tangent plane at the point $M(x_0, y_0, z_0)$ of the surface is

$$Z-z_0=f'_x(x_0, y_0)(X-x_0)+f'_y(x_0, y_0)(Y-y_0). \qquad (1)$$

Here, $z_0=f(x_0, y_0)$ and X, Y, Z are the current coordinates of the point of the tangent plane.

The equations of the normal are of the form

$$\frac{X-x_0}{f'_x(x_0, y_0)}=\frac{Y-y_0}{f'_y(x_0, y_0)}=\frac{Z-z_0}{-1}, \qquad (2)$$

where X, Y, Z are the current coordinates of the point of the normal.

Example 1. Write the equations of the tangent plane and the normal to the surface $z=\frac{x^2}{2}-y^2$ at the point $M(2, -1, 1)$.

Solution. Let us find the partial derivatives of the given function and their values at the point M

$$\frac{\partial z}{\partial x}=x, \qquad \left(\frac{\partial z}{\partial x}\right)_M=2,$$

$$\frac{\partial z}{\partial y}=-2y, \qquad \left(\frac{\partial z}{\partial y}\right)_M=2.$$

Whence, applying formulas (1) and (2), we will have $z-1=2(x-2)+2(y+1)$ or $2x+2y-z-1=0$ which is the equation of the tangent plane and $\frac{x-2}{2}=\frac{y+1}{2}=\frac{z-1}{-1}$, which is the equation of the normal.

2°. Equations of the tangent plane and the normal for the case of implicit representation of a surface. When the equation of a surface is represented implicitly,

$$F(x, y, z)=0,$$

and $F(x_0, y_0, z_0)=0$, the corresponding equations will have the form

$$F'_x(x_0, y_0, z_0)(X-x_0)+F'_y(x_0, y_0, z_0)(Y-y_0)+F'_z(x_0, y_0, z_0)(Z-z_0)=0 \qquad (3)$$

which is the equation of the tangent plane, and

$$\frac{X-x_0}{F'_x(x_0, y_0, z_0)}=\frac{Y-y_0}{F'_y(x_0, y_0, z_0)}=\frac{Z-z_0}{F'_z(x_0, y_0, z_0)} \qquad (4)$$

which are the equations of the normal.

Example 2. Write the equations of the tangent plane and the normal to the surface $3xyz-z^3=a^3$ at a point for which $x=0$, $y=a$.

Solution. Find the z-coordinate of the point of tangency, putting $x=0$, $y=a$ into the equation of the surface: $-z^3=a^3$, whence $z=-a$. Thus, the point of tangency is $M(0, a, -a)$.

Denoting by $F(x, y, z)$ the left-hand side of the equation, we find the partial derivatives and their values at the point M:

$$\begin{aligned} F'_x&=3yz, & (F'_x)_M&=-3a^2,\\ F'_y&=3xz, & (F'_y)_M&=0,\\ F'_z&=3xy-3z^2, & (F'_z)_M&=-3a^2. \end{aligned}$$

Applying formulas (3) and (4), we get

$$-3a^2(x-0)+0(y-a)-3a^2(z+a)=0$$

or $x+z+a=0$, which is the equation of the tangent plane,

$$\frac{x-0}{-3a^2}=\frac{y-a}{0}=\frac{z+a}{-3a^2}$$

or $\frac{x}{1}=\frac{y-a}{0}=\frac{z+a}{1}$, which are the equations of the normal.

1981. Write the equation of the tangent plane and the equations of the normal to the following surfaces at the indicated points:

a) to the paraboloid of revolution $z=x^2+y^2$ at the point $(1, -2, 5)$;

b) to the cone $\frac{x^2}{16}+\frac{y^2}{9}-\frac{z^2}{8}=0$ at the point $(4, 3, 4)$;

c) to the sphere $x^2+y^2+z^2=2Rz$ at the point $(R\cos\alpha, R\sin\alpha, R)$.

1982. At what point of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

does the normal to it form equal angles with the coordinate axes?

1983. Planes perpendicular to the x- and y-axes are drawn through the point $M(3, 4, 12)$ of the sphere $x^2+y^2+z^2=169$. Write the equation of the plane passing through the tangents to the obtained sections at their common point M.

1984. Show that the equation of the tangent plane to the central surface (of order two)

$$ax^2+by^2+cz^2=k$$

at the point $M(x_0, y_0, z_0)$ has the form

$$ax_0x + by_0y + cz_0z = k.$$

1985. Draw to the surface $x^2 + 2y^2 + 3z^2 = 21$ tangent planes parallel to the plane $x + 4y + 6z = 0$.

1986. Draw to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ a tangent plane which cuts off equal segments on the coordinate axes.

1987. On the surface $x^2 + y^2 - z^2 - 2x = 0$ find points at which the tangent planes are parallel to the coordinate planes.

1988. Prove that the tangent planes to the surface $xyz = m^3$ form a tetrahedron of constant volume with the planes of the coordinates.

1989. Show that the tangent planes to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ cut off, on the coordinate axes, segments whose sum is constant.

1990. Show that the cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ and the sphere

$$x^2 + y^2 + \left(z - \frac{b^2 + c^2}{c}\right)^2 = \frac{b^2}{c^2}(b^2 + c^2)$$

are tangent at the points $(0, \pm b, c)$.

1991. The angle between the tangent planes drawn to given surfaces at a point under consideration is called the *angle between two surfaces* at the point of their intersection.

At what angle does the cylinder $x^2 + y^2 = R^2$ and the sphere $(x - R)^2 + y^2 + z^2 = R^2$ intersect at the point $M\left(\frac{R}{2}, \frac{R\sqrt{3}}{2}, 0\right)$?

1992. Surfaces are called *orthogonal* if they intersect at right angles at each point of the line of their intersection.

Show that the surfaces $x^2 + y^2 + z^2 = r^2$ (sphere), $y = x \tan \varphi$ (plane), and $z^2 = (x^2 + y^2)\tan^2 \psi$ (cone), which are the coordinate surfaces of the spherical coordinates r, φ, ψ, are mutually orthogonal.

1993. Show that all the planes tangent to the conical surface $z = xf\left(\frac{y}{x}\right)$ at the point $M(x_0, y_0, z_0)$, where $x_0 \neq 0$, pass through the coordinate origin.

1994*. Find the projections of the ellipsoid

$$x^2 + y^2 + z^2 - xy - 1 = 0$$

on the coordinate planes.

1995. Prove that the normal at any point of the surface of revolution $z = f(\sqrt{x^2 + y^2})$ $(f' \neq 0)$ intersect the axis of rotation.

Sec. 12. Taylor's Formula for a Function of Several Variables

Let a function $f(x, y)$ have continuous partial derivatives of all orders up to the $(n+1)$ th inclusive in the neighbourhood of a point (a, b). Then *Taylor's formula* will hold in the neighbourhood under consideration:

$$f(x, y)=f(a, b)+\frac{1}{1!}[f'_x(a, b)(x-a)+f'_y(a, b)(y-b)]+$$
$$+\frac{1}{2!}[f''_{xx}(a, b)(x-a)^2+2f''_{xy}(a, b)(x-a)(y-b)+f''_{yy}(a, b)(y-b)^2]+\ldots$$
$$\ldots+\frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^n f(a, b)+R_n(x, y), \quad (1)$$

where

$$R_n(x, y)=\frac{1}{(n+1)!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^{n+1} f[a+\theta(x-a), b+\theta(y-b)]$$
$$(0<\theta<1).$$

In other notation,

$$f(x+h, y+k)=f(x, y)+\frac{1}{1!}[hf'_x(x, y)+kf'_y(x, y)]+\frac{1}{2!}[h^2f''_{xx}(x, y)+$$
$$+2hkf''_{xy}(x, y)+k^2f''_{yy}(x, y)]+\ldots+\frac{1}{n!}\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]^n f(x, y)+$$
$$+\frac{1}{(n+1)!}\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]^{n+1} f(x+\theta h; y+\theta k), \quad (2)$$

or

$$\Delta f(x, y)=\frac{1}{1!}df(x, y)+\frac{1}{2!}d^2f(x, y)+\ldots$$
$$\ldots+\frac{1}{n!}d^nf(x, y)+\frac{1}{(n+1)!}d^{n+1}f(x+\theta h; y+\theta k) \quad (3)$$

The particular case of formula (1), when $a=b=0$, is called *Maclaurin's formula*.

Similar formulas hold for functions of three and a larger number of variables.

Example. Find the increment obtained by the function $f(x, y)=x^3-2y^3+3xy$ when passing from the values $x=1$, $y=2$ to the values $x_1=1+h$, $y_1=2+k$.

Solution. The desired increment may be found by applying formula (2). First calculate the successive partial derivatives and their values at the given point (1, 2):

$$f'_x(x, y)=3x^2+3y, \qquad f'_x(1, 2)=3\cdot1+3\cdot2=9,$$
$$f'_y(x, y)=-6y^2+3x, \qquad f'_y(1, 2)=-6\cdot4+3\cdot1=-21,$$
$$f''_{xx}(x, y)=6x, \qquad f''_{xx}(1, 2)=6\cdot1=6,$$
$$f''_{xy}(x, y)=3, \qquad f''_{xy}(1, 2)=3,$$
$$f''_{yy}(x, y)=-12y, \qquad f''_{yy}(1, 2)=-12\cdot2=-24,$$
$$f'''_{xxx}(x, y)=6, \qquad f'''_{xxx}(1, 2)=6,$$
$$f'''_{xxy}(x, y)=0, \qquad f'''_{xxy}(1, 2)=0,$$
$$f'''_{xyy}(x, y)=0, \qquad f'''_{xyy}(1, 2)=0,$$
$$f'''_{yyy}(x, y)=-12, \qquad f'''_{yyy}(1, 2)=-12.$$

All subsequent derivatives are identically zero. Putting these results into formula (2), we obtain:

$$\Delta f(x, y) = f(1+h, 2+k) - f(1, 2) = \frac{1}{1!}[h\cdot 9 + k(-21)] +$$

$$+\frac{1}{2!}[h^2\cdot 6 + 2hk\cdot 3 + k^2(-24)] + \frac{1}{3!}[h^3\cdot 6 + 3h^2k\cdot 0 + 3hk^2\cdot 0 + k^3(-12)] =$$

$$= 9h - 21k + 3h^2 + 3hk - 12k^2 + h^3 - 2k^3.$$

1996. Expand $f(x+h, y+k)$ in a series of positive integral powers of h and k if

$$f(x, y) = ax^2 + 2bxy + cy^2.$$

1997. Expand the function $f(x, y) = -x^2 + 2xy + 3y^2 - 6x - 2y - 4$ by Taylor's formula in the neighbourhood of the point $(-2, 1)$.

1998. Find the increment received by the function $f(x, y) = x^2y$ when passing from the values $x = 1$, $y = 1$ to

$$x_1 = 1 + h, \quad y_1 = 1 + k.$$

1999. Expand the function $f(x, y, z) = x^2 + y^2 + z^2 + 2xy - yz - 4x - 3y - z + 4$ by Taylor's formula in the neighbourhood of the point $(1, 1, 1)$.

2000. Expand $f(x+h, y+k, z+l)$ in a series of positive integral powers of h, k, and l, if

$$f(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz.$$

2001. Expand the following function in a Maclaurin's series up to terms of the third order inclusive:

$$f(x, y) = e^x \sin y.$$

2002. Expand the following function in a Maclaurin's series up to terms of order four inclusive:

$$f(x, y) = \cos x \cos y.$$

2003. Expand the following function in a Taylor's series in the neighbourhood of the point $(1, 1)$ up to terms of order two inclusive:

$$f(x, y) = y^x.$$

2004. Expand the following function in a Taylor's series in the neighbourhood of the point $(1, -1)$ up to terms of order three inclusive:

$$f(x, y) = e^{x+y}.$$

2005. Derive approximate formulas (accurate to second-order terms in α and β) for the expressions

a) $\arctan \frac{1+\alpha}{1-\beta}$; b) $\sqrt{\frac{(1+\alpha)^m+(1+\beta)^n}{2}}$,

if $|\alpha|$ and $|\beta|$ are small compared with unity.

2006*. Using Taylor's formulas up to second-order terms, approximate

a) $\sqrt{1.03}$; $\sqrt[3]{0.98}$; b) $(0.95)^{2.01}$.

2007. z is an implicit function of x and y defined by the equation $z^3-2xz+y=0$, which takes on the value $z=1$ for $x=1$ and $y=1$. Write several terms of the expansion of the function z in increasing powers of the differences $x-1$ and $y-1$.

Sec. 13. The Extremum of a Function of Several Variables

1°. **Definition of an extremum of a function.** We say that a function $f(x, y)$ has a *maximum* (*minimum*) $f(a, b)$ at the point $P(a, b)$, if for all points $P'(x, y)$ different from P in a sufficiently small neighbourhood of P the inequality $f(a, b) > f(x, y)$ [or, accordingly, $f(a, b) < f(x, y)$] is fulfilled. The generic term for maximum and minimum of a function is *extremum*. In similar fashion we define the extremum of a function of three or more variables.

2°. **Necessary conditions for an extremum.** The points at which a differentiable function $f(x, y)$ may attain an extremum (so-called *stationary points*) are found by solving the following system of equations:

$$f'_x(x, y)=0, \quad f'_y(x, y)=0 \tag{1}$$

(*necessary conditions* for an extremum). System (1) is equivalent to a single equation, $df(x, y)=0$. In the general case, at the point of the extremum $P(a, b)$, the function $f(x, y)$, or $df(a, b)=0$, or $df(a, b)$ does not exist.

3°. **Sufficient conditions for an extremum.** Let $P(a, b)$ be a stationary point of the function $f(x, y)$, that is, $df(a, b)=0$. Then: a) if $d^2f(a, b)<0$ for $dx^2+dy^2>0$, then $f(a, b)$ is the *maximum* of the function $f(x, y)$; b) if $d^2f(a, b)>0$ for $dx^2+dy^2>0$, then $f(a, b)$ is the *minimum* of the function $f(x, y)$; c) if $d^2f(a, b)$ changes sign, then $f(a, b)$ is not an extremum of $f(x, y)$.

The foregoing conditions are equivalent to the following: let $f'_x(a, b)=f'_y(a, b)=0$ and $A=f''_{xx}(a, b)$, $B=f''_{xy}(a, b)$, $C=f''_{yy}(a, b)$. We form the *discriminant*

$$\Delta=AC-B^2.$$

Then: 1) if $\Delta>0$, then the function has an extremum at the point $P(a, b)$, namely a maximum, if $A<0$ (or $C<0$), and a minimum, if $A>0$ (or $C>0$); 2) if $\Delta<0$, then there is no extremum at $P(a, b)$; 3) if $\Delta=0$, then the question of an extremum of the function at $P(a, b)$ remains open (which is to say, it requires further investigation).

4°. **The case of a function of many variables.** For a function of three or more variables, the necessary conditions for the existence of an extremum are similar to conditions (1), while the sufficient conditions are analogous to the conditions a), b), and c) 3°.

Example 1. Test the following function for an extremum:

$$z = x^3 + 3xy^2 - 15x - 12y.$$

Solution. Find the partial derivatives and form a system of equations (1):

$$\frac{\partial z}{\partial x} = 3x^2 + 3y^2 - 15 = 0; \quad \frac{\partial z}{\partial y} = 6xy - 12 = 0$$

or

$$\begin{cases} x^2 + y^2 - 5 = 0, \\ xy - 2 = 0. \end{cases}$$

Solving the system we get four stationary points:

$$P_1(1, 2); \quad P_2(2, 1); \quad P_3(-1, -2); \quad P_4(-2, -1).$$

Let us find the second derivatives

$$\frac{\partial^2 z}{\partial x^2} = 6x, \quad \frac{\partial^2 z}{\partial x\, \partial y} = 6y, \quad \frac{\partial^2 z}{\partial y^2} = 6x$$

and form the discriminant $\Delta = AC - B^2$ for each stationary point.

1) For the point P_1: $A = \left(\frac{\partial^2 z}{\partial x^2}\right)_{P_1} = 6$, $B = \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)_{P_1} = 12$, $C = \left(\frac{\partial^2 z}{\partial y^2}\right)_{P_1} = 6$, $\Delta = AC - B^2 = 36 - 144 < 0$. Thus, there is no extremum at the point P_1.

2) For the point P_2: $A = 12$, $B = 6$, $C = 12$; $\Delta = 144 - 36 > 0$, $A > 0$. At P_2 the function has a minimum. This minimum is equal to the value of the function for $x = 2$, $y = 1$:

$$z_{\min} = 8 + 6 - 30 - 12 = -28.$$

3) For the point P_3: $A = -6$, $B = -12$, $C = -6$; $\Delta = 36 - 144 < 0$. There is no extremum.

4) For the point P_4: $A = -12$, $B = -6$, $C = -12$; $\Delta = 144 - 36 > 0$, $A < 0$. At the point P_4 the function has a maximum equal to $z_{\max} = -8 - 6 + 30 + 12 = 28$

5°. Conditional extremum. In the simplest case, the *conditional extremum* of a function $f(x, y)$ is a maximum or minimum of this function which is attained on the condition that its arguments are related by the equation $\varphi(x, y) = 0$ (*coupling equation*). To find the conditional extremum of a function $f(x, y)$, given the relationship $\varphi(x, y) = 0$ we form the so-called *Lagrange function*

$$F(x, y) = f(x, y) + \lambda \cdot \varphi(x, y),$$

where λ is an undetermined multiplier, and we seek the ordinary extremum of this auxiliary function. The necessary conditions for the extremum reduce to a system of three equations:

$$\begin{cases} \frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0, \\ \frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0, \\ \varphi(x, y) = 0 \end{cases} \tag{2}$$

with three unknowns x, y, λ, from which it is, generally speaking, possible to determine these unknowns.

The question of the existence and character of a conditional extremum is solved on the basis of a study of the sign of the second differential of the Lagrange function:

$$d^2F(x, y) = \frac{\partial^2 F}{\partial x^2}dx^2 + 2\frac{\partial^2 F}{\partial x\,\partial y}dx\,dy + \frac{\partial^2 F}{\partial y^2}dy^2$$

for the given system of values of x, y, λ obtained from (2) or the condition that dx and dy are related by the equation

$$\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = 0 \quad (dx^2 + dy^2 \neq 0).$$

Namely, the function $f(x, y)$ has a conditional maximum, if $d^2F < 0$, and a conditional minimum, if $d^2F > 0$. As a particular case, if the discriminant Δ of the function $F(x, y)$ at a stationary point is positive, then at this point there is a conditional maximum of the function $f(x, y)$, if $A < 0$ (or $C < 0$), and a conditional minimum, if $A > 0$ (or $C > 0$)

In similar fashion we find the conditional extremum of a function of three or more variables provided there is one or several coupling equations (the number of which, however, must be less than the number of the variables) Here, we have to introduce into the Lagrange function as many undetermined multipliers factors as there are coupling equations.

Example 2. Find the extremum of the function

$$z = 6 - 4x - 3y$$

provided the variables x and y satisfy the equation

$$x^2 + y^2 = 1$$

Solution. Geometrically, the problem reduces to finding the greatest and least values of the z-coordinate of the plane $z = 6 - 4x - 3y$ for points of its intersection with the cylinder $x^2 + y^2 = 1$

We form the Lagrange function

$$F(x, y) = 6 - 4x - 3y + \lambda(x^2 + y^2 - 1).$$

We have $\frac{\partial F}{\partial x} = -4 + 2\lambda x$, $\frac{\partial F}{\partial y} = -3 + 2\lambda y$. The necessary conditions yield the following system of equations:

$$\begin{cases} -4 + 2\lambda x = 0, \\ -3 + 2\lambda y = 0, \\ x^2 + y^2 = 1. \end{cases}$$

Solving this system we find

$$\lambda_1 = \frac{5}{2}, \quad x_1 = \frac{4}{5}, \quad y_1 = \frac{3}{5},$$

and

$$\lambda_2 = -\frac{5}{2}, \quad x_2 = -\frac{4}{5}, \quad y_2 = -\frac{3}{5}.$$

Since

$$\frac{\partial^2 F}{\partial x^2} = 2\lambda, \quad \frac{\partial^2 F}{\partial x\,\partial y} = 0, \quad \frac{\partial^2 F}{\partial y^2} = 2\lambda,$$

it follows that

$$d^2F = 2\lambda(dx^2 + dy^2).$$

If $\lambda=\frac{5}{2}$, $x=\frac{4}{5}$ and $y=\frac{3}{5}$, then $d^2F>0$, and, consequently, the function has a conditional minimum at this point. If $\lambda=-\frac{5}{2}$, $x=-\frac{4}{5}$ and $y=-\frac{3}{5}$, then $d^2F<0$, and, consequently, the function at this point has a conditional maximum.

Thus,

$$z_{max}=6+\frac{16}{5}+\frac{9}{5}=11,$$
$$z_{min}=6-\frac{16}{5}-\frac{9}{5}=1.$$

6°. **Greatest and smallest values of a function.** A function that is differentiable in a limited closed region attains its greatest (smallest) value either at a stationary point or at a point of the boundary of the region.

Example 3. Determine the greatest and smallest values of the function

$$z=x^2+y^2-xy+x+y$$

in the region

$$x\leqslant 0,\ y\leqslant 0,\ x+y\geqslant -3$$

Solution. The indicated region is a triangle (Fig. 70).

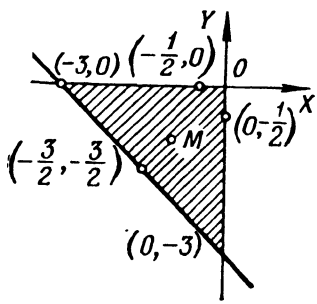

Fig. 70

1) Let us find the stationary points:

$$\begin{cases} z'_x\equiv 2x-y+1=0, \\ z'_y\equiv 2y-x+1=0; \end{cases}$$

whence $x=-1$, $y=-1$; and we get the point $M(-1, -1)$

At M the value of the function $z_M=-1$ It is not absolutely necessary to test for an extremum

2) Let us investigate the function on the boundaries of the region.

When $x=0$ we have $z=y^2+y$, and the problem reduces to seeking the greatest and smallest values of this function of one argument on the interval $-3\leqslant y\leqslant 0$. Investigating, we find that $(z_{gr})_{x=0}=6$ at the point $(0, -3)$; $(z_{sm})_{x=0}=-\frac{1}{4}$ at the point $(0, -^1/_2)$

When $y=0$ we get $z=x^2+x$. Similarly, we find that $(z_{gr})_{y=0}=6$ at the point $(-3, 0)$; $(z_{sm})_{y=0}=-\frac{1}{4}$ at the point $(-^1/_2, 0)$

When $x+y=-3$ or $y=-3-x$ we will have $z=3x^2+9x+6$. Similarly we find that $(z_{sm})_{x+y=-3}=-\frac{3}{4}$ at the point $\left(-\frac{3}{2}, -\frac{3}{2}\right)$; $(z_{gr})_{x+y=-3}=6$ metres coincides with $(z_{gr})_{x=0}$ and $(z_{gr})_{y=0}$. On the straight line $x+y=-3$ we could test the function for a conditional extremum without reducing to a function of one argument.

3) Correlating all the values obtained of the function z, we conclude that $z_{gr}=6$ at the points $(0, -3)$ and $(-3, 0)$; $z_{sm}=-1$ at the stationary point M.

Test for maximum and minimum the following functions of two variables:

2008. $z=(x-1)^2+2y^2$.

2009. $z=(x-1)^2-2y^2$.

2010. $z=x^2+xy+y^2-2x-y$.

2011. $z=x^3y^2(6-x-y)$ $(x>0,\ y>0)$.

2012. $z=x^4+y^4-2x^2+4xy-2y^2$.

2013. $z=xy\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$.

2014. $z=1-(x^2+y^2)^{2/3}$.

2015. $z=(x^2+y^2)e^{-(x^2+y^2)}$.

2016. $z=\frac{1+x-y}{\sqrt{1+x^2+y^2}}$.

Find the extrema of the following functions of three variables:

2017. $u=x^2+y^2+z^2-xy+x-2z$.

2018. $u=x+\frac{y^2}{4x}+\frac{z^2}{y}+\frac{2}{z}$ $(x>0,\ y>0,\ z>0)$.

Find the extrema of the following implicitly represented functions:

2019*. $x^2+y^2+z^2-2x+4y-6z-11=0$.

2020. $x^3-y^2-3x+4y+z^2+z-8=0$.

Determine the conditional extrema of the following functions:

2021. $z=xy$ for $x+y=1$.

2022. $z=x+2y$ for $x^2+y^2=5$.

2023. $z=x^2+y^2$ for $\frac{x}{2}+\frac{y}{3}=1$.

2024. $z=\cos^2 x+\cos^2 y$ for $y-x=\frac{\pi}{4}$.

2025. $u=x-2y+2z$ for $x^2+y^2+z^2=9$.

2026. $u=x^2+y^2+z^2$ for $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ $(a>b>c>0)$.

2027. $u=xy^2z^3$ for $x+y+z=12$ $(x>0, y>0, z>0)$.

2028. $u=xyz$ provided $x+y+z=5$, $xy+yz+zx=8$.

2029. Prove the inequality

$$\frac{x+y+z}{3}\geqslant\sqrt[3]{xyz},$$

if $x\geqslant 0$, $y\geqslant 0$, $z\geqslant 0$.

Hint: Seek the maximum of the function $u=xyz$ provided $x+y+z=S$.

2030. Determine the greatest value of the function $z=1+x+2y$ in the regions: a) $x\geqslant 0$, $y\geqslant 0$, $x+y\leqslant 1$; b) $x\geqslant 0$, $y\leqslant 0$, $x-y\leqslant 1$.

2031. Determine the greatest and smallest values of the functions a) $z=x^2y$ and b) $z=x^2-y^2$ in the region $x^2+y^2\leqslant 1$.

2032. Determine the greatest and smallest values of the function $z=\sin x+\sin y+\sin(x+y)$ in the region $0\leqslant x\leqslant\frac{\pi}{2}$, $0\leqslant y\leqslant\frac{\pi}{2}$.

2033. Determine the greatest and smallest values of the function $z=x^3+y^3-3xy$ in the region $0\leqslant x\leqslant 2$, $-1\leqslant y\leqslant 2$.

Sec. 14. Finding the Greatest and Smallest Values of Functions

Example 1. It is required to break up a positive number a into three nonnegative numbers so that their product should be the greatest possible.

Solution. Let the desired numbers be x, y, $a-x-y$. We seek the maximum of the function $f(x, y)=xy(a-x-y)$.

According to the problem, the function $f(x, y)$ is considered inside a closed triangle $x\geqslant 0$, $y\geqslant 0$, $x+y\leqslant a$ (Fig. 71).

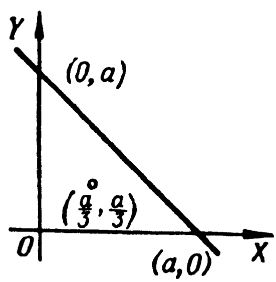

Fig. 71

Solving the system of equations

$$\begin{cases} f'_x(x, y)\equiv y(a-2x-y)=0, \\ f'_y(x, y)\equiv x(a-x-2y)=0, \end{cases}$$

we will have the unique stationary point $\left(\frac{a}{3}, \frac{a}{3}\right)$ for the interior of the triangle. Let us test the sufficiency conditions. We have

$$f''_{xx}(x, y)=-2y,\ f''_{xy}(x, y)=a-2x-2y,\ f''_{yy}(x, y)=-2x.$$

Consequently,

$$A=f''_{xx}\left(\frac{a}{3},\ \frac{a}{3}\right)=-\frac{2}{3}a,$$
$$B=f''_{xy}\left(\frac{a}{3},\ \frac{a}{3}\right)=-\frac{1}{3}a,$$
$$C=f''_{yy}\left(\frac{a}{3},\ \frac{a}{3}\right)=-\frac{2}{3}a \text{ and}$$
$$\Delta=AC-B^2>0,\ A<0.$$

And so at $\left(\frac{a}{3},\ \frac{a}{3}\right)$ the function reaches a maximum. Since $f(x,\ y)=0$ on the contour of the triangle, this maximum will be the greatest value, which is to say that the product will be greatest, if $x=y=a-x-y=\frac{a}{3}$, and the greatest value is equal to $\frac{a^3}{27}$.

Note The ploblem can also be solved by the methods of a conditional extremum, by seeking the maximum of the function $u=xyz$ on the condition that $x+y+z=a$.

2034. From among all rectangular parallelepipeds with a given volume V, find the one whose total surface is the least.

2035. For what dimensions does an open rectangular bathtub of a given capacity V have the smallest surface?

2036. Of all triangles of a given perimeter $2p$, find the one that has the greatest area.

2037. Find a rectangular parallelepiped of a given surface S with greatest volume.

2038. Represent a positive number a in the form of a product of four positive factors which have the least possible sum.

2039. Find a point $M(x,\ y)$, on an xy-plane, the sum of the squares of the distances of which from three straight lines ($x=0,\ y=0,\ x-y+1=0$) is the least possible.

2040. Find a triangle of a given perimeter $2p$, which, upon being revolved about one of its sides, generates a solid of greatest volume.

2041. Given in a plane are three material points $P_1(x_1,\ y_1)$, $P_2(x_2,\ y_2)$, $P_3(x_3,\ y_3)$ with masses m_1, m_2, m_3. For what position of the point $P(x,\ y)$ will the quadratic moment (the moment of inertia) of the given system of points relative to the point P (i.e., the sum $m_1P_1P^2+m_2P_2P^2+m_3P_3P^2$) be the least?

2042. Draw a plane through the point $M(a,\ b,\ c)$ to form a tetrahedron of least volume with the planes of the coordinates.

2043. Inscribe in an ellipsoid a rectangular parallelepiped of greatest volume.

2044. Determine the outer dimensions of an open box with a given wall thickness δ and capacity (internal) V so that the smallest quantity of material is used to make it.

2045. At what point of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

does the tangent line to it form with the coordinate axes a triangle of smallest area?

2046*. Find the axes of the ellipse

$$5x^2+8xy+5y^2=9.$$

2047. Inscribe in a given sphere a cylinder having the greatest total surface.

2048. The beds of two rivers (in a certain region) approximately represent a parabola $y=x^2$ and a straight line $x-y-2=0$. It is required to connect these rivers by a straight canal of least length. Through what points will it pass?

2049. Find the shortest distance from the point $M(1, 2, 3)$ to the straight line

$$\frac{x}{1}=\frac{y}{-3}=\frac{z}{2}.$$

2050*. The points A and B are situated in different optical media separated by a straight line (Fig. 72). The velocity of

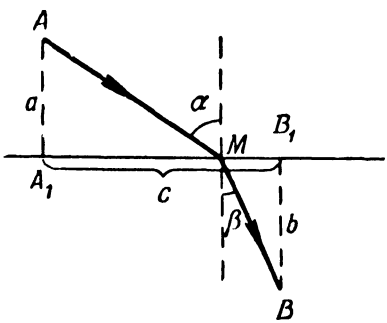

Fig. 72

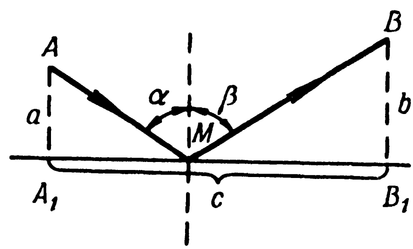

Fig. 73

light in the first medium is v_1, in the second, v_2. Applying the Fermat principle, according to which a light ray is propagated along a line AMB which requires the least time to cover, derive the law of refraction of light rays.

2051. Using the Fermat principle, derive the law of reflection of a light ray from a plane in a homogeneous medium (Fig. 73).

2052*. If a current I flows in an electric circuit containing a resistance R, then the quantity of heat released in unit time is proportional to I^2R. Determine how to divide the current I into currents I_1, I_2, I_3 by means of three wires, whose resistances are R_1, R_2, R_3, so that the generation of heat would be the least possible?

Sec. 15. Singular Points of Plane Curves

1°. Definition of a singular point. A point $M(x_0, y_0)$ of a plane curve $f(x, y)=0$ is called a *singular point* if its coordinates satisfy three equations at once:

$$f(x_0, y_0)=0, \quad f'_x(x_0, y_0)=0, \quad f'_y(x_0, y_0)=0.$$

2°. Basic types of singular points. At a singular point $M(x_0, y_0)$, let the second derivatives

$$A=f''_{xx}(x_0, y_0),$$
$$B=f''_{xy}(x_0, y_0),$$
$$C=f''_{yy}(x_0, y_0)$$

be not all equal to zero and

$$\Delta=AC-B^2,$$

then:

a) if $\Delta>0$, then M is an *isolated point* (Fig. 74);
b) if $\Delta<0$, then M is a *node (double point)* (Fig. 75);
c) if $\Delta=0$, then M is either a *cusp* of the first kind (Fig. 76) or of the second kind (Fig. 77), or an *isolated point*, or a *tacnode* (Fig. 78).


Fig. 74

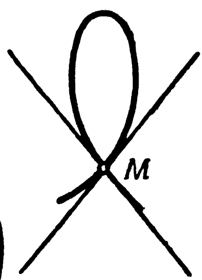

Fig. 75

When solving the problems of this section it is always necessary to draw the curves.

Example 1. Show that the curve $y^2=ax^2+x^3$ has a node if $a>0$; an isolated point if $a<0$; a cusp of the first kind if $a=0$.

Solution. Here, $f(x, y)\equiv ax^2+x^3-y^2$. Let us find the partial derivatives and equate them to zero:

$$f'_x(x, y)=2ax+3x^2=0,$$
$$f'_y(x, y)=-2y=0.$$

This system has two solutions: $O(0,\ 0)$ and $N\left(-\frac{2}{3}a,\ 0\right)$; but the coordinates of the point N do not satisfy the equation of the given curve. Hence, there is a unique singular point $O\ (0,\ 0)$.

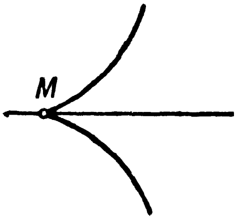

Fig. 76

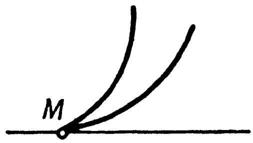

Fig. 77

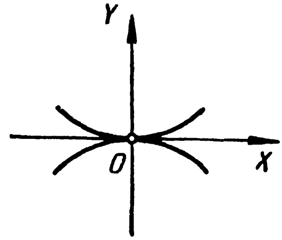

Fig. 78

Let us find the second derivatives and their values at the point O:

$$f''_{xx}(x,\ y)=2a+6x,\quad A=2a,$$
$$f''_{xy}(x,\ y)=0,\quad B=0,$$
$$f''_{yy}(x,\ y)=-2,\quad C=-2,$$
$$\Delta=AC-B^2=-4a.$$

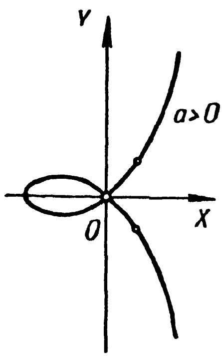

Fig. 79

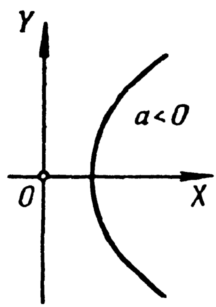

Fig. 80

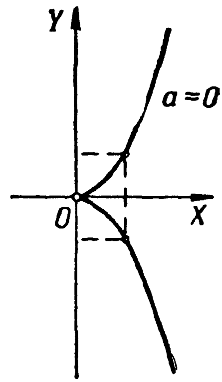

Fig. 81

Hence,

if $a>0$, then $\Delta<0$ and the point O is a node (Fig. 79);
if $a<0$, then $\Delta>0$ and O is an isolated point (Fig. 80);
if $a=0$, then $\Delta=0$. The equation of the curve in this case will be $y^2=x^3$ or $y=\pm\sqrt{x^3}$; $y=$exists only when $x\geqslant 0$; the curve is symmetric about the x-axis, which is a tangent. Hence, the point M is a cusp of the first kind (Fig. 81).

Determine the character of the singular points of the following curves:

2053. $y^2 = -x^2 + x^4$.

2054. $(y-x^2)^2 = x^5$.

2055. $a^4y^2 = a^2x^4 - x^6$.

2056. $x^2y^2 - x^2 - y^2 = 0$.

2057. $x^3 + y^3 - 3axy = 0$ *(folium of Descartes)*.

2058. $y^2(a-x) = x^3$ *(cissoid)*.

2059. $(x^2+y^2)^2 = a^2(x^2-y^2)$ *(lemniscate)*.

2060. $(a+x)y^2 = (a-x)x^2$ *(strophoid)*.

2061. $(x^2+y^2)(x-a)^2 = b^2x^2$ $(a>0,\ b>0)$ *(conchoid)*.

Consider three cases:

1) $a>b$, 2) $a=b$, 3) $a<b$.

2062. Determine the change in character of the singular point of the curve $y^2 = (x-a)(x-b)(x-c)$ depending on the values of a, b, c ($a \leqslant b \leqslant c$ are real).

Sec. 16. Envelope

1°. Definition of an envelope. The *envelope of a family of plane curves* is a curve (or a set of several curves) which is tangent to all lines of the given family, and at each point is tangent to some line of the given family.

2°. Equations of an envelope. If a family of curves

$$f(x, y, \alpha) = 0$$

dependent on a single variable parameter α has an envelope, then the parametric equations of the latter are found from the system of equations

$$\begin{cases} f(x, y, \alpha) = 0, \\ f'_\alpha(x, y, \alpha) = 0. \end{cases} \quad (1)$$

Eliminating the parameter α from the system (1), we get an equation of the form

$$D(x, y) = 0. \quad (2)$$

It should be pointed out that the formally obtained curve (2) (the so-called *"discriminant curve"*) may contain, in addition to an envelope (if there is one), a locus of singular points of the given family, which locus is not part of the envelope of this family.

When solving the problems of this section it is advisable to make drawings.

Example. Find the envelope of the family of curves

$$x\cos\alpha + y\sin\alpha - p = 0\ (p = \text{const},\ p > 0).$$

Solution. The given family of curves depends on the parameter α. Form the system of equations (1):

$$\begin{cases} x\cos\alpha + y\sin\alpha - p = 0, \\ -x\sin\alpha + y\cos\alpha = 0. \end{cases}$$

Solving the system for x and y, we obtain parametric equations of the envelope

$$x = p\cos\alpha, \quad y = p\sin\alpha.$$

Squaring both equations and adding, we eliminate the parameter α:

$$x^2 + y^2 = p^2.$$

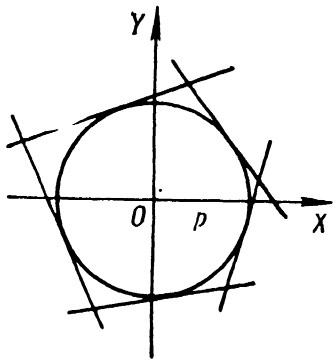

Fig. 82

Thus, the envelope of this family of straight lines is a circle of radius p with centre at the origin. This particular family of straight lines is a family of tangent lines to this circle (Fig. 82).

2063. Find the envelope of the family of circles

$$(x-a)^2 + y^2 = \frac{a^2}{2}.$$

2064. Find the envelope of the family of straight lines

$$y = kx + \frac{p}{2k}$$

(k is a variable parameter).

2065. Find the envelope of a family of circles of the same radius R whose centres lie on the x-axis.

2066. Find a curve which forms an envelope of a section of length l when its end-points slide along the coordinate axes.

2067. Find the envelope of a family of straight lines that form with the coordinate axes a triangle of constant area S.

2068. Find the envelope of ellipses of constant area S whose axes of symmetry coincide.

2069. Investigate the character of the "discriminant curves" of families of the following lines (C is a constant parameter):

a) cubic parabolas $y=(x-C)^3$;
b) semicubical parabolas $y^2=(x-C)^3$;
c) Neile parabolas $y^3=(x-C)^2$;
d) strophoids $(a+x)(y-C)^2=x^2(a-x)$.

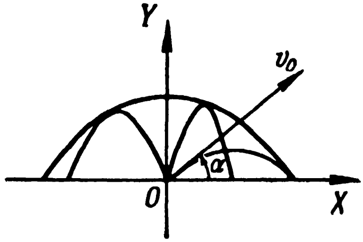

Fig. 83

2070. The equation of the trajectory of a shell fired from a point O with initial velocity v_0 at an angle α to the horizon (air resistance disregarded) is

$$y=x\tan\alpha-\frac{gx^2}{2v_0^2\cos^2\alpha}.$$

Taking the angle α as the parameter, find the envelope of all trajectories of the shell located in one and the same vertical plane ("*safety parabola*") (Fig. 83).

Sec. 17. Arc Length of a Space Curve

The *differential of an arc* of a space curve in rectangular Cartesian coordinates is equal to

$$ds=\sqrt{dx^2+dy^2+dz^2},$$

where x, y, z are the current coordinates of a point of the curve.

If

$$x=x(t),\quad y=y(t),\quad z=z(t)$$

are parametric equations of the space curve, then the arc length of a section of it from $t=t_1$ to $t=t_2$ is

$$s=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt.$$

In Problems 2071-2076 find the arc length of the curve:

2071. $x=t,\ y=t^2,\ z=\frac{2t^3}{3}$ from $t=0$ to $t=2$.

2072. $x=2\cos t,\ y=2\sin t,\ z=\frac{3}{\pi}t$ from $t=0$ to $t=\pi$.

2073. $x=e^t\cos t,\ y=e^t\sin t,\ z=e^t$ from $t=0$ to arbitrary t.

2074. $y=\frac{x^2}{2},\ z=\frac{x^3}{6}$ from $x=0$ to $x=6$.

2075. $x^2=3y,\ 2xy=9z$ from the point $O(0, 0, 0)$ to $M(3, 3, 2)$.

2076. $y=a\arcsin\frac{x}{a},\ z=\frac{a}{4}\ln\frac{a+x}{a-x}$ from the point $O(0, 0, 0)$ to the point $M(x_0, y_0, z_0)$.

2077. The position of a point for any time $t\ (t>0)$ is defined by the equations

$$x=2t,\quad y=\ln t,\quad z=t^2.$$

Find the mean velocity of motion between times $t=1$ and $t=10$.

Sec. 18. The Vector Function of a Scalar Argument

1°. The derivative of the vector function of a scalar argument. The *vector function* $\boldsymbol{a}=\boldsymbol{a}(t)$ may be defined by specifying three scalar functions $a_x(t)$, $a_y(t)$ and $a_z(t)$, which are its projections on the coordinate axes:

$$\boldsymbol{a}=a_x(t)\,\boldsymbol{i}+a_y(t)\,\boldsymbol{j}+a_z(t)\,\boldsymbol{k}.$$

The derivative of the vector function $\boldsymbol{a}=\boldsymbol{a}(t)$ with respect to the scalar argument t is a new vector function defined by the equality

$$\frac{d\boldsymbol{a}}{dt}=\lim_{\Delta t\to 0}\frac{\boldsymbol{a}(t+\Delta t)-\boldsymbol{a}(t)}{\Delta t}=\frac{da_x(t)}{dt}\boldsymbol{i}+\frac{da_y(t)}{dt}\boldsymbol{j}+\frac{da_z(t)}{dt}\boldsymbol{k}.$$

The modulus of the derivative of the vector function is

$$\left|\frac{d\boldsymbol{a}}{dt}\right|=\sqrt{\left(\frac{da_x}{dt}\right)^2+\left(\frac{da_y}{dt}\right)^2+\left(\frac{da_z}{dt}\right)^2}.$$

The end-point of the variable of the radius vector $\boldsymbol{r}=\boldsymbol{r}(t)$ describes in space the curve

$$\boldsymbol{r}=x(t)\,\boldsymbol{i}+y(t)\,\boldsymbol{j}+z(t)\,\boldsymbol{k},$$

which is called the *hodograph* of the vector $\boldsymbol{r}$.

The derivative $\frac{d\boldsymbol{r}}{dt}$ is a vector, tangent to the hodograph at the corresponding point; here,

$$\left|\frac{d\boldsymbol{r}}{dt}\right|=\frac{ds}{dt},$$

where s is the arc length of the hodograph reckoned from some initial point. For example, $\left|\frac{d\boldsymbol{r}}{ds}\right|=1$.

If the parameter t is the time, then $\frac{d\mathbf{r}}{dt}=\mathbf{v}$ is the *velocity vector* of the extremity of the vector $\mathbf{r}$, and $\frac{d^2\mathbf{r}}{dt^2}=\frac{d\mathbf{v}}{dt}=\mathbf{w}$ is the *acceleration vector* of the extremity of the vector $\mathbf{r}$.

2°. Basic rules for differentiating the vector function of a scalar argument.

1) $\frac{d}{dt}(\mathbf{a}+\mathbf{b}-\mathbf{c})=\frac{d\mathbf{a}}{dt}+\frac{d\mathbf{b}}{dt}-\frac{d\mathbf{c}}{dt}$;

2) $\frac{d}{dt}(m\mathbf{a})=m\frac{d\mathbf{a}}{dt}$, where m is a constant scalar;

3) $\frac{d}{dt}(\varphi\mathbf{a})=\frac{d\varphi}{dt}\mathbf{a}+\varphi\frac{d\mathbf{a}}{dt}$, where $\varphi(t)$ is a scalar function of t;

4) $\frac{d}{dt}(\mathbf{a}\mathbf{b})=\frac{d\mathbf{a}}{dt}\mathbf{b}+\mathbf{a}\frac{d\mathbf{b}}{dt}$;

5) $\frac{d}{dt}(\mathbf{a}\times\mathbf{b})=\frac{d\mathbf{a}}{dt}\times\mathbf{b}+\mathbf{a}\times\frac{d\mathbf{b}}{dt}$;

6) $\frac{d}{dt}\mathbf{a}[\varphi(t)]=\frac{d\mathbf{a}}{d\varphi}\cdot\frac{d\varphi}{dt}$;

7) $\mathbf{a}\frac{d\mathbf{a}}{dt}=0$, if $|\mathbf{a}|=\text{const}$.

Example 1. The radius vector of a moving point is at any instant of time defined by the equation

$$\mathbf{r}=\mathbf{i}-4t^2\mathbf{j}+3t^2\mathbf{k}. \tag{1}$$

Determine the trajectory of motion, the velocity and acceleration.

Solution. From (1) we have:

$$x=1, \quad y=-4t^2, \quad z=3t^2.$$

Eliminating the time t, we find that the trajectory of motion is a straight line:

$$\frac{x-1}{0}=\frac{y}{-4}=\frac{z}{3}.$$

From equation (1), differentiating, we find the velocity

$$\frac{d\mathbf{r}}{dt}=-8t\mathbf{j}+6t\mathbf{k}$$

and the acceleration

$$\frac{d^2\mathbf{r}}{dt^2}=-8\mathbf{j}+6\mathbf{k}.$$

The magnitude of the velocity is

$$\left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{(-8t)^2+(6t)^2}=10\,|t|.$$

We note that the acceleration is constant and is

$$\left|\frac{d^2\mathbf{r}}{dt^2}\right|=\sqrt{(-8)^2+6^2}=10.$$

2078. Show that the vector equation $\boldsymbol{r}-\boldsymbol{r}_1=(\boldsymbol{r}_2-\boldsymbol{r}_1)t$, where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are radius vectors of two given points, is the equation of a straight line.

2079. Determine which lines are hodographs of the following vector functions:

a) $\boldsymbol{r}=\boldsymbol{a}t+\boldsymbol{c}$; c) $\boldsymbol{r}=\boldsymbol{a}\cos t+\boldsymbol{b}\sin t$;

b) $\boldsymbol{r}=\boldsymbol{a}t^2+\boldsymbol{b}t$; d) $\boldsymbol{r}=\boldsymbol{a}\cosh t+\boldsymbol{b}\sinh t$,

where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are constant vectors; the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other.

2080. Find the derivative vector-function of the function $\boldsymbol{a}(t)=a(t)\boldsymbol{a}^\circ(t)$, where $a(t)$ is a scalar function, while $\boldsymbol{a}^\circ(t)$ is a unit vector, for cases when the vector $\boldsymbol{a}(t)$ varies: 1) in length only, 2) in direction only, 3) in length and in direction (general case). Interpret geometrically the results obtained.

2081. Using the rules of differentiating a vector function with respect to a scalar argument, derive a formula for differentiating a mixed product of three vector functions $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

2082. Find the derivative, with respect to the parameter t, of the volume of a parallelepiped constructed on three vectors:

$$\boldsymbol{a}=\boldsymbol{i}+t\boldsymbol{j}+t^2\boldsymbol{k};$$
$$\boldsymbol{b}=2t\boldsymbol{i}-\boldsymbol{j}+t^3\boldsymbol{k};$$
$$\boldsymbol{c}=-t^2\boldsymbol{i}+t^3\boldsymbol{j}+\boldsymbol{k}.$$

2083. The equation of motion is

$$\boldsymbol{r}=3\boldsymbol{i}\cos t+4\boldsymbol{j}\sin t,$$

where t is the time. Determine the trajectory of motion, the velocity and the acceleration. Construct the trajectory of motion and the vectors of velocity and acceleration for times, $t=0$, $t=\frac{\pi}{4}$ and $t=\frac{\pi}{2}$.

2084. The equation of motion is

$$\boldsymbol{r}=2\boldsymbol{i}\cos t+2\boldsymbol{j}\sin t+3\boldsymbol{k}t.$$

Determine the trajectory of motion, the velocity and the acceleration. What are the magnitudes of velocity and acceleration and what directions have they for time $t=0$ and $t=\frac{\pi}{2}$?

2085. The equation of motion is

$$\boldsymbol{r}=\boldsymbol{i}\cos\alpha\cos\omega t+\boldsymbol{j}\sin\alpha\cos\omega t+\boldsymbol{k}\sin\omega t,$$

where α and ω are constants and t is the time. Determine the trajectory of motion and the magnitudes and directions of the velocity and the acceleration.

2086. The equation of motion of a shell (neglecting air resistance) is

$$\mathbf{r} = \mathbf{v}_0 t - \frac{gt^2}{2}\mathbf{k},$$

where $\mathbf{v}_0 \{v_{ox}, v_{oy}, v_{oz}\}$ is the initial velocity. Find the velocity and the acceleration at any instant of time.

2087. Prove that if a point is in motion along the parabola $y = \frac{x^2}{a}$, $z = 0$ in such a manner that the projection of velocity on the x-axis remains constant $\left(\frac{dx}{dt} = \text{const}\right)$, then the acceleration remains constant as well.

2088. A point lying on the thread of a screw being screwed into a beam describes the spiral

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = h\theta,$$

where θ is the turning angle of the screw, a is the radius of the screw, and h is the height of rise in a rotation of one radian. Determine the velocity of the point.

2089. Find the velocity of a point on the circumference of a wheel of radius a rotating with constant angular velocity ω so that its centre moves in a straight line with constant velocity v_0.

Sec. 19. The Natural Trihedron of a Space Curve

At any nonsingular point $M(x, y, z)$ of a space curve $\mathbf{r} = \mathbf{r}(t)$ it is possible to construct a *natural trihedron* consisting of three mutually perpendicular planes (Fig. 84):

1) *osculating* plane MM_1M_2, containing the vectors $\frac{d\mathbf{r}}{dt}$ and $\frac{d^2\mathbf{r}}{dt^2}$;

2) *normal* plane MM_2M_3, which is perpendicular to the vector $\frac{d\mathbf{r}}{dt}$ and

3) *rectifying* plane MM_1M_3, which is perpendicular to the first two planes.

At the intersection we obtain three straight lines;

1) the *tangent* MM_1; 2) the *principal normal* MM_2; 3) the *binormal* MM_3, all of which are defined by the appropriate vectors:

1) $\mathbf{T} = \frac{d\mathbf{r}}{dt}$ (the *vector of the tangent line*);

2) $\mathbf{B} = \frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}$ (the *vector of the binormal*);

3) $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ (the *vector of the principal normal*);

The corresponding unit vectors

$$\boldsymbol{\tau} = \frac{\mathbf{T}}{|\mathbf{T}|}; \quad \boldsymbol{\beta} = \frac{\mathbf{B}}{|\mathbf{B}|}; \quad \boldsymbol{\nu} = \frac{\mathbf{N}}{|\mathbf{N}|}$$

may be computed from the formulas

$$\boldsymbol{\tau}=\frac{d\boldsymbol{r}}{ds};\quad \boldsymbol{\nu}=\frac{\frac{d\boldsymbol{\tau}}{ds}}{\left|\frac{d\boldsymbol{\tau}}{ds}\right|};\quad \boldsymbol{\beta}=\boldsymbol{\tau}\times\boldsymbol{\nu}.$$

If X, Y, Z are the current coordinates of the point of the tangent, then the equations of the tangent have the form

$$\frac{X-x}{T_x}=\frac{Y-y}{T_y}=\frac{Z-z}{T_z}, \tag{1}$$

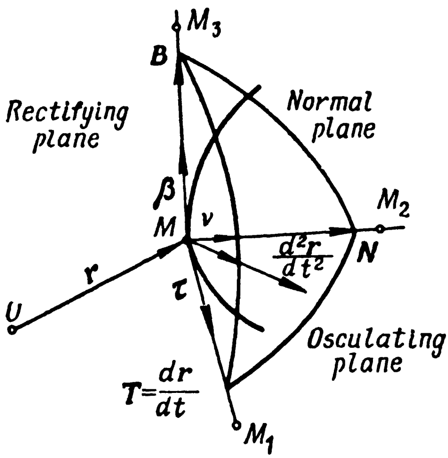

Fig. 84

where $T_x=\frac{dx}{dt}$; $T_y=\frac{dy}{dt}$, $T_z=\frac{dz}{dt}$; from the condition of perpendicularity of the line and the plane we get an equation of the normal plane:

$$T_x(X-x)+T_y(Y-y)+T_z(Z-z)=0. \tag{2}$$

If in equations (1) and (2), we replace T_x, T_y, T_z by B_x, B_y, B_z and N_x, N_y, N_z, we get the equations of the binormal and the principal normal and, respectively, the osculating plane and the rectifying plane.

Example 1. Find the basic unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ of the curve

$$x=t,\quad y=t^2,\quad z=t^3$$

at the point $t=1$.

Write the equations of the tangent, the principal normal and the binormal at this point.

Solution. We have

$$\boldsymbol{r}=t\boldsymbol{i}+t^2\boldsymbol{j}+t^3\boldsymbol{k}$$

and

$$\frac{d\boldsymbol{r}}{dt}=\boldsymbol{i}+2t\boldsymbol{j}+3t^2\boldsymbol{k},$$

$$\frac{d^2\boldsymbol{r}}{dt^2}=2\boldsymbol{j}+6t\boldsymbol{k}.$$

Whence, when $t=1$, we get

$$T=\frac{dr}{dt}=i+2j+3k;$$

$$B=\frac{dr}{dt}\times\frac{d^2r}{dt^2}=\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix}=6i-6j+2k;$$

$$N=B\times T=\begin{vmatrix} i & j & k \\ 6 & -6 & 2 \\ 1 & 2 & 3 \end{vmatrix}=-22i-16j+18k.$$

Consequently,

$$\tau=\frac{i+2j+3k}{\sqrt{14}}, \quad \beta=\frac{3i-3j+k}{\sqrt{19}}, \quad \nu=\frac{-11i-8j+9k}{\sqrt{266}}.$$

Since for $t=1$ we have $x=1$, $y=1$, $z=1$, it follows that

$$\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}$$

are the equations of the tangent,

$$\frac{x-1}{3}=\frac{y-1}{-3}=\frac{z-1}{1}$$

are the equations of the binormal and

$$\frac{x-1}{-11}=\frac{y-1}{-8}=\frac{z-1}{9}$$

are the equations of the principal normal.

If a space curve is represented as an intersection of two surfaces

$$F(x, y, z)=0, \quad G(x, y, z)=0,$$

then in place of the vectors $\frac{dr}{dt}$ and $\frac{d^2r}{dt^2}$ we can take the vectors $dr\{dx, dy, dz\}$ and $d^2r\{d^2x, d^2y, d^2z\}$; and one of the variables x, y, z may be considered independent and we can put its second differential equal to zero.

Example 2. Write the equation of the osculating plane of the circle

$$x^2+y^2+z^2=6, \quad x+y+z=0 \tag{3}$$

at its point $M(1, 1, -2)$.

Solution. Differentiating the system (3) and considering x an independent variable, we will have

$$x\,dx+y\,dy+z\,dz=0,$$
$$dx+dy+dz=0$$

and

$$dx^2+dy^2+y\,d^2y+dz^2+z\,d^2z=0,$$
$$d^2y+d^2z=0.$$

Putting $x=1$, $y=1$, $z=-2$, we get

$$dy=-dx; \quad dz=0;$$
$$d^2y=-\frac{2}{3}\,dx^2; \quad d^2z=\frac{2}{3}\,dx^2.$$

Hence, the osculating plane is defined by the vectors

$$\{dx,\ -dx,\ 0\} \quad \text{and} \quad \left\{0,\ -\frac{2}{3}dx^2,\ \frac{2}{3}dx^2\right\}$$

or

$$\{1,\ -1,\ 0\} \quad \text{and} \quad \{0,\ -1,\ 1\}.$$

Whence the normal vector of the osculating plane is

$$\boldsymbol{B}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix}=-\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$$

and, therefore, its equation is

$$-1(x-1)-(y-1)-(z+2)=0,$$

that is,

$$x+y+z=0,$$

as it should be, since our curve is located in this plane.

2090. Find the basic unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ of the curve

$$x=1-\cos t, \quad y=\sin t, \quad z=t$$

at the point $t=\frac{\pi}{2}$.

2091. Find the unit vectors of the tangent and the principal normal of the conic spiral

$$\boldsymbol{r}=e^t(\boldsymbol{i}\cos t+\boldsymbol{j}\sin t+\boldsymbol{k})$$

at an arbitrary point. Determine the angles that these lines make with the z-axis.

2092. Find the basic unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ of the curve

$$y=x^2, \quad z=2x$$

at the point $x=2$.

2093. For the screw line

$$x=a\cos t, \quad y=a\sin t, \quad z=bt$$

write the equations of the straight lines that form a natural trihedron at an arbitrary point of the line. Determine the direction cosines of the tangent line and the principal normal.

2094. Write the equations of the planes that form the natural trihedron of the curve

$$x^2+y^2+z^2=6, \quad x^2-y^2+z^2=4$$

at one of its points $M(1,\ 1,\ 2)$.

2095. Form the equations of the tangent line, the normal plane and the osculating plane of the curve $x=t$, $y=t^2$, $z=t^3$ at the point $M(2,\ 4,\ 8)$.

2096. Form the equations of the tangent, principal normal, and binormal at an arbitrary point of the curve

$$x=\frac{t^4}{4},\quad y=\frac{t^3}{3},\quad z=\frac{t^2}{2}.$$

Find the points at which the tangent to this curve is parallel to the plane $x+3y+2z-10=0$.

2097. Form equations of the tangent, the osculating plane, the principal normal and the binormal of the curve

$$x=t,\quad y=-t,\quad z=\frac{t^2}{2}$$

at the point $t=2$. Compute the direction cosines of the binormal at this point.

2098. Write the equations of the tangent and the normal plane to the following curves:

a) $x=R\cos^2 t$, $y=R\sin t\cos t$, $z=R\sin t$ for $t=\frac{\pi}{4}$;

b) $z=x^2+y^2$, $x=y$ at the point $(1, 1, 2)$;

c) $x^2+y^2+z^2=25$, $x+z=5$ at the point $(2, 2\sqrt{3}, 3)$.

2099 Find the equation of the normal plane to the curve $z=x^2-y^2$, $y=x$ at the coordinate origin.

2100. Find the equation of the osculating plane to the curve $x=e^t$, $y=e^{-t}$, $z=t\sqrt{2}$ at the point $t=0$.

2101. Find the equations of the osculating plane to the curves:

a) $x^2+y^2+z^2=9$, $x^2-y^2=3$ at the point $(2, 1, 2)$;

b) $x^2=4y$, $x^3=24z$ at the point $(6, 9, 9)$;

c) $x^2+z^2=a^2$, $y^2+z^2=b^2$ at any point of the curve (x_0, y_0, z_0).

2102. Form the equations of the osculating plane, the principal normal and the binormal to the curve

$$y^2=x,\quad x^2=z \text{ at the point } (1, 1, 1).$$

2103. Form the equations of the osculating plane, the principal normal and the binormal to the conical screw-line $x=t\cos t$, $y=t\sin t$, $z=bt$ at the origin. Find the unit vectors of the tangent, the principal normal, and the binormal at the origin.

Sec. 20. Curvature and Torsion of a Space Curve

1°. Curvature. By the *curvature* of a curve at a point M we mean the number

$$K=\frac{1}{R}=\lim_{\Delta s\to 0}\frac{\varphi}{\Delta s},$$

where φ is the angle of turn of the tangent line (*angle of contingence*) on a segment of the curve $\widehat{MN}$, Δs is the arc length of this segment of the curve. R is called the *radius of curvature*. If a curve is defined by the equation $\boldsymbol{r}=\boldsymbol{r}(s)$, where s is the arc length, then

$$\frac{1}{R}=\left|\frac{d^2\boldsymbol{r}}{ds^2}\right|.$$

For the case of a general parametric representation of the curve we have

$$\frac{1}{R}=\frac{\left|\frac{d\boldsymbol{r}}{dt}\times\frac{d^2\boldsymbol{r}}{dt^2}\right|}{\left|\frac{d\boldsymbol{r}}{dt}\right|^3}. \tag{1}$$

2°. Torsion. By *torsion* (*second curvature*) of a curve at a point M we mean the number

$$T=\frac{1}{\varrho}=\lim_{\Delta s\to 0}\frac{\theta}{\Delta s},$$

where θ is the angle of turn of the binormal (*angle of contingence of the second kind*) on the segment of the curve $\widehat{MN}$. The quantity ϱ is called the *radius of torsion* or the *radius of second curvature*. If $\boldsymbol{r}=\boldsymbol{r}(s)$, then

$$\frac{1}{\varrho}=\pm\left|\frac{d\boldsymbol{\beta}}{ds}\right|=\frac{\frac{d\boldsymbol{r}}{ds}\frac{d^2\boldsymbol{r}}{ds^2}\frac{d^3\boldsymbol{r}}{ds^3}}{\left(\frac{d^2\boldsymbol{r}}{ds^2}\right)^2},$$

where the minus sign is taken when the vectors $\frac{d\boldsymbol{\beta}}{ds}$ and $\boldsymbol{\nu}$ have the same direction, and the plus sign, when not the same.

If $\boldsymbol{r}=\boldsymbol{r}(t)$, where t is an arbitrary parameter, then

$$\frac{1}{\varrho}=\frac{\frac{d\boldsymbol{r}}{dt}\frac{d^2\boldsymbol{r}}{dt^2}\frac{d^3\boldsymbol{r}}{dt^3}}{\left(\frac{d\boldsymbol{r}}{dt}\times\frac{d^2\boldsymbol{r}}{dt^2}\right)^2}. \tag{2}$$

Example 1. Find the curvature and the torsion of the screw-line

$$\boldsymbol{r}=\boldsymbol{i}\,a\cos t+\boldsymbol{j}\,a\sin t+\boldsymbol{k}\,bt \qquad (a>0).$$

Solution. We have

$$\frac{d\boldsymbol{r}}{dt}=-\boldsymbol{i}\,a\sin t+\boldsymbol{j}\,a\cos t+\boldsymbol{k}b,$$

$$\frac{d^2\boldsymbol{r}}{dt^2}=-\boldsymbol{i}\,a\cos t-\boldsymbol{j}\,a\sin t,$$

$$\frac{d^3\boldsymbol{r}}{dt^3}=-\boldsymbol{i}\,a\sin t-\boldsymbol{j}\,a\cos t.$$

Whence

$$\frac{d\boldsymbol{r}}{dt}\times\frac{d^2\boldsymbol{r}}{dt^2}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ -a\sin t & a\cos t & b\\ -a\cos t & -a\sin t & 0\end{vmatrix}=\boldsymbol{i}\,ab\sin t-\boldsymbol{j}\,ab\cos t+a^2\boldsymbol{k}$$

and

$$\frac{dr}{dt}\frac{d^2r}{dt^2}\frac{d^3r}{dt^3}=\begin{vmatrix}-a\sin t & a\cos t & b\\ -a\cos t & -a\sin t & 0\\ a\sin t & -a\cos t & 0\end{vmatrix}=a^2b.$$

Hence, on the basis of formulas (1) and (2), we get

$$\frac{1}{R}=\frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}}=\frac{a}{a^2+b^2}$$

and

$$\frac{1}{\varrho}=\frac{a^2b}{a^2(a^2+b^2)}=\frac{b}{a^2+b^2}.$$

Thus, for a screw-line, the curvature and torsion are constants.

3° **Frenet formulas:**

$$\frac{d\tau}{ds}=\frac{\nu}{R},\quad \frac{d\nu}{ds}=-\frac{\tau}{R}+\frac{\beta}{\varrho},\quad \frac{d\beta}{ds}=-\frac{\nu}{\varrho}.$$

2104. Prove that if the curvature at all points of a line is zero, then the line is a straight line.

2105. Prove that if the torsion at all points of a curve is zero, then the curve is a plane curve.

2106. Prove that the curve

$$x=1+3t+2t^2,\ y=2-2t+5t^2,\ z=1-t^2$$

is a plane curve; find the plane in which it lies.

2107. Compute the curvature of the following curves:

a) $x=\cos t$, $y=\sin t$, $z=\cosh t$ at the point $t=0$;

b) $x^2-y^2+z^2=1$, $y^2-2x+z=0$ at the point (1, 1, 1).

2108. Compute the curvature and torsion at any point of the curves:

a) $x=e^t\cos t$, $y=e^t\sin t$, $z=e^t$;

b) $x=a\cosh t$, $y=a\sinh t$, $z=at$ (hyperbolic screw-line).

2109. Find the radii of curvature and torsion at an arbitrary point (x, y, z) of the curves:

a) $x^2=2ay$, $x^3=6a^2z$;

b) $x^3=3p^2y$, $2xz=p^2$.

2110. Prove that the tangential and normal components of acceleration $\boldsymbol{w}$ are expressed by the formulas

$$\boldsymbol{w}_\tau=\frac{dv}{dt}\boldsymbol{\tau},\qquad \boldsymbol{w}_\nu=\frac{v^2}{R}\boldsymbol{\nu},$$

where v is the velocity, R is the radius of curvature of the trajectory, $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are unit vectors of the tangent and principal normal to the curve.

2111. A point is in uniform motion along a screw-line $r = ia\cos t + ja\sin t + btk$ with velocity v. Compute its acceleration w.

2112. The equation of motion is

$$\boldsymbol{r} = t\boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k}.$$

Determine, at times $t = 0$ and $t = 1$: 1) the curvature of the trajectory and 2) the tangential and normal components of the acceleration.

Chapter VII

MULTIPLE AND LINE INTEGRALS

Sec. 1. The Double Integral in Rectangular Coordinates

1°. Direct computation of double integrals. The *double integral* of a continuous function $f(x, y)$ over a bounded closed region S is the limit of the corresponding two-dimensional integral sum

$$\int\limits_{(S)} f(x, y)\,dx\,dy = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_k \to 0}} \sum_i \sum_k f(x_i, y_k)\,\Delta x_i\,\Delta y_k, \tag{1}$$

where $\Delta x_i = x_{i+1} - x_i$, $\Delta y_k = y_{k+1} - y_k$ and the sum is extended over those values of i and k for which the points (x_i, y_k) belong to S.

2°. Setting up the limits of integration in a double integral. We distinguish two basic types of region of integration.

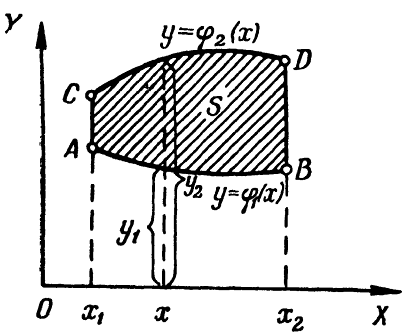

Fig. 85

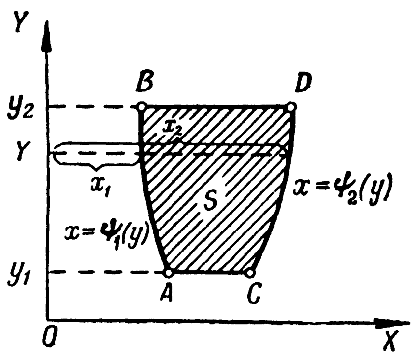

Fig. 86

1) The region of integration S (Fig. 85) is bounded on the left and right by the straight lines $x = x_1$ and $x = x_2$ $(x_2 > x_1)$, from below and from above by the continuous curves $y = \varphi_1(x)$ (AB) and $y = \varphi_2(x)$ (CD) $[\varphi_2(x) \geqslant \varphi_1(x)]$, each of which intersects the vertical $x = X$ $(x_1 \leqslant X \leqslant x_2)$ at only one point (see Fig. 85). In the region S, the variable x varies from x_1 to x_2, while the variable y (for x constant) varies from $y_1 = \varphi_1(x)$ to $y_2 = \varphi_2(x)$. The integral (1) may

be computed by reducing to an iterated integral by the formula

$$\iint\limits_{(S)} f(x, y)\,dx\,dy = \int\limits_{x_1}^{x_2} dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy,$$

where x is held constant when calculating $\int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy$.

2) The region of integration S is bounded from below and from above by the straight lines $y=y_1$ and $y=y_2\,(y_2>y_1)$, and from the left and the right by the continuous curves $x=\psi_1(y)$ (AB) and $x=\psi_2(y)$ (CD) $[\psi_2(y)\geqslant\psi_1(y)]$, each of which intersects the parallel $y=Y\,(y_1\leqslant Y\leqslant y_2)$ at only one point (Fig. 86).

As before, we have

$$\iint\limits_{(S)} f(x, y)\,dx\,dy = \int\limits_{y_1}^{y_2} dy \int\limits_{\psi_1(y)}^{\psi_2(y)} f(x, y)\,dx,$$

here, in the integral $\int\limits_{\psi_1(y)}^{\psi_2(y)} f(x, y)\,dx$ we consider y constant.

If the region of integration does not belong to any of the above-discussed types, then an attempt is made to break it up into parts, each of which does belong to one of these two types.

Example 1. Evaluate the integral

$$I=\int\limits_0^1 dx \int\limits_x^1 (x+y)\,dy.$$

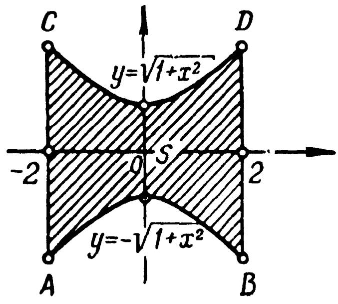

Fig. 87

Solution.

$$I=\int\limits_0^1\left(xy+\frac{y^2}{2}\right)\Bigg|_{y=x}^{y=1}dx=\int\limits_0^1\left[\left(x+\frac{1}{2}\right)-\left(x^2+\frac{x^2}{2}\right)\right]dx=\frac{1}{2}.$$

Example 2. Determine the limits of integration of the integral

$$\iint\limits_{(S)} f(x, y)\,dx\,dy$$

if the region of integration S (Fig. 87) is bounded by the hyperbola $y^2-x^2=1$ and by two straight lines $x=2$ and $x=-2$ (we have in view the region containing the coordinate origin).

Solution. The region of integration $ABCD$ (Fig. 87) is bounded by the straight lines $x=-2$ and $x=2$ and by two branches of the hyperbola

$$y=\sqrt{1+x^2} \quad \text{and} \quad y=-\sqrt{1+x^2};$$

that is, it belongs to the first type. We have:

$$\iint_{(S)} f(x, y)\,dx\,dy=\int_{-2}^{2} dx \int_{-\sqrt{1+x^2}}^{\sqrt{1+x^2}} f(x, y)\,dy.$$

Evaluate the following iterated integrals:

2113. $\int_0^2 dy \int_0^1 (x^2+2y)\,dx.$

2114. $\int_3^4 dx \int_1^2 \frac{dy}{(x+y)^2}.$

2115. $\int_0^1 dx \int_0^1 \frac{x^2dy}{1+y^2}.$

2116. $\int_1^2 dx \int_{\frac{1}{x}}^{x} \frac{x^2dy}{y^2}.$

2117. $\int_{-3}^{3} dy \int_{y^2-4}^{5} (x+2y)\,dx.$

2118. $\int_0^{2\pi} d\varphi \int_{a\sin\varphi}^{a} r\,dr.$

2119. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{3\cos\varphi} r^2\sin^2\varphi\,dr.$

2120. $\int_0^1 dx \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy.$

Write the equations of curves bounding regions over which the following double integrals are extended, and draw these regions:

2121. $\int_{-6}^{2} dy \int_{\frac{y^2}{4}-1}^{2-y} f(x, y)\,dx.$

2122. $\int_1^3 dx \int_{x^2}^{x+9} f(x, y)\,dy.$

2123. $\int_0^4 dy \int_y^{10-y} f(x, y)\,dx.$

2124. $\int_1^3 dx \int_{\frac{x}{3}}^{2x} f(x, y)\,dy.$

2125. $\int_0^3 dx \int_0^{\sqrt{25-x^2}} f(x, y)\,dy.$

2126. $\int_{-1}^{2} dx \int_{x^2}^{x+2} f(x, y)\,dy.$

Set up the limits of integration in one order and then in the other in the double integral

$$\iint_{(S)} f(x, y)\,dx\,dy$$

for the indicated regions S.

2127. S is a rectangle with vertices $O(0, 0)$, $A(2, 0)$, $B(2, 1)$, $C(0, 1)$.

2128. S is a triangle with vertices $O(0, 0)$, $A(1, 0)$, $B(1, 1)$.

2129. S is a trapezoid with vertices $O(0, 0)$, $A(2, 0)$, $B(1, 1)$, $C(0, 1)$.

2130. S is a parallelogram with vertices $A(1, 2)$, $B(2, 4)$, $C(2, 7)$, $D(1, 5)$.

2131. S is a circular sector OAB with centre at the point $O(0, 0)$, whose arc end-points are $A(1, 1)$ and $B(-1, 1)$ (Fig. 88).

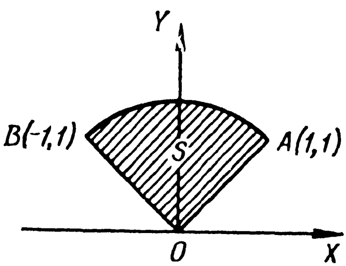

Fig. 88

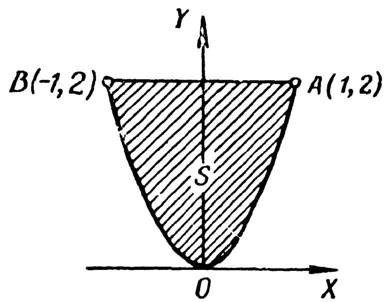

Fig. 89

2132. S is a right parabolic segment AOB bounded by the parabola BOA and a segment of the straight line BA connecting the points $B(-1, 2)$ and $A(1, 2)$ (Fig. 89).

2133. S is a circular ring bounded by circles with radii $r=1$ and $R=2$ and with common centre $O(0, 0)$.

2134. S is bounded by the hyperbola $y^2-x^2=1$ and the circle $x^2+y^2=9$ (the region containing the origin is meant).

2135. Set up the limits of integration in the double integral

$$\iint\limits_{(S)} f(x, y)\,dx\,dy$$

if the region S is defined by the inequalities

a) $x \geqslant 0$; $y \geqslant 0$; $x+y \leqslant 1$;

b) $x^2+y^2 \leqslant a^2$;

c) $x^2+y^2 \leqslant x$;

d) $y \geqslant x$; $x \geqslant -1$; $y \leqslant 1$;

e) $y \leqslant x \leqslant y+2a$; $0 \leqslant y \leqslant a$.

Change the order of integration in the following double integrals:

2136. $\int\limits_0^4 dx \int\limits_{3x^2}^{12x} f(x, y)\,dy$.

2137. $\int\limits_0^1 dx \int\limits_{2x}^{x} f(x, y)\,dy$.

2138. $\int_{0}^{a} dx \int_{\frac{a^2-x^2}{2a}}^{\sqrt{a^2-x^2}} f(x, y)\, dy.$

2139. $\int_{\frac{a}{2}}^{a} dx \int_{0}^{\sqrt{2ax-x^2}} f(x, y)\, dy.$

2140. $\int_{0}^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{4ax}} f(x, y)\, dy.$

2141. $\int_{0}^{1} dy \int_{-\sqrt{1-y^2}}^{1-y} f(x, y)\, dx.$

2142. $\int_{0}^{1} dy \int_{\frac{y^2}{2}}^{\sqrt{3-y^2}} f(x, y)\, dx.$

2143. $\int_{0}^{\frac{R\sqrt{2}}{2}} dx \int_{0}^{x} f(x, y)\, dy + \int_{\frac{R\sqrt{2}}{2}}^{R} dx \int_{0}^{\sqrt{R^2-x^2}} f(x, y)\, dy.$

2144. $\int_{0}^{\pi} dx \int_{0}^{\sin x} f(x, y)\, dy.$

Evaluate the following double integrals:

2145. $\iint\limits_{(S)} x\, dx\, dy$, where S is a triangle with vertices $O(0, 0)$, $A(1, 1)$, and $B(0, 1)$.

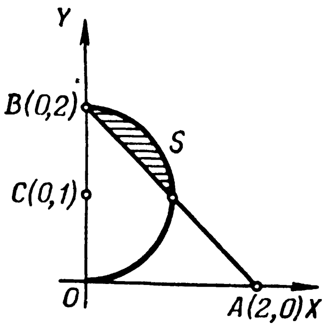

Fig. 90

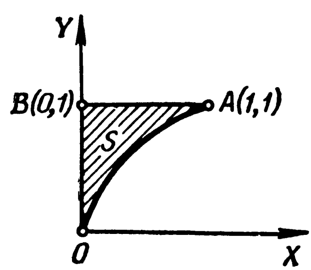

Fig. 91

2146. $\iint\limits_{(S)} x\, dx\, dy$, where the region of integration S is bounded by the straight line passing through the points $A(2, 0)$, $B(0, 2)$ and by the arc of a circle with centre at the point $C(0, 1)$, and radius 1 (Fig. 90).

2147. $\iint\limits_{(S)} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$, where S is a part of a circle of radius a with centre at $O(0, 0)$ lying in the first quadrant.

2148. $\iint\limits_{(S)} \sqrt{x^2-y^2}\,dx\,dy$, where S is a triangle with vertices $O(0, 0)$, $A(1, -1)$, and $B(1, 1)$.

2149. $\iint\limits_{(S)} \sqrt{xy-y^2}\,dx\,dy$, where S is a triangle with vertices $O(0, 0)$, $A(10, 1)$, and $B(1, 1)$.

2150. $\iint\limits_{(S)} e^{\frac{x}{y}}\,dx\,dy$, where S is a curvilinear triangle OAB bounded by the parabola $y^2=x$ and the straight lines $x=0$, $y=1$ (Fig. 91).

2151. $\iint\limits_{(S)} \frac{x\,dx\,dy}{x^2+y^2}$, where S is a parabolic segment bounded by the parabola $y=\frac{x^2}{2}$ and the straight line $y=x$.

2152. Compute the integrals and draw the regions over which they extend:

a) $\int\limits_0^{\pi} dx \int\limits_0^{1+\cos x} y^2 \sin x\,dy$;

b) $\int\limits_0^{\frac{\pi}{2}} dx \int\limits_{\cos x}^{1} y^4\,dy$;

c) $\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dy \int\limits_0^{3\cos y} x^2 \sin^2 y\,dx$.

When solving Problems 2153 to 2157 it is abvisable to make the drawings first.

2153. Evaluate the double integral

$$\iint\limits_{(S)} xy^2\,dx\,dy,$$

if S is a region bounded by the parabola $y^2=2px$ and the straight line $x=p$.

2154*. Evaluate the double integral

$$\iint\limits_{(S)} xy\,dx\,dy,$$

extended over the region S, which is bounded by the x-axis and an upper semicircle $(x-2)^2+y^2=1$.

2155. Evaluate the double integral

$$\iint\limits_{(S)} \frac{dx\,dy}{\sqrt{2a-x}},$$

where S is the area of a circle of radius a, which circle is tangent to the coordinate axes and lies in the first quadrant.

2156*. Evaluate the double integral

$$\iint\limits_{(S)} y\,dx\,dy,$$

where the region S is bounded by the axis of abscissas and an arc of the cycloid

$$x=R(t-\sin t),$$
$$y=R(1-\cos t).$$

2157. Evaluate the double integral

$$\iint\limits_{(S)} xy\,dx\,dy,$$

in which the region of integration S is bounded by the coordinate axes and an arc of the astroid

$$x=R\cos^3 t,\quad y=R\sin^3 t\ \left(0\leqslant t\leqslant \frac{\pi}{2}\right).$$

2158. Find the mean value of the function $f(x, y)=xy^2$ in the region $S\{0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 1\}$.

Hint. The *mean value of a function* $f(x, y)$ in the region S is the number

$$\bar{f}=\frac{1}{S}\iint\limits_{(S)} f(x, y)\,dx\,dy.$$

2159. Find the mean value of the square of the distance of a point $M(x, y)$ of the circle $(x-a)^2+y^2\leqslant R^2$ from the coordinate origin.

Sec. 2. Change of Variables in a Double Integral

1°. Double integral in polar coordinates. In a double integral, when passing from rectangular coordinates (x, y) to polar coordinates (r, φ), which are connected with rectangular coordinates by the relations

$$x=r\cos\varphi,\qquad y=r\sin\varphi,$$

we have the formula

$$\iint\limits_{(S)} f(x, y)\,dx\,dy=\iint\limits_{(S)} (r\cos\varphi,\ r\sin\varphi)\,r\,dr\,d\varphi, \qquad (1)$$

If the region of integration (S) is bounded by the half-lines $r=\alpha$ and $r=\beta\ (\alpha<\beta)$ and the curves $r=r_1(\varphi)$ and $r=r_2(\varphi)$, where $r_1(\varphi)$ and $r_2(\varphi)\ [r_1(\varphi)\leqslant r_2(\varphi)]$ are single-valued functions on the interval $\alpha\leqslant\varphi\leqslant\beta$, then the double integral may be evaluated by the formula

$$\iint\limits_{(S)} F(\varphi,\ r)\,r\,dr\,d\varphi=\int\limits_{\alpha}^{\beta} d\varphi \int\limits_{r_1(\varphi)}^{r_2(\varphi)} F(\varphi,\ r)\,r\,dr,$$

where $F(\varphi,\ r)=f(r\cos\varphi,\ r\sin\varphi)$. In evaluating the integral $\int\limits_{r_1(\varphi)}^{r_2(\varphi)} F(\varphi,\ r)\,r\,dr$ we hold the quantity φ constant.

If the region of integration does not belong to one of the kinds that has been examined, it is broken up into parts, each of which is a region of a given type.

2°. **Double integral in curvilinear coordinates.** In the more general case, if in the double integral

$$\iint\limits_{(S)} f(x,\ y)\,dx\,dy$$

it is required to pass from the variables $x,\ y$ to the variables $u,\ v$, which are connected with $x,\ y$ by the continuous and differentiable relationships

$$x=\varphi(u,\ v),\qquad y=\psi(u,\ v)$$

that establish a one-to-one (and, in both directions, continuous) correspondence between the points of the region S of the xy-plane and the points of some region S' of the UV-plane, and if the *Jacobian*

$$I=\frac{D(x,\ y)}{D(u,\ v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}$$

retains a constant sign in the region S, then the formula

$$\iint\limits_{(s)} f(x,\ y)\,dx\,dy=\iint\limits_{(s')} f[\varphi(u,\ v),\ \psi(u,\ v)]\,|I|\,du\,dv$$

holds true

The limits of the new integral are determined from general rules on the basis of the type of region S'

Example 1. In passing to polar coordinates, evaluate

$$\iint\limits_{(S)} \sqrt{1-x^2-y^2}\,dx\,dy,$$

where the region S is a circle of radius $R=1$ with centre at the coordinate origin (Fig 92).

Solution. Putting $x=r\cos\varphi,\ y=r\sin\varphi$, we obtain:

$$\sqrt{1-x^2-y^2}=\sqrt{1-(r\cos\varphi)^2-(r\sin\varphi)^2}=\sqrt{1-r^2}$$

Since the coordinate r in the region S varies from 0 to 1 for any φ, and φ varies from 0 to 2π, it follows that

$$\iint_{(S)} \sqrt{1-x^2-y^2}\,dx\,dy = \int_0^{2\pi} d\varphi \int_0^1 r\sqrt{1-r^2}\,dr = \frac{2}{3}\pi.$$

Pass to polar coordinates r and φ and set up the limits of integration with respect to the new variables in the following integrals:

2160. $\int_0^1 dx \int_0^1 f(x, y)\,dy.$

2161. $\int_0^2 dx \int_0^x f(\sqrt{x^2+y^2})\,dy.$

2162. $\iint_{(S)} f(x, y)\,dx\,dy,$

where S is a triangle bounded by the straight lines $y=x$, $y=-x$, $y=1$.

2163. $\int_{-1}^{1} dx \int_{x^2}^{1} f\left(\frac{y}{x}\right) dy.$

2164. $\iint_{(S)} f(x, y)\,dx\,dy$, where S is bounded by the lemniscate

$$(x^2+y^2)^2 = a^2(x^2-y^2).$$

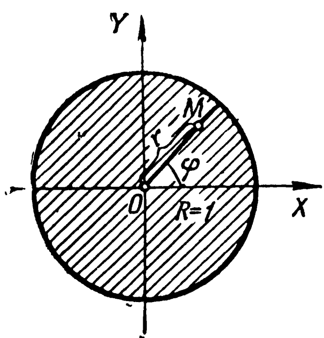

Fig. 92

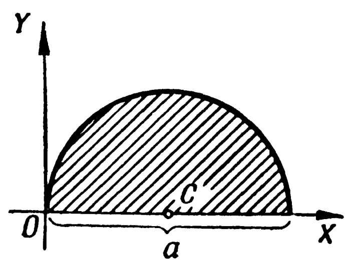

Fig. 93

2165. Passing to polar coordinates, calculate the double integral

$$\iint_{(S)} y\,dx\,dy,$$

where S is a semicircle of diameter a with centre at the point $C\left(\frac{a}{2}, 0\right)$ (Fig. 93).

2166. Passing to polar coordinates, evaluate the double integral

$$\iint\limits_{(S)} (x^2+y^2)\,dx\,dy,$$

extended over a region bounded by the circle $x^2+y^2=2ax$.

2167. Passing to polar coordinates, evaluate the double integral

$$\iint\limits_{(S)} \sqrt{a^2-x^2-y^2}\,dx\,dy,$$

where the region of integration S is a semicircle of radius a with centre at the coordinate origin and lying above the x-axis.

2168. Evaluate the double integral of a function $f(r, \varphi)=r$ over a region bounded by the cardioid $r=a(1+\cos\varphi)$ and the circle $r=a$. (This is a region that does not contain a pole.)

2169. Passing to polar coordinates, evaluate

$$\int\limits_0^a dx \int\limits_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy.$$

2170. Passing to polar coordinates, evaluate

$$\iint\limits_{(S)} \sqrt{a^2-x^2-y^2}\,dx\,dy,$$

where the region S is a loop of the lemniscate

$$(x^2+y^2)^2=a^2(x^2-y^2) \qquad (x \geqslant 0).$$

2171*. Evaluate the double integral

$$\iint\limits_{(S)} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy,$$

extended over the region S bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by passing to *generalized polar coordinates:*

$$\frac{x}{a}=r\cos\varphi,\ \frac{y}{b}=r\sin\varphi.$$

2172.** Transform

$$\int\limits_0^c dx \int\limits_{\alpha x}^{\beta x} f(x, y)\,dy$$

$(0<\alpha<\beta$ and $c>0)$ by introducing new variables $u=x+y$, $uv=y$.

2173*. Change the variables $u=x+y$, $v=x-y$ in the integral

$$\int_0^1 dx \int_0^1 f(x, y)\, dy.$$

2174.** Evaluate the double integral

$$\iint_{(S)} dx\, dy,$$

where S is a region bounded by the curve

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2}{h^2}-\frac{y^2}{k^2}.$$

Hint. Make the substitution

$$x=ar\cos\varphi, \quad y=br\sin\varphi.$$

Sec. 3. Computing Areas

1°. **Area in rectangular coordinates.** The *area of a plane region* S is

$$S=\iint_{(S)} dx\, dy.$$

If the region S is defined by the inequalities $a \leqslant x \leqslant b$, $\varphi(x) \leqslant y \leqslant \psi(x)$, then

$$S=\int_a^b dx \int_{\varphi(x)}^{\psi(x)} dy.$$

2°. **Area in polar coordinates.** If a region S in polar coordinates r and φ is defined by the inequalities $\alpha \leqslant \varphi \leqslant \beta$, $f(\varphi) \leqslant r \leqslant F(\varphi)$, then

$$S=\iint_{(S)} r\, d\varphi\, dr=\int_\alpha^\beta d\varphi \int_{f(\varphi)}^{F(\varphi)} r\, dr.$$

2175. Construct regions whose areas are expressed by the integrals

a) $\int_{-1}^{2} dx \int_{x^2}^{x+2} dy$; b) $\int_0^a dy \int_{a-y}^{\sqrt{a^2-y^2}} dx.$

Evaluate these areas and change the order of integration.

2176. Construct regions whose areas are expressed dy the integrals

a) $\int_{\frac{\pi}{4}}^{\arctan 2} d\varphi \int_0^{3\sec\varphi} r\, dr$; b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_a^{a(1+\cos\varphi)} r\, dr.$

Compute these areas.

2177. Compute the area bounded by the straight lines $x=y$, $x=2y$, $x+y=a$, $x+3y=a\ (a>0)$.

2178. Compute the area lying above the x-axis and bounded by this axis, the parabola $y^2=4ax$, and the straight line $x+y=3a$.

2179*. Compute the area bounded by the ellipse

$$(y-x)^2+x^2=1.$$

2180. Find the area bounded by the parabolas

$$y^2=10x+25 \text{ and } y^2=-6x+9.$$

2181. Passing to polar coordinates, find the area bounded by the lines

$$x^2+y^2=2x, \quad x^2+y^2=4x, \quad y=x, \quad y=0.$$

2182. Find the area bounded by the straight line $r\cos\varphi=1$ and the circle $r=2$. (The area is not to contain a pole.)

2183. Find the area bounded by the curves

$$r=a(1+\cos\varphi) \text{ and } r=a\cos\varphi\ (a>0).$$

2184. Find the area bounded by the line

$$\left(\frac{x^2}{4}+\frac{y^2}{9}\right)^2=\frac{x^2}{4}-\frac{y^2}{9}.$$

2185*. Find the area bounded by the ellipse

$$(x-2y+3)^2+(3x+4y-1)^2=100.$$

2186. Find the area of a curvilinear quadrangle bounded by the arcs of the parabolas $x^2=ay$, $x^2=by$, $y^2=\alpha x$, $y^2=\beta x\ (0<a<b,\ 0<\alpha<\beta)$.

Hint. Introduce the new variables u and v, and put

$$x^2=uy, \quad y^2=vx.$$

2187. Find the area of a curvilinear quadrangle bounded by the arcs of the curves $y^2=ax$, $y^2=bx$, $xy=\alpha$, $xy=\beta\ (0<a<b,\ 0<\alpha<\beta)$.

Hint. Introduce the new variables u and v, and put

$$xy=u, \quad y^2=vx.$$

Sec. 4. Computing Volumes

The *volume V of a cylindroid* bounded above by a continuous surface $z=f(x, y)$, below by the plane $z=0$, and on the sides by a right cylindrical surface, which cuts out of the xy-plane a region S (Fig. 94), is equal to

$$V=\iint\limits_{(S)} f(x, y)\,dx\,dy.$$

2188. Use a double integral to express the volume of a pyramid with vertices $O(0, 0, 0)$, $A(1, 0, 0)$, $B(1, 1, 0)$ and $C(0, 0, 1)$ (Fig. 95). Set up the limits of integration.

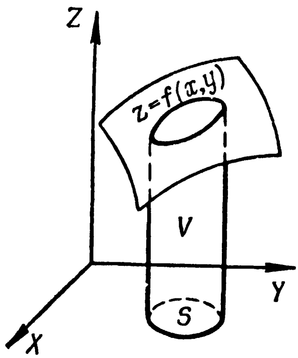

Fig. 94

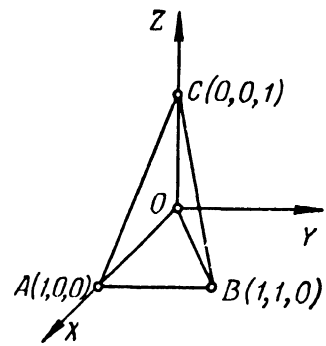

Fig. 95

In Problems 2189 to 2192 sketch the solid whose volume is expressed by the given double integral:

2189. $\int\limits_0^1 dx \int\limits_0^{1-x} (1-x-y)\,dy.$ 2191. $\int\limits_0^2 dx \int\limits_0^{\sqrt{1-x^2}} (1-x)\,dy.$

2190. $\int\limits_0^2 dx \int\limits_0^{2-x} (4-x-y)\,dy.$ 2192. $\int\limits_0^2 dx \int\limits_{2-x}^{2} (4-x-y)\,dy.$

2193. Sketch the solid whose volume is expressed by the integral $\int\limits_0^a dx \int\limits_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy$; reason geometrically to find the value of this integral.

2194. Find the volume of a solid bounded by the elliptical paraboloid $z=2x^2+y^2+1$, the plane $x+y=1$, and the coordinate planes.

2195. A solid is bounded by a hyperbolic paraboloid $z=x^2-y^2$ and the planes $y=0$, $z=0$, $x=1$. Compute its volume.

2196. A solid is bounded by the cylinder $x^2+z^2=a^2$ and the planes $y=0$, $z=0$, $y=x$. Compute its volume.

Find the volumes bounded by the following surfaces:

2197. $az=y^2$, $x^2+y^2=r^2$, $z=0$.

2198. $y=\sqrt{x}$, $y=2\sqrt{x}$, $x+z=6$, $z=0$.

2199. $z=x^2+y^2$, $y=x^2$, $y=1$, $z=0$.

2200. $x+y+z=a$, $3x+y=a$, $\frac{3}{2}x+y=a$, $y=0$, $z=0$.

2201. $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$, $y=\frac{b}{a}x$, $y=0$, $z=0$.

2202. $x^2+y^2=2ax$, $z=\alpha x$, $z=\beta x$ $(\alpha>\beta)$.

In Problems 2203 to 2211 use polar and generalized polar coordinates.

2203. Find the entire volume enclosed between the cylinder $x^2+y^2=a^2$ and the hyperboloid $x^2+y^2-z^2=-a^2$.

2204. Find the entire volume contained between the cone $2(x^2+y^2)-z^2=0$ and the hyperboloid $x^2+y^2-z^2=-a^2$.

2205. Find the volume bounded by the surfaces $2az=x^2+y^2$, $x^2+y^2-z^2=a^2$, $z=0$.

2206. Determine the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

2207. Find the volume of a solid bounded by the paraboloid $2az=x^2+y^2$ and the sphere $x^2+y^2+z^2=3a^2$. (The volume lying inside the paraboloid is meant.)

2208. Compute the volume of a solid bounded by the xy-plane, the cylinder $x^2+y^2=2ax$, and the cone $x^2+y^2=z^2$.

2209. Compute the volume of a solid bounded by the xy-plane, the surface $z=ae^{-(x^2+y^2)}$, and the cylinder $x^2+y^2=R^2$.

2210. Compute the volume of a solid bounded by the xy-plane, the paraboloid $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$, and the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\frac{x}{a}$.

2211. In what ratio does the hyperboloid $x^2+y^2-z^2=a^2$ divide the volume of the sphere $x^2+y^2+z^2\leqslant 3a^2$?

2212*. Find the volume of a solid bounded by the surfaces $z=x+y$, $xy=1$, $xy=2$, $y=x$, $y=2x$, $z=0$ $(x>0,\ y>0)$.

Sec. 5. Computing the Areas of Surfaces

The *area* σ of a smooth single-valued surface $z=f(x, y)$, whose projection on the xy-plane is the region S, is equal to

$$\sigma=\iint\limits_{(S)}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy.$$

2213. Find the area of that part of the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ which lies between the coordinate planes.

2214. Find the area of that part of the surface of the cylinder $x^2+y^2=R^2$ $(z \geqslant 0)$ which lies between the planes $z=mx$ and $z=nx$ $(m>n>0)$.

2215*. Compute the area of that part of the surface of the cone $x^2-y^2=z^2$ which is situated in the first octant and is bounded by the plane $y+z=a$.

2216. Compute the area of that part of the surface of the cylinder $x^2+y^2=ax$ which is cut out of it by the sphere $x^2+y^2+z^2=a^2$.

2217. Compute the area of that part of the surface of the sphere $x^2+y^2+z^2=a^2$ cut out by the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

2218. Compute the area of that part of the surface of the paraboloid $y^2+z^2=2ax$ which lies between the cylinder $y^2=ax$ and the plane $x=a$.

2219. Compute the area of that part of the surface of the cylinder $x^2+y^2=2ax$ which lies between the xy-plane and the cone $x^2+y^2=z^2$.

2220*. Compute the area of that part of the surface of the cone $x^2-y^2=z^2$ which lies inside the cylinder $x^2+y^2=2ax$.

2221*. Prove that the areas of the parts of the surfaces of the paraboloids $x^2+y^2=2az$ and $x^2-y^2=2az$ cut out by the cylinder $x^2+y^2=R^2$ are of equivalent size.

2222*. A sphere of radius a is cut by two circular cylinders whose base diameters are equal to the radius of the sphere and which are tangent to each other along one of the diameters of the sphere. Find the volume and the area of the surface of the remaining part of the sphere.

2223* An opening with square base whose side is equal to a is cut out of a sphere of radius a. The axis of the opening coincides with the diameter of the sphere. Find the area of the surface of the sphere cut out by the opening.

2224*. Compute the area of that part of the helicoid $z=c\arctan\frac{y}{x}$ which lies in the first octant between the cylinders $x^2+y^2=a^2$ and $x^2+y^2=b^2$.

Sec. 6. Applications of the Double Integral in Mechanics

1°. The mass and static moments of a lamina. If S is a region in an xy-plane occupied by a lamina, and $\varrho(x, y)$ is the surface density of the lamina at the point (x, y), then the mass M of the lamina and its static moments M_X and M_Y relative to the x- and y-axes are expressed by the double integrals

$$M=\iint_{(S)} \varrho(x, y)\,dx\,dy, \quad M_X=\iint_{(S)} y\varrho(x, y)\,dx\,dy,$$

$$M_Y=\iint_{(S)} x\varrho(x, y)\,dx\,dy. \qquad (1)$$

If the lamina is homogeneous, then $\varrho(x, y)=\text{const}$.

2°. **The coordinates of the centre of gravity of a lamina.** If $C(\bar{x}, \bar{y})$ is the centre of gravity of a lamina, then

$$\bar{x}=\frac{M_Y}{M}, \quad \bar{y}=\frac{M_X}{M},$$

where M is the mass of the lamina and M_X, M_Y are its static moments relative to the coordinate axes (see 1°). If the lamina is homogeneous, then in formulas (1) we can put $\varrho=1$.

3°. **The moments of inertia of a lamina.** The moments of inertia of a lamina relative to the x- and y-axes are, respectively, equal to

$$I_X=\iint_{(S)} y^2\varrho(x, y)\,dx\,dy, \quad I_Y=\iint_{(S)} x^2\varrho(x, y)\,dx\,dy. \qquad (2)$$

The moment of inertia of a lamina relative to the origin is

$$I_0=\iint_{(S)} (x^2+y^2)\,\varrho(x, y)\,dx\,dy=I_X+I_Y. \qquad (3)$$

Putting $\varrho(x, y)=1$ in formulas (2) and (3), we get the geometric moments of inertia of a plane figure.

2225. Find the mass of a circular lamina of radius R if the density is proportional to the distance of a point from the centre and is equal to δ at the edge of the lamina.

2226. A lamina has the shape of a right triangle with legs $OB=a$ and $OA=b$, and its density at any point is equal to the distance of the point from the leg OA. Find the static moments of the lamina relative to the legs OA and OB.

2227. Compute the coordinates of the centre of gravity of the area $OmAnO$ (Fig. 96), which is bounded by the curve $y=\sin x$ and the straight line OA that passes through the coordinate origin and the vertex $A\left(\frac{\pi}{2}, 1\right)$ of a sine curve.

2228. Find the coordinates of the centre of gravity of an area bounded by the cardioid $r=a(1+\cos\varphi)$.

2229. Find the coordinates of the centre of gravity of a circular sector of radius a with angle at the vertex 2α (Fig. 97).

2230. Compute the coordinates of the centre of gravity of an area bounded by the parabolas $y^2=4x+4$ and $y^2=-2x+4$.

2231. Compute the moment of inertia of a triangle bounded by the straight lines $x+y=2$, $x=2$, $y=2$ relative to the x-axis.

2232. Find the moment of inertia of an annulus with diameters d and D $(d<D)$: a) relative to its centre, and b) relative to its diameter.

2233. Compute the moment of inertia of a square with side a relative to the axis passing through its vertex perpendicularly to the plane of the square.

2234*. Compute the moment of inertia of a segment cut off the parabola $y^2=ax$ by the straight line $x=a$ relative to the straight line $y=-a$.

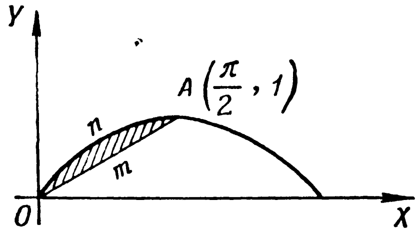

Fig. 96

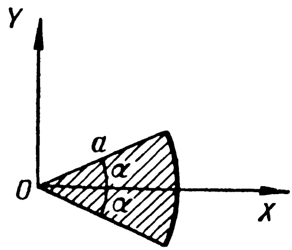

Fig. 97

2235*. Compute the moment of inertia of an area bounded by the hyperbola $xy=4$ and the straight line $x+y=5$ relative to the straight line $x=y$.

2236*. In a square lamina with side a, the density is proportional to the distance from one of its vertices. Compute the moment of inertia of the lamina relative to the side that passes through this vertex.

2237. Find the moment of inertia of the cardioid $r=a(1+\cos\varphi)$ relative to the pole.

2238. Compute the moment of inertia of the area of the lemniscate $r^2=2a^2\cos 2\varphi$ relative to the axis perpendicular to its plane in the pole.

2239*. Compute the moment of inertia of a homogeneous lamina bounded by one arc of the cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$ and the x-axis, relative to the x-axis.

Sec. 7. Triple Integrals

1°. Triple integrals in rectangular coordinates. The triple integral of the function $f(x, y, z)$ extended over the region V is the limit of the corresponding threefold iterated sum:

$$\iiint\limits_V f(x, y, z)\,dx\,dy\,dz=\lim_{\substack{\max \Delta x_i \to 0\\ \max \Delta y_j \to 0\\ \max \Delta z_k \to 0}} \sum_i \sum_j \sum_k f(x_i, y_j, z_k)\,\Delta x_i\,\Delta y_j\,\Delta z_k.$$

Evaluation of a triple integral reduces to the successive computation of the three ordinary (onefold iterated) integrals or to the computation of one double and one single integral.

Example 1. Compute

$$I=\iiint\limits_V x^3y^2z\,dx\,dy\,dz,$$

where the region V is defined by the inequalities

$$0\leqslant x\leqslant 1,\quad 0\leqslant y\leqslant x,\quad 0\leqslant z\leqslant xy.$$

Solution. We have

$$I=\int_0^1 dx\int_0^x dy\int_0^{xy} x^3y^2z\,dz=\int_0^1 dx\int_0^x x^3y^2\left.\frac{z^2}{2}\right|_0^{xy} dy=$$

$$=\int_0^1 dx\int_0^x \frac{x^5y^4}{2}\,dy=\int_0^1 \left.\frac{x^5}{2}\frac{y^5}{5}\right|_0^x dx=\int_0^1 \frac{x^{10}}{10}\,dx=\frac{1}{110}.$$

Example 2. Evaluate

$$\iiint\limits_{(V)} x^2\,dx\,dy\,dz,$$

extended over the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution.

$$\iiint\limits_{(V)} x^2\,dx\,dy\,dz=\int_{-a}^{a} x^2\,dx\iint\limits_{(S_{yz})} dy\,dz=\int_{-a}^{a} x^2S_{yz}\,dx,$$

where S_{yz} is the area of the ellipse $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1-\frac{x^2}{a^2}$, $x=\text{const}$, and is equal to

$$S_{yz}=\pi b\sqrt{1-\frac{x^2}{a^2}}\cdot c\sqrt{1-\frac{x^2}{a^2}}=\pi bc\left(1-\frac{x^2}{a^2}\right).$$

We therefore finally get

$$\iiint\limits_{(V)} x^2\,dx\,dy\,dz=\pi bc\int_{-a}^{a} x^2\left(1-\frac{x^2}{a^2}\right)dx=\frac{4}{15}\pi a^3bc.$$

2°. Change of variables in a triple integral. If in the triple integral

$$\iiint\limits_{(V)} f(x,\ y,\ z)\,dx\,dy\,dz$$

it is required to pass from the variables x, y, z to the variables u, v, w, which are connected with x, y, z by the relations $x=\varphi(u,\ v,\ w)$, $y=\psi(u,\ v,\ w)$, $z=\chi(u,\ v,\ w)$, where the functions φ, ψ, χ are:

1) continuous together with their partial first derivatives;

2) in one-to-one (and, in both directions, continuous) correspondence between the points of the region of integration V of xyz-space and the points of some region V' of UVW-space;

3) the functional determinant (Jacobian) of these functions

$$I=\frac{D(x, y, z)}{D(u, v, w)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

retains a constant sign in the region V, then we can make use of the formula

$$\iiint\limits_{(V)} f(x, y, z)\,dx\,dy\,dz=$$

$$=\iiint\limits_{(V')} f[\varphi(u, v, w),\ \psi(u, v, w),\ \chi(u, v, w)]\,|I|\,du\,dv\,dw.$$

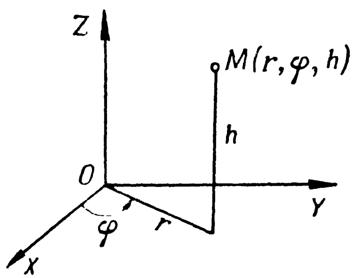

Fig. 98

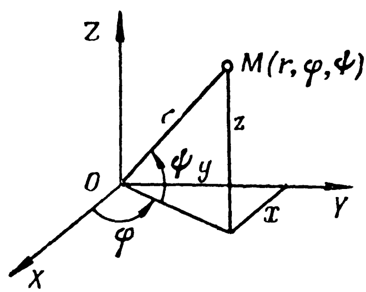

Fig. 99

In particular,

1) for cylindrical coordinates r, φ, h (Fig. 98), where

$$x=r\cos\varphi,\quad y=r\sin\varphi,\quad z=h,$$

we get $I=r$;

2) for spherical coordinates φ, ψ, r (φ is the longitude, ψ the latitude, r the radius vector) (Fig. 99), where

$$x=r\cos\psi\cos\varphi,\quad y=r\cos\psi\sin\varphi,\quad z=r\sin\psi,$$

we have $I=r^2\cos\psi$.

Example 3. Passing to spherical coordinates, compute

$$\iiint\limits_{(V)} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz,$$

where V is a sphere of radius R.

Solution. For a sphere, the ranges of the spherical coordinates φ (longitude), ψ (latitude), and r (radius vector) will be

$$0\leqslant\varphi\leqslant 2\pi,\quad -\frac{\pi}{2}\leqslant\psi\leqslant\frac{\pi}{2},\quad 0\leqslant r\leqslant R.$$

We therefore have

$$\iiint\limits_{(V)} \sqrt{x^2+y^2+z^2}\, dx\, dy\, dz = \int\limits_0^{2\pi} d\varphi \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int\limits_0^R r\, r^2 \cos\psi\, dr = \pi R^4.$$

3°. **Applications of triple integrals.** The *volume* of a region of three-dimensional xyz-space is

$$V = \iiint\limits_{(V)} dx\, dy\, dz.$$

The *mass* of a solid occupying the region V is

$$M = \iiint\limits_{(V)} \gamma(x, y, z)\, dx\, dy\, dz,$$

where $\gamma(x, y, z)$ is the density of the body at the point (x, y, z).

The *static moments* of the body relative to the coordinate planes are

$$M_{XY} = \iiint\limits_{(V)} \gamma(x, y, z)\, z\, dx\, dy\, dz;$$

$$M_{YZ} = \iiint\limits_{(V)} \gamma(x, y, z)\, x\, dx\, dy\, dz;$$

$$M_{ZX} = \iiint\limits_{(V)} \gamma(x, y, z)\, y\, dx\, dy\, dz.$$

The *coordinates of the centre of gravity* are

$$\bar{x} = \frac{M_{YZ}}{M}, \quad \bar{y} = \frac{M_{ZX}}{M}, \quad \bar{z} = \frac{M_{XY}}{M}.$$

If the solid is homogeneous, then we can put $\gamma(x, y, z) = 1$ in the formulas for the coordinates of the centre of gravity.

The *moments of inertia* relative to the coordinate axes are

$$I_X = \iiint\limits_{(V)} (y^2+z^2)\, \gamma(x, y, z)\, dx\, dy\, dz;$$

$$I_Y = \iiint\limits_{(V)} (z^2+x^2)\, \gamma(x, y, z)\, dx\, dy\, dz;$$

$$I_Z = \iiint\limits_{(V)} (x^2+y^2)\, \gamma(x, y, z)\, dx\, dy\, dz.$$

Putting $\gamma(x, y, z) = 1$ in these formulas, we get the geometric moments of inertia of the body.

A. Evaluating triple integrals

Set up the limits of integration in the triple integral

$$\iiint\limits_{(V)} f(x, y, z)\, dx\, dy\, dz$$

for the indicated regions V.

2240. V is a tetrahedron bounded by the planes

$$x+y+z=1, \quad x=0, \quad y=0, \quad z=0.$$

2241. V is a cylinder bounded by the surfaces

$$x^2+y^2=R^2, \quad z=0, \quad z=H.$$

2242*. V is a cone bounded by the surfaces

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}, \quad z=c.$$

2243. V is a volume bounded by the surfaces

$$z=1-x^2-y^2, \quad z=0.$$

Compute the following integrals:

2244. $\int\limits_0^1 dx \int\limits_0^1 dy \int\limits_0^1 \frac{dz}{\sqrt{x+y+z+1}}.$

2245. $\int\limits_0^2 dx \int\limits_0^{2\sqrt{x}} dy \int\limits_0^{\sqrt{\frac{4x-y^2}{2}}} x\,dz.$

2246. $\int\limits_0^a dx \int\limits_0^{\sqrt{a^2-x^2}} dy \int\limits_0^{\sqrt{a^2-x^2-y^2}} \frac{dz}{\sqrt{a^2-x^2-y^2-z^2}}.$

2247. $\int\limits_0^1 dx \int\limits_0^{1-x} dy \int\limits_0^{1-x-y} xyz\,dz.$

2248. Evaluate

$$\iiint\limits_{(V)} \frac{dx\,dy\,dz}{(x+y+z+1)^3},$$

where V is the region of integration bounded by the coordinate planes and the plane $x+y+z=1$.

2249. Evaluate

$$\iiint\limits_{(V)} (x+y+z)^2\,dx\,dy\,dz,$$

where V (the region of integration) is the common part of the paraboloid $2az \geqslant x^2+y^2$ and the sphere $x^2+y^2+z^2 \leqslant 3a^2$.

2250. Evaluate

$$\iiint\limits_{(V)} z^2\,dx\,dy\,dz.$$

where V (region of integration) is the common part of the spheres $x^2+y^2+z^2 \leqslant R^2$ and $x^2+y^2+z^2 \leqslant 2Rz$

2251. Evaluate

$$\iiint\limits_{(V)} z\,dx\,dy\,dz,$$

where V is a volume bounded by the plane $z=0$ and the upper half of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

2252. Evaluate

$$\iiint\limits_{(V)} \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) dx\,dy\,dz,$$

where V is the interior of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

2253. Evaluate

$$\iiint\limits_{(V)} z\,dx\,dy\,dz,$$

where V (the region of integration) is bounded by the cone $z^2=\frac{h^2}{R^2}(x^2+y^2)$ and the plane $z=h$.

2254. Passing to cylindrical coordinates, evaluate

$$\iiint\limits_{(V)} dx\,dy\,dz,$$

where V is a region bounded by the surfaces $x^2+y^2+z^2=2Rz$, $x^2+y^2=z^2$ and containing the point $(0,0,R)$.

2255. Evaluate

$$\int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2+y^2}\,dz,$$

first transforming it to cylindrical coordinates.

2256. Evaluate

$$\int_0^{2r} dx \int_{-\sqrt{2rx-x^2}}^{\sqrt{2rx-x^2}} dy \int_0^{\sqrt{4r^2-x^2-y^2}} dz,$$

first transforming it to cylindrical coordinates.

2257. Evaluate

$$\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_0^{\sqrt{R^2-x^2-y^2}} (x^2+y^2)\,dz,$$

first transforming it to spherical coordinates.

2258. Passing to spherical coordinates, evaluate the integral

$$\iiint\limits_{(V)} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz,$$

where V is the interior of the sphere $x^2+y^2+z^2 \leqslant x$.

B. *Computing volumes by means of triple integrals*

2259. Use a triple integral to compute the volume of a solid bounded by the surfaces

$$y^2 = 4a^2 - 3ax, \quad y^2 = ax, \quad z = \pm h.$$

2260.** Compute the volume of that part of the cylinder $x^2+y^2=2ax$ which is contained between the paraboloid $x^2+y^2=2az$ and the xy-plane.

2261*. Compute the volume of a solid bounded by the sphere $x^2+y^2+z^2=a^2$ and the cone $z^2=x^2+y^2$ (external to the cone).

2262*. Compute the volume of a solid bounded by the sphere $x^2+y^2+z^2=4$ and the paraboloid $x^2+y^2=3z$ (internal to the paraboloid).

2263. Compute the volume of a solid bounded by the xy-plane, the cylinder $x^2+y^2=ax$ and the sphere $x^2+y^2+z^2=a^2$ (internal to the cylinder).

2264. Compute the volume of a solid bounded by the paraboloid $\frac{y^2}{b^2}+\frac{z^2}{c^2}=2\frac{x}{a}$ and the plane $x=a$.

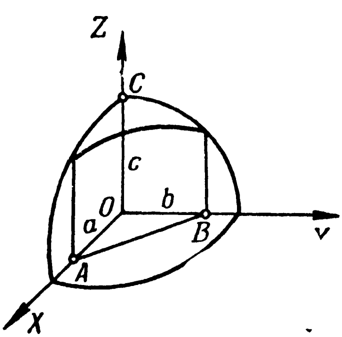

Fig. 100

C. *Applications of triple integrals to mechanics and physics*

2265. Find the mass M of a rectangular parallelepiped $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, $0 \leqslant z \leqslant c$, if the density at the point (x, y, z) is $\varrho(x, y, z) = x+y+z$.

2266. Out of an octant of the sphere $x^2+y^2+z^2 \leqslant c^2$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ cut a solid $OABC$ bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}=1$ $(a \leqslant c,\ b \leqslant c)$ (Fig. 100). Find the mass of this body if the density at each point (x, y, z) is equal to the z-coordinate of the point.

2267*. In a solid which has the shape of a hemisphere $x^2+y^2+z^2 \leqslant a^2$, $z \geqslant 0$, the density varies in proportion to the distance of the point from the centre. Find the centre of gravity of the solid.

2268. Find the centre of gravity of a solid bounded by the paraboloid $y^2+2z^2=4x$ and the plane $x=2$.

2269*. Find the moment of inertia of a circular cylinder, whose altitude is h and the radius of the base is a, relative to the axis which serves as the diameter of the base of the cylinder.

2270*. Find the moment of inertia of a circular cone (altitude, h, radius of base, a, and density ϱ) relative to the diameter of the base.

2271.** Find the force of attraction exerted by a homogeneous cone of altitude h and vertex angle α (in axial cross-section) on a material point containing unit mass and located at its vertex.

2272.** Show that the force of attraction exerted by a homogeneous sphere on an external material point does not change if the entire mass of the sphere is concentrated at its centre.

Sec. 8. Improper Integrals Dependent on a Parameter. Improper Multiple Integrals

1°. Differentiation with respect to a parameter. In the case of certain restrictions imposed on the functions $f(x, \alpha)$, $f'_\alpha(x, \alpha)$ and on the corresponding improper integrals we have the *Leibniz rule*

$$\frac{d}{d\alpha}\int_a^\infty f(x, \alpha)\,dx=\int_a^\infty f'_\alpha(x, \alpha)\,dx.$$

Example 1. By differentiating with respect to a parameter, evaluate

$$\int_0^\infty \frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x}\,dx \quad (\alpha>0,\ \beta>0).$$

Solution. Let

$$\int_0^\infty \frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x}\,dx=F(\alpha, \beta).$$

Then

$$\frac{\partial F(\alpha, \beta)}{\partial\alpha}=-\int_0^\infty xe^{-\alpha x^2}\,dx=\frac{1}{2\alpha}\,e^{-\alpha x^2}\Big|_0^\infty=-\frac{1}{2\alpha}.$$

Whence $F(\alpha, \beta)=-\frac{1}{2}\ln\alpha+C(\beta)$. To find $C(\beta)$, we put $\alpha=\beta$ in the latter equation. We have $0=-\frac{1}{2}\ln\beta+C(\beta)$.

Whence $C(\beta)=\frac{1}{2}\ln\beta$. Hence,

$$F(\alpha, \beta)=-\frac{1}{2}\ln\alpha+\frac{1}{2}\ln\beta=\frac{1}{2}\ln\frac{\beta}{\alpha}.$$

2°. Improper double and triple integrals.

a) An infinite region. If a function $f(x, y)$ is continuous in an unbounded region S, then we put

$$\iint\limits_{(S)} f(x, y)\,dx\,dy = \lim_{\sigma \to S} \iint\limits_{(\sigma)} f(x, y)\,dx\,dy, \qquad (1)$$

where σ is a finite region lying entirely within S, where $\sigma \to S$ signifies that we expand the region σ by an arbitrary law so that any point of S should enter it and remain in it. If there is a limit on the right and if it does not depend on the choice of the region σ, then the corresponding improper integral is called *convergent*, otherwise it is *divergent*.

If the integrand $f(x, y)$ is nonnegative $[f(x, y) \geqslant 0]$, then for the convergence of an improper integral it is necessary and sufficient for the limit on the right of (1) to exist at least for one system of regions σ that exhaust the region S.

b) A discontinuous function. If a function $f(x, y)$ is everywhere continuous in a bounded closed region S, except the point $P(a, b)$, then we put

$$\iint\limits_{(S)} f(x, y)\,dx\,dy = \lim_{\varepsilon \to 0} \iint\limits_{(S_\varepsilon)} f(x, y)\,dx\,dy, \qquad (2)$$

where S_ε is a region obtained from S by eliminating a small region of diameter ε that contains the point P. If (2) has a limit that does not depend on the type of small regions eliminated from S, the improper integral under consideration is called *convergent*, otherwise it is *divergent*.

If $f(x, y) \geqslant 0$, then the limit on the right of (2) is not dependent on the type of regions eliminated from S; for instance, such regions may be circles of radius $\frac{\varepsilon}{2}$ with centre at P.

The concept of improper double integrals is readily extended to the case of triple integrals.

Example 2. Test for convergence

$$\iint\limits_{(S)} \frac{dx\,dy}{(1+x^2+y^2)^p}, \qquad (3)$$

where S is the entire xy-plane.

Solution. Let σ be a circle of radius ϱ with centre at the coordinate origin. Passing to polar coordinates for $p \neq 1$, we have

$$I(\sigma) = \iint\limits_{(\sigma)} \frac{dx\,dy}{(1+x^2+y^2)^p} = \int_0^{2\pi} d\varphi \int_0^{\varrho} \frac{r\,dr}{(1+r^2)^p} =$$

$$= \int_0^{2\pi} \frac{1}{2} \frac{(1+r^2)^{1-p}}{1-p} \Bigg|_0^{\varrho} d\varphi = \frac{\pi}{1-p} \left[(1+\varrho^2)^{1-p} - 1\right].$$

If $p < 1$, then $\lim\limits_{\sigma \to S} I(\sigma) = \lim\limits_{\varrho \to \infty} I(\sigma) = \infty$ and the integral diverges. But if $p > 1$, then $\lim\limits_{\varrho \to \infty} I(\sigma) = \frac{\pi}{p-1}$ and the integral converges. For $p = 1$ we have

$I(\sigma)=\int\limits_0^{2\pi} d\varphi\int\limits_0^{\varrho}\frac{r\,dr}{1+r^2}=\pi\ln(1+\varrho^2)$; $\lim\limits_{\varrho\to\infty} I(\sigma)=\infty$, that is, the integral diverges.

Thus, the integral (3) converges for $p>1$.

2273. Find $f'(x)$, if

$$f(x)=\int\limits_x^{\infty} e^{-xy^2}\,dy \quad (x>0).$$

2274. Prove that the function

$$u=\int\limits_{-\infty}^{+\infty}\frac{xf(z)}{x^2+(y-z)^2}\,dz$$

satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

2275. The Laplace transformation $F(p)$ for the function $f(t)$ is defined by the formula

$$F(p)=\int\limits_0^{\infty} e^{-pt}f(t)\,dt.$$

Find $F(p)$, if: a) $f(t)=1$; b) $f(t)=e^{\alpha t}$; c) $f(t)=\sin\beta t$; d) $f(t)=\cos\beta t$.

2276. Taking advantage of the formula

$$\int\limits_0^1 x^{n-1}dx=\frac{1}{n}\ (n>0),$$

compute the integral

$$\int\limits_0^1 x^{n-1}\ln x\,dx.$$

2277*. Using the formula

$$\int\limits_0^{\infty} e^{-pt}\,dt=\frac{1}{p}\quad (p>0),$$

evaluate the integral

$$\int\limits_0^{\infty} t^2e^{-pt}\,dt.$$

Applying differentiation with respect to a parameter, evaluate the following integrals:

2278. $\int_0^\infty \frac{e^{-\alpha x}-e^{-\beta x}}{x}\,dx \quad (\alpha>0,\ \beta>0).$

2279. $\int_0^\infty \frac{e^{-\alpha x}-e^{-\beta x}}{x}\sin mx\,dx \quad (\alpha>0,\ \beta>0).$

2280. $\int_0^\infty \frac{\arctan \alpha x}{x(1+x^2)}\,dx.$

2281. $\int_0^1 \frac{\ln(1-\alpha^2x^2)}{x^2\sqrt{1-x^2}}\,dx \quad (|\alpha|<1).$

2282. $\int_0^\infty e^{-\alpha x}\frac{\sin\beta x}{x}\,dx \quad (\alpha\geqslant 0).$

Evaluate the following improper integrals:

2283. $\int_0^\infty dx\int_0^\infty e^{-(x+y)}\,dy.$

2284. $\int_0^1 dy\int_0^{y^2} e^{\frac{x}{y}}\,dx.$

2285. $\iint\limits_{(S)} \frac{dx\,dy}{x^4+y^2}$, where S is a region defined by the inequalities $x\geqslant 1,\ y\geqslant x^2$.

2286*. $\int_0^\infty dx\int_0^\infty \frac{dy}{(x^2+y^2+a^2)^2} \quad (a>0).$

2287. The *Euler-Poisson integral* defined by the formula $I=\int_0^\infty e^{-x^2}dx$ may also be written in the form $I=\int_0^\infty e^{-y^2}dy$. Evaluate I by multiplying these formulas and then passing to polar coordinates.

2288. Evaluate

$$\int_0^\infty dx\int_0^\infty dy\int_0^\infty \frac{dz}{(x^2+y^2+z^2+1)^2}.$$

Test for convergence the improper double integrals:

2289**. $\iint\limits_{(S)} \ln\sqrt{x^2+y^2}\,dx\,dy$, where S is a circle $x^2+y^2\leqslant 1$.

2290. $\iint\limits_{(S)} \frac{dx\,dy}{(x^2+y^2)^{\alpha}}$, where S is a region defined by the inequality $x^2+y^2\geqslant 1$ ("exterior" of the circle).

2291*. $\iint\limits_{(S)} \frac{dx\,dy}{\sqrt[3]{(x-y)^2}}$, where S is a square $|x|\leqslant 1$, $|y|\leqslant 1$.

2292. $\iiint\limits_{(V)} \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{\alpha}}$, where V is a region defined by the inequality $x^2+y^2+z^2\geqslant 1$ ("exterior" of a sphere).

Sec. 9. Line Integrals

1°. Line integrals of the first type. Let $f(x, y)$ be a continuous function and $y=\varphi(x)$ $[a\leqslant x\leqslant b]$ be the equation of some smooth curve C.

Let us construct a system of points $M_i(x_i, y_i)$ $(i=0, 1, 2, \ldots, n)$ that break up the curve C into elementary arcs $\overset{\frown}{M_{i-1}M_i}=\Delta s_i$ and let us form the integral sum $S_n=\sum\limits_{i=1}^{n} f(x_i, y_i)\,\Delta s_i$. The limit of this sum, when $n\to\infty$ and $\max\Delta s_i\to 0$, is called a *line integral of the first type*

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_i)\,\Delta s_i=\int\limits_C f(x, y)\,ds$$

(ds is the arc differential) and is evaluated from the formula

$$\int\limits_C f(x, y)\,ds=\int\limits_a^b f(x, \varphi(x))\sqrt{1+(\varphi'(x))^2}\,dx.$$

In the case of parametric representation of the curve C: $x=\varphi(t)$, $y=\psi(t)$ $[\alpha\leqslant t\leqslant\beta]$, we have

$$\int\limits_C f(x, y)\,ds=\int\limits_{\alpha}^{\beta} f(\varphi(t), \psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt.$$

Also considered are line integrals of the first type of functions of three variables $f(x, y, z)$ taken along a space curve. These integrals are evaluated in like fashion A line integral of the first type *does not depend on the direction of the path of integration*; if the integrand f is interpreted as a linear density of the curve of integration C, then this integral represents the *mass of the curve C*.

Example 1. Evaluate the line integral

$$\int\limits_C (x+y)\,ds,$$

where C is the contour of the triangle ABO with vertices $A\,(1,\ 0)$, $B\,(0,\ 1)$, and $O\,(0,\ 0)$ (Fig 101).

Solution. Here, the equation AB is $y=1-x$, the equation OB is $x=0$, and the equation OA is $y=0$. We therefore have

$$\int\limits_C (x+y)\,ds=\int\limits_{AB}(x+y)\,ds+\int\limits_{BO}(x+y)\,ds+\int\limits_{OA}(x+y)\,ds=$$

$$=\int\limits_0^1 \sqrt{2}\,dx+\int\limits_0^1 y\,dy+\int\limits_0^1 x\,dx=\sqrt{2}+1.$$

2°. **Line integrals of the second type.** If $P\,(x,\ y)$ and $Q\,(x,\ y)$ are continuous functions and $y=\varphi\,(x)$ is a smooth curve C that runs from a to b as

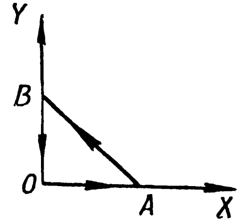

Fig. 101

x varies, then the corresponding *line integral of the second type* is expressed as follows:

$$\int\limits_C P\,(x,\ y)\,dx+Q\,(x,\ y)\,dy=\int\limits_a^b [P\,(x,\ \varphi\,(x))+\varphi'\,(x)\,Q\,(x,\ \varphi\,(x))]\,dx.$$

In the more general case when the curve C is represented parametrically: $x=\varphi\,(t)$, $y=\psi\,(t)$, where t varies from α to β, we have

$$\int\limits_C P\,(x,\ y)\,dx+Q\,(x,\ y)\,dy+\int\limits_\alpha^\beta [P\,(\varphi\,(t),\ \psi\,(t))\,\varphi'\,(t)+Q\,(\varphi\,(t),\ \psi\,(t))\,\psi'\,(t)]\,dt.$$

Similar formulas hold for a line integral of the second type taken over a space curve.

A line integral of the second type *changes sign when the direction of the path of integration is reversed*. This integral may be interpreted mechanically as the work of an appropriate variable force $\{P\,(x,\ y),\ Q\,(x,\ y)\}$ along the curve of integration C

Example 2. Evaluate the line integral

$$\int\limits_C y^2\,dx+x^2\,dy,$$

where C is the upper half of the ellipse $x=a\cos t$, $y=b\sin t$ traversed clockwise.

Solution. We have

$$\int\limits_C y^2\,dx+x^2\,dy=\int\limits_{\pi}^{0}[b^2\sin^2 t\cdot(-a\sin t)+a^2\cos^2 t\cdot b\cos t]\,dt=$$
$$=-ab^2\int\limits_{\pi}^{0}\sin^3 t\,dt+a^2b\int\limits_{\pi}^{0}\cos^3 t\,dt=\frac{4}{3}ab^2.$$

3°. The case of a total differential. If the integrand of a line integral of the second type is a total differential of some single-valued function $U=U(x, y)$, that is, $P(x, y)\,dx+Q(x, y)\,dy=dU(x, y)$, then this line integral is not dependent on the path of integration and we have the Newton-Leibniz formula

$$\int\limits_{(x_1, y_1)}^{(x_2, y_2)} P(x, y)\,dx+Q(x, y)\,dy=U(x_2, y_2)-U(x_1, y_1), \qquad (1)$$

where (x_1, y_1) is the initial and (x_2, y_2) is the terminal point of the path In particular, if the contour of integration C is closed, then

$$\oint\limits_C P(x, y)\,dx+Q(x, y)\,dy=0 \qquad (2)$$

If 1) the contour of integration C is contained entirely within some simply-connected region S and 2) the functions $P(x, y)$ and $Q(x, y)$ together with their partial derivatives of the first order are continuous in S, then a necessary and sufficient condition for the existence of the function U is the identical fulfilment (in S) of the equality

$$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y} \qquad (3)$$

(see integration of total differentials) If conditions one and two are not fulfilled, the presence of condition (3) does not guarantee the existence of a single-valued function U, and formulas (1) and (2) may prove wrong (see Problem 2332) We give a method of finding a function $U(x, y)$ from its total differential based on the use of line integrals (which is yet another method of integrating a total differential). For the contour of integration C let us take a broken line P_0P_1M (Fig 102), where $P_0(x_0, y_0)$ is a fixed point and $M(x, y)$ is a variable point. Then along P_0P_1 we have $y=y_0$ and $dy=0$, and along P_1M we have $dx=0$ We get:

$$U(x, y)-U(x_0, y_0)=\int\limits_{(x_0, y_0)}^{(x, y)} P(x, y)\,dx+Q(x, y)\,dy=$$
$$=\int\limits_{x_0}^{x} P(x, y_0)\,dx+\int\limits_{y_0}^{y} Q(x, y)\,dy.$$

Similarly, integrating with respect to P_0P_2M, we have

$$U(x, y)-U(x_0, y_0)=\int\limits_{y_0}^{y} Q(x_0, y)\,dy+\int\limits_{x_0}^{x} P(x, y)\,dx.$$

Example 3. $(4x+2y)\,dx+(2x-6y)\,dy=dU$. Find U.

Solution. Let $x_0=0$, $y_0=0$. Then

$$U(x, y)=\int_0^x 4x\,dx+\int_0^y (2x-6y)\,dy+C=2x^2+2xy-3y^2+C$$

or

$$U(x, y)=\int_0^y -6y\,dy+\int_0^x (4x+2y)\,dx+C=-3y^2+2x^2+3xy+C,$$

where $C=U(0, 0)$ is an arbitrary constant.

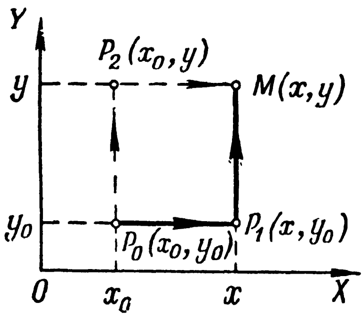

Fig. 102

4°. Green's formula for a plane. If C is the boundary of a region S and the functions $P(x, y)$ and $Q(x, y)$ are continuous together with their first-order partial derivatives in the closed region $S+C$, then *Green's formula* holds:

$$\oint_C P\,dx+Q\,dy=\iint_{(S)}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy,$$

here the circulation about the contour C is chosen so that the region S should remain to the left.

5°. Applications of line integrals.[1]) An *area* bounded by the closed contour C is

$$S=-\oint_C y\,dx=\oint_C x\,dy$$

(the direction of circulation of the contour is chosen counterclockwise).

The following formula for area is more convenient for application:

$$S=\frac{1}{2}\oint_C (x\,dy-y\,dx)=\frac{1}{2}\oint_C x^2\,d\left(\frac{y}{x}\right).$$

2) The *work of a force*, having projections $X=X(x, y, z)$, $Y=Y(x, y, z)$, $Z=Z(x, y, z)$ (or, accordingly, the work of a force field), along a path C is

expressed by the integral

$$A=\int_C X\,dx+Y\,dy+Z\,dz.$$

If the force has a potential, i.e., if there exists a function $U=U(x, y, z)$ (a potential function or a force function) such that

$$\frac{\partial U}{\partial x}=X,\quad \frac{\partial U}{\partial y}=Y,\quad \frac{\partial U}{\partial z}=Z,$$

then the work, irrespective of the shape of the path C, is equal to

$$A=\int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} X\,dx+Y\,dy+Z\,dz=\int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} dU=U(x_2, y_2, z_2)-U(x_1, y_1, z_1),$$

where (x_1, y_1, z_1) is the initial and (x_2, y_2, z_2) is the terminal point of the path.

A. *Line Integrals of the First Type*

Evaluate the following line integrals:

2293. $\int_C xy\,ds$, where C is the contour of the square $|x|+|y|=a$ $(a>0)$.

2294. $\int_C \frac{ds}{\sqrt{x^2+y^2+4}}$, where C is a segment of the straight line connecting the points $O(0, 0)$ and $A(1, 2)$.

2295. $\int_C xy\,ds$, where C is a quarter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ lying in the first quadrant.

2296. $\int_C y^2 ds$, where C is the first arc of the cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$.

2297. $\int_C \sqrt{x^2+y^2}\,ds$, where C is an arc of the involute of the circle $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$ $[0\leqslant t\leqslant 2\pi]$.

2298. $\int_C (x^2+y^2)^2\,ds$, where C is an arc of the logarithmic spiral $r=ae^{m\varphi}$ $(m>0)$ from the point $A(0, a)$ to the point $O(-\infty, 0)$.

2299. $\int_C (x+y)\,ds$, where C is the right-hand loop of the lemniscate $r^2=a^2\cos 2\varphi$.

2300. $\int_C (x+y)\,ds$, where C is an arc of the curve $x=t$, $y=\frac{3t^2}{\sqrt{2}}$, $z=t^3$ $(0\leqslant t\leqslant 1]$.

2301. $\int\limits_C \frac{ds}{x^2+y^2+z^2}$, where C is the first turn of the screw-line $x=a\cos t$, $y=a\sin t$, $z=bt$.

2302. $\int\limits_C \sqrt{2y^2+z^2}\,ds$, where C is the circle $x^2+y^2+z^2=a^2$, $x=y$.

2303*. Find the area of the lateral surface of the parabolic cylinder $y=\frac{3}{8}x^2$ bounded by the planes $z=0$, $x=0$, $z=x$, $y=6$.

2304. Find the arc length of the conic screw-line C $x=ae^t\cos t$, $y=ae^t\sin t$, $z=ae^t$ from the point $O(0, 0, 0)$ to the point $A(a, 0, a)$.

2305. Determine the mass of the contour of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if the linear density of it at each point $M(x, y)$ is equal to $|y|$.

2306. Find the mass of the first turn of the screw-line $x=a\cos t$, $y=a\sin t$, $z=bt$, if the density at each point is equal to the radius vector of this point.

2307. Determine the coordinates of the centre of gravity of a half-arc of the cycloid

$$x=a(t-\sin t),\quad y=a(1-\cos t)\quad [0\leqslant t\leqslant \pi].$$

2308. Find the moment of inertia, about the z-axis, of the first turn of the screw-line $x=a\cos t$, $y=a\sin t$, $z=bt$.

2309. With what force will a mass M distributed with uniform density over the circle $x^2+y^2=a^2$, $z=0$, act on a mass m located at the point $A(0, 0, b)$?

B. *Line Integrals of the Second Type*

Evaluate the following line integrals:

2310. $\int\limits_{AB} (x^2-2xy)\,dx+(2xy+y^2)\,dy$, where AB is an arc of the parabola $y=x^2$ from the point $A(1, 1)$ to the point $B(2, 4)$.

2311. $\int\limits_C (2a-y)\,dx+x\,dy$, where C is an arc of the first arch of the cycloid

$$x=a(t-\sin t),\quad y=a(1-\cos t)$$

which arc runs in the direction of increasing parameter t.

2312. $\int\limits_{OA} 2xy\,dx-x^2\,dy$ taken along different paths emanating from the coordinate origin $O(0, 0)$ and terminating at the point $A(2, 1)$ (Fig. 103):

a) the straight line OmA;

b) the parabola OnA, the axis of symmetry of which is the y-axis;
c) the parabola OpA, the axis of symmetry of which is the x-axis;
d) the broken line OBA;
e) the broken line OCA.

2313. $\int\limits_{OA} 2xy\,dx + x^2\,dy$ as in Problem 2312.

2314*. $\oint \frac{(x+y)\,dx-(x-y)\,dy}{x^2+y^2}$ taken along the circle $x^2+y^2=a^2$ counterclockwise.

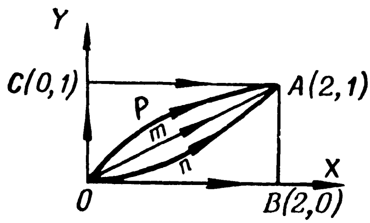

Fig. 103

2315. $\int\limits_{C} y^2\,dx+x^2\,dy$, where C is the upper half of the ellipse $x=a\cos t$, $y=b\sin t$ traced clockwise.

2316. $\int\limits_{AB} \cos y\,dx-\sin x\,dy$ taken along the segment AB of the bisector of the second quadrantal angle, if the abscissa of the point A is 2 and the ordinate of B is 2.

2317. $\oint\limits_{C} \frac{xy\,(y\,dx-x\,dy)}{x^2+y^2}$, where C is the right-hand loop of the lemniscate $r^2=a^2\cos 2\varphi$ traced counterclockwise.

2318. Evaluate the line integrals with respect to expressions which are total differentials:

a) $\int\limits_{(-1,\,2)}^{(2,\,3)} x\,dy+y\,dx$, b) $\int\limits_{(0,\,1)}^{(3,\,4)} x\,dx+y\,dy$, c) $\int\limits_{(0,\,0)}^{(1,\,1)} (x+y)(dx+dy)$,

d) $\int\limits_{(1,\,2)}^{(2,\,1)} \frac{y\,dx-x\,dy}{y^2}$ (along a path that does not intersect the x-axis),

e) $\int\limits_{\left(\frac{1}{2},\,\frac{1}{2}\right)}^{(x,\,y)} \frac{dx+dy}{x+y}$ (along a path that does not intersect the straight line $x+y=0$),

f) $\int\limits_{(x_1,\,y_1)}^{(x_2,\,y_2)} \varphi(x)\,dx+\psi(y)\,dy$.

2319. Find the antiderivative functions of the integrands and evaluate the integrals:

a) $\int\limits_{(-2,\,-1)}^{(3,\,0)} (x^4+4xy^3)\,dx+(6x^2y^2-5y^4)\,dy$,

b) $\int\limits_{(0,\,-1)}^{(1,\,0)} \frac{x\,dy-y\,dx}{(x-y)^2}$ (the integration path does not intersect the straight line $y=x$),

c) $\int\limits_{(1,\,1)}^{(3,\,1)} \frac{(x+2y)\,dx+y\,dy}{(x+y)^2}$ (the integration path does not intersect the straight line $y=-x$),

d) $\int\limits_{(0,\,0)}^{(1,\,1)} \left(\frac{x}{\sqrt{x^2+y^2}}+y\right)dx+\left(\frac{y}{\sqrt{x^2+y^2}}+x\right)dy$.

2320. Compute

$$I=\int \frac{x\,dx+y\,dy}{\sqrt{1+x^2+y^2}},$$

taken clockwise along the quarter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ that lies in the first quadrant.

2321. Show that if $f(u)$ is a continuous function and C is a closed piecewise-smooth contour, then

$$\oint\limits_C f(x^2+y^2)(x\,dx+y\,dy)=0.$$

2322. Find the antiderivative function U if:

a) $du=(2x+3y)\,dx+(3x-4y)\,dy$;

b) $du=(3x^2-2xy+y^2)\,dx-(x^2-2xy+3y^2)\,dy$;

c) $du=e^{x-y}[(1+x+y)\,dx+(1-x-y)\,dy]$;

d) $du=\frac{dx}{x+y}+\frac{dy}{x+y}$.

Evaluate the line integrals taken along the following space curves:

2323. $\int\limits_C (y-z)\,dx+(z-x)\,dy+(x-y)\,dz$, where C is a turn of the screw-line

$$\begin{cases} x=a\cos t, \\ y=a\sin t, \\ z=bt, \end{cases}$$

corresponding to the variation of the parameter t from 0 to 2π.

2324. $\oint\limits_C y\,dx+z\,dy+x\,dz$, where C is the circle

$$\begin{cases} x=R\cos\alpha\cos t, \\ y=R\cos\alpha\sin t, \\ z=R\sin\alpha \quad (\alpha=\text{const}), \end{cases}$$

traced in the direction of increasing parameter.

2325. $\int\limits_{OA} xy\,dx+yz\,dy+zx\,dz$, where OA is an arc of the circle

$$x^2+y^2+z^2=2Rx,\quad z=x,$$

situated on the side of the xz-plane where $y>0$.

2326. Evaluate the line integrals of the total differentials:

a) $\displaystyle\int\limits_{(1,\,0,\,-3)}^{(6,\,4,\,8)} x\,dx+y\,dy-z\,dz,$

b) $\displaystyle\int\limits_{(1,\,1,\,1)}^{(a,\,b,\,c)} yz\,dx+zx\,dy+xy\,dz,$

c) $\displaystyle\int\limits_{(0,\,0,\,0)}^{(3,\,4,\,5)} \frac{x\,dx+y\,dy+z\,dz}{\sqrt{x^2+y^2+z^2}},$

d) $\displaystyle\int\limits_{(1,\,1,\,1)}^{\left(x,\,y,\,\frac{1}{xy}\right)} \frac{yz\,dx+zx\,dy+xy\,dz}{xyz}$ (the integration path is situated in the first octant).

C. *Green's Formula*

2327. Using Green's formula, transform the line integral

$$I=\oint\limits_C \sqrt{x^2+y^2}\,dx+y\left[xy+\ln\left(x+\sqrt{x^2+y^2}\right)\right]dy,$$

where the contour C bounds the region S.

2328. Applying Green's formula, evaluate

$$I=\oint_C 2(x^2+y^2)\,dx+(x+y)^2\,dy,$$

where C is the contour of a triangle (traced in the positive direction) with vertices at the points $A(1, 1)$, $B(2, 2)$ and $C(1, 3)$. Verify the result obtained by computing the integral directly.

2329. Applying Green's formula, evaluate the integral

$$\oint_C -x^2y\,dx+xy^2\,dy,$$

where C is the circle $x^2+y^2=R^2$ traced counterclockwise.

2330. A parabola AmB, whose axis is the y-axis and whose chord is AnB, is drawn through the points $A(1, 0)$ and $B(2, 3)$. Find $\oint\limits_{AmBnA}(x+y)\,dx-(x-y)\,dy$ directly and by applying Green's formula.

2331. Find $\int\limits_{AmB} e^{xy}[y^2\,dx+(1+xy)\,dy]$, if the points A and B lie on the x-axis, while the area, bounded by the integration path AmB and the segment AB, is equal to S.

2332*. Evaluate $\oint\limits_C \frac{x\,dy-y\,dx}{x^2+y^2}$. Consider two cases:

a) when the origin is outside the contour C,

b) when the contour encircles the origin n times.

2333.** Show that if C is a closed curve, then

$$\oint_C \cos(X, n)\,ds=0,$$

where s is the arc length and n is the outer normal.

2334. Applying Green's formula, find the value of the integral

$$I=\oint_C [x\cos(X, n)+y\sin(X, n)]\,ds,$$

where ds is the differential of the arc and n is the outer normal to the contour C.

2335*. Evaluate the integral

$$\oint_C \frac{dx-dy}{x+y},$$

taken along the contour of a square with vertices at the points $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$, provided the contour is traced counterclockwise.

D. Applications of the Line Integral

Evaluate the areas of figures bounded by the following curves:

2336. The ellipse $x = a\cos t$, $y = b\sin t$.

2337. The astroid $x = a\cos^3 t$, $y = a\sin^3 t$.

2338. The cardioid $x = a\ (2\cos t - \cos 2t)$, $y = a\ (2\sin t - \sin 2t)$.

2339*. A loop of the folium of Descartes $x^3 + y^3 - 3axy = 0$ $(a > 0)$.

2340. The curve $(x + y)^3 = axy$.

2341*. A circle of radius r is rolling without sliding along a fixed circle of radius R and outside it. Assuming that $\frac{R}{r}$ is an integer, find the area bounded by the curve (epicycloid) described by some point of the moving circle. Analyze the particular case of $r = R$ (cardioid).

2342*. A circle of radius r is rolling without sliding along a fixed circle of radius R and inside it. Assuming that $\frac{R}{r}$ is an integer, find the area bounded by the curve (hypocycloid) described by some point of the moving circle. Analyze the particular case when $r = \frac{R}{4}$ (astroid).

2343. A field is generated by a force of constant magnitude F in the positive x-direction Find the work that the field does when a material point traces clockwise a quarter of the circle $x^2 + y^2 = R^2$ lying in the first quadrant.

2344. Find the work done by the force of gravity when a material point of mass m is moved from position $A\ (x_1,\ y_1,\ z_1)$ to position $B\,(x_2,\ y_2,\ z_2)$ (the z-axis is directed vertically upwards).

2345. Find the work done by an elastic force directed towards the coordinate origin if the magnitude of the force is proportional to the distance of the point from the origin and if the point of application of the force traces counterclockwise a quarter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ lying in the first quadrant.

2346. Find the potential function of a force $R\,\{X,\ Y,\ Z\}$ and determine the work done by the force over a given path if:

a) $X = 0$, $Y = 0$, $Z = -mg$ (force of gravity) and the material point is moved from position $A\,(x_1,\ y_1,\ z_1)$ to position $B\,(x_2,\ y_2,\ z_2)$;

b) $X = -\frac{\mu x}{r^3}$, $Y = -\frac{\mu y}{r^3}$, $Z = -\frac{\mu z}{r^3}$, where $\mu = \text{const}$ and $r = \sqrt{x^2 + y^2 + z^2}$ (Newton attractive force) and the material point moves from position $A\,(a,\ b,\ c)$ to infinity;

c) $X=-k^2x$, $Y=-k^2y$, $Z=-k^2z$, where $k=\text{const}$ (elastic force), and the initial point of the path is located on the sphere $x^2+y^2+z^2=R^2$, while the terminal point is located on the sphere $x^2+y^2+z^2=r^2$ $(R>r)$.

Sec. 10. Surface Integrals

1°. Surface integral of the first type. Let $f(x, y, z)$ be a continuous function and $z=\varphi(x, y)$ a smooth surface S.

The *surface integral of the first type* is the limit of the integral sum

$$\iint\limits_S f(x, y, z)\,dS=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta S_i,$$

where ΔS_i is the area of the ith element of the surface S, the point (x_i, y_i, z_i) belongs to this element, and the maximum diameter of elements of partition tends to zero.

The value of this integral is not dependent on the choice of side of the surface S over which the integration is performed.

If a projection σ of the surface S on the xy-plane is single-valued, that is, every straight line parallel to the z-axis intersects the surface S at only one point, then the appropriate surface integral of the first type may be calculated from the formula

$$\iint\limits_S f(x, y, z)\,dS=\iint\limits_{(\sigma)} f[x, y, \varphi(x, y)]\sqrt{1+\varphi_x'^2(x, y)+\varphi_y'^2(x, y)}\,dx\,dy.$$

Example 1. Compute the surface integral

$$\iint\limits_S (x+y+z)\,dS,$$

where S is the surface of the cube $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$, $0\leqslant z\leqslant 1$.

Let us compute the sum of the surface integrals over the upper edge of the cube $(z=1)$ and over the lower edge of the cube $(z=0)$:

$$\int_0^1\!\!\int_0^1 (x+y+1)\,dx\,dy+\int_0^1\!\!\int_0^1 (x+y)\,dx\,dy=\int_0^1\!\!\int_0^1 (2x+2y+1)\,dx\,dy=3.$$

The desired surface integral is obviously three times greater and equal to

$$\iint\limits_S (x+y+z)\,dS=9.$$

2°. Surface integral of the second type. If $P=P(x, y, z)$, $Q=Q(x, y, z)$, $R=R(x, y, z)$ are continuous functions and S^+ is a side of the smooth surface S characterized by the direction of the normal $n\{\cos\alpha, \cos\beta, \cos\gamma\}$, then the corresponding *surface integral of the second type* is expressed as follows:

$$\iint\limits_{S+} P\,dy\,dz+Q\,dz\,dx+R\,dx\,dy=\iint\limits_S (P\cos\alpha+Q\cos\beta+R\cos\gamma)\,dS.$$

When we pass to the other side, S^-, of the surface, this integral reverses sign.

If the surface S is represented implicitly, $F(x, y, z)=0$, then the direction cosines of the normal of this surface are determined from the formulas

$$\cos\alpha=\frac{1}{D}\frac{\partial F}{\partial x}, \quad \cos\beta=\frac{1}{D}\frac{\partial F}{\partial y}, \quad \cos\gamma=\frac{1}{D}\frac{\partial F}{\partial z},$$

where

$$D=\pm\sqrt{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2},$$

and the choice of sign before the radical should be brought into agreement with the side of the surface S.

3°. Stokes' formula. If the functions $P=P(x, y, z)$, $Q=Q(x, y, z)$, $R=R(x, y, z)$ are continuously differentiable and C is a closed contour bounding a two-sided surface S, we then have the *Stokes' formula*

$$\oint_C P\,dx+Q\,dy+R\,dz=$$

$$=\iint_S\left[\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\cos\alpha+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\cos\beta+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\cos\gamma\right]dS,$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the normal to the surface S, and the direction of the normal is defined so that on the side of the normal the contour S is traced counterclockwise (in a right-handed coordinate system).

Evaluate the following surface integrals of the first type:

2347. $\iint_S (x^2+y^2)\,dS$, where S is the sphere $x^2+y^2+z^2=a^2$.

2348. $\iint_S \sqrt{x^2+y^2}\,dS$ where S is the lateral surface of the cone $\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=0$ $[0 \leqslant z \leqslant b]$.

Evaluate the following surface integrals of the second type:

2349. $\iint_S yz\,dy\,dz+xz\,dz\,dx+xy\,dx\,dy$, where S is the external side of the surface of a tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=a$.

2350. $\iint_S z\,dx\,dy$, where S is the external side of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

2351. $\iint_S x^2\,dy\,dz+y^2\,dz\,dx+z^2\,dx\,dy$, where S is the external side of the surface of the hemisphere $x^2+y^2+z^2=a^2$ $(z \geqslant 0)$.

2352. Find the mass of the surface of the cube $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$, if the surface density at the point $M(x, y, z)$ is equal to xyz.

2353. Determine the coordinates of the centre of gravity of a homogeneous parabolic envelope $az = x^2 + y^2$ $(0 \leqslant z \leqslant a)$.

2354. Find the moment of inertia of a part of the lateral surface of the cone $z = \sqrt{x^2 + y^2}$ $[0 \leqslant z \leqslant h]$ about the z-axis.

2355. Applying Stokes' formula, transform the integrals:

a) $\oint\limits_C (x^2 - yz)\,dx + (y^2 - zx)\,dy + (z^2 - xy)\,dz$;

b) $\oint\limits_C y\,dx + z\,dy + x\,dz$.

Applying Stokes' formula, find the given integrals and verify the results by direct calculations:

2356. $\oint\limits_C (y + z)\,dx + (z + x)\,dy + (x + y)\,dz$, where C is the circle

$$x^2 + y^2 + z^2 = a^2, \quad x + y + z = 0.$$

2357. $\oint\limits_C (y - z)\,dx + (z - x)\,dy + (x - y)\,dz$, where C is the ellipse

$$x^2 + y^2 = 1, \quad x + z = 1.$$

2358. $\oint\limits_C x\,dx + (x + y)\,dy + (x + y + z)\,dz$, where C is the curve $x = a \sin t$, $y = a \cos t$, $z = a(\sin t + \cos t)$ $[0 \leqslant t \leqslant 2\pi]$.

2359. $\oint\limits_{ABCA} y^2\,dx + z^2\,dy + x^2\,dz$, where $ABCA$ is the contour of $\triangle ABC$ with vertices $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$.

2360. In what case is the line integral

$$I = \oint\limits_C P\,dx + Q\,dy + R\,dz$$

over any closed contour C equal to zero?

Sec. 11. The Ostrogradsky-Gauss Formula

If S is a closed smooth surface bounding the volume V, and $P = P(x, y, z)$, $Q = Q(x, y, z)$, $R = R(x, y, z)$ are functions that are continuous together with their first partial derivatives in the closed region V, then we have the *Ostrogradsky-Gauss formula*

$$\iint\limits_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS = \iiint\limits_{(V)} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz,$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the outer normal to the surface S

Applying the Ostrogradsky-Gauss formula, transform the following surface integrals over the closed surfaces S bounding the volume V ($\cos\alpha$, $\cos\beta$, $\cos\gamma$ are direction cosines of the outer normal to the surface S).

2361. $\iint\limits_S xy\,dx\,dy + yz\,dy\,dz + zx\,dz\,dx.$

2362. $\iint\limits_S x^2\,dy\,dz + y^2\,dz\,dx + z^2\,dx\,dy.$

2363. $\iint\limits_S \frac{x\cos\alpha + y\cos\beta + z\cos\gamma}{\sqrt{x^2+y^2+z^2}}\,dS.$

2364. $\iint\limits_S \left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma\right)dS.$

Using the Ostrogradsky-Gauss formula, compute the following surface integrals:

2365. $\iint\limits_S x^2\,dy\,dz + y^2\,dz\,dx + z^2\,dx\,dy$, where S is the external side of the surface of the cube $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant a$, $0 \leqslant z \leqslant a$.

2366. $\iint\limits_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$, where S is the external side of a pyramid bounded by the surfaces $x+y+z=a$, $x=0$, $y=0$, $z=0$.

2367. $\iint\limits_S x^3\,dy\,dz + y^3\,dz\,dx = z^3\,dx\,dy$, where S is the external side of the sphere $x^2+y^2+z^2=a^2$.

2368 $\iint\limits_S (x^2\cos\alpha + y^2\cos\beta + z^2\cos\gamma)\,dS$, where S is the external total surface of the cone

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=0 \qquad [0 \leqslant z \leqslant b].$$

2369. Prove that if S is a closed surface and l is any fixed direction, then

$$\iint\limits_S \cos(\boldsymbol{n},\ \boldsymbol{l})\,dS = 0,$$

where $\boldsymbol{n}$ is the outer normal to the surface S.

2370. Prove that the volume of the solid V bounded by the surface S is equal to

$$V = \frac{1}{3}\iint\limits_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\,dS,$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the outer normal to the surface S.

Sec. 12. Fundamentals of Field Theory

1°. Scalar and vector fields. A *scalar field* is defined by the scalar function of the point $u=f(P)=f(x, y, z)$, where $P(x, y, z)$ is a point of space. The surfaces $f(x, y, z)=C$, where $C=\text{const}$, are called *level surfaces* of the scalar field.

A *vector field* is defined by the vector function of the point $\boldsymbol{a}=\boldsymbol{a}(P)=\boldsymbol{a}(\boldsymbol{r})$, where P is a point of space and $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ is the radius vector of the point P. In coordinate form, $\boldsymbol{a}=a_x\boldsymbol{i}+a_y\boldsymbol{j}+a_z\boldsymbol{k}$, where $a_x=a_x(x, y, z)$, $a_y=a_y(x, y, z)$, and $a_z=a_z(x, y, z)$ are projections of the vector $\boldsymbol{a}$ on the coordinate axes. The *vector lines* (*force lines*, *flow lines*) of a vector field are found from the following system of differential equations

$$\frac{dx}{a_x}=\frac{dy}{a_y}=\frac{dz}{a_z}.$$

A scalar or vector field that does not depend on the time t is called *stationary;* if it depends on the time, it is called *nonstationary.*

2°. Gradient. The vector

$$\operatorname{grad} U(P)=\frac{\partial U}{\partial x}\boldsymbol{i}+\frac{\partial U}{\partial y}\boldsymbol{j}+\frac{\partial U}{\partial z}\boldsymbol{k}\equiv\nabla U,$$

where $\nabla=\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z}$ is the Hamiltonian operator (del, or nabla), is called the *gradient* of the field $U=f(P)$ at the given point P (cf. Ch. VI, Sec. 6). The gradient is in the direction of the normal n to the level surface at the point P and in the direction of increasing function U, and has length equal to

$$\frac{\partial U}{\partial n}=\sqrt{\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2}.$$

If the direction is given by the unit vector $\boldsymbol{l}\{\cos\alpha, \cos\beta, \cos\gamma\}$, then

$$\frac{\partial U}{\partial l}=\operatorname{grad}\boldsymbol{U}\cdot\boldsymbol{l}=\operatorname{grad}_l\boldsymbol{U}=\frac{\partial U}{\partial x}\cos\alpha+\frac{\partial U}{dy}\cos\beta+\frac{\partial U}{\partial z}\cos\gamma$$

(the derivative of the function U in the direction l).

3°. Divergence and rotation. The *divergence* of a vector field $\boldsymbol{a}(P)=a_x\boldsymbol{i}+a_y\boldsymbol{j}+a_z\boldsymbol{k}$ is the scalar $\operatorname{div}\boldsymbol{a}=\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\equiv\nabla\boldsymbol{a}$.

The *rotation* (curl) of a vector field $\boldsymbol{a}(P)=a_x\boldsymbol{i}+a_y\boldsymbol{j}+a_z\boldsymbol{k}$ is the vector

$$\operatorname{rot}\boldsymbol{a}=\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)\boldsymbol{k}\equiv\nabla\times\boldsymbol{a}.$$

4°. Flux of a vector. The *flux* of a vector field $\boldsymbol{a}(P)$ through a surface S in a direction defined by the unit vector of the normal $n\{\cos\alpha, \cos\beta, \cos\gamma\}$ to the surface S is the integral

$$\iint_S \boldsymbol{an}\,dS=\iint_S a_n\,dS=\iint_S(a_x\cos\alpha+a_y\cos\beta+a_z\cos\gamma)\,dS.$$

If S is a closed surface bounding a volume V, and n is a unit vector of the outer normal to the surface S, then the *Ostrogradsky-Gauss formula* holds, which in vector form is

$$\oiint\limits_{S} a_n\, dS = \iiint\limits_{(V)} \operatorname{div} \boldsymbol{a}\, dx\, dy\, dz.$$

5°. Circulation of a vector, the work of a field. The *line integral* of the vector $\boldsymbol{a}$ along the curve C is defined by the formula

$$\int\limits_C \boldsymbol{a}\, d\boldsymbol{r} = \int\limits_C a_s\, ds = \int\limits_C a_x\, dx + a_y dy + a_z\, dz \quad (1)$$

and represents the *work* done by the field $\boldsymbol{a}$ along the curve C (a_s is the projection of the vector $\boldsymbol{a}$ on the tangent to C).

If C is closed, then the line integral (1) is called the *circulation* of the vector field $\boldsymbol{a}$ around the contour C.

If the closed curve C bounds a two-sided surface S, then *Stokes' formula* holds, which in vector form has the form

$$\oint\limits_C \boldsymbol{a}\, d\boldsymbol{r} = \iint\limits_S \boldsymbol{n} \operatorname{rot} \boldsymbol{a}\, dS,$$

where $\boldsymbol{n}$ is the vector of the normal to the surface S; the direction of the vector should be chosen so that for an observer looking in the direction of $\boldsymbol{n}$ the circulation of the contour C should be counterclockwise in a right-handed coordinate system.

6°. **Potential and solenoidal fields.** The vector field $\boldsymbol{a}(\boldsymbol{r})$ is called *potential* if

$$\boldsymbol{a} = \operatorname{grad} U,$$

where $U = f(\boldsymbol{r})$ is a scalar function (the *potential* of the field).

For the potentiality of a field $\boldsymbol{a}$, given in a simply-connected domain, it is necessary and sufficient that it be *nonrotational*, that is, $\operatorname{rot} \boldsymbol{a} = 0$. In that case there exists a potential U defined by the equation

$$dU = a_x\, dx + a_y\, dy + a_z\, dz.$$

If the potential U is a single-valued function, then $\int\limits_{AB} \boldsymbol{a}\, d\boldsymbol{r} = U(B) - U(A)$;

in particular, the circulation of the vector $\boldsymbol{a}$ is equal to zero: $\oint\limits_C \boldsymbol{a}\, d\boldsymbol{r} = 0$.

A vector field $\boldsymbol{a}(\boldsymbol{r})$ is called *solenoidal* if at each point of the field $\operatorname{div} \boldsymbol{a} = 0$; in this case the flux of the vector through any closed surface is zero.

If the field is at the same time potential and solenoidal, then $\operatorname{div}(\operatorname{grad} U) = 0$ and the potential function U is harmonic; that is, it satisfies the Laplace equation $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$, or $\Delta U = 0$, where $\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator

2371. Determine the level surfaces of the scalar field $U = f(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. What will the level surfaces be of a field $U = F(\varrho)$, where $\varrho = \sqrt{x^2 + y^2}$?

2372. Determine the level surfaces ot the scalar field

$$U = \arc\sin \frac{z}{\sqrt{x^2+y^2}}.$$

2373. Show that straight lines parallel to a vector $\boldsymbol{c}$ are the vector lines of a vector field $\boldsymbol{a}(P)=\boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector.

2374. Find the vector lines of the field $\boldsymbol{a} = -\omega y\boldsymbol{i} + \omega x\boldsymbol{j}$, where ω is a constant.

2375. Derive the formulas:

a) $\operatorname{grad}(C_1U+C_2V)=C_1\operatorname{grad}U+C_2\operatorname{grad}V$, where C_1 and C_2 are constants;

b) $\operatorname{grad}(UV)=U\operatorname{grad}V+V\operatorname{grad}U$;

c) $\operatorname{grad}(U^2)=2U\operatorname{grad}U$;

d) $\operatorname{grad}\left(\frac{U}{V}\right)=\frac{V\operatorname{grad}U-U\operatorname{grad}V}{V^2}$;

e) $\operatorname{grad}\varphi(U)=\varphi'(U)\operatorname{grad}U$.

2376. Find the magnitude and the direction of the gradient of the field $U=x^3+y^3+z^3-3xyz$ at the point $A(2, 1, 1)$. Determine at what points the gradient of the field is perpendicular to the z-axis and at what points it is equal to zero.

2377. Evaluate grad U, if U is equal, respectively, to: a) r, b) r^2, c) $\frac{1}{r}$, d) $f(r)\ (r=\sqrt{x^2+y^2+z^2})$.

2378. Find the gradient of the scalar field $U=\boldsymbol{cr}$, where $\boldsymbol{c}$ is a constant vector. What will the level surfaces be of this field, and what will their position be relative to the vector $\boldsymbol{c}$?

2379. Find the derivative of the function $U=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$ at a given point $P(x, y, z)$ in the direction of the radius vector $\boldsymbol{r}$ of this point. In what case will this derivative be equal to the magnitude of the gradient?

2380. Find the derivative of the function $U=\frac{1}{r}$ in the direction of $\boldsymbol{l}\{\cos\alpha, \cos\beta, \cos\gamma\}$. In what case will this derivative be equal to zero?

2381. Derive the formulas:

a) $\operatorname{div}(C_1\boldsymbol{a}_1+C_2\boldsymbol{a}_2)=C_1\operatorname{div}\boldsymbol{a}_1+C_2\operatorname{div}\boldsymbol{a}_2$, where C_1 and C_2 are constants;

b) $\operatorname{div}(U\boldsymbol{c})=\operatorname{grad}U\cdot\boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector;

c) $\operatorname{div}(U\boldsymbol{a})=\operatorname{grad}U\cdot\boldsymbol{a}+U\operatorname{div}\boldsymbol{a}$.

2382. Evaluate $\operatorname{div}\left(\frac{\boldsymbol{r}}{r}\right)$.

2383. Find div $\boldsymbol{a}$ for the central vector field $\boldsymbol{a}(P)=f(r)\frac{\boldsymbol{r}}{r}$, where $r=\sqrt{x^2+y^2+z^2}$.

2384. Derive the formulas:

a) $\operatorname{rot}(C_1\boldsymbol{a}_1+C_2\boldsymbol{a}_2)=C_1\operatorname{rot}\boldsymbol{a}_1+C_2\operatorname{rot}\boldsymbol{a}_2$, where C_1 and C_2 are constants;

b) $\operatorname{rot}(U\boldsymbol{c})=\operatorname{grad}U\cdot\boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector;

c) $\operatorname{rot}(U\boldsymbol{a})=\operatorname{grad}U\cdot\boldsymbol{a}+U\operatorname{rot}\boldsymbol{a}$.

2385. Evaluate the divergence and the rotation of the vector $\boldsymbol{a}$ if $\boldsymbol{a}$ is, respectively, equal to: a) $\boldsymbol{r}$; b) $r\boldsymbol{c}$ and c) $f(r)\boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector.

2386. Find the divergence and rotation of the field of linear velocities of the points of a solid rotating counterclockwise with constant angular velocity ω about the z-axis.

2387. Evaluate the rotation of a field of linear velocities $\boldsymbol{v}=\boldsymbol{\omega}\cdot\boldsymbol{r}$ of the points of a body rotating with constant angular velocity $\boldsymbol{\omega}$ about some axis passing through the coordinate origin.

2388. Evaluate the divergence and rotation of the gradient of the scalar field U.

2389. Prove that $\operatorname{div}(\operatorname{rot}\boldsymbol{a})=0$.

2390. Using the Ostrogradsky-Gauss theorem, prove that the flux of the vector $\boldsymbol{a}=\boldsymbol{r}$ through a closed surface bounding an arbitrary volume v is equal to three times the volume.

2391. Find the flux of the vector $\boldsymbol{r}$ through the total surface of the cylinder $x^2+y^2\leqslant R^2$, $0\leqslant z\leqslant H$.

2392. Find the flux of the vector $\boldsymbol{a}=x^3\boldsymbol{i}+y^3\boldsymbol{j}+z^3\boldsymbol{k}$ through: a) the lateral surface of the cone $\frac{x^2+y^2}{R^2}\leqslant\frac{z^2}{H^2}$, $0\leqslant z\leqslant H$; b) the total surface of the cone.

2393*. Evaluate the divergence and the flux of an attractive force $\boldsymbol{F}=-\frac{m\boldsymbol{r}}{r^3}$ of a point of mass m, located at the coordinate origin, through an arbitrary closed surface surrounding this point.

2394. Evaluate the line integral of a vector $\boldsymbol{r}$ around one turn of the screw-line $x=R\cos t$; $y=R\sin t$; $z=ht$ from $t=0$ to $t=2\pi$.

2395. Using Stokes' theorem, evaluate the circulation of the vector $\boldsymbol{a}=x^2y^3\boldsymbol{i}+\boldsymbol{j}+z\boldsymbol{k}$ along the circumference $x^2+y^2=R^2$; $z=0$, taking the hemisphere $z=\sqrt{R^2-x^2-y^2}$ for the surface.

2396. Show that if a force $\boldsymbol{F}$ is central, that is, it is directed towards a fixed point 0 and depends only on the distance r from this point: $\boldsymbol{F}=f(r)\boldsymbol{r}$, where $f(r)$ is a single-valued continuous function, then the field is a potential field. Find the potential U of the field.

2397. Find the potential U of a gravitational field generated by a material point of mass m located at the origin of coordinates: $\boldsymbol{a}=-\frac{m}{r^3}\boldsymbol{r}$. Show that the potential U satisfies the Laplace equation $\Delta U=0$.

2398. Find out whether the given vector field has a potential U, and find U if the potential exists:

a) $\boldsymbol{a}=(5x^2y-4xy)\boldsymbol{i}+(3x^2-2y)\boldsymbol{j}$;
b) $\boldsymbol{a}=yz\boldsymbol{i}+zx\boldsymbol{j}+xy\boldsymbol{k}$;
c) $\boldsymbol{a}=(y+z)\boldsymbol{i}+(x+z)\boldsymbol{j}+(x+y)\boldsymbol{k}$.

2399. Prove that the central space field $\boldsymbol{a}=f(r)\boldsymbol{r}$ will be solenoidal only when $f(r)=\frac{k}{r^3}$, where k is constant.

2400. Will the vector field $\boldsymbol{a}=r(\boldsymbol{c}\times\boldsymbol{r})$ be solenoidal (where $\boldsymbol{c}$ is a constant vector)?

Chapter VIII

SERIES

Sec. 1. Number Series

1°. **Fundamental concepts.** A number series

$$a_1+a_2+\ldots+a_n+\ldots=\sum_{n=1}^{\infty} a_n \tag{1}$$

is called *convergent* if its *partial sum*

$$S_n=a_1+a_2+\ldots+a_n$$

has a finite limit as $n \longrightarrow \infty$. The quantity $S=\lim\limits_{n\to\infty} S_n$ is then called the *sum* of the series, while the number

$$R_n=S-S_n=a_{n+1}+a_{n+2}+\ldots$$

is called the *remainder* of the series. If the limit $\lim\limits_{n\to\infty} S_n$ does not exist (or is infinite), the series is then called *divergent*.

If a series converges, then $\lim\limits_{n\to\infty} a_n=0$ (*necessary condition for convergence*). The converse is not true.

For convergence of the series (1) it is necessary and sufficient that for any positive number ε it be possible to choose an N such that for $n>N$ and for any positive p the following inequality is fulfilled:

$$|a_{n+1}+a_{n+2}+\ldots+a_{n+p}|<\varepsilon$$

(*Cauchy's test*).

The convergence or divergence of a series is not violated if we add or subtract a finite number of its terms.

2°. **Tests of convergence and divergence of positive series.**

a) **Comparison test I.** If $0\leqslant a_n\leqslant b_n$ after a certain $n=n_0$, and the series

$$b_1+b_2+\ldots+b_n+\ldots=\sum_{n=1}^{\infty} b_n \tag{2}$$

converges, then the series (1) also converges. If the series (1) diverges, then (2) diverges as well.

It is convenient, for purposes of comparing series, to take a *geometric progression:*

$$\sum_{n=0}^{\infty} aq^n \qquad (a\neq 0),$$

which converges for $|q|<1$ and diverges for $|q| \geqslant 1$, and the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which is a divergent series.

Example 1. The series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\ldots+\frac{1}{n\cdot 2^n}+\ldots$$

converges, since here

$$a_n=\frac{1}{n\cdot 2^n}<\frac{1}{2^n},$$

while the geometric progression

$$\sum_{n=1}^{\infty} \frac{1}{2^n},$$

whose ratio is $q=\frac{1}{2}$, converges.

Example 2. The series

$$\frac{\ln 2}{2}+\frac{\ln 3}{3}+\ldots+\frac{\ln n}{n}+\ldots$$

diverges, since its general term $\frac{\ln n}{n}$ is greater than the corresponding term $\frac{1}{n}$ of the harmonic series (which diverges).

b) **Comparison test II.** If there exists a finite and nonzero limit $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ (in particular, if $a_n \sim b_n$), then the series (1) and (2) converge or diverge at the same time.

Example 3. The series

$$1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}+\ldots$$

diverges, since

$$\lim_{n\to\infty}\left(\frac{1}{2n-1}:\frac{1}{n}\right)=\frac{1}{2}\neq 0,$$

whereas a series with general term $\frac{1}{n}$ diverges.

Example 4. The series

$$\frac{1}{2-1}+\frac{1}{2^2-2}+\frac{1}{2^3-3}+\ldots+\frac{1}{2^n-n}+\ldots$$

converges, since

$$\lim_{n\to\infty}\left(\frac{1}{2^n-n}:\frac{1}{2^n}\right)=1, \quad \text{i.e.,} \quad \frac{1}{2^n-n}\sim\frac{1}{2^n},$$

while a series with general term $\frac{1}{2^n}$ converges.

c) **D'Alembert's test.** Let $a_n > 0$ (after a certain n) and let there be a limit

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = q.$$

Then the series (1) converges if $q < 1$, and diverges if $q > 1$. If $q = 1$, then it is not known whether the series is convergent or not.

Example 5. Test the convergence of the series

$$\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \ldots + \frac{2n-1}{2^n} + \ldots$$

Solution. Here,

$$a_n = \frac{2n-1}{2^n}, \quad a_{n+1} = \frac{2n+1}{2^{n+1}}$$

and

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(2n+1)\,2^n}{2^{n+1}(2n-1)} = \frac{1}{2} \lim_{n \to \infty} \frac{1 + \frac{1}{2n}}{1 - \frac{1}{2n}} = \frac{1}{2}.$$

Hence, the given series converges.

d) **Cauchy's test.** Let $a_n \geqslant 0$ (after a certain n) and let there be a limit

$$\lim_{n \to \infty} \sqrt[n]{a_n} = q.$$

Then (1) converges if $q < 1$, and diverges if $q > 1$. When $q = 1$, the question of the convergence of the series remains open.

e) **Cauchy's integral test.** If $a_n = f(n)$, where the function $f(x)$ is positive, monotonically decreasing and continuous for $x \geqslant a \geqslant 1$, the series (1) and the integral

$$\int_a^{\infty} f(x)\,dx$$

converge or diverge at the same time.

By means of the integral test it may be proved that the *Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \tag{3}$$

converges if $p > 1$, and diverges if $p \leqslant 1$. The convergence of a large number of series may be tested by comparing with the corresponding Dirichlet series (3)

Example 6. Test the following series for convergence

$$\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \ldots + \frac{1}{(2n-1)\,2n} + \ldots$$

Solution. We have

$$a_n = \frac{1}{(2n-1)\,2n} = \frac{1}{4n^2} \frac{1}{1 - \frac{1}{2n}} \sim \frac{1}{4n^2}.$$

Since the Dirichlet series converges for $p=2$, it follows that on the basis of comparison test II we can say that the given series likewise converges.

3°. Tests for convergence of alternating series. If a series

$$|a_1|+|a_2|+\ldots+|a_n|+\ldots, \tag{4}$$

composed of the absolute values of the terms of the series (1), converges, then (1) also converges and is called *absolutely convergent*. But if (1) converges and (4) diverges, then the series (1) is called *conditionally (not absolutely) convergent*.

For investigating the absolute convergence of the series (1), we can make use [for the series (4)] of the familiar convergence tests of positive series. For instance, (1) converges absolutely if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1 \quad \text{or} \quad \lim_{n\to\infty}\sqrt[n]{|a_n|}<1.$$

In the general case, the divergence of (1) does not follow from the divergence of (4). But if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ or $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}>1$, then not only does (4) diverge but the series (1) does also.

Leibniz test If for the alternating series

$$b_1-b_2+b_3-b_4+\ldots \quad (b_n\geqslant 0) \tag{5}$$

the following conditions are fulfilled: 1) $b_1\geqslant b_2\geqslant b_3\geqslant\ldots$; 2) $\lim\limits_{n\to\infty} b_n=0$, then (5) converges.

In this case, for the remainder of the series R_n the evaluation

$$|R_n|\leqslant b_{n+1}$$

holds.

Example 7. Test for convergence the series

$$1-\left(\frac{2}{3}\right)^2-\left(\frac{3}{5}\right)^3+\left(\frac{4}{7}\right)^4+\ldots+(-1)^{\frac{n(n-1)}{2}}\left(\frac{n}{2n-1}\right)^n+\ldots$$

Solution. Let us form a series of the absolute values of the terms of this series:

$$1+\left(\frac{2}{3}\right)^2+\left(\frac{3}{5}\right)^3+\left(\frac{4}{7}\right)^4+\ldots+\left(\frac{n}{2n-1}\right)^n+\ldots$$

Since

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n-1}\right)^n}=\lim_{n\to\infty}\frac{n}{2n-1}=\lim_{n\to\infty}\frac{1}{2-\frac{1}{n}}=\frac{1}{2},$$

the series converges absolutely.

Example 8. The series

$$1-\frac{1}{2}+\frac{1}{3}-\ldots+(-1)^{n+1}\cdot\frac{1}{n}+\ldots$$

converges, since the conditions of the Leibniz test are fulfilled. This series converges conditionally, since the series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

diverges (harmonic series).

Note. For the convergence of an alternating series it is not sufficient that its general term should tend to zero. The Leibniz test only states that an alternating series converges if the absolute value of its general term tends to zero *monotonically*. Thus, for example, the series

$$1-\frac{1}{5}+\frac{1}{2}-\frac{1}{5^2}+\frac{1}{3}-\ldots+\frac{1}{k}-\frac{1}{5^k}+\ldots$$

diverges despite the fact that its general term tends to zero (here, of course, the monotonic variation of the absolute value of the general term has been violated). Indeed, here, $S_{2k}=S_k'+S_k''$, where

$$S_k'=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k},\quad S_k''=-\left(\frac{1}{5}+\frac{1}{5^2}+\ldots+\frac{1}{5^k}\right),$$

and $\lim\limits_{k\to\infty} S_k'=\infty$ (S_k' is a partial sum of the harmonic series), whereas the limit $\lim\limits_{k\to\infty} S_k''$ exists and is finite (S_k'' is a partial sum of the convergent geometric progression), hence, $\lim\limits_{k\to\infty} S_{2k}=\infty$.

On the other hand, the Leibniz test is not necessary for the convergence of an alternating series: an alternating series may converge if the absolute value of its general term tends to zero in nonmonotonic fashion

Thus, the series

$$1-\frac{1}{2^2}+\frac{1}{3^3}-\frac{1}{4^2}+\ldots+\frac{1}{(2n-1)^3}-\frac{1}{(2n)^2}+\ldots$$

converges (and it converges absolutely), although the Leibniz test is not fulfilled: though the absolute value of the general term of the series tends to zero, it does not do so monotonically.

4°. Series with complex terms A series with the general term $c_n=a_n+ib_n$ ($i^2=-1$) converges if, and only if, the series with real terms $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ converge at the same time; in this case

$$\sum_{n=1}^{\infty} c_n=\sum_{n=1}^{\infty} a_n+i\sum_{n=1}^{\infty} b_n. \tag{6}$$

The series (6) definitely converges and is called *absolutely convergent*, if the series

$$\sum_{n=1}^{\infty} |c_n|=\sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2},$$

whose terms are the moduli of the terms of the series (6), converges.

5°. Operations on series.

a) A convergent series may be multiplied termwise by any number k; that is, if

$$a_1+a_2+\ldots+a_n+\ldots=S,$$

then

$$ka_1+ka_2+\ldots+ka_n+\ldots=kS.$$

b) By the *sum (difference)* of two convergent series

$$a_1+a_2+\ldots+a_n+\ldots=S_1, \quad (7)$$
$$b_1+b_2+\ldots+b_n+\ldots=S_2 \quad (8)$$

we mean a series

$$(a_1 \pm b_1)+(a_2 \pm b_2)+\ldots+(a_n \pm b_n)+\ldots=S_1 \pm S_2.$$

c) The *product* of the series (7) and (8) is the series

$$c_1+c_2+\ldots+c_n+\ldots, \quad (9)$$

where $c_n=a_1b_n+a_2b_{n-1}+\ldots+a_nb_1 (n=1, 2, \ldots)$.

If the series (7) and (8) converge absolutely, then the series (9) also converges absolutely and has a sum equal to S_1S_2.

d) If a series converges absolutely, its sum remains unchanged when the terms of the series are rearranged. This property is absent if the series converges conditionally.

Write the simplest formula of the nth term of the series using the indicated terms:

2401. $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots$

2402. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\ldots$

2403. $1+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\ldots$

2404. $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots$

2405. $\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\frac{6}{25}+\ldots$

2406. $\frac{2}{5}+\frac{4}{8}+\frac{6}{11}+\frac{8}{14}+\ldots$

2407. $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\ldots$

2408. $1+\frac{1\cdot 3}{1\cdot 4}+\frac{1\cdot 3\cdot 5}{1\cdot 4\cdot 7}+\frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 4\cdot 7\cdot 10}+\ldots$

2409. $1-1+1-1+1-1+\ldots$

2410. $1+\frac{1}{2}+3+\frac{1}{4}+5+\frac{1}{6}+\ldots$

In Problems 2411-2415 it is required to write the first 4 or 5 terms of the series on the basis of the known general term a_n.

2411. $a_n=\frac{3n-2}{n^2+1}$.

2412. $\frac{(-1)^n n}{2^n}$.

2413 $a_n=\frac{2+(-1)^n}{n^2}$.

2414. $a_n=\frac{1}{[3+(-1)^n]^n}$.

2415. $a_n=\frac{\left(2+\sin\frac{n\pi}{2}\right)\cos n\pi}{n!}$.

Test the following series for convergence by applying the comparison tests (or the necessary condition):

2416. $1-1+1-1+\ldots+(-1)^{n-1}+\ldots$

2417. $\frac{2}{5}+\frac{1}{2}\left(\frac{2}{5}\right)^2+\frac{1}{3}\left(\frac{2}{5}\right)^3+\ldots+\frac{1}{n}\left(\frac{2}{5}\right)^n+\ldots$

2418. $\frac{2}{3}+\frac{3}{5}+\frac{4}{7}+\ldots+\frac{n+1}{2n+1}+\ldots$

2419. $\frac{1}{\sqrt{10}}-\frac{1}{\sqrt[3]{10}}+\frac{1}{\sqrt[4]{10}}-\ldots+\frac{(-1)^{n+1}}{\sqrt[n+1]{10}}+\ldots$

2420. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}+\ldots$

2421. $\frac{1}{11}+\frac{1}{21}+\frac{1}{31}+\ldots+\frac{1}{10n+1}+\ldots$

2422. $\frac{1}{\sqrt{1\cdot 2}}+\frac{1}{\sqrt{2\cdot 3}}+\frac{1}{\sqrt{3\cdot 4}}+\ldots+\frac{1}{\sqrt{n(n+1)}}+\ldots$

2423. $2+\frac{2^2}{2}+\frac{2^3}{3}+\ldots+\frac{2^n}{n}+\ldots$

2424. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\ldots$

2425. $\frac{1}{2^2}+\frac{1}{5^2}+\frac{1}{8^2}+\ldots+\frac{1}{(3n-1)^2}+\ldots$

2426. $\frac{1}{2}+\frac{\sqrt[3]{2}}{3\sqrt{2}}+\frac{\sqrt[3]{3}}{4\sqrt{3}}+\ldots+\frac{\sqrt[3]{n}}{(n+1)\sqrt{n}}+\ldots$

Using d'Alembert's test, test the following series for convergence:

2427. $\frac{1}{\sqrt{2}}+\frac{3}{2}+\frac{5}{2\sqrt{2}}+\ldots+\frac{2n-1}{(\sqrt{2})^n}+\ldots$

2428. $\frac{2}{1}+\frac{2\cdot 5}{1\cdot 5}+\frac{2\cdot 5\cdot 8}{1\cdot 5\cdot 9}+\ldots+\frac{2\cdot 5\cdot 8\ldots(3n-1)}{1\cdot 5\cdot 9\ldots(4n-3)}+\ldots$

Test for convergence, using Cauchy's test:

2429. $\frac{2}{1}+\left(\frac{3}{3}\right)^2+\left(\frac{4}{5}\right)^3+\ldots+\left(\frac{n+1}{2n-1}\right)^n+\ldots$

2430. $\frac{1}{2}+\left(\frac{2}{5}\right)^3+\left(\frac{3}{8}\right)^5+\ldots+\left(\frac{n}{3n-1}\right)^{2n-1}+\ldots$

Test for convergence the positive series:

2431. $1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots$

2432. $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\ldots+\frac{1}{(n+1)^2-1}+\ldots$

2433. $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3n-2)(3n+1)}+\ldots$

2434. $\frac{1}{3}+\frac{4}{9}+\frac{9}{19}+\ldots+\frac{n^2}{2n^2+1}+\ldots$

2435. $\frac{1}{2}+\frac{2}{5}+\frac{3}{10}+\ldots+\frac{n}{n^2+1}+\ldots$

2436. $\frac{3}{2^2\cdot 3^3}+\frac{5}{3^2\cdot 4^2}+\frac{7}{4^2\cdot 5^2}+\ldots+\frac{2n+1}{(n+1)^2(n+2)^2}+\ldots$

2437. $\frac{3}{4}+\left(\frac{6}{7}\right)^2+\left(\frac{9}{10}\right)^3+\ldots+\left(\frac{3n}{3n+1}\right)^n+\ldots$

2438. $\left(\frac{3}{4}\right)^{\frac{1}{2}}+\frac{5}{7}+\left(\frac{7}{10}\right)^{\frac{3}{2}}+\ldots+\left(\frac{2n+1}{3n+1}\right)^{\frac{n}{2}}+\ldots$

2439. $\frac{1}{e}+\frac{8}{e^2}+\frac{27}{e^3}+\ldots+\frac{n^3}{e^n}+\ldots$

2440. $1+\frac{2}{2^2}+\frac{4}{3^3}+\ldots+\frac{2^{n-1}}{n^n}+\ldots$

2441. $\frac{1!}{2+1}+\frac{2!}{2^2+1}+\frac{3!}{2^3+1}+\ldots+\frac{n!}{2^n+1}+\ldots$

2442. $1+\frac{2}{1!}+\frac{4}{2!}+\ldots+\frac{2^{n-1}}{(n-1)!}+\ldots$

2443. $\frac{1}{4}+\frac{1\cdot 3}{4\cdot 8}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}+\ldots+\frac{1\cdot 3\cdot 5\ldots(2n-1)}{4\cdot 8\cdot 12\ldots 4n}+\ldots$

2444. $\frac{(1!)^2}{2!}+\frac{(2!)^2}{4!}+\frac{(3!)^2}{6!}+\ldots+\frac{(n!)^2}{(2n)!}+\ldots$

2445. $1000+\frac{1000\cdot 1002}{1\cdot 4}+\frac{1000\cdot 1002\cdot 1004}{1\cdot 4\cdot 7}+\ldots$

$\ldots+\frac{1000\cdot 1002\cdot 1004\ldots(998+2n)}{1\cdot 4\cdot 7\ldots(3n-2)}+\ldots$

2446. $\frac{2}{1}+\frac{2\cdot 5\cdot 8}{1\cdot 5\cdot 9}+\ldots+\frac{2\cdot 5\cdot 8\ldots(6n-7)(6n-4)}{1\cdot 5\cdot 9\ldots(8n-11)(8n-7)}+\ldots$

2447. $\frac{1}{2}+\frac{1\cdot 5}{2\cdot 4\cdot 6}+\ldots+\frac{1\cdot 5\ldots(4n-3)}{2\cdot 4\cdot 6\ldots(4n-4)(4n-2)}+\ldots$

2448. $\frac{1}{1!}+\frac{1\cdot 11}{3!}+\frac{1\cdot 11\cdot 21}{5!}+\ldots+\frac{1\cdot 11\cdot 21\ldots(10n-9)}{(2n-1)!}+\ldots$

2449. $1+\frac{1\cdot 4}{1\cdot 3\cdot 5}+\frac{1\cdot 4\cdot 9}{1\cdot 3\cdot 5\cdot 7\cdot 9}+\ldots+\frac{1\cdot 4\cdot 9\ldots n^2}{1\cdot 3\cdot 5\cdot 7\cdot 9\ldots(4n-3)}+\ldots$

2450. $\sum_{n=1}^{\infty}\arcsin\frac{1}{\sqrt{n}}$.

2451. $\sum_{n=1}^{\infty}\sin\frac{1}{n^2}$.

2452. $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)$.

2453. $\sum_{n=1}^{\infty}\ln\frac{n^2+1}{n^2}$.

2454. $\sum_{n=2}^{\infty}\frac{1}{\ln n}$.

2455. $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$.

2456. $\sum_{n=2}^{\infty}\frac{1}{n\ln^2 n}$.

2457. $\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln n\cdot\ln\ln n}$.

2458. $\sum_{n=2}^{\infty}\frac{1}{n^2-n}$.

2459. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$.

2460. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$.

2461. $\sum_{n=2}^{\infty} \frac{1}{n \ln n + \sqrt{\ln^3 n}}$.

2462. $\sum_{n=2}^{\infty} \frac{1}{n \sqrt[3]{n} - \sqrt{n}}$.

2463. $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{(2n-1)(5\sqrt[3]{n}-1)}$.

2464. $\sum_{n=1}^{\infty} \left(1 - \cos\frac{\pi}{n}\right)$.

2465. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

2466. $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$.

2467. $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$.

2468*. $\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$.

2469. Prove that the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln^q n}$:

1) converges for arbitrary q, if $p > 1$, and for $q > 1$, if $p = 1$;
2) diverges for arbitrary q, if $p < 1$, and for $q \leqslant 1$, if $p = 1$.

Test for convergence the following alternating series. For convergent series, test for absolute and conditional convergence.

2470. $1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{(-1)^{n-1}}{2n-1} + \ldots$

2471. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \ldots + \frac{(-1)^{n-1}}{\sqrt{n}} + \ldots$

2472. $1 - \frac{1}{4} + \frac{1}{9} - \ldots + \frac{(-1)^{n-1}}{n^2} + \ldots$

2473. $1 - \frac{2}{7} + \frac{3}{13} - \ldots + \frac{(-1)^{n-1} n}{6n-5} + \ldots$

2474. $\frac{3}{1\cdot 2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} - \ldots + (-1)^{n-1} \frac{2n+1}{n(n+1)} + \ldots$

2475. $-\frac{1}{2} - \frac{2}{4} + \frac{3}{8} + \frac{4}{16} - \ldots + (-1)^{\frac{n^2+n}{2}} \cdot \frac{n}{2^n} + \ldots$

2476. $-\frac{2}{2\sqrt{2}-1} + \frac{3}{3\sqrt{3}-1} - \frac{4}{4\sqrt{4}-1} + \ldots + (-1)^n \frac{n+1}{(n+1)\sqrt{n+1}-1} + \ldots$

2477. $-\frac{3}{4} + \left(\frac{5}{7}\right)^2 - \left(\frac{7}{10}\right)^3 + \ldots + (-1)^n \left(\frac{2n+1}{3n+1}\right)^n + \ldots$

2478. $\frac{3}{2} - \frac{3\cdot 5}{2\cdot 5} + \frac{3\cdot 5\cdot 7}{2\cdot 5\cdot 8} - \ldots + (-1)^{n-1} \frac{3\cdot 5\cdot 7 \ldots (2n+1)}{2\cdot 5\cdot 8 \ldots (3n-1)} + \ldots$

2479. $\frac{1}{7} - \frac{1\cdot 4}{7\cdot 9} + \frac{1\cdot 4\cdot 7}{7\cdot 9\cdot 11} - \ldots + (-1)^{n-1} \frac{1\cdot 4\cdot 7 \ldots (3n-2)}{7\cdot 9\cdot 11 \ldots (2n+5)} + \ldots$