2480. $\frac{\sin\alpha}{\ln 10}+\frac{\sin 2\alpha}{(\ln 10)^2}+\ldots+\frac{\sin n\alpha}{(\ln 10)^n}+\ldots$

2481. $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}$.

2482. $\sum_{n=1}^{\infty}(-1)^{n-1}\tan\frac{1}{n\sqrt{n}}$.

2483. Convince yourself that the d'Alembert test for convergence does not decide the question of the convergence of the series $\sum_{n=1}^{\infty}a_n$, where

$$a_{2k-1}=\frac{2^{k-1}}{3^{k-1}},\quad a_{2k}=\frac{2^{k-1}}{3^k}\quad (k=1,\ 2,\ \ldots),$$

whereas by means of the Cauchy test it is possible to establish that this series converges.

2484*. Convince yourself that the Leibniz test cannot be applied to the alternating series a) to d). Find out which of these series diverge, which converge conditionally and which converge absolutely:

a) $\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{4}-1}-\frac{1}{\sqrt{4}+1}+\ldots$

$$\left(a_{2k-1}=\frac{1}{\sqrt{k+1}-1},\quad a_{2k}=-\frac{1}{\sqrt{k+1}+1}\right);$$

b) $1-\frac{1}{3}+\frac{1}{2}-\frac{1}{3^3}+\frac{1}{2^2}-\frac{1}{3^5}+\ldots$

$$\left(a_{2k-1}=\frac{1}{2^{k-1}},\quad a_{2k}=-\frac{1}{3^{2k-1}}\right);$$

c) $1-\frac{1}{3}+\frac{1}{3}-\frac{1}{3^2}+\frac{1}{5}-\frac{1}{3^3}+\ldots$

$$\left(a_{2k-1}=\frac{1}{2k-1},\quad a_{2k}=-\frac{1}{3^k}\right);$$

d) $\frac{1}{3}-1+\frac{1}{7}-\frac{1}{5}+\frac{1}{11}-\frac{1}{9}+\ldots$

$$\left(a_{2k-1}=\frac{1}{4k-1},\quad a_{2k}=-\frac{1}{4k-3}\right).$$

Test the following series with complex terms for convergence:

2485. $\sum_{n=1}^{\infty}\frac{n(2+i)^n}{2^n}$.

2486. $\sum_{n=1}^{\infty}\frac{n(2i-1)^n}{3^n}$.

2487. $\sum_{n=1}^{\infty}\frac{1}{n(3+i)^n}$.

2488. $\sum_{n=1}^{\infty}\frac{i^n}{n}$.

2489. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+i}$.

2490. $\sum_{n=1}^{\infty}\frac{1}{(n+i)\sqrt{n}}$.

2491. $\sum_{n=1}^{\infty} \frac{1}{[n+(2n-1)\,i]^2}$.   2492. $\sum_{n=1}^{\infty} \left[\frac{n(2-i)+1}{n(3-2i)-3i}\right]^n$.

2493. Between the curves $y=\frac{1}{x^3}$ and $y=\frac{1}{x^2}$ and to the right of their point of intersection are constructed segments parallel to the y-axis at an equal distance from each other. Will the sum of the lengths of these segments be finite?

2494. Will the sum of the lengths of the segments mentioned in Problem 2493 be finite if the curve $y=\frac{1}{x^2}$ is replaced by the curve $y=\frac{1}{x}$?

2495. Form the sum of the series $\sum_{n=1}^{\infty} \frac{1+n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n-n}{3^n}$. Does this sum converge?

2496. Form the difference of the divergent series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ and $\sum_{n=1}^{\infty} \frac{1}{2n}$ and test it for convergence.

2497. Does the series formed by subtracting the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ from the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

2498. Choose two series such that their sum converges while their difference diverges.

2499. Form the product of the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$. Does this product converge?

2500. Form the series $\left(1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n-1}}+\ldots\right)^2$. Does this series converge?

2501. Given the series $1+\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^n}{n!}+\ldots$ Estimate the error committed when replacing the sum of this series with the sum of the first four terms, the sum of the first five terms. What can you say about the signs of these errors?

2502*. Estimate the error due to replacing the sum of the series

$$\frac{1}{2}+\frac{1}{2!}\left(\frac{1}{2}\right)^2+\frac{1}{3!}\left(\frac{1}{2}\right)^3+\ldots+\frac{1}{n!}\left(\frac{1}{2}\right)^n+\ldots$$

by the sum of its first n terms.

2503. Estimate the error due to replacing the sum of the series

$$1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots$$

by the sum of its first n terms. In particular, estimate the accuracy of such an approximation for $n=10$.

2504.** Estimate the error due to replacing the sum of the series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$$

by the sum of its first n terms. In particular, estimate the accuracy of such an approximation for $n=1{,}000$.

2505.** Estimate the error due to replacing the sum of the series

$$1+2\left(\frac{1}{4}\right)^2+3\left(\frac{1}{4}\right)^4+\ldots+n\left(\frac{1}{4}\right)^{2n-2}+\ldots$$

by the sum of its first n terms.

2506. How many terms of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ does one have to take to compute its sum to two decimal places? to three decimals?

2507. How many terms of the series $\sum_{n=1}^{\infty}\frac{n}{(2n+1)5^n}$ does one have to take to compute its sum to two decimal places? to three? to four?

2508*. Find the sum of the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}+\ldots$

2509. Find the sum of the series

$$\sqrt[3]{x}+(\sqrt[5]{x}-\sqrt[3]{x})+(\sqrt[7]{x}-\sqrt[5]{x})+\ldots+(\sqrt[2k+1]{x}-\sqrt[2k-1]{x})+\ldots$$

Sec. 2. Functional Series

1°. **Region of convergence.** The set of values of the argument x for which the functional series

$$f_1(x)+f_2(x)+\ldots+f_n(x)+\ldots \qquad (1)$$

converges is called the *region of convergence* of this series. The function

$$S(x)=\lim_{n\to\infty} S_n(x),$$

where $S_n(x)=f_1(x)+f_2(x)+\ldots+f_n(x)$, and x belongs to the region of convergence, is called the *sum* of the series; $R_n(x)=S(x)-S_n(x)$ is the *remainder* of the series.

In the simplest cases, it is sufficient, when determining the region of convergence of a series (1), to apply to this series certain convergence tests, holding x constant.

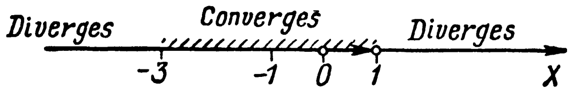

Fig. 104

Example 1. Determine the region of convergence of the series

$$\frac{x+1}{1\cdot 2}+\frac{(x+1)^2}{2\cdot 2^2}+\frac{(x+1)^3}{3\cdot 2^3}+\ldots+\frac{(x+1)^n}{n\cdot 2^n}+\ldots \qquad (2)$$

Solution. Denoting by u_n the general term of the series, we will have

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim_{n\to\infty}\frac{|x+1|^{n+1}2^n n}{2^{n+1}(n+1)|x|^n}=\frac{|x+1|}{2}.$$

Using d'Alembert's test, we can assert that the series converges (and converges absolutely), if $\frac{|x+1|}{2}<1$, that is, if $-3<x<1$; the series diverges, if $\frac{|x+1|}{2}>1$, that is, if $-\infty<x<-3$ or $1<x<\infty$ (Fig. 104). When $x=1$ we get the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\ldots$, which diverges, and when $x=-3$ we have the series $-1+\frac{1}{2}-\frac{1}{3}+\ldots$, which (in accord with the Leibniz test) converges (conditionally).

Thus, the series converges when $-3\leqslant x<1$.

2°. Power series. For any *power series*

$$c_0+c_1(x-a)+c_2(x-a)^2+\ldots+c_n(x-a)^n+\ldots \qquad (3)$$

(c_n and a are real numbers) there exists an interval (the *interval of convergence*) $|x-a|<R$ with centre at the point $x=a$, with in which the series (3) converges absolutely; for $|x-a|>R$ the series diverges. In special cases, the *radius of convergence* R may also be equal to 0 and ∞. At the end-points of the interval of convergence $x=a\pm R$, the power series may either converge or diverge. The interval of convergence is ordinarily determined with the help of the d'Alembert or Cauchy tests, by applying them to a series, the terms of which are the absolute values of the terms of the given series (3).

Applying to the series of absolute values

$$|c_0|+|c_1||x-a|+\ldots+|c_n||x-a|^n+\ldots$$

the convergence tests of d'Alembert and Cauchy, we get, respectively, for the radius of convergence of the power series (3), the formulas

$$R=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|c_n|}} \quad \text{and} \quad R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|.$$

However, one must be very careful in using them because the limits on the right frequently do not exist. For example, if an infinitude of coefficients c_n

vanishes [as a particular instance, this occurs if the series contains terms with only even or only odd powers of $(x-a)$], one cannot use these formulas. It is then advisable, when determining the interval of convergence, to apply the d'Alembert or Cauchy tests directly, as was done when we investigated the series (2), without resorting to general formulas for the radius of convergence.

If $z=x+iy$ is a complex variable, then for the power series

$$c_0+c_1(z-z_0)+c_2(z-z_0)^2+\ldots+c_n(z-z_0)^n+\ldots \tag{4}$$

($c_n=a_n+ib_n$, $z_0=x_0+iy_0$) there exists a certain circle *(circle of convergence)* $|z-z_0|<R$ with centre at the point $z=z_0$, inside which the series converges absolutely; for $|z-z_0|>R$ the series diverges. At points lying on the circumference of the circle of convergence, the series (4) may both converge and diverge. It is customary to determine the circle of convergence by means of the d'Alembert or Cauchy tests applied to the series

$$|c_0|+|c_1|\cdot|z-z_0|+|c_2|\cdot|z-z_0|^2+\ldots+|c_n|\cdot|z-z_0|^n+\ldots,$$

whose terms are absolute values of the terms of the given series. Thus, for example, by means of the d'Alembert test it is easy to see that the circle of convergence of the series

$$\frac{z+1}{1\cdot 2}+\frac{(z+1)^2}{2\cdot 2^2}+\frac{(z+1)^3}{3\cdot 2^3}+\ldots+\frac{(z+1)^n}{n\cdot 2^n}+\ldots$$

is determined by the inequality $|z+1|<2$ [it is sufficient to repeat the calculations carried out on page 305 which served to determine the interval of convergence of the series (2), only here x is replaced by z]. The centre of the circle of convergence lies at the point $z=-1$, while the radius R of this circle (the radius of convergence) is equal to 2.

3°. **Uniform convergence.** The functional series (1) converges uniformly on some interval if, no matter what $\varepsilon>0$, it is possible to find an N such that does not depend on x and that when $n>N$ for all x of the given interval we have the inequality $|R_n(x)|<\varepsilon$, where $R_n(x)$ is the remainder of the given series.

If $|f_n(x)|\leqslant c_n$ $(n=1, 2, \ldots)$ when $a\leqslant x\leqslant b$ and the number series $\sum\limits_{n=1}^{\infty} c_n$ converges, then the functional series (1) converges on the interval $[a, b]$ absolutely and uniformly *(Weierstrass' test)*.

The power series (3) converges absolutely and uniformly on any interval lying within its interval of convergence. The power series (3) may be termwise differentiated and integrated within its interval of convergence (for $|x-a|<R$); that is, if

$$c_0+c_1(x-a)+c_2(x-a)^2+\ldots+c_n(x-a)^n+\ldots=f(x), \tag{5}$$

then for any x of the interval of convergence of the series (3), we have

$$c_1+2c_2(x-a)+\ldots+nc_n(x-a)^{n-1}+\ldots=f'(x), \tag{6}$$

$$\int_{x_0}^{x} c_0\,dx+\int_{x_0}^{x} c_1(x-a)\,dx+\int_{x_0}^{x} c_2(x-a)^2\,dx+\ldots+\int_{x_0}^{x} c_n(x-a)^n\,dx+\ldots=$$

$$=\sum_{n=0}^{\infty} c_n\,\frac{(x-a)^{n+1}(x_0-a)^{n+1}}{n+1}=\int_{x_0}^{x} f(x)\,dx \tag{7}$$

[the number x_0 also belongs to the interval of convergence of the series (3)]. Here, the series (6) and (7) have the same interval of convergence as the series (3).

Find the region of convergence of the series:

2510. $\sum_{n=1}^{\infty} \frac{1}{n^x}$.

2511. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^x}$.

2512. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{\ln x}}$.

2513. $\sum_{n=1}^{\infty} \frac{\sin (2n-1) x}{(2n-1)^2}$.

2514. $\sum_{n=0}^{\infty} 2^n \sin \frac{x}{3^n}$.

2515**. $\sum_{n=0}^{\infty} \frac{\cos nx}{e^{nx}}$.

2516. $\sum_{n=0}^{\infty} (-1)^{n+1} e^{-n \sin x}$.

2517. $\sum_{n=1}^{\infty} \frac{n!}{x^n}$.

2518. $\sum_{n=1}^{\infty} \frac{1}{n!\, x^n}$.

2519. $\sum_{n=1}^{\infty} \frac{1}{(2n-1) x^n}$.

2520. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(x-2)^n}$.

2521. $\sum_{n=0}^{\infty} \frac{2n+1}{(n+1)^5 x^{2n}}$.

2522. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 3^n (x-5)^n}$.

2523. $\sum_{n=1}^{\infty} \frac{n^n}{x^{n^n}}$.

2524*. $\sum_{n=1}^{\infty} \left(x^n + \frac{1}{2^n x^n}\right)$.

2525. $\sum_{n=-1}^{\infty} x^n$.

Find the interval of convergence of the power series and test the convergence at the end-points of the interval of convergence:

2526. $\sum_{n=0}^{\infty} x^n$.

2527. $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 2^n}$.

2528. $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$.

2529. $\sum_{n=1}^{\infty} \frac{2^{n-1} x^{2n-1}}{(4n-3)^2}$.

2530. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$.

2531. $\sum_{n=0}^{\infty} \frac{(n+1)^5 x^{2n}}{2n+1}$.

2532. $\sum_{n=0}^{\infty} (-1)^n (2n+1)^2 x^n$.

2533. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$.

2534. $\sum_{n=1}^{\infty} n!\, x^n$.

2535. $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$.

2536. $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^{2n-1}x^n$.

2537. $\sum_{n=0}^{\infty}3^{n^2}x^{n^2}$.

2538. $\sum_{n=1}^{\infty}\frac{n}{n+1}\left(\frac{x}{2}\right)^n$.

2539. $\sum_{n=1}^{\infty}\frac{n!\,x^n}{n^n}$.

2540. $\sum_{n=2}^{\infty}\frac{x^{n-1}}{n\cdot 3^n\cdot\ln n}$.

2541. $\sum_{n=1}^{\infty}x^{n!}$.

2542**. $\sum_{n=1}^{\infty}n!\,x^{n!}$.

2543*. $\sum_{n=1}^{\infty}\frac{x^{n^4}}{2^{n-1}n^n}$.

2544*. $\sum_{n=1}^{\infty}\frac{x^{n^n}}{n^n}$.

2545. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-5)^n}{n\cdot 3^n}$.

2546. $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n\cdot 5^n}$.

2547. $\sum_{n=1}^{\infty}\frac{(x-1)^{2n}}{n\cdot 9^n}$.

2548. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-2)^{2n}}{2n}$.

2549. $\sum_{n=1}^{\infty}\frac{(x+3)^n}{n^2}$.

2550. $\sum_{n=1}^{\infty}n^n(x+3)^n$.

2551. $\sum_{n=1}^{\infty}\frac{(x+5)^{2n-1}}{2n\cdot 4^n}$.

2552. $\sum_{n=1}^{\infty}\frac{(x-2)^n}{(2n-1)\,2^n}$.

2553. $\sum_{n=1}^{\infty}(-1)^{n+1}\times$
$\times\frac{(2n-1)^{2n}(x-1)^n}{(3n-2)^{2n}}$.

2554. $\sum_{n=1}^{\infty}\frac{n!\,(x+3)^n}{n^n}$.

2555. $\sum_{n=1}^{\infty}\frac{(x+1)^n}{(n+1)\ln^2(n+1)}$.

2556. $\sum_{n=1}^{\infty}\frac{(x-3)^{2n}}{(n+1)\ln(n+1)}$.

2557. $\sum_{n=1}^{\infty}(-1)^{n+1}\times$
$\times\frac{(x-2)^n}{(n+1)\ln(n+1)}$.

2558. $\sum_{n=1}^{\infty}\frac{(x+2)^{n^2}}{n^n}$.

2559*. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}(x-1)^n$.

2560. $\sum_{n=1}^{\infty}\frac{(2n-1)^n(x+1)^n}{2^{n-1}\cdot n^n}$.

2561. $\sum_{n=0}^{\infty}(-1)^n\frac{\sqrt[3]{n+2}}{n+1}\times$
$\times(x-2)^n$.

2562. $\sum_{n=0}^{\infty}\frac{(3n-2)(x-3)^n}{(n+1)^2\,2^{n+1}}$.

2563. $\sum_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{(2n+1)\sqrt{n+1}}$.

Determine the circle of convergence:

2564. $\sum_{n=0}^{\infty} i^n z^n$.

2565. $\sum_{n=0}^{\infty} (1+ni) z^n$.

2566. $\sum_{n=1}^{\infty} \frac{(z-2i)^n}{n\cdot 3^n}$.

2567. $\sum_{n=0}^{\infty} \frac{z^{2n}}{2^n}$.

2568. $(1+2i)+(1+2i)(3+2i)z+\ldots+$
$$+(1+2i)(3+2i)\ldots(2n+1+2i)z^n+\ldots$$

2569. $1+\frac{z}{1-i}+\frac{z^2}{(1-i)(1-2i)}+\ldots$
$$\ldots+\frac{z^n}{(1-i)(1-2i)\ldots(1-ni)}+\ldots$$

2570. $\sum_{n=0}^{\infty}\left(\frac{1+2ni}{n+2i}\right)^n z^n$.

2571. Proceeding from the definition of uniform convergence, prove that the series

$$1+x+x^2+\ldots+x^n+\ldots$$

does not converge uniformly in the interval $(-1, 1)$, but converges uniformly on any subinterval within this interval.

Solution. Using the formula for the sum of a geometric progression, we get, for $|x|<1$,

$$R_n(x)=x^{n+1}+x^{n+2}+\ldots=\frac{x^{n+1}}{1-x}.$$

Within the interval $(-1, 1)$ let us take a subinterval $[-1+\alpha, 1-\alpha]$, where α is an arbitrarily small positive number. In this subinterval $|x|\leqslant 1-\alpha$, $|1-x|\geqslant\alpha$ and, consequently,

$$|R_n(x)|\leqslant\frac{(1-\alpha)^{n+1}}{\alpha}.$$

To prove the uniform convergence of the given series over the subinterval $[-1+\alpha, 1-\alpha]$, it must be shown that for any $\varepsilon>0$ it is possible to choose an N dependent only on ε such that for any $n>N$ we will have the inequality $|R_n(x)|<\varepsilon$ for *all* x of the subinterval under consideration.

Taking any $\varepsilon>0$, let us require that $\frac{(1-\alpha)^{n+1}}{\alpha}<\varepsilon$; whence $(1-\alpha)^{n+1}<\varepsilon\alpha$, $(n+1)\ln(1-\alpha)<\ln(\varepsilon\alpha)$, that is, $n+1>\frac{\ln(\varepsilon\alpha)}{\ln(1-\alpha)}$ [since $\ln(1-\alpha)<0$] and $n>\frac{\ln(\varepsilon\alpha)}{\ln(1-\alpha)}-1$. Thus, putting $N=\frac{\ln(\varepsilon\alpha)}{\ln(1-\alpha)}-1$, we are convinced that when $n>N$, $|R_n(x)|$ is indeed less than ε for all x of the subinterval $[-1+\alpha, 1-\alpha]$ and the uniform convergence of the given series on any subinterval within the interval $(-1, 1)$ is thus proved.

As for the entire interval $(-1, 1)$, it contains points that are arbitrarily close to $x=1$, and since $\lim_{x\to 1} R_n(x)=\lim_{x\to 1}\frac{x^{n+1}}{1-x}=\infty$, no matter how large n is, points x will be found for which $R_n(x)$ is greater than any arbitrarily large number Hence, it is impossible to choose an N such that for $n > N$ we would have the inequality $|R_n(x)| < \varepsilon$ at *all* points of the interval $(-1, 1)$, and this means that the convergence of the series in the interval $(-1, 1)$ is not uniform.

2572. Using the definition of uniform convergence, prove that:

a) the series

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots$$

converges uniformly in any finite interval;

b) the series

$$\frac{x^2}{1}-\frac{x^4}{2}+\frac{x^6}{3}-\ldots+\frac{(-1)^{n-1}x^{2n}}{n}+\ldots$$

converges uniformly throughout the interval of convergence $(-1, 1)$;

c) the series

$$1+\frac{1}{2^x}+\frac{1}{3^x}+\ldots+\frac{1}{n^x}+\ldots$$

converges uniformly in the interval $(1+\delta, \infty)$ where δ is any positive number;

d) the series

$$(x^2-x^4)+(x^4-x^6)+(x^6-x^8)+\ldots+(x^{2n}-x^{2n+2})+\ldots$$

converges not only within the interval $(-1, 1)$, but at the extremities of this interval, however the convergence of the series in $(-1, 1)$ is nonuniform.

Prove the uniform convergence of the functional series in the indicated intervals:

2573. $\sum\limits_{n=1}^{\infty}\frac{x^n}{n^2}$ on the interval $[-1, 1]$.

2574. $\sum\limits_{n=1}^{\infty}\frac{\sin nx}{2^n}$ over the entire number scale.

2575. $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{\sqrt{n}}$ on the interval $[0, 1]$.

Applying termwise differentiation and integration, find the sums of the series:

2576. $x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}+\ldots$

2577. $x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots$

2578. $x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots+\frac{x^{2n-1}}{2n-1}+\ldots$

2579. $x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots+(-1)^{n-1}\frac{x^{2n-1}}{2n-1}+\ldots$

2580. $1+2x+3x^2+\ldots+(n+1)x^n+\ldots$

2581. $1-3x^2+5x^4-\ldots+(-1)^{n-1}(2n-1)x^{2n-2}+\ldots$

2582. $1\cdot2+2\cdot3x+3\cdot4x^2+\ldots+n(n+1)x^{n-1}+\ldots$

Find the sums of the series:

2583. $\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}+\ldots+\frac{n}{x^n}+\ldots$

2584. $x+\frac{x^5}{5}+\frac{x^9}{9}+\ldots+\frac{x^{4n-3}}{4n-3}+\ldots$

2585*. $1-\frac{1}{3\cdot3}+\frac{1}{5\cdot3^2}-\frac{1}{7\cdot3^3}+\ldots+\frac{(-1)^{n-1}}{(2n-1)\,3^{n-1}}+\ldots$

2586. $\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\ldots+\frac{2n-1}{2^n}+\ldots$

Sec. 3. Taylor's Series

1°. Expanding a function in a power series. If a function $f(x)$ can be expanded, in some neighbourhood $|x-a|<R$ of the point a, in a series of powers of $x-a$, then this series (called *Taylor's series*) is of the form

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots \qquad (1)$$

When $a=0$ the Taylor series is also called a *Maclaurin's series*. Equation (1) holds if when $|x-a|<R$ the *remainder term* (or simply remainder) of the Taylor series

$$R_n(x)=f(x)-\left[f(a)\sum_{k=1}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k\right]\longrightarrow 0$$

as $n\longrightarrow\infty$.

To evaluate the remainder, one can make use of the formula

$$R_n(x)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a+\theta(x-a)],\ \text{where } 0<\theta<1 \qquad (2)$$

(Lagrange's form).

Example 1. Expand the function $f(x)=\cosh x$ in a series of powers of x.

Solution. We find the derivatives of the given function $f(x)=\cosh x$, $f'(x)=\sinh x$, $f''(x)=\cosh x$, $f'''(x)=\sinh x$, ...; generally, $f^{(n)}(x)=\cosh x$, if n is even, and $f^{(n)}(x)=\sinh x$, if n is odd. Putting $a=0$, we get $f(0)=1$, $f'(0)=0$, $f''(0)=1$, $f'''(0)=0$, ...; generally, $f^{(n)}(0)=1$, if n is even, and $f^{(n)}(0)=0$ if n is odd. Whence, from (1), we have:

$$\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+\frac{x^{2n}}{(2n)!}+\ldots \qquad (3)$$

To determine the interval of convergence of the series (3) we apply the d'Alembert test. We have

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}:\frac{x^{2n}}{(2n)!}\right|=\lim_{n\to\infty}\frac{x^2}{(2n+1)(2n+2)}=0$$

for any x. Hence, the series converges in the interval $-\infty < x < \infty$. The remainder term, in accord with formula (2), has the form:

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} \cosh \theta x, \text{ if } n \text{ is odd, and}$$

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} \sinh \theta x, \text{ if } n \text{ is even.}$$

Since $0 > \theta > 1$, it follows that

$$|\cosh \theta x| = \frac{e^{\theta x} + e^{-\theta x}}{2} \leqslant e^{|x|}, \quad |\sinh \theta x| = \left| \frac{e^{\theta x} - e^{-\theta x}}{2} \right| \leqslant e^{|x|},$$

and therefore $|R_n(x)| \leqslant \frac{|x|^{n+1}}{(n+1)!} e^{|x|}$. A series with the general term $\frac{|x|^n}{n!}$ converges for any x (this is made immediately evident with the help of d'Alembert's test); therefore, in accord with the necessary condition for convergence,

$$\lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0,$$

and consequently $\lim\limits_{n \to \infty} R_n(x) = 0$ for any x. This signifies that the sum of the series (3) for any x is indeed equal to $\cosh x$.

2°. **Techniques employed for expanding in power series.** Making use of the principal expansions

I. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \quad (-\infty < x < \infty)$,

II. $\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots \quad (-\infty < x < \infty)$,

III. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \quad (-\infty < x < \infty)$,

IV. $(1+x)^m = 1 + \frac{m}{1!} x + \frac{m(m-1)}{2!} x^2 + \ldots$

$\ldots + \frac{m(m-1)\ldots(m-n+1)}{n!} x^n + \ldots \quad (-1 < x < 1)$ *),

V $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1} \frac{x^n}{n} + \ldots \quad (-1 < x \leqslant 1)$,

and also the formula for the sum of a geometric progression, it is possible, in many cases, simply to obtain the expansion of a given function in a power series, without having to investigate the remainder term. It is sometimes advisable to make use of termwise differentiation or integration when expanding a function in a series. When expanding rational functions in power series it is advisable to decompose these functions into partial fractions.

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*) On the boundaries of the interval of convergence (i. e., when $x = -1$ and $x = 1$) the expansion IV behaves as follows: for $m \geqslant 0$ it converges absolutely on both boundaries; for $0 > m > -1$ it diverges when $x = -1$ and conditionally converges when $x = 1$; for $m \leqslant -1$ it diverges on both boundaries.

Example 2. Expand in powers of x *) the function

$$f(x)=\frac{3}{(1-x)(1+2x)}.$$

Solution. Decomposing the function into partial fractions, we will have

$$f(x)=\frac{1}{1-x}+\frac{2}{1+2x}.$$

Since

$$\frac{1}{1-x}=1+x+x^2+\ldots=\sum_{n=0}^{\infty} x^n \tag{4}$$

and

$$\frac{1}{1+2x}=1-2x+(2x)^2-\ldots=\sum_{n=0}^{\infty}(-1)^n 2^n x^n, \tag{5}$$

it follows that we finally get

$$f(x)=\sum_{n=0}^{\infty} x^n+2\sum_{n=0}^{\infty}(-1)^n 2^n x^n=\sum_{n=0}^{\infty}\left[1+(-1)^n 2^{n+1}\right]x^n. \tag{6}$$

The geometric progressions (4) and (5) converge, respectively, when $|x|<1$ and $|x|<\frac{1}{2}$; hence, formula (6) holds for $|x|<\frac{1}{2}$, i.e., when $-\frac{1}{2}<x<\frac{1}{2}$.

3°. Taylor's series for a function of two variables. Expanding a function of two variables $f(x, y)$ into a *Taylor's series* in the neighbourhood of a point (a, b) has the form

$$f(x, y)=f(a, b)+\frac{1}{1!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]f(a, b)+\frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x}+\right.$$
$$\left.+(y-b)\frac{\partial}{\partial y}\right]^2 f(a, b)+\ldots+\frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^n f(a, b)+\ldots \tag{7}$$

If $a=b=0$, the Taylor series is then called a *Maclaurin's series*. Here the notation is as follows:

$$\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]f(a, b)=\left.\frac{\partial f(x, y)}{\partial x}\right|_{\substack{x=a\\y=b}}(x-a)+\left.\frac{\partial f(x, y)}{\partial y}\right|_{\substack{x=a\\y=b}}(y-b);$$

$$\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^2 f(a, b)=\left.\frac{\partial^2 f(x, y)}{\partial x^2}\right|_{\substack{x=a\\y=b}}(x-a)^2+$$
$$+2\left.\frac{\partial^2 f(x, x)}{\partial x\,\partial y}\right|_{\substack{x=a\\y=b}}(x-a)(y-b)+\left.\frac{\partial^2 f(x, y)}{\partial y^2}\right|_{\substack{x=a\\y=b}}(y-b)^2 \text{ and so forth.}$$

*) Here and henceforward we mean "in positive integral powers".

The expansion (7) occurs if the remainder term of the series

$$R_n(x, y)=f(x, y)-\left\{f(a, b)+\sum_{k=1}^{n}\frac{1}{k!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^k f(a, b)\right\}\longrightarrow 0$$

as $n\longrightarrow\infty$. The remainder term may be represented in the form

$$R_n(x, y)+\frac{1}{(n+1)!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^{n+1} f(x, y)\Bigg|_{\substack{x=a+\theta(x-a)\\ y=b+\theta(y-b)}},$$

where $0<\theta<1$.

Expand the indicated functions in positive integral powers of x, find the intervals of convergence of the resulting series and investigate the behaviour of their remainders:

2587. $a^x\ (a>0)$.

2588. $\sin\left(x+\frac{\pi}{4}\right)$.

2589. $\cos(x+a)$.

2590. $\sin^2 x$.

2591*. $\ln(2+x)$.

Making use of the principal expansions I-V and a geometric progression, write the expansion, in powers of x, of the following functions, and indicate the intervals of convergence of the series:

2592. $\frac{2x-3}{(x-1)^2}$.

2593. $\frac{3x-5}{x^2-4x+3}$.

2594. xe^{-2x}.

2595. e^{x^2}.

2596. $\sinh x$.

2597. $\cos 2x$.

2598. $\cos^2 x$.

2599. $\sin 3x+x\cos 3x$.

2600. $\frac{x}{9+x^2}$.

2601. $\frac{1}{\sqrt{4-x^2}}$.

2602. $\ln\frac{1+x}{1-x}$.

2603. $\ln(1+x-2x^2)$.

Applying differentiation, expand the following functions in powers of x, and indicate the intervals in which these expansions occur:

2604. $(1+x)\ln(1+x)$.

2605. $\arctan x$

2606. $\arcsin x$.

2607. $\ln(x+\sqrt{1+x^2})$.

Applying various techniques, expand the given functions in powers of x and indicate the intervals in which these expansions occur:

2608. $\sin^2 x\cos^2 x$.

2609. $(1+x)e^{-x}$.

2610. $(1+e^x)^3$.

2611. $\sqrt[3]{8+x}$.

2612. $\frac{x^2-3x+1}{x^2-5x+6}$.

2613. $\cosh^3 x$.

2614. $\frac{1}{4-x^4}$.

2615. $\ln(x^2+3x+2)$.

2616. $\int\limits_0^x \frac{\sin x}{x}\,dx$.

2617. $\int\limits_0^x e^{-x^2}\,dx$.

2618. $\int\limits_0^x \frac{\ln(1+x)\,dx}{x}$.

2619. $\int\limits_0^x \frac{dx}{\sqrt{1-x^4}}$.

Write the first three nonzero terms of the expansion of the following functions in powers of x:

2620. $\tan x$.
2621. $\tanh x$.
2622. $e^{\cos x}$.
2623. $\sec x$.
2624. $\ln\cos x$.
2625. $e^x \sin x$.

2626*. Show that for computing the length of an ellipse it is possible to make use of the approximate formula

$$s \approx 2\pi a\left(1-\frac{\varepsilon^2}{4}\right),$$

where ε is the eccentricity and $2a$ is the major axis of the ellipse.

2627. A heavy string hangs, under its own weight, in a catenary line $y=a\cosh\frac{x}{a}$, where $a=\frac{H}{q}$ and H is the horizontal tension of the string, while q is the weight of unit length. Show that for small x, to the order of x^4, it may be taken that the string hangs in a parabola $y=a+\frac{x^2}{2a}$.

2628. Expand the function x^3-2x^2-5x-2 in a series of powers of $x+4$.

2629. $f(x)=5x^3-4x^2-3x+2$. Expand $f(x+h)$ in a series of powers of h

2630. Expand $\ln x$ in a series of powers of $x-1$.

2631. Expand $\frac{1}{x}$ in a series of powers of $x-1$.

2632. Expand $\frac{1}{x^2}$ in a series of powers of $x+1$.

2633. Expand $\frac{1}{x^2+3x+2}$ in a series of powers of $x+4$.

2634. Expand $\frac{1}{x^2+4x+7}$ in a series of powers of $x+2$.

2635. Expand e^x in a series of powers of $x+2$.

2636. Expand $\sqrt{x}$ in a series of powers of $x-4$.

2637. Expand $\cos x$ in a series of powers of $x-\frac{\pi}{2}$.

2638. Expand $\cos^2 x$ in a series of powers of $x-\frac{\pi}{4}$.

2639*. Expand $\ln x$ in a series of powers of $\frac{1-x}{1+x}$.

2640. Expand $\frac{x}{\sqrt{1+x}}$ in a series of powers of $\frac{x}{1+x}$.

2641. What is the magnitude of the error if we put approximately

$$e \approx 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}?$$

2642. To what degree of accuracy will we calculate the number $\frac{\pi}{4}$, if we make use of the series

$$\operatorname{arc\,tan} x = x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots,$$

by taking the sum of its first five terms when $x=1$?

2643*. Calculate the number $\frac{\pi}{6}$ to three decimals by expanding the function $\arcsin x$ in a series of powers of x (see Example 2606).

2644. How many terms do we have to take of the series

$$\cos x = 1-\frac{x^2}{2!}+\ldots,$$

in order to calculate $\cos 18°$ to three decimal places?

2645. How many terms do we have to take of the series

$$\sin x = x-\frac{x^3}{3!}+\ldots,$$

to calculate $\sin 15°$ to four decimal places?

2646. How many terms of the series

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots$$

have to be taken to find the number e to four decimal places?

2647. How many terms of the series

$$\ln(1+x) = x-\frac{x^2}{2}+\ldots,$$

do we have to take to calculate $\ln 2$ to two decimals? to 3 decimals?

2648. Calculate $\sqrt[3]{7}$ to two decimals by expanding the function $\sqrt[3]{8+x}$ in a series of powers of x.

2649. Find out the origin of the approximate formula $\sqrt{a^2+x} \approx a+\frac{x}{2a}$ $(a>0)$, evaluate it by means of $\sqrt{23}$, putting $a=5$, and estimate the error.

2650. Calculate $\sqrt[4]{19}$ to three decimals.

2651. For what values of x does the approximate formula

$$\cos x \approx 1 - \frac{x^2}{2}$$

yield an error not exceeding 0.01? 0.001? 0.0001?

2652. For what values of x does the approximate formula

$$\sin x \approx x$$

yield an error that does not exceed 0.01? 0.001?

2653. Evaluate $\int_0^{1/2} \frac{\sin x}{x} dx$ to four decimals.

2654. Evaluate $\int_0^1 e^{-x^2} dx$ to four decimals.

2655. Evaluate $\int_0^1 \sqrt[3]{x} \cos x \, dx$ to three decimals.

2656. Evaluate $\int_0^1 \frac{\sin x}{\sqrt{x}} dx$ to three decimals.

2657. Evaluate $\int_0^{1/4} \sqrt{1+x^3} \, dx$ to four decimals.

2658. Evaluate $\int_0^{1/9} \sqrt{x} e^x \, dx$ to three decimals.

2659. Expand the function $\cos(x-y)$ in a series of powers of x and y, find the region of convergence of the resulting series and investigate the remainder.

Write the expansions, in powers of x and y, of the following functions and indicate the regions of convergence of the series:

2660. $\sin x \cdot \sin y$.

2661. $\sin(x^2+y^2)$.

2662*. $\frac{1-x+y}{1+x-y}$.

2663*. $\ln(1-x-y+xy)$.

2664*. $\arctan \frac{x+y}{1-xy}$.

2665. $f(x, y) = ax^2 + 2bxy + cy^2$. Expand $f(x+h, y+k)$ in powers of h and k.

2666. $f(x, y) = x^3 - 2y^3 + 3xy$. Find the increment of this function when passing from the values $x=1$, $y=2$ to the values $x=1+h$, $y=2+k$.

2667. Expand the function e^{x+y} in powers of $x-2$ and $y+2$.

2668. Expand the function $\sin(x+y)$ in powers of x and $y-\frac{\pi}{2}$.

Write the first three or four terms of a power-series expansion in x and y of the functions:

2669. $e^x \cos y$.

2670. $(1+x)^{1+y}$.

Sec. 4. Fourier Series

1°. Dirichlet's theorem. We say that a function $f(x)$ satisfies the *Dirichlet conditions* in an interval (a, b) if, in this interval, the function

1) is uniformly bounded; that is $|f(x)| \leqslant M$ when $a < x < b$, where M is constant;

2) has no more than a finite number of points of discontinuity and all of them are of the first kind [i.e., at each discontinuity ξ the function $f(x)$ has a finite limit on the left $f(\xi - 0) = \lim\limits_{\varepsilon \to 0} f(\xi - \varepsilon)$ and a finite limit on the right $f(\xi + 0) = \lim\limits_{\varepsilon \to 0} f(\xi + \varepsilon)$ $(\varepsilon > 0)$];

3) has no more than a finite number of points of strict extremum.

Dirichlet's theorem asserts that a function $f(x)$, which in the interval $(-\pi, \pi)$ satisfies the Dirichlet conditions at any point x of this interval at which $f(x)$ is continuous, may be expanded in a trigonometric *Fourier series*:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots + a_n \cos nx + b_n \sin nx + \ldots, \quad (1)$$

where the *Fourier coefficients* a_n and b_n are calculated from the formulas

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx \; (n = 0, 1, 2, \ldots); \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx \; (n = 1, 2, \ldots).$$

If x is a point of discontinuity, belonging to the interval $(-\pi, \pi)$, of a function $f(x)$, then the sum of the Fourier series $S(x)$ is equal to the arithmetical mean of the left and right limits of the function:

$$S(x) = \frac{1}{2} [f(x-0) + f(x+0)].$$

At the end-points of the interval $x = -\pi$ and $x = \pi$,

$$S(-\pi) = S(\pi) = \frac{1}{2} [f(-\pi + 0) + f(\pi - 0)].$$

2°. Incomplete Fourier series. If a function $f(x)$ is even [i. e., $f(-x) = f(x)$], then in formula (1)

$$b_n = 0 \quad (n = 1, 2, \ldots)$$

and

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx \quad (n = 0, 1, 2, \ldots).$$

If a function $f(x)$ is odd [i.e., $f(-x)=-f(x)$], then $a_n=0$ ($n=0, 1, 2 \ldots$) and

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \quad (n=1, 2, \ldots).$$

A function specified in an interval $(0, \pi)$ may, at our discretion, be continued in the interval $(-\pi, 0)$ either as an even or an odd function; hence, it may be expanded in the interval $(0, \pi)$ in an incomplete Fourier series of sines or of cosines of multiple arcs.

3°. **Fourier series of a period $2l$.** If a function $f(x)$ satisfies the Dirichlet conditions in some interval $(-l, l)$ of length $2l$, then at the discontinuities of the function belonging to this interval the following expansion holds:

$$f(x)=\frac{a_0}{2}+a_1\cos\frac{\pi x}{l}+b_1\sin\frac{\pi x}{l}+a_2\cos\frac{2\pi x}{l}+b_2\sin\frac{2\pi x}{l}+\ldots$$
$$\ldots+a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}+\ldots,$$

where

$$\left.\begin{aligned} a_n&=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \quad (n=0, 1, 2, \ldots),\\ b_n&=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \quad (n=1, 2, \ldots).\end{aligned}\right\} \qquad (2)$$

At the points of discontinuity of the function $f(x)$ and at the end-points $x=\pm l$ of the interval, the sum of the Fourier series is defined in a manner similar to that which we have in the expansion in the interval $(-\pi, \pi)$.

In the case of an expansion of the function $f(x)$ in a Fourier series in an arbitrary interval $(a, a+2l)$ of length $2l$, the limits of integration in formulas (2) should be replaced respectively by a and $a+2l$

Expand the following functions in a Fourier series in the interval $(-\pi, \pi)$, determine the sum of the series at the points of discontinuity and at the end-points of the interval ($x=-\pi$, $x=\pi$), construct the graph of the function itself and of the sum of the corresponding series [outside the interval $(-\pi, \pi)$ as well]:

2671. $f(x)=\begin{cases} c_1 \text{ when } -\pi<x\leqslant 0,\\ c_2 \text{ when } 0<x<\pi.\end{cases}$

Consider the special case when $c_1=-1$, $c_2=1$.

2672. $f(x)=\begin{cases} ax \text{ when } -\pi<x\leqslant 0,\\ bx \text{ when } 0\leqslant x<\pi.\end{cases}$

Consider the special cases: a) $a=b=1$; b) $a=-1$, $b=1$; c) $a=0$, $b=1$; d) $a=1$, $b=0$.

2673. $f(x)=x^2$.

2674. $f(x)=e^{ax}$.

2675. $f(x)=\sin ax$.

2676. $f(x)=\cos ax$.

2677. $f(x)=\sinh ax$.

2678. $f(x)=\cosh ax$.

2679. Expand the function $f(x)=\frac{\pi-x}{2}$ in a Fourier series in the interval $(0, 2\pi)$.

2680. Expand the function $f(x)=\frac{\pi}{4}$ in sines of multiple arcs in the interval $(0, \pi)$. Use the expansion obtained to sum the number series:

a) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$; b) $1+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\ldots$;

c) $1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\ldots$

Take the functions indicated below and expand them, in the interval $(0, \pi)$, into incomplete Fourier series: a) of sines of multiple arcs, b) of cosines of multiple arcs. Sketch graphs of the functions and graphs of the sums of the corresponding series in their domains of definition.

2681. $f(x)=x$. Find the sum of the following series by means of the expansion obtained:

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

2682. $f(x)=x^2$. Find the sums of the following number series by means of the expansion obtained:

$$1)\ 1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots;\quad 2)\ 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$$

2683. $f(x)=e^{ax}$.

2684. $f(x)=\begin{cases} 1 \text{ when } 0<x<\frac{\pi}{2}, \\ 0 \text{ when } \frac{\pi}{2}\leqslant x<\pi. \end{cases}$

2685. $f(x)=\begin{cases} x \text{ when } 0<x\leqslant\frac{\pi}{2}, \\ \pi-x \text{ when } \frac{\pi}{2}<x<\pi. \end{cases}$

Expand the following functions, in the interval $(0, \pi)$, in sines of multiple arcs:

2686. $f(x)=\begin{cases} x \text{ when } 0<x\leqslant\frac{\pi}{2}, \\ 0 \text{ when } \frac{\pi}{2}<x<\pi. \end{cases}$

2687. $f(x)=x(\pi-x)$.

2688. $f(x)=\sin\frac{x}{2}$.

Expand the following functions, in the interval $(0, \pi)$, in cosines of multiple arcs:

2689. $f(x)=\begin{cases} 1 \text{ when } 0<x\leqslant h, \\ 0 \text{ when } h<x<\pi. \end{cases}$

2690. $f(x)=\begin{cases} 1-\frac{x}{2h} & \text{when } 0<x\leqslant 2h, \\ 0 & \text{when } 2h<x<\pi. \end{cases}$

2691. $f(x)=x\sin x$.

2692. $f(x)=\begin{cases} \cos x & \text{when } 0<x\leqslant \frac{\pi}{2}, \\ -\cos x & \text{when } \frac{\pi}{2}<x<\pi. \end{cases}$

2693. Using the expansions of the functions x and x^2 in the interval $(0, \pi)$ in cosines of multiple arcs (see Problems 2681 and 2682), prove the equality

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\frac{3x^2-6\pi x+2\pi^2}{12} \quad (0\leqslant x\leqslant \pi).$$

2694.** Prove that if the function $f(x)$ is even and we have $f\left(\frac{\pi}{2}+x\right)=-f\left(\frac{\pi}{2}-x\right)$, then its Fourier series in the interval $(-\pi, \pi)$ represents an expansion in cosines of odd multiple arcs, and if the function $f(x)$ is odd and $f\left(\frac{\pi}{2}+x\right)=f\left(\frac{\pi}{2}-x\right)$, then in the interval $(-\pi, \pi)$ it is expanded in sines of odd multiple arcs.

Expand the following functions in Fourier series in the indicated intervals:

2695. $f(x)=|x| \quad (-1<x<1)$.
2696. $f(x)=2x \quad (0<x<1)$.
2697. $f(x)=e^x \quad (-l<x<l)$.
2698. $f(x)=10-x \quad (5<x<15)$.

Expand the following functions, in the indicated intervals, in incomplete Fourier series: a) in sines of multiple arcs, and b) in cosines of multiple arcs:

2699. $f(x)=1 \quad (0<x<1)$.
2700. $f(x)=x \quad (0<x<l)$.
2701. $f(x)=x^2 \quad (0<x<2\pi)$.
2702. $f(x)=\begin{cases} x & \text{when } 0<x\leqslant 1, \\ 2-x & \text{when } 1<x<2. \end{cases}$

2703. Expand the following function in cosines of multiple arcs in the interval $\left(\frac{3}{2}, 3\right)$:

$$f(x)=\begin{cases} 1 & \text{when } \frac{3}{2}<x\leqslant 2, \\ 3-x & \text{when } 2<x<3. \end{cases}$$

Chapter IX
DIFFERENTIAL EQUATIONS

Sec. 1. Verifying Solutions. Forming Differential Equations of Families of Curves. Initial Conditions

1°. Basic concepts. An equation of the type

$$F(x, y, y', \ldots, y)^{(n)}=0, \tag{1}$$

where $y=y(x)$ is the sought-for function, is called a *differential equation of order n*. The function $y=\varphi(x)$, which converts equation (1) into an identity, is called the *solution* of the equation, while the graph of this function is called an *integral curve*. If the solution is represented implicitly, $\Phi(x, y)=0$, then it is usually called an *integral*

Example 1. Check that the function $y=\sin x$ is a solution of the equation

$$y''+y=0.$$

Solution. We have:

$$y'=\cos x, \quad y''=-\sin x$$

and, consequently,

$$y''+y=-\sin x+\sin x\equiv 0.$$

The integral

$$\Phi(x, y, C_1, \ldots, C_n)=0 \tag{2}$$

of the differential equation (1), which contains n independent arbitrary constants $C_1, \ldots, C_n$ and is equivalent (in the given region) to equation (1), is called the *general integral* of this equation (in the respective region). By assigning definite values to the constants $C_1, \ldots, C_n$ in (2), we get *particular integrals.*

Conversely, if we have a family of curves (2) and eliminate the parameters $C_1, \ldots, C_n$ from the system of equations

$$\Phi=0, \quad \frac{d\Phi}{dx}=0, \quad \ldots, \quad \frac{d^n\Phi}{dx^n}=0,$$

we, generally speaking, get a differential equation of type (1) whose general integral in the corresponding region is the relation (2).

Example 2. Find the differential equation of the family of parabolas

$$y=C_1(x-C_2)^2. \tag{3}$$

Solution. Differentiating equation (3) twice, we get:

$$y'=2C_1(x-C_2) \quad \text{and} \quad y''=2C_1. \tag{4}$$

Eliminating the parameters C_1 and C_2 from equations (3) and (4), we obtain the desired differential equation

$$2yy''=y'^2.$$

It is easy to verify that the function (3) converts this equation into an identity.

2°. **Initial conditions.** If for the desired particular solution $y=y(x)$ of a differential equation

$$y^{(n)}=f(x, y, y', \ldots, y^{(n-1)}) \tag{5}$$

the *initial conditions*

$$y(x_0)=y_0, \quad y'(x_0)=y_0', \quad \ldots, \quad y^{(n-1)}(x_0)=y_0^{(n-1)}$$

are given and we know the *general solution* of equation (5)

$$y=\varphi(x, C_1, \ldots, C_n),$$

then the arbitrary constants $C_1, \ldots, C_n$ are determined (if this is possible) from the system of equations

$$\left.\begin{array}{l} y_0=\varphi(x_0, C_1, \ldots, C_n), \\ y_0'=\varphi_x'(x_0, C_1, \ldots, C_n), \\ \ldots\ldots\ldots\ldots \\ y_0^{(n-1)}=\varphi_{x^{n-1}}^{(n-1)}(x_0, C_1, \ldots, C_n). \end{array}\right\}$$

Example 3. Find the curve of the family

$$y=C_1e^x+C_2e^{-2x}, \tag{6}$$

for which $y(0)=1$, $y'(0)=-2$.

Solution. We have:

$$y'=C_1e^x-2C_2e^{-2x} \tag{7}$$

Putting $x=0$ in formulas (6) and (7), we obtain

$$1=C_1+C_2, \quad -2=C_1-2C_2,$$

whence

$$C_1=0, \quad C_2=1$$

and, hence,

$$y=e^{-2x}.$$

Determine whether the indicated functions are solutions of the given differential equations:

2704. $xy'=2y$, $y=5x^2$.

2705. $y'^2=x^2+y^2$, $y=\frac{1}{x}$.

2706. $(x+y)\,dx+x\,dy=0$, $y=\frac{C^2-x^2}{2x}$.

2707. $y''+y=0$, $y=3\sin x-4\cos x$.

2708. $\frac{d^2x}{dt^2}+\omega^2x=0$, $x=C_1\cos\omega t+C_2\sin\omega t$.

2709. $y''-2y'+y=0$; a) $y=xe^x$, b) $y=x^2e^x$.

2710. $y''-(\lambda_1+\lambda_2)y'+\lambda_1\lambda_2y=0$,
$y=C_1e^{\lambda_1x}+C_2e^{\lambda_2x}$.

Show that for the given differential equations the indicated relations are integrals:

2711. $(x-2y)y'=2x-y$, $x^2-xy+y^2=C^2$.

2712. $(x-y+1)y'=1, \quad y=x+Ce^{y}$.
2713. $(xy-x)y''+xy'^2+yy'-2y'=0, \quad y=\ln(xy)$.

Form differential equations of the given families of curves (C, C_1, C_2, C_3 are arbitrary constants):

2714. $y=Cx$.
2715. $y=Cx^2$.
2716. $y^2=2Cx$.
2717. $x^2+y^2=C^2$.
2718. $y=Ce^{x}$.
2719. $x^3=C(x^2-y^2)$.
2720. $y^2+\frac{1}{x}=2+Ce^{-\frac{y^2}{2}}$.
2721. $\ln\frac{x}{y}=1+ay$ (a is a parameter).
2722. $(y-y_0)^2=2px$ (y_0, p are parameters).
2723. $y=C_1e^{2x}+C_2e^{-x}$.
2724. $y=C_1\cos 2x+C_2\sin 2x$.
2725. $y=(C_1+C_2x)e^{x}+C_3$.

2726. Form the differential equation of all straight lines in the xy-plane.

2727. Form the differential equation of all parabolas with vertical axis in the xy-plane.

2728. Form the differential equation of all circles in the xy-plane.

For the given families of curves find the lines that satisfy the given initial conditions:

2729. $x^2-y^2=C, \quad y(0)=5$.
2730. $y=(C_1+C_2x)e^{2x}, \quad y(0)=0, \quad y'(0)=1$.
2731. $y=C_1\sin(x-C_2), \quad y(\pi)=1, \quad y'(\pi)=0$.
2732. $y=C_1e^{-x}+C_2e^{x}+C_3e^{2x}$;
$y(0)=0, \quad y'(0)=1, \quad y''(0)=-2$.

Sec. 2. First-Order Differential Equations

1°. Types of first-order differential equations. A differential equation of the first order in an unknown function y, solved for the derivative y', is of the form

$$y'=f(x, y), \tag{1}$$

where $f(x, y)$ is the given function. In certain cases it is convenient to consider the variable x as the sought-for function, and to write (1) in the form

$$x'=g(x, y), \tag{1'}$$

where $g(x, y)=\frac{1}{f(x, y)}$.

Taking into account that $y'=\frac{dy}{dx}$ and $x'=\frac{dx}{dy}$, the differential equations (1) and (1') may be written in the symmetric form

$$P(x, y)\,dx+Q(x, y)\,dy=0, \tag{2}$$

where $P(x, y)$ and $Q(x, y)$ are known functions.

By solutions to (2) we mean functions of the form $y=\varphi(x)$ or $x=\psi(y)$ that satisfy this equation. The general integral of equations (1) and (1'), or equation (2), is of the form

$$\Phi(x,\ y,\ C)=0,$$

where C is an arbitrary constant.

2°. **Direction field.** The set of directions

$$\tan\alpha=f(x,\ y)$$

is called a direction field of the differential equation (1) and is ordinarily depicted by means of short lines or arrows inclined at an angle α.

Curves $f(x,\ y)=k$, at the points of which the inclination of the field has a constant value, equal to k, are called *isoclines*. By constructing the isoclines and direction field, it is possible, in the simplest cases, to give a

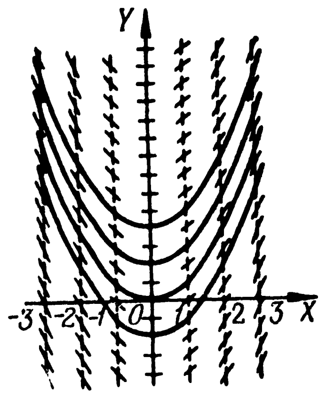

Fig 105

rough sketch of the field of integral curves, regarding the latter as curves which at each point have the given direction of the field.

Example 1. Using the method of isoclines, construct the field of integral curves of the equation

$$y'=x.$$

Solution. By constructing the isoclines $x=k$ (straight lines) and the direction field, we obtain approximately the field of integral curves (Fig. 105). The family of parabolas

$$y=\frac{x^2}{2}+C$$

is the general solution.

Using the method of isoclines, make approximate constructions of fields of integral curves for the indicated differential equations:

2733. $y'=-x$.

2734. $y'=-\frac{x}{y}$.

2735. $y'=1+y^2$.

2736. $y'=\frac{x+y}{x-y}$.

2737. $y'=x^2+y^2$.

3°. Cauchy's theorem. If a function $f(x, y)$ is continuous in some region $U\{a<x<A,\ b<y<B\}$ and in this region has a bounded derivative $f'_y(x, y)$, then through each point (x_0, y_0) that belongs to U there passes one and only one integral curve $y=\varphi(x)$ of the equation (1) $[\varphi(x_0)=y_0]$.

4°. Euler's broken-line method. For an approximate construction of the integral curve of equation (1) passing through a given point $M_0(x_0, y_0)$, we replace the curve by a broken line with vertices $M_i(x_i, y_i)$, where

$$x_{i+1}=x_i+\Delta x_i, \quad y_{i+1}=y_i+\Delta y_i,$$
$$\Delta x_i=h \quad \text{(one step of the process)},$$
$$\Delta y_i=hf(x_i, y_i) \quad (i=0, 1, 2, \ldots).$$

Example 2. Using Euler's method for the equation

$$y'=\frac{xy}{2},$$

find $y(1)$, if $y(0)=1$ $(h=0.1)$.
We construct the table:

i	x_i	y_i	$\Delta y_i=\frac{x_i y_i}{20}$
0	0	1	0
1	0.1	1	0 005
2	0.2	1.005	0.010
3	0.3	1 015	0 015
4	0 4	1.030	0 021
5	0.5	1.051	0 026
6	0.6	1.077	0 032
7	0.7	1.109	0.039
8	0.8	1.148	0 046
9	0.9	1.194	0.054
10	1.0	1.248	

Thus, $y(1)=1.248$. For the sake of comparison, the exact value is $y(1)=e^{\frac{1}{4}}\approx 1.284$

Using Euler's method, find the particular solutions to the given differential equations for the indicated values of x:

2738. $y'=y$, $y(0)=1$; find $y(1)$ $(h=0.1)$.

2739. $y'=x+y$, $y(1)=1$; find $y(2)$, $(h=0.1)$.

2740. $y'=-\frac{y}{1+x}$, $y(0)=2$; find $y(1)$ $(h=0.1)$.

2741. $y'=y-\frac{2x}{y}$, $y(0)=1$; find $y(1)$ $(h=0.2)$.

Sec. 3. First-Order Differential Equations with Variables Separable. Orthogonal Trajectories

1°. First-order equations with variables separable. An equation with *variables separable* is a first-order equation of the type

$$y' = f(x)\, g(y) \tag{1}$$

or

$$X(x)\, Y(y)\, dx + X_1(x)\, Y_1(y)\, dy = 0 \tag{1'}$$

Dividing both sides of equation (1) by $g(y)$ and multiplying by dx, we get $\frac{dy}{g(y)} = f(x)\, dx$ Whence, by integrating, we get the general integral of equation (1) in the form

$$\int \frac{dy}{g(y)} = \int f(x)\, dx + C \tag{2}$$

Similarly, dividing both sides of equation (1′) by $X_1(x)\, Y(y)$ and integrating, we get the general integral of (1′) in the form

$$\int \frac{X(x)}{X_1(x)}\, dx + \int \frac{Y_1(y)}{Y(y)}\, dy = C \tag{2'}$$

If for some value $y = y_0$ we have $g(y_0) = 0$, then the function $y = y_0$ is also (as is directly evident) a solution of equation (1) Similarly, the straight lines $x = a$ and $y = b$ will be the integral curves of equation (1′), if a and b are, respectively, the roots of the equations $X_1(x) = 0$ and $Y(y) = 0$, by the left sides of which we had to divide the initial equation.

Example 1. Solve the equation

$$y' = -\frac{y}{x}. \tag{3}$$

In particular, find the solution that satisfies the initial conditions

$$y(1) = 2$$

Solution. Equation (3) may be written in the form

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Whence, separating variables, we have

$$\frac{dy}{y} = -\frac{dx}{x}$$

and, consequently,

$$\ln|y| = -\ln|x| + \ln C_1,$$

where the arbitrary constant $\ln C_1$ is taken in logarithmic form. After taking antilogarithms we get the general solution

$$y = \frac{C}{x}, \tag{4}$$

where $C = \pm C_1$.

When dividing by y we could lose the solution $y = 0$, but the latter is contained in the formula (4) for $C = 0$.

Utilizing the given initial conditions, we get $C=2$; and, hence, the desired particular solution is

$$y=\frac{2}{x}.$$

2° Certain differential equations that reduce to equations with variables separable. Differential equations of the form

$$y'=f(ax+by+c) \quad (b\neq 0)$$

reduce to equations of the form (1) by means of the substitution $u=ax+by+c$, where u is the new sought-for function

3° Orthogonal trajectories are curves that intersect the lines of the given family $\Phi(x, y, a)=0$ (a is a parameter) at a right angle. If $F(x, y, y')=0$ is the differential equation of the family, then

$$F\left(x,\ y,\ -\frac{1}{y'}\right)=0$$

is the differential equation of the orthogonal trajectories.

Example 2. Find the orthogonal trajectories of the family of ellipses

$$x^2+2y^2=a^2. \tag{5}$$

Solution Differentiating the equation (5), we find the differential equation of the family

$$x+2yy'=0.$$

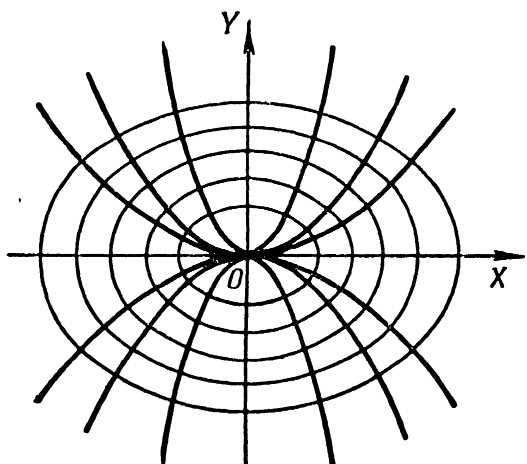

Fig. 106

Whence, replacing y' by $-\frac{1}{y'}$, we get the differential equation of the orthogonal trajectories

$$x-\frac{2y}{y'}=0 \quad \text{or} \quad y'=\frac{2y}{x}.$$

Integrating, we have $y=Cx^2$ (family of parabolas) (Fig. 106).

4°. Forming differential equations. When forming differential equations in geometrical problems, we can frequently make use of the geometrical meaning of the derivative as the tangent of an angle formed by the tangent line to the curve in the positive x-direction. In many cases this makes it possible straightway to establish a relationship between the ordinate y of the desired curve, its abscissa x, and the tangent of the angle of the tangent line y', that is to say, to obtain the differential equation. In other instances (see Problems 2783, 2890, 2895), use is made of the geometrical significance of the definite integral as the area of a curvilinear trapezoid or the length of an arc. In this case, by hypothesis we have a simple integral equation (since the desired function is under the sign of the integral); however, we can readily pass to a differential equation by differentiating both sides.

Example 3. Find a curve passing through the point (3,2) for which the segment of any tangent line contained between the coordinate axes is divided in half at the point of tangency.

Solution. Let $M(x,y)$ be the mid-point of the tangent line AB, which by hypothesis is the point of tangency (the points A and B are points of intersection of the tangent line with the y- and x-axes). It is given that $OA=2y$ and $OB=2x$. The slope of the tangent to the curve at $M(x, y)$ is

$$\frac{dy}{dx}=-\frac{OA}{OB}=-\frac{y}{x}.$$

This is the differential equation of the sought-for curve. Transforming, we get

$$\frac{dx}{x}+\frac{dy}{y}=0$$

and, consequently,

$$\ln x+\ln y=\ln C \text{ or } xy=C.$$

Utilizing the initial condition, we determine $C=3\cdot 2=6$. Hence, the desired curve is the hyperbola $xy=6$.

Solve the differential equations:

2742. $\tan x\sin^2 y\,dx+\cos^2 x\cot y\,dy=0$.

2743. $xy'-y=y^3$.

2744. $xyy'=1-x^2$.

2745. $y-xy'=a(1+x^2y')$.

2746. $3e^x\tan y\,dx+(1-e^x)\sec^2 y\,dy=0$.

2747. $y'\tan x=y$.

Find the particular solutions of equations that satisfy the indicated initial conditions:

2748. $(1+e^x)\,y\,y'=e^x$; $y=1$ when $x=0$.

2749. $(xy^2+x)\,dx+(x^2y-y)\,dy=0$; $y=1$ when $x=0$.

2750. $y'\sin x=y\ln y$; $y=1$ when $x=\frac{\pi}{2}$.

Solve the differential equations by changing the variables:

2751. $y'=(x+y)^2$.

2752. $y'=(8x+2y+1)^2$.

2753. $(2x+3y-1)\,dx+(4x+6y-5)\,dy=0$.

2754. $(2x-y)\,dx+(4x-2y+3)\,dy=0$.

In Examples 2755 and 2756, pass to polar coordinates:

2755. $y' = \frac{\sqrt{x^2+y^2}-x}{y}$.

2756. $(x^2+y^2)\,dx - xy\,dy = 0$.

2757*. Find a curve whose segment of the tangent is equal to the distance of the point of tangency from the origin.

2758. Find the curve whose segment of the normal at any point of a curve lying between the coordinate axes is divided in two at this point.

2759. Find a curve whose subtangent is of constant length a.

2760. Find a curve which has a subtangent twice the abscissa of the point of tangency.

2761*. Find a curve whose abscissa of the centre of gravity of an area bounded by the coordinate axes, by this curve and the ordinate of any of its points is equal to 3/4 the abscissa of this point.

2762. Find the equation of a curve that passes through the point (3,1), for which the segment of the tangent between the point of tangency and the x-axis is divided in half at the point of intersection with the y-axis.

2763. Find the equation of a curve which passes through the point (2,0), if the segment of the tangent to the curve between the point of tangency and the y-axis is of constant length 2.

Find the orthogonal trajectories of the given families of curves (a is a parameter), construct the families and their orthogonal trajectories.

2764. $x^2+y^2=a^2$.

2765. $y^2=ax$.

2766. $xy=a$.

2767. $(x-a)^2+y^2=a^2$.

Sec. 4. First-Order Homogeneous Differential Equations

1°. Homogeneous equations. A differential equation

$$P(x, y)\,dx + Q(x, y)\,dy = 0 \qquad (1)$$

is called *homogeneous*, if $P(x, y)$ and $Q(x, y)$ are homogeneous functions of the same degree. Equation (1) may be reduced to the form

$$y' = f\left(\frac{y}{x}\right);$$

and by means of the substitution $y = xu$, where u is a new unknown function, it is transformed to an equation with variables separable. We can also apply the substitution $x = yu$.

Example 1. Find the general solution to the equation

$$y' = e^{\frac{y}{x}} + \frac{y}{x}.$$

Solution. Put $y=ux$; then $u+xu'=e^u+u$ or

$$e^{-u}du=\frac{dx}{x}.$$

Integrating, we get $u=-\ln\ln\frac{C}{x}$, whence

$$y=-x\ln\ln\frac{C}{x}.$$

2°. Equations that reduce to homogeneous equations. If

$$y'=f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right) \qquad (2)$$

and $\delta=\begin{vmatrix} a_1b_1 \\ a_2b_2 \end{vmatrix}\neq 0$, then, putting into equation (2) $x=u+\alpha$, $y=v+\beta$, where the constants α and β are found from the following system of equations,

$$a_1\alpha+b_1\beta+c_1=0, \quad a_2\alpha+b_2\beta+c_2=0,$$

we get a homogeneous differential equation in the variables u and v. If $\delta=0$, then, putting in (2) $a_1x+b_1y=u$, we get an equation with variables separable.

Integrate the differential equations:

2768. $y'=\frac{y}{x}-1$.

2769. $y'=-\frac{x+y}{x}$.

2770. $(x-y)y\,dx-x^2dy=0$.

2771. For the equation $(x^2+y^2)\,dx-2xy\,dy=0$ find the family of integral curves, and also indicate the curves that pass through the points (4,0) and (1,1), respectively.

2772. $y\,dx+(2\sqrt{xy}-x)\,dy=0$.

2773. $x\,dy-y\,dx=\sqrt{x^2+y^2}dx$.

2774. $(4x^2+3xy+y^2)\,dx+(4y^2+3xy+x^2)\,dy=0$.

2775. Find the particular solution of the equation $(x^2-3y^2)\,dx+2xy\,dy=0$, provided that $y=1$ when $x=2$.

Solve the equations:

2776. $(2x-y+4)\,dy+(x-2y+5)\,dx=0$.

2777. $y'=\frac{1-3x-3y}{1+x+y}$.

2778. $y'=\frac{x+2y+1}{2x+4y+3}$.

2779. Find the equation of a curve that passes through the point (1,0) and has the property that the segment cut off by the tangent line on the y-axis is equal to the radius vector of the point of tangency.

2780**. What shape should the reflector of a search light have so that the rays from a point source of light are reflected as a parallel beam?

2781. Find the equation of a curve whose subtangent is equal to the arithmetic mean of the coordinates of the point of tangency.

2782. Find the equation of a curve for which the segment cut off on the y-axis by the normal at any point of the curve is equal to the distance of this point from the origin.

2783*. Find the equation of a curve for which the area contained between the x-axis, the curve and two ordinates, one of which is a constant and the other a variable, is equal to the ratio of the cube of the variable ordinate to the appropriate abscissa.

2784. Find a curve for which the segment on the y-axis cut off by any tangent line is equal to the abscissa of the point of tangency.

Sec. 5. First-Order Linear Differential Equations. Bernoulli's Equation

1°. Linear equations. A differential equation of the form

$$y' + P(x)\cdot y = Q(x) \tag{1}$$

of degree one in y and y' is called *linear*.

If a function $Q(x) \equiv 0$, then equation (1) takes the form

$$y' + P(x)\cdot y = 0 \tag{2}$$

and is called a *homogeneous linear* differential equation. In this case, the variables may be separated, and we get the general solution of (2) in the form

$$y = C\cdot e^{-\int P(x)\,dx}. \tag{3}$$

To solve the inhomogeneous linear equation (1), we apply a method that is called *variation of parameters*, which consists in first finding the general solution of the respective homogeneous linear equation, that is, relationship (3). Then, assuming here that C is a function of x, we seek the solution of the inhomogeneous equation (1) in the form of (3). To do this, we put into (1) y and y' which are found from (3), and then from the differential equation thus obtained we determine the function $C(x)$. We thus get the general solution of the inhomogeneous equation (1) in the form

$$y = C(x)\cdot e^{-\int P(x)\,dx}.$$

Example 1. Solve the equation

$$y' = \tan x\cdot y + \cos x. \tag{4}$$

Solution. The corresponding homogeneous equation is

$$y' - \tan x\cdot y = 0.$$

Solving it we get:

$$y = C\cdot\frac{1}{\cos x}.$$

Considering C as a function of x, and differentiating, we find;

$$y=\frac{1}{\cos x}\cdot\frac{dC}{dx}+\frac{\sin x}{\cos^2 x}\cdot C.$$

Putting y and y' into (4). we get:

$$\frac{1}{\cos x}\cdot\frac{dC}{dx}+\frac{\sin x}{\cos^2 x}\cdot C=\tan x\cdot\frac{C}{\cos x}+\cos x,\quad\text{or}\quad\frac{dC}{dx}=\cos^2 x,$$

whence

$$C(x)=\int\cos^2 x dx=\frac{1}{2}x+\frac{1}{4}\sin 2x+C_1.$$

Hence, the general solution of equation (4) has the form

$$y=\left(\frac{1}{2}x+\frac{1}{4}\sin 2x+C_1\right)\cdot\frac{1}{\cos x}.$$

In solving the linear equation (1) we can also make use of the substitution

$$y=uv, \tag{5}$$

where u and v are functions of x. Then equation (1) will have the form

$$[u'+P(x)u]v+v'u=Q(x). \tag{6}$$

If we require that

$$u'+P(x)u=0, \tag{7}$$

then from (7) we find u, and from (6) we find v; hence, from (5) we find y.

2°. Bernoulli's equation. A first-order equation of the form

$$y'+P(x)y=Q(x)y^{\alpha},$$

where $\alpha\neq 0$ and $\alpha\neq 1$, is called *Bernoulli's equation*. It is reduced to a linear equation by means of the substitution $z=y^{1-\alpha}$. It is also possible to apply directly the substitution $y=uv$, or the method of variation of parameters.

Example 2. Solve the equation

$$y'=\frac{4}{x}y+x\sqrt{y}.$$

Solution. This is Bernoulli's equation. Putting

$$y=u\cdot v,$$

we get

$$u'v+v'u=\frac{4}{x}uv+x\sqrt{uv}\quad\text{or}\quad v\left(u'-\frac{4}{x}u\right)+v'u=x\sqrt{uv}. \tag{8}$$

To determine the function u we require that the relation

$$u'-\frac{4}{x}u=0$$

be fulfilled, whence we have

$$u=x^4.$$

Putting this expression into (8), we get

$$v'x^4=x\sqrt{vx^4},$$

whence we find v:

$$v=\left(\frac{1}{2}\ln x+c\right)^2,$$

and, consequently, the general solution is obtained in the form

$$y=x^4\left(\frac{1}{2}\ln x+C\right)^2.$$

Find the general integrals of the equations:

2785. $\frac{dy}{dx}-\frac{y}{x}=x.$

2786. $\frac{dy}{dx}+\frac{2y}{x}=x^3.$

2787*. $(1+y^2)\,dx=(\sqrt{1+y^2}\sin y-xy)\,dy.$

2788. $y^2dx-(2xy+3)\,dy=0.$

Find the particular solutions that satisfy the indicated conditions:

2789. $xy'+y-e^x=0$; $y=b$ when $x=a$.

2790. $y'-\frac{y}{1-x^2}-1-x=0$; $y=0$ when $x=0$.

2791. $y'-y\tan x=\frac{1}{\cos x}$; $y=0$ when $x=0$.

Find the general solutions of the equations:

2792. $\frac{dy}{dx}+\frac{y}{x}=-xy^2.$

2793. $2xy\frac{dy}{dx}-y^2+x=0.$

2794. $y\,dx+\left(x-\frac{1}{2}x^3y\right)dy=0.$

2795. $3x\,dy=y(1+x\sin x-3y^3\sin x)\,dx.$

2796. Given three particular solutions y, y_1, y_2 of a linear equation. Prove that the expression $\frac{y_2-y}{y-y_1}$ remains unchanged for any x. What is the geometrical significance of this result?

2797. Find the curves for which the area of a triangle formed by the x-axis, a tangent line and the radius vector of the point of tangency is constant.

2798. Find the equation of a curve, a segment of which, cut off on the x-axis by a tangent line, is equal to the square of the ordinate of the point of tangency.

2799. Find the equation of a curve, a segment of which, cut off on the y-axis by a tangent line, is equal to the subnormal.

2800. Find the equation of a curve, a segment of which, cut off on the y-axis by a tangent line, is proportional to the square of the ordinate of the point of tangency.

2801. Find the equation of the curve for which the segment of the tangent is equal to the distance of the point of intersection of this tangent with the x-axis from the point $M(0,a)$.

Sec. 6. Exact Differential Equations. Integrating Factor

1°. Exact differential equations. If for the differential equation

$$P(x, y)\,dx + Q(x, y)\,dy = 0 \qquad (1)$$

the equality $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ is fulfilled, then equation (1) may be written in the form $dU(x, y) = 0$ and is then called an *exact differential equation*. The general integral of equation (1) is $U(x, y) = C$. The function $U(x, y)$ is determined by the technique given in Ch. VI, Sec. 8, or from the formula

$$U = \int_{x_0}^{x} P(x, y)\,dx + \int_{y_0}^{y} Q(x_0, y)\,dy$$

(see Ch. VII, Sec. 9).

Example 1. Find the general integral of the differential equation

$$(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0.$$

Solution. This is an exact differential equation, since $\frac{\partial(3x^2 + 6xy^2)}{\partial y} = \frac{\partial(6x^2y + 4y^3)}{\partial x} = 12xy$ and, hence, the equation is of the form $dU = 0$.

Here,

$$\frac{\partial U}{\partial x} = 3x^2 + 6xy^2 \quad \text{and} \quad \frac{\partial U}{\partial y} = 6x^2y + 4y^3;$$

whence

$$U = \int (3x^2 + 6xy^2)\,dx + \varphi(y) = x^3 + 3x^2y^2 + \varphi(y).$$

Differentiating U with respect to y, we find $\frac{\partial U}{\partial y} = 6x^2y + \varphi'(y) = 6x^2y + 4y^3$ (by hypothesis); from this we get $\varphi'(y) = 4y^3$ and $\varphi(y) = y^4 + C_0$. We finally get $U(x, y) = x^3 + 3x^2y^2 + y^4 + C_0$, consequently, $x^3 + 3x^2y^2 + y^4 = C$ is the sought-for general integral of the equation.

2°. Integrating factor. If the left side of equation (1) is not a total (exact) differential and the conditions of the Cauchy theorem are fulfilled, then there exists a function $\mu = \mu(x, y)$ *(integrating factor)* such that

$$\mu(P\,dx + Q\,dy) = dU. \qquad (2)$$

Whence it is found that the function μ satisfies the equation

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q).$$

The integrating factor μ is readily found in two cases:

1) $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = F(x)$, then $\mu = \mu(x)$;

2) $\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = F_1(y)$, then $\mu = \mu(y)$.

Example 2. Solve the equation $\left(2xy+x^2y+\frac{y^3}{3}\right)dx+(x^2+y^2)\,dy=0$.

Solution. Here $P=2xy+x^2y+\frac{y^3}{3}$, $Q=x^2+y^2$

and $\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=\frac{2x+x^2+y^2-2x}{x^2+y^2}=1$, hence, $\mu=\mu(x)$.

Since $\frac{\partial(\mu P)}{\partial y}=\frac{\partial(\mu Q)}{\partial x}$ or $\mu\frac{\partial P}{\partial y}=\mu\frac{\partial Q}{\partial x}+Q\frac{d\mu}{dx}$,
it follows that

$$\frac{d\mu}{\mu}=\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)dx=dx \text{ and } \ln\mu=x,\ \mu=e^x.$$

Multiplying the equation by $\mu=e^x$, we obtain

$$e^x\left(2xy+x^2y+\frac{y^3}{3}\right)dx+e^x(x^2+y^2)\,dy=0$$

which is an exact differential equation. Integrating it, we get the general integral

$$ye^x\left(x^2+\frac{y^2}{3}\right)=C.$$

Find the general integrals of the equations:

2802 $(x+y)\,dx+(x+2y)\,dy=0$.

2803. $(x^2+y^2+2x)\,dx+2xy\,dy=0$.

2804. $(x^3-3xy^2+2)\,dx-(3x^2y-y^2)\,dy=0$.

2805. $x\,dx-y\,dy=\frac{x\,dy-y\,dx}{x^2+y^2}$.

2806. $\frac{2x\,dx}{y^3}+\frac{y^2-3x^2}{y^4}\,dy=0$.

2807. Find the particular integral of the equation

$$\left(x+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0,$$

which satisfies the initial condition $y(0)=2$.

Solve the equations that admit of an integrating factor of the form $\mu=\mu(x)$ or $\mu=\mu(y)$:

2808. $(x+y^2)\,dx-2xy\,dy=0$.

2809. $y(1+xy)\,dx-x\,dy=0$.

2810. $\frac{y}{x}dx+(y^3-\ln x)dy=0$.

2811. $(x\cos y-y\sin y)\,dy+(x\sin y+y\cos y)\,dx=0$.

Sec. 7. First-Order Differential Equations not Solved for the Derivative

1°. First-order differential equations of higher powers. If an equation

$$F(x, y, y') = 0, \quad (1)$$

which for example is of degree two in y', then by solving (1) for y' we get two equations:

$$y' = f_1(x, y), \quad y' = f_2(x, y). \quad (2)$$

Thus, generally speaking, through each point $M_0(x_0, y_0)$ of some region of a plane there pass two integral curves. The general integral of equation (1) then, generally speaking, has the form

$$\Phi(x, y, C) \equiv \Phi_1(x, y, C)\,\Phi_2(x, y, C) = 0, \quad (3)$$

where Φ_1 and Φ_2 are the general integrals of equations (2).

Besides, there may be a *singular integral* for equation (1). Geometrically, a singular integral is the envelope of a family of curves (3) and may be obtained by eliminating C from the system of equations

$$\Phi(x, y, C) = 0, \quad \Phi'_C(x, y, C) = 0 \quad (4)$$

or by eliminating $p = y'$ from the system of equations

$$F(x, y, p) = 0, \quad F'_p(x, y, p) = 0. \quad (5)$$

We note that the curves defined by the equations (4) or (5) are not always solutions of equation (1); therefore, in each case, a check is necessary.

Example 1. Find the general and singular integrals of the equation

$$xy'^2 + 2xy' - y = 0.$$

Solution. Solving for y' we have two homogeneous equations:

$$y' = -1 + \sqrt{1 + \frac{y}{x}}, \quad y' = -1 - \sqrt{1 + \frac{y}{x}},$$

defined in the region

$$x(x+y) > 0,$$

the general integrals of which are

$$\left(\sqrt{1+\frac{y}{x}} - 1\right)^2 = \frac{C}{x}, \quad \left(\sqrt{1+\frac{y}{x}} + 1\right)^2 = \frac{C}{x}$$

or

$$(2x + y - C) - 2\sqrt{x^2 + xy} = 0, \quad (2x + y - C) + 2\sqrt{x^2 + xy} = 0.$$

Multiplying, we get the general integral of the given equation

$$(2x + y - C)^2 - 4(x^2 + xy) = 0$$

or

$$(y - C)^2 = 4Cx$$

(a family of parabolas).

Differentiating the general integral with respect to C and eliminating C, we find the singular integral

$$y + x = 0.$$

(It may be verified that $y + x = 0$ is the solution of this equation.)

It is also possible to find the singular integral by differentiating $xp^2+2xp-y=0$ with respect to p and eliminating p.

2°. **Solving a differential equation by introducing a parameter.** If a first-order differential equation is of the form

$$x=\varphi(y,\ y'),$$

then the variables y and x may be determined from the system of equations

$$\frac{1}{p}=\frac{\partial\varphi}{\partial y}+\frac{\partial\varphi}{\partial}\frac{dp}{dy},\quad x=\varphi(y,\ p),$$

where $p=y'$ plays the part of a parameter.

Similarly, if $y=\psi(x,\ y')$, then x and y are determined from the system of equations

$$p=\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial p}\frac{dp}{dx},\qquad y=\psi(x,\ p).$$

Example 2. Find the general and singular integrals of the equation

$$y=y'^2-xy'+\frac{x^2}{2}.$$

Solution. Making the substitution $y'=p$, we rewrite the equation in the form

$$y=p^2-xp+\frac{x^2}{2}.$$

Differentiating with respect to x and considering p a function of x, we have

$$p=2p\frac{dp}{dx}-p-x\frac{dp}{dx}+x$$

or $\frac{dp}{dx}(2p-x)=(2p-x)$, or $\frac{dp}{dx}=1$. Integrating we get $p=x+C$. Substituting into the original equation, we have the general solution

$$y=(x+C)^2-x(x+C)+\frac{x^2}{2}\ \text{or}\ y=\frac{x^2}{2}+Cx+C^2.$$

Differentiating the general solution with respect to C and eliminating C, we obtain the singular solution: $y=\frac{x^2}{4}$. (It may be verified that $y=\frac{x^2}{4}$ is the solution of the given equation.)

If we equate to zero the factor $2p-x$, which was cancelled out, we get $p=\frac{x}{2}$ and, putting p into the given equation, we get $y=\frac{x^2}{4}$, which is the same singular solution.

Find the general and singular integrals of the equations: (In Problems 2812 and 2813 construct the field of integral curves.)

2812. $y'^2-\frac{2y}{x}y'+1=0.$

2813. $4y'^2-9x=0.$

2814. $yy'^2-(xy+1)y'+x=0$.

2815. $yy'^2-2xy'+y=0$.

2816. Find the integral curves of the equation $y'^2+y^2=1$ that pass through the point $M\left(0, \frac{1}{2}\right)$.

Introducing the parameter $y'=p$, solve the equations:

2817. $x=\sin y'+\ln y'$.

2818. $y=y'^2 e^{y'}$.

2819. $y=y'^2+2\ln y'$.

2820. $4y=x^2+y'^2$.

2821. $e^x=\frac{y^2+y'^2}{2y'}$.

Sec. 8. The Lagrange and Clairaut Equations

1°. Lagrange's equation. An equation of the form

$$y=x\varphi(p)+\psi(p), \tag{1}$$

where $p=y'$ is called *Lagrange's equation* Equation (1) is reduced to a linear equation in x by differentiation and taking into consideration that $dy=p\,dx$:

$$p\,dx=\varphi(p)\,dx+[x\varphi'(p)+\psi'(p)]\,dp. \tag{2}$$

If $p\not\equiv\varphi(p)$, then from (1) and (2) we get the general solution in parametric form:

$$x=Cf(p)+g(p),\quad y=[Cf(p)+g(p)]\,\varphi(p)+\psi(p),$$

where p is a parameter and $f(p)$, $g(p)$ are certain known functions. Besides, there may be a singular solution that is found in the usual way.

2°. Clairaut's equation. If in equation (1) $p\equiv\varphi(p)$, then we get *Clairaut's equation*

$$y=xp+\psi(p).$$

Its general solution is of the form $y=Cx+\psi(C)$ (a family of straight lines). There is also a *particular solution* (envelope) that results by eliminating the parameter p from the system of equations

$$\begin{cases} x=-\psi'(p), \\ y=px+\psi(p). \end{cases}$$

Example. Solve the equation

$$y=2y'x+\frac{1}{y'}. \tag{3}$$

Solution. Putting $y'=p$ we have $y=2px+\frac{1}{p}$; differentiating and replacing dy by $p\,dx$, we get

$$p\,dx=2p\,dx+2x\,dp-\frac{dp}{p^2}$$

or

$$\frac{dx}{dp}=-\frac{2}{p}x+\frac{1}{p^3}.$$

Solving this linear equation, we will have

$$x=\frac{1}{p^2}(\ln p+C).$$

Hence, the general integral will be

$$\begin{cases} x=\frac{1}{p^2}(\ln p+C), \\ y=2px+\frac{1}{p}. \end{cases}$$

To find the singular integral, we form the system

$$y=2px+\frac{1}{p}, \quad 0=2x-\frac{1}{p^2}$$

in the usual way. Whence

$$x=\frac{1}{2p^2}, \quad y=\frac{2}{p}$$

and, consequently,

$$y=\pm 2\sqrt{2x}.$$

Putting y into (3) we are convinced that the function obtained is not a solution and, therefore, equation (3) does not have a singular integral.

Solve the Lagrange equations:

2822. $y=\frac{1}{2}x\left(y'+\frac{y}{y'}\right)$.

2823. $y=y'+\sqrt{1-y'^2}$.

2824. $y=(1+y')x+y'^2$.

2825*. $y=-\frac{1}{2}y'(2x+y')$.

Find the general and singular integrals of the Clairaut equations and construct the field of integral curves:

2826. $y=xy'+y'^2$.

2827. $y=xy'+y'$.

2828. $y=xy'+\sqrt{1+(y')^2}$.

2829. $y=xy'+\frac{1}{y'}$.

2830. Find the curve for which the area of a triangle formed by a tangent at any point and by the coordinate axes is constant.

2831. Find the curve if the distance of a given point to any tangent to this curve is constant.

2832. Find the curve for which the segment of any of its tangents lying between the coordinate axes has constant length l.

Sec. 9. Miscellaneous Exercises on First-Order Differential Equations

2833. Determine the types of differential equations and indicate methods for their solution:

a) $(x+y)y'=x\arctan\frac{y}{x}$;

b) $(x-y)y'=y^2$;

c) $y'=2xy+x^3$;

d) $y'=2xy+y^3$;

e) $xy'+y=\sin y$;

f) $(y-xy')^2=y'^3$;

g) $y=xe^{y'}$;

h) $(y'-2xy)\sqrt{y}=x^3$;

i) $y' = (x+y)^2$;

j) $x \cos y' + y \sin y' = 1$;

k) $(x^2 - xy)\,y' = y^4$;

l) $(x^2 + 2xy^3)\,dx + (y^2 + 3x^2y^2)\,dy = 0$;

m) $(x^3 - 3xy)\,dx + (x^2 + 3)\,dy = 0$;

n) $(xy^3 + \ln x)\,dx = y^2 dy$.

Solve the equations:

2834. a) $\left(x - y\cos\frac{y}{x}\right)dx + x\cos\frac{y}{x}\,dy = 0$;

b) $x \ln\frac{x}{y}\,dy - y\,dx = 0$.

2835. $x\,dx = \left(\frac{x^2}{y} - y^3\right)dy$.

2836. $(2xy^2 - y)\,dx + x\,dy = 0$.

2837. $xy' + y = xy^2 \ln x$.

2838. $y = xy' + y' \ln y'$.

2839. $y = xy' + \sqrt{-ay'}$.

2840. $x^2(y+1)\,dx + (x^3 - 1)(y-1)\,dy = 0$.

2841. $(1+y^2)(e^{2x}\,dx - e^y\,dy) - (1+y)\,dy = 0$.

2842. $y' - y\frac{2x-1}{x^2} = 1$.

2843. $ye^y = (y^3 + 2xe^y)\,y'$.

2844. $y' + y\cos x = \sin x \cos x$.

2845. $(1-x^2)\,y' + xy = a$.

2846. $xy' - \frac{y}{x+1} - x = 0$.

2847. $y'(x\cos y + a\sin 2y) = 1$.

2848. $(x^2y - x^2 + y - 1)\,dx + (xy + 2x - 3y - 6)\,dy = 0$.

2849. $y' = \left(1 + \frac{y-1}{2x}\right)^2$.

2850. $xy^3\,dx = (x^2y + 2)\,dy$.

2851. $y' = \frac{3x^2}{x^3 + y + 1}$.

2852. $2dx + \sqrt{\frac{x}{y}}\,dy - \sqrt{\frac{y}{x}}\,dx = 0$.

2853. $y' = \frac{y}{x} + \tan\frac{y}{x}$.

2854. $yy' + y^2 = \cos x$.

2855. $x\,dy + y\,dx = y^2\,dx$.

2856. $y'(x + \sin y) = 1$.

2857. $y\frac{dp}{dy} = -p + p^2$.

2858. $x^3\,dx - (x^4 + y^3)\,dy = 0$.

2859. $x^2y'^2 + 3xyy' + 2y^2 = 0$.

2860. $\frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} + \frac{x\,dy - y\,dx}{y^2} = 0$.

2861. $e^y\,dx + (xe^y - 2y)\,dy = 0$.

2862. $y = 2xy' + \sqrt{1 + y'^2}$.

2863. $y' = \frac{y}{x}(1 + \ln y - \ln x)$.

2864. $(2e^x + y^4)\,dy - ye^x\,dx = 0$.

2865. $y' = 2\left(\frac{y+2}{x+y-1}\right)^2$.

2866. $xy(xy^2 + 1)\,dy - dx = 0$.

2867. $a(xy' + 2y) = xyy'$.

2868. $x\,dy - y\,dx = y^2\,dx$.

2869. $(x^2-1)^{3/2}\,dy+(x^3+3xy\sqrt{x^2-1})\,dx=0.$

2870. $\tan x\frac{dy}{dx}-y=a.$

2871. $\sqrt{a^2+x^2}\,dy+(x+y-\sqrt{a^2+x^2})\,dx=0.$

2872. $xyy'^2-(x^2+y^2)y'+xy=0.$

2873. $y=xy'+\frac{1}{y'^2}.$

2874. $(3x^2+2xy-y^2)\,dx+(x^2-2xy-3y^2)\,dy=0.$

2875. $2yp\frac{dp}{dy}=3p^2+4y^2.$

Find solutions to the equations for the indicated initial conditions:

2876. $y'=\frac{y+1}{x}$; $y=0$ for $x=1$.

2877. $e^{x-y}y'=1$; $y=1$ for $x=1$.

2878. $\cot x y'+y=2$; $y=2$ for $x=0$.

2879. $e^y(y'+1)=1$; $y=0$ for $x=0$.

2880. $y'+y=\cos x$; $y=\frac{1}{2}$ for $x=0$.

2881. $y'-2y=-x^2$; $y=\frac{1}{4}$ for $x=0$.

2882. $y'+y=2x$; $y=-1$ for $x=0$.

2883. $xy'=y$; a) $y=1$ for $x=1$; b) $y=0$ for $x=0$.

2884. $2xy'=y$; a) $y=1$ for $x=1$; b) $y=0$ for $x=0$.

2885. $2xyy'+x^2-y^2=0$; a) $y=0$ for $x=0$; b) $y=1$ for $x=0$; c) $y=0$ for $x=1$.

2886. Find the curve passing through the point (0, 1), for which the subtangent is equal to the sum of the coordinates of the point of tangency.

2887. Find a curve if we know that the sum of the segments cut off on the coordinate axes by a tangent to it is constant and equal to $2a$.

2888. The sum of the lengths of the normal and subnormal is equal to unity. Find the equation of the curve if it is known that the curve passes through the coordinate origin.

2889*. Find a curve whose angle formed by a tangent and the radius vector of the point of tangency is constant.

2890. Find a curve knowing that the area contained between the coordinate axes, this curve and the ordinate of any point on it is equal to the cube of the ordinate.

2891. Find a curve knowing that the area of a sector bounded by the polar axis, by this curve and by the radius vector of any point of it is proportional to the cube of this radius vector.

2892. Find a curve, the segment of which, cut off by the tangent on the x-axis, is equal to the length of the tangent.

2893. Find the curve, of which the segment of the tangent contained between the coordinate axes is divided into half by the parabola $y^2 = 2x$.

2894. Find the curve whose normal at any point of it is equal to the distance of this point from the origin.

2895*. The area bounded by a curve, the coordinate axes, and the ordinate of some point of the curve is equal to the length of the corresponding arc of the curve. Find the equation of this curve if it is known that the latter passes through the point (0, 1).

2896. Find the curve for which the area of a triangle formed by the x-axis, a tangent, and the radius vector of the point of tangency is constant and equal to a^2.

2897. Find the curve if we know that the mid-point of the segment cut off on the x-axis by a tangent and a normal to the curve is a constant point $(a, 0)$.

When forming first-order differential equations, particularly in physical problems, it is frequently advisable to apply the so-called *method of differentials*, which consists in the fact that approximate relationships between infinitesimal increments of the desired quantities (these relationships are accurate to infinitesimals of higher order) are replaced by the corresponding relationships between their differentials. This does not affect the result.

Problem. A tank contains 100 litres of an aqueous solution containing 10 kg of salt. Water is entering the tank at the rate of 3 litres per minute, and the mixture is flowing out at 2 litres per minute. The concentration is maintained uniform by stirring. How much salt will the tank contain at the end of one hour?

Solution. The concentration c of a substance is the quantity of it in unit volume. If the concentration is uniform, then the quantity of substance in volume V is cV.

Let the quantity of salt in the tank at the end of t minutes be x kg. The quantity of solution in the tank at that instant will be $100+t$ litres, and, consequently, the concentration $c=\frac{x}{100+t}$ kg per litre.

During time dt, $2dt$ litres of the solution flows out of the tank (the solution contains $2c\,dt$ kg of salt). Therefore, a change of dx in the quantity of salt in the tank is given by the relationship

$$-dx = 2c\,dt = \frac{2x}{100+t}\,dt.$$

This is the sought-for differential equation. Separating variables and integrating, we obtain

$$\ln x = -2\ln(100+t) + \ln C$$

or

$$x = \frac{C}{(100+t)^2}.$$

The constant C is found from the fact that when $t=0$, $x=10$, that is, $C=100{,}000$. At the expiration of one hour, the tank will contain $x=\frac{100{,}000}{160^2} \approx 3.9$ kilograms of salt.

2898*. Prove that for a heavy liquid rotating about a vertical axis the free surface has the form of a paraboloid of revolution.

2899*. Find the relationship between the air pressure and the altitude if it is known that the pressure is 1 kgf on 1 cm^2 at sea level and 0.92 kgf on 1 cm^2 at an altitude of 500 metres.

2900*. According to Hooke's law an elastic band of length l increases in length $klF\,(k=\text{const})$ due to a tensile force F. By how much will the band increase in length due to its weight W if the band is suspended at one end? (The initial length of the band is l.)

2901. Solve the same problem for a weight P suspended from the end of the band.

When solving Problems 2902 and 2903, make use of Newton's law, by which the rate of cooling of a body is proportional to the difference of temperatures of the body and the ambient medium.

2902. Find the relationship between the temperature T and the time t if a body, heated to T_0 degrees, is brought into a room at constant temperature (a degrees).

2903. During what time will a body heated to 100° cool off to 30° if the temperature of the room is 20° and during the first 20 minutes the body cooled to 60°?

2904. The retarding action of friction on a disk rotating in a liquid is proportional to the angular velocity of rotation. Find the relationship between the angular velocity and time if it is known that the disk began rotating at 100 rpm and after one minute was rotating at 60 rpm.

2905*. The rate of disintegration of radium is proportional to the quantity of radium present. Radium disintegrates by one half in 1600 years. Find the percentage of radium that has disintegrated after 100 years.

2906*. The rate of outflow of water from an aperture at a vertical distance h from the free surface is defined by the formula

$$v=c\sqrt{2gh},$$

where $c\approx 0.6$ and g is the acceleration of gravity.

During what period of time will the water filling a hemispherical boiler of diameter 2 metres flow out of it through a circular opening of radius 0.1 m in the bottom.

2907*. The quantity of light absorbed in passing through a thin layer of water is proportional to the quantity of incident light and to the thickness of the layer. If one half of the original quantity of light is absorbed in passing through a three-metre-thick layer of water, what part of this quantity will reach a depth of 30 metres?

2908*. The air resistance to a body falling with a parachute is proportional to the square of the rate of fall. Find the limiting velocity of descent.

2909*. The bottom of a tank with a capacity of 300 litres is covered with a mixture of salt and some insoluble substance. Assuming that the rate at which the salt dissolves is proportional to the difference between the concentration at the given time and the concentration of a saturated solution (1 kg of salt per 3 litres of water) and that the given quantity of pure water dissolves 1/3 kg of salt in 1 minute, find the quantity of salt in solution at the expiration of one hour.

2910*. The electromotive force e in a circuit with current i, resistance R and self-induction L is made up of the voltage drop Ri and the electromotive force of self-induction $L\frac{di}{dt}$. Determine the current i at time t if $e=E\sin\omega t$ (E and ω are constants) and $i=0$ when $t=0$.

Sec. 10. Higher-Order Differential Equations

1°. The case of direct integration. If

$$y^{(n)}=f(x),$$

then

$$y=\underbrace{\int dx\int \ldots \int}_{n \text{ times}} f(x)\,dx+C_1x^{n-1}+C_2x^{n-2}+\ldots+C_n.$$

2°. Cases of reduction of order. 1) If a differential equation does not contain y explicitly, for instance,

$$F(x,\ y',\ y'')=0,$$

then, assuming $y'=p$, we get an equation of an order one unit lower:

$$F(x,\ p,\ p')=0.$$

Example 1. Find the particular solution of the equation

$$xy''+y'+x=0,$$

that satisfies the conditions

$$y=0,\ y'=0 \text{ when } x=0.$$

Solution. Putting $y'=p$, we have $y''=p'$, whence

$$xp'+p+x=0.$$

Solving the latter equation as a linear equation in the function p, we get

$$px=C_1-\frac{x^2}{2}.$$

From the fact that $y'=p=0$ when $x=0$, we have $0=C_1-0$, i.e., $C_1=0$. Hence,

$$p=-\frac{x}{2}$$

or

$$\frac{dy}{dx}=-\frac{x}{2},$$

whence, integrating once again, we obtain

$$y=-\frac{x^2}{4}+C_2$$

Putting $y=0$ when $x=0$, we find $C_2=0$. Hence, the desired particular solution is

$$y=-\frac{1}{4}x^2.$$

2) If a differential equation does not contain x explicitly, for instance,

$$F(y, y', y'')=0$$

then, putting $y'=p$, $y''=p\frac{dp}{dy}$, we get an equation of an order one unit lower:

$$F\left(y, p, p\frac{dp}{dy}\right)=0.$$

Example 2. Find the particular solution of the equation

$$yy''-y'^2=y^4$$

provided that $y=1$, $y'=0$ when $x=0$.

Solution. Put $y'=p$, then $y''=p\frac{dp}{dy}$ and our equation becomes

$$yp\frac{dp}{dy}-p^2=y^4.$$

We have obtained an equation of the Bernoulli type in p (y is considered the argument). Solving it, we find

$$p=\pm y\sqrt{C_1+y^2}.$$

From the fact that $y'=p=0$ when $y=1$, we have $C_1=-1$. Hence,

$$p=\pm y\sqrt{y^2-1}$$

or

$$\frac{dy}{dx}=\pm y\sqrt{y^2-1}.$$

Integrating, we have

$$\arccos\frac{1}{y}\pm x=C_2.$$

Putting $y=1$ and $x=0$, we obtain $C_2=0$, whence $\frac{1}{y}=\cos x$ or $y=\sec x$.

Solve the following equations:

2911. $y'' = \frac{1}{x}$.

2912. $y'' = -\frac{2}{2y^3}$.

2913. $y'' = 1 - y'^2$.

2914. $xy'' + y' = 0$.

2915. $yy'' = y'^2$.

2916. $yy'' + y'^2 = 0$.

2917. $(1 + x^2) y'' + y'^2 + 1 = 0$.

2918. $y' (1 + y'^2) = ay''$.

2919. $x^2 y'' + xy' = 1$.

2920. $yy'' = y^2 y' + y'^2$.

2921. $yy'' - y' (1 + y') = 0$.

2922. $y'' = -\frac{x}{y'}$.

2923. $(x + 1) y'' - (x + 2) y' + x + 2 = 0$.

2924. $xy'' = y' \ln \frac{y'}{x}$.

2925. $y' + \frac{1}{4} y''^2 = xy''$.

2926. $xy''' + y'' = 1 + x$.

2927. $y'''^2 + y''^2 = 1$.

Find the particular solutions for the indicated initial conditions:

2928. $(1 + x^2) y'' - 2xy' = 0$; $y = 0$, $y' = 3$ for $x = 0$.

2929. $1 + y'^2 = 2yy''$; $y = 1$, $y' = 1$ for $x = 1$.

2930. $yy'' + y'^2 = y'^3$; $y = 1$, $y' = 1$ for $x = 0$.

2931. $xy'' = y'$; $y = 0$, $y' = 0$ for $x = 0$.

Find the general integrals of the following equations:

2932. $yy' = \sqrt{y^2 + y'^2}\, y'' - y'y''$.

2933. $yy'' = y'^2 + y' \sqrt{y^2 + y'^2}$.

2934. $y'^2 - yy'' = y^2 y'$.

2935. $yy'' + y'^2 - y'^3 \ln y = 0$.

Find solutions that satisfy the indicated conditions:

2936. $y'' y^3 = 1$; $y = 1$, $y' = 1$ for $x = \frac{1}{2}$.

2937. $yy'' + y'^2 = 1$; $y = 1$, $y' = 1$ for $x = 0$.

2938. $xy'' = \sqrt{1 + y'^2}$; $y = 0$ for $x = 1$; $y = 1$ for $x = e^2$.

2939. $y'' (1 + \ln x) + \frac{1}{x} \cdot y' = 2 + \ln x$; $y = \frac{1}{2}$, $y' = 1$ for $x = 1$.

2940. $y'' = \frac{y'}{x} \left(1 + \ln \frac{y'}{x}\right)$; $y = \frac{1}{2}$, $y' = 1$ for $x = 1$.

2941. $y'' - y'^2 + y' (y - 1) = 0$; $y = 2$, $y' = 2$ for $x = 0$.

2942. $3y'y'' = y + y'^3 + 1$; $y = -2$, $y' = 0$ for $x = 0$.

2943. $y^2 + y'^2 - 2yy'' = 0$; $y = 1$, $y' = 1$ for $x = 0$.

2944. $yy' + y'^2 + yy'' = 0$; $y = 1$ for $x = 0$ and $y = 0$ for $x = -1$.

2945. $2y' + (y'^2 - 6x) \cdot y'' = 0$; $y = 0$, $y' = 2$ for $x = 2$.
2946. $y'y^2 + yy'' - y'^2 = 0$; $y = 1$, $y' = 2$ for $x = 0$.
2947. $2yy'' - 3y'^2 = 4y^2$; $y = 1$, $y' = 0$ for $x = 0$.
2948. $2yy'' + y^2 - y'^2 = 0$; $y = 1$, $y' = 1$ for $x = 0$.
2949. $y'' = y'^2 - y$; $y = -\frac{1}{4}$, $y' = \frac{1}{2}$ for $x = 1$.
2950. $y'' + \frac{1}{y^2} e^{y^2} y' - 2yy'^2 = 0$; $y = 1$, $y' = e$ for $x = -\frac{1}{2e}$.
2951. $1 + yy'' + y'^2 = 0$; $y = 0$, $y' = 1$ for $x = 1$.
2952. $(1 + yy')y'' = (1 + y'^2)y'$; $y = 1$, $y' = 1$ for $x = 0$.
2953. $(x + 1)y'' + xy'^2 = y'$; $y = -2$, $y' = 4$ for $x = 1$.

Solve the equations:

2954. $y' = xy''^2 + y''^2$.
2955. $y' = xy'' + y'' - y''^2$.
2956. $y'''^2 = 4y''$.

2957. $yy'y'' = y'^3 + y''^2$. Choose the integral curve passing through the point (0, 0) and tangent, at it, to the straight line $y + x = 0$.

2958. Find the curves of constant radius of curvature.

2959. Find a curve whose radius of curvature is proportional to the cube of the normal.

2960. Find a curve whose radius of curvature is equal to the normal.

2961. Find a curve whose radius of curvature is double the normal.

2962. Find the curves whose projection of the radius of curvature on the y-axis is a constant.

2963. Find the equation of the cable of a suspension bridge on the assumption that the load is distributed uniformly along the projection of the cable on a horizontal straight line. The weight of the cable is neglected.

2964*. Find the position of equilibrium of a flexible nontensile thread, the ends of which are attached at two points and which has a constant load q (including the weight of the thread) per unit length.

2965*. A heavy body with no initial velocity is sliding along an inclined plane. Find the law of motion if the angle of inclination is α, and the coefficient of friction is μ.

(**Hint.** The frictional force is μN, where N is the force of reaction of the plane.)

2966*. We may consider that the air resistance in free fall is proportional to the square of the velocity. Find the law of motion if the initial velocity is zero.

2967*. A motor-boat weighing 300 kgf is in rectilinear motion with initial velocity 66 m/sec. The resistance of the water is proportional to the velocity and is 10 kgf at 1 metre/sec. How long will it be before the velocity becomes 8 m/sec?

Sec. 11. Linear Differential Equations

1°. **Homogeneous equations.** The functions $y_1=\varphi_1(x)$, $y_2=\varphi_2(x)$, ..., $y_n=\varphi_n(x)$ are called *linearly dependent* if there are constants $C_1, C_2, \ldots, C_n$ not all equal to zero, such that

$$C_1y_1+C_2y_2+\ldots+C_ny_n\equiv 0;$$

otherwise, these functions are called *linearly independent*.

The general solution of a *homogeneous linear* differential equation

$$y^{(n)}+P_1(x)\,y^{(n-1)}+\ldots+P_n(x)\,y=0 \tag{1}$$

with continuous coefficients $P_i(x)$ $(i=1, 2, \ldots, n)$ is of the form

$$y=C_1y_1+C_2y_2+\ldots+C_ny_n,$$

where $y_1, y_2, \ldots, y_n$ are linearly independent solutions of equation (1) (*fundamental system of solutions*).

2°. **Inhomogeneous equations.** The general solution of an *inhomogeneous linear* differential equation

$$y^{(n)}+P_1(x)\,y^{(n-1)}+\ldots+P_n(x)\,y=f(x) \tag{2}$$

with continuous coefficients $P_i(x)$ and the right side $f(x)$ has the form

$$y=y_0+Y,$$

where y_0 is the general solution of the corresponding homogeneous equation (1) and Y is a particular solution of the given inhomogeneous equation (2).

If the fundamental system of solutions $y_1, y_2, \ldots, y_n$ of the homogeneous equation (1) is known, then the general solution of the corresponding inhomogeneous equation (2) may be found from the formula

$$y=C_1(x)\,y_1+C_2(x)\,y_2+\ldots+C_n(x)\,y_n,$$

where the functions $C_i(x)$ $(i=1, 2, \ldots, n)$ are determined from the following system of equations:

$$\left.\begin{aligned}
&C_1'(x)\,y_1+C_2'(x)\,y_2+\ldots+C_n'(x)\,y_1=0,\\
&C_1'(x)\,y_1'+C_2'(x)\,y_2'+\ldots+C_n'(x)\,y_n'=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&C_1'(x)\,y_1^{(n-2)}+C_2'(x)\,y_2^{(n-2)}+\ldots+C_n'(x)\,y_n^{(n-2)}=0,\\
&C_1'(x)\,y_1^{(n-1)}+C_2'(x)\,y_2^{(n-1)}+\ldots+C_n'(x)\,y_n^{(n-1)}=f(x)
\end{aligned}\right\} \tag{3}$$

(the *method of variation of parameters*).

Example. Solve the equation

$$xy''+y'=x^2. \tag{4}$$

Solution. Solving the homogeneous equation

$$xy''+y'=0,$$

we get

$$y=C_1\ln x+C_2. \qquad (5)$$

Hence, it may be taken that

$$y_1=\ln x \text{ and } y_2=1$$

and the solution of equation (4) may be sought in the form

$$y=C_1(x)\ln x+C_2(x).$$

Forming the system (3) and taking into account that the reduced form of the equation (4) is $y''+\frac{y'}{x}=x$, we obtain

$$\begin{cases} C_1'(x)\ln x+C_2'(x)\,1=0, \\ C_1'(x)\,\frac{1}{x}+C_2'(x)\,0=x. \end{cases}$$

Whence

$$C_1(x)=\frac{x^3}{3}+A \quad \text{and} \quad C_2(x)=-\frac{x^3}{3}\ln x+\frac{x^3}{9}+B$$

and, consequently,

$$y=\frac{x^3}{9}+A\ln x+B,$$

where A and B are arbitrary constants.

2968. Test the following systems of functions for linear relationships:

a) $x,\ x+1$;
b) $x^2,\ -2x^2$;
c) $0,\ 1,\ x$;
d) $x,\ x+1,\ x+2$;
e) $x,\ x^2,\ x^3$;
f) $e^x,\ e^{2x},\ e^{3x}$;
g) $\sin x,\ \cos x,\ 1$;
h) $\sin^2 x,\ \cos^2 x,\ 1$.

2969. Form a linear homogeneous differential equation, knowing its fundamental system of equations:

a) $y_1=\sin x,\ y_2=\cos x$;
b) $y_1=e^x,\ y_2=xe^x$;
c) $y_1=x,\ y_2=x^2$,
d) $y_1=e^x,\ y_2=e^x\sin x,\ y_3=e^x\cos x$.

2970. Knowing the fundamental system of solutions of a linear homogeneous differential equation

$$y_1=x,\ y_2=x^2,\ y_3=x^3,$$

find its particular solution y that satisfies the initial conditions

$$y|_{x=1}=0,\quad y'|_{x=1}=-1,\quad y''|_{x=1}=2.$$

2971*. Solve the equation

$$y'' + \frac{2}{x}y' + y = 0,$$

knowing its particular solution $y_1 = \frac{\sin x}{x}$.

2972. Solve the equation

$$x^2(\ln x - 1)y'' - xy' + y = 0,$$

knowing its particular solution $y_1 = x$.

By the method of variation of parameters, solve the following inhomogeneous linear equations.

2973. $x^2y'' - xy' = 3x^3$.

2974*. $x^2y'' + xy' - y = x^2$.

2975. $y''' + y' = \sec x$.

Sec. 12. Linear Differential Equations of Second Order with Constant Coefficients

1°. Homogeneous equations. A second-order linear equation with constant coefficients p and q without the right side is of the form

$$y'' + py' + qy = 0 \tag{1}$$

If k_1 and k_2 are roots of the characteristic equation

$$\varphi(k) \equiv k^2 + pk + q = 0, \tag{2}$$

then the general solution of equation (1) is written in one of the following three ways:

1) $y = C_1e^{k_1x} + C_2e^{k_2x}$ if k_1 and k_2 are real and $k_1 \neq k_2$;
2) $y = e^{k_1x}(C_1 + C_2x)$ if $k_1 = k_2$;
3) $y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$ if $k_1 = \alpha + \beta i$ and $k_2 = \alpha - \beta i$ $(\beta \neq 0)$.

2°. Inhomogeneous equations. The general solution of a linear inhomogeneous differential equation

$$y'' + py' + qy = f(x) \tag{3}$$

may be written in the form of a sum:

$$y = y_0 + Y,$$

where y_0 is the general solution of the corresponding equation (1) without right side and determined from formulas (1) to (3), and Y is a particular solution of the given equation (3).

The function Y may be found by the *method of undetermined coefficients* in the following simple cases:

1. $f(x) = e^{ax}P_n(x)$, where $P_n(x)$ is a polynomial of degree n.

If a is not a root of the characteristic equation (2), that is, $\varphi(a) \neq 0$, then we put $Y = e^{ax}Q_n(x)$ where $Q_n(x)$ is a polynomial of degree n with undetermined coefficients.

If a is a root of the characteristic equation (2), that is, $\varphi(a) = 0$, then $Y = x^re^{ax}Q_n(x)$, where r is the multiplicity of the root a ($r = 1$ or $r = 2$).

2. $f(x) = e^{ax}[P_n(x) \cos bx + Q_m(x) \sin bx]$.

If $\varphi(a \pm bi) \neq 0$, then we put

$$Y = e^{ax}[S_N(x)\cos bx + T_N(x)\sin bx],$$

where $S_N(x)$ and $T_N(x)$ are polynomials of degree N-max $\{n, m\}$.

But if $\varphi(a \pm bi) = 0$, then

$$Y = x^r e^{ax}[S_N(x)\cos bx + T_N(x)\sin bx],$$

where r is the multiplicity of the roots $a \pm bi$ (for second-order equations, $r = 1$).

In the general case, the *method of variation of parameters* (see Sec. 11) is used to solve equation (3).

Example 1. Find the general solution of the equation $2y'' - y' - y = 4xe^{2x}$.

Solution. The characteristic equation $2k^2 - k - 1 = 0$ has roots $k_1 = 1$ and $k_2 = -\frac{1}{2}$. The general solution of the corresponding homogeneous equation (first type) is $y_0 = C_1 e^x + C_2 e^{-\frac{x}{2}}$. The right side of the given equation is $f(x) = 4xe^{2x} = e^{ax}P_n(x)$. Hence, $Y = e^{2x}(Ax + B)$, since $n = 1$ and $r = 0$. Differentiating Y twice and putting the derivatives into the given equation, we obtain:

$$2e^{2x}(4Ax + 4B + 4A) - e^{2x}(2Ax + 2B + A) - e^{2x}(Ax + B) = 4xe^{2x}.$$

Cancelling out e^{2x} and equating the coefficients of identical powers of x and the absolute terms on the left and right of the equality, we have $5A = 4$ and $7A + 5B = 0$, whence $A = \frac{4}{5}$ and $B = -\frac{28}{25}$.

Thus, $Ye^{2x}\left(\frac{4}{5}x - \frac{28}{25}\right)$, and the general solution of the given equation is

$$y = C_1 e^x + C_2 e^{-\frac{1}{2}} + e^{2x}\left(\frac{4}{5}x - \frac{28}{25}\right).$$

Example 2. Find the general solution of the equation $y'' - 2y' + y = xe^x$.

Solution. The characteristic equation $k^2 - 2k + 1 = 0$ has a double root $k = 1$. The right side of the equation is of the form $f(x) = xe^x$; here, $a = 1$ and $n = 1$. The particular solution is $Y = x^2 e^x(Ax + B)$, since a coincides with the double root $k = 1$ and, consequently, $r = 2$.

Differentiating Y twice, substituting into the equation, and equating the coefficients, we obtain $A = \frac{1}{6}$, $B = 0$. Hence, the general solution of the given equation will be written in the form

$$y = (C_1 + C_2 x)e^x + \frac{1}{6}x^3 e^x.$$

Example 3. Find the general solution of the equation $y'' + y = x \sin x$.

Solution. The characteristic equation $k^2 + 1 = 0$ has roots $k_1 = i$ and $k_2 = -i$. The general solution of the corresponding homogeneous equation will [see 3, where $\alpha = 0$ and $\beta = 1$] be

$$y_0 = C_1 \cos x + C_2 \sin x.$$

The right side is of the form

$$f(x) = e^{ax}[P_n(x)\cos bx + Q_m(x)\sin bx],$$

where $a=0$, $b=1$, $P_n(x)=0$, $Q_m(x)=x$. To this side there corresponds the particular solution Y,

$$Y=x[(Ax+B)\cos x+(Cx+D)\sin x]$$

(here, $N=1$, $a=0$, $b=1$, $r=1$).

Differentiating twice and substituting into the equation, we equate the coefficients of both sides in $\cos x$, $x\cos x$, $\sin x$, and $x\sin x$. We then get four equations $2A+2D=0$, $4C=0$, $-2B+2C=0$, $-4A=1$, from which we determine $A=-\frac{1}{4}$, $B=0$, $C=0$, $D=\frac{1}{4}$. Therefore, $Y=-\frac{x^2}{4}\cos x+\frac{x}{4}\sin x$.

The general solution is

$$y=C_1\cos x+C_2\sin x-\frac{x^2}{4}\cos x+\frac{x}{4}\sin x.$$

3°. **The principle of superposition of solutions.** If the right side of equation (3) is the sum of several functions

$$f(x)=f_1(x)+f_2(x)+\ldots+f_n(x)$$

and Y_i $(i=1, 2, 3, \ldots, n)$ are the corresponding solutions of the equations

$$y''+py'+qy=f_i(x) \qquad (i=1, 2, \ldots, n),$$

then the sum

$$y=Y_1+Y_2+\ldots+Y_n$$

is the solution of equation (3).

Find the general solutions of the equations:

2976. $y''-5y'+6y=0$.

2977. $y''-9y=0$.

2978. $y''-y'=0$.

2979. $y''+y=0$.

2980. $y''-2y'+2y=0$.

2981. $y''+4y'+13y=0$.

2982. $y''+2y'+y=0$.

2983. $y''-4y'+2y=0$.

2984. $y''+ky=0$.

2985. $y=y''+y'$.

2986. $\frac{y'-y}{y''}=3$.

Find the particular solutions that satisfy the indicated conditions:

2987. $y''-5y'+4y=0$; $y=5$, $y'=8$ for $x=0$

2988. $y''+3y'+2y=0$; $y=1$, $y'=-1$ for $x=0$.

2989. $y''+4y=0$; $y=0$, $y'=2$ for $x=0$.

2990. $y''+2y'=0$; $y=1$, $y'=0$ for $x=0$

2991. $y''=\frac{y}{a^2}$; $y=a$, $y'=0$ for $x=0$.

2992. $y''+3y'=0$; $y=0$ for $x=0$ and $y=0$ for $x=3$.

2993. $y''+\pi^2y=0$; $y=0$ for $x=0$ and $y=0$ for $x=1$.

2994. Indicate the type of particular solutions for the given inhomogeneous equations:

a) $y''-4y=x^2e^{2x}$;

b) $y''+9y=\cos 2x$;

c) $y''-4y'+4y=\sin 2x+e^{2x}$;
d) $y''+2y'+2y=e^{x}\sin x$;
e) $y''-5y'+6y=(x^2+1)e^{x}+xe^{2x}$;
f) $y''-2y'+5y=xe^{x}\cos 2x-x^2e^{x}\sin 2x$.

Find the general solutions of the equations:

2995. $y''-4y'+4y=x^2$.
2996. $y''-y'+y=x^3+6$.
2997. $y''+2y'+y=e^{2x}$.
2998. $y''-8y'+7y=14$.
2999. $y''-y=e^{x}$.
3000. $y''+y=\cos x$.
3001. $y''+y'-2y=8\sin 2x$.
3002. $y''+y'-6y=xe^{2x}$.
3003. $y''-2y'+y=\sin x+\sinh x$.
3004. $y''+y'=\sin^2 x$.
3005. $y''-2y'+5y=e^{x}\cos 2x$.
3006. Find the solution of the equation $y''+4y=\sin x$ that satisfies the conditions $y=1$, $y'=1$ for $x=0$.

Solve the equations:

3007. $\frac{d^2x}{dt^2}+\omega^2x=A\sin pt$. Consider the cases: 1) $p\neq\omega$; 2) $p=\omega$.
3008. $y''-7y'+12y=-e^{4x}$.
3009. $y''-2y'=x^2-1$.
3010. $y''-2y'+y=2e^{x}$.
3011. $y''-2y'=e^{2x}+5$.
3012. $y''-2y'-8y=e^{x}-8\cos 2x$.
3013. $y''+y'=5x+2e^{x}$.
3014. $y''-y'=2x-1-3e^{x}$.
3015. $y''+2y'+y=e^{x}+e^{-x}$.
3016. $y''-2y'+10y=\sin 3x+e^{x}$.
3017. $y''-4y'+4y=2e^{2x}+\frac{x}{2}$.
3018. $y''-3y'=x+\cos x$.
3019. Find the solution to the equation $y''-2y'=e^{2x}+x^2-1$ that satisfies the conditions $y=\frac{1}{8}$, $y'=1$ for $x=0$.

Solve the equations:

3020. $y''-y=2x\sin x$.
3021. $y''-4y=e^{2x}\sin 2x$.
3022. $y''+4y=2\sin 2x-3\cos 2x+1$.
3023. $y''-2y'+2y=4e^{x}\sin x$.
3024. $y''=xe^{x}+y$.
3025. $y''+9y=2x\sin x+xe^{3x}$.

3026. $y''-2y'-3y=x(1+e^{3x})$.
3027. $y''-2y'=3x+2xe^{x}$.
3028. $y''-4y'+4y=xe^{2x}$.
3029. $y''+2y'-3y=2xe^{-3x}+(x+1)e^{x}$.
3030*. $y''+y=2x\cos x\cos 2x$.
3031. $y''-2y=2xe^{x}(\cos x-\sin x)$.

Applying the method of variation of parameters, solve the following equations:

3032. $y''+y=\tan x$.

3033. $y''+y=\cot x$.

3034. $y''-2y'+y=\frac{e^{x}}{x}$.

3035. $y''+2y'+y=\frac{e^{-x}}{x}$.

3036. $y''+y=\frac{1}{\cos x}$.

3037. $y''+y=\frac{1}{\sin x}$.

3038. a) $y''-y=\tanh x$.

b) $y''-2y=4x^{2}e^{x^{2}}$.

3039. Two identical loads are suspended from the end of a spring. Find the equation of motion that will be performed by one of these loads if the other falls.

Solution. Let the increase in the length of the spring under the action of one load in a state of rest be a and the mass of the load, m. Denote by x the coordinate of the load reckoned vertically from the position of equilibrium in the case of a single load. Then

$$m\frac{d^2x}{dt^2}=mg-k(x+a),$$

where, obviously, $k=\frac{mg}{a}$ and, consequently, $\frac{d^2x}{dt^2}=-\frac{g}{a}x$. The general solution is $x=C_1\cos\sqrt{\frac{g}{a}}\,t+C_2\sin\sqrt{\frac{g}{a}}\,t$. The initial conditions yield $x=a$ and $\frac{dx}{dt}=0$ when $t=0$; whence $C_1=a$ and $C_2=0$; and so

$$x=a\cos\sqrt{\frac{g}{a}}\,t.$$

3040*. The force stretching a spring is proportional to the increase in its length and is equal to 1 kgf when the length increases by 1 cm. A load weighing 2 kgf is suspended from the spring. Find the period of oscillatory motion of the load if it is pulled downwards slightly and then released.

3041*. A load weighing $P=4$ kgf is suspended from a spring and increases the length of the spring by 1 cm. Find the law of motion of the load if the upper end of the spring performs a vertical harmonic oscillation $y=2\sin 30t$ cm and if at the initial instant the load was at rest (resistance of the medium is neglected).

3042. A material point of mass m is attracted by each of two centres with a force proportional to the distance (the constant of proportionality is k). Find the law of motion of the point knowing that the distance between the centres is $2b$, at the initial instant the point was located on the line connecting the centres (at a distance c from its midpoint) and had a velocity of zero.

3043. A chain of length 6 metres is sliding from a support without friction. If the motion begins when 1 m of the chain is hanging from the support, how long will it take for the entire chain to slide down?

3044*. A long narrow tube is revolving with constant angular velocity ω about a vertical axis perpendicular to it. A ball inside the tube is sliding along it without friction. Find the law of motion of the ball relative to the tube, considering that

a) at the initial instant the ball was at a distance a from the axis of rotation; the initial velocity of the ball was zero;

b) at the initial instant the ball was located on the axis of rotation and had an initial velocity v_0.

Sec. 13. Linear Differential Equations of Order Higher than Two with Constant Coefficients

1°. **Homogeneous equations.** The fundamental system of solutions y_1, y_2, ..., y_n of a homogeneous linear equation with constant coefficients

$$y^{(n)}+a_1y^{(n-1)}+\ldots+a_{n-1}y'+a_ny=0 \qquad (1)$$

is constructed on the basis of the character of the roots of the *characteristic equation*

$$k^n+a_1k^{n-1}+\ldots+a_{n-1}k+a_n=0. \qquad (2)$$

Namely, 1) if k is a real root of the equation (2) of multiplicity m, then to this root there correspond m linearly independent solutions of equation (1):

$$y_1=e^{kx},\ y_2=xe^{kx},\ \ldots,\ y_m=x^{m-1}e^{kx};$$

2) if $\alpha\pm\beta i$ is a pair of complex roots of equation (2) of multiplicity m, then to the latter there correspond $2m$ linearly independent solutions of equation (1):

$$y_1=e^{\alpha x}\cos\beta x,\ y_2=e^{\alpha x}\sin\beta x,\ y_3=xe^{\alpha x}\cos\beta x,\ y_4=xe^{\alpha x}\sin\beta x,\ \ldots$$
$$\ldots,\ y_{2m-1}=x^{m-1}e^{\alpha x}\cos\beta x,\ y_{2m}=x^{m-1}e^{\alpha x}\sin\beta x.$$

2°. **Inhomogeneous equations.** A particular solution of the inhomogeneous equation

$$y^{(n)}+a_1y^{(n-1)}+\ldots+a_{n-1}y'+a_ny=f(x) \qquad (3)$$

is sought on the basis of rules 2° and 3° of Sec. 12.

Find the general solutions of the equations:

3045. $y''' - 13y'' + 12y' = 0$.

3046. $y''' - y' = 0$.

3047. $y''' + y = 0$.

3048. $y^{IV} - 2y'' = 0$.

3049. $y''' - 3y'' + 3y' - y = 0$.

3050. $y^{IV} + 4y = 0$.

3051. $y^{IV} + 8y'' + 16y = 0$.

3052. $y^{IV} + y' = 0$.

3053. $y^{IV} - 2y'' + y = 0$.

3054. $y^{IV} - a^4 y = 0$.

3055. $y^{IV} - 6y'' + 9y = 0$.

3056. $y^{IV} + a^2 y'' = 0$.

3057. $y^{IV} + 2y''' + y'' = 0$.

3058. $y^{IV} + 2y'' + y = 0$.

3059. $y^{(n)} + \frac{n}{1} y^{(n-1)} + \frac{n(n-1)}{1\cdot 2} y^{(n-2)} + \ldots + \frac{n}{1} y' + y = 0$.

3060. $y^{IV} - 2y''' + y'' = e^x$.

3061. $y^{IV} - 2y''' + y'' = x^3$.

3062. $y''' - y = x^3 - 1$.

3063. $y^{IV} + y''' = \cos 4x$.

3064. $y''' + y'' = x^2 + 1 + 3xe^x$.

3065. $y''' + y'' + y' + y = xe^x$.

3066. $y''' + y' = \tan x \sec x$.

3067. Find the particular solution of the equation

$$y''' + 2y'' + 2y' + y = x$$

that satisfies the initial conditions $y(0) = y'(0) = y''(0) = 0$.

Sec. 14. Euler's Equations

A linear equation of the form

$$(ax+b)^n y^{(n)} + A_1 (ax+b)^{n-1} y^{(n-1)} + \ldots + A_{n-1}(ax+b) y + A_n y = f(x), \quad (1)$$

where a, b, A_1, ..., A_{n-1}, A_n are constants, is called *Euler's equation*.

Let us introduce a new independent variable t, putting

$$ax + b = e^t.$$

Then

$$y' = ae^{-t}\frac{dy}{dt}, \quad y'' = a^2 e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right),$$

$$y''' = a^3 e^{-3t}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right) \text{ and so forth}$$

and Euler's equation is transformed into a linear equation with constant coefficients.

Example 1. Solve the equation $x^2y'' + xy' + y = 1$.

Solution. Putting $x = e^t$, we get

$$\frac{dy}{dx} = e^{-t}\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Consequently, the given equation takes on the form

$$\frac{d^2y}{dt^2} + y = 1,$$

whence

$$y = C_1 \cos t + C_2 \sin t + 1$$

or

$$y = C_1 \cos(\ln x) + C_2 \sin(\ln x) + 1.$$

For the homogeneous Euler equation

$$x^n y^{(n)} + A_1 x^{n-1} y^{(n-1)} + \ldots + A_{n-1} x y' + A_n y = 0 \qquad (2)$$

the solution may be sought in the form

$$y = x^k. \qquad (3)$$

Putting into (2) $y, y', \ldots, y^{(n)}$ found from (3), we get a characteristic equation from which we can find the exponent k.

If k is a real root of the characteristic equation of multiplicity m, then to it correspond m linearly independent solutions

$$y_1 = x^k,\ y_2 = x^k \cdot \ln x,\ y_3 = x^k (\ln x)^2,\ \ldots,\ y_m = x^k (\ln x)^{m-1}.$$

If $\alpha \pm \beta i$ is a pair of complex roots of multiplicity m, then to it there correspond $2m$ linearly independent solutions

$$y_1 = x^\alpha \cos(\beta \ln x),\ y_2 = x^\alpha \sin(\beta \ln x),\ y_3 = x^\alpha \ln x \cos(\beta \ln x),$$
$$y_4 = x^\alpha \ln x \cdot \sin(\beta \ln x),\ \ldots,\ y_{2m-1} = x^\alpha (\ln x)^{m-1} \cos(\beta \ln x),$$
$$y_{2m} = x^\alpha (\ln x)^{m-1} \sin(\beta \ln x).$$

Example 2. Solve the equation

$$x^2 y'' - 3xy' + 4y = 0.$$

Solution. We put

$$y = x^k,\quad y' = kx^{k-1},\quad y'' = k(k-1)x^{k-2}.$$

Substituting into the given equation, after cancelling out x^k, we get the characteristic equation

$$k^2 - 4k + 4 = 0.$$

Solving it we find

$$k_1 = k_2 = 2.$$

Hence, the general solution will be

$$y = C_1 x^2 + C_2 x^2 \ln x.$$

Solve the equations:

3068. $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = 0.$

3069. $x^2 y'' - xy' - 3y = 0.$

3070. $x^2 y'' + xy' + 4y = 0.$

3071. $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0.$

3072. $(3x + 2) y'' + 7y' = 0.$

3073. $y'' = \frac{2y}{x^2}.$

3074. $y'' + \frac{y'}{x} + \frac{y}{x^2} = 0.$

3075. $x^2 y'' - 4xy' + 6y = x.$

3076. $(1 + x)^2 y'' - 3(1 + x) y' + 4y = (1 + x)^3.$

3077. Find the particular solution of the equation

$$x^2y''-xy'+y=2x$$

that satisfies the initial conditions $y=0$, $y'=1$ when $x=1$.

Sec. 15. Systems of Differential Equations

Method of elimination. To find the solution, for instance, of a normal system of two first-order differential equations, that is, of a system of the form

$$\frac{dy}{dx}=f(x, y, z), \qquad \frac{dz}{dx}=g(x, y, z), \tag{1}$$

solved for the derivatives of the desired functions, we differentiate one of them with respect to x. We have, for example,

$$\frac{d^2y}{dx^2}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}f+\frac{\partial f}{\partial z}g. \tag{2}$$

Determining z from the first equation of the system (1) and substituting the value found,

$$z=\varphi\left(x, y, \frac{dy}{dx}\right) \tag{3}$$

into equation (2), we get a second-order equation with one unknown function y. Solving it, we find

$$y=\psi(x, C_1, C_2), \tag{4}$$

where C_1 and C_2 are arbitrary constants. Substituting function (4) into formula (3), we determine the function z without new integrations. The set of formulas (3) and (4), where y is replaced by ψ, yields the *general solution of the system* (1).

Example. Solve the system

$$\begin{cases} \dfrac{dy}{dx}+2y+4z=1+4x, \\ \dfrac{dz}{dx}+y-z=\dfrac{3}{2}x^2. \end{cases}$$

Solution. We differentiate the first equation with respect to x:

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4\frac{dz}{dx}=4.$$

From the first equation we determine $z=\frac{1}{4}\left(1+4x-\frac{dy}{dx}-2y\right)$ and then from the second we will have $\frac{dz}{dx}=\frac{3}{2}x^2+x+\frac{1}{4}-\frac{3}{2}y-\frac{1}{4}\frac{dy}{dx}$. Putting z and $\frac{dz}{dx}$ into the equation obtained after differentiation, we arrive at a second-order equation in one unknown y:

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=-6x^2-4x+3.$$

Solving it we find:

$$y = C_1e^{2x} + C_2e^{-3x} + x^2 + x,$$

and then

$$z = \frac{1}{4}\left(1 + 4x - \frac{dy}{dx} - 2y\right) = -C_1e^{2x} + \frac{C_2}{4}e^{-3x} - \frac{1}{2}x^2.$$

We can do likewise in the case of a system with a larger number of equations.

Solve the systems:

3078. $\begin{cases} \frac{dy}{dx} = z, \\ \frac{dz}{dx} = -y. \end{cases}$

3079. $\begin{cases} \frac{dy}{dx} = y + 5z, \\ \frac{dz}{dx} + y + 3z = 0. \end{cases}$

3080. $\begin{cases} \frac{dy}{dx} = -3y - z, \\ \frac{dz}{dx} = y - z. \end{cases}$

3081. $\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = z, \\ \frac{dz}{dt} = x. \end{cases}$

3082. $\begin{cases} \frac{dx}{dt} = y + z, \\ \frac{dy}{dt} = x + z, \\ \frac{dz}{dt} = x + y. \end{cases}$

3083. $\begin{cases} \frac{dy}{dx} = y + z, \\ \frac{dz}{dx} = x + y + z. \end{cases}$

3084. $\begin{cases} \frac{dy}{dx} + 2y + z = \sin x, \\ \frac{dz}{dx} - 4y - 2z = \cos x. \end{cases}$

3085. $\begin{cases} \frac{dy}{dx} + 3y + 4z = 2x, \\ \frac{dz}{dx} - y - z = x, \end{cases}$

$y = 0$, $z = 0$ when $x = 0$.

3086. $\begin{cases} \frac{dx}{dt} - 4x - y + 36t = 0, \\ \frac{dy}{dt} + 2x - y + 2e^t = 0, \end{cases}$

$x = 0$, $y = 1$ when $t = 0$.

3087. $\begin{cases} \frac{dy}{dx} = \frac{y^2}{z}, \\ \frac{dz}{dx} = \frac{1}{2}y. \end{cases}$

3088*. a) $\frac{dx}{x^3 + 3xy^2} = \frac{dy}{2y^3} = \frac{dz}{2y^2z}$;

b) $\frac{dx}{x - y} = \frac{dy}{x + y} = \frac{dz}{z}$;

c) $\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y}$,

isolate the integral curve passing through the point (1, 1, —2).

3089. $\begin{cases} \frac{dy}{dx} + z = 1, \\ \frac{dz}{dx} + \frac{2}{x^2}y = \ln x. \end{cases}$

3090. $\begin{cases} \frac{d^2y}{dx^2} + 2y + 4z = e^x, \\ \frac{d^2z}{dx^2} - y - 3z = -x. \end{cases}$

3091.** A shell leaves a gun with initial velocity v_0 at an angle α to the horizon. Find the equation of motion if we take the air resistance as proportional to the velocity.

3092*. A material point is attracted by a centre O with a force proportional to the distance. The motion begins from point A at a distance a from the centre with initial velocity v_0 perpendicular to OA. Find the trajectory.

Sec. 16. Integration of Differential Equations by Means of Power Series

If it is not possible to integrate a differential equation with the help of elementary functions, then in some cases its solution may be sought in the form of a power series:

$$y=\sum_{n=0}^{\infty} c_n (x-x_0)^n. \tag{1}$$

The undetermined coefficients c_n $(n=1, 2, \ldots)$ are found by putting the series (1) into the equation and equating the coefficients of identical powers of the binomial $x-x_0$ on the left-hand and right-hand sides of the resulting equation.

We can also seek the solution of the equation

$$y'=f(x, y); \quad y(x_0)=y_0 \tag{2}$$

in the form of the Taylor's series

$$y(x)=\sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x-x_0)^n, \tag{3}$$

where $y(x_0)=y_0$, $y'(x_0)=f(x_0, y_0)$ and the subsequent derivatives $y^{(n)}(x_0)$ $(n=2, 3, \ldots)$ are successively found by differentiating equation (2) and by putting x_0 in place of x

Example 1. Find the solution of the equation

$$y''-xy=0,$$

if $y=y_0$, $y'=y_0'$ for $x=0$.

Solution. We put

$$y=c_0+c_1x+\ldots+c_nx^n+\ldots,$$

whence, differentiating, we get

$$y''=2\cdot 1c_2+3\cdot 2c_3x+\ldots+n(n-1)c_nx^{n-2}+(n+1)nc_{n+1}x^{n-1}+ \\ +(n+2)(n+1)c_{n+2}x^n+\ldots$$

Substituting y and y'' into the given equation, we arrive at the identity

$$[2\cdot 1c_2+3\cdot 2c_3x+\ldots+n(n-1)c_nx^{n-2}+(n+1)nc_{n+1}x^{n-1}+ \\ +(n+2)(n+1)c_{n+2}x^n+\ldots]-x[c_0+c_1x+\ldots+c_nx^n+\ldots]\equiv 0.$$

Collecting together, on the left of this equation, the terms with identical powers of x and equating to zero the coefficients of these powers, we will have

$$c_2=0;\quad 3\cdot 2c_3-c_0=0,\quad c_3=\frac{c_0}{3\cdot 2};\quad 4\cdot 3c_4-c_1=0,\ c_4=\frac{c_1}{4\cdot 3};\quad 5\cdot 4c_5-c_2=0,$$

$$c_5=\frac{c_2}{5\cdot 4} \text{ and so forth.}$$

Generally,

$$c_{3k}=\frac{c_0}{2\cdot 3\cdot 5\cdot 6\cdot\ldots\cdot(3k-1)\,3k},\qquad c_{3k+1}=\frac{c_1}{3\cdot 4\cdot 6\cdot 7\cdot\ldots\cdot 3k\,(3k+1)},$$

$$c_{3k+2}=0\quad (k=1,\ 2,\ 3,\ \ldots).$$

Consequently,

$$y=c_0\left(1+\frac{x^3}{2\cdot 3}+\frac{x^6}{2\cdot 3\cdot 5\cdot 6}+\ldots+\frac{x^{3k}}{2\cdot 3\cdot 5\cdot 6\cdot\ldots\cdot(3k-1)\,3k}+\ldots\right)+$$
$$+c_1\left(x+\frac{x^4}{3\cdot 4}+\frac{x^7}{3\cdot 4\cdot 6\cdot 7}+\ldots+\frac{x^{3k+1}}{3\cdot 4\cdot 6\cdot 7\cdot\ldots\cdot 3k\,(3k+1)}+\ldots\right), \quad (4)$$

where $c_0=y_0$ and $c_1=y_0'$.

Applying d'Alembert's test, it is readily seen that series (4) converges for $-\infty<x<+\infty$.

Example 2. Find the solution of the equation

$$y'=x+y;\quad y_0=y\,(0)=1.$$

Solution. We put

$$y=y_0+y_0'x+\frac{y_0''}{2!}x^2+\frac{y_0'''}{3!}x^3+\ldots$$

We have $y_0=1$, $y_0'=0+1=1$. Differentiating equation $y'=x+y$, we successively find $y''=1+y'$, $y_0''=1+1=2$, $y'''=y''$, $y_0'''=2$, etc. Consequently,

$$y=1+x+\frac{2}{2!}x^2+\frac{2}{3!}x^3+\ldots$$

For the example at hand, this solution may be written in final form as

$$y=1+x+2\,(e^x-1-x) \text{ or } y=2e^x-1-x.$$

The procedure is similar for differential equations of higher orders. Testing the resulting series for convergence is, generally speaking, complicated and is not obligatory when solving the problems of this section.

With the help of power series, find the solutions of the equations for the indicated initial conditions.

In Examples 3097, 3098, 3099, 3101, test the solutions obtained for convergence.

3093. $y'=y+x^2$; $y=-2$ for $x=0$.

3094. $y'=2y+x-1$; $y=y_0$ for $x=1$.

3095. $y'=y^2+x^3$; $y=\frac{1}{2}$ for $x=0$.

3096. $y'=x^2-y^2$; $y=0$ for $x=0$.

3097. $(1-x)\,y'=1+x-y$; $y=0$ for $x=0$.

3098*. $xy''+y=0$; $y=0$, $y'=1$ for $x=0$.

3099. $y''+xy=0$; $y=1$, $y'=0$ for $x=0$.

3100*. $y''+\frac{2}{x}y'+y=0$; $y=1$, $y'=0$ for $x=0$.

3101*. $y''+\frac{1}{x}y'+y=0$; $y=1$, $y'=0$ for $x=0$.

3102. $\frac{d^2x}{dt^2}+x\cos t=0$; $x=a$; $\frac{dx}{dt}=0$ for $t=0$.

Sec. 17. Problems on Fourier's Method

To find the solutions of a linear homogeneous partial differential equation by Fourier's method, first seek the particular solutions of this special-type equation, each of which represents the product of functions that are dependent on one argument only. In the simplest case, there is an infinite set of such solutions u_n ($n=1, 2, \ldots$), which are linearly independent among themselves in any finite number and which satisfy the given *boundary conditions*. The desired solution u is represented in the form of a series arranged according to these particular solutions:

$$u=\sum_{n=1}^{\infty} C_n u_n. \qquad (1)$$

The coefficients C_n which remain undetermined are found from the *initial conditions*.

Problem. A transversal displacement $u=u(x, t)$ of the points of a string with abscissa x satisfies, at time t, the equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}, \qquad (2)$$

where $a^2=\frac{T_0}{\varrho}$ (T_0 is the tensile force and ϱ is the linear density of the string). Find the form of the string at time t if its ends $x=0$ and $x=l$ are

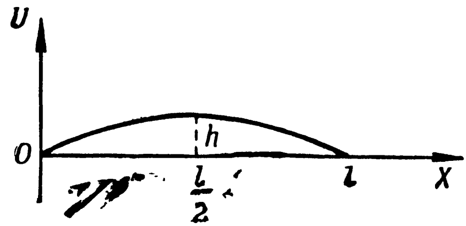

Fig. 107

fixed and at the initial instant, $t=0$, the string had the form of a parabola $u=\frac{4h}{l^2}x(l-x)$ (Fig. 107) and its points had zero velocity.

Solution. It is required to find the solution $u=u(x, t)$ of equation (2) that satisfies the boundary conditions

$$u(0, t)=0, \quad u(l, t)=0 \qquad (3)$$

and the initial conditions

$$u(x, 0)=\frac{4h}{l^2}x(l-x), \quad u'_t(x, 0)=0. \tag{4}$$

We seek the nonzero solutions of equation (2) of the special form

$$u=X(x)\,T(t).$$

Putting this expression into equation (2) and separating the variables, we get

$$\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}. \tag{5}$$

Since the variables x and t are independent, equation (5) is possible only when the general quantity of relation (5) is constant Denoting this constant by $-\lambda^2$, we find two ordinary differential equations:

$$T''(t)+(a\lambda)^2\cdot T(t)=0 \text{ and } X''(x)+\lambda^2X(x)=0.$$

Solving these equations, we get

$$T(t)=A\cos a\lambda t+B\sin a\lambda t,$$
$$X(x)=C\cos\lambda x+D\sin\lambda x,$$

where A, B, C, D are arbitrary constants. Let us determine the constants. From condition (3) we have $X(0)=0$ and $X(l)=0$; hence, $C=0$ and $\sin\lambda l=0$ (since D cannot be equal to zero at the same time as C is zero). For this reason, $\lambda_k=\frac{k\pi}{l}$, where k is an integer. It will readily be seen that we do not lose generality by taking for k only positive values ($k=1, 2, 3, \ldots$).

To every value λ_k there corresponds a particular solution

$$u_k=\left(A_k\cos\frac{ka\pi}{l}t+B_k\sin\frac{ka\pi}{l}t\right)\sin\frac{k\pi x}{l}$$

that satisfies the boundary conditions (3).

We construct the series

$$u=\sum_{k=1}^{\infty}\left(A_k\cos\frac{ka\pi t}{l}+B_k\sin\frac{ka\pi t}{l}\right)\sin\frac{k\pi x}{l},$$

whose sum obviously satisfies equation (2) and the boundary conditions (3).

We choose the constants A_k and B_k so that the sum of the series should satisfy the initial conditions (4). Since

$$\frac{\partial u}{\partial t}=\sum_{k=1}^{\infty}\frac{ka\pi}{l}\left(-A_k\sin\frac{ka\pi t}{l}+B_k\cos\frac{ka\pi t}{l}\right)\sin\frac{k\pi x}{l},$$

it follows that, by putting $t=0$, we obtain

$$u(x, 0)=\sum_{k=1}^{\infty}A_k\sin\frac{k\pi x}{l}\equiv\frac{4h}{l^2}x(l-x)$$

and

$$\frac{\partial u(x, 0)}{\partial t}=\sum_{k=1}^{\infty}\frac{ka\pi}{l}B_k\sin\frac{k\pi x}{l}\equiv 0.$$

Hence, to determine the coefficients A_k and B_k it is necessary to expand in a Fourier series, in sines only, the function $u(x, 0)=\frac{4h}{l^2}x(l-x)$ and the function $\frac{\partial u(x, 0)}{\partial t}\equiv 0$.

From familiar formulas (Ch. VIII, Sec. 4,3°) we have

$$A_k=\frac{2}{l}\int_0^l \frac{4h}{l^2}x(l-x)\sin\frac{k\pi x}{l}\,dx=\frac{32h}{\pi^3k^3},$$

if k is odd, and $A_k=0$ if k is even;

$$\frac{ka\pi}{l}B_k=\frac{2}{l}\int_0^l 0\sin\frac{k\pi x}{l}\,dx=0,\quad B_k=0.$$

The sought-for solution will be

$$u=\frac{32h}{\pi^3}\sum_{n=0}^{\infty}\frac{\cos\frac{(2n+1)a\pi t}{l}}{(2n+1)^3}\sin\frac{(2n+1)\pi x}{l}.$$

3103*. At the initial instant $t=0$, a string, attached at its ends, $x=0$ and $x=l$, had the form of the sine curve $u=A\sin\frac{\pi x}{l}$, and the points of it had zero velocity. Find the form of the string at time t.

3104*. At the initial time $t=0$, the points of a straight string $0<x<l$ receive a velocity $\frac{\partial u}{\partial t}=1$. Find the form of the string at time t if the ends of the string $x=0$ and $x=l$ are fixed (see Problem 3103).

3105*. A string of length $l=100$ cm and attached at its ends, $x=0$ and $x=l$, is pulled out to a distance $h=2$ cm at point $x=50$ cm at the initial time, and is then released without any impulse. Determine the shape of the string at any time t.

3106*. In longitudinal vibrations of a thin homogeneous and rectilinear rod, whose axis coincides with the x-axis, the displacement $u=u(x, t)$ of a cross-section of the rod with abscissa x satisfies, at time t, the equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2},$$

where $a^2=\frac{E}{\varrho}$ (E is Young's modulus and ϱ is the density of the rod). Determine the longitudinal vibrations of an elastic horizontal rod of length $l=100$ cm fixed at the end $x=0$ and pulled back at the end $x=100$ by $\Delta l=1$ cm, and then released without impulse.

3107*. For a rectilinear homogeneous rod whose axis coincides with the x-axis, the temperature $u = u(x, t)$ in a cross-section with abscissa x at time t, in the absence of sources of heat, satisfies the equation of heat conduction

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2},$$

where a is a constant. Determine the temperature distribution for any time t in a rod of length 100 cm if we know the initial temperature distribution

$$u(x, 0) = 0.01\,x\,(100 - x).$$

Chapter X

APPROXIMATE CALCULATIONS

Sec. 1. Operations on Approximate Numbers

1°. **Absolute error.** The *absolute error* of an approximate number a which replaces the exact number A is the absolute value of the difference between them. The number Δ, which satisfies the inequality

$$|A-a| \leqslant \Delta, \tag{1}$$

is called the *limiting absolute error*. The exact number A is located within the limits $a-\Delta \leqslant A \leqslant a+\Delta$ or, more briefly, $A=a \pm \Delta$

2°. **Relative error.** By the *relative error* of an approximate number a replacing an exact number A $(A>0)$ we understand the ratio of the absolute error of the number a to the exact number A. The number δ, which satisfies the inequality

$$\frac{|A-a|}{A} \leqslant \delta, \tag{2}$$

is called the *limiting relative error* of the approximate number a. Since in actual practice $A \approx a$, we often take the number $\delta=\frac{\Delta}{a}$ for the limiting relative error.

3°. **Number of correct decimals.** We say that a positive approximate number a written in the form of a decimal expansion has *n correct decimal places in a narrow sense* if the absolute error of this number does not exceed one half unit of the nth decimal place. In this case, when $n>1$ we can take, for the limiting relative error, the number

$$\delta=\frac{1}{2k}\left(\frac{1}{10}\right)^{n-1},$$

where k is the first significant digit of the number a. And conversely, if it is known that $\delta \leqslant \frac{1}{2(k+1)}\left(\frac{1}{10}\right)^{n-1}$, then the number a has n correct decimal places in the narrow meaning of the word. In particular, the number a definitely has n correct decimals in the narrow meaning if $\delta \leqslant \frac{1}{2}\left(\frac{1}{10}\right)^{n}$.

If the absolute error of an approximate number a does not exceed a unit of the last decimal place (such, for example, are numbers resulting from measurements made to a definite accuracy), then it is said that all decimal places of this approximate number are *correct in a broad sense*. If there is a larger number of significant digits in the approximate number, the latter (if it is the final result of calculations) is ordinarily rounded off so that all the remaining digits are correct in the narrow or broad sense.

Henceforth, we shall assume that all digits in the initial data are correct (if not otherwise stated) in the narrow sense. The results of intermediate calculations may contain one or two reserve digits.

We note that the examples of this section are, as a rule, the results of final calculations, and for this reason the answers to them are given as approximate numbers with only correct decimals.

4°. Addition and subtraction of approximate numbers. The limiting absolute error of an algebraic sum of several numbers is equal to the sum of the limiting absolute errors of these numbers. Therefore, in order to have, in the sum of a small number of approximate numbers (all decimal places of which are correct), only correct digits (at least in the broad sense), all summands should be put into the form of that summand which has the smallest number of decimal places, and in each summand a reserve digit should be retained. Then add the resulting numbers as exact numbers, and round off the sum by one decimal place

If we have to add approximate numbers that have not been rounded off, they should be rounded off and one or two reserve digits should be retained. Then be guided by the foregoing rule of addition while retaining the appropriate extra digits in the sum up to the end of the calculations.

Example 1. $215.21 + 14.182 + 21.4 = 215.2(1) + 14.1(8) + 21.4 = 250.8$.

The relative error of a sum of positive terms lies between the least and greatest relative errors of these terms.

The relative error of a difference is not amenable to simple counting. Particularly unfavourable in this sense is the difference of two close numbers.

Example 2. In subtracting the approximate numbers 6.135 and 6.131 to four correct decimal places, we get the difference 0.004. The limiting relative error is $\delta = \dfrac{\frac{1}{2}0.001 + \frac{1}{2}0.001}{0.004} = \dfrac{1}{4} = 0.25$. Hence, not one of the decimals of the difference is correct. Therefore, it is always advisable to avoid subtracting close approximate numbers and to transform the given expression, if need be, so that this undesirable operation is omitted.

5°. Multiplication and division of approximate numbers. The limiting relative error of a product and a quotient of approximate numbers is equal to the sum of the limiting relative errors of these numbers Proceeding from this and applying the rule for the number of correct decimals (3°), we retain in the answer only a definite number of decimals

Example 3. The product of the approximate numbers $25.3 \cdot 4.12 = 104.236$.

Assuming that all decimals of the factors are correct, we find that the limiting relative error of the product is

$$\delta = \frac{1}{2 \cdot 2} 0.01 + \frac{1}{4 \cdot 2} 0.01 \approx 0.003.$$

Whence the number of correct decimals of the product is three and the result, if it is final, should be written as follows: $25.3 \cdot 4.12 = 104$, or more correctly, $25.3 \cdot 4.12 = 104.2 \pm 0.3$.

6°. Powers and roots of approximate numbers. The limiting relative error of the mth power of an approximate number a is equal to the m-fold limiting relative error of this number

The limiting relative error of the mth root of an approximate number a is the $\frac{1}{m}$th part of the limiting relative error of the number a.

7°. Calculating the error of the result of various operations on approximate numbers. If $\Delta a_1, \ldots, \Delta a_n$ are the limiting absolute errors of the approximate numbers $a_1, \ldots, a_n$, then the limiting absolute error ΔS of the result

$$S = f(a_1, \ldots, a_n)$$

may be evaluated approximately from the formula

$$\Delta S = \left|\frac{\partial f}{\partial a_1}\right| \Delta a_1 + \ldots + \left|\frac{\partial f}{\partial a_n}\right| \Delta a_n.$$

The limiting relative error S is then equal to

$$\delta S = \frac{\Delta S}{|S|} = \left|\frac{\partial f}{\partial a_1}\right| \cdot \frac{\Delta a_1}{|f|} + \ldots + \left|\frac{\partial f}{\partial a_n}\right| \frac{\Delta a_n}{|f|} =$$
$$= \frac{\partial \ln f}{\partial a_1} \left| \Delta a_1 + \ldots + \left|\frac{\partial \ln f}{\partial a_n}\right| \Delta a_n.\right.$$

Example 4. Evaluate $S = \ln(10.3 + \sqrt{4.4})$; the approximate numbers 10.3 and 4.4 are correct to one decimal place.

Solution. Let us first compute the limiting absolute error ΔS in the general form: $S = \ln(a + \sqrt{b})$, $\Delta S = \frac{1}{a + \sqrt{b}}\left(\Delta a + \frac{1}{2}\frac{\Delta b}{\sqrt{b}}\right)$. We have $\Delta a = \Delta b \approx \frac{1}{20}$; $\sqrt{4.4} = 2.0976...$; we leave 2.1, since the relative error of the approximate number $\sqrt{4.4}$ is equal to $\approx \frac{1}{2} \cdot \frac{1}{40} = \frac{1}{80}$; the absolute error is then equal to $\approx 2\frac{1}{80} = \frac{1}{40}$; we can be sure of the first decimal place. Hence,

$$\Delta S = \frac{1}{10.3 + 2.1}\left(\frac{1}{20} + \frac{1}{2} \cdot \frac{1}{20 \cdot 2.1}\right) = \frac{1}{12.4 \cdot 20}\left(1 + \frac{1}{4.2}\right) = \frac{13}{2604} \approx 0.005.$$

Thus, two decimal places will be correct.

Now let us do the calculations with one reserve decimal:

$\log(10.3 + \sqrt{4\ 4}) \approx \log 12\ 4 = 1.093$, $\ln(10\ 3 + \sqrt{4.4}) \approx 1.093 \cdot 2.303 = 2.517$. And we get the answer: 2 52

8°. Establishing admissible errors of approximate numbers for a given error in the result of operations on them. Applying the formulas of 7° for the quantities ΔS or δS given us and considering all particular differentials $\left|\frac{\partial f}{\partial a_k}\right| \Delta a_k$ or the quantities $\left|\frac{\partial f}{\partial a_k}\right| \frac{\Delta a_k}{|f|}$ equal, we calculate the admissible absolute errors $\Delta a_1, \ldots, \Delta a_n, \ldots$ of the approximate numbers $a_1, \ldots, a_n, \ldots$ that enter into the operations (the *principle of equal effects*).

It should be pointed out that sometimes when calculating the admissible errors of the arguments of a function it is not advantageous to use the principle of equal effects, since the latter may make demands that are practically unfulfilable In these cases it is advisable to make a reasonable redistribution of errors (if this is possible) so that the overall total error does not exceed a specified quantity. Thus, strictly speaking, the problem thus posed is indeterminate.

Example 5. The volume of a "cylindrical segment", that is, a solid cut off a circular cylinder by a plane passing through the diameter of the base (equal to $2R$) at an angle α to the base, is computed from the formula $V = \frac{2}{3} R^3 \tan \alpha$. To what degree of accuracy should we measure the radius

$R \approx 60$ cm and the angle of inclination α so that the volume of the cylindrical segment is found to an accuracy up to 1%?

Solution. If ΔV, ΔR and $\Delta\alpha$ are the limiting absolute errors of the quantities V, R and α, then the limiting relative error of the volume V that we are calculating is

$$\delta = \frac{3\Delta R}{R} + \frac{2\Delta\alpha}{\sin 2\alpha} \leqslant \frac{1}{100}.$$

We assume $\frac{3\Delta R}{R} \leqslant \frac{1}{200}$ and $\frac{2\Delta\alpha}{\sin 2\alpha} \leqslant \frac{1}{200}$. Whence

$$\Delta R \leqslant \frac{R}{600} \approx \frac{60 \text{ cm}}{600} = 1 \text{ mm};$$

$$\Delta\alpha \leqslant \frac{\sin 2\alpha}{400} \leqslant \frac{1}{400} \text{ radian} \approx 9'.$$

Thus, we ensure the desired accuracy in the answer to 1% if we measure the radius to 1 mm and the angle of inclination α to 9'.

3108. Measurements yielded the following approximate numbers that are correct in the broad meaning to the number of decimal places indicated:

a) 12°07′14″; b) 38.5 cm; c) 62.215 kg.

Compute their absolute and relative errors.

3109. Compute the absolute and relative errors of the following approximate numbers which are correct in the narrow sense to the decimal places indicated:

a) 241.7; b) 0.035; c) 3.14.

3110. Determine the number of correct (in the narrow sense) decimals and write the approximate numbers:

a) 48.361 for an accuracy of 1%;

b) 14.9360 for an accuracy of 1%;

c) 592.8 for an accuracy of 2%.

3111. Add the approximate numbers, which are correct to the indicated decimals:

a) $25.386 + 0.49 + 3.10 + 0.5$;

b) $1.2 \cdot 10^2 + 41.72 + 0.09$;

c) $38.1 + 2.0 + 3.124$.

3112. Subtract the approximate numbers, which are correct to the indicated decimals:

a) $148.1 - 63.871$; b) $29.72 - 11.25$; c) $34.22 - 34.21$.

3113*. Find the difference of the areas of two squares whose measured sides are 15.28 cm and 15.22 cm (accurate to 0.05 mm).

3114. Find the product of the approximate numbers, which are correct to the indicated decimals:

a) $3.49 \cdot 8.6$; b) $25.1 \cdot 1.743$; c) $0.02 \cdot 16.5$. Indicate the possible limits of the results.

3115. The sides of a rectangle are 4.02 and 4.96 m (accurate to 1 cm). Compute the area of the rectangle.

3116. Find the quotient of the approximate numbers, which are correct to the indicated decimals:

a) 5.684 : 5.032; b) 0.144 : 1.2; c) 216:4.

3117. The legs of a right triangle are 12.10 cm and 25.21 cm (accurate to 0.01 cm). Compute the tangent of the angle opposite the first leg.

3118. Compute the indicated powers of the approximate numbers (the bases are correct to the indicated decimals):

a) 0.4158^2; b) 65.2^3; c) 1.5^2.

3119. The side of a square is 45.3 cm (accurate to 1 mm). Find the area.

3120. Compute the values of the roots (the radicands are correct to the indicated decimals):

a) $\sqrt{2.715}$; b) $\sqrt[3]{65.2}$; c) $\sqrt{81.1}$.

3121. The radii of the bases and the generatrix of a truncated cone are $R = 23.64$ cm ± 0.01 cm; $r = 17.31$ cm ± 0.01 cm; $l = 10.21$ cm ± 0.01 cm; $\pi = 3.14$. Use these data to compute the total surface of the truncated cone. Evaluate the absolute and relative errors of the result.

3122. The hypotenuse of a right triangle is 15.4 cm $\pm$ 0.1 cm; one of the legs is 6.8 cm $\pm$ 0.1 cm. To what degree of accuracy can we determine the second leg and the adjacent acute angle? Find their values.

3123. Calculate the specific weight of aluminium if an aluminium cylinder of diameter 2 cm and altitude 11 cm weighs 93.4 gm. The relative error in measuring the lengths is 0.01, while the relative error in weighing is 0.001.

3124. Compute the current if the electromotive force is equal to 221 volts $\pm$ 1 volt and the resistance is 809 ohms $\pm$ 1 ohm.

3125. The period of oscillation of a pendulum of length l is equal to

$$T = 2\pi \sqrt{\frac{l}{g}},$$

where g is the acceleration of gravity. To what degree of accuracy do we have to measure the length of the pendulum, whose period is close to 2 sec, in order to obtain its oscillation period with a relative error of 0.5%? How accurate must the numbers π and g be taken?

3126. It is required to measure, to within 1%, the lateral surface of a truncated cone whose base radii are 2 m and 1 m, and the generatrix is 5 m (approximately). To what degree of accuracy do we have to measure the radii and the generatrix and to how many decimal places do we have to take the number π?

3127. To determine Young's modulus for the bending of a rod of rectangular cross-section we use the formula

$$E=\frac{1}{4}\cdot\frac{l^3P}{d^3bs},$$

where l is the rod length, b and d are the basis and altitude of the cross-section of the rod, s is the sag, and P the load. To what degree of accuracy do we have to measure the length l and the sag s so that the error E should not exceed 5.5%, provided that the load P is known to 0.1%, and the quantities d and b are known to an accuracy of 1%, $l\approx 50$ cm, $s\approx 2.5$ cm?

Sec. 2. Interpolation of Functions

1°. Newton's interpolation formula. Let $x_0, x_1, \ldots, x_n$ be the tabular values of an argument, the difference of which $h=\Delta x_i$ ($\Delta x_i=x_{i+1}-x_i$; $i=0, 1, \ldots, n-1$) is constant (*table interval*) and $y_0, y_1, \ldots, y_n$ are the corresponding values of the function y. Then the value of the function y for an intermediate value of the argument x is approximately given by *Newton's interpolation formula*

$$y=y_0+q\cdot\Delta y_0+\frac{q(q-1)}{2!}\Delta^2y_0+\ldots+\frac{q(q-1)\ldots(q-n+1)}{n!}\Delta^ny_0, \qquad (1)$$

where $q=\frac{x-x_0}{h}$ and $\Delta y_0=y_1-y_0$, $\Delta^2y_0=\Delta y_1-\Delta y_0$, ... are successive finite differences of the function y. When $x=x_i$ ($i=0, 1, \ldots, n$), the polynomial (1) takes on, accordingly, the tabular values y_i ($i=0, 1, \ldots, n$). As particular cases of Newton's formula we obtain: for $n=1$, *linear interpolation*; for $n=2$, *quadratic interpolation*. To simplify the use of Newton's formula, it is advisable first to set up a table of finite differences.

If $y=f(x)$ is a polynomial of degree n, then

$$\Delta^ny_i=\text{const and } \Delta^{n+1}y_i=0$$

and, hence, formula (1) is exact.

In the general case, if $f(x)$ has a continuous derivative $f^{(n+1)}(x)$ on the interval $[a, b]$, which includes the points $x_0, x_1, \ldots, x_n$ and x, then the error of formula (1) is

$$R_n(x)=y-\sum_{i=0}^{n}\frac{q(q-1)\ldots(q-i+1)}{i!}\Delta^iy_0=$$

$$=h^{n+1}\frac{q(q-1)\ldots(q-n)}{(n+1)!}f^{(n+1)}(\xi), \qquad (2)$$

where ξ is some intermediate value between x_i ($i=0, 1, \ldots, n$) and x. For practical use, the following approximate formula is more convenient:

$$R_n(x)\approx\frac{\Delta^{n+1}y_0}{(n+1)!}q(q-1)\ldots(q-n).$$

If the number n may be any number, then it is best to choose it so that the difference $\Delta^{n+1}y_0 \approx 0$ within the limits of the given accuracy; in other words, the differences $\Delta^n y_0$ should be constant to within the given places of decimals

Example 1. Find sin 26°15′ using the tabular data sin 26° = 0.43837, sin 27° = 0.45399, sin 28° = 0.46947.

Solution. We set up the table

i	x_i	y_i	Δy_i	$\Delta^2 y_i$
0	26°	0 43837	1562	—14
1	27°	0 45399	1548	
2	28°	0 46947		

Here, $h=60'$, $q=\frac{26°15'-26°}{60'}=\frac{1}{4}$.

Applying formula (1) and using the first horizontal line of the table, we have

$$\sin 26°15' = 0.43837 + \frac{1}{4}\,0.01562 + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\cdot(-0.00014) = 0.44229.$$

Let us evaluate the error R_2 Using formula (2) and taking into account that if $y=\sin x$, then $|y^{(n)}| \leqslant 1$, we will have:

$$|R_2| \leqslant \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{3!}\left(\frac{\pi}{180}\right)^3 = \frac{7}{128}\cdot\frac{1}{57.3^3} \approx \frac{1}{4}\cdot 10^{-6}.$$

Thus, all the decimals of sin 26°15′ are correct.

Using Newton's formula, it is also possible, from a given intermediate value of the function y, to find the corresponding value of the argument x (*inverse interpolation*). To do this, first determine the corresponding value q by the method of successive approximation, putting

$$q^{(0)} = \frac{y-y_0}{\Delta y_0}$$

and

$$q^{(i+1)} = q^{(0)} - \frac{q^{(i)}(q^{(i)}-1)}{2!}\cdot\frac{\Delta^2 y_0}{\Delta y_0} - \ldots - \frac{q^{(i)}(q^{(i)}-1)\ldots(q^{(i)}-n+1)}{n!}\,\frac{\Delta^n y_0}{\Delta y_0}$$
$$(i=0,\ 1,\ 2,\ \ldots).$$

Here, for q we take the common value (to the given accuracy!) of two successive approximations $q^{(m)} = q^{(m+1)}$. Whence $x = x_0 + q\cdot h$.

Example 2. Using the table

x	$y=\sinh x$	Δy	$\Delta^2 y$
2 2	4.457	1.009	0.220
2 4	5.466	1 229	
2 6	6.695		

approximate the root of the equation $\sinh x = 5$.

Solution. Taking $y_0=4.457$, we have

$$q^{(0)}=\frac{5-4.457}{1.009}=\frac{0.543}{1.009}=0.538;$$

$$q^{(1)}=q^{(0)}+\frac{q^{(0)}(1-q^{(0)})}{2}\cdot\frac{\Delta^2 y_0}{\Delta y_0}=0.538+\frac{0.538\cdot 0.462}{2}\cdot\frac{0.220}{1.009}=$$
$$=0.538+0.027=0.565;$$

$$q^{(2)}=0.538+\frac{0.565\cdot 0.435}{2}\cdot\frac{0.220}{1.009}=0.538+0.027=0.565.$$

We can thus take

$$x=2.2+0.565\cdot 0.2=2.2+0.113=2.313.$$

2°. Lagrange's interpolation formula. In the general case, a polynomial of degree n, which for $x=x_i$ takes on given values y_i ($i=0, 1, \ldots, n$), is given by the *Lagrange interpolation formula*

$$y=\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1+\ldots$$
$$\ldots+\frac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}y_k+\ldots$$
$$\ldots+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n.$$

3128. Given a table of the values of x and y:

x	1	2	3	4	5	6
y	3	10	15	12	9	5

Set up a table of the finite differences of the function y.

3129. Set up a table of differences of the function $y=x^3-5x^2+x-1$ for the values $x=1, 3, 5, 7, 9, 11$. Make sure that all the finite differences of order 3 are equal.

3130*. Utilizing the constancy of fourth-order differences, set up a table of differences of the function $y=x^4-10x^3+2x^2+3x$ for integral values of x lying in the range $1\leqslant x\leqslant 10$.

3131. Given the table

$$\log 1=0.000,$$
$$\log 2=0.301,$$
$$\log 3=0.477,$$
$$\log 4=0.602,$$
$$\log 5=0.699.$$

Use linear interpolation to compute the numbers: $\log 1.7$, $\log 2.5$, $\log 3.1$, and $\log 4.6$.

3132. Given the table

$$\sin 10^\circ = 0.1736, \quad \sin 13^\circ = 0.2250,$$
$$\sin 11^\circ = 0.1908, \quad \sin 14^\circ = 0.2419,$$
$$\sin 12^\circ = 0.2079, \quad \sin 15^\circ = 0.2588.$$

Fill in the table by computing (with Newton's formula, for $n=2$) the values of the sine every half degree.

3133. Form Newton's interpolation polynomial for a function represented by the table

x	0	1	2	3	4
y	1	4	15	40	85

3134*. Form Newton's interpolation polynomial for a function represented by the table

x	2	4	6	8	10
y	3	11	27	50	83

Find y for $x=5.5$. For what x will $y=20$?

3135. A function is given by the table

x	−2	1	2	4
y	25	−8	−15	−23

Form Lagrange's interpolation polynomial and find the value of y for $x=0$.

3136. Experiment has yielded the contraction of a spring (x mm) as a function of the load (P kg) carried by the spring:

x	5	10	15	20	25	30	35	40
P	49	105	172	253	352	473	619	793

Find the load that yields a contraction of the spring by 14 mm.

3137. Given a table of the quantities x and y

x	0	1	3	4	5
y	1	-3	25	129	381

Compute the values of y for $x=0.5$ and for $x=2$: a) by means of linear interpolation; b) by Lagrange's formula.

Sec. 3. Computing the Real Roots of Equations

1°. Establishing initial approximations of roots. The approximation of the roots of a given equation

$$f(x)=0 \tag{1}$$

consists of two stages: 1) *separating the roots*, that is, establishing the intervals (as small as possible) within which lies one and only one root of equation (1); 2) *computing the roots* to a given degree of accuracy

If a function $f(x)$ is defined and continuous on an interval $[a, b]$ and $f(a)\cdot f(b)<0$, then on $[a, b]$ there is at least one root ξ of equation (1). This root will definitely be the only one if $f'(x)>0$ or $f'(x)<0$ when $a<x<b$.

In approximating the root ξ it is advisable to use millimetre paper and construct a graph of the function $y=f(x)$. The abscissas of the points of intersection of the graph with the x-axis are the roots of the equation $f(x)=0$. It is sometimes convenient to replace the given equation with an equivalent equation $\varphi(x)=\psi(x)$. Then the roots of the equation are found as the abscissas of points of intersection of the graphs $y=\varphi(x)$ and $y=\psi(x)$.

2°. The rule of proportionate parts (chord method). If on an interval $[a, b]$ there is a unique root ξ of the equation $f(x)=0$, where the function $f(x)$ is continuous on $[a, b]$, then by replacing the curve $y=f(x)$ by a chord passing through the points $[a, f(a)]$ and $[b, f(b)]$, we obtain the first approximation of the root

$$c_1=a-\frac{f(a)}{f(b)-f(a)}(b-a). \tag{2}$$

To obtain a second approximation c_2, we apply formula (2) to that one of the intervals $[a, c_1]$ or $[c_1, b]$ at the ends of which the function $f(x)$ has values of opposite sign. The succeeding approximations are constructed in the same manner. The sequence of numbers c_n $(n=1, 2, \ldots)$ converges to the root ξ, that is,

$$\lim_{n\to\infty} c_n=\xi.$$

Generally speaking, we should continue to calculate the approximations c_1, c_2, ..., until the decimals retained in the answer cease to change (in accord with the specified degree of accuracy!); for intermediate calculations, take one or two reserve decimals This is a general remark.

If the function $f(x)$ has a nonzero continuous derivative $f'(x)$ on the interval $[a, b]$, then to evaluate the absolute error of the approximate root

c_n, we can make use of the formula

$$|\xi - c_n| \leqslant \frac{|f(c_n)|}{\mu},$$

where $\mu = \min_{a \leqslant x \leqslant b} |f'(x)|$.

3°. Newton's method (method of tangents). If $f'(x) \neq 0$ and $f''(x) \neq 0$ for $a \leqslant x \leqslant b$, where $f(a) f(b) < 0$, $f(a) f''(a) > 0$, then the successive approximations x_n $(n = 0, 1, 2, \ldots)$ to the root ξ of an equation $f(x) = 0$ are computed from the formulas

$$x_0 = a, \quad x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad (n = 1, 2, \ldots). \tag{3}$$

Under the given assumptions, the sequence x_n $(n = 1, 2, \ldots)$ is monotonic and

$$\lim_{n \to \infty} x_n = \xi.$$

To evaluate the errors we can use the formula

$$|x_n - \xi| \leqslant \frac{|f(x_n)|}{\mu},$$

where $\mu = \min_{a \leqslant x \leqslant b} |f'(x)|$.

For practical purposes it is more convenient to use the simpler formulas

$$x_0 = a, \quad x_n = x_{n-1} - \alpha f(x_{n-1}) \quad (n = 1, 2, \ldots), \tag{3'}$$

where $\alpha = \frac{1}{f'(a)}$, which yield the same accuracy as formulas (3).

If $f(b) f''(b) > 0$, then in formulas (3) and (3') we should put $x_0 = b$.

4°. Iterative method. Let the given equation be reduced to the form

$$x = \varphi(x), \tag{4}$$

where $|\varphi'(x)| \leqslant r < 1$ (r is constant) for $a \leqslant x \leqslant b$. Proceeding from the initial value x_0, which belongs to the interval $[a, b]$, we build a sequence of numbers $x_1, x_2, \ldots$ according to the following law:

$$x_1 = \varphi(x_0), \quad x_2 = \varphi(x_1), \quad \ldots, \quad x_n = \varphi(x_{n-1}), \quad \ldots \tag{5}$$

If $a \leqslant x_n \leqslant b$ $(n = 1, 2, \ldots)$, then the limit

$$\xi = \lim_{n \to \infty} x_n$$

is the *only root* of equation (4) on the interval $[a, b]$; that is, x_n are *successive approximations* to the root ξ.

The evaluation of the absolute error of the nth approximation to x_n is given by the formula

$$|\xi - x_n| \leqslant \frac{|x_{n+1} - x_n|}{1 - r}.$$

Therefore, if x_n and x_{n+1} coincide to within ε, then the limiting absolute error for x_n will be $\frac{\varepsilon}{1-r}$.

In order to transform equation $f(x) = 0$ to (4), we replace the latter with an equivalent equation

$$x = x - \lambda f(x),$$

where the number $\lambda \neq 0$ is chosen so that the function $\frac{d}{dx}[x - \lambda f(x)] = 1 - \lambda f'(x)$

should be small in absolute value in the neighbourhood of the point x_0 [for example, we can put $1-\lambda f'(x_0)=0$].

Example 1. Reduce the equation $2x-\ln x-4=0$ to the form (4) for the initial approximation to the root $x_0=2.5$.

Solution. Here, $f(x)=2x-\ln x-4$; $f'(x)=2-\frac{1}{x}$. We write the equivalent equation $x=x-\lambda(2x-\ln x-4)$ and take 0.5 as one of the suitable values of λ; this number is close to the root of the equation $\left|1-\lambda\left(2-\frac{1}{x}\right)\right|_{x=2.5}=0$, that is, close to $\frac{1}{1.6}\approx 0.6$.

The initial equation is reduced to the form

$$x=x-0.5\,(2x-\ln x-4)$$

or

$$x=2+\frac{1}{2}\ln x.$$

Example 2. Compute, to two decimal places, the root ξ of the preceeding equation that lies between 2 and 3.

Computing the root by the iterative method. We make use of the result of Example 1, putting $x_0=2.5$. We carry out the calculations using formulas (5) with one reserve decimal.

$$x_1=2+\frac{1}{2}\ln 2.5\approx 2.458,$$

$$x_2=2+\frac{1}{2}\ln 2.458\approx 2.450,$$

$$x_3=2+\frac{1}{2}\ln 2.450\approx 2.448,$$

$$x_4=2+\frac{1}{2}\ln 2.448\approx 2.448.$$

And so $\xi\approx 2\ 45$ (we can stop here since the third decimal place has become fixed)

Let us now evaluate the error. Here,

$$\varphi(x)=2+\frac{1}{2}\ln x \quad\text{and}\quad \varphi'(x)=\frac{1}{2x}.$$

Considering that all approximations to x_n lie in the interval [2.4, 2.5], we get

$$r=\max|\varphi'(x)|=\frac{1}{2\cdot 2.4}=0.21.$$

Hence, the limiting absolute error in the approximation to x_3 is, by virtue of the remark made above,

$$\Delta=\frac{0.001}{1-0.21}=0.0012\approx 0.001.$$

Thus, the exact root ξ of the equation lies within the limits

$$2\ 447<\xi<2.449;$$

we can take $\xi\approx 2.45$, and all the decimals of this approximate number will be correct in the narrow sense.

Calculating the root by Newton's method. Here,

$$f(x)=2x-\ln x-4, \qquad f'(x)=2-\frac{1}{x}, \qquad f''(x)=\frac{1}{x^2}.$$

On the interval $2 \leqslant x \leqslant 3$ we have: $f'(x)>0$ and $f''(x)>0$; $f(2)f(3)<0$; $f(3)f''(3)>0$. Hence, the conditions of 3° for $x_0=3$ are fulfilled.

We take

$$\alpha=\left(2-\frac{1}{3}\right)^{-1}=0.6.$$

We carry out the calculations using formulas (3′) with two reserve decimals:

$$x_1=3-0.6\,(2\cdot 3-\ln 3-4)=2.4592;$$
$$x_2=2.4592-0.6\,(2\cdot 2.4592-\ln 2.4592-4)=2.4481;$$
$$x_3=2.4481-0.6\,(2\cdot 2.4481-\ln 2.4481-4)=2.4477;$$
$$x_4=2.4477-0.6\,(2\cdot 2.4477-\ln 2.4477-4)=2.4475.$$

At this stage we stop the calculations, since the third decimal place does not change any more. The answer is: the root $\xi=2.45$. We omit the evaluation of the error.

5°. **The case of a system of two equations.** Let it be required to calculate the real roots of a system of two equations in two unknowns (to a given degree of accuracy):

$$\begin{cases} f(x, y)=0, \\ \varphi(x, y)=0, \end{cases} \tag{6}$$

and let there be an initial approximation to one of the solutions (ξ, η) of this system $x=x_0$, $y=y_0$.

This initial approximation may be obtained, for example, graphically, by plotting (in the same Cartesian coordinate system) the curves $f(x, y)=0$ and $\varphi(x, y)=0$ and by determining the coordinates of the points of intersection of these curves.

a) **Newton's method.** Let us suppose that the functional determinant

$$I=\frac{\partial(f, \varphi)}{\partial(x, y)}$$

does not vanish near the initial approximation $x=x_0$, $y=y_0$. Then by Newton's method the first approximate solution to the system (6) has the form $x_1=x_0+\alpha_0$, $y_1=y_0+\beta_0$, where α_0, β_0 are the solution of the system of two linear equations

$$\begin{cases} f(x_0, y_0)+\alpha_0 f'_x(x_0, y_0)+\beta_0 f'_y(x_0, y_0)=0, \\ \varphi(x_0, y_0)+\alpha_0\varphi'_x(x_0, y_0)+\beta_0\varphi'_y(x_0, y_0)=0. \end{cases}$$

The second approximation is obtained in the very same way:

$$x_2=x_1+\alpha_1, \qquad y_2=y_1+\beta_1,$$

where α_1, β_1 are the solution of the system of linear equations

$$\begin{cases} f(x_1, y_1)+\alpha_1 f'_x(x_1, y_1)+\beta_1 f'_y(x_1, y_1)=0, \\ \varphi(x_1, y_1)+\alpha_1\varphi'_x(x_1, y_1)+\beta_1\varphi'_y(x_1, y_1)=0. \end{cases}$$

Similarly we obtain the third and succeeding approximations.

b) Iterative method. We can also apply the iterative method to solving the system of equations (6), by transforming this system to an equivalent one

$$\begin{cases} x=F(x, y), \\ y=\Phi(x, y) \end{cases} \tag{7}$$

and assuming that

$$|F'_x(x,y)|+|\Phi'_x(x, y)| \leqslant r < 1; \quad |F'_y(x, y)|+|\Phi'_y(x, y)| \leqslant r < 1 \tag{8}$$

in some two-dimensional neighbourhood U of the initial approximation (x_0, y_0), which neighbourhood also contains the exact solution (ξ, η) of the system.

The sequence of approximations (x_n, y_n) $(n=1, 2, \ldots)$, which converges to the solution of the system (7) or, what is the same thing, to the solution of (6), is constructed according to the following law:

$$\begin{aligned} &x_1=F(x_0, y_0), \quad y_1=\Phi(x_0, y_0), \\ &x_2=F(x_1, y_1), \quad y_2=\Phi(x_1, y_1), \\ &x_3=F(x_2, y_2), \quad y_3=\Phi(x_2, y_2), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

If all (x_n, y_n) belong to U, then $\lim\limits_{n\to\infty} x_n=\xi$, $\lim\limits_{n\to\infty} y_n=\eta$.

The following technique is advised for transforming the system of equations (6) to (7) with condition (8) observed. We consider the system of equations

$$\begin{cases} \alpha f(x, y)+\beta\varphi(x, y)=0, \\ \gamma f(x, y)+\delta\varphi(x, y)=0, \end{cases}$$

which is equivalent to (6) provided that $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \neq 0$. Rewrite it in the form

$$\begin{aligned} x&=x+\alpha f(x, y)+\beta\varphi(x, y) \equiv F(x, y), \\ y&=y+\gamma f(x, y)+\delta\varphi(x, y) \equiv \Phi(x, y). \end{aligned}$$

Choose the parameters α, β, γ, δ such that the partial derivatives of the functions $F(x, y)$ and $\Phi(x, y)$ will be equal or close to zero in the initial approximation; in other words, we find α, β, γ, δ as approximate solutions of the system of equations

$$\begin{cases} 1+\alpha f'_x(x_0, y_0)+\beta\varphi'_x(x_0, y_0)=0, \\ \alpha f'_y(x_0, y_0)+\beta\varphi'_y(x_0, y_0)=0, \\ \gamma f'_x(x_0, y_0)+\delta\varphi'_x(x_0, y_0)=0, \\ 1+\gamma f'_y(x_0, y_0)+\delta\varphi'_y(x_0, y_0)=0. \end{cases}$$

Condition (8) will be observed in such a choice of parameters α, β, γ, δ on the assumption that the partial derivatives of the functions $f(x, y)$ and $\varphi(x, y)$ do not vary very rapidly in the neighbourhood of the initial approximation (x_0, y_0).

Example 3. Reduce to the form (7) the system of equations

$$\begin{cases} x^2+y^2-1=0, \\ x^3-y=0 \end{cases}$$

given the initial approximation to the root $x_0=0.8$, $y_0=0.55$.

Solution. Here, $f(x, y)=x^2+y^2-1$, $\varphi(x, y)=x^3-y$; $f'_x(x_0, y_0)=1.6$, $f'_y(x_0, y_0)=1.1$; $\varphi'_x(x_0, y_0)=1.92$, $\varphi'_y(x_0, y_0)=-1$.

Write down the system (that is equivalent to the initial one)

$$\begin{cases} \alpha(x^2+y^2-1)+\beta(x^3-y)=0, \\ \gamma(x^2+y^2-1)+\delta(x^3-y)=0 \end{cases} \quad \left(\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} \neq 0\right)$$

in the form

$$x=x+\alpha(x^2+y^2-1)+\beta(x^3-y),$$
$$y=y+\gamma(x^2+y^2-1)+\delta(x^3-y).$$

For suitable numerical values of α, β, γ and δ choose the solution of the system of equations

$$\begin{cases} 1+1.6\alpha+1.92\beta=0, \\ 1.1\alpha-\beta=0, \\ 1.6\gamma+1.92\delta=0, \\ 1+1.1\gamma-\delta=0; \end{cases}$$

i. e., we put $\alpha\approx-0.3$, $\beta\approx-0.3$, $\gamma\approx-0.5$, $\delta\approx0.4$.

Then the system of equations

$$\begin{cases} x=x-0.3(x^2+y^2-1)-0.3(x^3-y), \\ y=y-0.5(x^2+y^2-1)+0\ 4(x^3-y), \end{cases}$$

which is equivalent to the initial system, has the form (7); and in a sufficiently small neighbourhood of the point (x_0, y_0) condition (8) will be fulfilled.

Isolate the real roots of the equations by trial and error, and by means of the rule of proportional parts compute them to two decimal places.

3138. $x^3-x+1=0$.

3139. $x^4+0\ 5x-1.55=0$.

3140. $x^3-4x-1=0$.

Proceeding from the graphically found initial approximations, use Newton's method to compute the real roots of the equations to two decimal places:

3141. $x^3-2x-5=0$.

3142. $2x-\ln x-4=0$.

3143. $2^x=4x$.

3144. $\log x=\frac{1}{x}$.

Utilizing the graphically found initial approximations, use the iterative method to compute the real roots of the equations to two decimal places:

3145. $x^3-5x+0.1=0$.

3146. $4x=\cos x$.

3147. $x^5-x-2=0$.

Find graphically the initial approximations and compute the real roots of the equations and systems to two decimals:

3148. $x^3-3x+1=0$.

3149. $x^3-2x^2+3x-5=0$.

3150. $x^4+x^2-2x-2=0$.

3151. $x\cdot\ln x-14=0$.

3152. $x^3+3x-0.5=0$.

3153. $4x-7\sin x=0$.

3154. $x^x + 2x - 6 = 0$.

3155. $e^x + e^{-3x} - 4 = 0$.

3156. $\begin{cases} x^2 + y^2 - 1 = 0, \\ x^3 - y = 0. \end{cases}$

3157. $\begin{cases} x^2 + y - 4 = 0, \\ y - \log x - 1 = 0. \end{cases}$

3158. Compute to three decimals the smallest positive root of the equation $\tan x = x$.

3159. Compute the roots of the equation $x \cdot \tanh x = 1$ to four decimal places.

Sec. 4. Numerical Integration of Functions

1°. Trapezoidal formula. For the approximate evaluation of the integral

$$\int_a^b f(x)\,dx$$

[$f(x)$ is a function continuous on $[a, b]$] we divide the interval of integration $[a, b]$ into n equal parts and choose the *interval of calculations* $h = \frac{b-a}{n}$. Let $x_i = x_0 + ih$ ($x_0 = a$, $x_n = b$, $i = 0, 1, 2, \ldots, n$) be the abscissas of the partition points, and let $y_i = f(x_i)$ be the corresponding values of the integrand $y = f(x)$. Then the *trapezoidal formula* yields

$$\int_a^b f(x)\,dx \approx h\left(\frac{y_0 + y_n}{2} + y_1 + y_2 + \ldots + y_{n-1}\right) \tag{1}$$

with an absolute error of

$$R_n \leqslant \frac{h^2}{12}(b-a) \cdot M_2,$$

where $M_2 = \max |f''(x)|$ when $a \leqslant x \leqslant b$.

To attain the specified accuracy ε when evaluating the integral, the interval h is found from the inequality

$$h^2 \leqslant \frac{12\varepsilon}{(b-a)\,M_2}. \tag{2}$$

That is, h must be of the order of $\sqrt{\varepsilon}$. The value of h obtained is rounded off to the smaller value so that

$$\frac{b-a}{h} = n$$

should be an integer; this is what gives us the number of partitions n. Having established h and n from (1), we compute the integral by taking the values of the integrand with one or two reserve decimal places.

2°. Simpson's formula (parabolic formula). If n is an even number, then in the notation of 1° *Simpson's formula*

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\right] \tag{3}$$

holds with an absolute error of

$$R_n \leqslant \frac{h^4}{180}(b-a)\,M_4, \tag{4}$$

where $M_4 = \max|f^{IV}(x)|$ when $a \leqslant x \leqslant b$.

To ensure the specified accuracy ε when evaluating the integral, the interval of calculations h is determined from the inequality

$$\frac{h^4}{180}(b-a)\,M_4 \leqslant \varepsilon. \tag{5}$$

That is, the interval h is of the order $\sqrt[4]{\varepsilon}$. The number h is rounded off to the smaller value so that $n = \frac{b-a}{n}$ is an even integer.

Remark. Since, generally speaking, it is difficult to determine the interval h and the number n associated with it from the inequalities (2) and (5), in practical work h is determined in the form of a rough estimate. Then, after the result is obtained, the number n is doubled; that is, h is halved. If the new result coincides with the earlier one to the number of decimal places that we retain, then the calculations are stopped, otherwise the procedure is repeated, etc.

For an approximate calculation of the absolute error R of Simpson's quadrature formula (3), use can also be made of the *Runge principle*, according to which

$$R = \frac{|\Sigma - \overline{\Sigma}|}{15},$$

where Σ and $\overline{\Sigma}$ are the results of calculations from formula (3) with interval h and $H = 2h$, respectively.

3160. Under the action of a variable force $\overline{F}$ directed along the x-axis, a material point is made to move along the x-axis from $x = 0$ to $x = 4$. Approximate the work A of a force $\overline{F}$ if a table is given of the values of its modulus F:

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
F	1.50	0.75	0.50	0.75	1.50	2.75	4.50	6.75	10.00

Carry out the calculations by the trapezoidal formula and by the Simpson formula.

3161. Approximate $\int_0^1 (3x^2 - 4x)\,dx$ by the trapezoidal formula putting $n = 10$. Evaluate this integral exactly and find the absolute and relative errors of the result. Establish the upper limit Δ of absolute error in calculating for $n = 10$, utilizing the error formula given in the text.

3162. Using the Simpson formula, calculate $\int_0^1 \frac{x\,dx}{x+1}$ to four decimal places, taking $n=10$. Establish the upper limit Δ of absolute error, using the error formula given in the text.

Calculate the following definite integrals to two decimals:

3163. $\int_0^1 \frac{dx}{1+x}$.

3164. $\int_0^1 \frac{dx}{1+x^2}$.

3165. $\int_0^1 \frac{dx}{1+x^3}$.

3166. $\int_1^2 x \log x\,dx$.

3167. $\int_1^2 \frac{\log x}{x}\,dx$.

3168. $\int_0^2 \frac{\sin x}{x}\,dx$.

3169. $\int_0^\pi \frac{\sin x}{x}\,dx$.

3170. $\int_1^2 \frac{\cos x}{x}\,dx$.

3171. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+x}\,dx$.

3172. $\int_0^1 e^{-x^2}\,dx$.

3173. Evaluate to two decimal places the improper integral $\int_1^\infty \frac{dx}{1+x^2}$ by applying the substitution $x=\frac{1}{t}$. Verify the calculations by applying Simpson's formula to the integral $\int_1^b \frac{dx}{1+x^2}$, where b is chosen so that $\int_b^{+\infty} \frac{dx}{1+x^2} < \frac{1}{2}\cdot 10^{-2}$.

3174. A plane figure bounded by a half-wave of the sine curve $y=\sin x$ and the x-axis is in rotation about the x-axis. Using the Simpson formula, calculate the volume of the solid of rotation to two decimal places.

3175*. Using Simpson's formula, calculate to two decimal places the length of an arc of the ellipse $\frac{x^2}{1}+\frac{y^2}{(0.6222)^2}=1$ situated in the first quadrant.

Sec. 5. Numerical Integration of Ordinary Differential Equations

1°. A method of successive approximation (Picard's method). Let there be given a first-order differential equation

$$y' = f(x, y) \tag{1}$$

subject to the initial condition $y=y_0$ when $x=x_0$.

The solution $y(x)$ of (1), which satisfies the given initial condition, can, generally speaking, be represented in the form

$$y(x)=\lim_{i\to\infty} y_i(x) \tag{2}$$

where the *successive approximations* $y_i(x)$ are determined from the formulas

$$y_0(x)=y_0,$$
$$y_i(x)=y_0+\int_{x_0}^{x} f(x,\ y_{i-1}(x))\,dx$$
$$(i=0,\ 1,\ 2,\ \ldots).$$

If the right side $f(x, y)$ is defined and continuous in the neighbourhood

$$R\{\,|x-x_0|\leqslant a,\quad |y-y_0|\leqslant b\}$$

and satisfies, in this neighbourhood, the *Lipschitz condition*

$$|f(x,\ y_1)-f(x,\ y_2)|\leqslant L\,|y_1-y_2|$$

(L is constant), then the process of successive approximation (2) definitely converges in the interval

$$|x-x_0|\leqslant h,$$

where $h=\min\limits_R\left(a,\ \dfrac{b}{M}\right)$ and $M=\max\limits_R|f(x,\ y)|$. And the error here is

$$R_n=|y(x)-y_n(x)|\leqslant ML^n\frac{|x-x_0|^{n+1}}{(n+1)!},$$

if

$$|x-x_0|\leqslant h.$$

The method of successive approximation *(Picard's method)* is also applicable, with slight modifications, to normal systems of differential equations. Differential equations of higher orders may be written in the form of systems of differential equations.

2°. The Runge-Kutta method. Let it be required, on a given interval $x_0\leqslant x\leqslant X$, to find the solution $y(x)$ of (1) to a specified degree of accuracy ε.

To do this, we choose the interval of calculations $h=\dfrac{X-x_0}{n}$ by dividing the interval $[x_0,\ X]$ into n equal parts so that $h^4<\varepsilon$. The partition points x_i are determined from the formula

$$x_i=x_0+ih\quad (i=0,\ 1,\ 2,\ \ldots,\ n).$$

By the *Runge-Kutta method*, the corresponding values $y_i=y(x_i)$ of the desired function are successively computed from the formulas

$$y_{i+1}=y_i+\Delta y_i,$$
$$\Delta y_i=\frac{1}{6}\left(k_1^{(i)}+2k_2^{(i)}+2k_3^{(i)}+k_4^{(i)}\right),$$

where

$$i=0,\ 1,\ 2,\ \ldots,\ n \text{ and}$$

$$\left.\begin{aligned} k_1^{(i)} &= f(x_i,\ y_i)\,h, \\ k_2^{(i)} &= f\left(x_i+\frac{h}{2},\ y_i+\frac{k_1^{(i)}}{2}\right)h, \\ k_3^{(i)} &= f\left(x_i+\frac{h}{2},\ y_i+\frac{k_2^{(i)}}{2}\right)h, \\ k_4^{(i)} &= f\left(x_i+h,\ y_i+k_3^{(i)}\right)h. \end{aligned}\right. \qquad (3)$$

To check the correct choice of the interval h it is advisable to verify the quantity

$$\theta=\left|\frac{k_2^{(i)}-k_3^{(i)}}{k_1^{(i)}-k_2^{(i)}}\right|.$$

The fraction θ should amount to a few hundredths, otherwise h has to be reduced.

The Runge-Kutta method is accurate to the order of h^4. A rough estimate of the error of the Runge-Kutta method on the given interval $[x_0,\ X]$ may be obtained by proceeding from the Runge principle:

$$R=\frac{|y_{2m}-\tilde{y}_m|}{15},$$

where $n=2m$, y_{2m} and $\tilde{y}_m$ are the results of calculations using the scheme (3) with interval h and interval $2h$.

The Runge-Kutta method is also applicable for solving systems of differential equations

$$y'=f(x,\ y,\ z),\quad z'=\varphi(x,\ y,\ z) \qquad (4)$$

with given initial conditions $y=y_0$, $z=z_0$ when $x=x_0$.

3°. **Milne's method.** To solve (1) by the *Milne method*, subject to the initial conditions $y=y_0$ when $x=x_0$, we in some way find the successive values

$$y_1=y(x_1),\quad y_2=y(x_2),\quad y_3=y(x_3)$$

of the desired function $y(x)$ [for instance, one can expand the solution $y(x)$ in a series (Ch. IX, Sec. 17) or find these values by the method of successive approximation, or by using the Runge-Kutta method, and so forth]. The approximations $\overline{y}_i$ and $\overline{\overline{y}}_i$ for the following values of y_i ($i=4,\ 5,\ \ldots,\ n$) are successively found from the formulas

$$\left.\begin{aligned} \overline{y}_i &= y_{i-4}+\frac{4h}{3}(2f_{i-3}-f_{i-2}+2f_{i-1}), \\ \overline{\overline{y}}_i &= y_{i-2}+\frac{h}{3}(\overline{f}_i+4f_{i-1}+f_{i-2}), \end{aligned}\right\} \qquad (5)$$

where $f_i=f(x_i,\ y_i)$ and $\overline{f}_i=f(x_i,\ \overline{y}_i)$. To check we calculate the quantity

$$\varepsilon_i=\frac{1}{29}|\overline{y}_i-\overline{\overline{y}}_i|. \qquad (6)$$

If ε_i does not exceed the unit of the last decimal 10^{-m} retained in the answer for $y(x)$, then for y_i we take $\bar{y}_i$ and calculate the next value y_{i+1}, repeating the process. But if $\varepsilon_i > 10^{-m}$, then one has to start from the beginning and reduce the interval of calculations. The magnitude of the initial interval is determined approximately from the inequality $h^4 < 10^{-m}$.

For the case of a solution of the system (4), the Milne formulas are written separately for the functions $y(x)$ and $z(x)$. The order of calculations remains the same.

Example 1. Given a differential equation $y' = y - x$ with the initial condition $y(0) = 1.5$. Calculate to two decimal places the value of the solution of this equation when the argument is $x = 1.5$. Carry out the calculations by a combined Runge-Kutta and Milne method.

Solution. We choose the initial interval h from the condition $h^4 < 0.01$. To avoid involved writing, let us take $h = 0.25$. Then the entire interval of integration from $x = 0$ to $x = 1.5$ is divided into six equal parts of length 0.25 by means of points x_i $(i = 0, 1, 2, 3, 4, 5, 6)$; we denote by y_i and y'_i the corresponding values of the solution y and the derivative y'.

We calculate the first three values of y (not counting the initial one) by the Runge-Kutta method [from formulas (3)]; the remaining three values — y_4, y_5, y_6 — we calculate by the Milne method [from formulas (5)]

The value of y_6 will obviously be the answer to the problem.

We carry out the calculations with two reserve decimals according to a definite scheme consisting of two sequential Tables 1 and 2. At the end of Table 2 we obtain the answer.

Calculating the value y_1. Here, $f(x, y) = -x + y$, $x_0 = 0$, $y_0 = 1.5$

$$h = 0.25. \quad \Delta y_0 = \frac{1}{6}\left(k_1^{(0)} + 2k_2^{(0)} + 2k_3^{(0)} + k_4^{(0)}\right) =$$

$$= \frac{1}{6}(0.3750 + 2\cdot 0.3906 + 2\cdot 0.3926 + 0.4106) = 0.3920;$$

$$k_1^{(0)} = f(x_0, y_0)\,h = (-0 + 1.5000)0.25 = 0.3750;$$

$$k_2^{(0)} = f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1^{(0)}}{2}\right)h = (-0.125 + 1.5000 + 0.1875)\,0.25 = 0.3906;$$

$$k_3^{(0)} = f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2^{(0)}}{2}\right)h = (-0\ 125 + 1.5000 + 0.1953)\,0.25 = 0.3926;$$

$$k_4^{(0)} = f\left(x_0 + h,\ y_0 + k_3^{(0)}\right)h = (-0.25 + 1.5000 + 0.3926)\,0.25 = 0.4106;$$

$y_1 = y_0 + \Delta y_0 = 1.5000 + 0.3920 = 1.8920$ (the first three decimals in this approximate number are guaranteed).

Let us check:

$$\theta = \left|\frac{k_2^{(0)} - k_3^{(0)}}{k_1^{(0)} - k_2^{(0)}}\right| = \frac{|0.3906 - 0.3926|}{|0.3750 - 0.3906|} = \frac{20}{156} = 0.13.$$

By this criterion, the interval h that we chose was rather rough.

Similarly we calculate the values y_2 and y_3. The results are tabulated in Table 1.

Table 1. **Calculating y_1, y_2, y_3 by the Runge-Kutta Method.**

$f(x, y) = -x + y;\quad h = 0.25$

Value of i	x_i	y_i	$y'_i = f(x_i, y_i)$	$k_1^{(i)}$	$f\left(x_i + \frac{h}{2},\ y_i + \frac{k_1^{(i)}}{2}\right)$	$k_2^{(i)}$
0	0	1.5000	1.5000	0.3750	1.5625	0.3906
1	0.25	1.8920	1.6420	0.4105	1.7223	0.4306
2	0.50	2.3243	1.8243	0.4561	1.9273	0.4818
3	0 75	2.8084	2.0584	0.5146	2.1907	0.5477

Value of i	$f\left(x_i + \frac{h}{2},\ y_i + \frac{k_2^{(i)}}{2}\right)$	$k_3^{(i)}$	$f(x_i + h,\ y_i + k_3^{(i)})$	$k_4^{(i)}$	Δy_i	y_{i+1}
0	1.5703	0.3926	1.6426	0.4106	0.3920	1.8920
1	1.7323	0.4331	1.8251	0.4562	0.4323	2.3243
2	1.9402	0.4850	2.0593	0.5148	0.4841	2.8084
3	2.2073	0.5518	2.3602	0.5900	0.5506	3.3590

Calculating the value of y_4. We have: $f(x, y) = -x + y$, $h = 0.25$, $x_4 = 1$;

$$y_0 = 1.5000,\quad y_1 = 1.8920,\quad y_2 = 2.3243,\quad y_3 = 2.8084;$$

$$y'_0 = 1.5000,\quad y'_1 = 1.6420,\quad y'_2 = 1.8243,\quad y'_3 = 2.0584.$$

Applying formulas (5), we find

$$\bar{y}_4 = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) =$$

$$= 1.5000 + \frac{4 \cdot 0.25}{3}(2 \cdot 1.6420 - 1.8243 + 2 \cdot 2.0584) = 3.3588;$$

$$\bar{y}'_4 = f(x_4, \bar{y}_4) = -1 + 3.3588 = 2.3588;$$

$$\bar{\bar{y}}_4 = y_2 + \frac{h}{3}(\bar{y}'_4 + 4y'_3 + y'_2) = 2.3243 + \frac{0.25}{3}(2.3588 + 4 \cdot 2.0584 + 1.8243) = 3.3590;$$

$$\varepsilon_4 = \frac{|\bar{y}_4 - \bar{\bar{y}}_4|}{29} = \frac{|3.3588 - 3.3590|}{29} = \frac{0.0002}{29} \approx 7 \cdot 10^{-6} < \frac{1}{2} \cdot 0.001;$$

hence, there is no need to reconsider the interval of calculations.

We obtain $y_4=\overline{\overline{y}}_4=3.3590$ (in this approximate number the first three decimals are guaranteed).

Similarly we calculate the values of y_5 and y_6. The results are given in Table 2.

Thus, we finally have

$$y(1.5)=4.74.$$

4°. Adams' method. To solve (1) by the Adams method on the basis of the initial data $y(x_0)=y_0$ we in some way find the following three values of the desired function $y(x)$:

$$y_1=y(x_1)=y(x_0+h),\quad y_2=y(x_2)=y(x_0+2h),\quad y_3=y(x_3)=y(x_0+3h)$$

[these three values may be obtained, for instance, by expanding $y(x)$ in a power series (Ch IX, Sec. 16), or they may be found by the method of successive approximation (1°), or by applying the Runge-Kutta method (2°) and so forth].

With the help of the numbers x_0, x_1, x_2, x_3 and y_0, y_1, y_2, y_3 we calculate q_0, q_1, q_2, q_3, where

$$q_0=hy_0'=hf(x_0,\ y_0),\quad q_1=hy_1'=hf(x_1,\ y_1),$$
$$q_2=hy_2'=hf(x_2,\ y_2),\quad q_3=hy_3'=hf(x_3,\ y_3).$$

We then form a *diagonal table* of the finite differences of q:

x	y	$\Delta y=$ $=y_{n+1}-y_n$	$y'=f(x,\ y)$	$q=y'h$	$\Delta q=q_{n+1}-q_n$	$\Delta^2 q=$ $=\Delta q_{n+1}-\Delta q_n$	$\Delta^3 q=$ $=\Delta^2 q_{n+1}-$ $-\Delta^2 q_n$
x_0	y_0	Δy_0	$f(x_0,\ y_0)$	q_0	Δq_0	$\Delta^2 q_0$	$\Delta^3 q_0$
x_1	y_1	Δy_1	$f(x_1,\ y_1)$	q_1	Δq_1	$\Delta^2 q_1$	$\Delta^3 q_1$
x_2	y_2	Δy_2	$f(x_2,\ y_2)$	q_2	Δq_2	$\Delta^2 q_2$	$\Delta^3 q_2$
x_3	y_3	Δy_3	$f(x_3,\ y_3)$	q_3	Δq_3	$\Delta^2 q_3$	
x_4	y_4	Δy_4	$f(x_4,\ y_4)$	q_4	Δq_4		
x_5	y_5	Δy_5	$f(x_5,\ y_5)$	q_5			
x_6	y_6						

The *Adams method* consists in continuing the diagonal table of differences with the aid of the *Adams formula*

$$\Delta y_n = q_n + \frac{1}{2}\Delta q_{n-1} + \frac{5}{12}\Delta^2 q_{n-2} + \frac{3}{8}\Delta^3 q_{n-3}. \qquad (7)$$

Thus, utilizing the numbers q_3, Δq_2, $\Delta^2 q_1$, $\Delta^3 q_0$ situated diagonally in the difference table, we calculate, by means of formula (7) and putting $n=3$ in it, $\Delta y_3 = q_3 + \frac{1}{2}\Delta q_2 + \frac{5}{12}\Delta^2 q_1 + \frac{3}{8}\Delta^3 q_0$. After finding Δy_3, we calculate $y_4 = y_3 + \Delta y_3$. And when we know x_4 and y_4, we calculate $q_4 = hf(x_4, y_4)$, introduce y_4, Δy_3 and q_4 into the difference table and then fill into it the finite differences Δq_3, $\Delta^2 q_2$, $\Delta^3 q_1$, which are situated (together with q_4) along a new diagonal parallel to the first one.

Then, utilizing the numbers of the new diagonal, we use formula (8) (putting $n=4$ in it) to calculate Δy_4, y_5 and q_5 and obtain the next diagonal: q_5, Δq_4, $\Delta^2 q_3$, $\Delta^3 q_2$. Using this diagonal we calculate the value of y_6 of the desired solution $y(x)$, and so forth.

The Adams formula (7) for calculating Δy proceeds from the assumption that the third finite differences $\Delta^3 q$ are constant. Accordingly, the quantity h of the initial interval of calculations is determined from the inequality $h^4 < 10^{-m}$ [if we wish to obtain the value of $y(x)$ to an accuracy of 10^{-m}].

In this sense the Adams formula (7) is equivalent to the formulas of Milne (5) and Runge-Kutta (3).

Evaluation of the error for the Adams method is complicated and for practical purposes is useless, since in the general case it yields results with considerable excess. In actual practice, we follow the course of the third finite differences, choosing the interval h so small that the adjacent differences $\Delta^3 q_i$ and $\Delta^3 q_{i+1}$ differ by not more than one or two units of the given decimal place (not counting reserve desimals).

To increase the accuracy of the result, Adams' formula may be extended by terms containing fourth and higher differences of q, in which case there is an increase in the number of first values of the function y that are needed when we first fill in the table. We shall not here give the Adams formula for higher accuracy.

Example 2. Using the combined Runge-Kutta and Adams method, calculate to two decimal places (when $x=1.5$) the value of the solution of the differential equation $y' = y - x$ with the initial condition $y(0)=1.5$ (see Example 1).

Solution. We use the values y_1, y_2, y_3 that we obtained in the solution of Example 1. Their calculation is given in Table 1.

We calculate the subsequent values y_4, y_5, y_6 by the Adams method (see Tables 3 and 4).

The answer to the problem is $y_6 = 4.74$.

For solving system (4), the Adams formula (7) and the calculation scheme shown in Table 3 are applied separately for both functions $y(x)$ and $z(x)$.

Find three successive approximations to the solutions of the differential equations and systems indicated below.

3176. $y' = x^2 + y^2$; $y(0) = 0$.

3177. $y' = x + y + z$, $z' = y - z$; $y(0) = 1$, $z(0) = -2$.

3178. $y'' = -y$; $y(0) = 0$, $y'(0) = 1$.

Table 2. **Calculating y_4, y_5, y_6 by the Milne Method.**
$f(x, y) = -x + h$; $h = 0.25$. (Italicised figures are input data)

Value of i	x_i	y_i	$y'_i = f(x_i, y_i)$	$\overline{y}_i$	$\overline{y}'_i = f(x_i, \overline{y}_i)$	$\overline{\overline{y}}_i$	ε_i	y_i	$y'_i = f(x_i, y_i)$	Reconsider interval of calculations, following indications of formula (6).
0	*0*	*1.5000*	*1.5000*							
1	*0.25*	*1.8920*	*1.6420*							
2	*0.50*	*2.3243*	*1.8243*							
3	*0.75*	*2.8084*	*2.0584*							
4	1.00			3.3588	2.3588	3.3590	$\approx 7 \cdot 10^{-5}$	3.3590	2.3590	Do not reconsider
5	1.25			3.9947	2.7447	3.9950	$\approx 10^{-5}$	3.9950	2.7450	Do not reconsider
6	1.50			4.7402	3.2402	4.7406	$\approx 1.4 \cdot 10^{-5}$	4.7406		Do not reconsider
							Answer:	$y(1.5) = 4.74$		

Table 3. **Basic Table for Calculating y_4, y_5, y_6 by the Adams Method.**

$f(x, y) = -x + y$; $h = 0.25$

(Italicised figures are input data)

Value of i	x_i	y_i	Δy_i	$y'_i = f(x_i, y_i)$	$q_i = y'_i h$	Δq_i	$\Delta^2 q_i$	$\Delta^3 q_i$
0	*0*	*1.5000*		*1.5000*	0.3750	0 0355	0.0101	0 0028
1	*0.25*	*1.8920*		*1.6420*	0.4105	0.0456	0.0129	0 0037
2	*0 50*	*2.3243*		*1.8243*	0.4561	0.0585	0.0166	0.0047
3	*0.75*	*2.8084*	0.5504	2.0584	0.5146	0.0751	0.0213	
4	1 00	3.3588	0.6356	2.3588	0.5897	0.0964		
5	1 25	3.9944	0.7450	2.7444	0.6861			
6	1.50	4.7394						

Answer: 4.74

Table 4 **Auxiliary Table for Calculating by the Adams Method**

$$\Delta y_i = q_i + \frac{1}{2}\Delta q_{i-1} + \frac{5}{12}\Delta^2 q_{i-2} + \frac{3}{8}\Delta^3 q_{i-3}$$

Value of i	q_i	$\frac{1}{2}\Delta q_{i-1}$	$\frac{5}{12}\Delta^2 q_{i-2}$	$\frac{3}{8}\Delta^3 q_{i-3}$	Δy_i
3	0.5146	0.0293	0.0054	0.0011	0.5504
4	0 5897	0.0376	0.0069	0.0014	0.6356
5	0.6861	0.0482	0.0089	0.0018	0.7450

Putting the interval $h=0.2$, use the Runge-Kutta method to calculate approximately the solutions of the given differential equations and systems for the indicated intervals:

3179. $y'=y-x$; $y(0)=1.5$ $(0\leqslant x\leqslant 1)$.

3180. $y'=\frac{y}{x}-y^2$; $y(1)=1$ $(1\leqslant x\leqslant 2)$.

3181. $y'=z+1$, $z'=y-x$, $y(0)=1$, $z(0)=1$ $(0\leqslant x\leqslant 1)$.

Applying a combined Runge-Kutta and Milne method or Runge-Kutta and Adams method, calculate to two decimal places the solutions to the differential equations and systems indicated below for the indicated values of the argument;

3182. $y'=x+y$; $y=1$ when $x=0$. Compute y when $x=0.5$.

3183. $y'=x^2+y$; $y=1$ when $x=0$. Compute y when $x=1$.

3184. $y'=2y-3$; $y=1$ when $x=0$. Compute y when $x=0.5$.

3185. $\begin{cases} y'=-x+2y+z, \\ z'=x+2y+3z; \end{cases}$ $y=2$, $z=-2$ when $x=0$.

Compute y and z when $x=0.5$.

3186. $\begin{cases} y'=-3y-z, \\ z'=y-z; \end{cases}$ $y=2$, $z=-1$ when $x=0$.

Compute y and z when $x=0.5$.

3187. $y''=2-y$; $y=2$, $y'=-1$ when $x=0$.

Compute y when $x=1$.

3188. $y^3y''+1=0$; $y=1$, $y'=0$ when $x=1$.

Compute y when $x=1.5$.

3189. $\frac{d^2x}{dt^2}+\frac{x}{2}\cos 2t=0$; $x=0$, $x'=1$ when $t=0$.

Find $x(\pi)$ and $x'(\pi)$.

Sec. 6. Approximating Fourier Coefficients

Twelve-ordinate scheme. Let $y_n=f(x_n)$ $(n=0, 1, \ldots, 12)$ be the values of the function $y=f(x)$ at equidistant points $x_n=\frac{\pi n}{6}$ of the interval $[0,2\pi]$, and $y_0 - y_{12}$ We set up the tables:

	y_0	y_1	y_2	y_3	y_4	y_5	y_6
		y_{11}	y_{10}	y_9	y_8	y_7	
Sums ($\sum$)	u_0	u_1	u_2	u_3	u_4	u_5	u_6
Differences (Δ)		v_1	v_2	v_3	v_4	v_5	

	u_0	u_1	u_2	u_3
	u_6	u_5	u_4	
Sums	s_0	s_1	s_2	s_3
Differences	t_0	t_1	t_2	

	v_1	v_2	v_3
	v_5	v_4	
Sums	σ_1	σ_2	σ_3
Differences	τ_1	τ_2	

The Fourier coefficients a_n, b_n $(n=0, 1, 2, 3)$ of the function $y=f(x)$ may be determined approximately from the formulas:

$$\begin{aligned} 6a_0 &= s_0+s_1+s_2+s_3, & 6b_1 &= 0.5\sigma_1+0.866\sigma_2+\sigma_3, \\ 6a_1 &= t_0+0.866t_1+0.5t_2, & 6b_2 &= 0.866(\tau_1+\tau_2), \\ 6a_2 &= s_0-s_3+0.5(s_1-s_2), & 6b_3 &= \sigma_1-\sigma_3, \\ 6a_3 &= t_0-t_2, & & \end{aligned} \qquad (1)$$

where $0.866=\frac{\sqrt{3}}{2}\approx 1-\frac{1}{10}-\frac{1}{30}$.

We have

$$f(x)\approx\frac{a_0}{2}+\sum_{n=1}^{3}(a_n\cos nx+b_n\sin nx).$$

Other schemes are also used. Calculations are simplified by the use of *patterns*.

Example. Find the Fourier polynomial for the function $y=f(x)$ $(0\leqslant x\leqslant 2\pi)$ represented by the table

y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}
38	38	12	4	14	4	−18	−23	−27	−24	8	32

Solution. We set up the tables:

y	38	38	12	4	14	4	−18
		32	8	−24	−27	−23	
u	38	70	20	−20	−13	−19	−18
v		6	4	28	41	27	

u	38	70	20	−20
	−18	−19	−13	
s	20	51	7	−20
t	56	89	33	

v	6	4	28
	27	41	
σ	33	45	28
τ	−21	−37	

From formulas (1) we have

$$a_0=9.7;\ a_1=24.9;\ a_2=10.3;\ a_3=3.8;$$
$$b_1=13.9;\ b_2=-8.4;\ b_3=0.8.$$

Consequently,

$$f(x)\approx 4.8+(24.9\cos x+13.9\sin x)+(10.3\cos 2x-8.4\sin 2x)+$$
$$+(3.8\cos 3x+0.8\sin 3x).$$

Using the 12-ordinate scheme, find the Fourier polynomials for the following functions defined in the interval $(0,2\pi)$ by the tables of their values that correspond to the equidistant values of the argument.

3190.

$y_0 = -7200$	$y_3 = 4300$	$y_6 = 7400$	$y_9 = 7600$
$y_1 = 300$	$y_4 = 0$	$y_7 = -2250$	$y_{10} = 4500$
$y_2 = 700$	$y_5 = -5200$	$y_8 = 3850$	$y_{11} = 250$

3191.

$y_0 = 0$	$y_3 = 9.72$	$y_6 = 7.42$	$y_9 = 5.60$
$y_1 = 6.68$	$y_4 = 8.97$	$y_7 = 6.81$	$y_{10} = 4.88$
$y_2 = 9.68$	$y_5 = 8.18$	$y_8 = 6.22$	$y_{11} = 3.67$

3192.

$y_0 = 2.714$	$y_3 = 1.273$	$y_6 = 0.370$	$y_9 = -0.357$
$y_1 = 3.042$	$y_4 = 0.788$	$y_7 = 0.540$	$y_{10} = -0.437$
$y_2 = 2.134$	$y_5 = 0.495$	$y_8 = 0.191$	$y_{11} = 0.767$

3193. Using the 12-ordinate scheme, evaluate the first several Fourier coefficients for the following functions:

a) $f(x) = \frac{1}{2\pi^2}(x^3 - 3\pi x^2 + 2\pi^2 x) \quad (0 \leqslant x \leqslant 2\pi)$,

b) $f(x) = \frac{1}{\pi^2}(x - \pi)^2 \quad (0 \leqslant x \leqslant 2\pi)$.

ANSWERS

Chapter I

1. Solution. Since $a=(a-b)+b$, then $|a|\leqslant|a-b|+|b|$. Whence $|a-b|\geqslant|a|-|b|$ and $|a-b|=|b-a|\geqslant|b|-|a|$. Hence, $|a-b|\geqslant|a|-|b|$. Besides, $|a-b|=|a+(-b)|\leqslant|a|+|-b|=|a|+|b|$. **3.** a) $-2<x<4$; b) $x<-3$, $x>1$; c) $-1<x<0$; d) $x>0$. **4.** -24; -6; 0; 0; 0; 6. **5.** 1; $1\frac{1}{4}$; $\sqrt{1+x^2}$; $|x|^{-1}\sqrt{1+x^2}$; $1/\sqrt{1+x^2}$. **6.** π; $\frac{\pi}{2}$; 0. **7.** $f(x)=-\frac{5}{3}x+\frac{1}{3}$. **8.** $f(x)=\frac{7}{6}x^2-\frac{13}{6}x+1$. **9.** 0.4. **10.** $\frac{1}{2}(x+|x|)$. **11.** a) $-1\leqslant x<+\infty$; b) $-\infty<x<+\infty$. **12.** $(-\infty, -2)$, $(-2, 2)$, $(2, +\infty)$. **13.** a) $-\infty<x\leqslant-\sqrt{2}$, $\sqrt{2}\leqslant x<+\infty$; b) $x=0$, $|x|\geqslant\sqrt{2}$. **14.** $-1\leqslant x\leqslant 2$. **Solution.** It should be $2+x-x^2\geqslant 0$, or $x^2-x-2\leqslant 0$; that is, $(x+1)(x-2)\leqslant 0$. Whence either $x+1\geqslant 0$, $x-2\leqslant 0$, i. e., $-1\leqslant x\leqslant 2$ or $x+1\leqslant 0$, $x-2\geqslant 0$, i. e., $x\leqslant -1$, $x\geqslant 2$, but this is impossible. Thus, $-1\leqslant x\leqslant 2$. **15.** $-2<x\leqslant 0$. **16.** $-\infty<x\leqslant -1$, $0\leqslant x\leqslant 1$. **17.** $-2<x<2$. **18.** $-1<x<1$, $2<x<+\infty$. **19.** $-\frac{1}{3}\leqslant x\leqslant 1$. **20.** $1\leqslant x\leqslant 100$. **21.** $k\pi\leqslant x\leqslant k\pi+\frac{\pi}{2}$ $(k=0, \pm1, \pm2, \dots)$. **22.** $\varphi(x)=2x^4-5x^2-10$, $\psi(x)=-3x^3+6x$. **23.** a) Even, b) odd, c) even, d) odd, e) odd. **24. Hint.** Utilize the identity $f(x)=\frac{1}{2}[f(x)+f(-x)]+\frac{1}{2}[f(x)-f(-x)]$. **26.** a) Periodic, $T=\frac{2}{3}\pi$, b) periodic, $T=\frac{2\pi}{\lambda}$, c) periodic, $T=\pi$, d) periodic $T=\pi$, e) nonperiodic. **27.** $y=\frac{b}{c}x$, if $0\leqslant x\leqslant c$; $y=b$ if $c<x\leqslant a$; $S=\frac{b}{2c}x^2$ if $0\leqslant x\leqslant c$; $S=bx-\frac{bc}{2}$ if $c<x\leqslant a$. **28.** $m=q_1x$ when $0\leqslant x\leqslant l_1$; $m=q_1l_1+q_2(x-l_1)$ when $l_1<x\leqslant l_1+l_2$; $m=q_1l_1+q_2l_2+q_3(x-l_1-l_2)$ when $l_1+l_2<x\leqslant l_1+l_2+l_3=l$. **29.** $\varphi[\psi(x)]=2^{2^x}$; $\psi[\varphi(x)]=2^{x^2}$ **30.** x. **31.** $(x+2)^2$. **37.** $-\frac{\pi}{2}$; 0; $\frac{\pi}{4}$. **38.** a) $y=0$ when $x=-1$, $y>0$ when $x>-1$, $y<0$ when $x<-1$; b) $y=0$ when $x=-1$ and $x=2$, $y>0$ when $-1<x<2$, $y<0$ when $-\infty<x<-1$ and $2<x<+\infty$; c) $y>0$ when $-\infty<x<+\infty$; d) $y=0$ when $x=0$, $x=-\sqrt{3}$ and $x=\sqrt{3}$, $y>0$ when $-\sqrt{3}<x<0$ and $\sqrt{3}<x<+\infty$, $y<0$ when $-\infty<x<-\sqrt{3}$ and $0<x<\sqrt{3}$; e) $y=0$ when $x=1$, $y>0$ when $-\infty<x<-1$ and $1<x<+\infty$, $y<0$ when $0<x<1$ **39.** a) $x=\frac{1}{2}(y-3)$ $(-\infty<y<+\infty)$; b) $x=\sqrt{y+1}$ and $x=-\sqrt{y+1}$ $(-1\leqslant y<+\infty)$;

c) $x=\sqrt[3]{1-y^3}$ $(-\infty<y<+\infty)$; d) $x=2\cdot 10^y$ $(-\infty<y<+\infty)$; e) $x=$ $=\frac{1}{3}\tan y\left(-\frac{\pi}{2}<y<\frac{\pi}{2}\right)$. **40.** $x=y$ when $-\infty<y\leqslant 0$; $x=\sqrt{y}$ when $0<y<+\infty$. **41.** a) $y=u^{10}$, $u=2x-5$; b) $y=2^u$, $u=\cos x$; c) $y=\log u$, $u=\tan v$, $v=\frac{x}{2}$; d) $y=\arcsin u$, $u=3^v$, $v=-x^2$. **42.** a) $y=\sin^2 x$; b) $y=$ $=\arctan\sqrt{\log x}$; c) $y=2(x^2-1)$ if $|x|\leqslant 1$, and $y=0$ if $|x|>1$. **43.** a) $y=-\cos x^2$, $\sqrt{\pi}\leqslant|x|\leqslant\sqrt{2\pi}$; b) $y=\log(10-10^x)$, $-\infty<x<1$; c) $y=\frac{x}{3}$ when $-\infty<x<0$ and $y=x$ when $0\leqslant x<+\infty$. **46. Hint.** See Appendix VI, Fig. 1. **51. Hint.** Completing the square in the quadratic trinomial we will have $y=y_0+a(x-x_0)^2$ where $x_0=-b/2a$ and $y_0=(4ac-b^2)/4a$. Whence the desired graph is a parabola $y=ax^2$ displaced along the x-axis by x_0 and along the y-axis by y_0. **53. Hint.** See Appendix VI, Fig. 2. **58. Hint.** See Appendix VI, Fig. 3. **61. Hint.** The graph is a hyperbola $y=\frac{m}{x}$, shifted along the x-axis by x_0 and along the y-axis by y_0. **62. Hint.** Taking the integral part, we have $y=\frac{2}{3}-\frac{13}{9}\Big/\left(x+\frac{2}{3}\right)$ (Cf. 61*). **65. Hint.** See Appendix VI, Fig. 4. **67. Hint.** See Appendix VI, Fig. 5. **71. Hint.** See Appendix VI, Fig. 6. **72. Hint.** See Appendix VI, Fig. 7. **73. Hint.** See Appendix VI, Fig. 8. **75. Hint.** See Appendix VI, Fig. 19 **78. Hint.** See Appendix VI, Fig. 23. **80. Hint.** See Appendix VI, Fig. 9. **81. Hint.** See Appendix VI, Fig. 9. **82. Hint.** See Appendix VI, Fig. 10 **83. Hint.** See Appendix VI, Fig. 10. **84. Hint.** See Appendix VI, Fig 11. **85. Hint.** See Appendix VI, Fig. 11. **87. Hint.** The period of the function is $T=2\pi/n$. **89. Hint.** The desired graph is the sine curve $y=5\sin 2x$ with amplitude 5 and period π displaced rightwards along the x-axis by the quantity $1\frac{1}{2}$. **90. Hint.** Putting $a=A\cos\varphi$ and $b=-A\sin\varphi$, we will have $y=A\sin(x-\varphi)$ where $A=\sqrt{a^2+b^2}$ and $\varphi=\arctan\left(-\frac{b}{a}\right)$. In our case, $A=10$, $\varphi=0.927$. **92. Hint.** $\cos^2 x=\frac{1}{2}(1+\cos 2x)$. **93. Hint.** The desired graph is the sum of the graphs $y_1=x$ and $y_2=\sin x$. **94. Hint.** The desired graph is the product of the graphs $y_1=x$ and $y_2=\sin x$. **99. Hint.** The function is even For $x>0$ we determine the points at which 1) $y=0$; 2) $y=1$; and 3) $y=-1$. When $x\longrightarrow+\infty$, $y\longrightarrow 1$. **101. Hint.** See Appendix VI, Fig. 14. **102. Hint.** See Appendix VI, Fig. 15. **103. Hint.** See Appendix VI, Fig. 17. **104. Hint.** See Appendix VI, Fig. 17. **105. Hint.** See Appendix VI, Fig. 18. **107. Hint.** See Appendix VI, Fig. 18. **118. Hint.** See Appendix VI, Fig. 12. **119. Hint.** See Appendix VI, Fig. 12. **120. Hint.** See Appendix VI, Fig. 13. **121. Hint.** See Appendix VI, Fig. 13. **132. Hint.** See Appendix VI, Fig. 30. **133. Hint.** See Appendix VI, Fig. 32. **134. Hint.** See Appendix VI, Fig. 31. **138. Hint.** See Appendix VI, Fig. 33. **139. Hint.** See Appendix VI, Fig. 28. **140. Hint.** See Appendix VI, Fig. 25. **141. Hint.**

Form a table of values:

t	0	1	2	3	...	-1	-2	-3
x	0	1	8	27	..	-1	-8	-27
y	0	1	4	9	..	1	4	9

Constructing the points (x, y) obtained, we get the desired curve (see Appendix VI, Fig. 7). (Here, the parameter t cannot be laid off geometrically!) **142.** See Appendix VI, Fig. 19. **143.** See Appendix VI, Fig. 27. **144.** See Appendix VI, Fig. 29. **145.** See Appendix VI, Fig. 22 **150.** See Appendix VI, Fig. 28. **151.** **Hint.** Solving the equation for y, we get $y = \pm\sqrt{25 - x^2}$. It is now easy to construct the desired curve from the points. **153.** See Appendix VI, Fig. 21. **156.** See Appendix VI, Fig. 27. It is sufficient to construct the points (x, y) corresponding to the abscissas $x = 0, \pm\frac{a}{2}, \pm a$. **157.** **Hint.** Solving the equation for x, we have $x = 10 \log y - y^{(*)}$. Whence we get the points (x, y) of the sought-for curve, assigning to the ordinate y arbitrary values $(y > 0)$ and calculating the abscissa x from the formula $^{(*)}$ Bear in mind that $\log y \to -\infty$ as $y \to 0$. **159.** **Hint.** Passing to polar coordinates $r = \sqrt{x^2 + y^2}$ and $\tan\varphi = \frac{y}{x}$, we will have $r = e^{\varphi}$ (see Appendix VI, Fig 32) **160.** **Hint.** Passing to polar coordinates $x = r\cos\varphi$, and $y = r\sin\varphi$, we will have $r = \frac{3\sin\varphi\cos\varphi}{\cos^3\varphi + \sin^3\varphi}$ (see Appendix VI, Fig. 32) **161.** $F = 32 + 1, 8C$ **162.** $y = 0.6x(10 - x)$; $y_{\max} = 15$ when $x = 5$. **163.** $y = \frac{ab}{2}\sin x$; $y_{\max} = \frac{ab}{2}$ when $x = \frac{\pi}{2}$. **164.** a) $x_1 = \frac{1}{2}$, $x_2 = 2$; b) $x = 0.68$; c) $x_1 = 1.37$, $x_2 = 10$; d) $x = 0.40$; e) $x = 1.50$; f) $x = 0.86$. **165.** a) $x_1 = 2$, $y_1 = 5$; $x_2 = 5$, $y_2 = 2$; b) $x_1 = -3$, $y_1 = -2$; $x_2 = -2$, $y_2 = -3$; $x_3 = 2$, $y_3 = 3$; $x_4 = 3$, $y_4 = 2$; c) $x_1 = 2$, $y_1 = 2$; $x_2 \approx 3.1$, $y_2 \approx -2.5$; d) $x_1 \approx -3.6$, $y_1 \approx -3.1$; $x_2 \approx -2.7$, $y_2 \approx 2.9$; $x_3 \approx 2.9$, $y_3 \approx 1.8$; $x_4 \approx 3.4$, $y_4 \approx -1.6$; e) $x_1 = \frac{\pi}{4}$, $y_1 = \frac{\sqrt{2}}{2}$; $x_2 = \frac{5\pi}{4}$, $y_2 = -\frac{\sqrt{2}}{2}$. **166.** $n > \frac{1}{\sqrt{\varepsilon}}$. a) $n \geqslant 4$; b) $n > 10$; c) $n \geqslant 32$. **167.** $n > \frac{1}{\varepsilon} - 1 = N$. a) $N = 9$; b) $N = 99$; c) $N = 999$. **168.** $\delta = \frac{\varepsilon}{5}$ $(\varepsilon < 1)$. a) 0 02; b) 0 002; c) 0.0002. **169.** a) $\log x < -N$ when $0 < x < \delta(N)$; b) $2^x > N$ when $x > X(N)$; c) $|f(x)| > N$ when $|x| > X(N)$. **170.** a) 0; b) 1; c) 2; d) $\frac{7}{30}$. **171.** $\frac{1}{2}$. **172.** 1. **173.** $-\frac{3}{2}$. **174.** 1. **175.** 3. **176.** 1. **177.** $\frac{3}{4}$. **178.** $\frac{1}{3}$. **Hint.** Use the formula $1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$. **179.** 0. **180.** 0. **181.** 1. **182.** 0. **183.** ∞. **184.** 0. **185.** 72. **186.** 2. **187.** 2. **188.** ∞. **189.** 0. **190.** 1. **191.** 0. **192.** ∞. **193.** -2. **194.** ∞. **195.** $\frac{1}{2}$ **196.** $\frac{a-1}{3a^2}$. **197.** $3x^2$. **198.** -1. **199.** $\frac{1}{2}$.

200. 3. 201. $\frac{4}{3}$. 202. $\frac{1}{9}$. 203. $-\frac{1}{56}$. 204. 12. 205. $\frac{3}{2}$. 206. $-\frac{1}{3}$. 207. 1. 208. $\frac{1}{2\sqrt{x}}$. 209. $\frac{1}{3\sqrt[3]{x^2}}$. 210. $-\frac{1}{3}$. 211. 0. 212. $\frac{a}{2}$. 213. $-\frac{5}{2}$. 214. $\frac{1}{2}$. 215. 0. 216. a) $\frac{1}{2}\sin 2$; b) 0. 217. 3. 218. $\frac{5}{2}$. 219. $\frac{1}{3}$. 220. π. 221. $\frac{1}{2}$. 222. $\cos\alpha$. 223. $-\sin\alpha$. 224. π. 225. $\cos x$. 226. $-\frac{1}{\sqrt{2}}$. 227. a) 0; b) 1. 228. $\frac{2}{\pi}$. 229. $\frac{1}{2}$. 230. 0. 231. $-\frac{1}{\sqrt{3}}$. 232. $\frac{1}{2}(n^2-m^2)$. 233. $\frac{1}{2}$. 234. 1. 235. $\frac{2}{3}$. 236. $\frac{2}{\pi}$. 237. $-\frac{1}{4}$. 238. π. 239. $\frac{1}{4}$. 240. 1. 241. 1. 242. $\frac{1}{4}$. 243. 0 244. $\frac{3}{2}$. 245. 0. 246. e^{-1}. 247. e^2. 248. e^{-1}. 249. e^{-4}. 250. e^x. 251. e. 252. a) 1. **Solution.** $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x}} = \lim\limits_{x\to 0}[1-(1-\cos x)]^{\frac{1}{x}} =$

$$= \lim_{x\to 0}\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{x}} = \lim_{x\to 0}\left[\left(1-2\sin^2\frac{x}{2}\right)^{-\frac{1}{2\sin^2\frac{x}{2}}}\right]^{-\frac{2\sin^2\frac{x}{2}}{x}} = e^{\lim\limits_{x\to 0}\left(-\frac{2\sin^2\frac{x}{2}}{x}\right)}.$$

Since $\lim\limits_{x\to 0}\left(-\frac{2\sin^2\frac{x}{2}}{x}\right) = -2\lim\limits_{x\to 0}\left[\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\frac{x^2}{4x}\right] = -2\cdot 1\cdot\lim\limits_{x\to 0}\frac{x}{4} = 0$, it follows that $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x}} = e^0 = 1$. b) $\frac{1}{\sqrt{e}}$. **Solution.** As in the preceding case (see a), $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}} = e^{\lim\limits_{x\to 0}\left(\frac{-2\sin^2\frac{x}{2}}{x^2}\right)}$. Since $\lim\limits_{x\to 0}\left(\frac{-2\sin^2\frac{x}{2}}{x^2}\right) =$

$= -2\lim\limits_{x\to 0}\left[\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\frac{x^2}{4x^2}\right] = -\frac{1}{2}$, it follows that $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$. 253. $\ln 2$. 254. $10\log e$. 255. 1. 256. 1. 257. $-\frac{1}{2}$. 258. 1. **Hint.** Put $e^x-1=\alpha$, where $\alpha\to 0$. 259. $\ln a$. **Hint.** Utilize the identity $a=e^{\ln a}$. 260. $\ln a$ **Hint.** Put $\frac{1}{n}=\alpha$, where $\alpha\to 0$ (see Example 259) 261. $a-b$. 262. 1. 263. a) 1; b) $\frac{1}{2}$. 264. a) -1; b) 1. 265. a) -1; b) 1. 266. a) 1; b) 0. 267. a) 0; b) 1. 268. a) -1; b) 1. 269. a) -1; b) 1. 270. a) $-\infty$; b) $+\infty$.

271. Solution. If $x \neq k\pi$ ($k=0, \pm 1, \pm 2, \ldots$), then $\cos^2 x < 1$ and $y=0$; but if $x=k\pi$, then $\cos^2 x=1$ and $y=1$. **272.** $y=x$ when $0 \leqslant x < 1$; $y=\frac{1}{2}$ when $x=1$; $y=0$ when $x>1$ **273.** $y=|x|$. **274.** $y=-\frac{\pi}{2}$ when $x<0$; $y=0$ when $x=0$; $y=\frac{\pi}{2}$ when $x>0$. **275.** $y=1$ when $0 \leqslant x \leqslant 1$; $y=x$ when $1<x<+\infty$. **276.** $\frac{61}{450}$. **277.** $x_1 \to -\frac{c}{b}$; $x_2 \to \infty$. **278.** π. **279.** $2\pi R$. **280** $\frac{e}{e-1}$. **281.** $1\frac{1}{3}$. **282.** $\frac{\sqrt{e^{\pi}+1}}{e^{\frac{\pi}{2}}-1}$. **284.** $\lim\limits_{n\to\infty} AC_n=\frac{l}{3}$. **285.** $\frac{ab}{2}$. **286.** $k=1$, $b=0$; the straight line $y=x$ is the asymptote of the curve $y=\frac{x^3+1}{x^2+1}$.
287. $Q_t^{(n)}=Q_0\left(1+\frac{kt}{n}\right)^n$, where k is the proportionality factor (law of compound interest); $Q_t=Q_0e^{kt}$. **288.** $|x|>\frac{1}{\varepsilon}$, a) $|x|>10$; b) $|x|>100$; c) $|x|>1000$. **289.** $|x-1|<\frac{\varepsilon}{2}$ when $0<\varepsilon<1$; a) $|x-1|<0.05$; b) $|x-1|<0.005$; c) $|x-1|<0.0005$ **290.** $|x-2|<\frac{1}{N}=\delta$; a) $\delta=0.1$; b) $\delta=0.01$; c) $\delta=0.001$. **291.** a) Second, b) third. $\frac{1}{2}$, $\frac{3}{2}$. **292.** a) 1; b) 2; c) 3. **293** a) 1; b) $\frac{1}{4}$; c) $\frac{2}{3}$; d) 2; e) 3. **295.** No **296.** 15. **297.** —1. **298.** —1. **299.** 3. **300.** a) 1.03 (1.0296); b) 0.985 (0.9849); c) 3.167 (3.1623) **Hint.** $\sqrt{10}=\sqrt{9+1}=3\sqrt{1+\frac{1}{9}}$; d) 10.954 (10.954). **301.** 1) 0.98 (0.9804); 2) 1.03 (1.0309); 3) 0.0095 (0.00952); 4) 3.875 (3.8730); 5) 1.12 (1.125); 6) 0.72 (0.7480); 7) 0.043 (0.04139). **303.** a) 2; b) 4; c) $\frac{1}{2}$; d) $\frac{2}{3}$. **307.** **Hint.** If $x>0$, then when $|\Delta x|<x$ we have $|\sqrt{x+\Delta x}-\sqrt{x}|=$ $=|\Delta x|/(\sqrt{x+\Delta x}+\sqrt{x}) \leqslant |\Delta x|/\sqrt{x}$. **309.** **Hint.** Take advantage of the inequality $|\cos(x+\Delta x)-\cos x| \leqslant |\Delta x|$ **310.** a) $x \neq \frac{\pi}{2}+k\pi$, where k is an integer; b) $x \neq k\pi$, where k is an integer **311.** **Hint.** Take advantage of the inequality $||x+\Delta x|-|x|| \leqslant |\Delta x|$ **313.** $A=4$. **314.** $f(0)=1$. **315.** No **316.** a) $f(0)=n$; b) $f(0)=\frac{1}{2}$; c) $f(0)=2$; d) $f(0)=2$; e) $f(0)=0$; f) $f(0)=1$.
317. $x=2$ is a discontinuity of the second kind. **318.** $x=-1$ is a removable discontinuity. **319.** $x=-2$ is a discontinuity of the second kind; $x=2$ is a removable discontinuity **320.** $x=0$ is a discontinuity of the first kind. **321.** a) $x=0$ is a discontinuity of the second kind; b) $x=0$ is a removable discontinuity. **322.** $x=0$ is a removable discontinuity, $x=k\pi$ ($k=\pm 1, \pm 2, \ldots$) are infinite discontinuities **323.** $x=2\pi k \pm \frac{\pi}{2}$ ($k=0, \pm 1, \pm 2, \ldots$) are infinite discontinuities.
324. $x=k\pi$ ($k=0, \pm 1, \pm 2, \ldots$) are infinite discontinuities. **325.** $x=0$ is a discontinuity of the first kind. **326.** $x=-1$ is a removable discontinuity; $x=1$ is a point of discontinuity of the first kind. **327.** $x=-1$ is a discon-

tinuity of the second kind. **328.** $x=0$ is a removable discontinuity. **329.** $x=1$ is a discontinuity of the first kind. **330.** $x=3$ is a discontinuity of the first kind. **332.** $x=1$ is a discontinuity of the first kind. **333.** The function is continuous. **334.** a) $x=0$ is a discontinuity of the first kind; b) the function is continuous; c) $x=k\pi$ (k is integral) are discontinuities of the first kind. **335.** a) $x=k$ (k is integral) are discontinuities of the first kind; b) $x=k$ ($k\neq 0$ is integral) are points of discontinuity of the first kind. **337.** No, since the function $y=E(x)$ is discontinuous at $x=1$. **338.** 1.53. **339.** **Hint.** Show that when x_0 is sufficiently large, we have $P(-x_0)\,P(x_0)<0$.

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341. a) 3; b) 0.21; c) $2h+h^2$. **342.** a) 0.1; b) -3; c) $\sqrt[3]{a+h}-\sqrt[3]{a}$. **344.** a) 624; 1560; b) 0.01; 100; c) -1; 0.000011. **345.** a) $a\Delta x$; b) $3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3$; $3x^2+3x\,\Delta x+(\Delta x)^2$; c) $-\frac{2x\,\Delta x+(\Delta x)^2}{x^2(x+\Delta x)^2}$; $-\frac{2x+\Delta x}{x^2(x+\Delta x)^2}$; d) $\sqrt{x+\Delta x}-\sqrt{x}$; $\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}$; e) $2^x(2^{\Delta x}-1)$; $\frac{2^x(2^{\Delta x}-1)}{\Delta x}$; f) $\ln\frac{x+\Delta x}{x}$; $\frac{1}{\Delta x}\ln\left(1+\frac{\Delta x}{x}\right)$. **346.** a) -1; b) 0.1; c) $-h$; 0. **347.** 21. **348.** 15 cm/sec. **349.** 7.5. **350.** $\frac{f(x+\Delta x)-f(x)}{\Delta x}$. **351.** $f'(x)=\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$. **352.** a) $\frac{\Delta\varphi}{\Delta t}$; b) $\frac{d\varphi}{dt}=\lim\limits_{\Delta t\to 0}\frac{\Delta\varphi}{\Delta t}$, where φ is the angle of turn at time t. **353.** a) $\frac{\Delta T}{\Delta t}$; b) $\frac{dT}{dt}=\lim\limits_{\Delta t\to 0}\frac{\Delta T}{\Delta t}$, where T is the temperature at time t. **354.** $\frac{dQ}{dt}=\lim\limits_{\Delta t\to 0}\frac{\Delta Q}{\Delta t}$, where Q is the quantity of substance at time t. **355.** a) $\frac{\Delta m}{\Delta x}$; b) $\lim\limits_{\Delta x\to 0}\frac{\Delta m}{\Delta x}$ **356.** a) $-\frac{1}{6}\approx-0.16$; b) $-\frac{5}{21}\approx-0\ 238$; c) $-\frac{50}{201}\approx-0.249$; $y'_{x=2}=-0.25$. **357.** $\sec^2 x$. **Solution.**

$$y'=\lim_{\Delta x\to 0}\frac{\tan(x+\Delta x)-\tan x}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x\cos x\cos(x+\Delta x)}=\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}\times$$
$$\times\lim_{\Delta x\to 0}\frac{1}{\cos x\cos(x+\Delta x)}=\frac{1}{\cos^2 x}=\sec^2 x.$$

358. a) $3x^2$; b) $-\frac{2}{x^3}$; c) $\frac{1}{2\sqrt{x}}$; d) $\frac{-1}{\sin^2 x}$. **359.** $\frac{1}{12}$ **Solution.** $f'(8)=\lim\limits_{\Delta x\to 0}\frac{f(8+\Delta x)-f(8)}{\Delta x}=$

$$=\lim_{\Delta x\to 0}\frac{\sqrt[3]{8+\Delta x}-\sqrt[3]{8}}{\Delta x}=\lim_{\Delta x\to 0}\frac{8+\Delta x-8}{\Delta x\left[\sqrt[3]{(8+\Delta x)^2}+\sqrt[3]{(8+\Delta x)\,8}+\sqrt[3]{8^2}\right]}=$$
$$=\lim_{\Delta x\to 0}\frac{1}{\sqrt[3]{(8+\Delta x)^2}+2\sqrt[3]{8+\Delta x}+4}=\frac{1}{12}.$$

360. $f'(0)=-8$, $f'(1)=0$, $f'(2)=0$. **361.** $x_1=0$, $x_2=3$. **Hint.** For the given function the equation $f'(x)=f(x)$ has the form $3x^2=x^3$. **362.** 30m/sec. **363.** 1, 2. **364.** -1. **365.** $f'(x_0)=\frac{-1}{x_0^2}$. **366.** -1, 2, $\tan\varphi=3$. **Hint.** Use the results of Example 3 and Problem 365. **367.** **Solution.** a) $f'(0)=\lim\limits_{\Delta x\to 0}\frac{\sqrt[3]{(\Delta x)^2}}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{1}{\sqrt[3]{\Delta x}}=\pm\infty$;