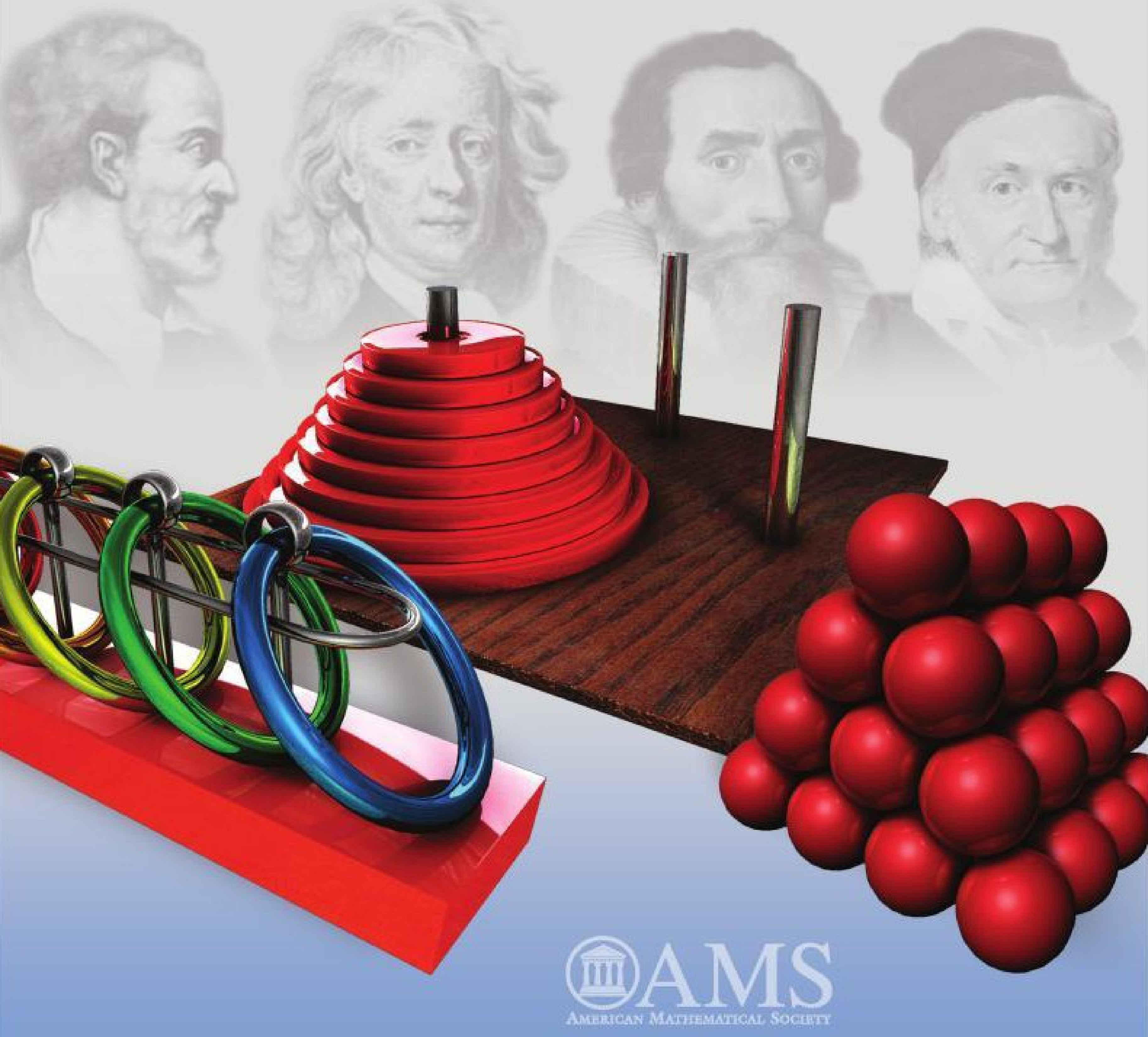
FAMOUS PUZZLES
of Great Mathematicians

Miodrag S. Petković



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To my family

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PREFACE

Mathematics is too serious and, therefore, no opportunity should be missed to make it amusing.

Blaise Pascal

Mathematical puzzles and games have been in evidence ever since man first began posing mathematical problems. The history of mathematics is replete with examples of puzzles, games, and entertaining problems that have fostered the development of new disciplines and sparked further research. Important connections exist between problems originally meant to amuse and mathematical concepts critical to graph theory, geometry, optimization theory, combinatorics, and number theory, to name only a few.

As a motivating force, then, the inclination to seek diversion and entertainment has resulted in the unintended revelation of mathematical truths while also tempering mathematical logic. In fact, Bertrand Russell (1872–1970) once noted: *"A logical theory may be tested by its capacity for dealing with puzzles, and it is a wholesome plan, in thinking about logic, to stock the mind with as many puzzles as possible, since these serve much the same purpose as is served by experiments in physical science."*

Perhaps the popularity of mathematical puzzles and games endures because they fulfill the need for diversion, the desire to achieve mastery over challenging subject matter or simply to test our intellectual capacities. Of equal importance, mathematical amusements also offer an ample playing field to both the amateur and the professional mathematician. That mathematicians from antiquity to the present have always taken interest and delighted in puzzles and diversions might lend credence to the notion that creative stimulus and aesthetic considerations are closely interwoven. Edvard Kasner and James Newman in their essay *Pastimes of past and present times* (in *The World of Mathematics*, Vol. 4 (ed. James Newman), Dover, Mineola 2000) declare: *" ... No branch of intellectual activity is a more appropriate subject for discussion than puzzles and paradoxes .... Puzzles* *in one sense, better than any other single branch of mathematics, reflect its always youthful, unspoiled, and inquiring spirit . . . . Puzzles are made of the things that the mathematician, no less than the child, plays with, and dreams and wonders about, for they are made of the things and circumstances of the world he lives in."*

In attempting to bring the reader closer to the distinguished mathematicians, I have selected 127 problems from their works. Another 50 related problems have been added to this collection. The majority of these mathematical diversions find their basis in number theory, graph theory and probability. Others find their basis in combinatorial and chess problems, and still others in geometrical and arithmetical puzzles. Noteworthy mathematicians ranging from Archimedes, Cardano, Kepler, Pascal, Huygens, Newton, Euler, Gauss, Hamilton, Cayley, Sylvester, to von Neumann, Banach, Erdös and others, have all communicated brilliant ideas, methodological approaches leavened with humor, and absolute genius in mathematical thought by using recreational mathematics as a framework.

This book also explores the brain-teasing and puzzling contributions of contemporary scientists and mathematicians such as John E. Littlewood, John von Neumann, Stephen Banach, Paul Erdös, (H. S. M.) Donald Coxeter, the Nobel-Prize winning physicist Paul Dirac, the famous mathematical physicist Roger Penrose, the eminent mathematician and puzzle composer John Horton Conway and the great computer scientist and mathematician Donald Knuth.

I have purposely selected problems that do not require advanced mathematics in order to make them accessible to a variety of readers. The tools are simple: nothing but pencil and paper. What's required is patience and persistence, the same qualities that make for good careful mathematical research. Restricting problems to only those requiring the use of elementary mathematics consequently forces the omission of other equally celebrated problems requiring higher mathematical knowledge or familiarity with other mathematical disciplines not usually covered at the high school level. Even so, I have made several exceptions in the application of certain nonstandard yet elementary techniques in order to solve some problems. To help readers, I have provided outlines in the book's four appendices because I believe that the time and effort needed to master any additional material are negligible when compared to the reader's enjoyment in solving those problems.

At some point when writing a book of this kind, most authors must limit their choices. The dilemma I most frequently confronted as I selected problems was this: *What determines whether a task is recreational or not?* As already mentioned, in centuries past almost all mathematical problems (excluding, of course, real-life problems of measurement and counting) existed chiefly for intellectual pleasure and stimulation. Ultimately, however, deciding the recreational merits of a given problem involves imposing arbitrary distinctions and artificial boundaries. Over time, a significant number of recreational mathematics problems have become integral to the development of entirely new branches in the field. Furthermore, scientific journals often take as their subject of study problems having the same features as those that characterize recreational mathematics problems. If the reader takes pleasure in squaring off with the problems included here, then the author may regard his selections as satisfactory.

Although several tasks may appear trivial to today's amateur mathematician, we must recall that several centuries ago, most of these problems were not easy to solve. While including such problems provides historical insight into mathematical studies, we must also remain alert to their historical context.

As this book is intended principally to amuse and entertain (and only incidentally to introduce the general reader to other intriguing mathematical topics), without violating mathematical exactitude, it does not always strictly observe the customary rigorous treatment of mathematical details, definitions, and proofs. About 65 intriguing problems, marked by *, are given as exercises to the readers.

I note that, in some instances, difficulties arose with respect to reproducing exact quotes from various sources. However, I trust that these minor inconveniences will not detract from the book's overall worth.

Last, a few comments regarding the arrangement of materials. The table of contents lists the tasks by their title, followed by the author's name in parentheses. Mathematicians whose tasks are included appear in the book's index in boldface. Brief biographies of these contributors appear in chronological order on pages 299–310. The page location indicating a particular biography is given in the text behind the name of the contributor and his puzzle (for example, → p. 299). Furthermore, when introducing the tasks themselves, I have included sometimes amusing anecdotal material since I wanted to underscore the informal and recreational character of the book. Given that the majority of terms, mathematical or otherwise, are familiar to readers, there is no subject index.

Acknowledgments. In the course of writing this book, I received great support from my family, friends and editors.

An especially warm thank you goes to my wife and colleague Ljiljana Petković, and my elder son Ivan, for their comments during the preparation of the manuscript, and their never-failing support and love.

Most of all, I thank my younger son Vladimir Petković for his eye-catching illustrations. A very talented computer artist and designer, Vladimir not only created computer artworks that are both aesthetically attractive and mathematically flavored, but he also solved several intricate problems discussed in this book.

I have made use of personal materials collected over a twenty year period from university libraries in Freiburg, Oldenburg, and Kiel, Germany; London, England; Strasbourg (Université Louis Pasteur), France; Tsukuba, Japan; Minneapolis, Minnesota, Columbia University; Vienna, Austria; the Department of Mathematics, Novi Sad, Serbia and the Institute of Mathematics, Belgrade, Serbia. I wish to thank the staff of these libraries for their assistance.

I would also like to say thanks to dear friends Dr. Martyn Durrant and Professor Biljana Mišić-Ilić, who read a great deal of the manuscript and suggested some improvements in language and style.

Many thanks go to Sergei Gelfand, the Publisher of the American Mathematical Society, for his helpful comments, suggestions and encouragement, and to Christine M. Thivierge, Associate Editor for Proceedings of the American Mathematical Society, for her great effort and persistence in providing the photos of great mathematicians and several very valuable illustrations, and Luann Cole, a production editor at the American Mathematical Society, who read the manuscript from beginning to end and made a number of improvements. I am also thankful to the anonymous referees; their helpful suggestions, remarks and criticism greatly improved the exposition.

I especially wish to acknowledge the assistance and exceptional efforts of Margaret Alice Hanson who helped me rewrite and edit sections of the manuscript to prepare it for publication.

Miodrag S. Petković
University of Niš, Serbia
February 2009

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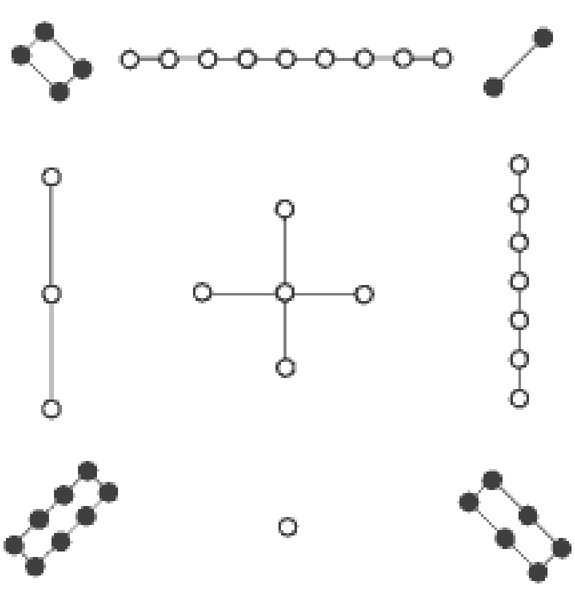
Chapter 1 RECREATIONAL MATHEMATICS

Recreational problems have survived, not because they were fostered by the textbook writers, but because of their inherent appeal to our love of mystery.

Vera Sanford

Before taking up the noteworthy mathematical thinkers and their memorable problems, a brief overview of the history of mathematical recreations may benefit the reader. For more historical details see, *e.g.*, the books [6], [118], [133, Vol. 4], [153, Ch. VI], [167, Vol. II]. According to V. Sanford [153, Ch. VI], recreational mathematics comprises two principal divisions: those that depend on object manipulation and those that depend on computation.



4	9	2
3	5	7
8	1	6

FIGURE 1.1. The oldest magic square—lo-shu

Perhaps the oldest known example of the first group is the magic square shown in the figure above. Known as *lo-shu* to Chinese mathematicians around 2200 B.C., the magic square was supposedly constructed during the reign of the Emperor Yii (see, *e.g.*, [61, Ch. II], or [167, Vol. I, p. 28]). Chinese myth [27] holds that Emperor Yii saw a tortoise of divine creation swimming in the Yellow River with the *lo-shu*, or magic square figure, adorning its shell. The figure on the left shows the *lo-shu* configuration where the numerals from 1 to 9 are composed of knots in strings with black knots for even and white knots for odd numbers.

The Rhind (or Ahmes) papyrus,[1] dating to around 1650 B.C., suggests that the early Egyptians based their mathematics problems in puzzle form. As these problems had no application to daily life, perhaps their main purpose was to provide intellectual pleasure. One of the earliest instances, named "*As I was going to St. Ives*", has the form of a nursery rhyme (see [153]):

"Seven houses; in each are 7 cats; each cat kills 7 mice; each mouse would have eaten 7 ears of spelt; each ear of spelt will produce 7 hekat. What is the total of all of them?"[2]

The ancient Greeks also delighted in the creation of problems strictly for amusement. One name familiar to us is that of Archimedes, whose the *cattle problem* appears on pages 41 to 43. It is one of the most famous problems in number theory, whose complete solution was not found until 1965 by a digital computer.

The classical Roman poet Virgil (70 B.C.–19 B.C.) described in the *Aeneid* the legend of the Phoenician princess Dido. After escaping tyranny in her home country, she arrived on the coast of North Africa and asked the local ruler for a small piece of land, only as much land as could be encompassed by a bull's hide. The clever Dido then cut the bull's hide into the thinnest possible strips, enclosed a large tract of land and built the city of Carthage that would become her new home. Today the problem of enclosing the maximum area within a fixed boundary is recognized as a classical *isoperimetric problem*. It is regarded as the first problem in a new mathematical discipline, established 17 centuries later, as calculus of variations. Jacob Steiner's elegant solution of Dido's problem is included in this book.

Another of the problems from antiquity is concerned with a group of men arranged in a circle so that if every kth man is removed going around the circle, the remainder shall be certain specified (favorable) individuals. This problem, appearing for the first time in Ambrose of Milan's book *ca.* 370, is known as the Josephus problem, and it found its way not just into later European manuscripts, but also into Arabian and Japanese books. Depending on the time and location where the particular version of the Josephus problem was raised, the survivors and victims were sailors and

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[1]Named after Alexander Henry Rhind (1833–1863), a Scottish antiquarian, layer and Egyptologist who acquired the papyrus in 1858 in Luxor (Egypt).

[2]T. Eric Peet's translation of *The Rhind Mathematical Papyrus*, 1923.

smugglers, Christians and Turks, sluggards and scholars, good guys and bad guys, and so on. This puzzle attracted attention of many outstanding scientists, including Euler, Tait, Wilf, Graham, and Knuth.

As Europe emerged from the Dark Ages, interest in the arts and sciences reawakened. In eighth-century England, the mathematician and theologian Alcuin of York wrote a book in which he included a problem that involved a man wishing to ferry a wolf, a goat and a cabbage across a river. The solution shown on pages 240–242 demonstrates how one can solve the problem accurately by using graph theory. River-crossing problems under specific conditions and constraints were very popular in medieval Europe. Alcuin, Tartaglia, Trenchant and Leurechon studied puzzles of this type. A variant involves how three couples should cross the river in a boat that cannot carry more than two people at a time. The problem is complicated by the jealousy of the husbands; each husband is too jealous to leave his wife in the company of either of the other men.

Four centuries later, mathematical puzzles appear in the third section of Leonardo Fibonacci's *Liber Abaci*, published in 1202. This medieval scholar's most famous problem, the *rabbit problem*, gave rise to the unending sequence that bears his name: the Fibonacci sequence, or Fibonacci numbers as they are also known, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... (see pages 12–13).

Yet another medieval mathematician, ibn Khallikan (1211–1282), formulated a brain teaser requiring the calculation of the total number of wheat grains placed on a standard 8×8 chessboard. The placement of the grains must respect the following distribution: 1 grain is placed on the first square, 2 grains on the second, 4 on the third, 8 on the fourth, and so on, doubling the number for each successive square. The resulting number of grains is $2^{64} - 1$, or 18,446,744,073,709,551,615. Ibn Khallikan presented this problem in the form of the tale of the Indian king Shirham who wanted to reward the Grand Vizier Sissa ben Dahir for having invented chess. Sissa asked for the number of grains on the chessboard if each successive position is the next number in a geometric progression. However, the king could not fulfill Sissa's wish; indeed, the number of grains is so large that it is far greater than the world's annual production of wheat grains. Speaking in broad terms, ibn Khallikan's was one of the earliest chess problems.

Ibn Kallikan's problem of the number of grains is a standard illustration of geometric progressions, copied later by Fibonacci, Pacioli, Clavius and Tartaglia. Arithmetic progressions were also used in these entertaining problems. One of the most challenging problems appeared in Buteo's book *Logistica* (Lyons, 1559, 1560):[3]

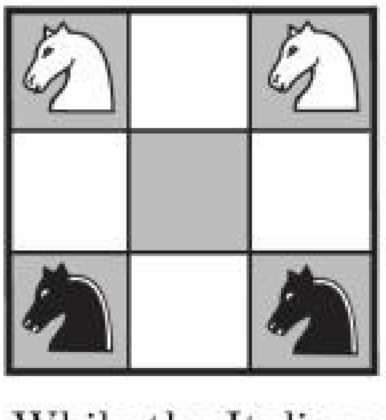
[3]The translation from Latin is given in [153], p. 64.

"A mouse is at the top of a poplar tree 60 braccia[4] high, and a cat is on the ground at its foot. The mouse descends 1/2 of a braccia a day and at night it turns back 1/6 of a braccia. The cat climbs one braccia a day and goes back 1/4 of a braccia each night. The tree grows 1/4 of a braccia between the cat and the mouse each day and it shrinks 1/8 of a braccia every night. In how many days will the cat reach the mouse and how much has the tree grown in the meantime, and how far does the cat climb?"

At about the same time Buteo showed enviable knowledge of the general laws of permutations and combinations; moreover, he constructed a combination lock with movable cylinders displayed in Figure 1.2.[5]

Figure 1.2. Buteo's combination lock (1559)

In 1512 Guarini devised a chessboard problem in which the goal is to effect the exchange of two black and two white knights, with each pair placed at the corners of a 3×3 chessboard (see figure left), in the minimum number of moves. The solution of this problem by using graph theory is shown on pages 274–276. People's interest in chess problems and the challenge they provide has lasted from the Middle Ages, through the Renaissance and to the present day.



While the Italian mathematicians Niccolo Tartaglia (1500–1557) and Girolamo Cardano (1501–1576) labored jointly to discover the explicit formula for the solution of cubic algebraic equations, they also found time for recreational problems and games in their mathematical endeavors. Tartaglia's *General Trattato* (1556) described several interesting tasks; four of which,

[4] *Braccia* is an old Italian unit of length.

[5] Computer artwork, sketched according to the illustration from Buteo's *Logistica* (Lyons, 1559, 1560).

the weighing problem, the division of 17 horses, the wine and water problem, and the ferryboat problem, are described on pages 20, 24, 25 and 173.

Girolamo Cardano was one the most famous scientists of his time and an inventor in many fields. Can you believe that the joint connecting the gear box to the rear axle of a rear wheel drive car is known to the present day by a version of his name—the cardan shaft? In an earlier book, *De Subtilitate* (1550), Cardano presented a game, often called the *Chinese ring puzzle* (Figure 1.3), that made use of a bar with several rings on it that remains popular even now. The puzzle's solution is closely related to Gray's error-correcting binary codes introduced in the 1930s by the engineer Frank Gray. The Chinese ring puzzle also bears similarities to the *Tower of Hanoi*, invented in 1883 by Edouard Lucas (1842–1891), which is also discussed later in the book.

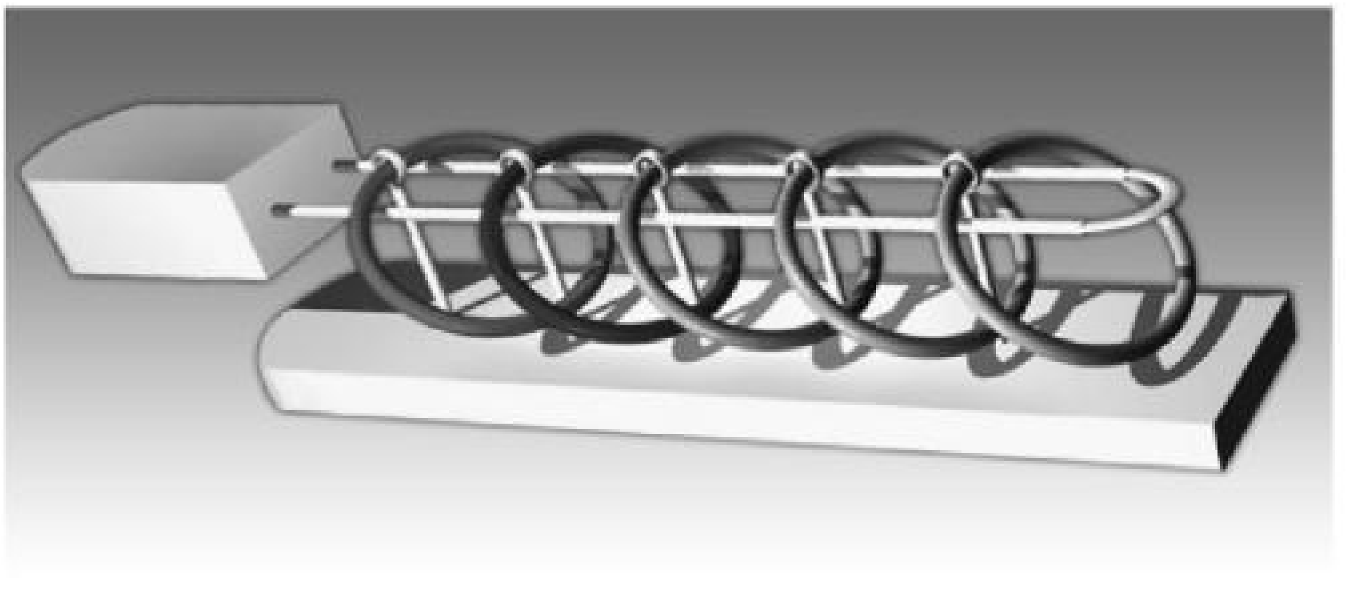


FIGURE 1.3. Chinese ring puzzle

Many scholars consider *Problèms Plaisans et Délectables*, by Claude Gaspar Bachet (1581–1638), to be the first book on mathematical puzzles and tricks. Most of the famous puzzles and curious problems invented before the seventeenth century may be found in Bachet's delightful book. In addition to Bachet's original "delectable" problems, the book contains puzzles by Alcuin of York, Pacioli, Tartaglia and Cardano, and other puzzles of Asian origin. Bachet's book, first published in 1612 and followed by the second edition published in 1624, probably served as the inspiration for subsequent works devoted to mathematical recreation.

Other important writers on the subject include the Jesuit scholar Jean Leurechon (1591–1670), who published under the name of Hendrik van Etten, and Jacques Ozanam (1640–1717). Etten's work, *Mathematical Recreations, or a Collection of Sundry Excellent Problems Out of Ancient and Modern Philosophers Both Useful and Recreative*, first published in French in 1624 with an English translation appearing in 1633, is a compilation of mathematical problems interspersed with mechanical puzzles and experiments in hydrostatics and optics that most likely borrowed heavily from Bachet's work.

Leonhard Euler (1707–1783), one of the world's greatest mathematicians whose deep and exacting investigations led to the foundation and development of new mathematical disciplines, often studied mathematical puzzles and games. Euler's results from the *seven bridges of Königsberg* problem (pages 230–232) presage the beginnings of graph theory. The *thirty-six officers problem* and orthogonal Latin squares (or Eulerian squares), discussed by Euler and later mathematicians, have led to important work in combinatorics. Euler's conjecture on the construction of mutually orthogonal squares found resolution nearly two hundred years after Euler himself initially posed the problem. These problems, and his examination of the chessboard *knight's re-entrant tour problem*, are described on pages 188 and 258. A knight's reentrant path consists of moving a knight so that it moves successively to each square once and only once and finishes its tour on the starting square. This famous problem has a long history and dates back to the sixth century in India. P. M. Roget's half-board solution (1840), shown in Figure 1.4, offers a remarkably attractive design.

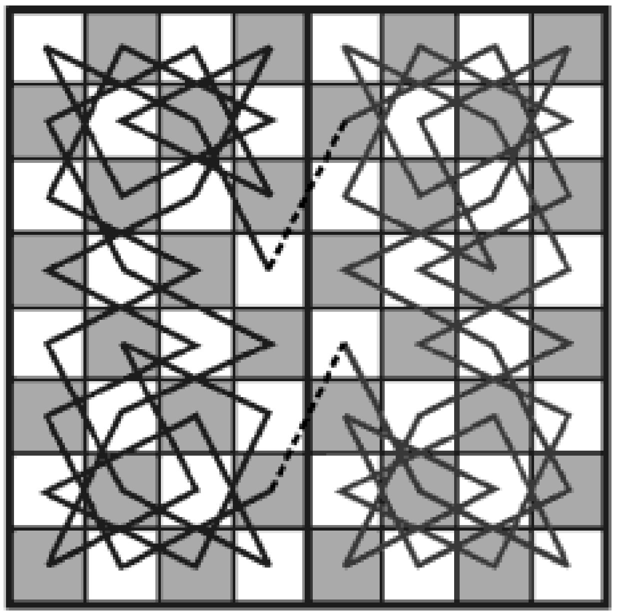


Figure 1.4. Knight's re-entrant path—Roget's solution

In 1850 Franz Nauck posed another classic chess problem, the *eight queens problem*, that calls for the number of ways to place eight queens on a chessboard so that no two queens attack each other. Gauss gave a solution of this problem, albeit incomplete in the first attempts. Further details about the *eight queens problem* appear on pages 269–273. In that same year, Thomas P. Kirkman (1806–1895) put forth the *schoolgirls problem* presented on pages

189 to 192. Several outstanding mathematicians, Steiner, Cayley and Sylvester among them, dealt with this combinatorial problem and other related problems. Although some of these problems remain unsolved even now, the subject continues to generate important papers on combinatorial design theory.

In 1857 the eminent Irish mathematician William Hamilton (1788–1856) invented the *icosian game* in which one must locate a path along the edges of a regular dodecahedron that passes through each vertex of the dodecahedron once and only once (see pages 234–237). As in Euler's Königsberg bridges problem, the Hamiltonian game is related to graph theory. In modern terminology, this task requires a Hamiltonian cycle in a certain graph and it is one of the most important open problems not only in graph theory but in the whole mathematics. The Hamiltonian cycle problem is closely connected to the famous *traveling salesman problem* that asks for an optimal route between some places on a map with given distances.

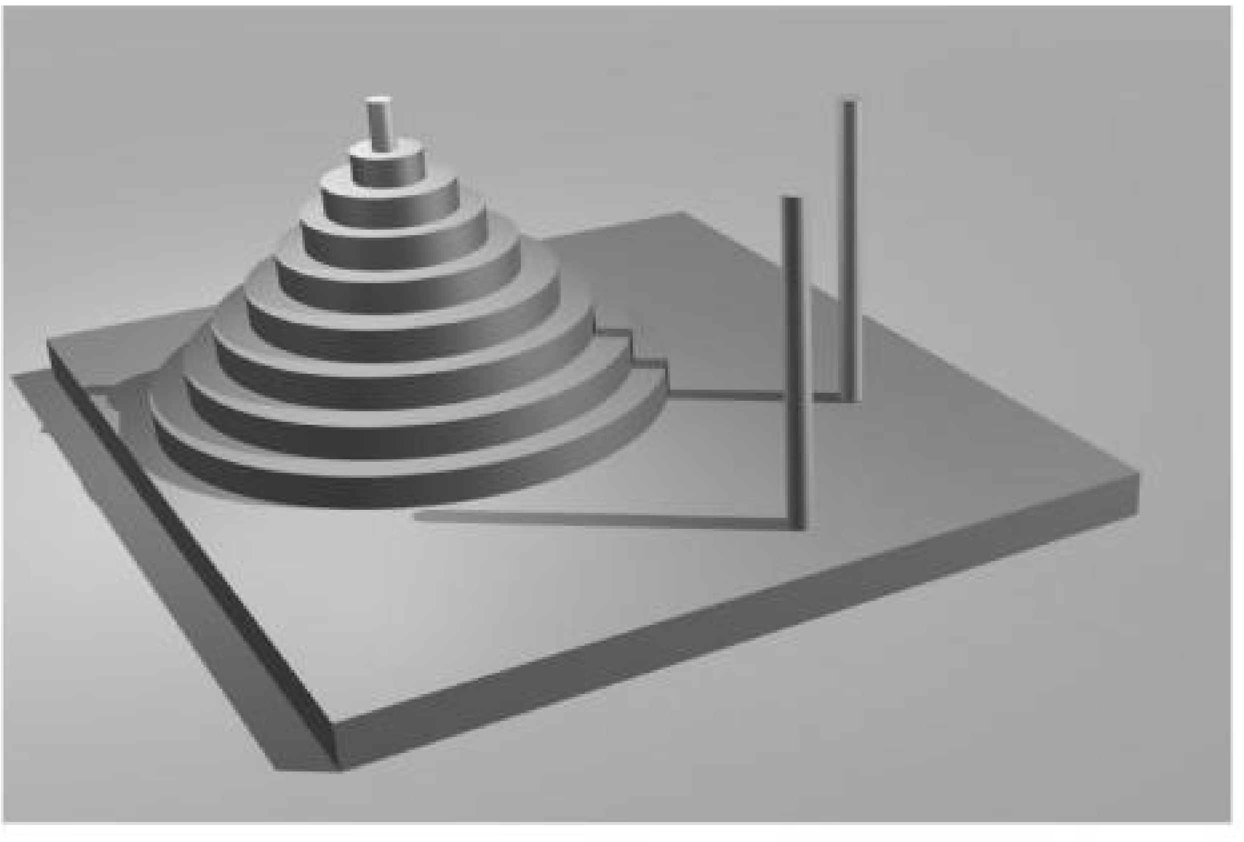
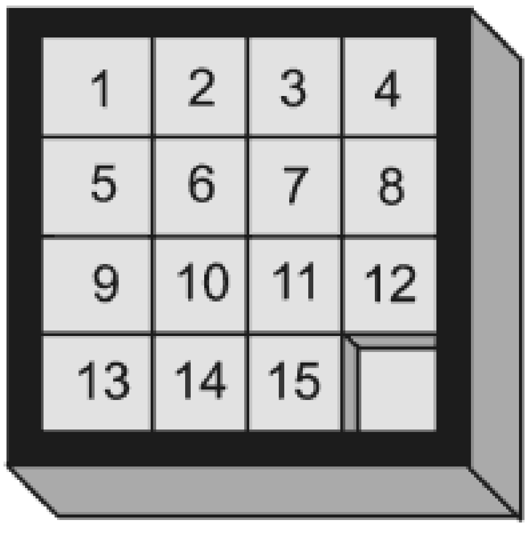


FIGURE 1.5. The Tower of Hanoi

The French mathematician François Edouard Lucas, best known for his results in number theory, also made notable contributions to recreational mathematics, among them, as already mentioned, the *Tower of Hanoi* (Figure 1.5), which is covered on pages 196–199, and many other amusing puzzles. Lucas' four-volume book *Récréations Mathématiques* (1882–94), together with Rouse Ball's, *Mathematical Recreations and Problems*, published in 1892, have become classic works on recreational mathematics.

No discussion of recreational mathematics would be complete without including Samuel Loyd (1841–1911) and Henry Ernest Dudeney (1857–1931), two of the most renowned creators of mathematical diversions. Loyd and Dudeney launched an impressive number of games and puzzles that remain as popular now as when they first appeared. Loyd's ingenious toy-puzzle the "15 Puzzle" (known also as the "Boss Puzzle", or "Jeu de Taquin") is popular even today. The "15 Puzzle" (figure below) consists of a square divided into 16 small squares and holds 15 square blocks numbered from 1 to 15. The task is to start from a given initial arrangement and set these numbered blocks into the required positions (say, from 1 to 15), using the vacant square for moving blocks. For many years after its appearance in 1878, people all over the world were obsessed by this toy-puzzle. It was played in taverns, factories, in homes, in the streets, in the royal palaces, even in the Reichstag (see page 2430 in [133, Vol. 4].



Martin Gardner (b. 1914 Tulsa, OK), most certainly deserves mention as perhaps the greatest twentieth-century popularizer of mathematics and mathematical recreations. During the twenty-five years in which he wrote his *Mathematical Games* column for the *Scientific American*, he published quantities of amusing problems either posed or solved by notable mathematicians.

Chapter 2 ARITHMETICS

What would life be without arithmetic,
but a scene of horrors?
Sydney Smith

In the arithmetic of love,
one plus one equals everything,
and two minus one equals nothing.
Mignon McLaughlin

In elementary school we first encounter "the three R's," a basic skills oriented education program: **r**eading, w**r**iting and a**r**ithmetic. We start to count to ten, to add to hundred, then gradually increase our operation ability using addition, subtraction, multiplication, division, and finally root extraction, congruence calculation, factorization, and power computation. This is arithmetic (or arithmetics), the oldest branch of mathematics, which records elementary properties of the above operations on numbers.

Arithmetics is all around us and is used by almost everyone. It is essential to almost every profession for tasks ranging from simple everyday counting to business calculations and scientific research. A mnemonic for the spelling of "arithmetic", that I found on an Internet site, reads: "a rat in the house may eat the ice cream."

Puzzles and entertaining problems whose solutions depend entirely on basic arithmetic operations have been in evidence from ancient times to the present. The early puzzle problems were difficult at that time due to the lack of good symbolism and they lost their mystery when the algebraic relations had been developed. Modern arithmetic puzzles are often based on tricks or relations hidden under misleading statements. Some of them are entertaining because they are unsolvable upon given information. Here is an example: *"If three eagles catch three hares in three days, how many eagles will catch* 100 *hares in* 100 *days?"* The question is well known and seems innocent at first glance. Actually, the correct answer cannot be given without more information.

Most of the arithmetic puzzle problems presented in this chapter belong to medieval mathematicians such as Diophantus, Alcuin of York, Mahāvira, Fibonacci, Bachet, Tartaglia, Recorde and Viète. The reasons for such selection are twofold: first, the problems of classic arithmetics and higher arithmetics (that is, number theory, the name used by some authors) are separated into two chapters—on number theory and arithmetics, respectively. Second, after the development of the power algebraic methods and algorithms, mathematicians of modern era have dealt with new theories and disciplines rather than with problems of elementary arithmetics. Of course, there were exceptions and one of them, ascribed to the great Newton, is included in this chapter.

*

* *

Diophantus of Alexandria (*ca.* 200–*ca.* 284) (→ p. 299)

Diophantus' age

Diophantus' contribution to mathematics is better known than the facts of his life (see, *e.g.*, Katz [113, pp. 173–183]). Some details can be concluded from the collection of puzzles in the *Greek Anthology* compiled by Metrodorus around 500 A.D., which contains the following puzzle [37]:

Problem 2.1. *Diophantus' boyhood lasted $\frac{1}{6}$ of his life; he married after $\frac{1}{7}$ more; his beard grew after $\frac{1}{12}$ more, and his son was born 5 years later; the son lived to half his father's age, and the father died 4 years after the son.*

This task served to determine how long Diophantus lived and to identify other important dates in his life. If x denotes the number of years Diophantus lived, according to the above information we form the equation

$$\frac{x}{6} + \frac{x}{7} + \frac{x}{12} + 5 + \frac{x}{2} + 4 = x,$$

wherefrom $x = 84$. Therefore, Diophantus married at the age of 33 and had a son who died at the age of 42, four years before Diophantus himself died at the age of 84.

Today every pupil in elementary school (well, almost everyone) can easily solve this problem. However, problems of this sort were formulated in antiquity only by words and the lack of a good symbolism made them difficult to represent and solve. Speaking about the ages of great mathematicians, recall that a similar question was posed much later. Once asked about his age, the eminent British mathematician Augustus de Morgan answered: "*I was x years old in the year x^2.*" Knowing that de Morgan was born in 1806, we form the quadratic equation $x^2 = 1806 + x$ and find the positive solution $x = 43$—de Morgan's age.

Mahāvira (*ca.* 800–*ca.* 870) (→ p. 300)

Mahāvira was a ninth-century Indian mathematician who wrote on elementary mathematics, combinatorics and integer solutions of first degree indeterminate equations. In this book we give three problems (see below and Chapters 3 and 6) that can be found in Wells' book [186].[1]

Number of arrows

Problem 2.2.* *Arrows, in the form of thin cylinders with circular cross-section, can be packed in hexagonal bundles. If there exist eighteen circumferential arrows, determine the total number of the arrows to be found* (*in the bundle*) *within the quiver.*

Leonardo Pisano (Fibonacci) (1170–1250) (→ p. 300)

Leonardo Pisano, better known today by his nickname Fibonacci, sometimes went by the name "Bigollo", which may possibly have meant "good-for-nothing" or "traveler". As a matter of fact, he traveled widely with his father who held a diplomatic post in North Africa. Throughout his many journeys, Fibonacci encountered numerous mathematical problem-solving techniques, in particular, the new Hindu–Arabic numerals, already known in China, India, and the Islamic world. He figured out that the use of the symbols 0 to 9 was much more convenient than the Roman numerals.

Fibonacci
1170–1250

After his return to Pisa, Fibonacci wrote a number of important texts as well as his masterpiece, published in 1202, *Liber Abaci* (Book of Calculation). *Liber Abaci*, which gathered a wealth of practical material and an assortment of problems, was the most influential mathematical work in Europe for at least three centuries, especially in bringing a positional numbering system and the new form of number notation (Hindu–Arabic numerals instead of Roman numerals, still in common use in Europe at that time).

[1] Wells took these problems from Mahāvira's book *The Ganita Sara–Sangraha by Mahāvira*, translated by M. Rangacarya and published by Government Press (Madras, India) in 1912.

How many rabbits?

A problem in the third section of Fibonacci's famous book *Liber Abaci* (1202), led to the introduction of the Fibonacci numbers and the Fibonacci sequence for which he is best remembered today.

Problem 2.3. *A man bought a pair of rabbits. How many pairs of rabbits can be produced from the original pair in a year if it is assumed that every month each pair begets a new pair that can reproduce after two months?*

The solution to Fibonacci's problem is given in Figure 2.1, where the total number of pairs of rabbits is presented by the number of black points, counting from the top to the considered month. The numbers of rabbit pairs make a sequence 1, 1, 2, 3, 5, 8, 13, ... It is easy to conclude that any member of this sequence is equal to the sum of the previous two members. Therefore, the next members are 21, 34, 55, 89, 144, 233, ... and hence, the total number of rabbit pairs in a year is 233.

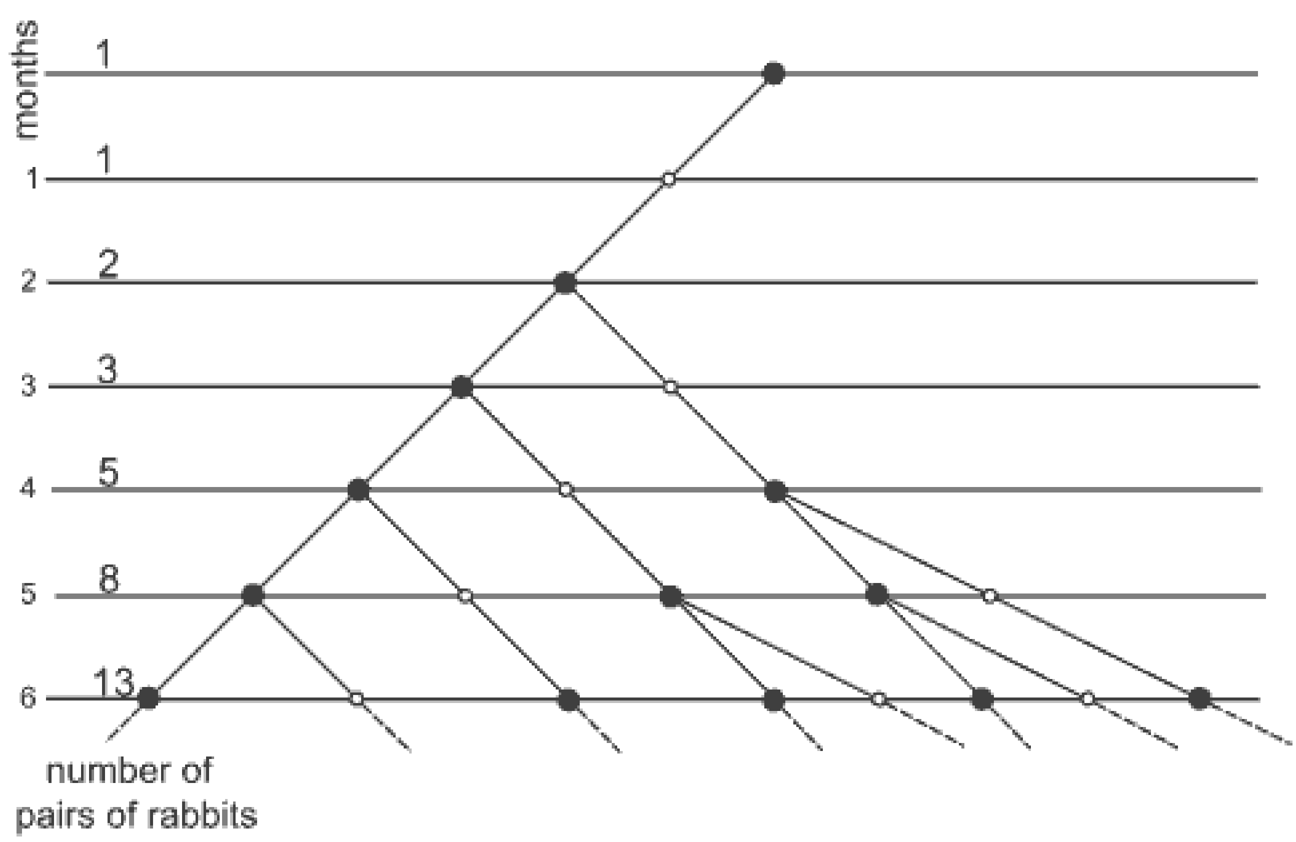


FIGURE 2.1. A graphical solution of Fibonacci's rabbit problem

In the middle of the nineteenth century, the French mathematician Edouard Lucas (1843–1891) named the resulting sequence "Fibonacci's sequence".[2] It is interesting to note that the Fibonacci numbers 1, 2, 3, 5, 8, 13,... had been mentioned explicitly in discussions of Indian scholars, such as Gopāla

[2]Some witty guys have proposed "rabbit sequence".

and Hemachandra, before the appearance of Fibonacci's book, *Liber Abaci* (see [115]).

As mentioned, Fibonacci's sequence satisfies the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 3.$$

The explicit formula for the nth term F_n (assuming $F_1 = F_2 = 1$) is given by

$$F_n = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big]$$

(see (D.6), Appendix D).[3]

When using a calculator or a computer, it is easy to compute the nth member F_n. Since the influence of the second term $((1-\sqrt{5})/2)^n/\sqrt{5}$ is negligible because $|(1-\sqrt{5})/2| < 1$, it is sufficient to calculate the first term $((1+\sqrt{5})/2)^n/\sqrt{5}$ and round off the result to the nearest integer to obtain the exact (integer) value of F_n. To be more exact,

$$F_n = \left\lfloor \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^n + \frac{1}{2}\Big]\right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. For example, for $n = 20$ one obtains

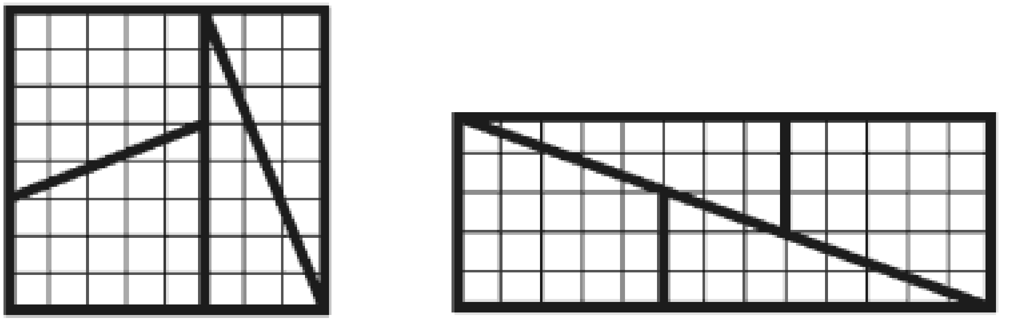
$$\frac{1}{\sqrt{5}}\Big(\frac{1+\sqrt{5}}{2}\Big)^{20} + \frac{1}{2} = 6{,}765.5000295, \quad \text{so that} \quad F_{20} = 6{,}765.$$

Let us note that the Fibonacci sequence possesses a number of interesting and remarkable properties (see, *e.g.*, [31, Ch. 2], [43], [88], [182], [185]). The *Fibonacci Quarterly*, a mathematical journal launched in 1963, is devoted to studying mathematics related to the Fibonacci sequence. It turns out that this sequence is extremely versatile and appears in many different areas of mathematics, such as optimization theory and the analysis of algorithms; it also appears in physics, chemistry, biology, architecture, and even in poetry and music. The first eight Fibonacci numbers in a scrambled order 13, 3, 2, 21, 1, 1, 8, 5 appear in the very popular novel *The Da Vinci Code* by Dan Brown.

[3] Daniel Bernoulli (1700–1782) published this formula in 1728. It seems that Abracham de Moivre (1667–1754) also knew this formula, which is often attributed to the French mathematician Jacques Binet (1786–1856).

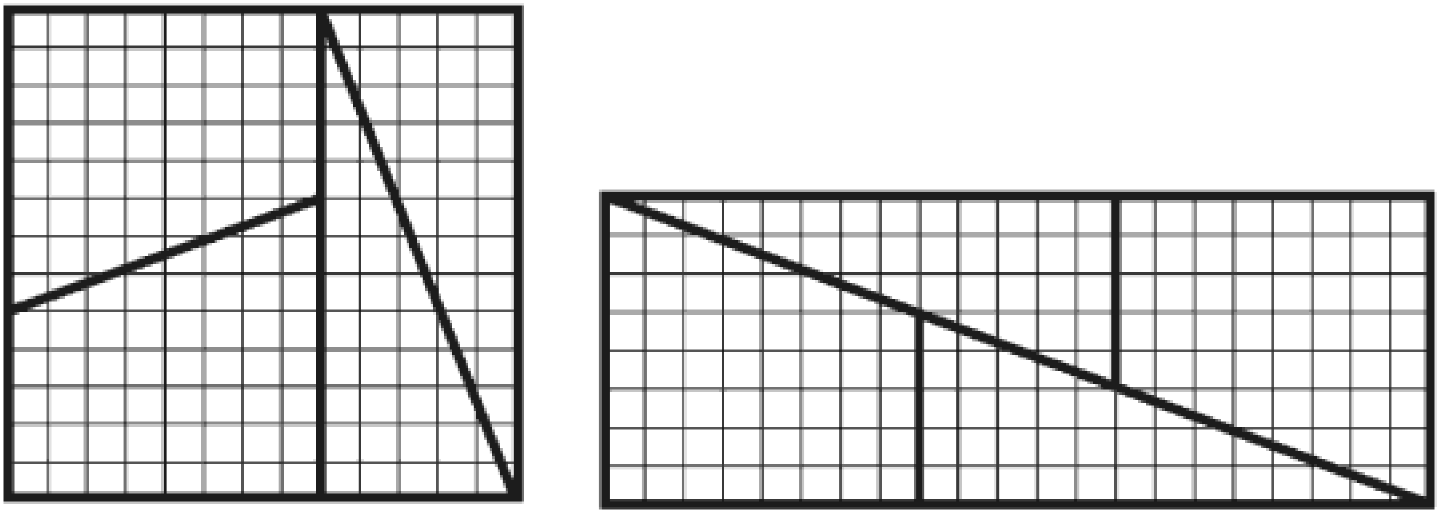
Fibonacci's sequence gave rise to a multitude of amusing problems to claim the attention of many mathematicians. Below we present two well-known geometric paradoxes. Perhaps a little bird will tell you a solution.

Problem 2.4.* *Let us consider an ordinary* 8×8 *chessboard cut into four pieces—two trapezoidals and two triangles—as shown in Figure* 2.2(*a*). *If these four pieces are put together in the shape of a rectangle as shown in Figure* 2.2(*b*), *one obtains a* 5×13 *rectangle containing* 65 *small squares* (*one square more*). *Therefore,* $64 = 65$?! *Similarly, if we cut a* 13×13 *chessboard into four pieces and reassemble the pieces, we obtain a rectangle* 8×21 *containing* 168 *small squares* (*one square less, see Figure* 2.3). *In this case we obtain* $169 = 168$?! *Can you explain these paradoxes?*



a) b)

FIGURE 2.2. Chessboard paradox: 64=65!?



a) b)

FIGURE 2.3. Chessboard paradox: 169=168!?

The first paradox "64=65" can be found in a great variety of books (see, *e.g.*, [88], [138]). It seems that the earliest reference appeared in the work *Rational Recreations* (1774), written by William Hooper. This paradox was later discussed in *Zeitschrift für Mathematik und Physik* (Leipzig, 1868).

The Italian artist Mario Merz (b. 1925) has been obsessed for many years by the Fibonacci numbers. In 1994 he decorated the chimney of the Energy Power Plant in Turku (Finland), as an environment art project, designing the Fibonacci numbers by neon tubes (Figure 2.4).

FIGURE 2.4. Fibonacci's numbers on the chimney in Turku (Finland)

Another example shows that Fibonacci's numbers can be applied to compose plane figures.

Problem 2.5.* *Make a rectangle without any gaps by using small squares whose sides are the Fibonacci numbers* 1, 1, 2, 3, 5, 8, 13 *and* 21.

Square numbers problem

In his book, *Liber Quadratorum* (Book of Squares) (1225), Master John of Palermo, a member of Emperor Frederick II's entourage, posed the following problem to Fibonacci.

Problem 2.6. *Find a square number such that, when five is added or subtracted, the result is again a square number.*

Master John's problem can be interpreted as finding integer solutions of the system of equations $x^2 + 5 = y^2$, $x^2 - 5 = z^2$. Fibonacci succeeded in solving a more general problem in which he introduced what he called *congruous* numbers, that is numbers n of the form

$$ab(a+b)(a-b) \quad \text{when} \quad a+b \text{ is even}$$

and

$$4ab(a+b)(a-b) \quad \text{when} \quad a+b \text{ is odd.}$$

He showed that congruous numbers are always divisible by 24. He also showed that integer solutions of

$$x^2 + n = y^2 \quad \text{and} \quad x^2 - n = z^2$$

can be found only if n is congruous. Since Master John's problem is obtained for $n = 5$ and since 5 is not congruous, it follows that this problem is not solvable in integers. However, a solution to the problem exists that uses rational numbers. From the facts that $720 = 12^2 \cdot 5$ is a congruous number (with $a = 5$ and $b = 4$), and that $41^2 + 720 = 49^2$ and $41^2 - 720 = 31^2$, it follows by dividing both equations by 12^2 that

$$x = \frac{41}{12}, \quad y = \frac{49}{12}, \quad z = \frac{31}{12},$$

which is a solution in rational numbers to $x^2 + 5 = y^2$, $x^2 - 5 = z^2$.

Money in a pile

Problem 2.7.* *Three men **A**, **B**, and **C** each place money in a common pile, their shares being $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$ of the total amount, respectively. Next, each man takes some money from the pile until no money remains. Now **A** returns $\frac{1}{2}$ of what he took, **B** $\frac{1}{3}$, and **C** $\frac{1}{6}$. When the returned money is divided equally among the men, it turns out that each has what he possessed at the beginning, before removing money from the pile. How much money was in the original pile?*

Yang Hui (*ca.* 1238–*ca.* 1298) (→ p. 300)

Chinese mathematician Yang Hui was a minor official who lived in the thirteenth century. Yang Hui is best remembered as being the first to represent Pascal's triangle, basing his achievement on the work of another Chinese mathematician Jia Xian. He wrote two books, dated 1261 and 1275, which present works with decimal fractions, calculations of square and cube roots and the earliest extant documents in mathematics education in ancient China.

Magic configuration

Hui's second book *Yang Hui suanfa* (Yang Hui's Methods of Computation, 1275) contains the following magic configuration (see [186]).

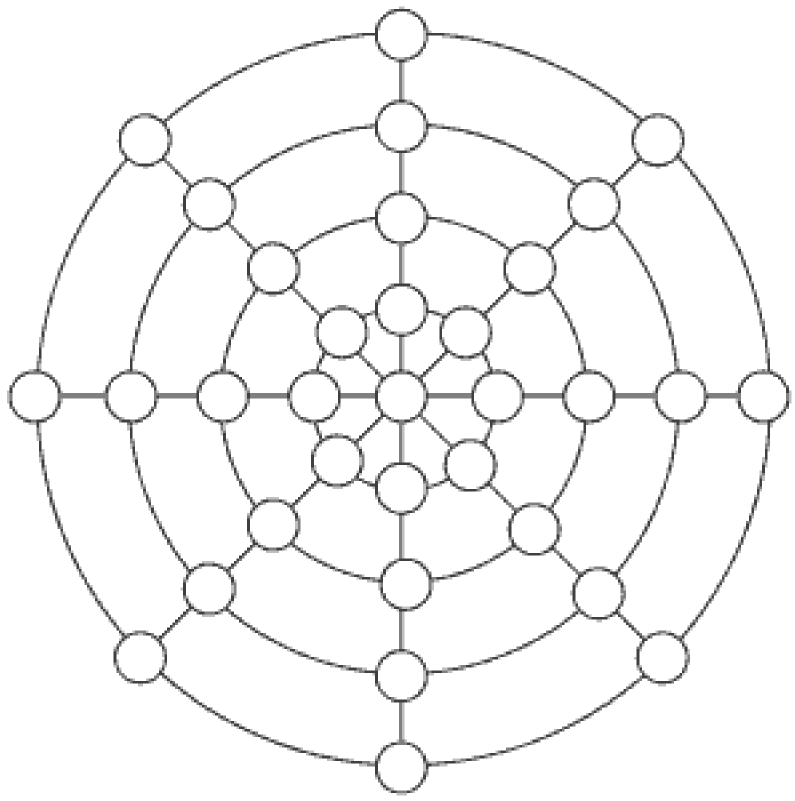


FIGURE 2.5. Magic configuration

Problem 2.8.* *Arrange the numbers* 1 *to* 33 *in the small circles shown in Figure* 2.5 *so that every circle including its center and every diameter has the same sum.*

Claude Gaspar Bachet (1581–1638) (→ p. 302)

As mentioned in the Preface, Claude Gaspar Bachet, the French mathematician, philosopher, poet, and a member of the Académie des Sciences, wrote the first books on mathematical puzzles and tricks ever published. His book *Problèmes Plaisants et Délectables* (1612) contains many mathematical puzzles, arithmetical tricks and recreational tasks.

Triangle with integer sides

A *Heronian triangle* is a triangle having side lengths and area expressed by rational numbers. Multiplying the three side lengths and area of such a triangle by their least common multiple, a new Heronian triangle with integer side lengths and area is obtained. Claude Bachet considered the following problem.

Problem 2.9. *Find a triangle with integer sides whose area is* 24.

Solution. The area of a triangle with sides a, b, and c can be calculated using the Heron formula

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter. Substituting s in the above formula, after squaring we obtain

$$A^2 = \frac{1}{16}(a+b+c)(b+c-a)(a+c-b)(a+b-c),$$

or, after some elementary manipulation,

$$16A^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4.$$

The last relation can be rearranged in the form

$$(4A)^2 + (b^2 + c^2 - a^2)^2 = (2bc)^2. \tag{2.1}$$

The relation (2.1) is of the form $x^2 + y^2 = z^2$, which enables us to connect the above problem with *Pythagorean triangles*, that is, right triangles whose sides are integers. The corresponding triples of numbers (x, y, z) are called *Pythagorean triples*. Some familiar Pythagorean triples are (3, 4, 5) and (5, 12, 13).

From number theory, we know that the number of Pythagorean triples is infinite. Namely, if (x, y, z) is any Pythagorean triple, then the triples (kx, ky, kz) are also Pythagorean triples for any natural number k. If m and n $(m > n)$ are natural numbers, then Pythagorean triples can be generated by the two-parameter formula

$$x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2. \tag{2.2}$$

Let us return to the relation (2.1) and, having in mind (2.2), we write

$$4A = k(m^2 - n^2), \quad b^2 + c^2 - a^2 = \pm k \cdot 2mn, \quad 2bc = k(m^2 + n^2).$$

In particular, we have

$$4A = 96 = k(m^2 - n^2) = k \cdot (m - n) \cdot (m + n).$$

The integer 96 can be factored into three factors in the following ways

$$\begin{aligned}96 &= \underline{1\cdot 1\cdot 96} = 1\cdot 2\cdot 48 = \underline{1\cdot 3\cdot 32} = 1\cdot 4\cdot 24\\ &= 1\cdot 6\cdot 16 = 1\cdot 8\cdot 12 = 2\cdot 2\cdot 24 = \underline{2\cdot 3\cdot 16}\\ &= 2\cdot 4\cdot 12 = 3\cdot 4\cdot 8 = 4\cdot 4\cdot 6,\end{aligned}$$

where the first digit stands for k. The underlined combinations should be excluded since the factors $m-n$ and $m+n$ are either both even or both odd. In this way we obtain the following triples (k,m,n) $(m>n)$ which are possible candidates to give integer sides $a,\ b,\ c$:

$$\begin{gathered}(1,25,23),\ (1,14,10),\ (1,11,5),\ (1,10,2),\\ (2,13,11),\ (2,8,4),\ (3,6,2),\ (4,5,1).\end{gathered}$$

By splitting $k(m^2+n^2)/2 = b\cdot c$ into two factors (when it is possible), we find the pairs (b,c) and calculate the third side a by the formula

$$a = \sqrt{b^2+c^2 \pm 2kmn}$$

selecting only integer values of a. After the checking procedure we find that only triples (2,8,4) and (4,5,1) give integer sides; the required Heronian triangles with the area equal to 24 are $(6,8,10)$ and $(4,13,15)$.

Carmichael [35] gave the parametric version of the complete integer solutions to Heronian triangles in the form

$$\begin{aligned}a &= n(m^2+k^2),\\ b &= m(n^2+k^2),\\ c &= (m+n)(mn-k^2),\\ A &= kmn(m+n)(mn-k^2),\end{aligned}$$

where integers $m,\ n$ and k satisfy the conditions $\gcd(m,n,k)=1,\ mn > k^2 > n^2n/(2m+n)$ and $m \ge n \ge 1$.[4] The above formulae generate at least one member (a,b,c) of a required class of Heronian triangles with $A = t^2A^*$, where t is an integer multiple and A^* is the originally given area. Then the required side lengths are $(a/t, b/t, c/t)$. For example, the parametric formulae directly produce the integer triangle $(6,8,10)$ with the given area $A^*=24$, but omit the other solution $(4,13,15)$. However, this solution is contained in

[4] $\gcd(p,q)$—greatest common divisor of two numbers p and q.

the triple $(24, 78, 90)$ with $A = 864 = 36 \cdot 24 = 6^2 \cdot A^*$, giving $t = 6$. Hence, the original triangle with the area 24 is $(24/6, 78/6, 90/6) = (4, 13, 15)$.

D. Wells mentions in [186] a trick method that combines the right triangles $(5, 12, 13)$ and $(9, 12, 15)$ to find triangle $(4, 13, 15)$. Next, let us fit together and then overlap these two triangles along their common side 12; see Figure 2.6. The obtuse triangle with sides 4, 13 and 15 (in bold face) yields the required area $54 - 30 = 24$.

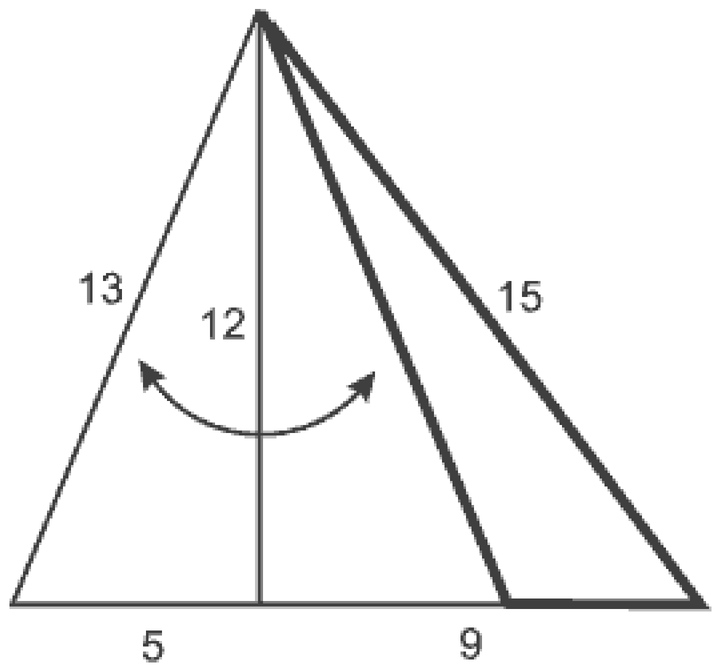


FIGURE 2.6. Obtuse triangle with integer sides

Leonardo Pisano (Fibonacci) (1170–1250) (→ p. 300)

Niccolo Tartaglia (1500–1557) (→ p. 301)

Claude Gaspar Bachet (1581–1638) (→ p. 302)

Weights problem

Bachet's highly-regarded book, *Problèms Plaisants et Délectables,* collects a multitude of problems many of which Bachet himself wrote, including this classic problem of weights.

Problem 2.10. *Determine the least number of weights necessary to weigh any integer number of pounds from* 1 *lb. to* 40 *lbs. inclusive.*[5]

Bachet gave two solutions describing two separate cases: (i) the weights may be placed in only one of the scale-pans; (ii) the weights may be placed in either of the scale-pans. In addition to Bachet, Fibonacci (Leonardo Pisano)

[5]**lb** is the abbreviation of *pound* ≈ 0.453 kg.

and Niccolo Tartaglia are often cited in literature as having solved variant (i) with the following series of weights: 1, 2, 4, 8, 16, and 32 lbs. In case (ii) Bachet found that the series of weights of 1, 3, 9, and 27 lbs. satisfied the solution. Assuming that there are no constraints in the problem posed, these solutions give the least possible number of weights required.

Let us explain now what reasoning leads to the above solutions. Let Q be any integer number of pounds ≤ 40 and let $t_1, t_2, \ldots, t_n$ be the required weights. Consider first the variant (i). The weights $t_1, t_2, \ldots, t_n$ have to be chosen so that the equality

$$Q = a_1 t_1 + a_2 t_2 + \cdots + a_n t_n \tag{2.3}$$

holds for every $Q = 1, 2, \ldots, 40$, assuming that every coefficient a_k ($k \in \{1, 2, \ldots, n\}$) is 1 if the weight t_k is placed on the scale-pan and 0 if not.

We observe that (2.3) is equivalent to the representation of numbers in the binary system. Indeed, if we take

$$t_1 = 2^0 = 1,\ t_2 = 2^1 = 2,\ t_3 = 2^2 = 4,\ \ldots,\ t_n = 2^{n-1},$$

we will obtain

$$Q = a_n \cdot 2^{n-1} + \cdots + a_3 \cdot 2^2 + a_2 \cdot 2^1 + a_1 \cdot 2^0 \quad (a_k \in \{0, 1\},\ k = 1, \ldots, n). \tag{2.4}$$

The largest number that can be expressed using (2.4) is $Q = (1 \cdots 111)_2$ (the number written in the binary system with n units), which is equal to

$$(1 \cdots 111)_2 = 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2^1 + 2^0 = 2^n - 1.$$

Taking $n = 6$, we obtain $(111111)_2 = 2^6 - 1 = 63 > 40$, so that the choice of 6 weights satisfies the condition of the task. Therefore, it is necessary (and sufficient) to take the weights of 1, 2, 4, 8, 16, and 32 lbs. Other solutions with unequal weights do not exist.

We see that the use of the binary system goes back to the middle ages and probably centuries before Fibonacci's day. However, the first person to study binary numbers was Gottfried Leibniz, the great German mathematician, philosopher, politician, theologist, physicist, lawyer and linguist. His treatment of binary was mainly philosophical. Peter Bentley wrote in *The Book of Numbers* [16]: *"He [Leinbiz] believed that the universe could be represented more elegantly through binary and its conflicting, yes-no, on-off nature such as male-female, light-dark, right-wrong."* Does this mean that Leibniz, a philosopher, may have been close to the concept of computing machines in his thinking, bearing in mind that the work of digital computers is based on the binary numbering system? Quite possibly that man of all professions and trades may have been the first computer scientist and information theorist. In 1671 Leibniz invented a calculating machine capable of performing basic arithmetic operations.

Let us return to the solution of Bachet's problem in case (ii). Bachet used the same idea as above (case (i)) and represented $Q \in \{1, \dots, 40\}$ in the form

$$Q = b_1 t_1 + b_2 t_2 + \cdots + b_m t_m, \tag{2.5}$$

where coefficients $b_1, b_2, \dots, b_m$ have the following values:

$b_k = -1$ if the weight t_k is placed in the same scale-pan as the weighed object;

$b_k = 0$ if the weight t_k is not placed in either of the scale-pans;

$b_k = 1$ if the weight t_k is placed in the scale-pan not loaded with the weighed object.

If the value -1 would be regarded as a "digit" of a number system, the representation (2.5) would suggest the use of the ternary system with the base 3. In that case, taking the weights $t_1 = 3^0,\ t_2 = 3^1 = 3,\ t_3 = 3^2 = 9$, and so on, we first perform the conversion of the integer value Q from the decimal system into the "number system" with the base 3, that is,

$$Q = b_m \cdot 3^{m-1} + \cdots + b_3 \cdot 3^2 + b_2 \cdot 3^1 + b_1 \cdot 3^0 = (b_m \cdots b_3 b_2 b_1)_3. \tag{2.6}$$

Here $b_k \in \{-1, 0, 1\}$ and the greatest integer number of the form (2.6) is $Q = (1 \cdots 111)_3$ (the number written in the ternary system with m units), which is equal to

$$(1 \cdots 111)_3 = 3^{m-1} + \cdots + 3^2 + 3^1 + 3^0 = \frac{3^m - 1}{2}.$$

Taking $m = 4$ we obtain $(1111)_3 = (3^4 - 1)/2 = 40$. Thus, the set of 1, 3, 9, and 27-pound weights satisfies the condition of the problem.

The procedure described in case (ii) will be illustrated in the example of $Q = 23$ lbs. Then

$$Q = 23 = 1 \cdot 27 + (-1) \cdot 3 + (-1) \cdot 1.$$

In this way decimal number 23 is transformed to form (2.6). According to the right-hand expression, we conclude that the 1-pound and 3-pound weights must be placed in the scale-pan together with the object weighed, while the 27-pound weight is in the other scale-pan.

In the *Quarterly Journal of Mathematics* (1886, Volume 21) the English mathematician P. A. MacMahon determined all conceivable sets of integer weights to weigh all loads from 1 to n. To solve this problem, he applied the method of generating functions discovered by Euler. In this way, MacMahon generalized Bachet's weight problem. Moreover, he completed the solution of the presented problem since Bachet's approach does not give all solutions. MacMahon found eight solutions:

$$\{1_{40}\}, \{1, 3_{13}\}, \{1_4, 9_4\}, \{1, 3, 9_4\}, \{1_{13}, 27\},$$
$$\{1, 3_4, 27\}, \{1_4, 9, 27\}, \{1, 3, 9, 27\}.$$

The notation w_k denotes that k weights are needed, each of which weighs w lbs. For example, the solution $\{1_4, 9_4\}$ denotes that 4 weights of 1 lb. and 4 weights of 9 lbs. are necessary. The last of the listed solutions belongs to Bachet; his solution requires the least number of weights and it is also the only one in which all weights are unequal. Rouse Ball and Coxeter [150] and Kraitchik [118] discussed some of the details of Bachet's weight problem.

Niccolo Tartaglia (1500–1557) (→ p. 301)

When Tartaglia was a boy of about thirteen, the French invaded his native city of Brescia and brutally massacred local inhabitants. Tartaglia was the victim of such severe injuries that he almost died. Somehow he survived but his injuries left him with a permanent speech defect that led to the nickname "Tartaglia", which means the stammerer. His poverty prevented him from receiving a proper education. However, despite a painfully difficult childhood, Tartaglia later became one of the most influential Italian mathematicians of the sixteenth century, known best today for his formula, the Cardano–Tartaglia formula for solving cubic equations. The question of priority of this formula caused a bitter quarrel between Tartaglia and Cardano. In fact, Scipione del Ferro and Tartaglia independently discovered the cubic formula, but Cardano published it in his book, *Ars Magna* (Great Skill). Although he fully credited del Ferro and Tartaglia for their discoveries, this formula was remembered most often under Cardano's name.

Included in his, *Quesiti et Inventioni Diverse* (1546) and *General Trattato* (1556), are several problems that were considered as challenging and serious in Tartaglia's time but which today have acquired a recreational character. Below we reproduce two of them, while two others, *weights problem* and *married couples cross the river*, are presented on pages 20 and 173.

Division of 17 horses

Problem 2.11. *A dying man leaves seventeen horses to be divided among his three sons in the proportions $\frac{1}{2} : \frac{1}{3} : \frac{1}{9}$. Can the brothers carry out their father's will?*

To simplify calculating with fractions, Tartaglia contrived an artificial method whereby a horse is borrowed. He then divided and obtained $18 : 2 = 9,\ 18 : 3 = 6,\ 18 : 9 = 2$, which meant that the sons received 9, 6 and 2 horses, for a total of 17 horses. After this division, the borrowed horse was returned to its owner and the problem was solved.

In general case, this contrivance can be implemented when n horses must be divided among three brothers in proportions of $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$. It can be shown that there exist seven possible values of $\boldsymbol{n}, a, b, c$:

$$(\mathbf{7},2,4,8),\ (\mathbf{11},2,4,6),\ (\mathbf{11},2,3,12),\ \ (\mathbf{17},2,3,9),$$
$$(\mathbf{19},2,4,5),\ (\mathbf{23},2,3,8),\ \ (\mathbf{41},2,3,7).$$

Actually, these natural numbers are the only integer solutions of the Diophantine equation[6]

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{n}{n+1}.$$

The above problem is closely connected with another interesting one about dividing 17 horses among brothers that can be found in literature devoted to popular and recreational mathematics. In this story, the brothers do not divide horses according to the proportions of $\frac{1}{2} : \frac{1}{3} : \frac{1}{9}$, but in such a way that the oldest brother receives $1/2$ of all horses, the next $1/3$, and the youngest brother $1/9$. Since such a division is impossible, the brothers remember to borrow one horse from their neighbor. After dividing 18 into the corresponding parts, they receive 9, 6 and 2 horses, which give a total of 17. Then they return the borrowed horse to the neighbor, quite satisfied with their "clever" solution of the division problem.

However, did the brothers' clever maneuver result in an acceptable solution according to the terms of the will? Actually, no, the division was not done in a proper way; moreover, it is not possible. Indeed, since $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} = \frac{17}{18} < 1$, any division gives a remainder. Owing to the fact of the borrowed horse, each of the brothers receives more since $9 > \frac{17}{2},\ 6 > \frac{17}{3},\ 2 > \frac{17}{9}$.

[6] A Diophantine equation is an undeterminate equation in which only integer solutions are required. The word *diophantine* refers to Diophantus of Alexandria (third century) who made a study of such equations.

Wine and water

Problem 2.12.* *A dishonest servant takes out* 3 *pints of wine from a barrel and replaces them with the same amount of water. He repeats his theft twice, removing in total* 9 *pints and replacing them with water. As a result of this swindle, the diluted wine remaining in the barrel lost half of its former strength. How much wine did the barrel originally hold?*

Robert Recorde (*ca.* 1510–1558) (→ p. 301)

Robert Recorde, one of the most influential English textbook writers of the sixteenth century, was a physician, mathematician and astronomer. In his book *The Whetstone of Witte* (1557) he introduced the modern symbol = for equality.[7] Although he served as a physician to Edward VI and Queen Mary and later became Comptroller of Mines and Monies in Ireland, Recorde was arrested under mysterious circumstances and died in prison in 1558. The following task is from Recorde's book *The Whetstone of Witte.*

Coins in hands

Problem 2.13.* *A man has in both hands* 8 *coins. The number of coins in his left hand is added to its square and its cube, and the same procedure applies to the number of coins in his right hand. Both sums obtained in this way make* 194. *How many coins are in each hand?*

François Viète (1540–1603) (→ p. 302)

The great Frenchman **François Viète** was not a mathematician by vocation. As did Fermat, Cayley, and Sylvester, Viète studied and practiced law; became a member of the Breton parliament; a master of requests in Paris, and even later a member of the king's Council. Although he engaged in mathematics only in his leisure time, he made important contributions to geometry, arithmetic, algebra and trigonometry. In addition, Viète was an extraordinary code-cracker. During the war occasioned by Spain's attempt to place a pretender on the French throne, he very quickly succeeded in breaking the Spanish army's most complicated ciphers. Viète's deciphering skills greatly influenced the final outcome of the war, and also provoked accusations from the Spanish that he was in collusion with the devil.

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[7]Katz [113, p. 356] quotes Recorde's explanation: *"To avoid the tedious repetition of these words—is equal to—I will set as I do often in work use, a pair of parallels, or gemow [twin] lines of one length, thus =, because no 2 things can be more equal."*

Sides of two cubes

Problem 2.14.* *The difference between the sides of two cubes is* 6 *and the difference of their volumes is* 504. *Find the sides of the cubes.*

When we solve quadratic equations in high school, we encounter the well-known Viète's rule: *If a and b are the roots of the quadratic equation $x^2 + \alpha x + \beta = 0$, then*

$$(x-a)(x-b) = x^2 - (a+b)x + ab, \quad \text{that is,} \quad \alpha = -(a+b),\ \beta = ab.$$

We would like to see your reaction to the following "generalization."

Problem 2.15.* *Calculate* $(x-a)(x-b)\cdots(x-z)$.

Isaac Newton (1624–1727) (→ p. 304)

Isaac Newton, the English mathematician and physicist of Jewish origin, is considered one of the greatest mathematicians and scientists of all historical time. Only two men can be compared to him by their achievements: Archimedes of ancient Greece and the German mathematician Carl Friedrich Gauss. Their combined scientific genius introduced revolutionary advances in mathematics and other branches of science, spurring rapid developments in numerous directions.

Isaac Newton
1624–1727

There is a vast literature about Newton's life and his monumental work in mathematics and physics, so that many well-known details are omitted. It is less known that he was obsessively occupied with alchemy and theology; it is hard to believe that he actually spent more time performing his experiments in laboratories than he ever did in mathematics. When he was buried in Westminster Abbey, a high concentration of Mercury was found in his body, most likely as the consequence of his alchemy experiments (see [16, p. 125]).

Newton was not a pleasant and modest man, his arrogance and difficult behaviour were well known. Nevertheless, he is credited with the following famous quotation having a modest flavor: *"If I have seen further than others, it is because I've stood on the shoulders of giants."*[8]

[8] I also like the following quip attributed to Harold Abelson, a professor of MIT: *"If I have not seen as far as others, it is because giants were standing on my shoulders."*

FIGURE 2.7. Newton's birth house at Woolsthorpe, England

Newton's book *Arithmetica Universalis*[9] contains, among its very influential contributions, several elementary problems. In this chapter we present a problem about animals, while some other problems are given in Chapter 11. Speaking about animals, it seems appropriate to give here a story about Newton and his cats. Newton is often credited with the invention of the "catflap". The anecdote that accompanies his invention demonstrates the great mathematician's absentmindedness. Newton had two cats: a small and a larger one. To enable his cats to come and go as they pleased without disturbing his work, he constructed two doors for them: a small one for the small cat and a larger one for the big cat. Had he considered the matter more thoroughly, he would surely have realized that one pet door of the proper dimensions would have sufficed for both cats.

Animals on a field

Although Newton laid the foundations of modern mathematics by solving difficult and challenging problems, a number of these problems could also be considered as recreational mathematics tasks. The following task is a typical one that appears in Newton's book *Universal Arithmetick*.

[9]Written in Latin in 1707 and translated under the title *Universal Arithmetick* in 1720 (edited by John Machin).

Problem 2.16. *In 4 weeks, 12 oxen consume $3\frac{1}{3}$ acres of pasture land; in 9 weeks, 21 oxen consume 10 acres of pasture land. Accounting for the uniform growth rate of grass, how many oxen will it require to consume 24 acres in a period of 18 weeks?*

George Pólya presented a general solution in his book *Mathematical Discovery* [142, Vol. I, p. 162].

Let us introduce the following quantities:

α – the quantity of grass per acre when the pasture is put into use;

β – the quantity of grass eaten by one ox in one week;

γ – the quantity of grass that grows in one acre in one week;

$a_1,\ a_2,\ a$ – the number of oxen;

$m_1,\ m_2,\ m_3$ – the number of acres;

$t_1,\ t_2,\ t$ – the numbers of weeks in the three cases considered, respectively.

According to the conditions given in the task we can form a system of three equations,

$$\begin{aligned} m_1(\alpha + t_1\gamma) &= a_1 t_1 \beta, \\ m_2(\alpha + t_2\gamma) &= a_2 t_2 \beta, \\ m(\alpha + t\gamma) &= a t \beta, \end{aligned} \tag{2.7}$$

where $a,\ \alpha/\beta,\ \gamma/\beta$ appear as unknowns. Solving the above system one obtains

$$a = \frac{m\big[m_1 a_2 t_2(t - t_1) - m_2 a_1 t_1(t - t_2)\big]}{m_1 m_2 t(t_2 - t_1)}.$$

Substituting the numerical data from Newton's original problem, we find $a = 36$. Therefore, 36 oxen will consume 24 acres in 18 weeks.

The above task has many variants. The three tasks that follow here appear frequently in literature, the first of which is given in Heinrich Dörrie's book [54].

Problem 2.17.* *What relation exists between the nine magnitudes x to z'' if:*

x cows graze y fields bare in z days,

x' cows graze y' fields bare in z' days,

x'' cows graze y'' fields bare in z'' days?

Another variation of Newton's problem reads:

Problem 2.18.* *Every day a flock of sheep graze grass in a field. If the field supports a flock of* 10 *sheep each day, then the flock will consume all of the grass in* 20 *days; if a flock contains three times as many sheep, then this larger flock will consume all of the grass in* 4 *days. Assuming a uniform daily rate of grass growth, in how many days will a flock of* 25 *sheep, grazing daily, consume all of the grass in the field?*

The following task includes a combination of familiar farm animals.

Problem 2.19.* *A cow, a goat, and a goose graze grass in a field. The cow eats the same quantity of grass as the goat and the goose together. The cow and the goat eat all of the grass in the field in* 45 *days, the cow and the goose in* 60 *days, and the goat and the goose in* 90 *days. In how many days will the cow, the goat, and the goose together eat all of the grass in the field, again assuming that the grass grows at the same daily rate?*

Alcuin of York (735–804) (→ p. 299)

The English scholar, mathematician, and churchman Alcuin of York spent his life at the court of the Emperor Charlemagne. He wrote a collection of fifty-three amusing problems, riddles and trick questions. *Propositiones ad Acuendos Juvenes* (Problems for Quickening the Mind),[10] the earliest known European collection of mathematical and logical puzzles written two centuries after Alcuin's death at the monastery of Augsburg, includes his tasks. Many puzzles and tasks have survived to the present day.

Gathering an army

Problem 2.20. *In making preparations for war, the king of a powerful country orders his servant to assemble an army from thirty shires in such a way that the servant will enlist the same number of men from each shire as he has collected until that point. The servant travels to the first shire alone; to the second in the company of one soldier... . How many soldiers will be collected in all?*

Let us first assume that the servant is not included in the count at each stage. In such a case, he would arrive at the first shire having gathered no

[10] Ozanam's *Récréations Mathématiques*, 1803, English edition, Vol. I, p. 171, included these problems.

men, and thus would gather none there, and so on; the total collected would be zero! Therefore, the servant must include himself as the first soldier, and the numbers on leaving each shire are 2, 4, 8,... . The total number of soldiers (together with the servant) on leaving the thirtieth shire is obtained by summing the geometrical progression,

$$1\,(\text{servant}) + 1 + 2 + 2^2 + \cdots + 2^{29} = 1 + \frac{2^{30}-1}{2-1} = 2^{30} = 1{,}073{,}741{,}824.$$

Let us note that the number of soldiers, not including the servant, is the same as the number of moves necessary to transfer 30 rings in Lucas' *Tower of Hanoi* puzzle (see page 195).

Answers to Problems

2.2. To find the number of arrows in a bundle, it is sufficient to sum, as far as necessary, the series $1 + 1\cdot 6 + 2\cdot 6 + 3\cdot 6 + \cdots + k\cdot 6 + \cdots$ (see Figure 2.8). If 18 arrows are visible in the package, the adding stops when member 18 of the series appears. Therefore, there are $1 + 6 + 12 + 18 = 37$ arrows in all.

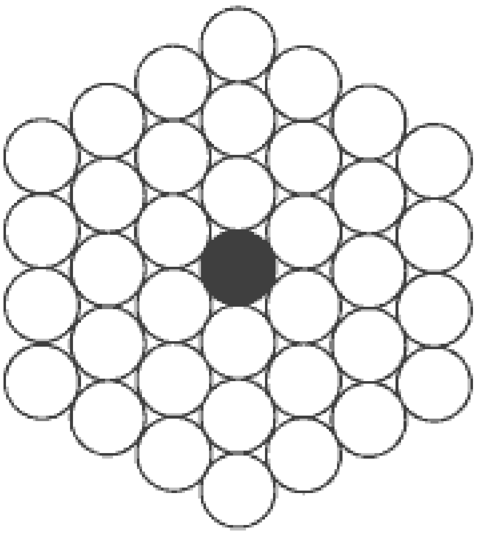


FIGURE 2.8. Number of arrows

2.4. Before we explain these paradoxes, we give one of the most beautiful relations on the Fibonacci numbers, known as Cassini's identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n \quad (n > 0). \tag{2.8}$$

Jean-Dominique Cassini, the French astronomer and mathematician, stated this identity in 1680.[11] We leave the proof to the reader although we note that complete induction is a convenient device.

[11] According to [88, p. 292], Johannes Kepler knew this formula already in 1608.

Let us note that the numbers 5, 8, 13, 21, which appear as dimensions in the above dissections, are Fibonacci's numbers $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$. In the first example we have $13 \times 5 - 8 \times 8 = 1$ (one square more), while the second example yields $21 \times 8 - 13 \times 13 = -1$ (one square less). In general, by taking Fibonacci's numbers F_{n-1}, F_n and F_{n+1}, we can dissect any $F_n \times F_n$ square into four pieces by using a similar construction that, after reassembling, form a rectangle $F_{n+1} \times F_{n-1}$. According to Cassini's identity (2.8), one square will be created (when n is even) or lost (when n is odd).

We shall explain this phenomenon on the example of the 8×8 chessboard. The paradox arises from the fact that the edges of the four pieces that lie along the diagonal of the formed rectangle 5×13, do not coincide exactly in direction. This diagonal *is not* a straight segment line but a small *lozenge* (diamond-shaped figure), whose acute angle is

$$\arctan \tfrac{2}{3} - \arctan \tfrac{3}{8} = \arctan \tfrac{1}{46} \approx 1\,\tfrac{1}{4}^{\circ}.$$

Only a very precise drawing enables us to distinguish such a small angle. Using analytic geometry or trigonometry, we can easily prove that the area of the "hidden" lozenge is equal to that of a small square of the chessboard.

2.5. The solution is given in Figure 2.9.

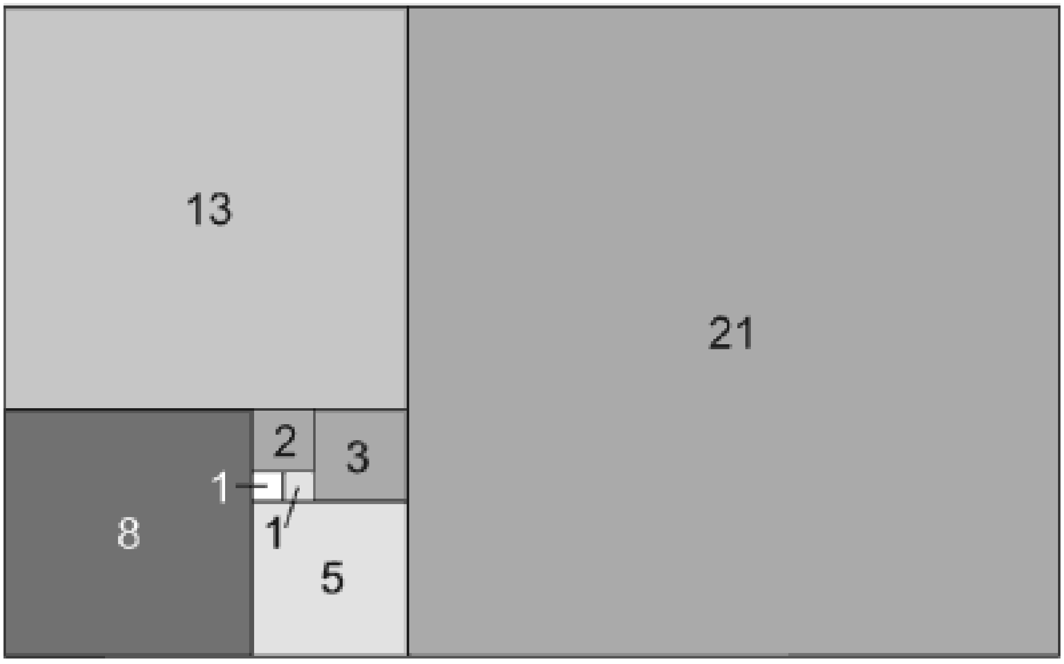


FIGURE 2.9. Fibonacci's rectangle

It is easy to observe that the constructions of larger and larger "Fibonacci rectangles" can be continued in an obvious way. The following challenging question is quite natural: *Can we obtain perfect squares by this process* (excepting the trivial case of the unit square)? J. H. E. Cohn [38] proved in 1964 that there exists the unique square with the requested property composed from the squares with sides 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 and 144.

2.7. Let s denote the original sum and x the sum returned (equally) to each man. Before the three men received this sum, they possessed $s/2 - x$, $s/3 - x$, $s/6 - x$. Since these are the sums possessed after putting back 1/2, 1/3, 1/6 of what they had first taken, the amounts first taken were distributed as follows:

$$\boldsymbol{A}: \quad 2(s/2 - x), \quad \boldsymbol{B}: \quad \frac{3}{2}(s/3 - x), \quad \boldsymbol{C}: \quad \frac{6}{5}(s/6 - x).$$

The sum of these amounts gives the original sum s, that is,

$$2(s/2 - x) + \frac{3}{2}(s/3 - x) + \frac{6}{5}(s/6 - x) = s,$$

which reduces to the indeterminate equation $7s = 47x$. Fibonacci took $s = 47$ and $x = 7$. Thus the sums taken from the original pile are 33, 13, and 1.

It is easy to show that the sum in the original pile, which provides that all transfers of money (taking, returning and sharing) are expressed in integers, has the form $s = 47 \cdot 6d = 282d$, where $d = 1, 2, \ldots$ is a natural number. Hence, the minimal sum with this property is $s = 282$.

2.8. The solution is given in Figure 2.10 (see Needham [131]).

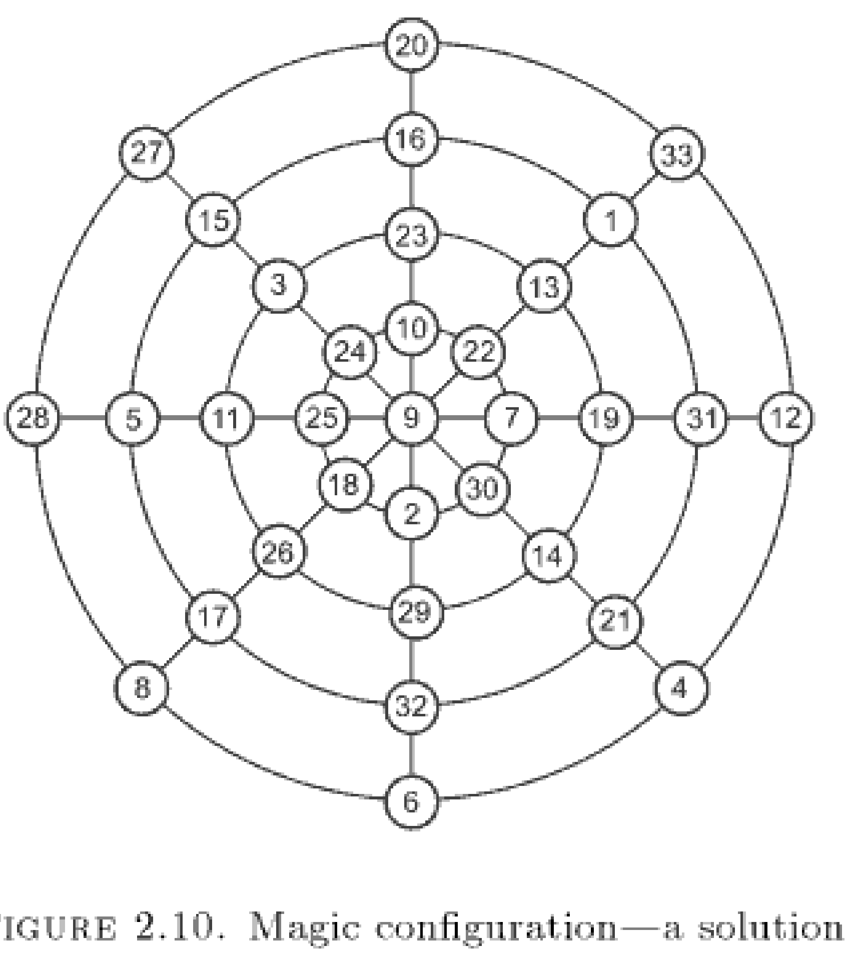


FIGURE 2.10. Magic configuration—a solution

2.12. Let the barrel originally contain x pints of wine. After one removal and replacement, the amount of wine and its strength are $a_1 = x - 3$ and $s_1 = a_1/x = (x-3)/x$. On the second removal the amount of wine removed will be $3s_1 = 3(x-3)/x$ and the amount of wine in the barrel is $a_2 = x - 3 - 3(x-3)/x$; its strength is

$$s_2 = a_2/x = \big(x - 3 - 3(x-3)/x\big)/x.$$

After the third removal, the wine removed will be

$$3s_2 = \frac{3\big(x - 3 - 3(x-3)/x\big)}{x},$$

while the amount of wine and its strength in the barrel are, respectively,

$$a_3 = \frac{3\big(x - 3 - 3(x-3)/x\big)}{x} = \frac{(x-3)^3}{x^2},$$
$$s_3 = \frac{a_3}{x} = \frac{(x-3)^3}{x^3}.$$

Since the strength of wine remaining in the barrel is $s_3 = 1/2$, from the last expression we come to the equation

$$2(x-3)^3 = x^3, \quad \text{or} \quad (1 - 3/x)^3 = 1/2.$$

Hence

$$x = \frac{3 \cdot 2^{1/3}}{2^{1/3} - 1} \approx 14.54 \text{ pints}.$$

2.13. Let x and y be the numbers of coins in hand. Then

$$x + y + x^2 + y^2 + x^3 + y^3 = 194,$$

or

$$(x+y) + (x+y)^2 - 2xy + (x+y)^3 - 3xy(x+y) = 194.$$

If we put $x + y = 8$ and $xy = t$, the last equation becomes

$$8 + 8^2 + 8^3 - 2t - 24t = 194,$$

wherefrom

$$t = xy = 15.$$

Introducing $y = 8 - x$, we come to the quadratic equation $x^2 - 8x + 15 = 0$ with the solutions 3 and 5 meaning that if $x = 3$, then $y = 5$, and conversely. Therefore, the required numbers of coins are 3 and 5.

2.14. Let x and y be the sides of the given cubes. Then

$$x - y = 6, \quad x^3 - y^3 = 504.$$

Since

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2) = (x-y)[(x-y)^2 + 3xy] = 504,$$

putting $x - y = 6$ and $xy = t$ we obtain

$$6(36 + 3t) = 504,$$

wherefrom $t = 16$. Solving the system

$$x - y = 6, \; xy = 16$$

we find $x = 8, \; y = 2$.

2.15. The solution of this tricky problem is very simple (if you have a flash of inspiration). The twenty-fourth factor in the product

$$P = (x-a)(x-b)\cdots(x-z)$$

is $(x - x)$ $(= 0)$ so that $P = 0$.

2.17. As in Dörrie's book, let the initial amount of grass contained by each field be M, the daily growth of each field m, and each cow's daily grass consumption of Q. We can then form the following system of linear homogeneous equations:

$$\begin{aligned} yM + yzm - zxQ &= 0, \\ y'M + y'z'm - z'x'Q &= 0, \\ y''M + y''z''m - z''x''Q &= 0. \end{aligned} \tag{2.9}$$

We recall one of the basic properties of the theory of linear systems of equations: *The determinant of a system of n linear homogeneous equations possessing n unknowns that do not all vanish (M, m, and Q in our case) must be equal to zero.* Therefore, for the system (2.9) we have

$$\begin{vmatrix} y & yz & -zx \\ y' & y'z' & -z'x' \\ y'' & y''z'' & -z''x'' \end{vmatrix} = 0.$$

After multiplying both sides with -1 we obtain the desired relation

$$\begin{vmatrix} y & yz & zx \\ y' & y'z' & z'x' \\ y'' & y''z'' & z''x'' \end{vmatrix} = 0.$$

2.18. This task can be easily resolved starting from system (2.7) of the three equations and assuming that $a = 25$ is known, t is the desired unknown, and weeks are replaced by days. However, we give another, somewhat shorter solution.

Let A denote all of the grass in the field, expressed in some unit, and let B represent the quantity of daily grass growth, also expressed in the same units. Then we can state two equations:

$$\begin{aligned} A + 20\,(\text{days}) \cdot B\,(\text{units of grass}) &= 20\,(\text{days}) \cdot 10\,(\text{sheep}), \\ A + 4\,(\text{days}) \cdot B\,(\text{units of grass}) &= 4\,(\text{days}) \cdot 30\,(\text{sheep}). \end{aligned}$$

From the system

$$A + 20B = 200, \quad A + 4B = 120,$$

we find $A = 100,\ B = 5$.

Let x be the required number of days necessary for 25 sheep to eat all of the grass in the field. As above, we form the equation

$$A + x\,(\text{days}) \cdot B\,(\text{units of grass}) = x\,(\text{days}) \cdot 25\,(\text{sheep}),$$

that is,

$$100 + 5x = 25x,$$

wherefrom we find $x = 5$. Thus, 25 sheep will eat all of the grass in 5 days.

2.19. Since the cow eats as much grass as the goat and the goose together, and the cow and the goat can eat all of the grass in 45 days, then two goats and one goose will eat the same quantity of grass in that same time, assuming a constant rate of growth for the grass. This period is twice as short as the time necessary for the goat and the goose to eat all the grass (90 days). Hence we conclude that the goat can eat all of the grass in the field in 90 days if during that time the goose eats only the grass growth.

According to the previous consideration, after a little thought we realize that, for one day, the cow eats 1/60 and the goat eats 1/90 of the initial supply of grass. Therefore, the cow and the goat together eat in one day

$$\frac{1}{60} + \frac{1}{90} = \frac{1}{36}$$

of the grass supply. Hence, the cow and the goat can eat all of the grass in 36 days while the goose eats only the grass growth in the same number of days. Answer: the cow, goat, and goose can eat all of the grass (that grows uniformly every day) in 36 days.

Chapter 3 NUMBER THEORY

Mathematics is the queen of the sciences and
number theory is the queen of mathematics.
Carl Friedrich Gauss

Why are numbers beautiful? It's like asking why
is Beethoven's Ninth Symphony beautiful.
If numbers aren't beautiful, nothing is.
Paul Erdös

Number theory is a vast and very attractive field of mathematics that studies the properties of whole numbers. Numbers have fascinated people from the dawn of civilization. Euclid (*ca.* 300 B.C.) showed that there are infinitely many prime numbers, the amicable (friendly) pair (220,284) was known to the early Pythagoreans. Today, primes and prime factorization, Diophantine equations and many functions (for example, Riemann zeta function) make up an especially important area in number theory.

Number theory is full of many profound, subtle and beautiful theorems. A number of results has a simple and comprehensible formulation, yet puzzling nature. For example, Goldbach's conjecture (stated in 1742) that *"Every even integer greater than* 2 *can be written as the sum of two primes"* has a very simple formulation, but it has not been resolved yet. On the other hand, the proofs of problems are often very difficult and lie in exceeding obscurity. For these reasons, it can be claimed that number theory possesses magical charm and inexhaustible wealth.

Number theory, more than any branch of mathematics, has set traps for mathematicians and caused even some eminent mathematicians to make a number of faulty assumptions. Recall that Fermat's Last Theorem from 1637 (*"If an integer n is greater than* 2, *then the equation $a^n + b^n = c^n$ has no solutions in nonzero integers a, b, and c."*) was solved after 357 years and many wrong "proofs".

From the beginning of the computer era, programmers have been testing their skills, the quality of the programs and the power of digital computers solving problems of number theory and discovering various curiosities in this field. Generating prime numbers is the ultimate test in the construction of digital computers because extensive calculations quickly point out different problems in their design.

No other branch of mathematics is as popular among mathematicians—amateurs as number theory. The main reason is that it does not require any lengthy preliminary training. Number theory is also the favorite science of leading mathematicians. A German mathematician Leopold Kronecker (1823–1891) said: *"Number theorists are like lotus-eaters—having tasted this food they can never give it up."* Lotus-eaters are a mythical people, mentioned in Homer's epic *The Odyssey.* Constantly eating lotusfruit, these people became oblivious to the outer world, and lived in contented indolence, with the only desire never to leave the Lotus-land. Kronecker is the same fellow who once said that, *"God made integers and all else is the work of man."*

This chapter attempts to present a few of the recreational gems in the theory of numbers. You will find mostly the problems of Diophantine's type, starting from the *cattle problem.* This is the most famous and oldest ancient problem of number theory, ascribed to Archimedes. While Archimedes thought about cattle, Dirac chose a monkey and sailors, Ramanujan houses, Bhāskara soldiers on the battlefield, Sylvester and Frobenius stamps and coins and Euler horses and bulls. An essay on amicable numbers and suitable generating formulae of ibn Qorra and Euler are also included.

*

* *

Archimedes (280 B.C.–220 B.C.) (→ p. 299)

As mentioned in the preface, many centuries ago the main purpose of most mathematical tasks, excepting some counting and measuring problems, was to provide intellectual pleasure and diversion. Archimedes, one of the greatest mathematicians who ever lived, was mainly occupied with real-life problems and geometry, although some of these problems sound today rather as recreational ones. The extant works of Archimedes can be found in Heath [100], Dijksterhuis [52] and Stein [169].

Some of ancient stories are, in fact, legends about Archimedes' contrivances made to aid the defense of his native city Syracuse against the siege directed by the Roman general Marcellus. According to legend, Archimedes constructed movable poles for dropping heavy weights and boiling liquids on enemy ships that approached to the city walls too closely, catapults with adjustable ranges and other defense weapons.[1]

[1] Another legend tells that Archimedes also constructed a large "burning glass mirror"

FIGURE 3.1. Catapult—Archimedes' defense weapon

We regard many of these problems from antiquity and the Middle Ages more as recreational mathematics. For this reason, we include them in this book, starting with some of Archimedes' most famous problems.

Archimedes
280 B.C–212 B.C

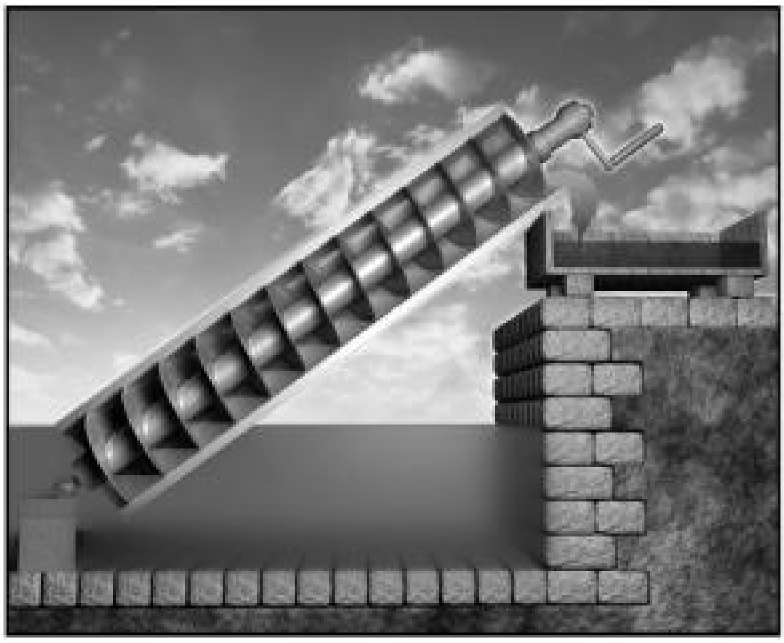
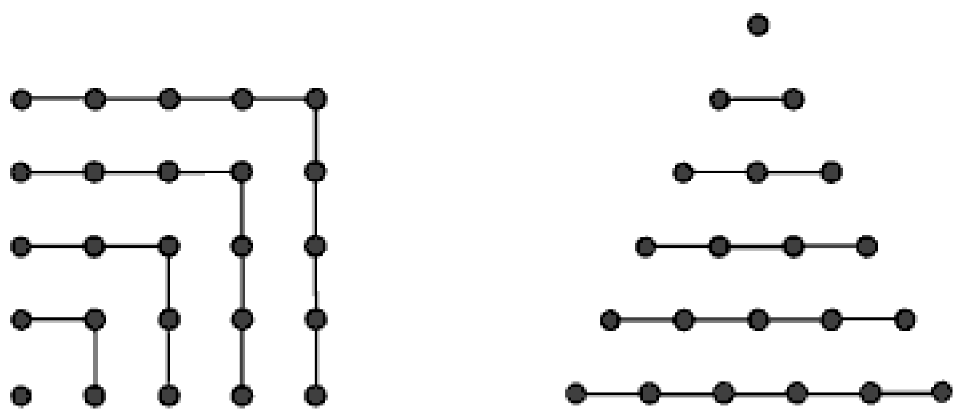


FIGURE 3.2. Archimedes' screw, still used in various parts of the world

in the shape of a paraboloid to set fire to enemy ships. Perhaps the previous legends are true, but the last story seems to be false. With a little help of physics and mathematics we find that the length of the *latus rectum* (the straight line through the focus perpendicular to the axis) is equal to the parameter p in the equation of parabola $y^2 = px$. Since the focus of the parabola $y^2 = px$ is at $(p/4, 0)$, assuming that an enemy ship is 50 meters from the city walls and situated just at the focus of Archimedes' burning mirror ($p/4 = 50$ meters), we find that the mirror's diameter (that is, the length of the latus rectum) would measure $p = 200$ meters, evidently an impossible length.

The ancient Greeks were deeply interested in numbers that resemble geometric forms, called figurative numbers. The Pythagoreans observed that the sum of successive odd natural numbers forms a square (Figure 3.3(a)), thus $S_n = 1 + 3 + \cdots + 2n - 1 = n^2$. Triangular numbers are the numbers 1, 3, 6, 10, ..., and can be obtained as the sum of the first n natural numbers (see Figure 3.3(b)). Therefore, the nth triangular number has the form $T_n = 1 + 2 + \cdots + n = n(n+1)/2$.



a) b)

FIGURE 3.3. Square and triangular numbers

There are infinitely many numbers that are simultaneously square and triangular; see Problem 3.2. Surprisingly, these numbers are very closely related to the solution of Ramanujan's Problem 3.11, given in this chapter.

The sums P_n of consecutive triangular numbers are the tetrahedral numbers

$$P_1 = 1,\ P_2 = 4,\ P_3 = 10,\ P_4 = 20,\ \ldots, P_n = \tfrac{1}{6}n(n+1)(n+2)$$

and these are three-dimensional analogies to the plane-figurative numbers (actually, the numbers of equal spheres that can be piled in pyramids, hence the denotation P). Are there numbers that are simultaneously square and tetrahedral except 1, that is, $S_n = P_m$ for some natural numbers n and m greater than 1? This is the so-called *cannonball problem* and the answer is yes, but there is *only* one number: 4,900. This unique solution of the Diophantine equation

$$n^2 = \frac{1}{6}m(m+1)(m+2)$$

was found by G. N. Watson in 1918.[2] The proof of this is difficult and we omit it.

[2] *The problem of the square pyramid*, Messenger of Mathematics 48 (1918), 1–22.

The first problem presented in this chapter most likely comes from Archimedes and involves square and triangular numbers.

Cattle problem

In 1773 Gotthold Ephraim Lessing discovered a Greek epigram made up of twenty-two distichs ascribed to Archimedes, in the Wolfenbüttel library. These verses state a problem, now commonly referred to as the *cattle problem*. This problem is the best known ancient Diophantine equation and reads:

Problem 3.1. *The sun god had a herd of cattle consisting of* W, X, Y, Z (*respectively*) *of white, black, spotted, and brown bulls and* w, x, y, z *cows of the same shades. These numbers satisfy the following nine equations:*

$$
\begin{aligned}
W &= \Big(\frac{1}{2}+\frac{1}{3}\Big)X + Z,\\
X &= \Big(\frac{1}{4}+\frac{1}{5}\Big)Y + Z,\\
Y &= \Big(\frac{1}{6}+\frac{1}{7}\Big)W + Z,\\
w &= \Big(\frac{1}{3}+\frac{1}{4}\Big)(X + x),\\
x &= \Big(\frac{1}{4}+\frac{1}{5}\Big)(Y + y),\\
y &= \Big(\frac{1}{5}+\frac{1}{6}\Big)(Z + z),\\
z &= \Big(\frac{1}{6}+\frac{1}{7}\Big)(W + w),\\
W + X &= \text{a square number},\\
Y + Z &= \text{a triangular number}.
\end{aligned}
$$

Determine the total number of cattle

$$T = W + X + Y + Z + w + x + y + z.$$

After somewhat tedious but elementary manipulations[3], the above system reduces to Pell's equation (see Appendix A)

$$u^2 - 4{,}729{,}494v^2 = 1.$$

[3]See, *e.g.*, B. Krumbiegel, A. Amthor, *Das Problema Bovinum des Archimedes*, Historisch-literarische Abteilung der Zeitschrift für Mathematik und Physik 25 (1880), pp. 121–136, 153–171, É. Callandreau, *Célèbres Problèmes Mathématiques*. Éditions Albin Michel, Paris 1949.

Solving this equation by the continued fraction method[4], one obtains

$$
\begin{aligned}
u &= 109{,}931{,}986{,}732{,}829{,}734{,}979{,}866{,}232{,}821{,}433{,}543{,}901{,}088{,}049,\\
v &= 50{,}549{,}485{,}234{,}315{,}033{,}074{,}477{,}819{,}735{,}540{,}408{,}986{,}340,
\end{aligned}
$$

leading to the solutions of the original problem, the smallest value of which contains 206,545 digits. A. Amthor (1880) was the first to determine the total number of digits of the smallest solution, although he did not find the solution. In 1889, undaunted by what lay before them, a civil engineer named A. H. Bell and two friends formed the Hillsboro Mathematical Club (Illinois) and started the computation. They spent four years at the job, computing 32 of the left-hand and 12 of the right-hand digits of the 206,531-digit number that directly leads to the required (least) solution consisting of 206,545 digits.

The final solution by H. C. Williams, R. A. German and C. R. Zarnke [187] was published in 1965. The authors performed calculations on the digital computers IBM 7040 and IBM 1620 using a special procedure for memorizing very large integers and converting them from the binary system to the decimal system. The total computing time required was 7 hours and 49 minutes and the obtained result consisting of 206,545 decimal digits was printed on 42 computer A4 sheets. In its abbreviated form, the solution reads

$$77602714\ldots237983357\ldots55081800,$$

each of the six dots representing 34,420 omitted digits. Using a CRAY-1 supercomputer, which greatly simplified the computations, H. L. Nelson [132] confirmed the result in 1980. Readers interested in the exact number of cattle may find all 206,545 digits in the paper [132].

Let us note that Lessing and also Nesselmann, Rouse Ball and others, disputed the authorship of Archimedes. Some of them have commented that the solution is senseless, giving remarkably huge numbers. According to Donald Knuth (Science 194 (1976)), the total number of protons and neutrons in the known universe is about 10^{125}, an "astronomically large" number, but actually it has only 126 digits; compare it with the above solution consisting of 206,545 digits. Beiler estimated in [14] that a sphere with a radius equal to the distance from the Earth to the Milky Way could contain only a small number of animals even if they were the smallest microbes. On the other hand, the outstanding Danish researcher of Archimedes' work, Johan L. Heiberg, as well as some other mathematicians, were convinced that the above problem should be attributed to Archimedes.

[4]For this method see, *e.g.*, Davenport's book [45]. See, also, Appendix A.

Further details about the *cattle problem* can be found in the recent work [123] of Lenstra. He presents all solutions to the cattle problem given below:

$$w = 300{,}426{,}607{,}914{,}281{,}713{,}365 \cdot \sqrt{609} + 84{,}129{,}507{,}677{,}858{,}393{,}258 \cdot \sqrt{7{,}766}\ ,$$

$$k_m = \frac{(w^{4,658\cdot m} - w^{-4,658\cdot m})^2}{368{,}238{,}304} \quad (m = 1, 2, 3, \dots)$$

mth solution	*bulls*	*cows*	*all cattle*
white	$10{,}366{,}482 \cdot k_m$	$7{,}206{,}360 \cdot k_m$	$17{,}572{,}842 \cdot k_m$
black	$7{,}460{,}514 \cdot k_m$	$4{,}893{,}246 \cdot k_m$	$12{,}353{,}760 \cdot k_m$
spotted	$7{,}358{,}060 \cdot k_m$	$3{,}515{,}820 \cdot k_m$	$10{,}873{,}880 \cdot k_m$
brown	$4{,}149{,}387 \cdot k_m$	$5{,}439{,}213 \cdot k_m$	$9{,}588{,}600 \cdot k_m$
all colors	$29{,}334{,}443 \cdot k_m$	$21{,}054{,}639 \cdot k_m$	$50{,}389{,}082 \cdot k_m$

Archimedes would be amazed at this result, but these days the cattle problem is an easy-as-pie task even on a standard personal computer lasting only a fraction of a second to solve the corresponding Pell's equation (see Appendix A).

Finally, let us mention an interesting discussion which took place in the pages of the journal *Historia Mathematica* about the solvability of the cattle problem. A controversy arises from an unclear translation of the Greek text. Namely, P. Schreiber [154] asserted that the known "solutions" contradict one of the conditions (which reads: *"In each sort of cattle there are many more bulls than cows"*) given in the wording of the problem. According to Schreiber, this condition fails for the brown cattle. W. C. Waterhouse [180] disputed this approach since he regards some parts of the Greek text only as a stylized poetic form which is mathematically irrelevant, while mathematical conditions are written in straightforward mathematical terms. At any rate, the task posed at the beginning of this essay is mathematically quite clear and precise[5] and has a solution.

Problem 3.2.* *Devise a formula that generates all numbers that are simultaneously square and triangular.*

Hint: Consider the corresponding Pell's equation and formulae given in Appendix A.

[5]Many mathematicians, even today, consider that the presented formulation of the cattle problem was created at some later period, many centuries after Archimedes; in their estimation, mathematicians from that time would have had great difficulties with this problem.

Diophantus of Alexandria (*ca.* 200–*ca.* 284) (→ p. 299)

Diophantus' book *Arithmetica* was written in the third century in 13 books (six survived in Greek, another four in medieval Arabic translation). It is a collection of 130 arithmetic problems with numerical solutions of (determinate and indeterminate) algebraic equations. The following two problems are selected from this book.

Dividing the square

Problem 3.3. *Divide a given square number into two squares.*

Solution. This is an indeterminate problem whose solution Diophantus expressed in the form of a quadratic polynomial which must be a square. Let b be a given rational number and let $x^2 + y^2 = b^2$, where x and y are rational solutions of the last equation. To ensure a rational solution, Diophantus introduced the substitution $y = ax - b$, where a is an arbitrary rational number.[6] Then

$$b^2 - x^2 = a^2x^2 - 2abx + b^2,$$

which reduces to $2abx = (a^2 + 1)x^2$. Hence

$$x = \frac{2ab}{a^2 + 1}.$$

The last formula generates as many solutions as desired. Taking $b = 4$ as in Diophantus' book, for $a = 2$ it follows that $x = 16/5$, $y = 12/5$, which satisfies the given equation:

$$\left(\frac{16}{5}\right)^2 + \left(\frac{12}{5}\right)^2 = \frac{400}{25} = 16.$$

Beside the fame of being the most prominent work on algebra in Greek mathematics, Diophantus' *Arithmetica* is also famous because of the note made in a copy of it by the renowned French mathematician Fermat in which he states the impossibility of dividing a cube into a sum of two cubes or, in general, any nth power ($n > 2$) into a sum of two nth powers. In the margin of the 1621 edition of this book, Fermat wrote: *"I have a truly marvelous proof of this proposition which this margin is too narrow to contain."* This

[6] Diophantus did not employ this exact notation since negative numbers and zero were not known in his time.

note was discovered after Fermat's death. It is believed today that Fermat did not actually have the correct proof of this conjecture. Peter Bentley, the author of *The Book of Numbers* [16] said: *"Today Fermat is remembered most for what he did not write down."*

Wine problem

Problem 3.4.* *A man bought several liters of two kinds of wines. He was paying* 8 *drachmas* (*Greek money*) *a liter of fine wine and* 5 *drachmas a liter of ordinary wine. The total sum of money he paid is equal to the square of a natural number and, added to* 60, *it gives the square of the total quantity of the wine expressed in liters. The task is to find the quantities* (*in liters*) *of each kind of wine.*

Tābit ibn Qorra (826–901) (→ p. 300)

The Arabian scientist Tābit ibn Qorra (or Qurra, following V. Katz [113, p. 249][7]), famous for his remarkable translations of Euclid (*Elements*), Apollonius, Archimedes, Ptolomy and Theodosius, wrote on elementary algebra, conics and astronomy, but also on fanciful topics such as magic squares and amicable (or friendly) numbers.

Amicable numbers

The discovery of *amicable* or *friendly numbers* is ascribed to Greek mathematicians. It is said that two numbers are *amicable* if and only if the sum of the proper divisors (the divisors excluding the number itself) of each of them is equal to the other. Pythagoras is attributed to finding one such pair **220** and **284**. Indeed, the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110, and their sum is 284, while the sum of the proper divisors 1, 2, 4, 71, 142 of 284 is just 220. This was the only known pair for some 2000 years. So, this extraordinary property of the two numbers was the cause of a superstition that a pair of talismans bearing these numbers would ensure a perfect friendship between the wearers.

Pythagoras set the amicable pair (220,284) (the only one known in his time) in the context of a discussion of friendship saying that a friend is *"one who is the other I such as are* 220 *and* 284.*"* According to Beiler [14], there was once an experiment performed by El Madschrity, an Arab of the eleventh

[7] Also Korrah, see Beiler [14].

century, who tested the erotic effect of the amicable pair (220,284) by giving one person 220 special cakes to eat, and himself eating 284 at the same time. Unfortunately, El Madschrity did not report the outcome of this experiment.

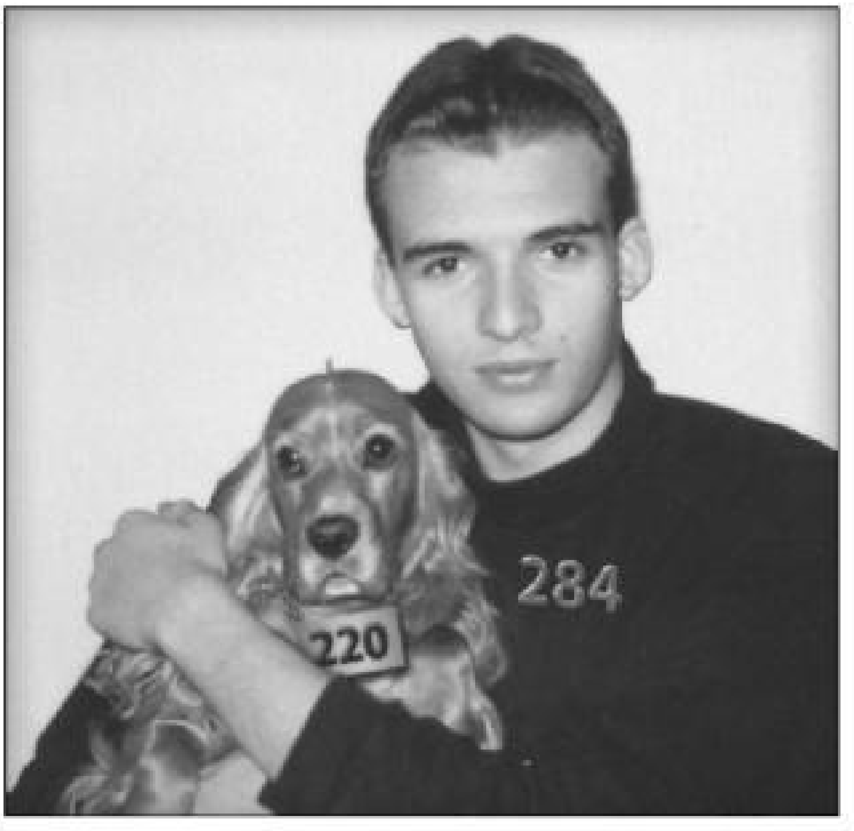


FIGURE 3.4. Amicable numbers 220 and 284

No advance in the field was made until the ninth century when Tābit ibn Qorra undertook the study of the following question:

Problem 3.5. *State a one-parameter formula which generates amicable numbers.*

As a result of ibn Qorra's investigation, the following theorem was proved (stated here in modern notation, see [32, p. 104] or [113, p. 266]):

Ibn Qorra's theorem. *For* $n > 1$, *let* $q_n = 3\cdot 2^n - 1$, $r_n = 9\cdot 2^{2n-1} - 1$. *If* q_{n-1}, q_n *and* r_n *are three prime numbers, then* $A = 2^n q_{n-1} q_n$ *and* $B = 2^n r_n$ *are amicable.*

Qorra's rule for generating amicable pairs was rediscovered by Fermat (1636) and Descartes (1638). The simplest case $n = 2$ gives primes $q_1 = 5$, $q_2 = 11$, $r_2 = 71$, and the resultant pair of amicable numbers is (**220**,**284**), which was already known to the Greeks. Despite the existence of such a fruitful formula, no new pair of amicable numbers was discovered until another Islamic mathematician Kamāl al-Dīn al-Fārisī (died *ca.* 1320) announced (**17,296**, **18,416**) as a second pair. Three centuries later the great French mathematician Pierre de Fermat rediscovered the same pair. This pair can be obtained from ibn Qorra's formula for $n = 4$. The next pair of amicable numbers generated by ibn Qorra's formula is (**9,363,584, 9,437,056**), which is obtained for $n = 7$.

Let us note that, after a systematic study of amicable numbers, Leonhard Euler gave (1747) a list of 30 pairs and later he extended it to 64 pairs (two of which were shown later to be unfriendly). Euler stated the following rule for generating amicable pairs (Dickson [51]):

If three natural numbers

$$\begin{aligned} p &= 2^m(2^{n-m}+1)-1,\\ q &= 2^n(2^{n-m}+1)-1,\\ r &= 2^{n+m}(2^{n-m}+1)^2-1 \end{aligned}$$

are all prime for some positive integer m *satisfying* $1 \le m \le n-1$, *then the numbers* $2^n pq$ *and* $2^n r$ *are an amicable pair.*

Euler's rule is a generalization of ibn Qorra's formula (the case $m = n-1$). The first few pairs (m, n) which satisfy Euler's rule are $(m, n) = (1, 2), (3, 4), (6, 7), \ldots$, generating the amicable pairs (**220**, **284**), (**17,296**, **18,416**), (**9,363,584**, **9,437,056**), the same ones found by ibn Qorra's formula. Another, more powerful, generating formula was discovered in 1972 by W. Borho [22].

Curiously enough, the above lists and other lists made up to 1866 do not include the relatively small pair of amicable numbers, **1,184** and **1,210**. This pair was discovered in 1866 by a sixteen-year-old Italian boy, Niccolo Paganini, not to be confused with the composer and violinist of the same name! This example, as well as many other examples, show that ibn Qorra's, Euler's and Borho's formula do not generate all pairs of amicable numbers.

In recent years, the number of known amicable numbers has grown explosively thanks to powerful computer machines. For example, there are 1,427 amicable pairs less than 10^{10} and 5,001 less than 3.06×10^{11}. Pomerance [140] proved that

$$[\text{amicable numbers} \le n] < n \exp\bigl(-[\ln n]^{1/3}\bigr)$$

for large enough n. In 2005 an improved bound is given by M. Kobayashi, P. Pollack and C. Pomerance [116]. In 2005 P. Jobling found the largest known amicable pair (at that time) each member of which has 24 073 decimal digits.

Problem 3.6.* *Find the fourth amicable pair produced by Euler's rule.*

Bhāskara (1114–1185) (→ p. 300)

Bhāskara (or Bhāskaracharya—"Bhaskara the Teacher"—as he is known in India) was the greatest Indian mathematician and astronomer of the twelfth century. He also worked in the astronomical observatory at Ujjain as his great predecessor Brahmagupta had done. Bhāskara wrote mainly on arithmetic, algebra, mensuration and astronomy. He tried to clear up basic arithmetic operations involving zero, the difficult task solved partially by Brahmagupta a half century previously. Actually, Brahmagupta could not understand what to do with division by zero; see [16]. Bhāskara claimed that, say, $7:0$ is infinity[8], an incorrect answer. The division of any number by zero produces an *undefined* result. Poor Bhāskara, one could comment, but recall that some modern textbooks also assert that the result is infinity.

Bhāskara's work *Lilāvati* (a woman's name meaning "lovely" or "beautiful") is considered one of the most influential contributions of his time. Indeed, much of our knowledge of Hindu arithmetic comes from this book.

There is a romantic myth connected with Bhāskara's book *Lilāvati*. According to a story which appears in the work of Fyzi (1587), a counselor of the Persian emperor Akbar, the astrologers predicted that Bhāskara's daughter *Lilāvati* would never marry. Bhāskara, however, being an expert astronomer and astrologer, divined a lucky moment for his daughter's marriage to fall at a certain hour on a certain propitious day. On that day he devised a water clock by floating a cup in a vessel of water. At the bottom of the cup he pierced a small hole in such a way that water would trickle in and sink the cup at the end of the hour. Shortly before the hour's end, as curious Lilāvati watched the rising water level sink the cup, a pearl from her headdress accidentally dropped into the water clock and, stopping up the hole in the cup, slowed the influx of water. Thus the hour expired without the cup sinking. The lucky moment passed unnoticed and Lilāvati was thus fated never to marry. To console his unhappy daughter, Bhāskara promised to write a book, saying:[9] *"I will write a book named in your honor that shall last until the end of time, for she who has a good name obtains a second life and this in turn shall lead to eternal existence."*

How many soldiers?

Problem 3.7. *An army consists of* 61 *phalanxes each containing a square number of soldiers, not counting the commander. During the battle this army*

[8]Today we denote infinity by the symbol ∞; some people call it "lazy eight." This symbol was introduced by the English mathematician John Wallis in 1655.

[9]D. E. Smith [167], Vol. I, p. 277.

can rearrange itself into a solid square, including its commander. Find the minimum number of soldiers in the army.

In mathematical terms, the equation $61y^2 + 1 = x^2$ must be solved in the smallest terms. This equation, written in a more familiar form

$$x^2 - 61y^2 = 1 \tag{3.1}$$

is known as Pell's equation (see Appendix A). Bhāskara gave the particular solutions of equations of the form $x^2 - py^2 = 1$ for $p = 8,\ 11,\ 32,\ 61$, and 67. For solving the equation (3.1) Bhāskara used a general method (see [113, pp. 223–225]) and found the solution

$$x = 1{,}776{,}319{,}049, \quad y = 22{,}615{,}390.$$

The above task has been included in Beiler's book [14, Ch. XXII] in the form of a short story about the Battle of Hastings in 1066. In this battle the Norman army led by William the Conqueror, Duke of Normandy, defeated King Harold's Anglo-Saxon army. After this victory the Normans occupied England and William became the first Norman king. We present this story in an adapted form below.

Problem 3.8. *King Harold's men arrayed themselves in their customary formation of sixty-one squares, each square containing the same number of soldiers. Any Norman soldier who dared to penetrate their fortifications put his life at great risk; one blow from a Saxon battle-axe would shatter a Norman lance and slice through a coat of mail... . When Harold and his men joined forces with the other Saxons, they entered the battle as one single great square... .*

Can you find the number of men in the Saxon horde? And don't for one moment think that the Battle of Hastings led to Pell's equation.

As mentioned above, Bhāskara found that the solution of $x^2 - 61y^2 = 1$ in its smallest terms is

$$1{,}766{,}319{,}049^2 - 61 \cdot 226{,}153{,}980^2 = 1.$$

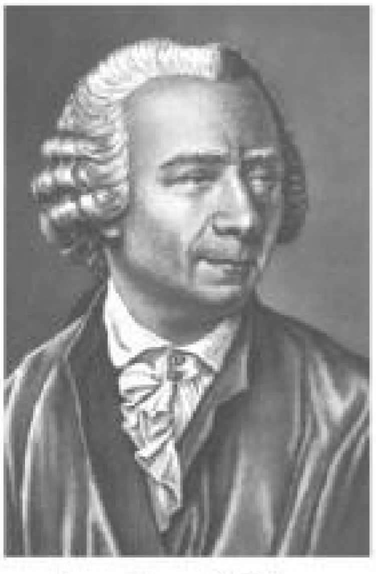
Including Harold, the Anglo-Saxon army must have consisted of

$$x^2 = (1{,}766{,}319{,}049)^2 = 3{,}119{,}882{,}982{,}860{,}264{,}401$$

soldiers. Using rough calculations and allowing for six soldiers per square meter, the solution would require a globe with a radius equal to the distance from the Earth to the Moon in order to accommodate the army on its surface.

Leonhard Euler (1707–1783) (→ p. 305)

In terms of the sheer volume of published papers, the Swiss mathematician Leonhard Euler ranks as the most prolific mathematician of all times. Euler's bibliography, including posthumous items, contains 886 entries which could fill some 80 large-format volumes. Euler worked with amazing ease under all sorts of conditions, even in the presence of all thirteen of his children (although only five survived to adulthood). He made remarkable contributions in almost all branches of mathematics and also in astronomy, mechanics, optics and many practical problems such as magnetism, cartography, fire engines and ship building.



Leonhard Euler
1707–1783

In 1766, an illness at the age of 59 left Leonhard Euler nearly blind. Yet thanks to his remarkable mental calculation skills and photographic memory (for example, Euler could repeat the Aeneid of Virgil from beginning to end without hesitation), he continued his work on mathematics, optics, and lunar motion for the last 17 years of his life, thus producing almost half of his total works. Asked after an unsuccessful eye operation how he would continue working, he answered: *"At least now, nothing will distract my attention."*

Not only did Leonhard Euler work in virtually all areas of mathematics, he was also greatly interested in recreational mathematics problems. His research on this topic led to the development of new methods as well as the development of new branches of mathematics.

Horses and bulls—a Diophantine equation

If a, b and c are integers and $ab \neq 0$, then a linear equation of the form

$$ax + by = c,$$

where unknowns x and y are also integers, is called a *linear Diophantine equation in two variables.* Euler developed an efficient method for solving these types of equations, which consist of multiple applications of a very simple procedure.[10] We illustrate the method in the solution of the following problem that Euler himself posed.

[10] According to F. Cajori [32, p. 95], the famous Hindu astronomer Āryabhata (born 476 A.D.) knew this method, known by the name *pulverizer.*

Problem 3.9. *A merchant purchases a number of horses and bulls for the sum of* 1,770 *talers. He pays* 31 *talers for each bull, and* 21 *talers for each horse. How many bulls and how many horses does the merchant buy?*

The linear Diophantine equation that corresponds to our problem reads

$$31x + 21y = 1{,}770,$$

where x and y stand for the number of bulls and horses, respectively. To solve this equation, we apply Euler's method by first starting with the unknown whose coefficient is smaller in magnitude, in this case y. Using the equalities

$$31 = 1 \cdot 21 + 10 \quad \text{and} \quad 1{,}770 = 84 \cdot 21 + 6,$$

we obtain

$$y = \frac{-31x + 1{,}770}{21} = -x + 84 + \frac{-10x + 6}{21}.$$

Since x and y are integers, then $(-10x + 6)/21$ should also be an integer. Denote this integer with t and represent the expression for t as a new Diophantine equation

$$21t = -10x + 6,$$

which has coefficients of smaller magnitude relative to the original equation. In fact, the essence of Euler's method is a permanent decrease of coefficients in Diophantine equations arising from the original equation. Repeating the described recursive procedure, we solve the previous equation in x (due to the smaller coefficient) and find that

$$x = \frac{-21t + 6}{10} = -2t + \frac{-t + 6}{10} = -2t + u,$$

where we put

$$u = \frac{-t + 6}{10}.$$

Hence we again obtain the Diophantine equation

$$t = -10u + 6.$$

Taking u as a parameter, we have an array of solutions,

$$\begin{aligned} t &= -10u + 6, \\ x &= -2t + u = 21u - 12, \\ y &= -x + 84 + t = -31u + 102. \end{aligned}$$

The last two equalities, where u is a nonnegative integer, give all solutions (x, y) of Euler's equation

$$31x + 21y = 1{,}770.$$

First, for $u = 0$, one obtains $x = -12,\ y = 102$, which is pointless because we need only positive solutions. Since $y < 0$ for $u > 3$, we consider only those solutions obtained for $u = 1, 2, 3$, and find

$$\begin{aligned}
\boldsymbol{u} &= 1, \quad x = 9, \quad\ \ y = 71;\\
\boldsymbol{u} &= 2, \quad x = 30, \quad y = 40;\\
\boldsymbol{u} &= 3, \quad x = 51, \quad y = 9.
\end{aligned}$$

Thus, the solution of Euler's equation is not unique, and all three pairs (9,71), (30,40), (51,9) satisfy this equation.

Paul Dirac (1902–1984) (→ p. 309)

The sailors, the coconuts, and the monkey

In 1926 the *Saturday Evening Post* printed a short story about some sailors and coconuts on a desert island that held a great deal of attraction for its readers. Martin Gardner presented a full discussion of a story similar to it in the form of a problem in [71].

Problem 3.10. *A shipwreck maroons five sailors and a monkey on a desert island. To survive, they spend each day collecting coconuts; at night of course, they sleep. During the night, one sailor wakes up and, doubtful of receiving his fair share, he divides the coconuts into five equal piles; he notices that one coconut remains. He gives this coconut to the monkey, hides his share and goes back to sleep. A little later, the second sailor wakes up, and, having the same doubts, divides the remaining coconuts into five equal piles, notices that there is one left over, which he gives to the monkey. The second sailor hides his share and returns to sleep. This happens for each sailor in turn. In the morning, the sailors awake and share the remaining coconuts into five equal piles and see that there is one left over, which they give to the monkey. Of course, all of the sailors notice that the pile of coconuts is smaller than the previous day, but they each feel guilty and say nothing. The question is: What is the smallest number of coconuts that the sailors could have collected?*

Supposing that the sailors collected n coconuts, we must then solve the following system of equations

$$\begin{aligned} n &= 5n_1 + 1, \\ 4n_1 &= 5n_2 + 1, \\ 4n_2 &= 5n_3 + 1, \\ 4n_3 &= 5n_4 + 1, \\ 4n_4 &= 5n_5 + 1, \\ 4n_5 &= 5n_6 + 1. \end{aligned} \tag{3.2}$$

The unit added to each equation represents the single remaining coconut that the sailors give to the monkey each time after dividing the pile. (Lucky monkey!) All the unknowns in this system are certainly positive integers.

FIGURE 3.5. Sailors, coconuts and a monkey

We can continue solving the problem using a number of approaches. The three methods that we describe each have interesting and instructive aspects.

Solution (I). By the process of elimination, we reduce the above system of equations to the single Diophantine equation

$$1{,}024n = 15{,}625n_6 + 11{,}529.$$

To solve this equation we use Euler's method described on page 51. First we find that

$$n = \frac{15{,}625n_6 + 11{,}529}{1{,}024} = 15n_6 + 11 + \frac{265n_6 + 265}{1{,}024} = 15n_6 + 11 + x,$$

where we put $x = (265n_6 + 165)/1{,}024$. Hence we obtain a new Diophantine equation

$$265n_6 + 265 = 1{,}024x. \tag{3.3}$$

After dividing (3.3) by 265 we find

$$n_6 = 3x - 1 + \frac{229x}{265} = 3x - 1 + y, \quad \text{with} \quad \frac{y}{x} = \frac{229}{265}.$$

Since the numbers 229 and 265 have no common factors, we must take

$$y = 229k, \ x = 265k \ (k \in \mathbb{N}),$$

and for $k = 1$ we obtain $x = 265$. Substituting this value into equation (3.3) we find $n_6 = 1{,}023$ so that the required number of coconuts is given by

$$n = 15n_6 + 11 + x = 15 \cdot 1{,}023 + 11 + 265 = 15{,}621.$$

Solution (II). David Sharpe [160] used congruences to solve the above system. We recall that, if a, b, c are integers with $c > 0$ and $a - b$ is divisible by c, then we can express this fact in the language of congruences as $a \equiv b \pmod{c}$. In other words, we say that a modulo c is equal to b. For example, 25 modulo 7 is equal to 4 (written as $25 \equiv 4 (\text{mod } 7)$) since $25 - 4$ is divisible by 7, or 4 is the remainder that results when 25 is divided by 7.

From the last equation of system (3.2) we see that $4n_5 - 1$ is divisible by 5. Then

$$4n_5 + n_5 - n_5 - 1 = 5n_5 - (n_5 + 1)$$

is also divisible by 5; hence it follows that the second addend $n_5 + 1$ must be divisible by 5. Using congruence notation we write $n_5 \equiv -1 \pmod{5}$. After multiplying by 5 one obtains $5n_5 \equiv -5 \pmod{25}$. Since $4n_4 = 5n_5 + 1$, from the last relation we find that

$$4n_4 \equiv -4 \pmod{25}.$$

The numbers 4 and 25 are relative primes (that is, having no common factors) so that 4 can be cancelled to yield

$$n_4 \equiv -1 \,(\mathrm{mod}\ 25).$$

Using the same argumentation we are able to write the following congruence relations:

$$\begin{aligned}
5n_4 &\equiv -5 \,(\mathrm{mod}\ 125),\\
4n_3 &\equiv -4 \,(\mathrm{mod}\ 125),\\
n_3 &\equiv -1 \,(\mathrm{mod}\ 125),\\
5n_3 &\equiv -5 \,(\mathrm{mod}\ 625),\\
4n_2 &\equiv -4 \,(\mathrm{mod}\ 625),\\
n_2 &\equiv -1 \,(\mathrm{mod}\ 625),\\
5n_2 &\equiv -5 \,(\mathrm{mod}\ 3{,}125),\\
4n_1 &\equiv -4 \,(\mathrm{mod}\ 3{,}125),\\
n_1 &\equiv -1 \,(\mathrm{mod}\ 3{,}125),\\
5n_1 &\equiv -5 \,(\mathrm{mod}\ 15{,}625),\\
n &\equiv -4 \,(\mathrm{mod}\ 15{,}625).
\end{aligned}$$

From the last relation we see that the number $n+4$ is divisible by 15,625. Since we search for the smallest number with this property, we take $n = 15{,}621$. Therefore, the smallest number of coconuts that the sailors could have collected is 15,621.

Solution (III). Certain references[11] attribute this fascinating solution to the celebrated English physicist Paul Dirac (1902–1984), with Erwin Schrödinger, winner of the 1933 Nobel Prize. Other sources (for example, [62, p. 134]) suggest the key idea's creator was Enrico Fermi (1901–1954), another Nobel Prize-winning physicist and the "father" of the atom bomb.

First, one should determine the smallest possible number M having these properties: (i) M is divisible by 5; (ii) the difference $M - M/5$ is also divisible by 5; (iii) the described process can be repeated six times consecutively. One can easily find that $M = 5^6 = 15{,}625$. We will obtain a set of solutions, including the smallest one, by adding this number, as well as any multiple $p \cdot 5^6$ ($p \in \mathbb{N}$), to any number satisfying the problem's conditions.

[11] Including the article *Paul Dirac and three fishermen*, published in the journal *Kvant*, No. 8 (1982), issued by the Academy of Science of the (former) USSR.

Dirac based his idea on manipulating the quantity of coconuts using negative numbers (!), and determined that -4 coconuts yielded a proper solution. Although it seems rather surprising, this number satisfies, at least in theory, all conditions of the problem. This is one of those "it doesn't-make-sense-but-it-works" ideas. Checking the solution is simple, and we leave further discussion to the reader. The number of coconuts is given by

$$n = -4 + p \cdot 15{,}625.$$

Taking $p = 1$ we obtain the smallest number of coconuts $n = 15{,}621$.

Problem 3.10 can be generalized; see Gardner [72, Ch. 9]. If s is the number of sailors and m is the number of coconuts which are left over to the monkey (after each division), then the number n of coconuts originally collected on a pile is given by

$$n = ks^{s+1} - m(s-1), \tag{3.4}$$

where k is an arbitrary integer. In particular (Problem 3.10), for $s = 5$ and $m = 1$ from (3.4) we obtain the smallest positive solution (taking $k = 1$) $n = 15{,}621$.

Beiler's book [14] contains a version of the coconuts problem in which the pile of coconuts remaining after five consecutive nights of reallocations, gets divided into five exactly equal piles, and the monkey goes without his share of coconuts. The least number of coconuts equals 3,121.

Srinivasa Ramanujan (1887–1920) (→ p. 308)

The life and work of the Indian self-taught genius Srinivasa Ramanujan certainly provides the history of mathematics with one of its most romantic and most spectacular stories. Although Ramanujan had no university education, his exceptional capacity in mathematics was known to Indian mathematicians. The famous British number theorist G. H. Hardy (1877–1947) observed his uncanny intuitive reasoning and incredible facility to state very deep and complicated number relations and brought Ramanujan to England. Despite Ramanujan's delicate health, they jointly authored several remarkable papers before Ramanujan's death at the young age of 33.

Srinivasa Ramanujan
1887–1920

In his book *Ramanujan* (1940) Hardy described an interesting story. When Hardy went to the hospital to visit Ramanujan who was lying ill, he told him that he had arrived in taxi-cab No. 1,729, and remarked that the number seemed to him a rather dull one, quite an unfavorable omen. *"No"*, Ramanujan answered immediately, *"it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways,* $1{,}729 = 1^3 + 12^3 = 9^3 + 10^3$."

Let us allow a short digression. The above splitting of the number 1,729 into the sum of two cubes resembles very much a problem by Henry Dudeney posed in his famous book, *Canterbury Puzzles* (1907). He asked for the number 9 to be expressed as the sum of two cubes of rational numbers in two different ways. One solution is pretty obvious: $9 = 1^3 + 2^3$. The other is extremely difficult:

$$\frac{415{,}280{,}564{,}497}{348{,}671{,}682{,}660} \quad \text{and} \quad \frac{676{,}702{,}467{,}503}{348{,}671{,}682{,}660}$$

(found by Dudeney). Truly mind-boggling results! Bearing in mind that Dudeney did not have any modern calculating devices, his solution arouses admiration. "Thus did Dudeney."

It is a high time to give the next intriguing problem.

Unknown address

Problem 3.11. *The houses with numbers* 1 *through* N *are situated in a row only on one side of a street. Find the address (house) numbered by* n *such that the house numbers on one side of the address* n *(that is, from* 1 *to* $n-1$*) add up exactly to the same sum as all numbers (from* $n+1$ *to* N*) on the other side of the address.*

According to the story presented by Robert Kanigel in his book *The Man Who Knew Infinity* [109], Ramanujan answered the "unknown address" problem in his head communicating the solution in the form of a continued fraction to his friend. Paul Weidlinger in [184] gives the solution to the problem and assumes that it coincides with Ramanujan's.

From the conditions of the problem, one obtains

$$1 + 2 + \cdots + (n-1) = (n+1) + \cdots + N.$$

By adding the sum $1 + 2 + \cdots + n$ to both sides we find that

$$2(1 + 2 + \cdots + n - 1) + n = 1 + 2 + \cdots + N,$$

which gives

$$2 \cdot \frac{n(n-1)}{2} + n = \frac{N(N+1)}{2} \tag{3.5}$$

or

$$n_k^2 = \frac{N_k^2 + N_k}{2}. \tag{3.6}$$

The subscript index k is introduced to emphasize that relation (3.5) has integer solutions only for some specific numbers n_k and N_k. Obviously, (3.6) is a Diophantine equation that defines the square of triangular numbers (for triangular numbers see page 40).

Solving the quadratic equation (3.6) in N_k, we find positive solutions for N_k :

$$N_k = \frac{-1 + \sqrt{1 + 8n_k^2}}{2}. \tag{3.7}$$

The expression $1 + 8n_k^2$ must be a perfect square, that is,

$$1 + 8n_k^2 = p_k^2 \quad \text{or} \quad p_k^2 - 8n_k^2 = 1,$$

which is a Pell equation (see Appendix A).

It is easy to see that the least solutions of the last Pell equation $x^2 - 8y^2 = 1$ are $x = 3$ and $y = 1$. From the discussion presented in Appendix A and the form of difference equation (A.6) given there, we conclude that the difference equation corresponding to the Pell equation above has the form

$$n_k - 6n_{k-1} + n_{k-2} = 0 \quad (k \geq 2). \tag{3.8}$$

Since $n_0 = N_0 = 0$ and $n_1 = N_1 = 1$ for obvious reasons, using these initial values we can find n_k from (3.8), and hence N_k given by (3.7).

From the difference equation (3.8) we have

$$c_k = \frac{n_k}{n_{k-1}} = 6 - \frac{1}{c_{k-1}}.$$

Hence, by a successive continuation, we come to the continued fraction

$$6 - \cfrac{1}{6 - \cfrac{1}{6 - \cfrac{1}{6 - \cdots}}}\,. \tag{3.9}$$

This means that $a_0 = a_1 = \cdots = a_k = \cdots = 6$ and $b_1 = b_2 = \cdots = b_k = \cdots = -1$ and (3.9) can be written in the form $\left[6; \frac{-1}{6}, \frac{-1}{6}, \cdots\right]$. At the same time, the difference equation (A.3) from Appendix A becomes $y_k - 6y_{k-1} + y_{k-2} = 0$, which coincides with (3.8). In particular, Euler's formulae (A.2) become

$$P_k = 6P_{k-1} - P_{k-2},$$
$$Q_k = 6Q_{k-1} - Q_{k-2},$$

and, therefore, satisfy the previous difference equation. Since Q_k also satisfies the initial conditions $Q_{-1} = 0$, $Q_0 = 1$, the values of Q_k coincide with n_k required in (3.6) and (3.8), and both of them give the solution to the problem.

Therefore, expression (3.9) is the continued fraction which Ramanujan most likely had in mind. Depending on the number of terms taken in the continued fraction (3.9), particular solutions can be obtained from (3.8) taking $n_0 = 0$ and $n_1 = 1$. Numerical values are given in the following table.

k	0	1	2	3	4	5	6
n_k	0	1	6	35	204	1,189	6,930
N_k	0	1	8	49	288	1,681	9,800

For example, for $n_k = 6$ we have $N_k = 8$ and the following arrangement of houses with the sum equal to 15:

$$\underbrace{1\quad 2\quad 3\quad 4\quad 5}_{15}\quad \boxed{6}\quad \underbrace{7\quad 8}_{15}$$

The value of n_k can be found by solving the difference equation (3.8). Bearing in mind the initial conditions $n_0 = 0$, $n_1 = 1$, formula (A.5) (Appendix A) yields the required solution (with $p = 3$, $q = 1$) and thus reads

$$n_k = \frac{1}{4\sqrt{2}}\Big[(3+2\sqrt{2})^k - (3-2\sqrt{2})^k\Big] \quad (k = 0, 1, \dots).$$

Since $|3-2\sqrt{2}\,| \approx 0.172$ is less than 1, then $(3-2\sqrt{2})^k$ becomes exponentially small when k increases, and its effect is almost negligible. For this reason n_k can be calculated as (see page 13)

$$n_k = \lfloor (3+2\sqrt{2})^k/4\sqrt{2} + 1/2 \rfloor,$$

where $\lfloor a \rfloor$ presents the greatest integer less than or equal to a number a.[12] For example,

$$n_4 = \lfloor (3+2\sqrt{2})^4/4\sqrt{2} \rfloor = \lfloor 204.000153 \rfloor = 204.$$

[12] The *floor* function $\lfloor x \rfloor$ was introduced by Kenneth E. Iverson early in the 1960s.

Ferdinand Georg Frobenius (1849–1917) (→ p. 307)

James Joseph Sylvester (1814–1897) (→ p. 307)

We end this section with an interesting task known in the literature as the Frobenius' coin problem or Sylvester's stamp problem, or sometimes the money-changing problem. Its first version has a puzzle character, but its later generalizations led to numerous challenging problems in number theory that have appeared as the subject of many papers even today.

Georg Frobenius, a professor at the University of Berlin, was one of the leading mathematicians of his day. He continuously endeavored to preserve very high standards in sciences and education at the university. To carry out his plans, he was occasionally unpleasant, choleric and quarrelsome. He paid attention only to pure mathematics and had an aversion to applied mathematical disciplines.[13] Despite his primary orientation to serious mathematics, Frobenius' work finds a place in this book thanks to his research on the coin problem belonging to number theory which bears his name.

James Joseph Sylvester, one of Britain's greatest mathematicians of the nineteenth-century, was a genius in mathematical investigation, but his absentmindedness often complicated matters for him. Sylvester not only encountered great difficulties in memorizing other mathematicians' results, but he also had troubles remembering his own. Once he even denied the truth of a theorem that he himself had formulated. Due to his perpetual absentmindedness, his lectures did not enjoy great popularity, concentrating, as they did, mainly on his own work and its results. Consequently, unsolved problems or unclear details from Sylvester's unpublished manuscripts inevitably caused delays sometimes lasting for weeks or even permanently diverting him from his original subject.


James J. Sylvester
1814–1897

Sylvester entertained a great deal of interest in many branches of mathematics, yet despite devoting immense activity to solving serious mathematical problems,[14] he managed to find a little time to spare for mathematical

[13]It sounds paradoxical, but the representation theory of finite groups, developed by Frobenius—the man who loved only pure mathematics—later found important application in quantum mechanics.

[14]Moreover, Sylvester was also interested in poetry and music as an amateur performer. Sometimes, the famous French composer Charles Gounod gave him singing lessons.

amusements. In this book we present several of his problems, the first of which is given below.

Stamp combinations

Sylvester submitted the following simple but interesting contribution to the *Educational Times* (1884). A similar problem was considered by Georg Frobenius.

Problem 3.12. *A man possesses a large quantity of stamps of only two denominations: 5-cent stamps and 17-cent stamps. What is the largest (finite) amount of cents which the man cannot make up with a combination of these stamps?*

Clearly, we assume that very large amounts of money (theoretically, infinitely large amounts) are not taken into account since they could not be paid out for lack of a sufficient supply of stamps.

Let z be the amount that should be paid off by a combination of stamps which are 5 and 17 pounds worth. Then

$$z = 5a + 17b, \tag{3.10}$$

and this is a linear Diophantine equation. The amount z is a natural number and a and b are nonnegative integers. We need to determine the greatest number z_0 which cannot be represented in the form (3.10).

Instead of the described particular problem we will consider a more general case, namely, we will prove the following assertion:

If p and q are relative prime natural numbers,[15] *then $pq - p - q$ is the greatest integer which cannot be represented in the form $pa + qb$, $a, b \in \mathbb{N}_0$.*

To solve this problem, we will prove: 1° the number $pq - p - q$ cannot be represented in the mentioned form; 2° every number $n > pq - p - q$ can be represented in the form $pa + qb$.

1° Let us assume, contrary to our proposition, that there exist numbers $a, b \in \mathbb{N}_0$, such that $pq - p - q = pa + qb$. Since p and q are relatively prime, it follows that p *divides* $b + 1$ and q *divides* $a + 1$. Thus, it must be $b + 1 \neq p$,

[15]Two natural numbers p and q are relatively prime if they have no common positive divisors except 1, that is, if the *greatest common divisor* (gcd) of p and q is 1, briefly $\gcd(p, q) = 1$.

otherwise $a < 0$. Similarly, it must be $a + 1 \neq q$. Therefore, $p < b + 1$ and $q < a + 1$, that is, $b > p - 1$ and $a > q - 1$, so that

$$pq - p - q = pa + qb > p(q-1) + q(p-1) = 2pq - p - q.$$

Hence, $pq > 2pq$, which is impossible. The contradiction proves the statement 1°.

2° Let $n > pq - p - q$ and let (a_0, b_0) be an arbitrary integer solution of the Diophantine equation $pa + qb = n + p + q$. Since p and q are relatively prime, *i.e.* $\gcd(p,q) = 1$, this solution exists according to the theorem of the existence of solutions of linear Diophantine equations ($\gcd(p,q)$ divides the right side of equation). The general solution of the above Diophantine equation is given by

$$a = a_0 + qt, \quad b = b_0 - pt, \quad t \in \mathbb{Z}.$$

Let us choose t so that $0 \leq a < q$. Then $t < \dfrac{q - a_0}{q}$ and

$$b > b_0 - p\frac{q - a_0}{q} = \frac{a_0 p + b_0 q - pq}{q} = \frac{n - (pq - p - q)}{q} > 0.$$

We have proved that the number n can be represented in the described manner. Indeed, from the Diophantine equation we have $n = p(a-1) + q(b-1) = pa' + qb'$ $(a', b' \in \mathbb{N}_0)$.

In the special case (Sylvester's problem), taking $p = 5$ and $q = 17$, we find that the greatest number (amount) which cannot be obtained by a linear combination of the numbers 5 and 17 is equal to $z_0 = pq - p - q = 5 \cdot 17 - 5 - 17 = 63$, and this is the solution of Sylvester's problem.

The above puzzle just begs to be generalized. Let $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ be given mutually relatively prime natural numbers such that $0 < a_1 < a_2 < \cdots < a_n$. Let us assume that the values a_i represent the denomination of n different coins and introduce the function

$$g(a_1, a_2, \ldots, a_n) = a_1x_1 + a_1x_2 + \cdots + a_nx_n,$$

where x_i are unknown nonnegative integers. Then the following question is very intriguing.

Problem 3.13. *Find the largest value $\mathcal{F}_n$ of the function $g(a_1, a_2, \ldots, a_n)$ for which the equation* (sometimes called Frobenius equation)

$$a_1x_1 + a_1x_2 + \cdots + a_nx_n = \mathcal{F}_n$$

has no solution in terms of nonnegative integers x_i.

The solution $\mathcal{F}_n$, if it exists, is called the *Frobenius number.* Problem 3.13 can be appropriately interpreted in terms of coins of denominations $a_1, \ldots, a_n$; the Frobenius number is simply the largest amount of money which cannot be collected using these coins.

As shown above, Sylvester found

$$\mathcal{F}_2 = g(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1 = a_1 a_2 - a_1 - a_2$$

and, in particular, for $a_1 = 5$ and $a_2 = 17$, one obtains $\mathcal{F}_2 = 63$. This theory cries out for a practical example as follows: *A gambler in a local casino in Nevada won* 375 *dollars. The croupier can take the money by putting chips worth* 12 *and* 35 *dollars into an automatic money-machine. Can he pay off the gambler?* Sylvester's results leads to the answer. The largest amount of money which the croupier cannot pay off is $\mathcal{F}_2 = g(12, 35) = (12 - 1)(35 - 1) - 1 = 373$ dollars. Therefore, the required payment of \$375 is feasible: 9 chips of \$35 and 5 chips of \$12 are needed.

Is it possible to find the Frobenius number $\mathcal{F}_n$ for $n \geq 3$? There are many papers that consider this question; see, *e.g.*, [4], [89], [91], [151], [158]. Explicit formulae for $n \geq 3$ have not yet been stated. However, Seimer and Beyer [159] came to a solution expressed by a continued fraction algorithm for $n = 3$. Their result was simplified by Rödseth [151] and later by Greenberg [89]. Davison [46] derived the relatively sharp lower bound

$$\mathcal{F}_3 \geq \sqrt{3a_1 a_2 a_3} - a_1 - a_2 - a_3. \tag{3.11}$$

A refined upper bound is given by Beck and Zacks [12].

However, no general formula is known for $n \geq 4$; there are only (relatively) fast algorithms that can solve higher order cases. Nevertheless, these algorithms become more complicated and slower as n increases, so that even the most powerful computers cannot find the result within a reasonable time for large n.

You are probably hungry after intriguing stories about Frobenius' coins and Sylvester's stamps and this makes a good excuse to visit a McDonald's restaurant and buy some boxes of McDonald's Chicken McNuggets.[16] We will find there that the boxes contain 6, 9 or 20 nuggets. If you take many friends with you, you will want a lot of nuggets. There is no problem at all with the fulfilling this requirement, as according to Schur's theorem any sufficiently large number can be expressed as a linear combination of relatively

[16] Chicken McNuggets are a fast food product available in the restaurant chain McDonald's.

prime numbers, and 6, 9 and 20 are relatively prime. Now we come to the problem of Frobenius' type.

Problem 3.14.* *Determine the largest (finite) number of nuggets which cannot be delivered using the boxes of* 6, 9 *and* 20 *nuggets.*

Answers to Problems

3.2. We have to solve the Diophantine equation $S_n = T_m$, that is,

$$n^2 = \frac{m(m+1)}{2}.$$

Hence we obtain the quadratic equation $m^2 + m - 2n^2 = 0$ whose positive solution is

$$m = \frac{-1 + \sqrt{1 + 8n^2}}{2}.$$

The number m will be a natural number if $1+8n^2$ is a perfect square, say, $1 + 8n^2 = r^2$, or $r^2 - 8n^2 = 1$, which is, surprisingly, the same Pell equation that appears in the solution of Ramanujan's Problem 3.11. This equation can be solved using the facts given in Appendix A, but we use the solution of Problem 3.11 that reads thus:

$$n_k = \frac{1}{4\sqrt{2}}\Big[(3 + 2\sqrt{2})^k - (3 - 2\sqrt{2})^k\Big] \quad (k = 1, 2, \dots) \tag{3.12}$$

(see page 59). The superscript index k indicates that there are infinitely many solutions of the considered Pell equation. It remains to find n_k^2. Squaring (3.12), we find the required formula for the kth square triangular number:

$$n_k^2 = \frac{\big(17 + 12\sqrt{2}\big)^k + \big(17 - 12\sqrt{2}\big)^k - 2}{32} \quad (k = 1, 2, \dots).$$

Hence we obtain the infinite series: 1, 36, 1,225, 41,616, 1,413,721, ...

3.4. Let x and y denote the quantities of wine expressed in liters. Then we can form the following system of equations:

$$\begin{aligned} 5x + 8y &= n^2, \\ n^2 + 60 &= z^2, \\ z &= x + y, \end{aligned}$$

which reduces to the Diophantine equation

$$z^2 - 5z - 60 = 3y.$$

We have to determine integer solutions z and y of the last equation.

Since $y < z$ we have

$$z^2 - 5z - 60 < 3z, \quad \text{that is,} \quad z^2 - 8z - 60 < 0,$$

wherefrom we find that z must satisfy $z < 12.717...$. The lower bound follows from the inequality

$$z^2 - 5z - 60 \geq 3 \cdot 1, \quad \text{that is,} \quad z^2 - 5z - 63 \geq 0,$$

which gives $z > 10.821...$. The only integers in the interval $[10.821, 12.717]$ are $z = 11$ and $z = 12$. For these values of z, we find

$$y = 8, \ x = 4 \quad \text{and} \quad y = 2, \ x = 9.$$

Therefore, there are two pairs of solutions to the wine problem.

3.6. The fourth amicable pair generated by Euler's rule is obtained for $m = 1$ and $n = 8$. We first calculate $p = 257$, $q = 33,023$ and $r = 8,520,191$ and then find the amicable pair (**2,172,649,216**, **2,181,168,896**).

3.14. To shorten our solution procedure, we use Davison's lower bound (3.11) and find

$$\mathcal{F}_3 \geq \sqrt{3 \cdot 6 \cdot 9 \cdot 20} - 6 - 9 - 20 \approx 21.92 > 21.$$

Therefore, we will search for candidates only among numbers greater than 21.

A graphical approach is convenient for finding $\mathcal{F}_3$. Let x be any positive multiple of 20 and let y (> 21) be any positive multiple of 3 (excepting 3 itself). Obviously, y can be represented as a linear combination of 6 and 9 for $y \geq 6$. Now we form the grid of points in the xy coordinate plane with the coordinates (x, y). Each straight line $x + y = k$ represents a total number of k nuggets; see Figure 3.6. A full line passes through at least one grid point, say (x^*, y^*), indicating that $k = x^* + y^*$ is the number of nuggets that can be delivered. On the other hand, a dashed line does not go through any grid point so that the corresponding number k of nuggets cannot be delivered.

From Figure 3.6 we observe that the "highest" dashed line is $x + y = 43$, so that the number 43 is a candidate for $\mathcal{F}_3$.

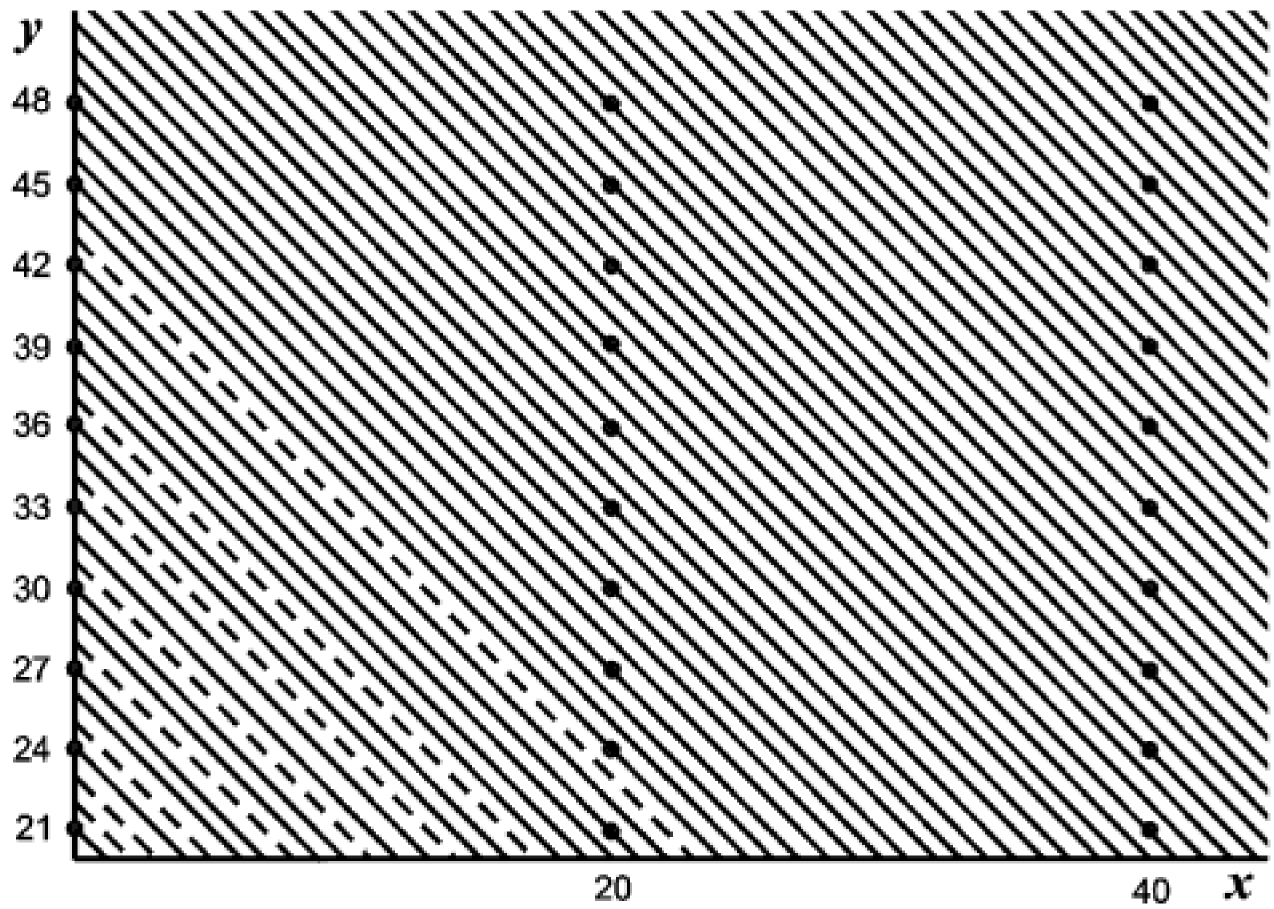


FIGURE 3.6. Graphical solution of the Chicken McNuggets problem

It remains to check some numbers greater than 43. Let $A_1 = \{44, 45, 46, 47, 48, 49\}$ be the first group of six consecutive numbers. The following linear combinations of 6, 9 and 20 are possible (not necessarily uniquely):

$$\begin{aligned}
44 &= 6 + 9 + 9 + 20,\\
45 &= 9 + 9 + 9 + 9 + 9,\\
46 &= 6 + 20 + 20,\\
47 &= 9 + 9 + 9 + 20,\\
48 &= 6 + 6 + 9 + 9 + 9,\\
49 &= 9 + 20 + 20.
\end{aligned}$$

Evidently, the elements of the next group $A_2 = \{50, 51, 52, 53, 54, 55\}$ can be obtained by adding 6 to the corresponding numbers from A_1, the group A_3 can be formed in the same way from A_2, and so on. Therefore, the number 43 is the largest number (=Frobenius number) of nuggets that can indeed not be delivered.

Chapter 4 GEOMETRY

Nature is an infinite sphere
whose center is everywhere and
whose circumference is nowhere.
Blaise Pascal

Equations are just the boring part of mathematics.
I attempt to see things in terms of geometry.
Stephen Hawking

Elementary geometry that we learn in the primary and high school is essentially Euclidean geometry axiomatized in Euclid's masterpiece the *Elements* written *ca.* 300 B.C. It is interesting to note that Euclid did not use the word "geometry" in his treatise to keep his distance from problems related to land measure. Namely, the term geometry ($\gamma\epsilon\omega\mu\epsilon\tau\rho\iota a$ in Ancient Greek) is combined of the Greek words *geo*= "earth" and *metria*= "measure." Another curiosity is that Euclid's *Elements* was the first printed mathematical book ever, published in Venice in 1482.

Geometry, one of the oldest sciences, studies problems concerned with size, shape, and relative position of lines, circles, triangles and polygons (plane geometry) as well as spheres and polyhedrons (solid geometry) and with properties of space. Among the oldest recreational problems are those which belong to plane and solid geometry. In this chapter we present a diverse set of geometrical puzzles and paradoxes posed and/or solved by mathematicians from Archimedes' time to the present day.

Most of the presented problems require merely the knowledge of elementary geometry and arithmetic. We did not include puzzles involving advance mathematics or contemporary geometry that studies more abstract spaces than the familiar Euclidean space. Geometrical fallacies are also excluded since their origin is unknown, although some historians of mathematics believe that Euclid prepared a collection of fallacies.

This chapter starts with the famous Archimedes' figure known as "shoemaker's knife" or *arbelos*. One of the related problems is Pappus' task on arbelos. In order to present its solution in an elegant and concise form, a geometric transformation called inversion is used. For the reader's convenience, some basic properties of the geometric inversion are given in Appendix B. Heron's problem on minimal distance demonstrates the law of reflection, established sixteen centuries later by Fermat and Snell. Heron's principle can be applied for solving Steiner's problem on the minimal sum of distances in a triangle, given later in this chapter. This problem was solved earlier by Fermat, Torricelli and Cavalieri.

You will also find several challenging problems of medieval mathematicians Brahmagupta, Mahāvira, Tābit ibn Qorra, Abu'l-Wafa (dissection of squares and triangles), Alhazen (circular billiard problem), Regiomontanus (distance of optimal viewpoint) and Kepler. Famous Dido's extremal problem or the classical isoperimetric problem, described by Rome's poet Virgil, is regarded as a forerunner of the much later established calculus of variations. We present Steiner's elegant solution of this problem. The geometric puzzle, popularly called "kissing circles", that attracted the attention of Descartes, Kowa, Soddy and Coxeter, is shown together with Coxeter's solution. Finally, we give Pólya's problem on the bisection of the area of a given planar region.

*

* *

Archimedes (280 B.C.–220 B.C.) (→ p. 299)

Arbelos problem

The following interesting elementary problem may be found in Archimedes' *Book of Lemmas* (see, *e.g.*, [61, p. 167]).

Problem 4.1. *The "shoemaker's knife" or arbelos is the region bounded by the three semicircles that touch each other. The task is to find the area which lies inside the largest semicircle and outside the two smallest (Figure 4.1, shaded portion).*

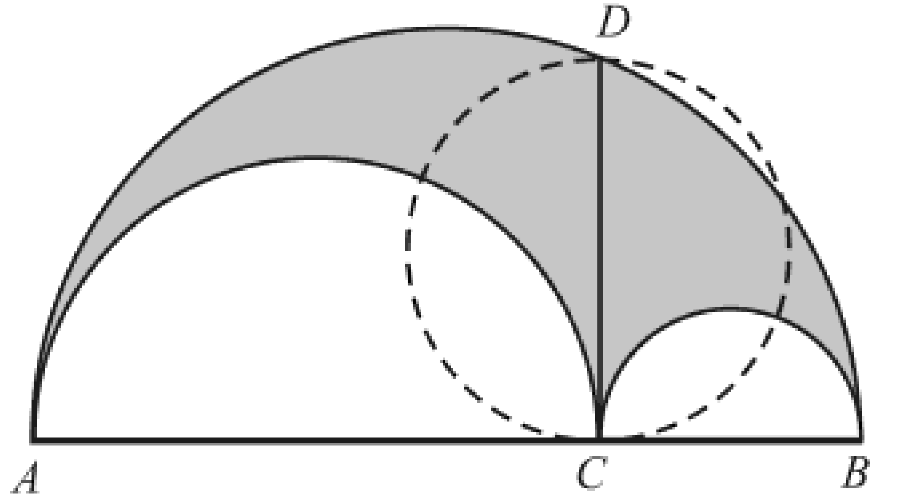


Figure 4.1. Arbelos problem

In the above-mentioned book, Archimedes demonstrated in Proposition 4 that if CD is perpendicular to AB, then the area of the circle with diameter CD is equal to the area of the arbelos.

This assertion is easy to prove. The triangle ABD is right-angled since the angle at D is 90° (as a peripheral angle which corresponds to the diameter AB). Then

$$(|AC| + |CB|)^2 = |AD|^2 + |DB|^2 = |AC|^2 + |DC|^2 + |CB|^2 + |DC|^2,$$

wherefrom $|DC|^2 = |AC| \cdot |CB|$, which is the well-known result related to the geometrical mean. Here $|\cdot|$ denotes the length of a segment.

Let $P(\frown)$ and $P(\bigcirc)$ be the areas of the arbelos and the circle with the diameter CD. According to the above relation we obtain

$$\begin{aligned} P(\frown) &= \frac{\pi}{8}|AB|^2 - \frac{\pi}{8}|AC|^2 - \frac{\pi}{8}|CB|^2 \\ &= \frac{\pi}{8}\Big((|AC| + |CB|)^2 - |AC|^2 - |CB|^2\Big) \\ &= \frac{\pi}{4}|AC| \cdot |CB| = \frac{\pi}{4}|CD|^2 = P(\bigcirc). \end{aligned}$$

Many amazing and beautiful geometric properties of the arbelos ($\acute{\alpha}\rho\beta\eta\lambda o\zeta$ in Greek) and its variants have fascinated mathematicians over the centuries. There is a vast literature concerning this subject; see, *e.g.*, [78], [173], as well as the papers by Bankoff [8], [9], Boas [21] and Dodge *et al.* [53], and references cited there. In this chapter we give some arbelos-like problems.

The next problem introduces circles called *Archimedean circles* or *Archimedean twins.*

Problem 4.2.* *Let us inscribe the circles K_1 and K_2 in the shaded regions ACD and CDB, respectively, where CD is perpendicular to AB; see Figure 4.2. Prove that the circles K_1 and K_2 have the same radii.*

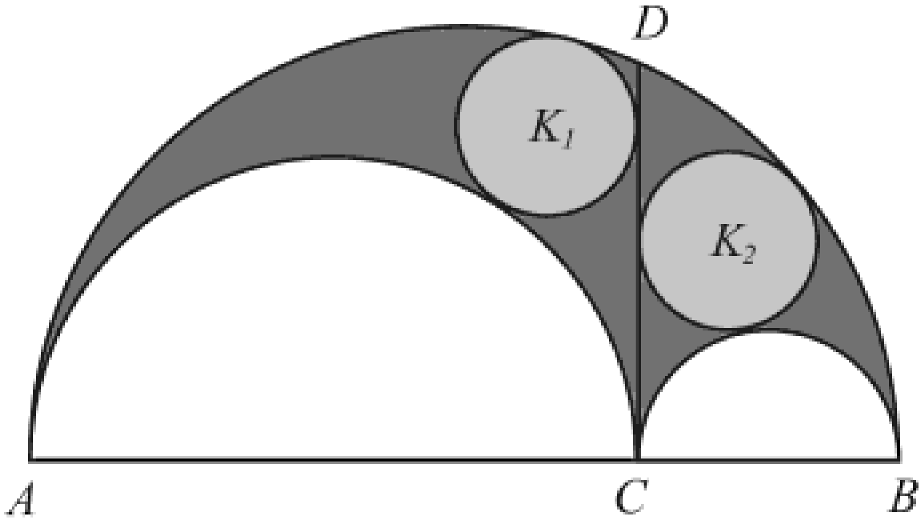


Figure 4.2. Archimedean twins

There are another amazing properties of Archimedean twins. We select the following one:

Problem 4.3.* *Let K be the smallest circumcircle of Archimedean twins K_1 and K_2, as shown in Figure 4.3. Prove that the area of K is equal to the area of the arbelos (or, according to Problem 4.1, the diameter of K is equal to $|CD|$).*

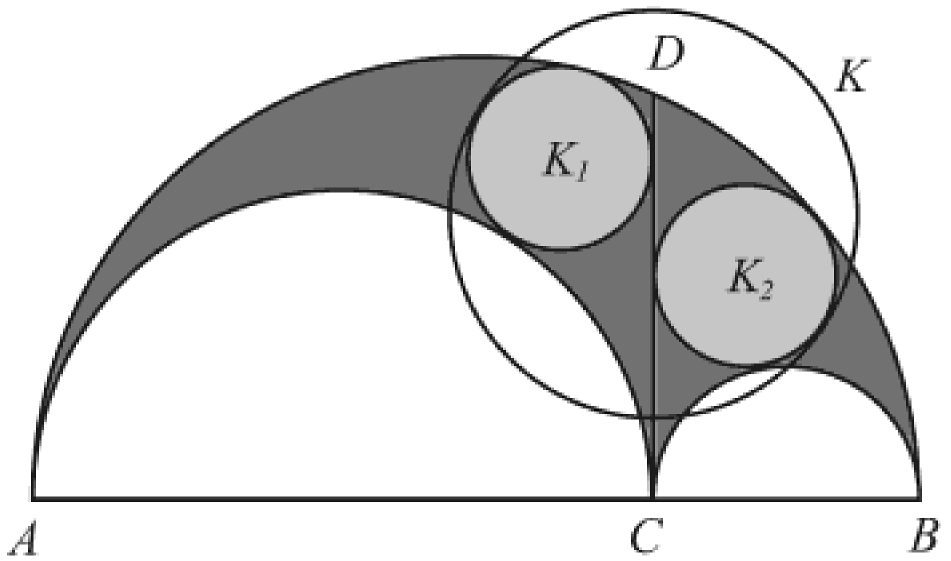


FIGURE 4.3. A variant of arbelos problem

We leave to the reader the pleasant work of finding the proof.

The well-known book *Synagoge* (Collection), written by Pappus of Alexandria who lived during the reign of Diocletian (284–305), contains another task on arbelos.

Problem 4.4. *Let the circles $k_1, k_2, k_3, \dots, k_n, \dots$ be inscribed successively as in Figure 4.4 so that they are all tangent to the semicircles over AB and AC, and successively to each other. Find the perpendicular distance from the center of the nth circle k_n to the base line AC.*

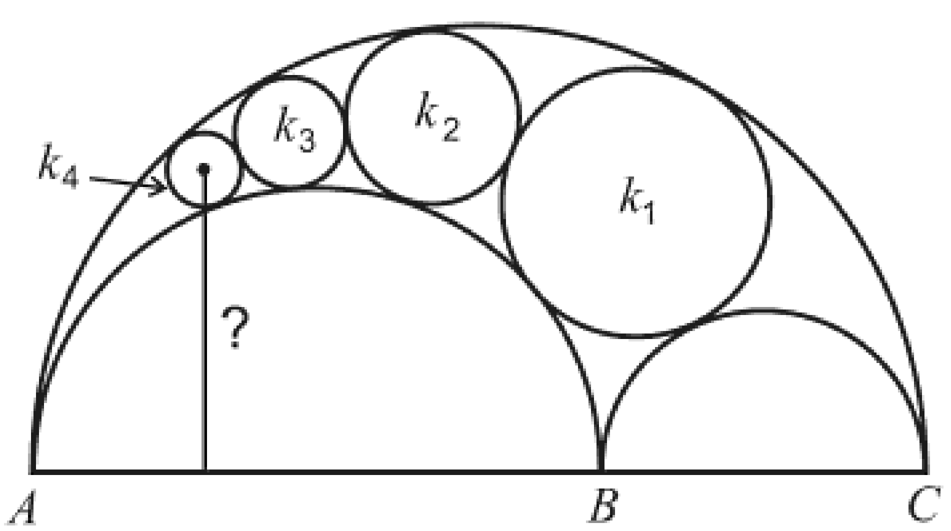


FIGURE 4.4. Pappus' arbelos-like problem

Pappus offered a proof that is too long to be presented here. Leon Bankoff, a mathematician who wrote many marvelous articles about the arbelos and its variants, gave a discussion of how Pappus proved this results in his paper, *How did Pappus do it?* [10]. The arbelos problem was considered in short by R. A. Johnson.[1] Thanks to a new geometric transformation called *inversion*, introduced by J. Steiner in 1824[2] and L. J. Magnus in 1831, we are able to demonstrate a very short and concise proof. The review of the basic properties of the geometric inversion, necessary for our proof, is given in Appendix B. For more details see, *e.g.*, Chapter 6 of H. S. M. Coxeter's book *Introduction to Geometry* [43].

To solve the arbelos problem above, we use the properties (1)–(6) of the inversion given in Appendix B. Since the circles p and q pass through the same point A (see Figure 4.5), this point will be chosen for the center of inversion. We take the radius ρ of the circle of inversion k in such a way that the nth circle k_n stays fixed. To do this, it is sufficient to choose the circle of inversion k to be orthogonal to the circle k_n (according to the property (2)).

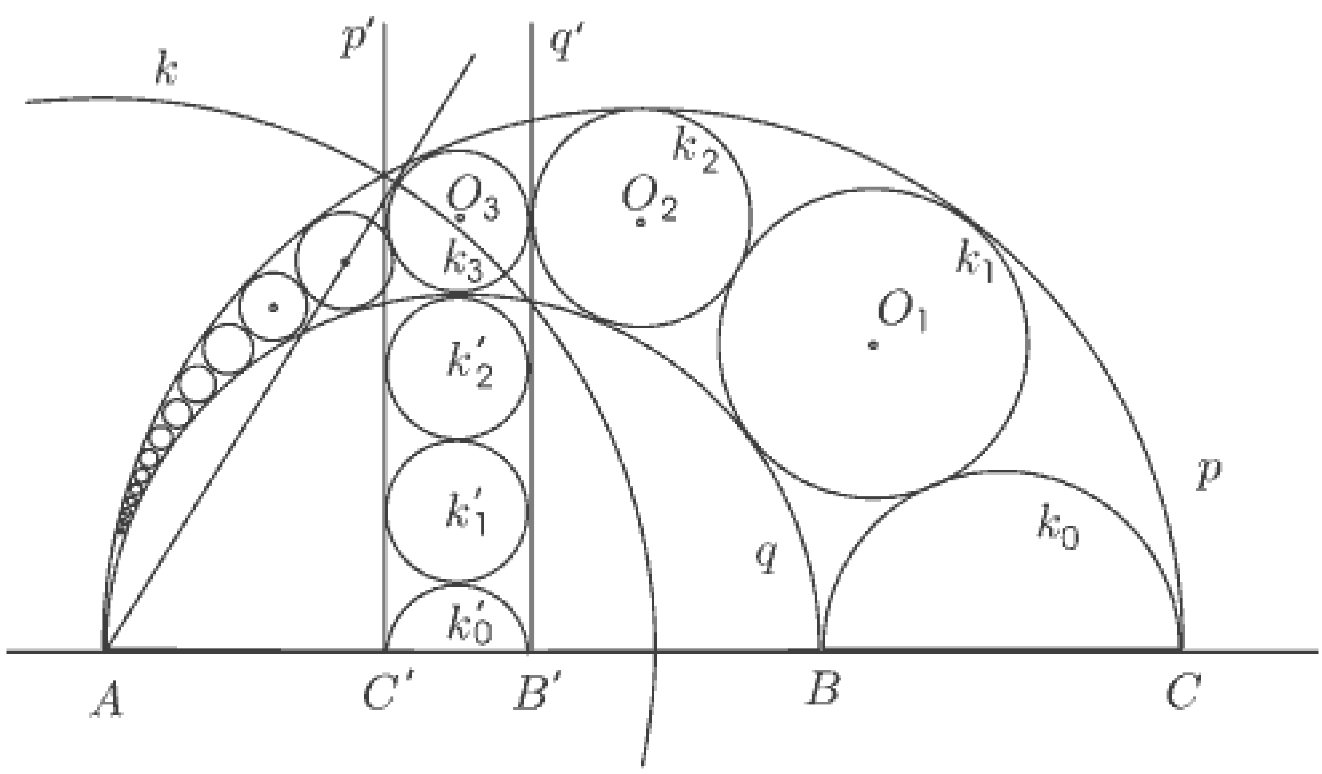


FIGURE 4.5. Solution of arbelos problem by inversion

For clarity, the idea of the proof will be demonstrated through a special example where $n = 3$ (Figure 4.5). The presented proof is, in essence, the same for arbitrary n.

[1] *Modern Geometry*, New York, Houghton Mifflin, 1929, p. 117.

[2] Steiner never published his idea on inversion, and its transformation was discovered independently by other contributors, among them the physicist William Thomson (Lord Kelvin) and the geometer Luigi Cremona (who gave a more general transformation).

Since the circles passing through the center of inversion invert into straight lines (property (4)), circles p and q will be transformed into straight lines that pass through the intersecting points of these circles with the circle of inversion k. Besides, both circles p and q touch the circle k_3 and they are orthogonal to the straight line AB. With regard to the property (6) it follows that these circles invert into the straight lines that touch simultaneously the fixed circle k_3. In addition, these lines are orthogonal to the straight line AB (that inverts into itself because of the property (1)). Hence we conclude that the circles p and q are transformed into the straight lines p' and q' as in Figure 4.5. We note that the points B and C are transformed into the points B' and C'.

Furthermore, the circles k_2, k_1 and k_0 touch the circles p and q so that, according to the property (6), they invert into the circles k_2', k_1' and k_0' touching the lines p' and q'. In this way we find the image k_2' of the circle k_2, that must touch k_3' and has the lines p' and q' as tangents. Similarly, the circle k_1' touches k_2' and k_0' touches k_1'. Their joint tangents p' and q' are parallel so that $r_3 = r_2' = r_1' = r_0'$, where r_m and r_m' denote the radii of the circle k_m and its inverse circle k_m'. (In general, $r_n = r_{n-1}' = \ldots = r_2' = r_1' = r_0'$). Therefore, the distance of the center O_3 of the circle k_3 from the straight line AC is

$$d_3 = r_3 + 2r_2' + 2r_1' + r_0' = 6r_3.$$

Following the same technique, for arbitrary nth circle k_n we can apply the inversion that leaves this circle fixed and inverts the circles $k_0, k_1, k_2, \ldots, k_{n-1}$ into circles of the same radius, arranged vertically under the nth circle k_n. Hence, the desired distance is

$$d_n = r_n + 2r_{n-1}' + \cdots + 2r_1' + r_0' = 2nr_n.$$

It is interesting to note that a similar problem of tangent circles appeared in a manuscript by Iwasaki Toshihisa (*ca.* 1775); see [167, Vol. II, p. 537].

Arbelos-like problems have attracted the attention of mathematicians from the time of early Greeks to the present; see Boas' paper [21]. George Pólya discussed the following variant in [142, pp. 42–44].

Problem 4.5.* *Two disjoint smaller circles lie inside a third, larger circle. Each of the three circles is tangent to the other two and their centers are along the same straight line. A chord of the larger circle is drawn to tangent both of the smaller circles. If the length of this chord is t, find the area inside the larger circle but outside the two smaller ones.*

At the end of this story of arbelos, we give a simple but challenging problem which resembles the arbelos, posed and solved by Li C. Tien in *The Mathematical Intelligencer* [175].

Problem 4.6.* *The centers of two small semicircles K_3 and K_4 of equal size lie on the diameter of a larger semicircle K. Two other circles K_1 and K_2 with diameters d_1 and d_2 touch each other, are both tangent to the big semicircle K and to one of the smaller circles K_3 and K_4 (see Figure 4.6). Prove that* $d_1 + d_2 = \text{const}$.

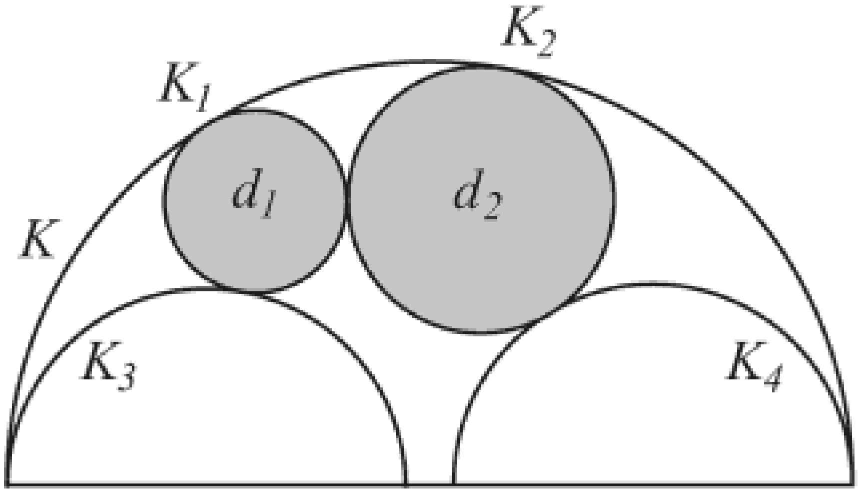


FIGURE 4.6. $d_1 + d_2 = \text{constant}$

Heron of Alexandria (*ca.* 65 A.D.–*ca.* 125 A.D.) (→ p. 299)

Minimal distance

Using a simple geometrical argument, the outstanding mathematician Heron of Alexandria demonstrated the law of reflection (known earlier to Euclid and Aristotle) in the work *Catoptrica* (Reflection). This book was written most likely in the first century A.D. (no sample of this book was kept). Heron showed that the quality of the angles of incidence and reflection is a consequence of the Aristotelian principle that "nature does nothing in vain,"[3] *i.e.*, that the path of the light ray from its source via the mirror to the eye must be the shortest possible. Heron considered the following problem which can be found today in almost any geometry textbook.

Problem 4.7. *A and B are two given points on the same side of a straight line k. Find the point C on k such that the sum of the distances from A to C and from C to B is minimal.*

[3] I. Thomas, *Selections Illustrating the History of Greek Mathematics*, Harvard University Press, Cambridge 1941, p. 497.

A geometrical solution of Heron's problem is as follows (see Figure 4.7).

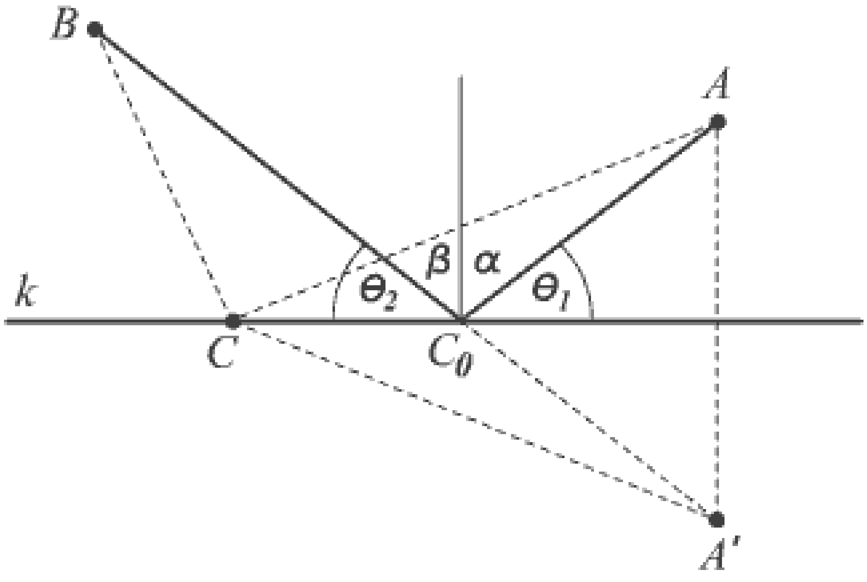


FIGURE 4.7. Heron's problem

Let A' be the point symmetric to A with respect to the straight line k, and let C be an arbitrary point on the line k. Join B to A' and denote with C_0 the intersecting point of the straight line BA' and the straight line k. Since the triangle $\triangle CAA'$ is isosceles, we have

$$|BC| + |CA| = |BC| + |CA'|.$$

Then

$$|BC| + |CA| = |BC| + |CA'| \geq |BA'| = |BC_0| + |C_0A|,$$

where the equality holds if C coincides with C_0.

The above relation expresses the solution of Heron's problem. The required point that provides the minimal sum of distances is the intersection of the given straight line k and the straight line drawn through the given point B and the point A' symmetrical to the second point A with respect to the straight line k.

In his book *On mirrors* Heron investigated the laws of reflection of light and applied his conclusions to problems related to the properties of mirrors. Consider a physical situation that illustrates Heron's problem. Let A be a light source, B a light receiver (for instance, the observer's eye), and the line k an intersection of a plane containing the points A and B and perpendicular plane of a mirror that serves as a reflecting surface. A light ray from A reflects and travels to the receiver B, but also seems to originate from the point A', symmetric to the point A with respect to the line k (the mirror image).

Heron confirmed in fact that a light ray reflects from a mirror in such a way that its path between a light source and a light receiver is the shortest distance possible. The required point C_0 in Heron's problem has the property that angle θ_1 is equal to angle θ_2. Moreover, angle α is equal to angle β, or, as is usually said, *the angle of incidence is equal to the angle of reflection.* This property is, in fact, *Fermat's principle of minimal time* or the *law of refraction* of light (established earlier experimentally by Dutch physicist and mathematician Willebrord Snell (1580–1626)). Historians of science consider that Heron's problem and its solution constituted the first application of the *extreme principle* in describing natural phenomena (see, for instance, Hildebrandt's book [102]).

Note that some modern textbooks cast Heron's problem as a practical problem. For example, the straight line k is turned into a rectilinear section of a railroad track, points A and B become towns, point C is called a railroad platform and the question is: *Where should one build the platform so that the combined length of the rectilinear roads linking the towns is minimal?*

In connection with Heron's problem, we present another geodesic problem posed by Henry E. Dudeney in his book *Modern Puzzles* (1926).

Problem 4.8. *A fly sits on the outside surface of a cylindrical drinking glass. It must crawl to a drop of honey situated inside the glass. Find the shortest path possible* (*disregarding the thickness of the glass*).

Denote the position of the fly and the drop of honey with points A and B. If these points lie on the same generatrix, the solution is obvious. Assume that this is not the case. We can "unroll" the surface of the glass to get a rectangle (see Figure 4.8(a)).

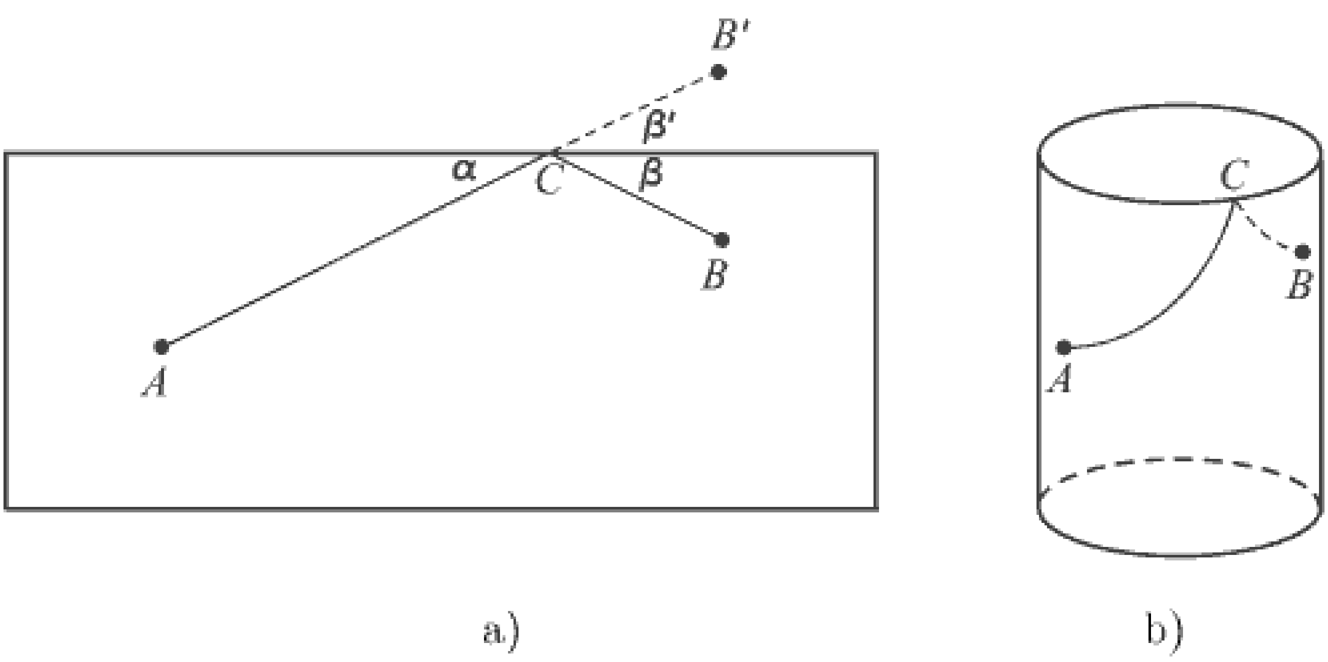


FIGURE 4.8. A fly and a drop of honey

The problem now reduces to the determination of the shortest path ACB, where C is the point on the edge of the glass. Actually, this is Heron's problem and the solution uses a point B' symmetric to B with respect to the edge of the glass. The angle of incidence α is equal to the angle of reflection β $(= \beta')$. The fly will crawl over the surface of the glass along two arcs of a cylindrical spiral (Figure 4.8(b)), crossing the edge of the glass at point C.

Heron's reflection principle can be successfully applied to find the shortest tour in the following scenario.

Problem 4.9.* *A young adventurer, fed up with the big city rush and TV soap operas, in his search for a peaceful and quite place went to spend his vacation in Africa. After a short time wandering around, he found a beautiful place just in the spot where two rivers met each other forming a peninsula in the shape of an acute angle and built a little cottage there. Every day the adventurer leaves his humble cottage C, goes to the river bank A to watch a waterpolo game between local tribes, then goes to the other bank B to pick up some lotus flowers and spend a short time petting his favorite hippo, and then he returns to his cottage C (see Figure 4.9).*

Which tour should the adventurer take so that the total traversed distance $CABC$ would be the shortest possible, assuming that each of his routes is rectilinear?

FIGURE 4.9. Adventurer on the peninsula

Brahmagupta (*ca.* 598–*ca.* 670) (→ p. 299)

The most influential Indian mathematician and astronomer of the seventh century was Brahmagupta, who lived and worked in the town Ujjain (or Ujjayinī), the great astronomical center of Hindu science. Among many important results from astronomy, arithmetic and algebra, let us note his formula for the area P of a cyclic quadrilateral having sides a, b, c, d and semiperimeter s,

$$P = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

which is a remarkable extension of Heron's well-known formula for the area of a triangle.

The first recorded research of *zero* in history is attributed to Brahmagupta (628 A.D.) who defined *zero* as the result of subtracting a number from itself (see Bentley [16]). Another of his great contributions was the study of indeterminate equations of the form $x^2 - Ay^2 = 1$, known as Pell's equations (see Appendix A). His definition of a mathematician is worth citing: *"A mathematician is a person who can solve the equation $x^2 - 92y^2 = 1$ within a year."* See Beiler [14, Ch. XXII]. Brahmagupta is credited with being the first person to solve a linear Diophantine equation in a general form $ax + by = c$, where a, b, c are integers.

Now let us take a close look at recreational mathematics of Brahmagupta. The following two fanciful problems of Brahmagupta can be found in Smith's book *History of Mathematics* [167, Vol. I, p. 159].

The same distance of traversed paths

Problem 4.10. *On a cliff of height h, lived two ascetics. One day one of them descended the cliff and walked to the village inn which was distance d from the base of the cliff. The other, being a wizard, first flew up height x and then flew in a straight line to the village inn. If they both traversed the same distance, what is x?*

Solution. Let $h = |AB|$, where h is the height of a cliff (Figure 4.10, right). From this figure, one can easily derive the equation

$$h + d = x + \sqrt{(h+x)^2 + d^2} \quad \text{or} \quad h + d - x = \sqrt{(h+x)^2 + d^2}.$$

After squaring and arranging the equation, one obtains $2hd - 2dx - 2hx = 2hx$, wherefrom

$$x = \frac{hd}{2h+d}.$$

FIGURE 4.10. Two ascetics' paths

Broken bamboo

Problem 4.11.* *A bamboo* 18 *cubits high was broken by the wind. Its top touched the ground* 6 *cubits from the root. Determine the lengths of the bamboo segments.*[4]

Mahāvira (*ca.* 800–*ca.* 870) (→ p. 300)

Height of a suspended string

This old problem can be found in many books, even recent ones discussing recreational mathematics and teaching.

Problem 4.12. *Two vertical pillars are of a known height, say a and b. Two strings are tied, each of them from the top of one pillar to the bottom of the other, as shown in Figure* 4.11. *From the point where the two strings meet, another string is suspended vertically till it touches the ground. The*

[4] There is an older Chinese version of this problem, given about 176 B.C. by Ch'ang Ts'ang in his *K'iu-ch'ang Suan-shu* (Arithmetic in Nine Sections); see, also, Y. Mikami, *The Development of Mathematics in China and Japan* (1913), p. 23.

distance between the pillars is not known. It is required to determine the height of this suspended string.

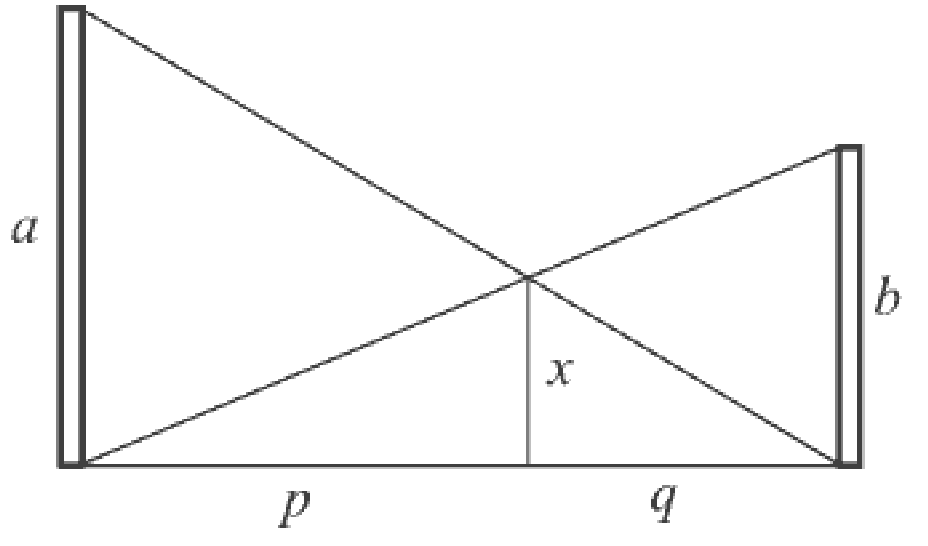


FIGURE 4.11. The height of suspended string

Solution. Let p and q be the lengths of distances obtained after dividing the horizontal distance between pillars by the suspended string. Let x be the unknown height of suspended string. From Figure 4.11 we have

$$\frac{p}{x} = \frac{p+q}{b}, \quad \frac{q}{x} = \frac{p+q}{a}.$$

Hence

$$\frac{1}{x} = \frac{1+q/p}{b}, \quad \frac{1}{x} = \frac{1+p/q}{a} \tag{4.1}$$

or

$$\frac{b}{x} - 1 = \frac{q}{p}, \quad \frac{a}{x} - 1 = \frac{p}{q}.$$

After multiplying these two relations, we obtain

$$\left(\frac{b}{x} - 1\right)\left(\frac{a}{x} - 1\right) = 1,$$

which reduces to

$$\frac{ab}{x^2} - \frac{a+b}{x} = 0.$$

Hence we find the sought height

$$x = \frac{ab}{a+b}\ .$$

As can be seen from the solution, the height x does not depend on the distance between the pillars. With this value of x it is easy to find from

(4.1) that $q/p = b/a$, which means that the horizontal distance between the pillars is divided by the point at which the suspended string touches the ground in the ratio of their heights.

Tābit ibn Qorra (826–901) (→ p. 300)

The diameter of the material sphere

The Arabian mathematicians were interested in constructions on a spherical surface. The following problem is sometimes ascribed to Tābit ibn Qorra, the same fellow who gave a formula for generating amicable numbers (see Chapter 2).

Problem 4.13. *Using Euclidean tools (straightedge and compass) find the diameter of a given material sphere.*

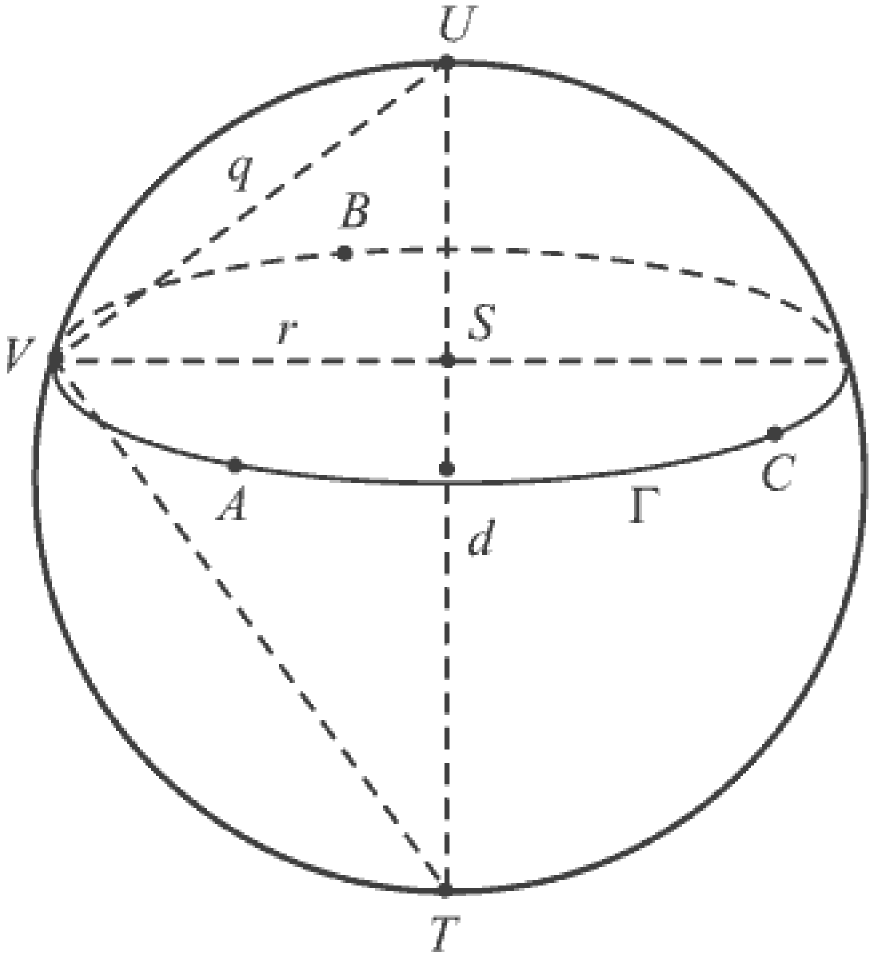


Figure 4.12. Finding the diameter of a material sphere

Solution. Using the compass fixed at an arbitrary point U on the sphere, draw a circle Γ on the sphere. Mark any three points A, B, C on the circumference of Γ and construct a triangle on a plane (using a separate sheet) congruent to the triangle $\triangle ABC$. Obviously, this construction can be easily performed translating the distances AB, BC, CA measured by the compass. Then construct the circle describing the triangle $\triangle ABC$. This circle has the same radius, say r, as the circle Γ drawn on the sphere.

Now we construct a right triangle $\triangle UVT$ which has the radius r as an altitude and the diameter $d = |UT|$ as hypotenuse (see Figure 4.12). The length q of the leg UV of this right triangle can be measured by the compass—it is the same opening used in drawing the circle Γ at the first step.

Knowing q and r, from the right triangle $\triangle UVT$ we find by proportion

$$\frac{r}{q} = \frac{\sqrt{d^2 - q^2}}{d};$$

therefore the diameter of the material sphere is

$$d = \frac{q^2}{\sqrt{q^2 - r^2}}.$$

Mohammad Abu'l-Wafa al-Buzjani (940–998)

(→ p. 300)

Abu'l-Wafa, probably the most prominent Islamic mathematician of the tenth century, together with Omar Khayyam (1048–1131) and Nasīr al-Dīn (1202–1274), all of them being born in the Persian mountain region Khorâsân, made the famous "Khorâsân trio." Apart from significant contributions on a number of mathematical topics, Abu'l-Wafa is also remembered today as a creator of many amusing dissection puzzles.

Dissection of three squares

Wafa's well-known dissection of squares reads:

Problem 4.14. *Dissect three identical squares into nine parts and compose a larger square from these parts.*

Abu'l-Wafa's solution is given in Figure 4.13. Two squares are cut along their diagonals and then the four resulting triangles are arranged around one uncut square. The dashed lines show how to perform four additional cuttings and insert them to make a large square (see the marked maps of congruent parts).

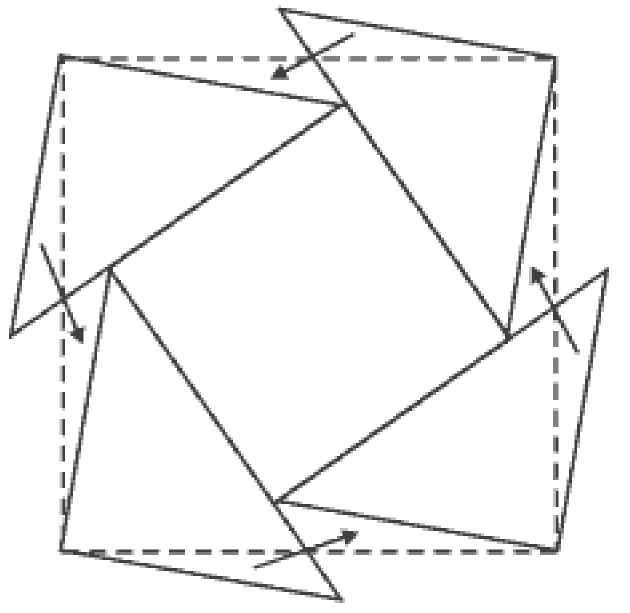


FIGURE 4.13. Solution to Abu'l-Wafa dissection problem

However, the greatest English creator of mathematical puzzles and games Henry E. Dudeney solved Abu'l-Wafa's problem using only six cuts. His amazing solution is shown in Figure 4.14. Point B is the intersection of the straight line through C and G and the arc with the center at A and the radius $|AD|$. The points E and F are determined so that $|BC| = |DE| = |FG|$. The desired square can be composed from the parts 1, 2, 3, 4, 5 and 6; parts 1 and 4 (joined) slide right down along HE, below parts 5 and 6, and parts 2 and 3 (joined) fill the gap.

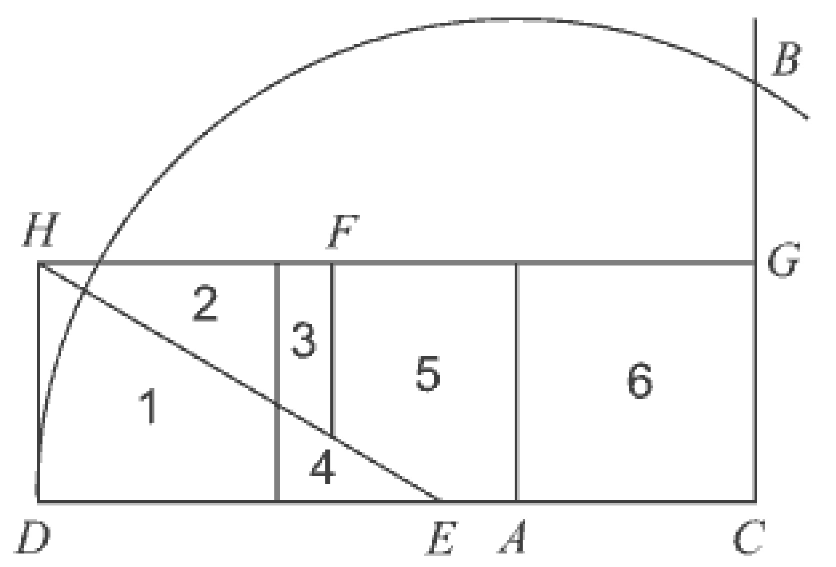


FIGURE 4.14. Dudeney's solution to Abu'l-Wafa dissection problem

As mentioned in [186], Wafa's idea can be also applied if among three squares two larger squares are identical while the third one is smaller in size. As in the solution presented in Figure 4.13, the larger squares are bisected and their pieces are placed symmetrically around the smaller square. The four small pieces, lying outside the square drawn by the dotted lines, fit the spaces inside its boundary exactly.

Dissection of four triangles

Problem 4.15.* *How can three identical triangles and one smaller triangle, similar to them in shape, be dissected into seven pieces which fit together to make one (larger) triangle?*

Ibn al-Haytham (Alhazen) (965–1039) (→ p. 300)

Although born in the southern Iraqi city of Basra, the Islamic physicist and mathematician Ibn al-Haytham, better known in Europe as Alhazen, spent the greater part of his life in Egypt. He traveled to Cairo at the invitation of the caliph al-Hakim to demonstrate his ideas on a flood-control project for the Nile. In the version presented by H. Eves [61, p. 194], Alhazen received the invitation owing to his boast that he could construct a machine that would successfully regulate the annual flooding of the Nile. Although Alhazen arrived in Cairo, he soon experienced misgivings concerning the success of his plan in practice. Fearing al-Hakim's anger, he feigned madness knowing that the insane received special treatment and protection. Alhazen simulated insanity so perfectly that he managed to keep any trouble at bay until Hakim's death in 1021. Alhazen did not, however, spend his time in Egypt in vain; while there he wrote his great scientific work, *Optics*, and attained fame.

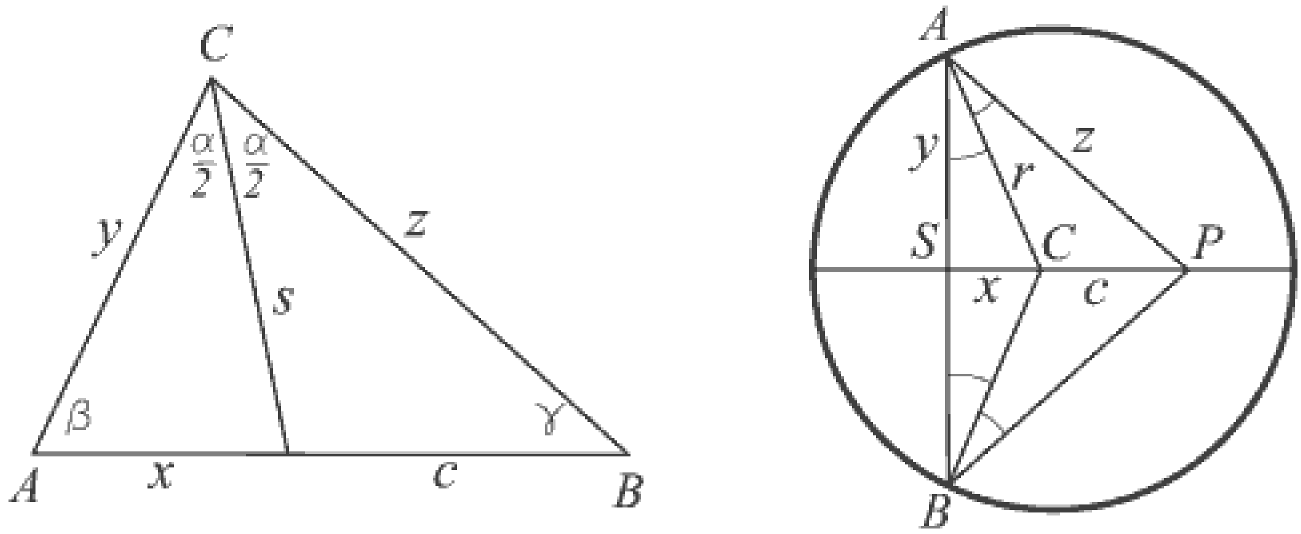
Alhazen's fame as a mathematician also derives from his investigation of problems named for him concerning the points obtained in certain reflecting processes associated with a variety of reflecting surfaces ("Alhazen's problems"). Eminent mathematicians who studied these problems after Alhazen, included Huygens, Barrow, de L'Hôspital, Riccati, and Quételet. Solutions to these problems can be found in the works of É. Callandreau [33, pp. 305–308], H. Dörrie [54, pp. 197–200] and J. S. Madachy [127, pp. 231–241]. Here we present one of Alhazen's problems, as posed and solved by him.

Billiard problem

Problem 4.16. *How does one strike a ball lying on a circular billiard table in such a way that after striking the cushion twice, the ball returns to its original position?*

Solution. Let us represent the billiard table as a circle in the plane with the radius r and the center C (Figure 4.15(b)). We denote the initial position of the ball by P, and assume that the distance $|CP| = c$ is known. Let the ball first strike the edge of the circle at point A, make a ricochet, cross the extension of PC (diameter of the circle) at a right angle to the line PC at S, then strike the circle at B and return to P. Knowing that the angle of incidence and the angle of reflection must be equal, we have

$$\measuredangle PAC = \measuredangle CAS = \measuredangle PBC = \measuredangle CBS.$$



a) b)

FIGURE 4.15. Billiard problem

Let us recall the angle bisector theorem as we continue solving the problem. If s is the bisector of the angle $\measuredangle ACB = \alpha$ (see Figure 4.15(a)), then applying the sine theorem we obtain

$$\frac{s}{\sin\beta} = \frac{x}{\sin(\alpha/2)} \quad \text{and} \quad \frac{s}{\sin\gamma} = \frac{c}{\sin(\alpha/2)}.$$

Hence

$$\frac{x}{c} = \frac{\sin\gamma}{\sin\beta}.$$

From the triangle $\triangle ABC$ (Figure 4.15(a)) we have

$$\frac{y}{\sin\gamma} = \frac{z}{\sin\beta}, \quad \text{that is,} \quad \frac{y}{z} = \frac{\sin\gamma}{\sin\beta}.$$

Therefore, it follows that

$$\frac{y}{z} = \frac{x}{c}. \tag{4.2}$$

Let us return to Figure 4.15(b). Setting

$$|CS| = x, \quad |SA| = y, \quad |AP| = z$$

and applying the angle bisector theorem, we obtain just (4.2). In regard to the Pythagorean theorem,

$$r^2 = x^2 + y^2 \quad \text{and} \quad z^2 = y^2 + (x+c)^2. \tag{4.3}$$

Eliminating y and z from (4.2) and (4.3) we obtain the quadratic equation

$$2cx^2 + r^2x = cr^2$$

for the unknown x. In the sequel we will show how to construct x $(= |CS|)$ knowing the values of r and c.

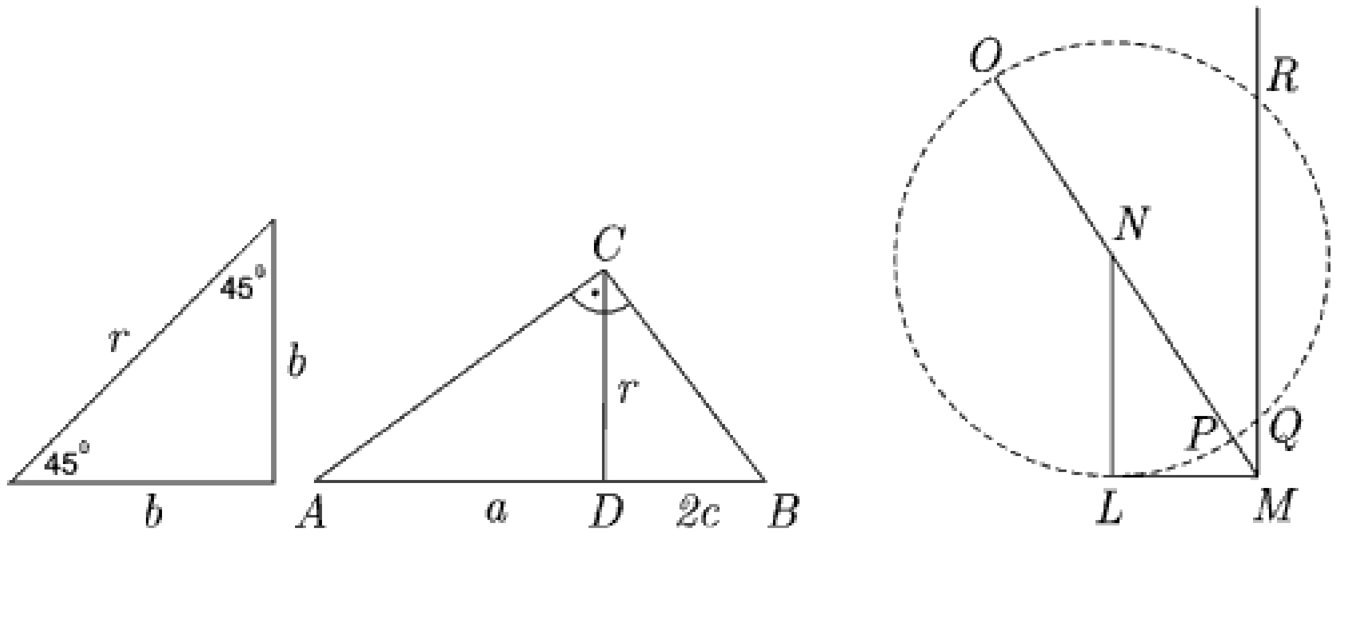


FIGURE 4.16. Geometric solution of the quadratic equation

We rewrite the last quadratic equation in the form

$$x^2 = -ax + b^2, \quad \text{where} \quad a = \frac{r^2}{2c}, \; b = \frac{r}{\sqrt{2}}. \tag{4.4}$$

From Figure 4.16(a) we see how to find b; it is the side of a square with the diagonal r. With the sides $2c$ and r we construct the right triangle $\triangle DBC$ and draw a line through the vertex C perpendicular to CB to find the intersecting point A with the line through DB. Then the line segment AD gives a; indeed, we have $r = \sqrt{a \cdot 2c}$ (Figure 4.16(b)).

Now we will apply Descartes' method to construct the solution of the quadratic equation $x^2 = -ax + b^2$ (see, *e.g.*, Katz [113, pp. 437–438]). First, we construct a right triangle $\triangle NLM$ with $|LM| = b$ and $|LN| = \frac{1}{2}a$ (Figure 4.16(c)). Then we construct a circle centered at N with radius $|NL|$. Let P

be the intersection of this circle with the hypotenuse $|NL|$, then $|MP|$ is the required value x. Indeed,

$$|MP| = |MN| - |PN| = \sqrt{|LM|^2 + |LN|^2} - |LN| = \sqrt{\frac{1}{4}a^2 + b^2} - \frac{a}{2},$$

which is exactly the value x found from (4.4).

Johann Müller (Regiomontanus) (1436–1476)

(→ p. 301)

Distance of the optimal viewpoint

Often when we view an object such as a painting or a statue in a gallery, or a monument (as in Figure 4.17), we notice, whether consciously or unconsciously, that the object appears at its best when viewed from a particular distance. Johann Müller, the fifteenth-century German mathematician and astronomer known as Regiomontanus, found this phenomenon to be of great significance. In 1471, he presented the problem to Christian Roder, a professor from the city of Erfurt. Many historians of mathematics regard Müller's work in the field of optimization as being the first such work done since antiquity.

FIGURE 4.17. The best view of the Statue of Liberty

We present Regiomontanus' maximum problem in an abstract geometric form with these restrictions: the formulation of the problem is restricted to an imaginary picture plane, and our object, in this case a rod, is understood as a vertical line segment.

Problem 4.17. *A rod of length d is suspended vertically at height h from an observer's eye level. Find the horizontal distance from the rod hanging vertically to the observer's eye in such a way to see the rod the best.*

Solution. We present a solution constructed entirely in the picture plane. The solution is similar to that published in *Zeitschrift für Mathematik und Physik* by Ad. Lorsch.

We assume that the optimal viewing distance for the object corresponds to the maximum viewing angle $\measuredangle BCA$ from the observer's eye (presented by the point C) towards the rod AB (with $|AB| = d$); see Figure 4.18. The straight line p, perpendicular to the vertical line (of hanging) OAB, denotes the viewing level.

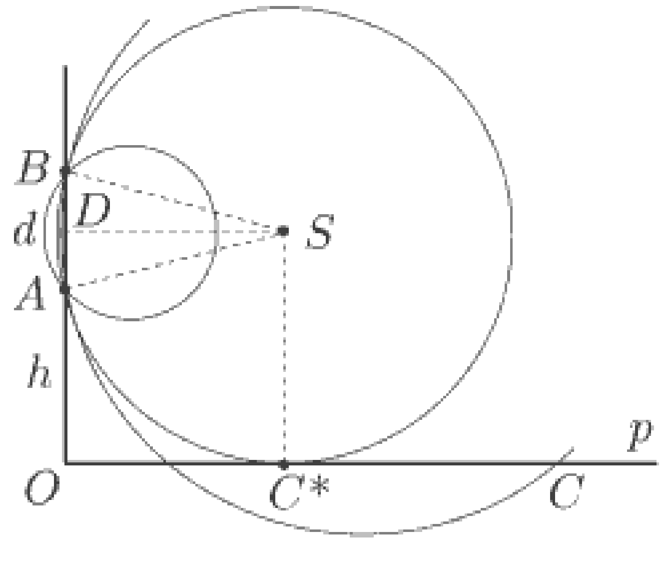


FIGURE 4.18. Distance of the best view

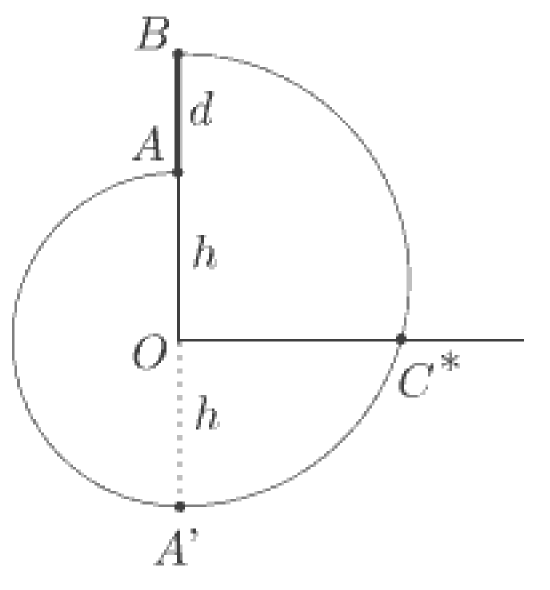


FIGURE 4.19.

Let us draw a family of circles, all of them passing through the points A and B. We observe the rod AB from a point in the plane that results in a viewing angle which is the same for all points that lie on a fixed circle belonging to this family (since the peripheral angles corresponding to a fixed chord of a given circle are the same). Evidently, the value of these angles will be larger if the radius of the circle is smaller. However, we are interested only in those points that lie on the straight line p (at the observer's eye level). Therefore, the solution is that point which lies both on the straight line p and on a circle through A and B with the radius as small as possible. This point is the point C^* of the tangency of the circle through A and B and the straight line p. Therefore, the point of tangency C^* is the desired point, and $x = |OC^*|$ is the distance affording the most desirable view.

How does one find the distance $x = |OC^*|$? Denoting $h = |OA|$ and recalling that $|AB| = d$, we find

$$x = |OC^*| = |DS| = \sqrt{|AS|^2 - |DA|^2} = \sqrt{|SC^*|^2 - |DA|^2}$$
$$= \sqrt{\left(h + \frac{d}{2}\right)^2 - \left(\frac{d}{2}\right)^2} = \sqrt{(h+d)h}.$$

Therefore, the optimal viewing distance is the geometric mean of the distances of the upper and lower ends of the rod from the viewing level. The critical point C^* can be found by the geometrical construction shown on Figure 4.19, where $|A'O| = |OA| = h$ and $A'C^*B$ is the semicircle with the diameter $|A'B| = 2h + d$.

More details about "extreme viewpoint problem" may be found in [7], [54], [121], [134], [141] and [190]. Lester G. Lange considered in [121] a more general problem in which the picture is tilted out from the wall. David Wells [186] considered the following variant of the "best-view problem" of practical interest.

Figure 4.20 shows part of a rugby union football field with the goal line passing through the goal posts marked by little black circles. According to the rules of the game, an attempt at a conversion must be taken on a line extending backwards from the point of touchdown, perpendicular to the goal line. Find the point T on this line from which the conversion should be taken so that the angle subtended by the goal posts is the largest. This problem applies only when the attempt is not scored between the posts.

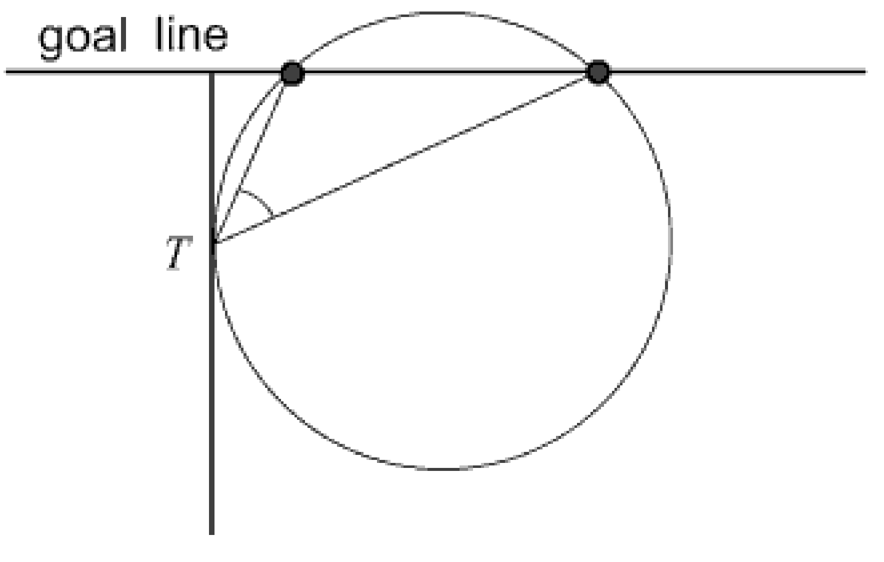


FIGURE 4.20. Rugby football problem

Obviously, the solution to the rugby conversion problem bears a great similarity to the solution of Regiomontanus' "best-view problem". Since the peripheral angles corresponding to a fixed chord of a given circle are the same, the same angle will be subtended by the goal posts at any point on this circle; the choice of any point outside it will subtend a smaller angle and any point inside it, a larger angle. Choosing the circle to touch the line exactly at point T implies that all other points of the line are *outside* the circle. Therefore, the conversion should be taken from the touching point T, as shown in Figure 4.20.

The *Saturn problem* is an interesting space-variant of Regiomontanus' problem.

Problem 4.18. *At what latitude does Saturn's ring system appear widest?*

We recall that latitude is the distance on the sphere surface toward or away from the equator expressed as an arc of meridian. Hermann Martus was probably the first to pose this problem, and it can be found in Heinrich Dörie's book, *100 Great Problems of Elementary Mathematics* [54, pp. 370–371]. We give here the outline of the solution to the Saturn problem; for more details, see Dörrie's book [54].

FIGURE 4.21. The widest view of Saturn's ring

To solve this problem we assume that Saturn is a sphere with the radius of 56,900 km, and that Saturn's ring system lies in its equatorial plane, with an inner radius of 88,500 km and an outer radius of 138,800 km.

Figures 4.21 and 4.22 aid us in visualizing this situation. The arc m represents a meridian, M is the midpoint of Saturn, $|AB|$ the width of the ring, $|MA| = a$ being the outer radius, and $|MB| = b$ the inner radius of the ring. Now, let $|MC| = r$ be the equatorial radius of Saturn on MA and let O be the point situated at the latitude $\varphi = \measuredangle CMO$ at which the ring width appears largest, so that $\measuredangle AOB = \psi$ is maximal.

Using Figure 4.22 we will find the solution. First we draw the circle Ω that passes through points A and B and which is tangent to the meridian m. Then the point of tangency O is the point from which the ring appears to be greatest.

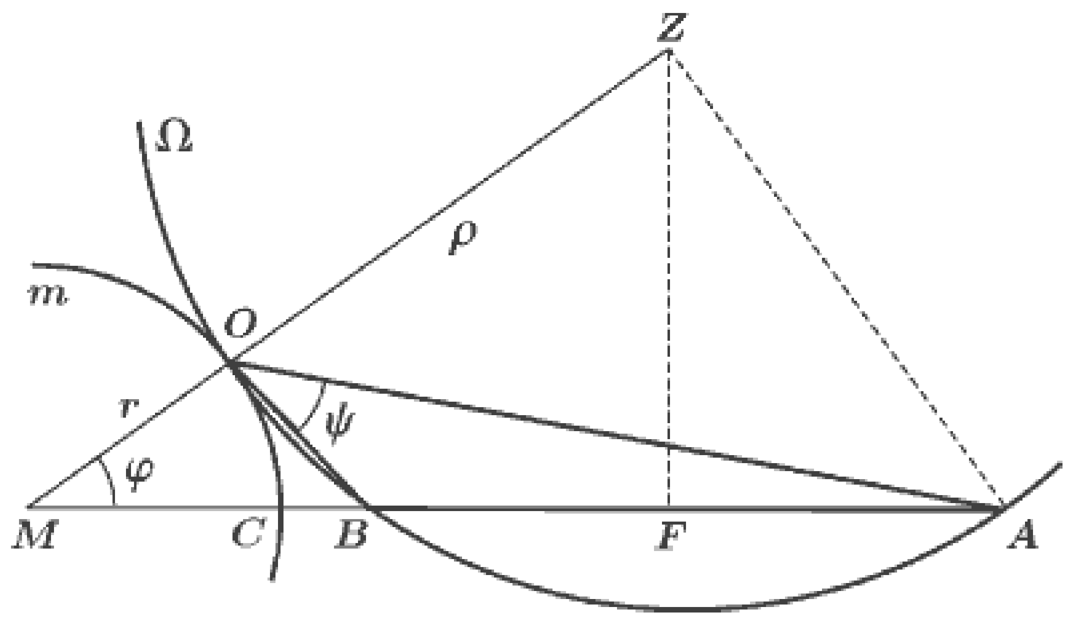


FIGURE 4.22. The widest view of Saturn's ring

The latitude φ of O and the maximum ψ, we will calculate from the right triangles $\triangle MZF$ and $\triangle AZF$, where Z is the center of the circle Ω and F the middle of AB. Denoting the radius of Ω by ρ, we obtain therefrom

$$\cos\varphi = \frac{|MF|}{|MZ|} = \frac{a+b}{2(r+\rho)},$$
$$\sin\psi = \frac{|AF|}{|AZ|} = \frac{a-b}{2\rho}.$$

It can be shown that the value of ρ is given by

$$\rho = \frac{ab-r^2}{2r}$$

(see [54, p. 371]), so that we finally obtain

$$\cos\varphi = \frac{(a+b)r}{ab+r^2},$$
$$\sin\psi = \frac{(a-b)r}{ab-r^2}.$$

Hence, for the specific values of a, b, and r, it follows

$$\varphi \approx 33^\circ\ 33'\ 54'', \quad \psi \approx 18^\circ\ 26'\ 40''.$$

According to Smart [165], Saturn is considerably deformed from the ideal sphere on which the above derivation is based, having an equatorial radius of 59,650 km and a polar radius of 53,850 km. The solution in this real case was given by John Trainin (Granada, Spain) in *The Mathematical Gazette*. Assuming that the cross-section of Saturn through its poles is an ellipse with semi-major and -minor axes p and q, respectively, the relevant geocentric latitude is found from the cubic equation

$$(p^2 - q^2)\cos^3 t - (2p^2 - q^2 + ab)\cos t + p(a+b) = 0,$$

where $\tan\varphi = (q/p)\tan t$. This yields $29^\circ\ 39'\ 10''$ as an approximate value for φ. Note how this solution reverts to that mentioned earlier when $p = q = r$.

Jacob Steiner (1796–1867) (→ p. 306)

Pierre de Fermat (1601–1665) (→ p. 303)

Evangelista Torricelli (1608–1647) (→ p. 303)

Bonaventura Cavalieri (1598–1647) (→ p. 302)

Evangelista Torricelli and Bonaventura Cavalieri were very influential seventeenth-century Italian scientists. Toricelli won fame for his discoveries in physics, while Cavalieri was successful in optics and astronomy. Both collaborated with Galileo; both were forerunners of infinitesimal methods, and finally, they both studied the challenging problem that follows.

The minimal sum of distances in a triangle

The formulation of this well-known problem is very simple:

Problem 4.19. *In the plane of a triangle, find the point whose sum of distances from the vertices of the triangle is minimal.*

The point with this property is now known as the *isogonic center* of the triangle. This problem frequently appears in many mathematics books, recreational and course-related, that include challenging optimization tasks such as:

Three villages A, B, and C agree to build a joint project, for example, a department store, a warehouse, an oil pipeline, or a casino. Find the location D of this joint venture such that the inhabitants of these three villages traverse a minimal route. In other words, determine the minimal combined lengths of the rectilinear roads linking the common point D to A, B, and C.

Obviously, problems of this type are not restricted to mathematical recreation only, but more importantly, they provide accurate models of real-life situations, and their optimal solutions have a direct economic impact.

The problem posed at the beginning of this section has a long history. According to some historians of mathematics, the great French mathematician Pierre de Fermat was interested in it, and, moreover, he challenged the eminent Italian physicist and mathematician Evangelista Torricelli with this very problem. Torricelli's student Vincenzo Viviani (1622–1703) published his solution in probably the first work on this subject, *On maximal and minimal values* (1659). For these reasons, the point where the required minimum is attained is often called the *Torricelli point* or, sometimes, the *Fermat point*. Bonaventura Cavalieri, another great Italian mathematician, also studied this problem.

In the nineteenth century Jacob Steiner, the famous German-Swiss geometrician, considered the above problem and a series of similar problems in detail. For this reason they are frequently called *Steiner's problems*. In the sequel, we will present the well-known geometric solution of Steiner's problem of the minimal sum of distances for triangles whose angles do not exceed 120°. Otherwise, knowing that obtuse angles exceed 120°, the isogonic center just coincides with the vertex of the obtuse angle.

Assume that the angle $\measuredangle C$ is $\geq 60^\circ$ (Figure 4.23). Let T be a point inside $\triangle ABC$ (a candidate for the isogonic center), and connect T with vertices A, B, and C. Rotate $\triangle ABC$ about C through 60° into the position $A'B'C$. Let T' be the image of T under the performed rotation. By construction, we observe that $\triangle CTT'$ is equilateral, $|CT| = |CT'| = |TT'|$, and $|AT| = |A'T'|$. Then the sum of lengths $|AT| + |BT| + |CT|$ is equal to the sum $|A'T'| + |CT| + |BT| = |A'T'| + |T'T| + |BT|$. The last sum is the length of the polygonal line $A'T'TB$ and it is not shorter than the straight line $A'B$. Hence we conclude that the length $|A'T'TB|$ will reach its minimum if the points A', T', T, B are colinear. Therefore, the isogonic center, denoted by T^*, lies on the straight line $A'B$. Furthermore, using the symmetry and the equality of crossing angles related to T^*, it is easy to show that all sides of the triangle ABC are seen at the angle of $120°$. This property gives us an idea about the location of T^* (see Figure 4.24).

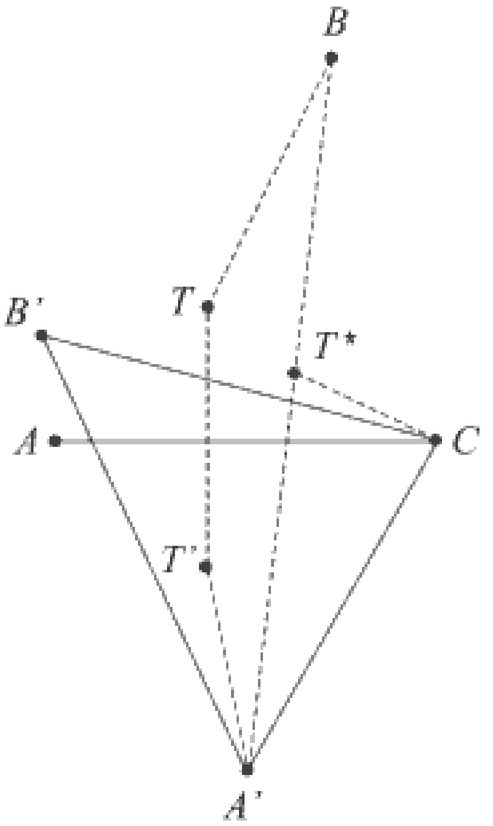


FIGURE 4.23. Minimal sum of distances in a triangle

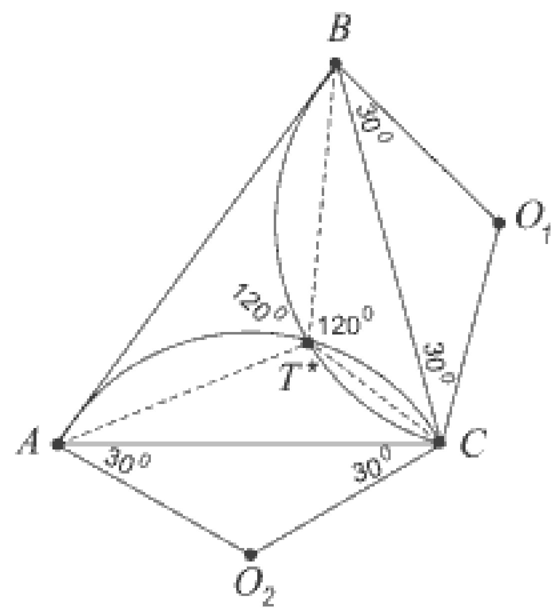


FIGURE 4.24. Location of Torricelli's point

First, we draw straight lines under the angle of $30°$ related to two sides of $\triangle ABC$, say BC and CA, and find the points O_1 and O_2. Then we draw two arcs with the radii $|O_1B|$ and $|O_2C|$ taking O_1 and O_2 as the centers; the intersection of these arcs gives the desired isogonic center T^*. The discussion of this construction is left to the reader.

Problem 4.20.* *Find the isogonic center (Torricelli point) using the Heron reflection principle.*

Hint: See Problem 4.7.

Johannes Kepler (1571–1630) (→ p. 302)

The great German astronomer and mathematician Johan Kepler is best known for his discovery of the three mathematical laws of planetary motion. What is most surprising, but also very impressive, is that he was not aware that the cause of the motion of the planets is gravity, a fact explained a few decades later by Newton in his monumental work *Philosophiae Naturalis*

Principia Mathematica (1687). In the next chapter we will see that Kepler dealt with some problems which could be considered today as recreational mathematics. So it is not strange that this inquisitive and imaginative man was the author of probably the first ever science fiction story, called *Somnium* (The Dream).

Johannes Kepler
1571–1639

Kepler's personal life, however, was plagued by misfortune. Kepler's favorite child died of smallpox at the age of seven; his mother was tried and sentenced for practicing witchcraft; and his first wife Barbara went mad and died. Kepler himself was accused of heterodoxy since he was a profound Lutheran. He and his children found themselves forced to leave Prague because the Emperor Matthias did not tolerate Protestantism. Kepler lived a great deal of his life in poverty and while on a trip apparently to recover his permanently unpaid salary, he died.

Volumes of cylinders and spheres

In his book *New solid geometry of wine barrels*, Johannes Kepler described the methods for simple measurement of the volumes of barrels of different shape; see [161]. Kepler advanced new ideas for the solution of maximum and minimum problems to clarify the basis of his methods, which in turn have touched upon the essence of differential and integral calculus. Theorem V is one of the results contained in Kepler's book: *"Of all cylinders with the same diagonal, the largest and most capacious is that in which the ratio of the base diameter to the height is $\sqrt{2}$."* In other words, this theorem gives the solution of the following problem:

Problem 4.21. *Inscribe a cylinder of maximal volume in a given sphere.*

This problem can be solved using elementary mathematics. Let R be the radius of the sphere and let x be half the height h of the cylinder. Then the restriction $0 \le x \le R$ is obvious (see Figure 4.25). The base radius of the cylinder is $\sqrt{R^2 - x^2}$ and its volume is

$$V_c(x) = \pi r^2 h = 2\pi(R^2 - x^2)x = 2\pi f(x). \tag{4.5}$$

The aim is to find x that provides maximal volume V_c of the cylinder.

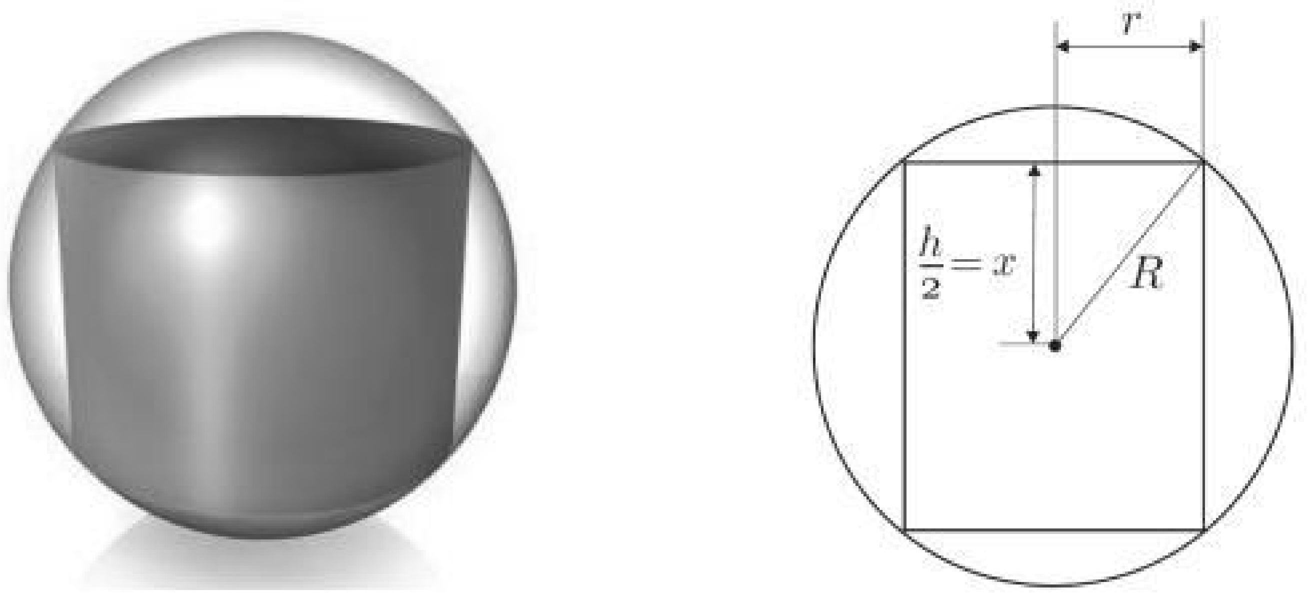


FIGURE 4.25. Largest cylinder in a sphere

The problem can be easily solved by using differential calculus. Here we use the suitable elementary manipulations to avoid differential calculus. For our purposes, it is sufficient to consider the function

$$f(x) = (R^2 - x^2)x$$

appearing in (4.5). First we factor $f(x)$ in the following way:

$$\begin{aligned} f(x) &= R^2x - x^3 = R^2x - x^3 - \frac{2R^3}{3\sqrt{3}} + \frac{2R^3}{3\sqrt{3}} \\ &= -\Big(x - \frac{R}{\sqrt{3}}\Big)^2\Big(x + \frac{2R}{\sqrt{3}}\Big) + \frac{2R^2}{3\sqrt{3}}. \end{aligned}$$

We note that the function f reaches its maximum when the first term vanishes, that is, when $x = R/\sqrt{3}$. For this value of x the height and the base radius of the cylinder are

$$h = 2x = 2R/\sqrt{3} \quad \text{and} \quad r = \sqrt{R^2 - x^2} = R\sqrt{2/3}.$$

Hence we see that the ratio of the base diameter of the extreme cylinder to the height is $\sqrt{2}$, a fact established by Kepler.

Jacob Steiner (1796–1863) (→ p. 306)

Although he did not learn to read or to write until the age of 14, Jacob Steiner later became a professor at the University of Berlin and one of the greatest geometers ever. Poncelet-Steiner's theorem, one of his remarkable results, asserts that all Euclidean constructions can be carried out with a straightedge alone in the presence of one circle and its center drawn on the plane of the construction.[5] Mathematical recreation problems often make use of constructions of this sort and variations on them.[6]

Dido's problem

Virgil, one of ancient Rome's greatest poets, describes in the *Aeneid* the aventure of the legendary Phoenician princess Dido (also known as Elissa) and her role in the founding of the city of Carthage.

As Dido fled from persecution by her brother, she followed the Mediterranean coastline. Once she had put enough distance between them to assure her safety, she endeavored to find a suitable plot of land on which to settle. However, Yarb, the local ruler, was not willing to sell her the land. Nevertheless Dido, being a capable negotiator, managed to persuade Yarb to fulfill a "modest" wish: to sell her as much land as could be "encircled with a bull's hide." Clever Dido then cut a bull's hide into narrow strips, tied them together, and in that way enclosed a large tract of land. She built a fortress on land next to what is today named the Bay of Tunis, and near it, she built the city of Carthage.

This prompts one to ask: *How much land can a bull's hide enclose?* Using the terms of modern mathematics, let us consider the problem as follows:

Problem 4.22. *Among all closed plane curves of a given length, find the one that encloses the largest area.*

This question is known as *Dido's problem*, or the *classical isoperimetric problem*.[7] Today we understand that the desired curve is a circle, and many historians are of the opinion that this was the first discussion in literature of an extremal problem.[8] The literature devoted to Dido's problem is vast; some historical references can be found in the book *Kreis und Kugel* (Circle and Sphere) (Leipzig 1916) of the German geometer W. Blaschke.[9] One of the most ingenious proofs is due to the great German-Swiss geometer Jacob

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[5] In 1904 the Italian Francesco Severi showed that an arc (of arbitrary size) and its center are sufficient to make possible all Euclidean constructions with a straightedge alone [61, p. 98].

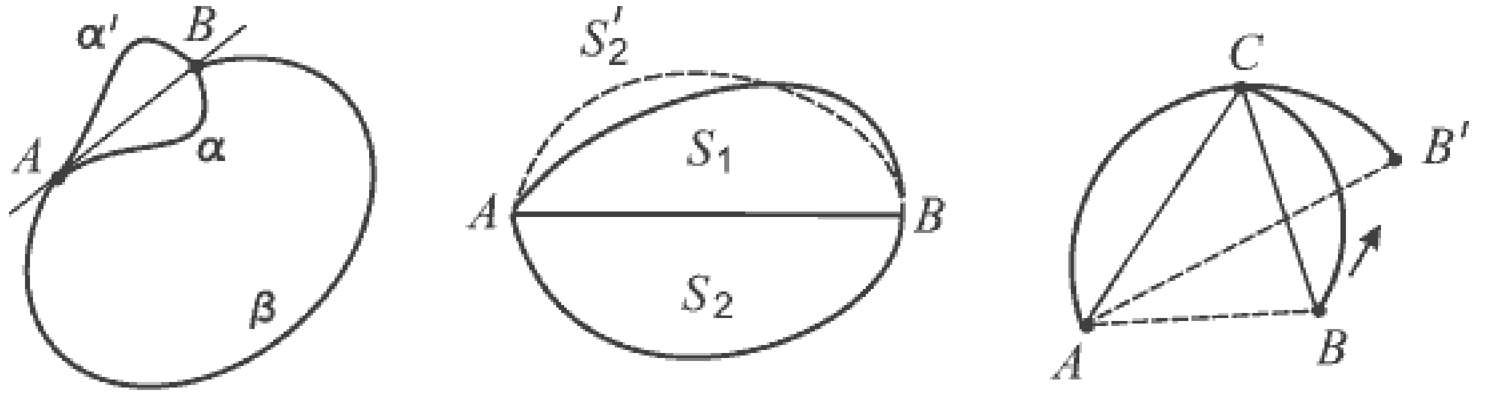
[6] See, for instance, Abu'l-Wafa's problems in this book.

[7] Isoperimetric figures are figures having the same perimeter.

[8] S. Hildebrandt, *Variationsrechung heute*, Rheinisch-Westfälische Academie der Wissenschaften, 1986.

[9] Reprinted by Auflage, DeGruyter, Berlin 1956.

Steiner. Steiner, however, failed to prove the actual existence of a maximum; he assumed the existence of a curve that would solve the isoperimetric problem. Weierstrass later gave a rigorous proof.[10] We present Steiner's proof below as it appears in V. M. Tikhomirov's book [176]. Steiner's solution of the isoperimetric problem follows from the three auxiliary assertions (Lemmas (I), (II) and (III)).



a) b) c)

FIGURE 4.26. Steiner's solution of Dido's problem

(I) *The extremal curve is convex.*[11]

Proof. Suppose that the desired curve is not convex. In that case, it contains two points A and B such that both arcs $A\alpha B$ and $A\beta B$ joining A and B lie on the same side of the straight line through A and B (see Figure 4.26(a)). By replacing one of these arcs, say $A\alpha B$, with its image $A\alpha' B$ under reflection in AB, one obtains a new curve $A\alpha' B\beta A$ of the same length that encloses a larger area than $A\alpha B\beta A$.

(II) *If points A and B halve the perimeter of the extremal curve into equal lengths, then the chord $\overline{AB}$ halves the area of this curve into equal parts.*

Proof. Assume that the chord $\overline{AB}$ divides the area into unequal parts S_1, S_2, area $S_1 <$ area S_2, both with the same length. But then we take the image S_2' of the larger part S_2 under reflection in the diameter AB instead of the smaller part S_1, and add it up to S_2 (which has the same length as S_1). In this way we would obtain a figure with the same length but the larger area (Figure 4.26(b)).

(III) *Suppose that points A and B halve the extremal curve. If C is any point on the curve, then the angle ACB is a right angle.*

[10] W. Blaschke, *Kreis und Kugel*, Leipzig 1916, pp. 1–42.

[11] A convex region U is one with the property that if any pair of points (P, Q) lies inside U, then the line segment $\overline{PQ}$ connecting P and Q lies inside U.

Proof. Assume there is a point C such that ACB is not a right angle. The area S_{ABC} bounded by the arc ACB and the diameter AB splits into three parts: the triangle ABC and the segments adjacent to the sides AC and CB with the areas S_{AC} and S_{BC} (see Figure 4.26(c)).

Now, rotate segment S_{CB} around point C until point B arrives at position B' in such a way that ACB' is a right angle, as shown in Figure 4.26(c). Since

$$\begin{aligned} S_{ABC} &= S_{AC} + S_{CB} + S_{\triangle ABC} = S_{AC} + S_{CB} + \frac{1}{2}|AC||BC|\sin C \\ &\le S_{AC} + S_{CB} + \frac{1}{2}|AC||BC| = S_{AC} + S_{CB} + S_{\triangle AB'C}, \end{aligned}$$

we conclude that the area S_{ABC} will reach its maximum when the angle ACB becomes a right angle ($\sin C = 1$). The figure obtained by reflecting the curve ACB' in the chord AB' has the same perimeter but a larger area than the original figure, which proves the assertion.

We see that the extremal figure consists of all points C from which a chord that halves the length of the extremal curve is seen at a right angle. This illustrates a well-known property of a circle: the peripheral angle over the diameter of a circle is always 90°, which means that the curve in question is a circle.

Division of space by planes

The following problem and many others, some included in this section, fall somewhere between serious mathematics and recreational mathematics. This is not unusual; we have already emphasized that the study of many mathematical tasks and games led to the development of new mathematical branches or to the discovery of important mathematical results. On the other hand, the solutions of some recreational problems require very powerful mathematical tools. The famous geometer Jacob Steiner considered several such problems of dual character. The following problem was first solved in 1826 by Steiner.[12]

Problem 4.23. *What is the maximum number of parts into which a space can be divided by n planes?*

Perhaps this problem has a more combinatorial flavor, but we included it in the chapter on geometry since Steiner was a great geometer.[13] Is this a sufficiently good reason?

[12] *Journal für die reine und angewandte Mathematik* 1 (1826), 349–364.

[13] Although he also gave a great contribution to combinatorial designs; recall a Steiner triple system.

In solving this problem we will use the well-known formulae

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$
$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

To find the desired maximum number, let us first solve the same type of planimetric problem that we present here: *What is the maximum number of parts into which a plane can be divided by n straight lines?*

Obviously, to obtain the maximum number of parts, we must exclude (i) parallel lines and situations in which (ii) three or more lines pass through one point. In what follows, we will assume that these two conditions are satisfied.

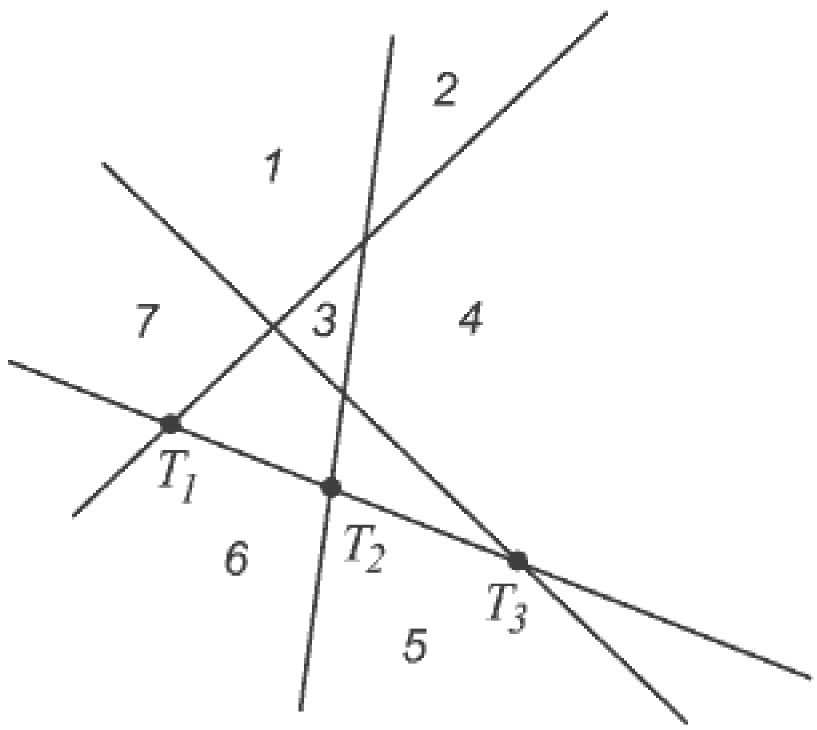


FIGURE 4.27. Dividing the plane by straight lines

Let P_n denote the number of parts in the plane, generated by n lines. Taking into account that those particular cases (i) and (ii) are excluded, one additional line will be cut by the previous n lines into n points, say $T_1, T_2, \dots, T_n$. The line segments ($n-1$ in total) $T_1T_2,\ T_2T_3, \dots, T_{n-1}T_n$ and the two semistraight lines with the ends at T_1 and T_n belong to the various parts of the plane and their number is obviously $n+1$. Each of these traversed parts is divided into two parts so that the (n+1)th line increases the number of parts by $n+1$ (see Figure 4.27 where $n = 3$). According to this simple consideration we have

$$P_{n+1} = P_n + n + 1.$$

Telescoping summation[14] yields

$$\sum_{k=0}^{n-1}\Big(P_{k+1}-P_k\Big)=1+2+\cdots+n,$$

that is,

$$P_n-P_0=1+2+\cdots+n.$$

Since $P_0=1$ and $1+2+\cdots+n=n(n+1)/2$, we finally find

$$P_n=1+\frac{n(n+1)}{2}=\frac{n^2+n+2}{2}. \tag{4.6}$$

Let us return to the space problem originally posed. To attain the maximum number of partial spaces we require that (i) the lines of intersection of no more than two planes are parallel and (ii) no more than three planes intersect at one point.

Let S_n denote the required maximum number of partial spaces obtained by n planes. A new additional plane is cut by the first n planes into n lines in such a way that conditions (i) and (ii) are satisfied. Therefore, the new (n+1)th plane is divided by the n lines into P_n planar sections. Each of these sections cuts the traversed partial space into two smaller spaces, which means that the additional $(n+1)$th plane increases the number of the partial spaces by P_n. This consideration leads to the relation

$$S_{n+1}=S_n+P_n. \tag{4.7}$$

According to (4.7) we obtain

$$\sum_{k=1}^{n-1}\Big(S_{k+1}-S_k\Big)=S_n-S_1=P_1+P_2+\cdots+P_{n-1}.$$

Since $S_1=S_0+P_0=1+1=2$, it follows that

$$S_n=2+P_1+P_2+\cdots+P_{n-1}. \tag{4.8}$$

---

[14]Sums of the form $\sum_{m\le k\le n}\big(a(k+1)-a(k)\big)$ are often called *telescoping* by analogy with a telescope whose thickness is determined by the radii of the outermost and innermost tubes of the telescope. By the way, Σ-notation was introduced by Joseph Fourier in 1820.

By (4.6) and the formulae for the sums of natural numbers and squares of natural numbers, given above, we find that

$$\sum_{k=1}^{n-1} P_k = \frac{1}{2}\sum_{k=1}^{n-1}(k^2+k+2) = n-1+\frac{1}{2}\left[\sum_{k=1}^{n-1} k^2 + \sum_{k=1}^{n-1} k\right]$$
$$= n-1+\frac{1}{2}\left[\frac{n(n-1)(2n-1)}{6}+\frac{(n-1)n}{2}\right] = \frac{n^3+5n-6}{6}.$$

Using this result, from (4.8) we find that the required maximum number of parts, obtained by dividing a space by n planes, is given by

$$S_n = \frac{n^3+5n+6}{6}.$$

It is interesting to note that the numbers P_n and S_n can be expressed by the entries of the Pascal triangle. S_n is equal to the sum of the first *three* entries of the nth row (figure below left, $P_5 = 16$) and S_n is the sum of the first *four* entries of the nth row (figure below right, $S_5 = 26$).

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
\qquad
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

The sum P_n The sum S_n

The first step in solving Problem 4.23 was the determination of the maximum number of parts into which a plane is divided by n straight lines. We have seen that some of these parts are infinite, while others are bounded. The following natural question arises (see [88]):

Problem 4.24.* *A plane is divided into (infinite and bounded) parts by n lines. Determine the maximum possible number of bounded parts (regions).*

Road system in a square

The following problem, which can be found in many books, is ascribed again to Jacob Steiner.

Problem 4.25. *Four villages (hamlets, airport terminals, warehouses, or whatever), each being a vertex of a square, should be connected by a road network so that the total length of the road system is minimal.*

Solution. Let V_1, V_2, V_3, and V_4 be the vertices of the square of side length equal to 2 units. AB and CD are perpendicular bisectors; see Figure 4.28. A road system is composed of the solid lines EV_1, EV_2, EF, FV_3, and FV_4, with the unknown angle θ which should be determined so that this road system has minimal length. Using the triangle inequality it is easy to prove that the symmetric network $\asymp$ shown in Figure 4.28 has smaller total length than any nonsymmetric network $\times$. Other configurations will be discussed later.

Referring to Figure 4.28, the length of the desired road network is $S = 2(2x + z)$. Since

$$y = \tan\theta, \quad x = \frac{1}{\cos\theta}, \quad z = 1 - y = 1 - \tan\theta, \quad \theta \in [0, \pi/4],$$

we have

$$S(\theta) = 2\Big(1 - \tan\theta + \frac{2}{\cos\theta}\Big) = 2 + 2f(\theta),$$

where we introduce

$$f(\theta) = \frac{2}{\cos\theta} - \tan\theta.$$

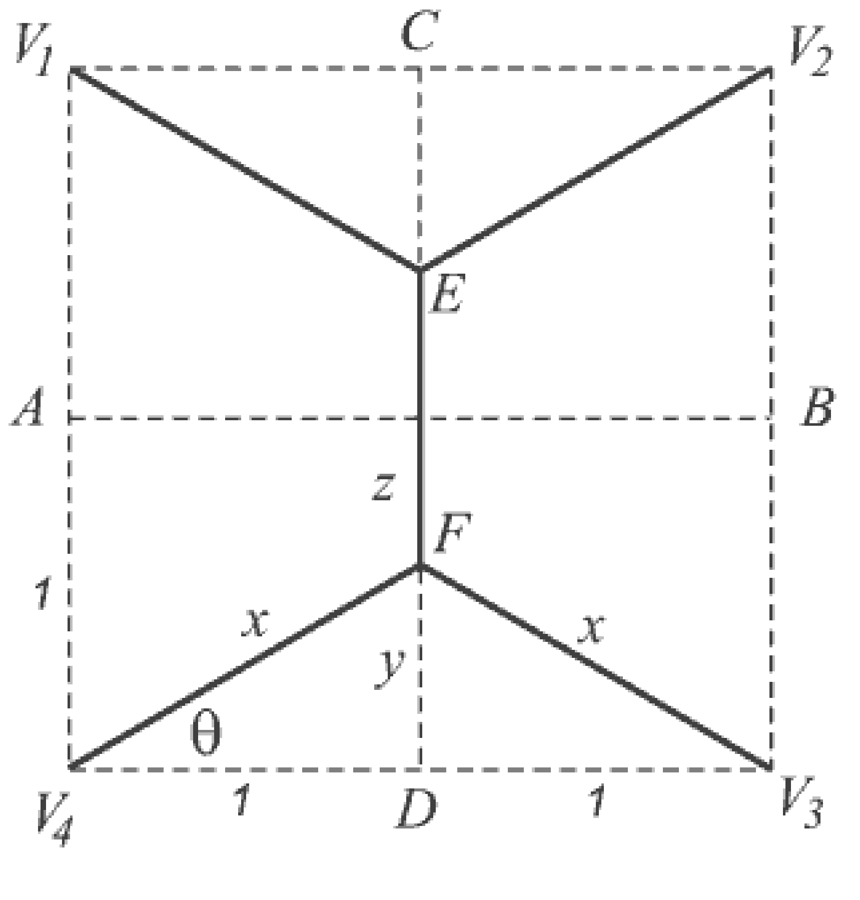


FIGURE 4.28. Minimum road system

Since $f(\theta) = (2 - \sin\theta)/\cos\theta > 0$ for $\theta \in [0, \pi/4]$, to find the minimum of $S(\theta)$ it is sufficient to determine the minimum of the function $f(\theta)$ on the interval $[0, \pi/4]$. This problem can be easily solved by differential calculus, but we wish to avoid it and demonstrate an elementary method, more familiar to the wider circle of readers.

First, we prove the inequality

$$f(\theta) = \frac{2}{\cos\theta} - \tan\theta \geq \sqrt{3}, \quad \theta \in [0, \pi/4]. \tag{4.9}$$

Assume that (4.9) holds true. Since $\cos\theta > 0$, the inequality (4.9) can be written in the form

$$\sin\theta + \sqrt{3}\cos\theta \leq 2. \tag{4.10}$$

Using the well-known formulae

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{1-t}{2}}, \quad \cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1+t}{2}},$$

where we set $t = \cos 2\theta \in [0, 1]$, from (4.10) we get

$$\sqrt{\frac{1-t}{2}} + \sqrt{3}\sqrt{\frac{1+t}{2}} \leq 2.$$

Hence, after squaring and rearrangement, we obtain

$$t + \sqrt{3}\sqrt{1 - t^2} \leq 2 \quad \text{or} \quad \sqrt{3}\sqrt{1 - t^2} \leq 2 - t.$$

Squaring once again yields

$$1 - 4t + 4t^2 = (1 - 2t)^2 \geq 0.$$

Therefore, the inequality (4.10) is proved. From the last expression we see that the minimum of $f(\theta)$ is obtained for $t = \cos 2\theta = 1/2$, that is, $2\theta = 60^\circ$. Hence, the sought value of the angle θ is 30°.

The length of the minimal road network is

$$S(30^\circ) = 2 + 2f(30^\circ) = 2 + 2\sqrt{3} \approx 5.4641.$$

For comparison, let us note that the length of the two diagonals of the square (case $\theta = 45^\circ$) is $2\sqrt{8} \approx 5.6568$.

René Descartes (1596–1650) (→ p. 302)

Frederick Soddy (1877–1956) (→ p. 308)

Seki Kowa (1642–1708) (→ p. 303)

Harold Scott MacDonald Coxeter (1907–2003) (→ p. 309)

Scientists frequently conduct simultaneous yet independent research on the same problems. It is less seldom for researchers from disparate time periods to pay their attention to the same problem. René Descartes, the French philosopher, soldier and mathematician, Seki Kowa, the Japanese mathematician, mechanist and samurai (!), Frederick Soddy, the British physicist and Nobel Prize-winning chemist, and the famous long-lived geometer Harold (always known as Donald) Coxeter, a professor at the University of Toronto, all studied the problem of "kissing" (or touching) circles at one time in their lives.

Kissing circles

Not more than four circles can be placed in a plane so that each circle touches every other circle, with every pair touching at a different point. There are two possible situations: either three circles surround a smaller one (Figure 4.29(a)) or one larger circle contains three smaller circles inside it (Figure 4.29(b)). Thus appears the following challenging question.

Problem 4.26. *Find a relation that involves the radii of four adjoining circles which allows easy calculation of the radii of the fourth touching circle, knowing the radii of the remaining three circles.*

Frederick Soddy derived a simple formula for the radii and expressed it in an unusual way in the stanzas of his poem, *The Kiss Precise*, published in *Nature* (Vol. 137, June 20, 1936, p. 1021). Let a, b, c, and d be the reciprocals of the radii of four "kissing" circles taken with the plus sign if a circle is touched on the outside (all circles on Figure 4.29(a)), and the minus sign if a circle is touched on the inside (the circle 4 on Figure 4.29(b)). Then Soddy's formula reads

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

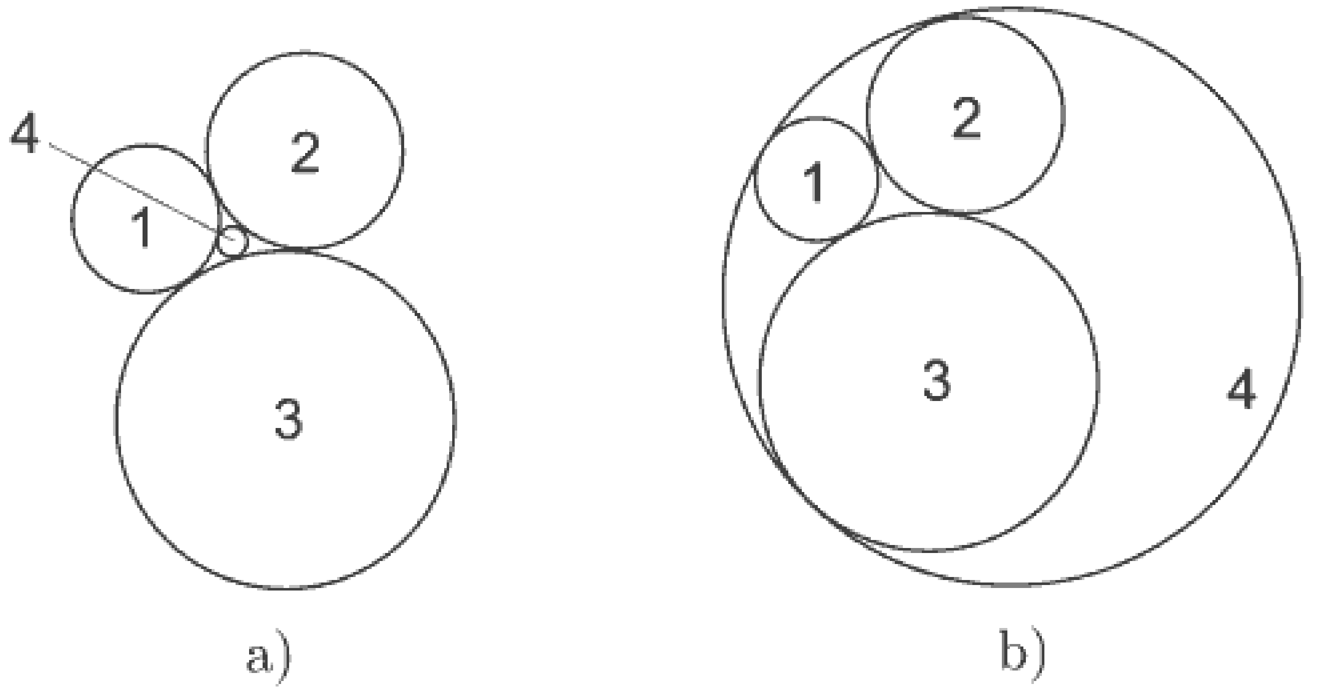


Figure 4.29. Kissing circles

The reciprocal of a circle's radius is customarily referred to as its *curvature*. Soddy's poem mentions this quantity as the *bend* as the middle verse states clearly:

> Four circles to the kissing come.
> The smaller are the benter.
> The bend is just the inverse of
> The distance from the center.
> Though their intrigue left Euclid dumb
> There's now no need for rule of thumb.
> Since zero bend's a dead straight line
> And concave bends have minus sign,
> *The sum of the squares of all four bends*
> *Is half the square of their sum.*

However, after the above formula's publication, it was discovered that Descartes already knew of this formula. For this reason, the above formula is sometimes referred to as the Descartes–Soddy formula.

We will now discuss in short the derivation of the Descartes–Soddy formula; for more details see Coxeter's book [43, pp. 13–15].

Let K_a, K_b and K_c be three mutually tangent circles with centers A, B, C. Let us form the triangle $\triangle ABC$ with sides a, b, c and the corresponding angles α, β, γ (see Figure 4.30). If $s = (a + b + c)/2$ is the semiperimeter of the triangle $\triangle ABC$, then it is easy to conclude that the lengths $s-a$, $s-b$, $s-c$ are the radii of the circles K_a, K_b, K_c, respectively.

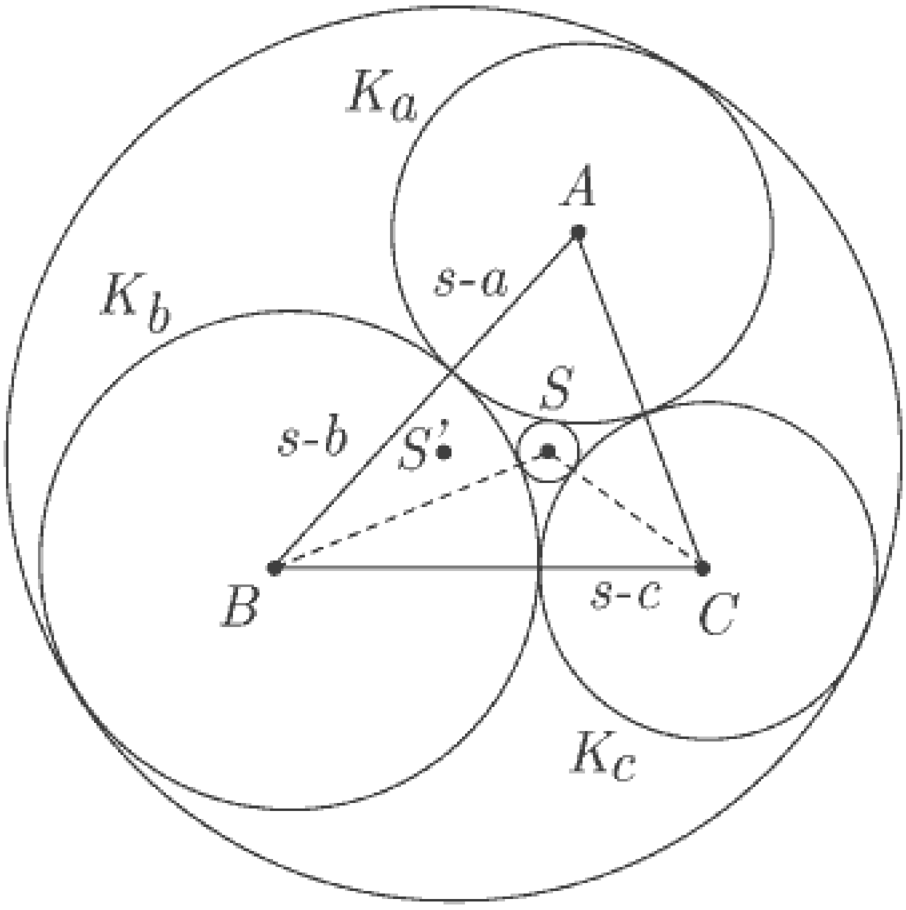


FIGURE 4.30. Coxeter's solution for the kissing circles

From Figure 4.30 we see that two circles can be drawn to touch all three circles K_a, K_b, K_c: a small inner circle K, and a larger outer circle K'. Let these two circles have centers S, S' and radii r, r'. Then

$$|SA| = r + s - a,\ |SB| = r + s - b,\ |SC| = r + s - c.$$

Let φ_a, φ_b, φ_c denote the angles at center S of the smallest circle K in the three triangles $\triangle SBC$, $\triangle SCA$, $\triangle SAB$. Applying the cosine theorem to triangle $\triangle SBC$, we obtain

$$|BC|^2 = |SB|^2 + |SC|^2 - 2|SB| \cdot |SC| \cos\varphi_a,$$

wherefrom, after a brief rearrangement,

$$\cos^2\frac{\varphi_a}{2} = \frac{1+\cos\varphi_a}{2} = \frac{(r+a)r}{(r+s-b)(r+s-c)}. \tag{4.11}$$

Hence,

$$\sin^2\frac{\varphi_a}{2} = 1 - \cos^2\frac{\varphi_a}{2} = \frac{(s-b)(s-c)}{(r+s-b)(r+s-c)}. \tag{4.11'}$$

Analogously, for the angles φ_b and φ_c we obtain

$$\cos^2\frac{\varphi_b}{2} = \frac{(r+b)r}{(r+s-a)(r+s-c)}, \tag{4.12}$$

$$\sin^2\frac{\varphi_b}{2} = \frac{(s-a)(s-c)}{(r+s-a)(r+s-c)}, \tag{4.12$'$}$$

and

$$\cos^2\frac{\varphi_c}{2} = \frac{(r+c)r}{(r+s-a)(r+s-b)}, \tag{4.13}$$

$$\sin^2\frac{\varphi_c}{2} = \frac{(s-a)(s-b)}{(r+s-a)(r+s-b)}. \tag{4.13$'$}$$

Combining the sine theorem and the cosine theorem for any triangle with the angles α, β, γ, we can derive the following identity:

$$\sin^2\alpha - \sin^2\beta - \sin^2\gamma + 2\sin\beta\sin\gamma\cos\alpha = 0. \tag{4.14}$$

This relation holds for any three angles whose sum is 180°. Since the angles $\frac{1}{2}\varphi_a$, $\frac{1}{2}\varphi_b$, $\frac{1}{2}\varphi_c$ satisfy this condition, from (4.14) we have

$$\sin^2\frac{\varphi_a}{2} - \sin^2\frac{\varphi_b}{2} - \sin^2\frac{\varphi_c}{2} + 2\sin\frac{\varphi_b}{2}\sin\frac{\varphi_c}{2}\cos\frac{\varphi_a}{2} = 0. \tag{4.15}$$

For the sake of brevity, let

$$\begin{aligned} f_a &= (r+s-b)(r+s-c), \\ f_b &= (r+s-a)(r+s-c), \\ f_c &= (r+s-a)(r+s-b). \end{aligned}$$

Using (4.11), (4.11$'$), (4.12), (4.12$'$), (4.13) and (4.13$'$), from identity (4.15) we obtain

$$\frac{(s-b)(s-c)}{f_a} - \frac{(s-c)(s-a)}{f_b} - \frac{(s-a)(s-b)}{f_c} + 2\sqrt{\frac{(s-c)(s-a)}{f_b}\cdot\frac{(s-a)(s-b)}{f_c}\cdot\frac{r(r+a)}{f_a}} = 0.$$

Hence

$$\frac{r+s-a}{s-a} - \frac{r+s-b}{s-b} - \frac{r+s-c}{s-c} + 2\sqrt{\frac{r(r+s-b+s-c)}{(s-b)(s-c)}} = 0.$$

After dividing by r and using abbreviations

$$\sigma_1 = \frac{1}{s-a}, \quad \sigma_2 = \frac{1}{s-b}, \quad \sigma_3 = \frac{1}{s-c}, \quad \sigma_4 = \frac{1}{r},$$

we come to the relation

$$\sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 + 2\sqrt{\sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_2} = 0.$$

After squaring, we get

$$\begin{aligned}(\sigma_1 - \sigma_2 - \sigma_3 - \sigma_4)^2 &= \big(2\sigma_1 - (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)\big)^2 \\ &= 4(\sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_2).\end{aligned}$$

Hence, after elementary manipulations, we finally obtain

$$2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2) = (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)^2. \tag{4.16}$$

The reciprocals σ_1, σ_2, σ_3, σ_4 of the radii are called their *curvatures*. Solving the quadratic equation (4.16) for σ_4 (the curvature of either K or K'), we obtain two roots

$$\sigma_1 + \sigma_2 + \sigma_3 \pm 2\sqrt{\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2}\ .$$

The upper sign yields the larger curvature, that is, the smaller circle. Therefore, the radii of circles K and K' are

$$\begin{aligned} r &= \Big[\sigma_1 + \sigma_2 + \sigma_3 + 2\sqrt{\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2}\Big]^{-1}, \\ r' &= \Big[\sigma_1 + \sigma_2 + \sigma_3 - 2\sqrt{\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2}\Big]^{-1}. \end{aligned}$$

In 1670, three decades after the death of Descartes, on the other side of the world, Sawaguchi Kazuyuki, a Japanese mathematician, wrote a work entitled, *Kokon Sampō-ki* (Old and New Method in Mathematics). Kazuyuki formulated a problem that incorporated four kissing circles (see [167, Vol. I, p. 439], [168]).

Problem 4.27. *Three circles are inscribed within a circle, each tangent to the other two and to the original circle (Figure* 4.31*). They cover all but* 120 *square units of the circumscribing circle. The diameters of the two smaller circles are equal and each is* 5 *units less than the diameter of the next larger one. Find the diameters of the three inscribed circles.*

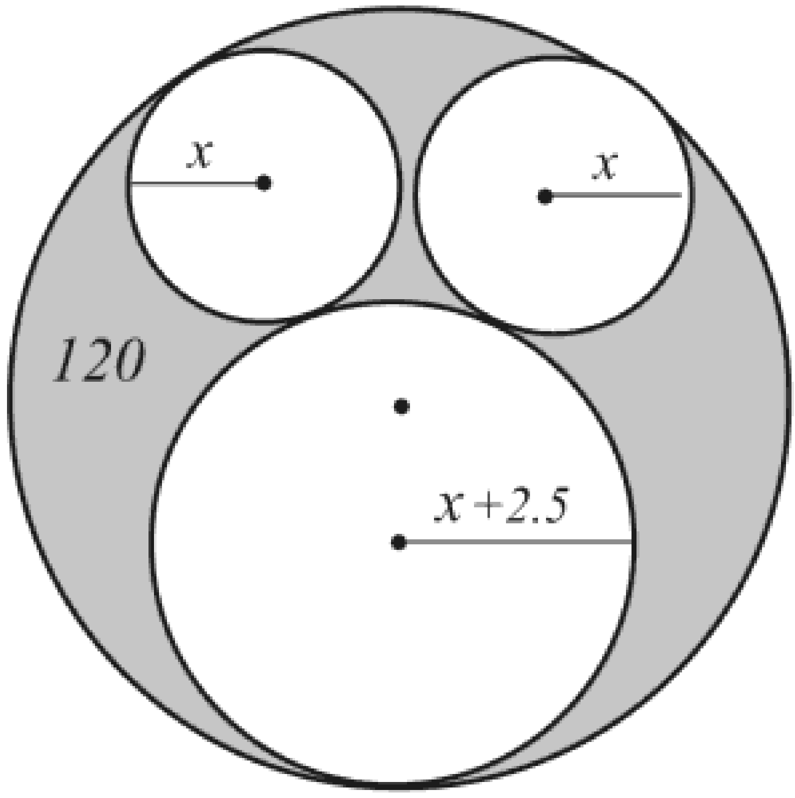


Figure 4.31. Kazuyuki's kissing circles problem

According to Smith and Mikami [168], Seki Kowa, the most distinguished mathematician of seventeenth-century Japan, solved this problem. Seki Kowa used only words to express his method of solution, and arrived at an equation of the sixth degree [168, pp. 96–97]. Did Seki Kowa have knowledge of Descartes–Soddy's formula? Through analysis of Kazuyuki's solution of the problem, we might perhaps answer this question in part.

Let x $(= r_1 = r_2)$ be the radius of the two smallest circles and y $(= r_4)$ the radius of the largest circle. Then the radius of the remaining circle is $r_3 = x + 2.5$. Using the condition concerning the covered area, we can form the equation

$$2x^2 + (x + 2.5)^2 = y^2 - \frac{120}{\pi}. \tag{4.17}$$

But now the crucial question arises: How to eliminate one of the two variables in (4.17)? Seki Kowa had to have an auxiliary relation necessary for the elimination. According to the reconstruction of Seki's solution given by his pupil Takabe [168, pp. 96–100], and considering the analysis of the prominent historians of mathematics Smith and Mikami, the required relation was most probably the Descartes–Soddy formula. The fact that a wooden table engraved with the formula was found in Tokyo's prefecture in 1796 (see [68]) appears to substantiate this.

In his poem *The Kiss Precise*, Frederick Soddy expanded the kissing circles formula to five mutually kissing spheres,

$$3(a^2 + b^2 + c^2 + d^2 + e^2) = (a + b + c + d + e)^2, \tag{4.18}$$

where a, b, c, d, and e are reciprocals of the radii of five spheres taken with the corresponding signs. Is the reader nervous about n-spheres ($n > 3$)? "Mission impossible?" What is most astonishing is that there exists a beautiful formula for this general case. Thorold Gosset of Cambridge University made further generalization on n-spaces which he published in *Nature* (Vol. 139, January 1937). Gosset has shown that in the case of n-dimensional spheres at most $n+2$ spheres "touch" each other and satisfy the following generalized formula:

$$n\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\cdots+\frac{1}{r_{n+2}^2}\right)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\cdots+\frac{1}{r_{n+2}}\right)^2.$$

The last formula is like a shining star in the sky of geometry. Let us stop here: formulae are given, and a diligent reader is welcome to solve the following simple problem.

Problem 4.28.* *Four touching unit spheres are arranged in the form of "rounded" tetrahedron leaving a space for a fifth sphere. Find the radius of that sphere.*

George Pólya (1887–1985) (→ p. 308)

George Pólya was a versatile mathematician whose output of mathematical publications was not only voluminous, but also wide ranging. As well as working in many mathematical disciplines, he also taught; his popular works such as *How to Solve It*, *Mathematical Discovery* (two volumes), and *Mathematics and Plausible Reasoning* (two volumes) contain a wealth of useful, pleasing, and amusing examples. We have selected one from his *Mathematics and Plausible Reasoning* [141, Vol. 2].

The shortest bisecting arc of area

Problem 4.29. *Let us define a bisecting arc as a simple curve which bisects the area of a given region. The shortest bisecting arc of a circle is its diameter and the shortest bisecting arc of a square is its altitude through the center. What is the shortest bisecting arc of an equilateral triangle?*

Many mathematicians, amateurs and professionals alike respond with the standard answer of a chord parallel to the base since it is shorter than the angle bisector (altitude). This answer, however, is incorrect.

To answer correctly, let us recall the solution of a classic *isoperimetric problem*, isoperimetric figures being figures that have the same perimeter.

Among all closed plane curves of a given length, a circle encloses the largest area.

Actually, this is Jacob Steiner's solution of Dido's problem, found on pages 96–98 of this book.

Let us return to Pólya's problem. Let ω be an eventual bisecting arc for a given equilateral triangle $\triangle$. By five reflections of a given equilateral triangle $\triangle$, performed in such a way that the next (copied) bisecting arc starts at the end of the former bisecting arc, we create a regular hexagon as shown in Figure 4.32.

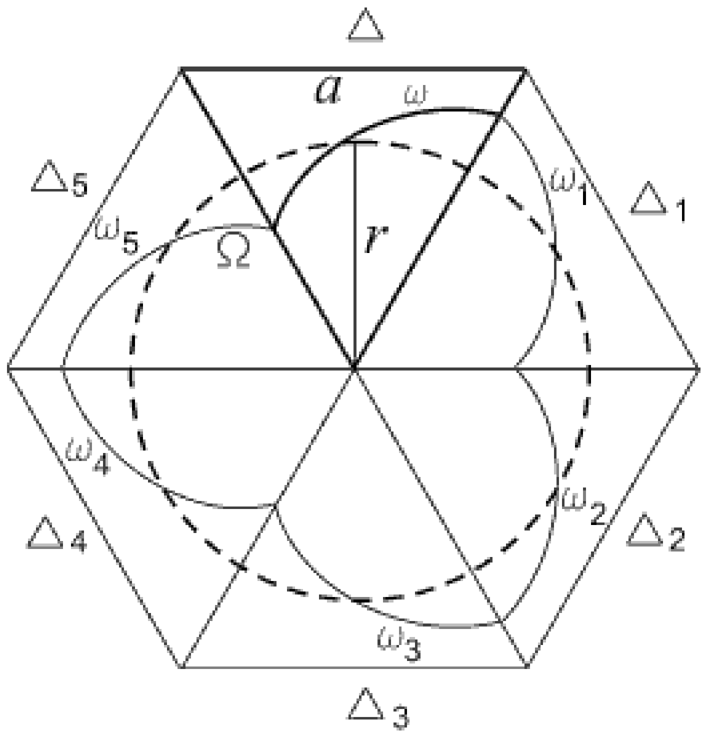


FIGURE 4.32. Shortest bisecting arc of area

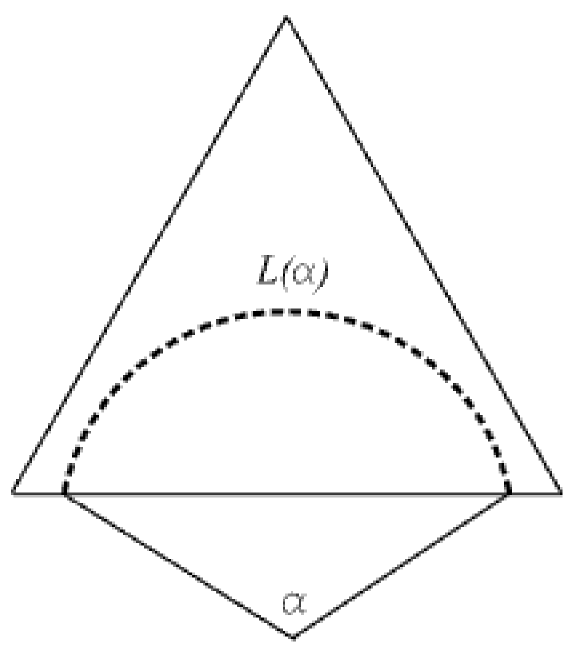


FIGURE 4.33. Arc on one side

The new curve $\Omega = \omega \cup \omega_1 \cup \cdots \cup \omega_5$, where $\omega_1, \ldots, \omega_5$ are the copies of ω, is closed and it is a candidate for the bisecting closed curve of the hexagon. The bisecting arc ω must have such a shape so that the curve Ω encloses half of the hexagon's area. Thus, to find ω, we are looking for a shape of Ω such as that having the shortest perimeter.

According to the solution of the isoperimetric problem given above, a circle has the shortest perimeter among all shapes with the given area. Therefore, the closed curve Ω is a circle, denoted by a dashed line, and, consequently, the shortest bisecting arc of the equilateral triangle $\triangle$ is a 60° arc of a circle. From the relation $r^2\pi/6 = a^2\sqrt{3}/8$, where a is the side of the equilateral triangle $\triangle$, we find that the radius of this circle is

$$r = a\sqrt{\frac{3\sqrt{3}}{4\pi}} \approx 0.643\,a.$$

Finally, we show that the length of a circular arc, which bisects the area of the triangle and has its ends on one side of the triangle (Figure 4.33), is always longer than the bisecting arc presented above in Figure 4.32. Of course, from the aforementioned reasons, a curve of a different shape than a circular arc is always longer. Taking $a = 1$, we present on Figure 4.34 the dependence of the length $L(\alpha)$ of a circular arc on the central angle $\alpha \in (0, \pi)$. The straight line $L = 0.6833$ is the length of the shortest bisecting arc given in Figure 4.32, shown for the purpose of comparison. End of proof, end of mystery.

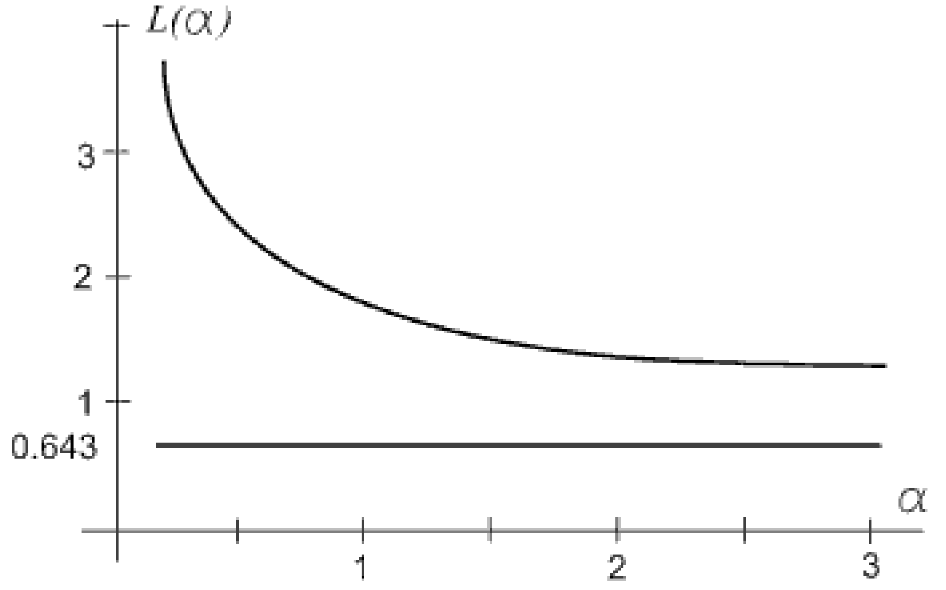


FIGURE 4.34. The length of a circular arc against the central angle

Answers to Problems

4.2. The proof is elementary and uses only the Pythagorean theorem. Let O, O_1 and O_2 be the centers of the semicircles with the diameters AB, AC and CB, respectively, and let

$$\frac{|AB|}{2} = r, \quad \frac{|AC|}{2} = r_1, \quad \frac{|CB|}{2} = r_2, \quad r = r_1 + r_2.$$

Let OF be the radius line of the greatest semicircle passing through the centers O and G_1 and draw the lines O_1G_1 and OG_1. Finally, draw two perpendiculars from G_1 to the lines CD and AB to the point H, as shown in Figure 4.35. Denote the radii of the circles centered at G_1 and G_2 with ρ_1 and ρ_2. Then we have

$$|O_1G_1| = r_1 + \rho_1, \quad |OG_1| = r - \rho_1 = r_1 + r_2 - \rho_1,$$
$$|O_1H| = r_1 - \rho_1, \quad |OH| = r - 2r_2 - \rho_1 = r_1 - r_2 - \rho_1.$$

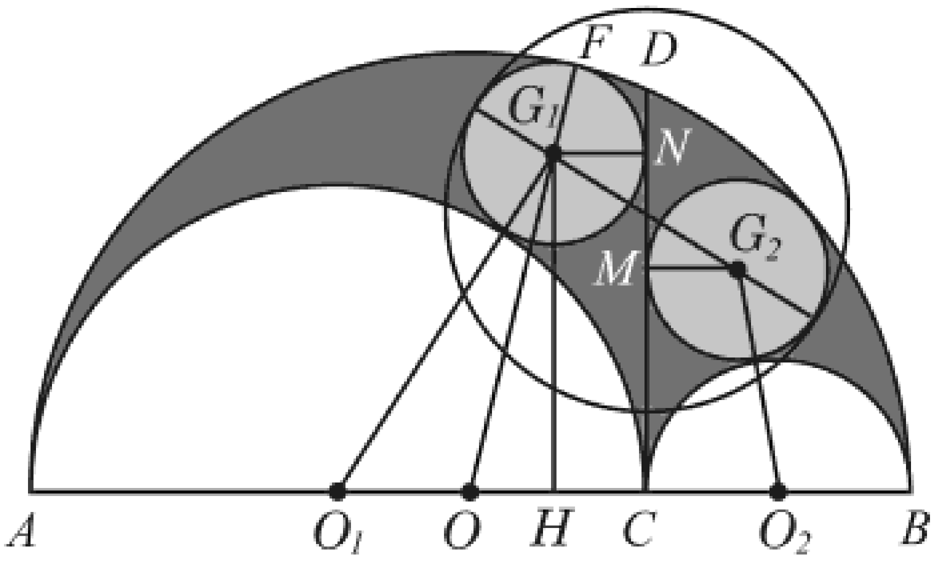


Figure 4.35. The Arhimedean circles—solution

From the right triangles O_1G_1H and OG_1H we obtain by the Pythagorean theorem

$$|HG_1|^2 = (r_1 + \rho_1)^2 - (r_1 - \rho_1)^2,$$
$$|HG_1|^2 = (r_1 + r_2 - \rho_1)^2 - (r_1 - r_2 - \rho_1)^2.$$

After equalizing the right-hand sides and short arrangement, we get

$$4r_1\rho_1 = 4(r_1 - \rho_1)r_2,$$

hence

$$\rho_1 = \frac{r_1 r_2}{r_1 + r_2}.$$

Applying a similar method to the circle centered at G_2 (simply exchanging the subscript indices 1 and 2), we obtain the same result,

$$\rho_2 = \frac{r_1 r_2}{r_1 + r_2} = \rho_1. \tag{4.19}$$

Therefore, the twin circles of Archimedes have the same radius equal to half the harmonic mean of the radii r_1 and r_2.

4.3. We will use the notation from the solution of Problem 5.2 and Figure 4.35. The Archimedean twins shown in Figure 4.35 have the same radii $\rho = \rho_1 = \rho_2 = r_1r_2/(r_1 + r_2)$; see (4.19). By the Pythagorean theorem we find that

$$|MC| = \sqrt{|G_2O_2|^2 - (r_2 - \rho)^2} = \sqrt{(r_2 + \rho)^2 - (r_2 - \rho)^2} = 2\sqrt{r_2\rho}.$$

Similarly, we obtain $|NC| = 2\sqrt{r_1\rho}$. Hence, $|NM|^2 = (|NC| - |MC|)^2 = 4\rho(\sqrt{r_1} - \sqrt{r_2})^2$.

The quadrilateral G_1NG_2M is a parallelogram so that, using (4.19),

$$\begin{aligned}|G_1G_2| &= 2\sqrt{(|NM|/2)^2 + \rho^2} = 2\sqrt{\rho(\sqrt{r_1} - \sqrt{r_2})^2 + \rho^2}\\ &= 2\sqrt{r_1r_2 - 2\sqrt{r_1r_2} + \rho^2} = 2(\sqrt{r_1r_2} - \rho) \quad (\sqrt{r_1r_2} > \rho).\end{aligned}$$

Since the smallest circle tangents the twin circles, its diameter d is then given by

$$d = 2\rho + |G_1G_2| = 2\rho + 2(\sqrt{r_1r_2} - \rho) = 2\sqrt{r_1r_2} = \sqrt{|AC| \cdot |CB|} = |CD|.$$

4.5. It is easy to notice that this problem is equivalent to Problem 4.1. Indeed, halving the given circles, shown in Figure 4.36, by the diameter perpendicular to the tangent (of the length t), we obtain Figure 4.1 with $|CD| = t/2$. Hence, the area of Pólya's figure is

$$S = 2 \cdot \frac{\pi}{4}|CD|^2 = \frac{\pi}{2}\left(\frac{t}{2}\right)^2 = \frac{\pi t^2}{8}.$$

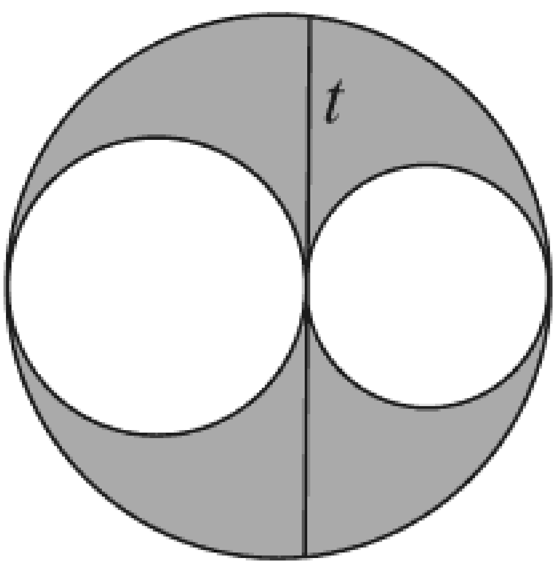


FIGURE 4.36. The arbelos problem of George Pólya

Pólya's problem itself is more intriguing if it is considered independently of the arbelos problem because, at first sight, it seems that the given data are insufficient to find the required area.

4.6. The presented proof follows the argumentation given in Tien's paper [175]. Let r be the radius of two small semicircles K_3 and K_4, and R the radius of a larger semicircle K. The radii $r_1 = d_1/2$ and $r_2 = d_2/2$ of the circles K_1 and K_2 are changeable unlike the fixed entries r and R. A "backwards" method of drawing will be used to solve the problem.

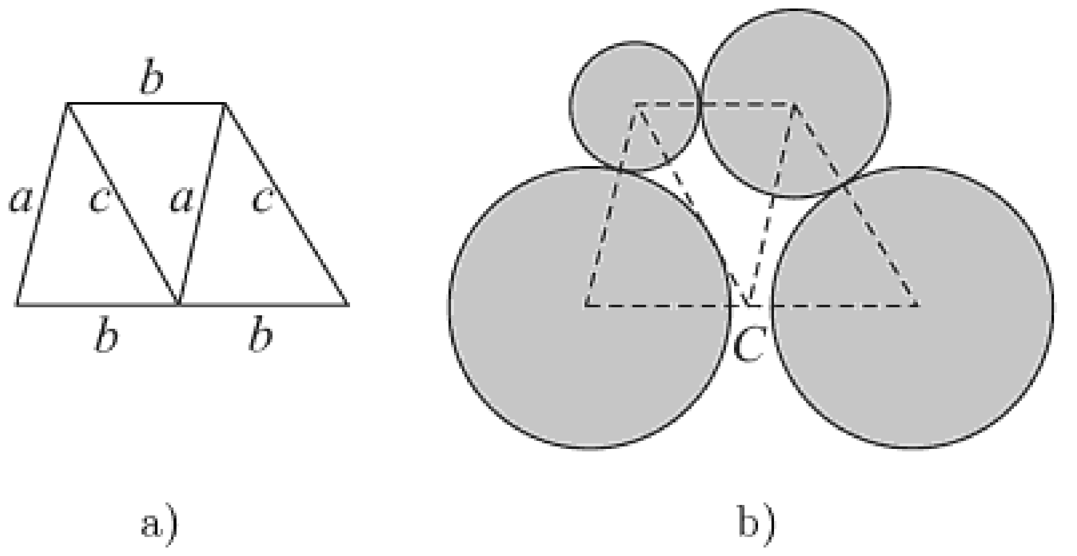


Figure 4.37.

Let us construct a quadrilateral composed of three equal triangles with sides $a = r + r_1$, $b = r_1 + r_2$, and $c = r + r_2$, as in Figure 4.37(a). Obviously, this quadrilateral is a trapezoid. Next draw circles of radii r, r_1, r_2, and again r, centered at the vertices of these triangles, except for the common central vertex C (Figure 4.37(b)). It is easy to see that the radii and sides are chosen in such a way that these four circles touch each other (except for K_3 and K_4). Since

$$c + r_1 = r + r_2 + r_1, \quad a + r_2 = r + r_1 + r_2$$

and

$$2R = 2r + 2b = 2r + d_1 + d_2,$$

it follows that $c + r_1 = a + r_2 = R$. This means that the circles K_1 and K_2 touch the big circle K centered in the central vertex C. Thus, we return to our starting point, Figure 4.6.

Therefore, if $r + r_1 + r_2 = R$, then the described configuration with touching circles is possible. To complete the proof, we need to prove the opposite claim. Assume that for the configuration displayed in Figure 4.6, we have $r + r_1 + r_2 \neq R$. Let us change r_1, for example, so that a new value r_1' satisfies $r + r_1' + r_2 = R$. According to the previous analysis, this would mean that there is a new configuration formed by touching circles of radii r, r_1' $(\neq r_1)$, r_2 and r inside a circle of radius R. Now we have two configurations where all circles are the same except for one (K_1). Yet such a situation is impossible since the "hole" determines uniquely the size that fits into it. In this way we obtain that $r + r_1 + r_2 = R$ holds, wherefrom $d_1 + d_2 = 2R - 2r = \text{const}$. This completes the proof.

4.9. Let the half-straight lines p_A and p_B denote the river banks and let the point C mark the position of the cottage. Let C_A and C_B be the points symmetric to C with respect to the half-straight lines p_A and p_B, respectively (Figure 4.38). Join C_A to C_B and denote with A_0 and B_0 the intersecting points of the straight line C_AC_B and the half-straight lines p_A and p_B. The intersecting points A_0 and B_0 determine the places on the river banks that provide the shortest tour CA_0B_0C of the adventurer. Indeed, if A' and B' are arbitrary points on p_A and p_B, then

$$\begin{aligned}|CA'| + |A'B'| + |B'C| &= |C_AA'| + |A'B'| + |B'C_B| \\ &\geq |C_AA_0| + |A_0B_0| + |B_0C_B| = |C_AC_B|.\end{aligned}$$

In other words, the shortest way between the points C_A and C_B goes in a straight line.

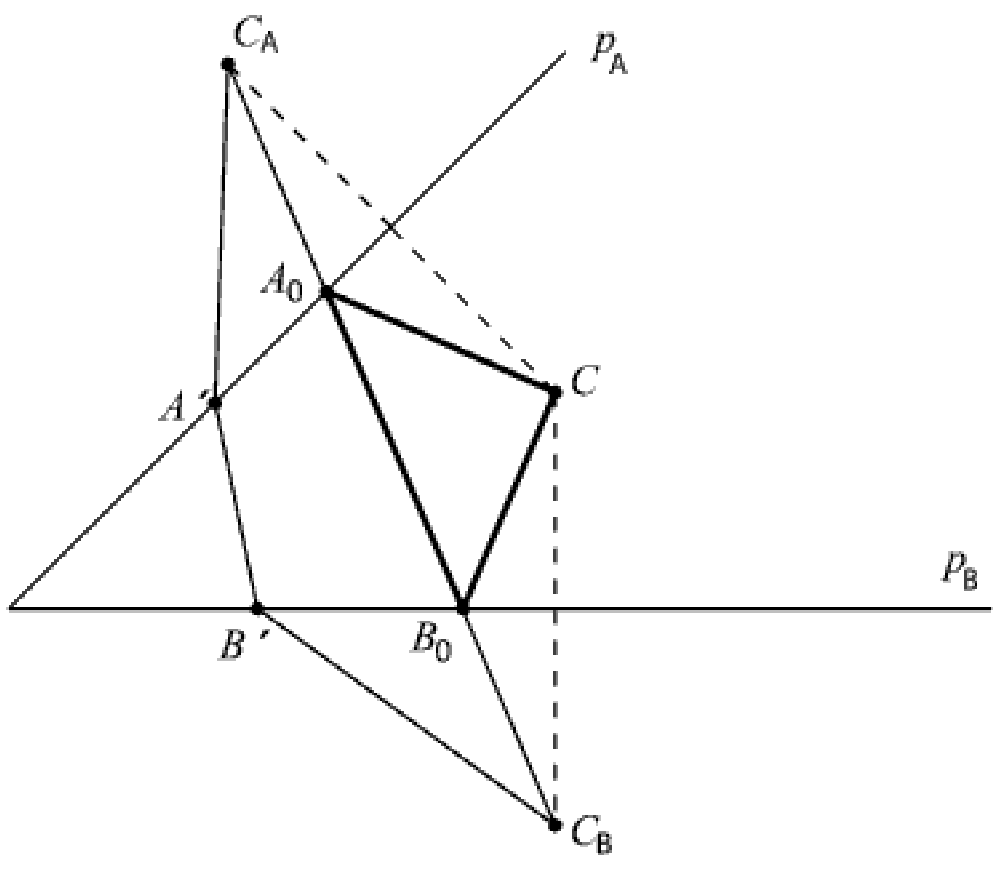


FIGURE 4.38. The shortest tour

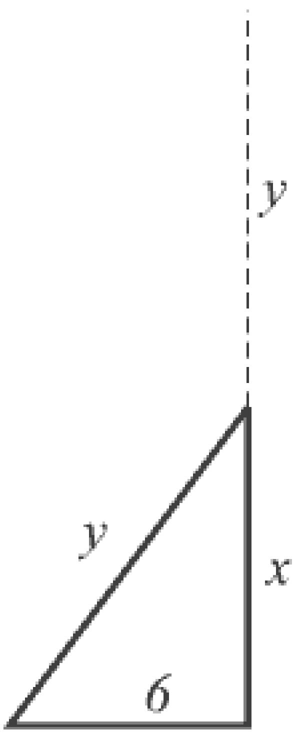


FIGURE 4.39. Broken bamboo

4.11. Let x and y be the lengths of segments of the broken bamboo. From the system of equations

$$x + y = 18, \quad y^2 - x^2 = 36$$

(see Figure 4.39), we find $x = 8,\ y = 10$, or opposite.

4.15. Wafa's idea, exposed in solving Problem 4.14, can be also usefully applied in this case. First, we arrange three larger triangles around the small triangle as shown in Figure 4.40. Joining vertices by the dotted lines, we construct a larger triangle. The three pieces outside this triangle fit the spaces inside its boundary exactly (see [186, p. 192]).

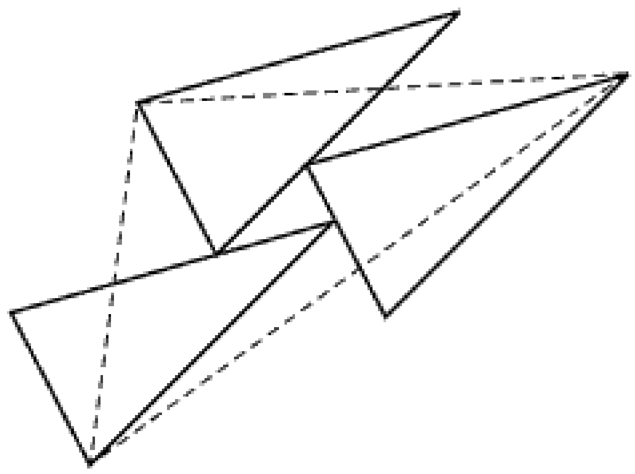


FIGURE 4.40. Dissection of four triangles

4.20. Let A, B and C be the vertices of a given triangle $\triangle ABC$, we have to find a point X in the plane determined by the points A, B, C so that the sum of lengths $|XA| + |XB| + |XC$ is minimal.

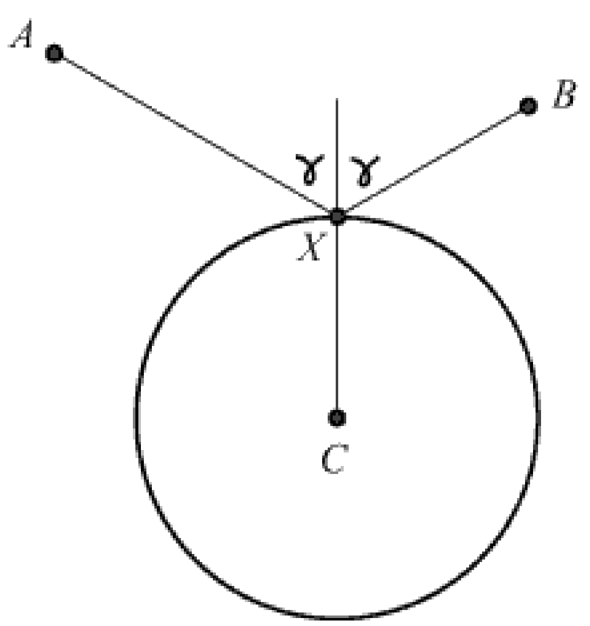


FIGURE 4.41.

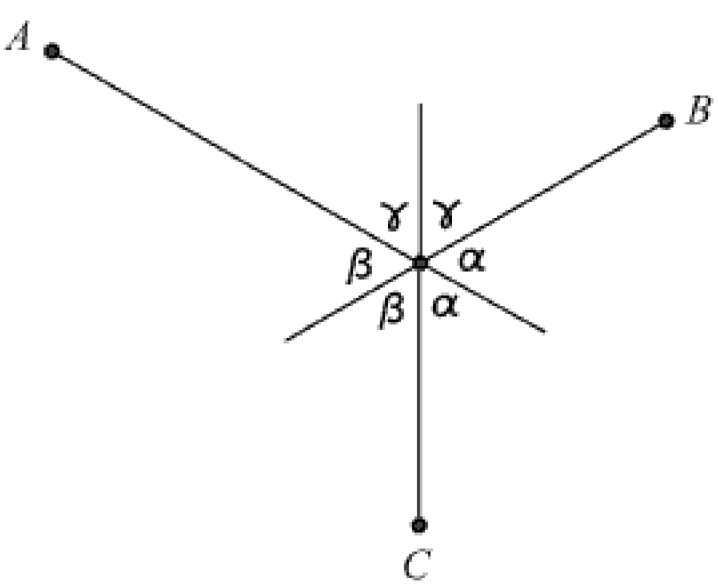


FIGURE 4.42.

Assume for a while that the distance $|CX|$ is fixed and equal to (say) r. Then we can apply the Heron principle of minimum considered in Problem 4.7 in a specific way. Namely, with the distance $|CX|$ fixed, let us find the minimum of the sum $|AX| + |BX|$, but with the point X traversing along a circle (with radius r and centered in C) instead of the line (Figure 4.41). One can presume that in this task we have an image in the "round mirror". Let A be the source of a light ray, which reflects from the round mirror in its way towards the receiver B. Since the angle of incidence is equal to the angle of reflection, the straight line passing through A and X must be the bisector of the angle $\measuredangle AXB$ to provide minimal distance.

For symmetry, the straight lines through B and X, and through A and X, should be also the bisectors of the angles $\measuredangle CXA$ and $\measuredangle BXC$, respectively. Three straight lines, connecting the point X to the points A, B and C, form six angles with the common vertex X. It is obvious from the Figure 4.42 that all these angles are mutually equal (according to the equality of opposite angles), hence it follows that each of them is equal to 60°. Consequently, $\measuredangle AXB = \measuredangle BXC = \measuredangle CXA = 120^\circ$, which means that all sides of the triangle ABC are seen at the angle of 120°, the same result obtained as before. The location of the Torricelli point X is given in the discussion of Problem 4.7.

4.24. Let P_n^* be the required maximum possible number of bounded parts. From the solution of Problem 4.23 we conclude that the mth line $(m \geq 3)$ intersects the previous lines in $m-1$ distinct points increasing the number of parts by m, two new infinite parts and $m-2$ bounded parts. Therefore, we have

$$P_3^* = 1,\ P_4^* = P_3^* + (4-2),\ P_5^* = P_4^* + (5-2),\ \ldots\ , P_n^* = P_{n-1}^* + (n-2).$$

The telescoping summation gives

$$\sum_{m=3}^{n-1} \left(P_{m+1}^* - P_m^*\right) = P_n^* - P_3^* = 2 + \cdots + (n-2).$$

Since $P_3^* = 1$, it follows that

$$P_n^* = 1 + 2 + \cdots + (n-2) = \frac{(n-1)(n-2)}{2}.$$

4.28. Let x be the curvature of the fifth sphere. From formula (4.18) for five kissing spheres we have

$$3(1^2 + 1^2 + 1^2 + 1^2 + x^2) = (1 + 1 + 1 + 1 + x)^2,$$

hence the quadratic equation $x^2 - 4x - 2 = 0$ follows. The solutions of this equation give the values of curvatures $x_1 = 2 + \sqrt{6} > 0$ and $x_2 = 2 - \sqrt{6} < 0$. Since we are concerned with the smaller sphere, we choose x_1 (all five spheres are touched outside, all the corresponding signs of curvatures are plus) and find the required radius $r_1 = 1/x_1 = -1 + \sqrt{6}/2 \approx 0.2247$. The other value $r_2 = |1/x_2| = |-1 - \sqrt{6}/2| \approx 2.2247$ is the radius of the larger sphere which encloses and touches four given spheres.

Chapter 5 TILING AND PACKING

Sphere packing is one of the most fascinating and challenging subjects in mathematics.
Chuanming Zong

In this chapter we consider the filling of a plane and a (3-dimensional) space. If the plane figures fill a plane with no overlaps and no gaps, then we most often use the terms *tessellation* or *tiling* of the plane for it. It is needless to say that, from the ancient times to the present, the tessellation has been an everyday job in creating mosaics and a variety of decorations for building walls, windows, floors and yards. After all, the word *tessella* comes from Latin to denote a small cubical piece of clay, stone or glass used to make mosaics. It is worth noting that the patterns of tessellations can be found in nature. For example, hexagonal columns of lava, resulting from an ancient volcanic eruption, can be seen at the Giant's Causeway in Ireland. Another example is the so-called Tessellated pavement in Tasmania, an unusual sedimentary rock formation, with the rock fracturing into rectangular blocks.

In the first part we present regular tessellations (consisting of congruent regular polygons) studied by Johannes Kepler and semiregular tessellations which permit more kinds of regular polygons (one of eight patterns found by Kepler). Both of these tessellations are periodic giving a region that tiles the plane by translation. It is well known that the Dutch artist M. C. Escher made many famous pieces of art with periodic tessellations with shapes resembling living things.

You will also find nonperiodic tessellation of the eminent British mathematical physicist Roger Penrose. Penrose's tilings, which use two different polygons, are the most famous examples of tessellations creating nonperiodic patterns. We have also included the very recent Conway and Radin's pinwheel tiling.

The filling of the 3-dimensional space is the subject of the second part of this chapter. Clearly, a tessellation of a space (often referred to as honeycombs) is a special case of filling the space since the latter permits the appearance of gaps. However, in this book we focus on those puzzle problems that have been posed and/or solved by great mathematicians. So, we omit the tessellation of the 3-dimensional space and turn our attention to another fascinating recreational subject: packing spheres.

We can find mathematical patterns all around us. Consider, for example, the familiar arrangement of fruits at the market, and look closely at the piles of fruits. How are they arranged? Is the arrangement of spherically-shaped fruits, oranges perhaps, the most efficient in terms of optimal density? In other words, does the packing result in the maximum number of oranges that can fit into the available space?

The problem familiar to greengrocers and market vendors of stacking oranges and the distinctly mathematical pattern seen in such arrangements, is known as *sphere packing*. What is the most efficient way to pack identical spheres? Johannes Kepler was the first who tried to give the answer to this question (in 1611), known as Kepler's conjecture.

When David Hilbert announced his famous list of 23 problems in Paris at the International Congress of Mathematicians of 1900, he noted that the first criterion for a good problem was that it could be explained *"to the first person one meets in the street."* According to its very simple formulation, the sphere-packing problem certainly satisfies Hilbert's criterion for a good problem. However, *"the problem is so simple that there is some hope that some day we may be able to solve it"*, as the Swiss mathematician Peter Henrici once said for some classes of simply-formulated problems. Indeed, despite extensive investigations dating back at least as far as Kepler's seventeenth-century work, the basic question of the most efficient packing remained unanswered until 2005 when Thomas Hales solved Kepler's sphere-packing problem.

David Gregory and Isaac Newton discussed another type of sphere-packing problem in 1694, known as the Gregory–Newton problem which concerns the number of touching spheres. Problems such as these and others related to them, have attracted the attention not only of outstanding mathematicians such as Dirichlet, Gauss, Hermite, Lagrange, and Minkowski, but also that of several contemporary mathematicians.

Conway and Sloane [41], Fejes Tóth [63], Rogers [148], Zong [194], and Szpiro [172] have all written books that deal with various aspects of sphere packing. These books, and a number of citations within them, suggest that sphere packing has developed into an important discipline. In his book [194], C. Zong says: *"As work on the classical sphere packing problems has progressed many exciting results have been obtained, ingenious methods have been created, related challenging problems have been proposed and investigated, and surprising connections with other subjects have been found."*

In this chapter we will present brief essays of the two best known problems related to sphere packing: the *kissing spheres* of David Gregory and Newton and the *densest sphere packing* of Kepler. The story about the latter problem includes a short discussion on the very recent Hales' solution of Kepler's conjecture. Knuth's pentamino-puzzle, Penrose nonperiodical tiling and Conway's cube puzzles are also considered.

*

* *

Johannes Kepler (1571–1630) (→ p. 302)

Mosaics

In addition to sphere-packing problems and space-filling problems with regular polyhedra, Kepler was also interested in mosaics—the filling of a plane with regular, although not necessarily like, polygons. Here we present some interesting tasks concerning this subject, listed in [63, p. 273], [74, Ch. 17] and [137, Ch. 3].

(I) Let us consider the filling of the plane with congruent regular polygons and let n be the number of sides of each polygon. Then the interior angle at each vertex of such a polygon is $(n-2)180°/n$.

(II) If the vertex of one polygon cannot lie along the side another, then the number $V(n)$ of polygons at each vertex is given by (using (I))

$$V(n) = \frac{360n}{180(n-2)} = 2 + \frac{4}{n-2} .$$

Hence it follows that n must be 3, 4 or 6; in other words, at each vertex we can join $V(3) = 6$ equilateral triangles, $V(4) = 4$ squares and $V(6) = 3$ regular hexagons. The creation of mosaics by these patterns is called a *tessellation.* A tessellation is *regular* if it is made up from regular polygons of the same kind. As shown above, there are only three such tessellations: mosaics of equilateral triangles, squares and hexagons; see Figure 5.1. *Semiregular* tessellations permit combinations of two or more kinds of regular polygons, not necessarily different.

Problem 5.1. *Assume that we have a mosaic composed of regular polygons of three different kinds at each vertex. Find all patterns of regular polygons that tile the plane.*

If the three kinds of polygons have $n_1,\ n_2,\ n_3$ sides, then the following is valid:

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{1}{2} \tag{5.1}$$

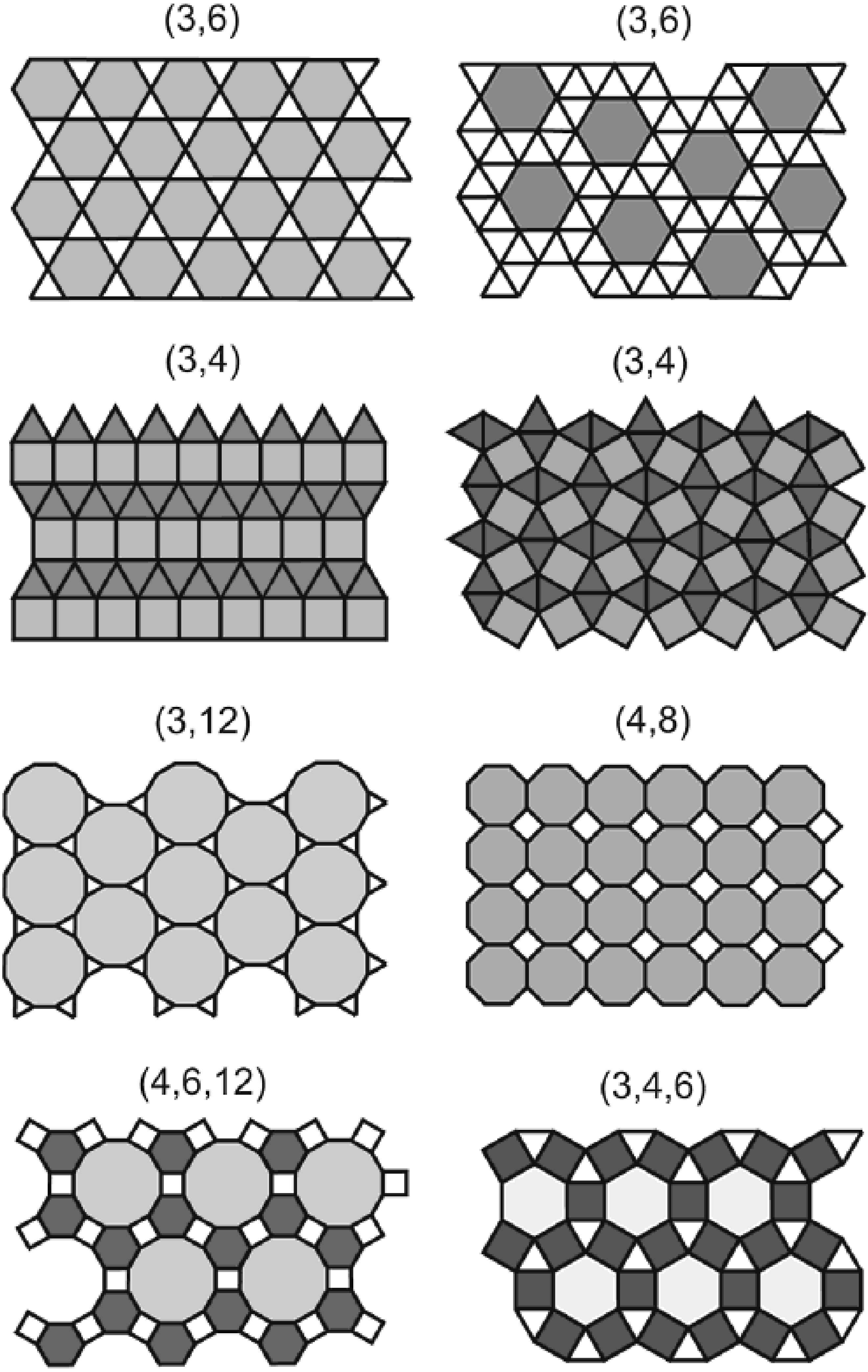


Figure 5.1. Eight patterns of regular polygons that tile the plane

(it follows from $\alpha_1+\alpha_2+\alpha_3 = 360^\circ$, where $\alpha_k = (n_k-2)180^\circ/n_k$ $(k = 1,2,3)$ (from (I)). Let us find integral solutions of the equation (5.1). From (5.1) we find

$$n_3 = 2 + \frac{4}{t-2}, \quad \text{where } t = \frac{n_1 n_2}{n_1 + n_2}\,. \tag{5.2}$$

Obviously, t must be 3, 4, or 6. From the equation

$$\frac{n_1 n_2}{n_1 + n_2} = t$$

we find

$$n_1 = t + \frac{t^2}{n_2 - t}, \quad t \in \{3, 4, 6\}.$$

The quantity n_2 takes those values from the interval $[t+1, t^2+t]$ which provide that the ratio $t^2/(n_2 - t)$ is a positive integer. Considering the three cases $t = 3$, $t = 4$ and $t = 6$, we find n_2 and n_1 and, finally, from (5.2) we determine n_3. All integral solutions of the equation (5.1), given by the triples (n_1, n_2, n_3), are

$$(3,7,42),\ (3,8,24),\ (3,9,18),\ (3,10,15),\ (4,5,20),\ (4,6,12).$$

These six triples of integral solutions are only candidates for composing a mosaic. In other words, relation (5.1) gives only necessary conditions. It was shown that $(4,6,12)$ is the only triple of *different* congruent regular polygons that can form a mosaic under the given conditions;[1] thus, we can compose a mosaic combining congruent squares, congruent regular hexagons, and congruent regular dodecagons. This pattern is displayed in Figure 5.1, in the last row (left).

An interesting fact is that there are precisely eight semi-regular tessellations, composed of different combinations of triangles, squares, hexagons, octagons and dodecagons (see Figure 5.1). A pattern $(3,6)$ in the upper right-hand corner was first described by Johannes Kepler and it is the only one (among the eight) with the property that it is changed by mirror reflection. The fourth pattern $(3,4)$ on the right side of the second row, served as the inspiration for a Salvador Dali painting that he titled simply *Fifty Abstract Pictures Which as Seen from Two Yards Change into Three Lenins Masquerading as Chinese and as Seen from Six Yards Appear as the Head of a Royal Bengal Tiger.*[2]

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[1] The pattern (3,4,6) has two squares at each vertex.

[2] See M. Gardner [74, p. 207].

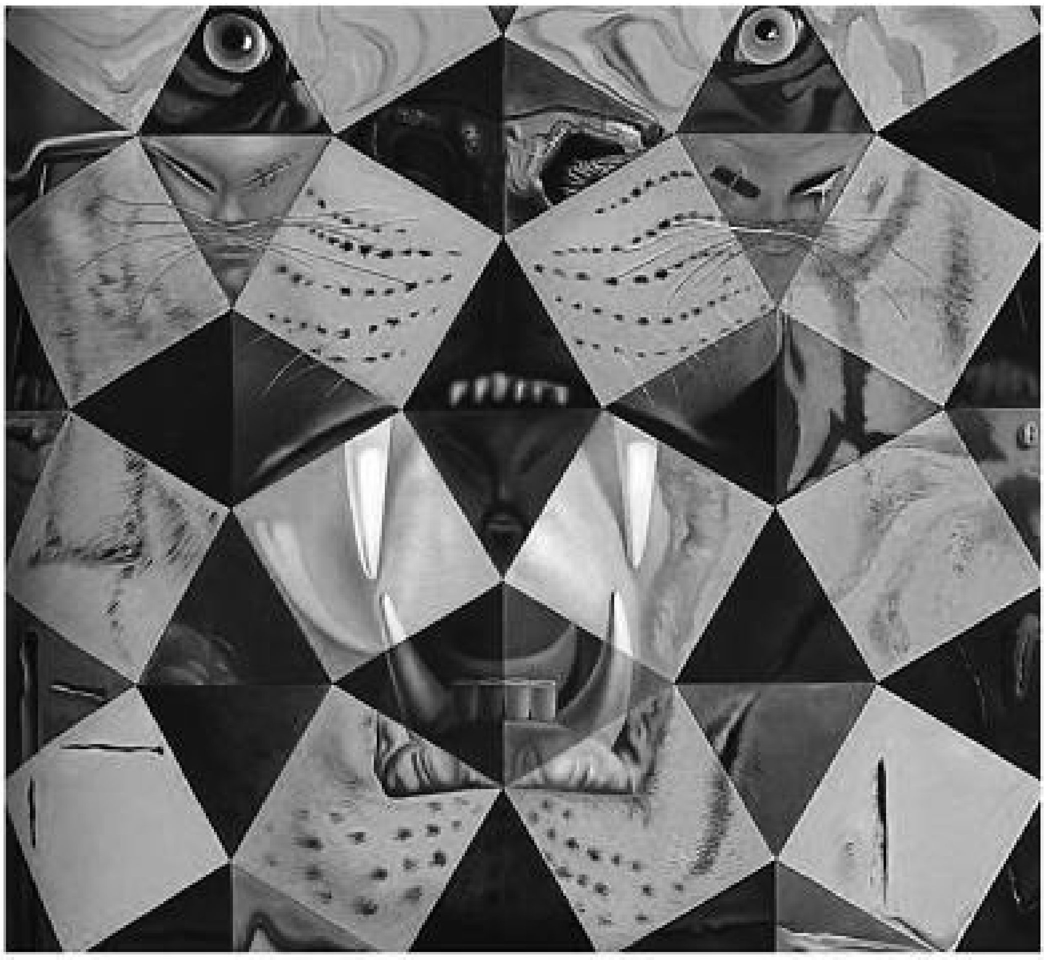


FIGURE 5.2.
The pattern (3,4)
Salvador Dali's painting *Fifty Abstract Pictures Which as Seen from Two Yards Change into Three Lenins Masquerading as Chinese and as Seen from Six Yards Appear as the Head of a Royal Bengal Tiger.*

J. Kepler and his successors were the pioneering researchers into mosaics and tessellations. It is worth noting that tessellations are not only the subject of recreational mathematics, they are also a useful tool in making models in crystallography, coding theory, cellular structure of plants, etc.

The classification of tessellations of the plane using tiles[3] can be performed in various ways, depending on the type of tiling; thus we have regular, semiregular, or irregular tiling, periodic or aperiodic tiling, symmetric or asymmetric tiling, and so on. Many details can be found in the book, *Tilings and Patterns* [90] by B. Grünbaum and G. C. Shephard.

In the early twentieth century, the Russian crystallographer E. S. Fedorov enumerated exactly 17 essentially different types of periodic symmetry patterns. The symmetric patterns in the Alhambra, a Moorish palace in Granada (Spain), are one of the finest examples of the mathematical art of thirteenth-century Islamic artists, and probably the best known architectural use of symmetric patterns.

Speaking about tessellation, it is impossible to overlook the Dutch artist Maurits C. Escher (1898–1972). He was impressed by the patterns in the Alhambra and George Pólya's academic paper on plane symmetry groups. Although Escher's understanding of mathematics was mainly visual and

[3]Tiling is a more restrictive category of repeating patterns because it is usually related to patterns of polygons. However, most authors do not make a difference between tessellation and tiling.

intuitive—he had no formal training in mathematics—he fast caught on to the concept of the 17 plane symmetry groups and created splendid periodic tilings with woodcuts and colored drawings using this concept. Escher created many pictures using periodic tessellation in which living things such as birds, reptiles and fish are used as tiles instead of polygons. Figure 5.3 named *Symmetric drawing E67* (1946) is one of his typical images of this kind. Adjacent horsemen in two colors constitute the basic image; the tiling of the plane is performed by translation and repetition of this image.

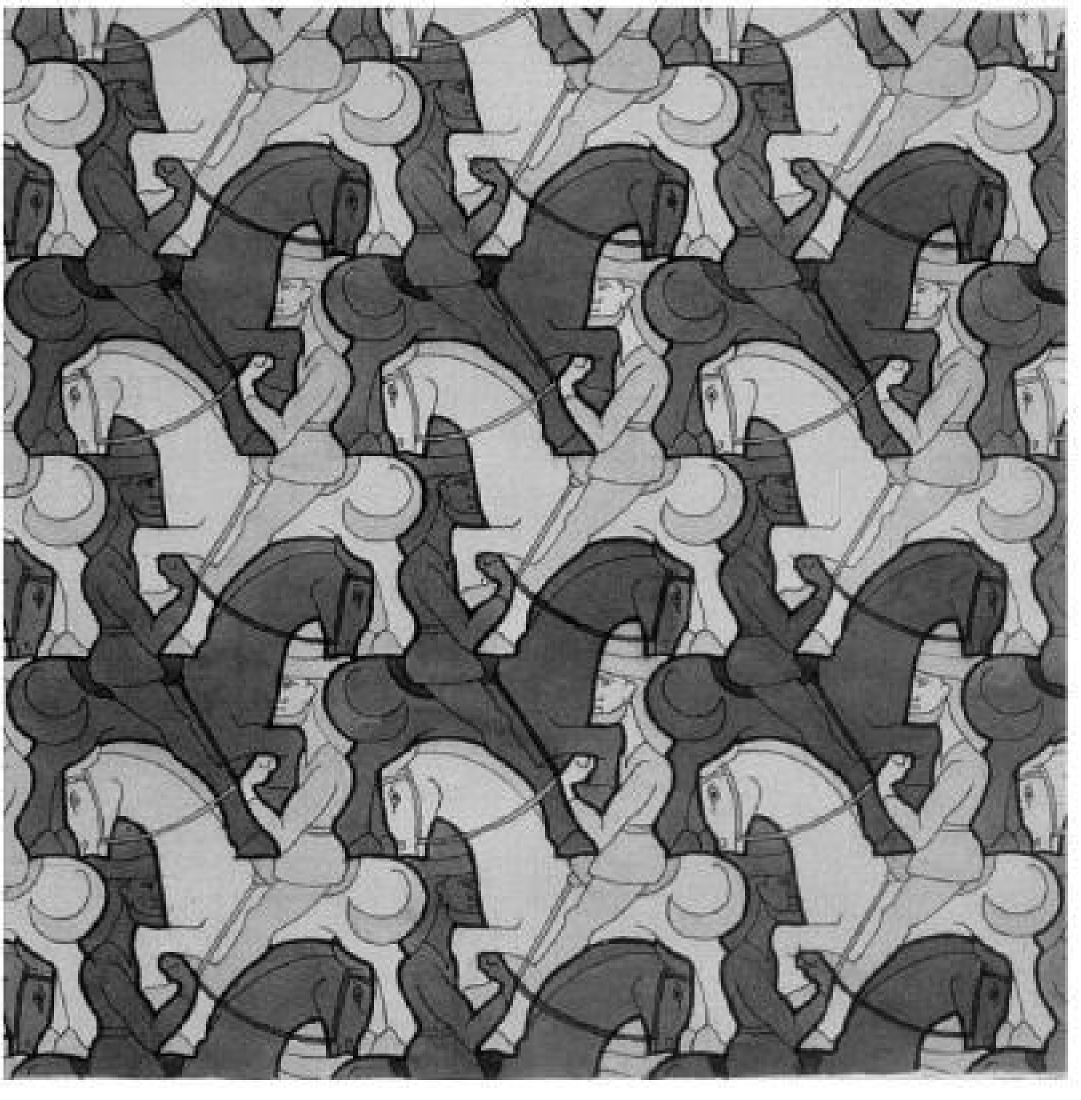


Figure 5.3. *Symmetry drawing E67* (1946); a periodic tessellation by M. C. Escher

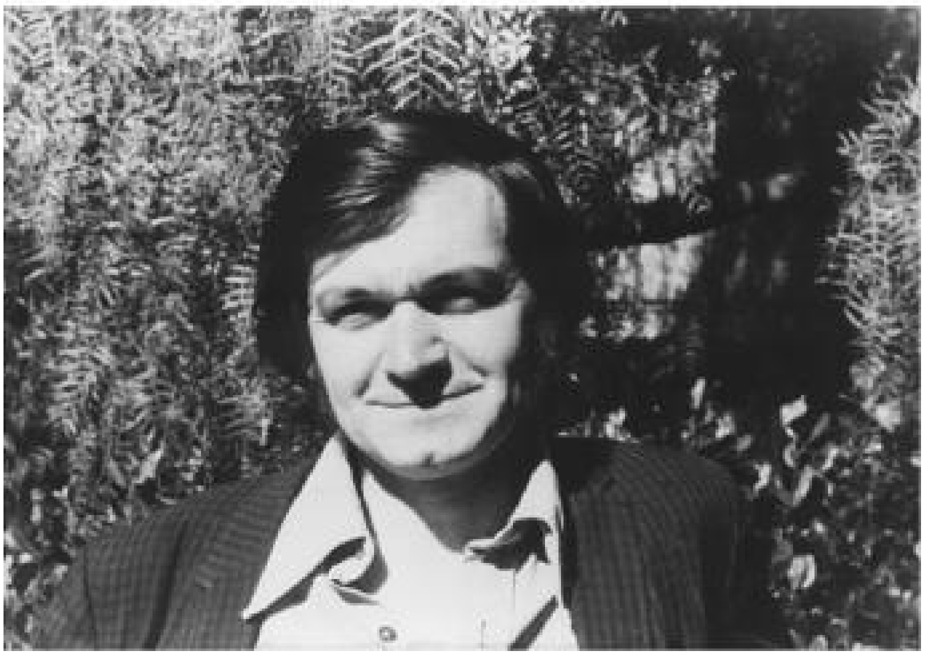
In addition to his creations using tessellations, M. C. Escher created many ingenious illusions, impossible building structures and geometric distortion in 3-space, which often had a flavor of "mystery and absurdity". Escher's lithographs, woodcuts and graphic arts, involving mathematical relationships among figures and space, demonstrated his effort to incorporate mathematics into art, and vice versa. M. C. Escher was highly appreciated by eminent scientists, crystallographers and mathematicians (including the famous Canadian geometer H. S. M. Coxeter, the man who loved symmetry).

Roger Penrose (1931–) (→ p. 310)

John Horton Conway (1937–) (→ p. 310)

Nonperiodic tiling

Sir Roger Penrose, the outstanding British mathematical physicist and cosmologist, is probably best known for his books on popular science and the 1974 discovery of nonperiodical tilings of the plane. Being also a recreational mathematician, he searched for sets of tiles that tile only nonperiodically. For a long time experts were convinced that such a set does not exist. However, in 1964 Robert Berger constructed such a set using more than 20,000 Wang-like dominoes, named after Hao Wang who introduced in 1961 sets of unit squares with colored edges (see Gardner [82] for details). Later Berger reduced the number to 104, and Donald Knuth found an even smaller set of 92. Dramatic improvement was made by Raphael M. Robinson who constructed six tiles that enable a nonperiodic tiling.



Roger Penrose
1931–

Finally, Roger Penrose came to the tiling scene. In 1974 he found a set of only two tiles that force nonperiodicity. John Horton Conway, another famous British mathematician and also a top expert in tiling problems, named these tiles "kite" and "dart". They are derived by cutting a rhombus with angles of 72 and 108 degrees and the sides equal to the golden ration $\phi = (1+\sqrt{5})/2 \approx 1.618$, as shown in Figure 5.4. What is really surprising is that such tile patterns were later observed in the arrangement of atoms in quasi-crystals.

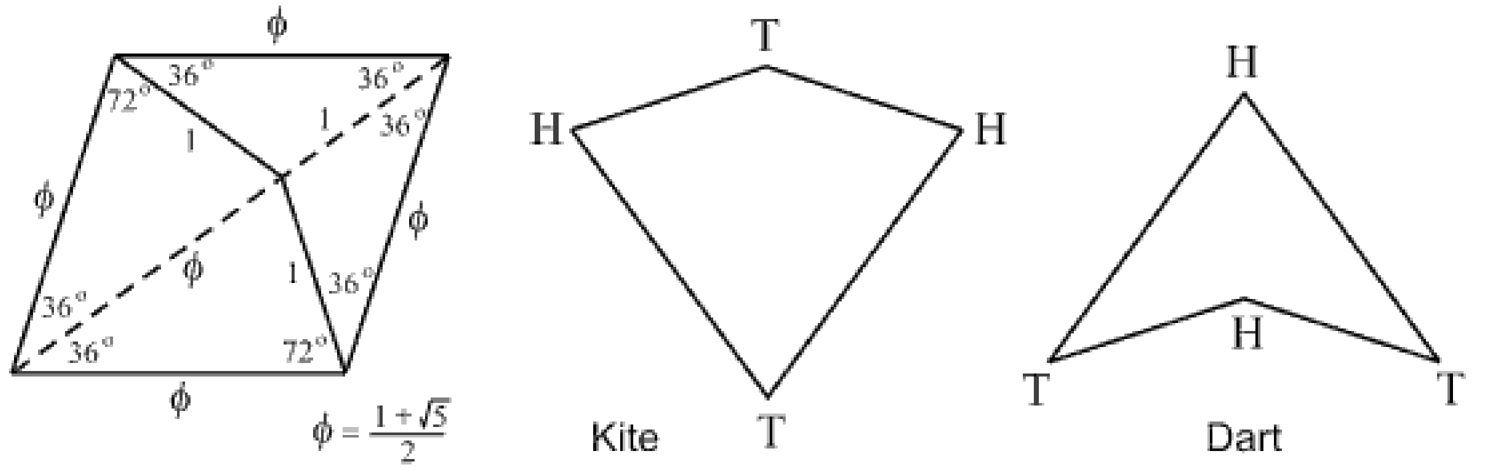


FIGURE 5.4. Construction of Penrose's tiles, kite and dart

Now we have tools, kites and darts. Wouldn't it be nice if we could joint these tiles in such a way as to avoid periodicity, and also bumps and dents? Penrose proposed a simple way. Let us mark the corners H (head) and T (tail) as is shown in Figure 5.4 right. Then, to provide nonperiodic tiling, it is sufficient to fit tiles so that only corners of the same letter may touch. Penrose's proof that the tiling is nonperiodic is based on the fact that the ratio of the numbers of pieces of the two shapes is golden ratio $\phi = 1.618...$, that is, an irrational number. The complete proof is beyond the scope of this book.

The described tiling with kites and darts produces remarkable Penrose patterns which make in turn Penrose universes. These universes contain amazing patterns and shapes, and possess surprising features that take your breath away. Most of them were discovered and studied by Roger Penrose and John Horton Conway. It is interesting to note that Penrose, following Escher's transformation of polygonal tiles into animal shapes, generated nonperiodical tiling patterns by changing his kites and darts into chickens! Many details can be found in Chapter 7 of Gardner's book *The Colossal Book of Mathematics* [82].

Using translations and rotations of Penrose's kites and darts, it is possible to create numerous extraordinary patterns. Playing with these tiles is very pleasurable and you can generate remarkable pictures that will really astonish your friends. Here is one easy but interesting task.

Problem 5.2.* *Armed with kites and darts, try to construct the truncated rhombus shown in Figure* 5.5 *together with its dimensions, where* $\phi = 1.618...$ *is the golden ratio.*

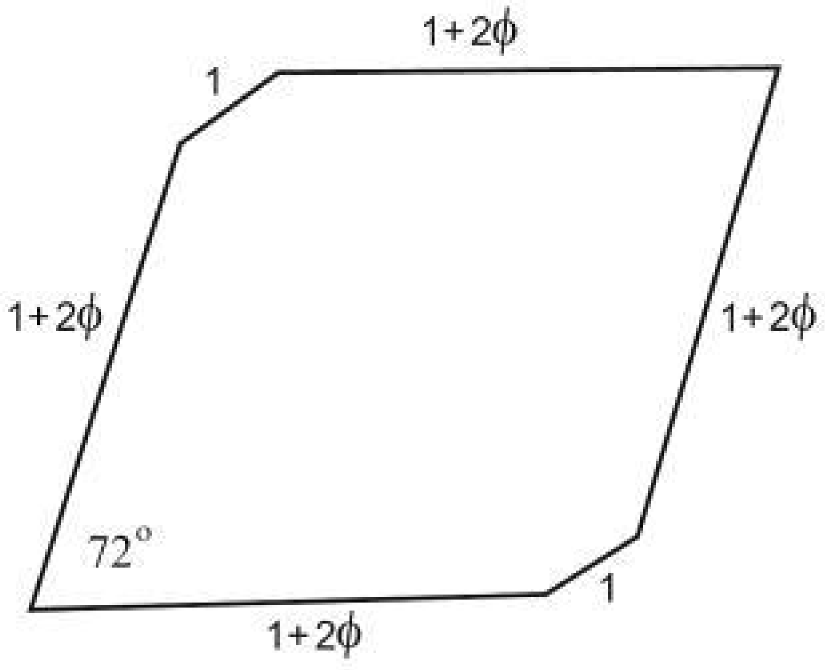


FIGURE 5.5. Penrose's tiling of the truncated rhombus

John Horton Conway has been mentioned above in connection with Penrose's nonperiodic tiling. Together with Charles Radin, Conway experimented with another kind of nonperiodic tiling, the so-called *pinwheel tiling*. Pinwheel tiling uses only one tile, a right-angled triangle with sides 1, 2, $\sqrt{5}$, but it allows infinitely many orientations of this unusual tile; see [143]. The tiles can match edge-to-edge, but also vertex-to-edge. An example of the pinwheel tiling is shown in Figure 5.6.

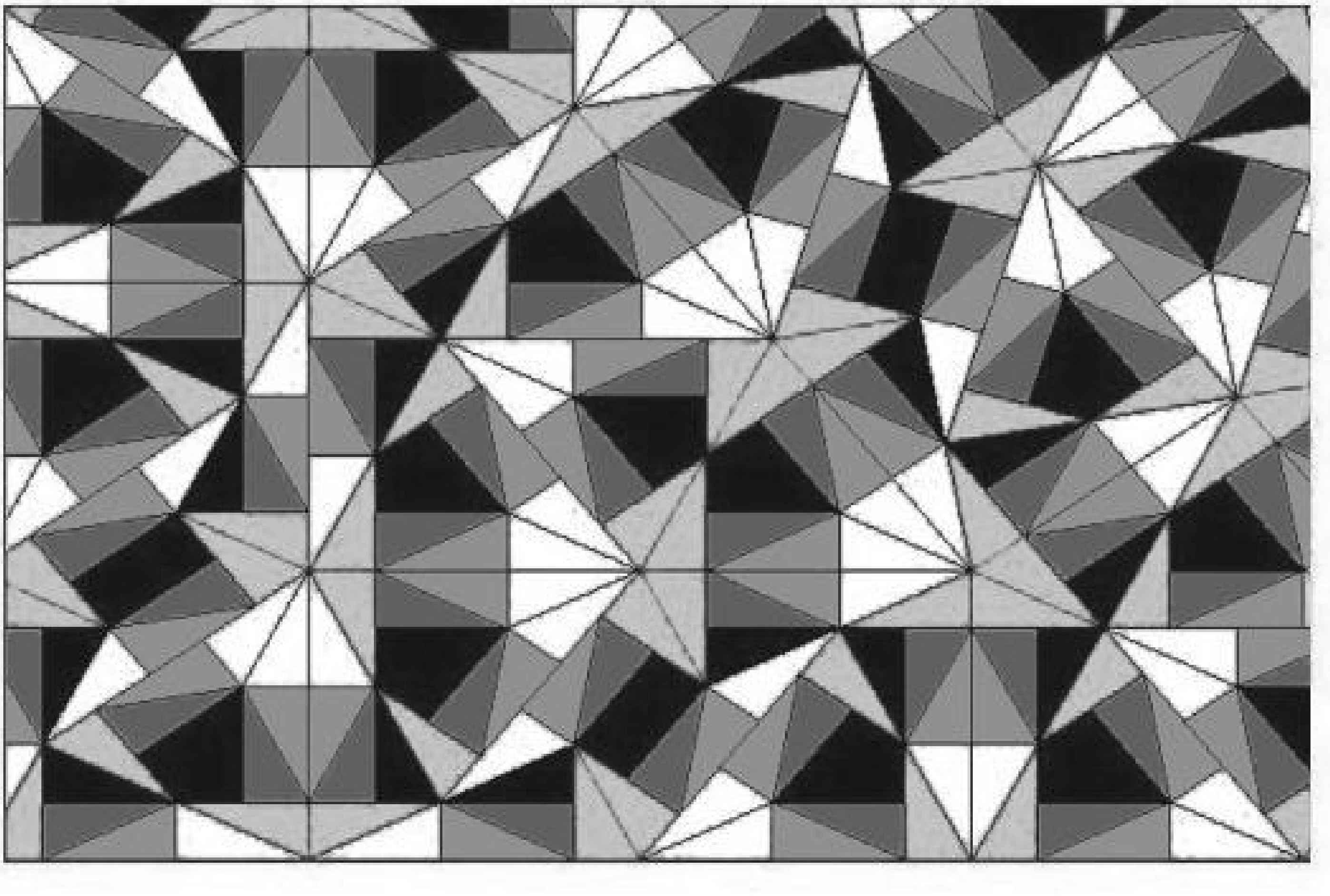
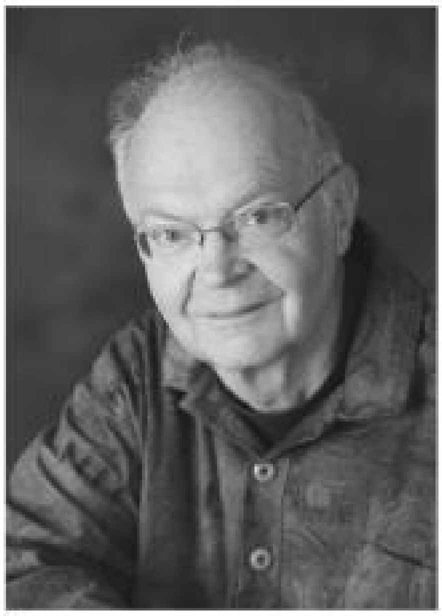


FIGURE 5.6. The pinwheel tiling

Donald Knuth (1938–) (→ p. 310)



Donald Knuth
1938–

Perhaps the most innovative computer scientist ever, Donald Knuth (Stanford University), the author of the three-volume masterpiece *The Art of Computer Programming* and the creator of TEX, the revolutionary typesetting program, has solved and analyzed a multitude of mathematical problems by using computer techniques. Knuth has given significant contribution to recreational mathematics collaborating, among many authors, with Martin Gardner in his famous column *Mathematical Games* in the journal *Scientific American* and the famous mathematicians John Horton Conway and Ronald Graham.

Aside from the extremely complex problems that he has considered, he has also tackled various challenging tasks and puzzles, three of which we present in this book (this chapter, and Chapters 7 and 10).

Maximum area by pentaminoes

A polyomino is any figure consisting of a set of edge-connected unit squares. For instance, a domino is the rectangle formed by two unit squares. Five-square figures, called pentaminoes, are useful in constructing various games. Many different types of polyominoes can be found in Golomb's excellent book, *Polyominoes* [87]. There are exactly 12 pentamino figures, each of which, to a certain extent, resembles a letter of the alphabet, as Figure 5.7 illustrates.

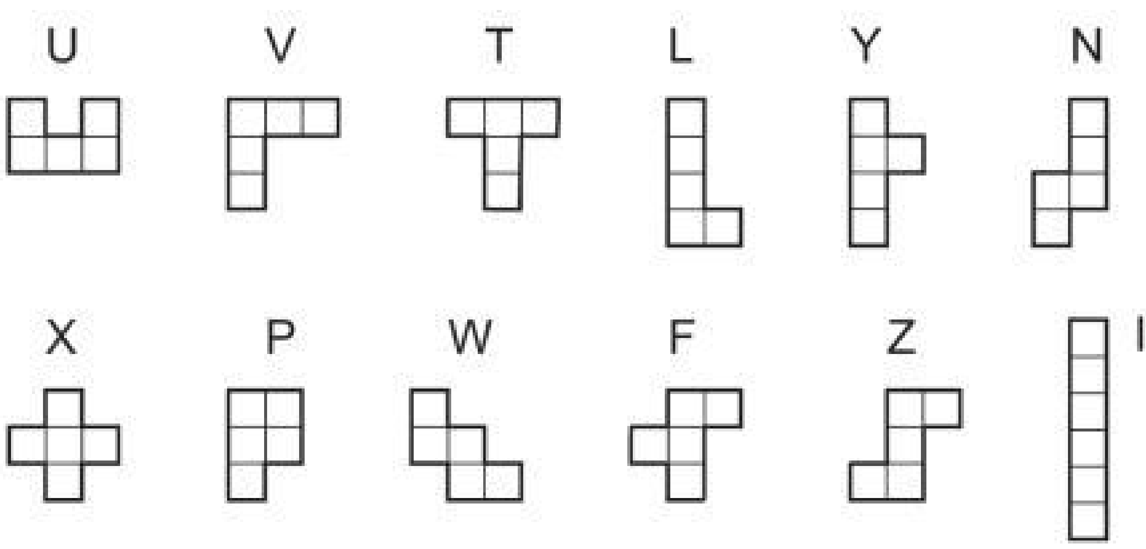


FIGURE 5.7. Twelve pentaminoes denoted by the letters of the alphabet

We have selected the following puzzle among the many where pentaminoes figure as the main object.

Problem 5.3. *Using the full set of* 12 *pentominoes, form a fence of any shape around the largest possible field of any shape.*

Figure 5.8 shows a pentamino fence that encloses a shape with a maximum area of 128 unit composite squares. Donald E. Knuth proposed the solution and proved that the number of 128 squares cannot be exceeded [75, p. 109].

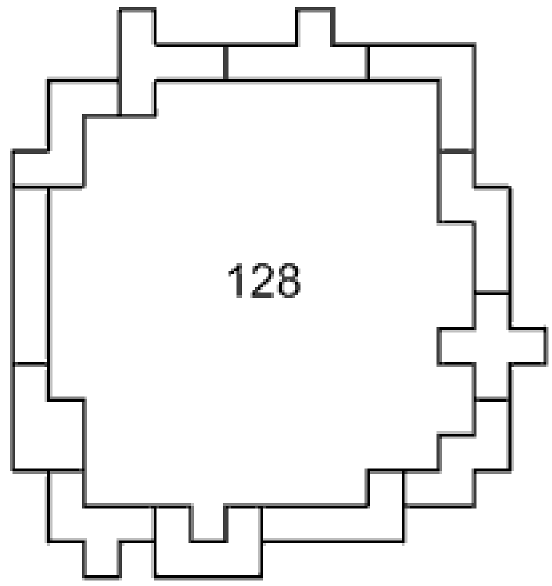


FIGURE 5.8. Pentamino fence of the maximum area

Ten years after Knuth published his solution in the *Scientific American*, Pablo E. Coll from Argentina posed the same problem once again in the *Journal of Recreational Mathematics*, No. 1 (1983). Among many submitted solutions, two of them attracted particular attention. Using the *corner-to-corner* configuration, S. Vejmola from Prague found the solution displayed in Figure 5.9 with a hole whose area is an even 161 units.

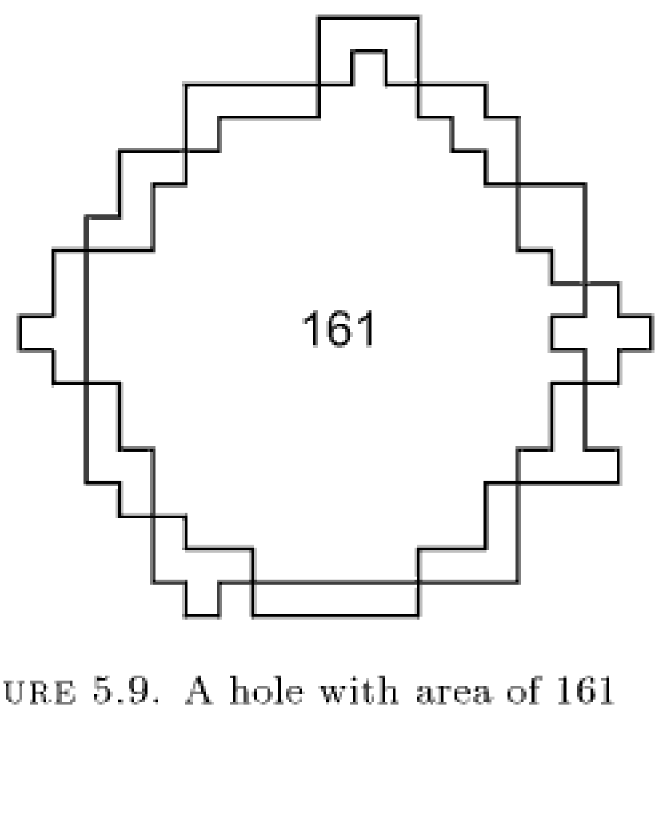


FIGURE 5.9. A hole with area of 161

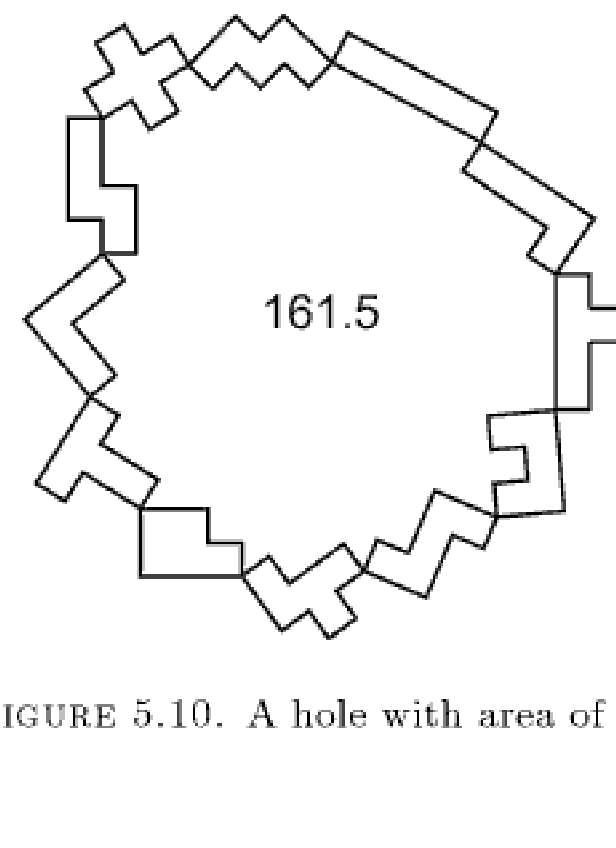


FIGURE 5.10. A hole with area of 161.5

T. M. Campbell of Feilding, New Zealand, constructed a second solution shown in Figure 5.10 configuring an area of 161.5. Campbell did not place the pentaminoes in the *expected* perpendicular or horizontal position since the task had never explicitly stipulated placement conditions; he preferred employing a "disheveled" style. Quite probably, such a style would lead to an increase in the area of the hole. The competition ended after a heated debate that "oblique" solutions such as Campbell's should not be regarded seriously. Although never actually expressed, in solving similar polyomino problems one has always understood that the pentaminoes should be placed perpendicularly or horizontally; the corners of the pieces must coincide with the lattice point of the plane. Subsequent to the extensive debate referred to above, Knuth's record shown in Figure 5.8 remains unbeaten.

Here is a similar puzzle with all 12 pentaminoes, posed by Pablo Coll from Argentina in *Journal of Recreational Mathematics* (Problem 1347). One assumes that the side of each pentamino has the length 1.

Problem 5.4.* *Join two different "empty" pentaminoes to form a hole of the area* 10 *units, and construct around the hole a rectangle using all* 12 *pentaminoes under the following conditions:*

1) *Each of the* 12 *pentaminoes must border the hole.*

2) *The perimeter of a rectangle should be at most* 34.

Can you construct such a rectangle?

Hint: The problem is difficult and we give a hint as a help to the reader: the hole inside a rectangle looks like this: 

Problem 5.5.* *Using all* 12 *pentaminoes* (*shown in Figure* 5.7) *solve the following two tiling problems:*

a) *Cover the* 8×8 *chessboard with all* 12 *pentaminoes so that one square tetramino* (2×2) *could be placed at the center of the board* (*the squares d4, d5 e4, e5*).

b) *Cover an incomplete* 8×8 *chessboard except the four corner squares* (*a1, a8, h1, h8*) *with all* 12 *pentaminoes.*

Now we present several problems concerning the tiling of plane polygons by other plane polygons. The next example is probably familiar to the reader.

Problem 5.6.* *Two diagonally opposite squares of a standard chessboard* 8×8 *have been cut out. Can you cover the given truncated board, consisting* *of* 62 *squares, with dominoes whose size is such that each of them covers exactly two small squares? See Figure* 5.11.

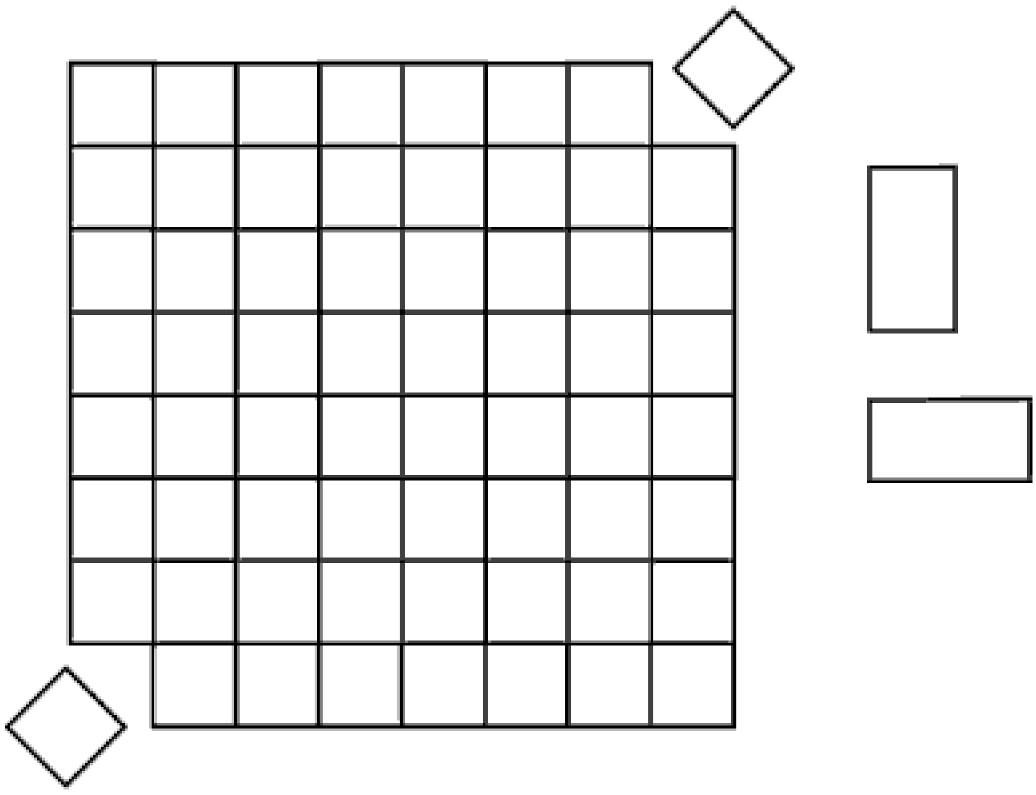


FIGURE 5.11. Covering by 2×1 dominoes

Problem 5.7* (Fuchs and Tabachnikov [67]). *Is it possible to tile a* 10×10 *square composed of* 100 *unit squares with L-shaped tiles consisting of* 4 *unit squares. Notice that the L-tiles may have eight different orientations.*

The reader may remember Heronian triangles from Chapter 2 (Problem 2.9), they have rational sides. It is quite natural to focus our attention to a more interesting subject, squares with rational sides, often called rational squares.[4] *Squaring the square* is the problem of tiling one integral square using only other integral squares. The problem becomes more challenging and more difficult if all constituent squares are required to be of different sizes. Then we speak about the *perfect* squared square. The number of the contained small squares is called the *order* of a perfect squared square. This intriguing problem, which belongs to combinatorial geometry, has been extensively studied for more than sixty decades. Mathematicians and computer scientists have searched for the answer to the following question of great interest.

Problem 5.8. *Find the perfect squared square of lowest order.*

[4]Squares with integral sides are more interesting and challenging than rational squares; consequently, most of problems directly handle integral squares.

The first perfect squared square, with the order 55, was constructed by Roland Sprague in 1939. This record was beaten several times during the later forty years period. At the beginning of the 1960s, the Dutch computer scientist Adrianus J. W. Duijvestijn (1927–1998) started to investigate the problem. His doctoral thesis, *Electronic computation of squared rectangles* (1962) was an introduction to his search for simple perfect squared squares. Although he had straightforward ideas, the limit of the power and memory of computers at that time kept him from any further research.

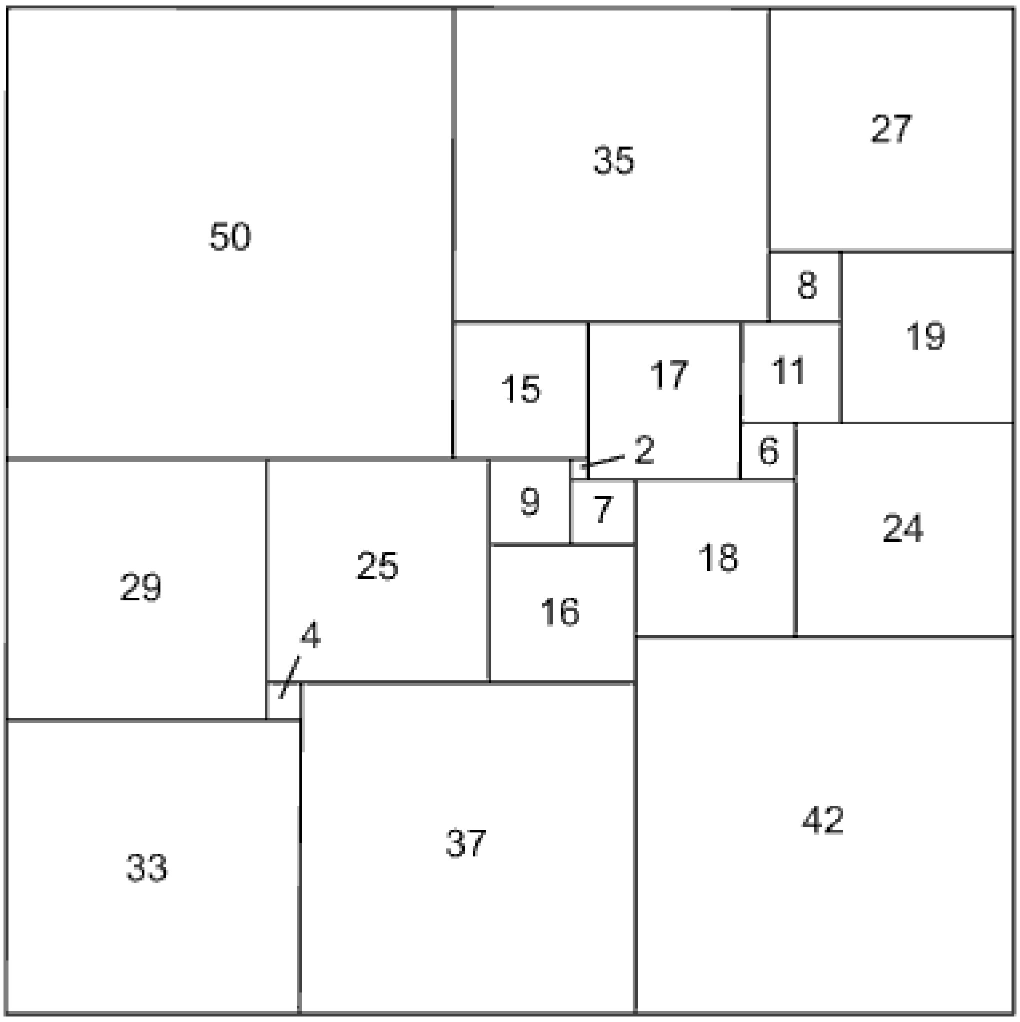


FIGURE 5.12. Perfect squared square of the lowest order

When the computer storage and processing power had grown significantly, Duijvestijn returned to the problem and finally, using a computer program, he found the lowest order perfect squared square on March 22, 1978; see [57]. This square, composed of 21 smaller squares, is carved into the black granite surface of a monument erected at the University of Göttingen, Germany. Duijvestijn's perfect square of minimal order is shown in Figure 5.12.

It is worth mentioning that a related problem of tiling the plane with squares of different sizes, unsolved for many years, was recently solved. F. V. Henle and J. M. Henle proved that the plane can be tiled with a set of integral squares such that each natural number is used exactly once as the size of a square tile; see [101].

H. Reichert and H. Töpkin proved in 1940 that a rectangle cannot be dissected into fewer than nine different integral squares. Here is a problem that requires squaring a rectangle.

Problem 5.9.* *Can you dissect the rectangle* 32×33 *into nine squares of different sizes?*

Hint: The largest constituent square is 18×18.

David Gregory (1659–1708) (→ p. 304)

Isaac Newton (1643–1727) (→ p. 304)

David Gregory was the nephew of the more famous mathematician James Gregory (1638–1675). He became a professor of mathematics at the age of 24 at the University of Edinburgh where he lectured on Newtonian theories. David Gregory supported Newton strongly in the Newton–Leibniz priority dispute over the discovery of integral and differential calculus. Newton reciprocated by assisting Gregory in his efforts to build a successful career.

Kissing spheres

The problem of the thirteen spheres, or the problem of "kissing spheres", a reference to billiard terminology, arose as a result of a famous conversation between David Gregory and Isaac Newton at Oxford in 1694.

Problem 5.10. *How many unit spheres can simultaneously touch a given sphere of the same size?*

Newton thought that the maximum was 12 spheres, while Gregory believed the answer was 13. However, neither of them had proof for their statements. In honor of their discussion, the problem is sometimes referred to as the Gregory–Newton problem (see, *e.g.*, Aigner and Ziegler [3, Ch. 12], Leech [122], Shüttle and Waerden [155], Zong [194, pp. 10–13]).

Let $\kappa(n)$ denote the maximal number of n-dimensional spheres that touch a given sphere in the n-dimensional space, in which all of the spheres are identical in size. The number $\kappa(n)$ is sometimes called the *Newton number* or *kissing number*. Then, obviously, one has $\kappa(1) = 2$ and $\kappa(2) = 6$, as shown in Figures 5.14(a) and 5.14(b).

In 3-dimensional space an arrangement of 12 spheres is possible; for example, one can place the touching spheres at the vertices of a regular icosahedron, as one may see in Figure 5.14(c). David Gregory and many after him considered that this configuration, as well as some other configurations, leaves "a lot of space" for the thirteenth sphere. Thus, $\kappa(3) = 12$, or $\kappa(3) = 13$: which is the true answer?

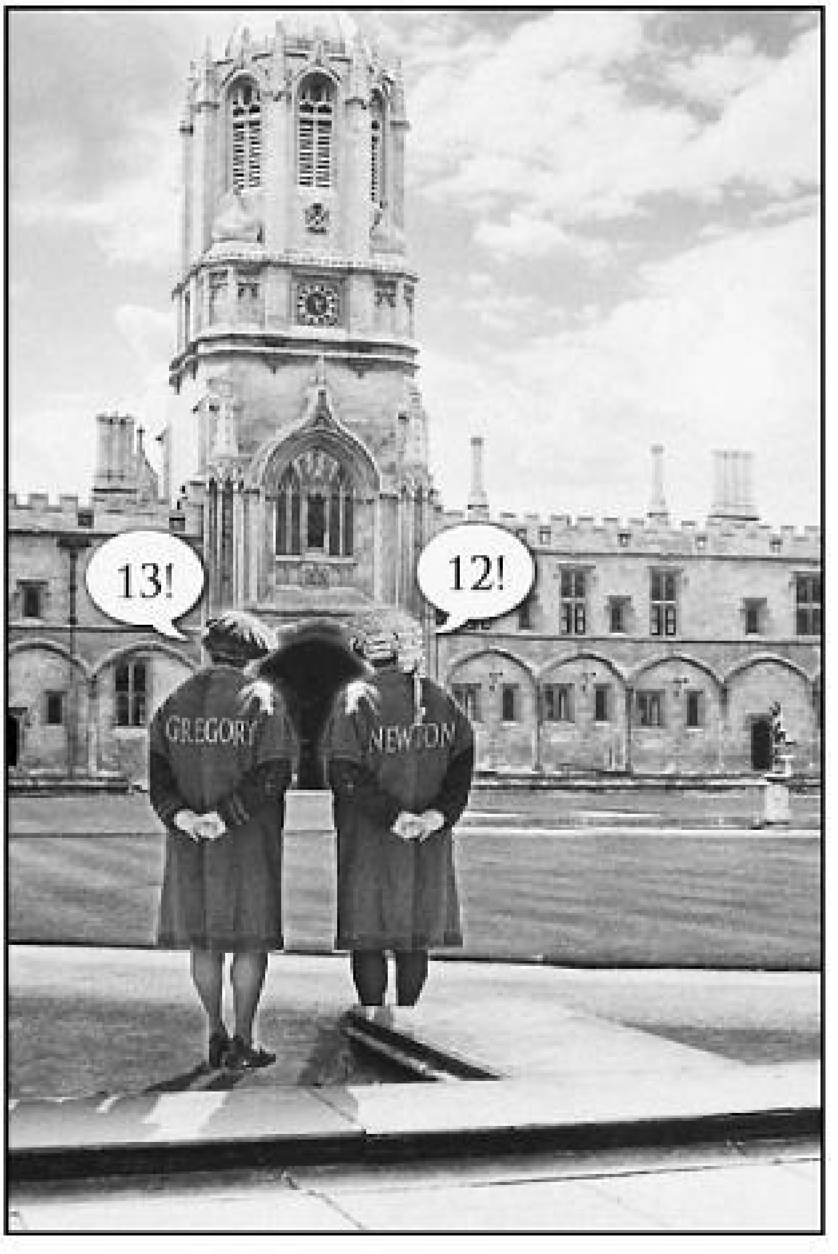


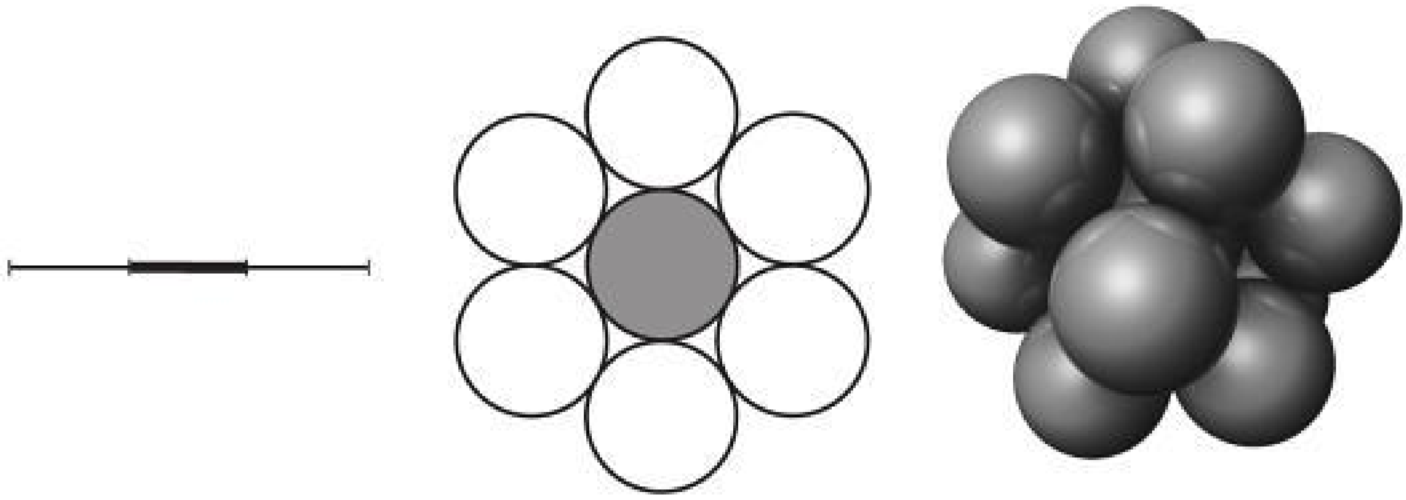
FIGURE 5.13. The Oxford discussion of David Gregory and Isaac Newton on "kissing spheres"

For more than 250 years, this fascinating problem remained unsolved although several "solutions" were advanced in literature covering physics problems.[5] Finally, in 1953, Kurt Schüttle and Bartel L. van der Waerden [155] settled the problem definitely.

No more than 12 *unit spheres can be placed simultaneously in such a way that all of them touch a given sphere of the same size.*

[5] See, *e.g.*, C. Bender, *Bestimmung der grössten Anzahl gleicher Kugeln, weiche sich auf eine Kugel von demselben Radius, wie die übrigen, auflegen lassen,* Archiv Math. Physik (Grunert) 56 (1874), 302–306, R. Hoppe, *Bemerkung der Redaction*, Archiv Math. Physik (Grunert) 56 (1874), 307–312, and S. Günther, *Ein stereometrusches Problems,* Archiv Math. Physik (Grunert) 57 (1875), 209–215. More details can be found in [41, Ch. 13].

In other words, $\kappa(3) = 12$. This means that even without a proof, Newton supplied the correct answer.



a) $\kappa(1) = 2$ b) $\kappa(2) = 6$ c) $\kappa(3) = 12$

FIGURE 5.14. "Kissing spheres" in n-dimensional space

Three years after Schüttle and van der Waerden's 1953 solution, John Leech [122] provided a simple solution that a relatively wide circle of readers could understand. The knowledge of spherical geometry and graph theory suffices in order to comprehend Leech's solution. Chapter 12 of Aigner and Ziegler's book [3, Ch. 12] also contains this solution. However, this solution is beyond the scope of this book and we omit it.

Is it possible to find $\kappa(n)$ for a general dimension $n > 3$? It is quite surprising that the problem of kissing spheres was solved in very high dimension 24 giving $\kappa(24) = 196{,}560$ (Andrew Odlyzko and Neil Sloane, and independently Vladimir Levenstein, all in 1979). John Horton Conway, an outstanding Cambridge mathematician now at Princeton, has remarked [82]: *"There is a lot of room up there."* However, for a long time the problem has been open even in dimension 4.

Philippe Delsarte (Philips Research Labs) found in 1972 that $24 \le \kappa(4) \le 25$ (see [41, §1.2], [47], [139]), which leads to the *problem of the* 24 *spheres*. Finally, the Russian mathematician Oleg Musin, who lives in Los Angeles, found that $\kappa(4) = 24$ (see [139]). In this moment, the exact values for lattice packing are known for $n = 1$ to 8 and $n = 24$; see Table 5.1 (according to Internet site `www.research.att.com/~njas/lattices/kiss.html`).

n	1	2	3	4	5	6	7	8	24
$\kappa(n)$	2	6	12	24	40	72	126	240	196,560

TABLE 5.1. Kissing numbers in dimension n

Johannes Kepler (1571–1639) (→ p. 302)

Carl Friedrich Gauss (1777–1855) (→ p. 305)

The densest sphere packing

The mentioned Kepler problem of the densest sphere packing, is considerably more difficult than the kissing spheres problem. Indeed, it was not solved until very recently (2005) by Thomas Hales, by making extensive use of computer calculations.

The quantity that measures the efficiency of any packing is surely the "density," the total area or volume of the objects being packed divided by the total area or volume of the container. In this calculation the method of limits should be applied assuming that the container boundary tends to infinity. Let $\delta(n)$ denote the density of sphere packing in general dimension n, and let $\delta_{max}(n)$ be the density of the most efficient packing. The classic sphere-packing problem thus reads:

Problem 5.11. *Find the densest packing of equal spheres in the three-dimensional space.*

Johannes Kepler started with the sphere packing problem in 1611. He was inspired not by piles of fruit but by an equally real phenomenon—the shape of a snowflake. He wrote a little booklet, *Strena Seu de Nive Sexangula*[6] (The Six-Cornered Snowflake), that influenced the direction of crystallography for the next two centuries. Kepler's interest to study arrangements of spheres has also reinforced a result of his correspondence with the English mathematician and astronomer Thomas Harriot in 1606. At that time Harriot studied cannonball packing and developed an early version of atomic theory.

Kepler sought the answer in geometry and based his study on the fact that nature imposes a regular geometric structure on the growth of such seemingly diverse objects as snowflakes, honeycombs, and pomegranates. Stated simply, he observed that nature always adopts the most efficient means to achieve its end. Kepler believed that *face-centered cubic packing*, shown in Figure 5.16(b), is the tightest possible (with the density of $\pi/\sqrt{18}$). Today we know Kepler's statement as

Kepler's conjecture. $\delta_{max}(3) = \dfrac{\pi}{\sqrt{18}}$.

[6] Kepler's essay on why snow flakes are hexagonal, was issued by Gottfried, Tampach, Frankfurt 1611. This essay was written in the form of a letter as a New Year's gift for his friend Johannes von Wackenfels.

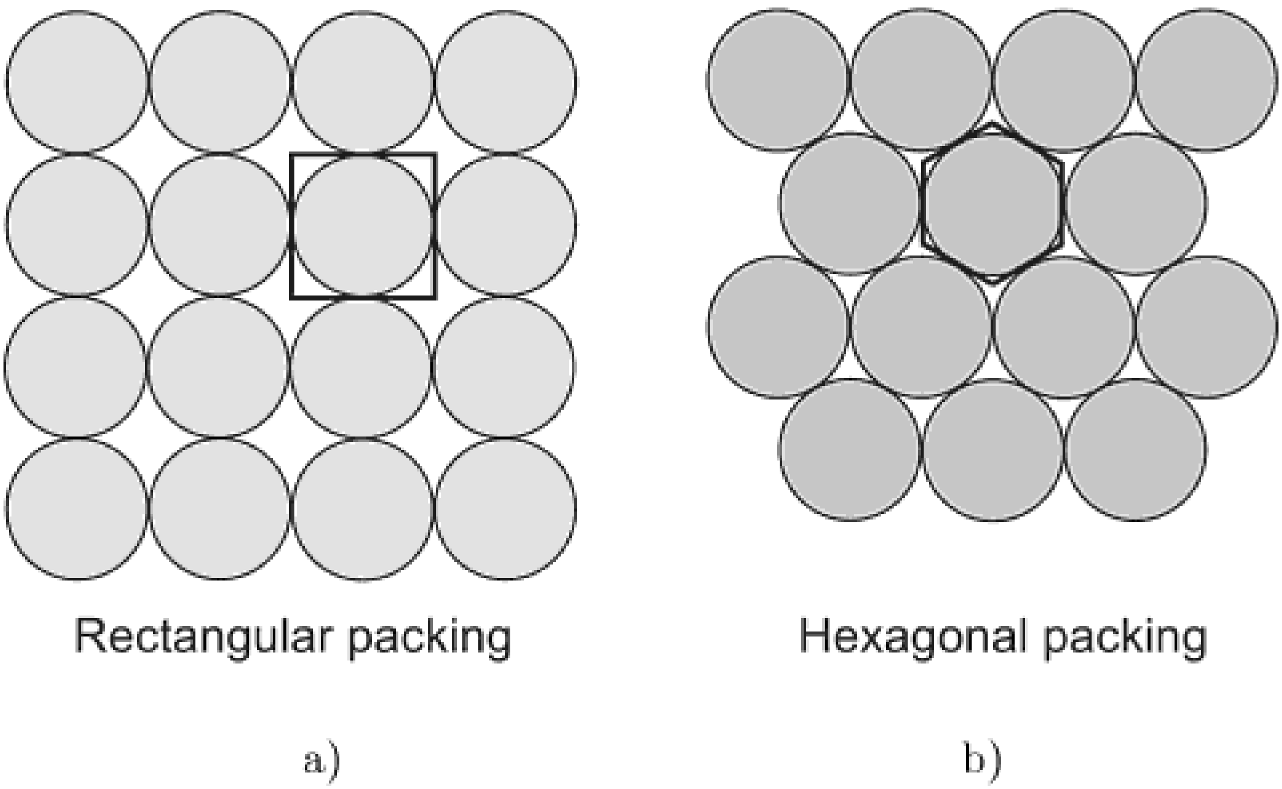


FIGURE 5.15. The two familiar ways of circle packing

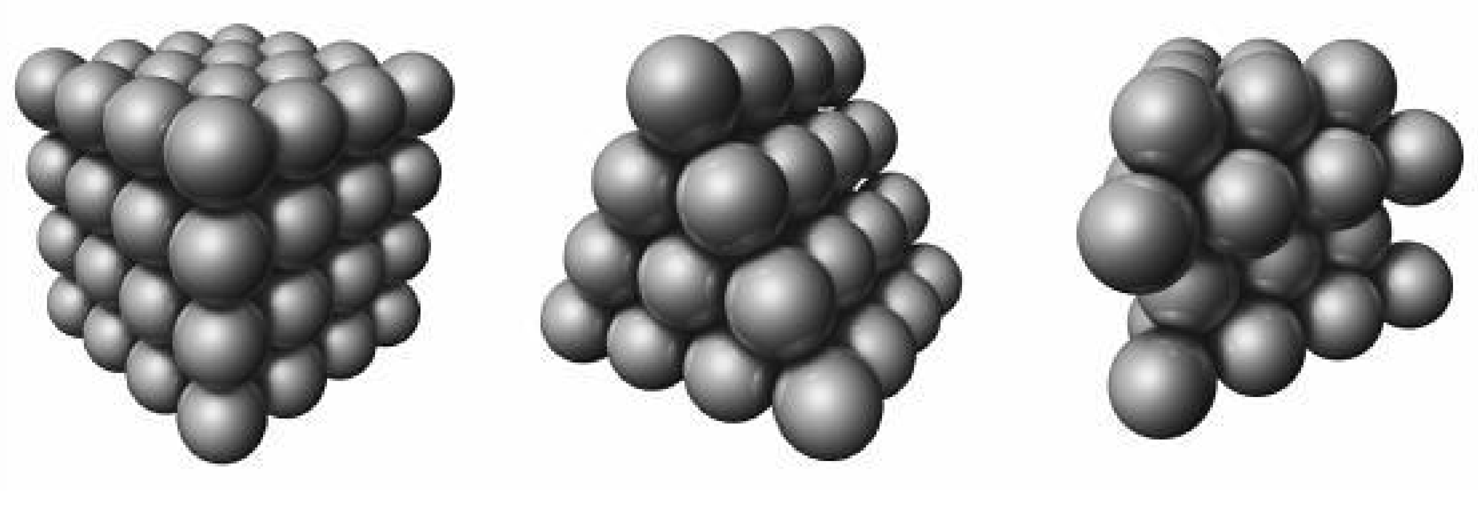
Before discussing Kepler's conjecture, let us consider first the case of disk packing in a plane. Two familiar ways to arrange identical disks in order to fill a plane are shown in Figure 5.15. Figure 5.15(a) presents "rectangular packing" with the density $\delta(2) = \pi/4 \approx 0.785$ and Figure 5.15(b) presents "hexagonal packing" with the density $\delta(2) = \pi/\sqrt{12} \approx 0.907$. The terminology refers to the figures formed by common tangents to the circles. Therefore, hexagonal packing yields a greater density, which is evident from the picture that indicates that this kind of packing leaves less space between adjacent disks.

But it is less obvious that hexagonal packing is *the* most efficient among *all possible* ways of packings, regular or irregular. In 1831 Gauss showed that hexagonal packing is the densest among lattice packings. Within a plane, a lattice is a collection of points arranged at the vertices of a regular, two-dimensional grid. The grid may be square, rectangular, or in the form of identical parallelograms.

A lattice packing of disks is one in which the centers of the disks form a lattice. Rectangular and hexagonal packings are clearly lattice packings. Gauss left open the question of the most efficient of all possible disk packings. The Norwegian mathematician Alex Thue (1863–1922) is often credited with the proof that hexagonal packing is the most efficient of all disk packings, regular or otherwise, that is, $\delta_{max}(2) = \pi/\sqrt{12} \approx 0.907$. Thue presented the first proof to the members of the Scandinavian Society of Natural Science, published in 1892 (in Norwegian). Eighteen years later, he gave quite a different proof in the 1910 paper (in German), reprinted in [130]. However, George Szpiro explains in his book *Kepler's Conjecture* [172] that Thue did not present complete rigorous proofs, only outlines of possible proofs. The Hungarian mathematician László Fejes Tóth was the first who gave a rigorous proof in 1940: the hexagonal packing is the most efficient, reaching $\delta_{max}(2) = \pi/\sqrt{12}$.

Following Gauss, the first strategies in searching for the densest packing of spheres in three-dimensional cases were concentrated on lattice packing, where the centers of the spheres form a three-dimensional lattice, a regular, three-dimensional grid. The first major advance came in 1848 when the French botanist and physicist Auguste Bravais proved that there are exactly fourteen distinct kinds of three-dimensional lattices. We recommend Devlin's book [49, p. 156] to readers interested in these 14 regular lattices in 3-space.

One obvious way to arrange spheres in a regular, lattice fashion is to build up the arrangement layer by layer. It seems reasonable to arrange each layer so that the centers of the spheres are one of the planar lattice formations considered above, the rectangular and the hexagonal. The resulting packings are shown in Figure 5.16, where the first two arrangements possess rectangular layers.



a) Cubic lattice b) Face-centered-cubic lattice c) Hexagonal lattice

FIGURE 5.16. Three different ways of sphere packing by stacking regular layers

Kepler computed the density associated with each of the three lattice packings shown in Figure 5.16. The densities found by Kepler were $\delta(3) = \pi/6 \approx 0.5236$ for the cubic lattice (Figure 5.16(a)), $\delta(3) = \pi/\sqrt{27} \approx 0.6046$ for the hexagonal lattice (Figure 5.16(c)), and $\delta(3) = \pi/\sqrt{18} \approx 0.7404$ for the face-centered cubic lattice (Figure 5.16(b)). Thus, the face-centered cubic lattice (the orange-pile arrangement that one often notices in the market) is the most efficient packing of the three. But, is this way of packing the most efficient of *all* lattice packings? More generally, is it the most efficient of *all* packings, regular or otherwise? Kepler's answer was *"yes"*.

Gauss answered the first of these two questions not long after he had solved an analogous problem in two dimensions. The second problem, however, remains unsolved to this day.

The stacking of solid spheres in pyramids, known as face-centered cubic packing (Figure 5.16(b)), is familiar to chemists. This kind of packing is also known as cannonball packing, because it is commonly used for that purpose at war memorials. An example of a cannonball pyramid dating from the sixteenth century is situated in front of the City Museum of Munich.

Nonetheless, this kind of packing does not impress greengrocers. *"Such a way of packing oranges is quite common for us"*, they say, *"but we have troubles with the arrangement of artichokes."*[7] Until 1993, the best known bound on density, found by D. J. Muder [129], was 0.773055, and hence we can conclude that the market pile of oranges is very close to the best. We will see later that, after Hales' proof, this arrangement of oranges is really the most efficient.

The difficulty of the problem under consideration stems from the fact that there is an unwieldy number of possible configurations and that uniform estimates are rarely found. The question of whether there is an irregular packing that might be denser than the packing based on a lattice is a very important one. In 1900 the very difficult Kepler problem was included in David Hilbert's list of twenty-three important unsolved problems of mathematics (the International Congress of Mathematicians at Paris).

From time to time, some researcher or another announces his solution, but until recently, it has turned out in all cases that some gaps were present in the "proof". The very respectable *Encyclopedia Britannica* announced in its 1992 yearbook [93]: *"Without doubt the mathematical event of* 1991 *was the likely solution of Kepler's sphere-packing problem by Wu-Yi Hsiang."* Wu-Yi Hsiang (the University of California, Berkeley), announced his solution through four preprints, *i.e.*, a revision of a revision of a revision, in 1990/91, yet each contained some flaws (for more details see [93]). His final proof was disputed by some experts on sphere packing, claiming that he gave insufficient support for some of his assertions. One of Hsiang's harshest critics was Thomas Hales, then at the University of Michigan, who was working at the time on his own proof.

[7] According to Thomas Hales [94].

Thomas C. Hales, a professor of mathematics at the University of Michigan, Ann Arbor, and currently Andrew Mellon Professor of mathematics at the University of Pittsburgh, has made the most recent attempt. In his article, *Cannonballs and Honeycombs*, published in [94], Hales asserts that with the help of Samuel P. Ferguson, he has completed the proof of the following statement.

Theorem 5.1 (T. C. Hales). *No packing of balls of the same radius in three dimensions has a density greater than that of face-centered cubic packing, that is,* $\delta_{max}(3) = \pi/\sqrt{18}$.

Thomas Hales exposed a detailed plan of the proof and later gave a broad outline of it. He stated that the proof is rather long and every aspect of it is based on even longer computer calculations applying methods from the theory of global optimization, linear programming, and interval arithmetic. To find the maximum density, Hales had to minimize a function with 150 variables. The computer files require more than 3 gigabytes of space for storage and consist of 40,000 lines. The proof relies on lengthy computer calculations checking many individual cases.

Although the proof was a computer-aided proof, the editors of the journal *Annals of Mathematics* accepted the report of a jury of twelve referees who investigated the proof for four years, and published this proof in an abridged form (126 pages); see [95]. The referees did not raise doubts about the overall correctness of the proof, although they could not check the correctness of all of the computer codes. The unabridged Hales' paper (265 pages) was published in *Discrete and Computational Geometry* [96].

It is perhaps that greengrocers would not be impressed by Hales' proof, but experts are generally optimistic and accepted it with a great deal of euphoria. However, there is a small group of sceptics who both point to the possibility of software and hardware errors and also consider that absolute certainty is required in a proof, as achieved by a structured sequence of arguments and logic, known among mathematicians as a "formal proof". They say: *"If historical precedent counts for anything, and having in mind the scepticism provoked by the number of proofs that contained deficiencies, gaps or inconsistencies, discovered only after their publication, then one might wait for years until Kepler's conjecture is accepted as a theorem."*

In response to the difficulties in verifying computer codes in his proof and computer-aided proofs in general, Thomas Hales launched the "Flyspeck project" (the name derived from the acronym FPK, for the Formal Proof of the Kepler conjecture), a fundamental system for doing mathematical proofs on a computer; see [97].

John Horton Conway (1937–) (→ p. 310)

John Horton Conway
1937–

John Horton Conway, the outstanding British professor of mathematics at Cambridge and later at Princeton, and a member of the Royal Society of London, is widely known among mathematicians and amateur mathematicians for his remarkable contributions to combinatorial game theory and many branches of recreational mathematics. His book *Winning Ways for Your Mathematical Plays*, written with E. R. Berlekamp and R. K. Guy (published in 2 volumes in 1982 and 4 volumes in the second edition in 2001), has attracted the attention of a wide audience for many years.

J. H. Conway became instantly famous when he launched in 1970 the Game of Life, a kind of artificial simulation of life. Martin Gardner, who first published Conway's invention in his *Mathematical Games* column in *Scientific American* (October 1970), said: *"The game opened up a whole new field of mathematical research, the field of cellular automata... . Because of Life's analogies with the rise, fall and alterations of a society of living organisms, it belongs to a growing class of what are called 'simulation games' (games that resemble real life processes)."* Researching certain games, Conway came to a new system of numbers, named *surreal numbers* by Donald Knuth. John Conway proposed many mathematical puzzles, including cube-packing puzzles, the subject of our final exposition in this chapter.

Cube-packing puzzles

One of the first cube-packing puzzles appeared in 1970 in a book by the Dutch architects Jan Slothouber and William Graatsma. This puzzle, sometimes called Slothouber–Graatsma–Conway cube puzzle, is presented in this chapter as Problem 5.13. However, the versatile genius John Horton Conway had wanted to design more difficult cube-packing puzzles and, as may be expected, he was very successful as usual; today there are many variants of Convey's cubes. One of them is given in Figure 5.17.

Problem 5.12.* *It is required to assemble thirteen* $1 \times 2 \times 4$ *blocks, three* $1 \times 1 \times 3$ *blocks, one* $1 \times 2 \times 2$ *block, and one* $2 \times 2 \times 2$ *block into a* $5 \times 5 \times 5$ *cube* (*Figure* 5.17).

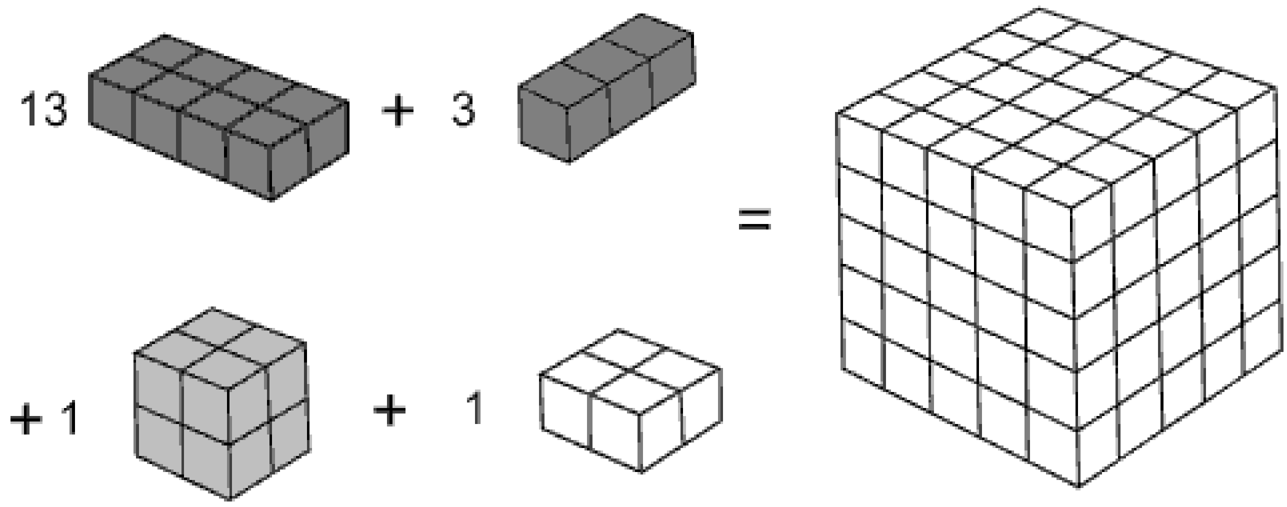


FIGURE 5.17. Conway's cube-packing puzzle

This puzzle is very difficult because there is an immense number of arrangements of blocks, especially if you try to solve it by trial and error. You immediately ask yourself which blocks must be placed in the first moves. Well, *"the beginning is always the most difficult,"* the old stories tell us, and it is more than true in this particular problem. However, there is a nice and attractive theory proposed by Gardner [76] and Honsberger [106] based on the coloring of the unit cubes. To save space and relax the reader, we omit it; nevertheless, the basic principle may be found as a part of the solution of Problem 5.13.

If you are still desperate to know the solution key, here it is! It is the structure created by the smallest blocks; see Figure 5.18.

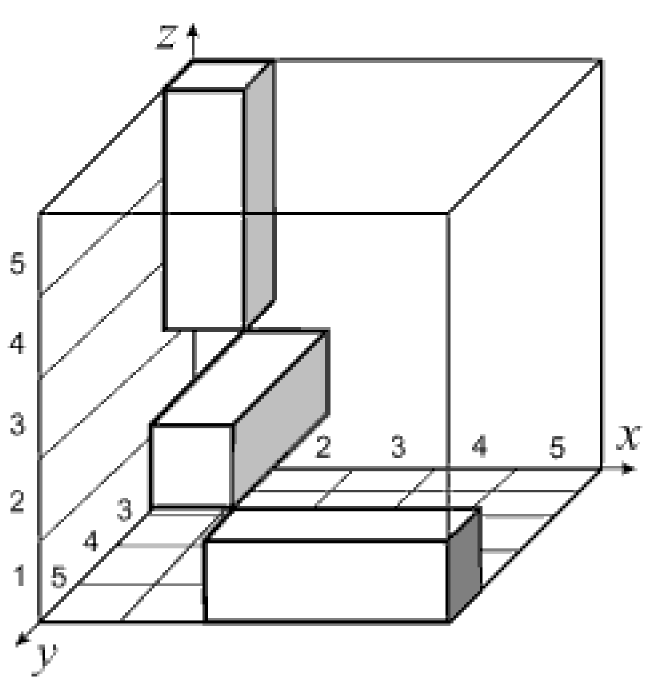


FIGURE 5.18. The starting position of Conway's cube puzzle

The smallest blocks $1 \times 1 \times 3$ must occupy all 15 layers 5×5 in all three perspectives, as shown in Figure 5.18. Upon using this base it is not particularly difficult to find at least one of many ways of packing the larger blocks $1\times2\times4$, $1\times2\times2$ and $2\times2\times2$ around the already positioned smallest bricks $1\times1\times3$. Impatient and nervous readers can find the complete solution at the end of this chapter (**5.12.**).

We end this chapter with the already mentioned Slothouber–Graatsma–Conway cube puzzle.

Problem 5.13.* *Assemble six* $1 \times 2 \times 2$ *blocks and three* $1 \times 1 \times 1$ *blocks into a* $3 \times 3 \times 3$ *cube.*

Before answering some questions posed in this chapter, let us mention two variants of Conways' $5 \times 5 \times 5$ cube that may be of interest to the reader. The blocks to be used in the first variant are fourteen $1 \times 2 \times 4$ blocks, three $1 \times 1 \times 3$ blocks and one $1 \times 2 \times 2$ block. In another version it is required to assemble thirty $1 \times 2 \times 2$ blocks and five $1 \times 1 \times 1$ blocks into a $5 \times 5 \times 5$ cube.

Answers to Problems

5.2. One solution is shown in Figure 5.19.

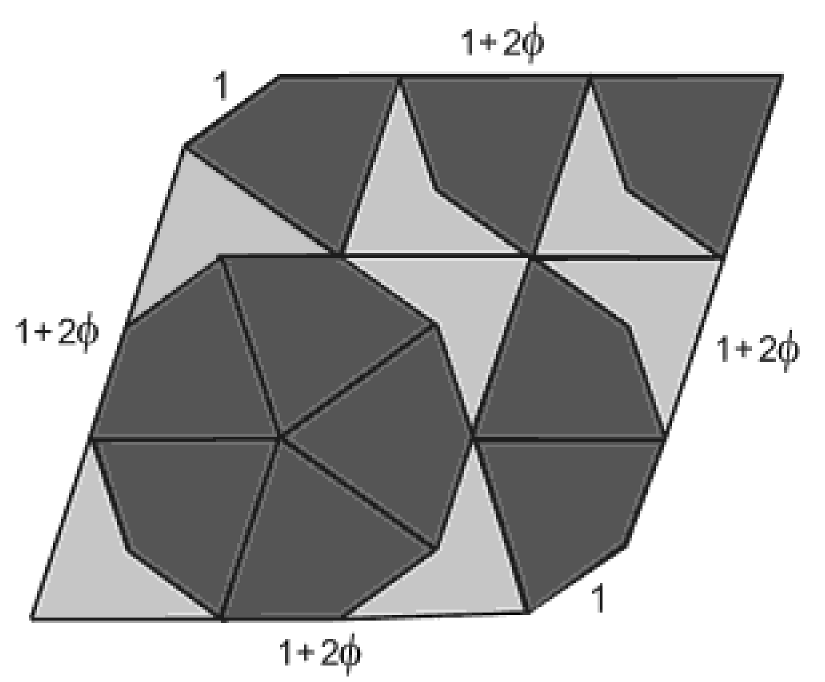


FIGURE 5.19. Kites and darts tile the truncated rhombus

5.4. Figure 5.20 shows the absolute minimum solution: the perimeter of 34 units cannot be decreased. The displayed solution was first given by P. J. Torbijn in 1984. The problem can be efficiently solved by a computer program.

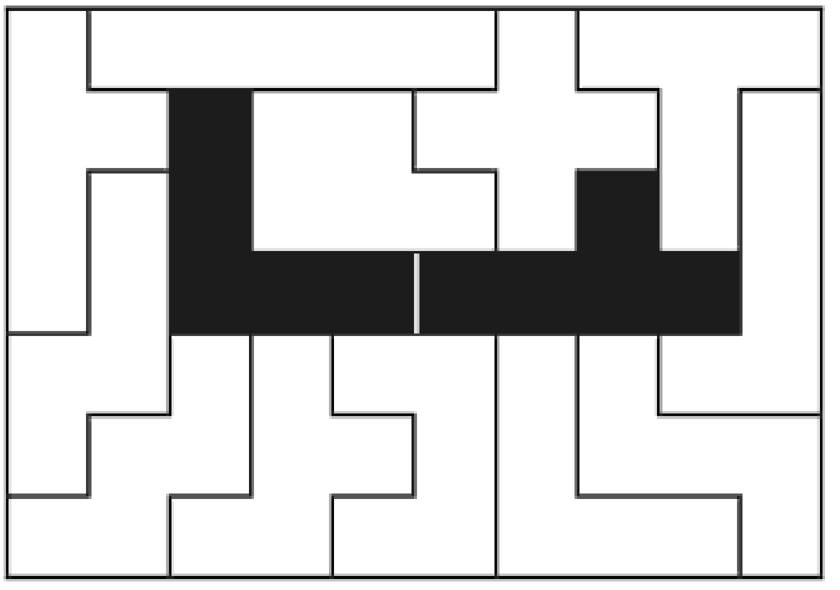


Figure 5.20. Pentamino-rectangle with minimal perimeter

5.5. One solution of the first problem is shown in Figure 5.21. T. R. Dawson, a famous English composer of chess (and other) problems, proved that this covering problem has a solution for the arbitrary position of the square tetramino. In 1958 Dana Scott, a mathematician from Princeton University, used a computer program to find 65 different solutions (excluding those solutions obtainable by rotations and reflections) with the square in the center of the board. However, the first solution for covering a chessboard with all 12 pentaminoes (not requiring the tetramino in the center) comes from H. Dudeney [55]. In his solution the square tetramino is located on the edge of the board.

One solution of the second problem is displayed in Figure 5.22.

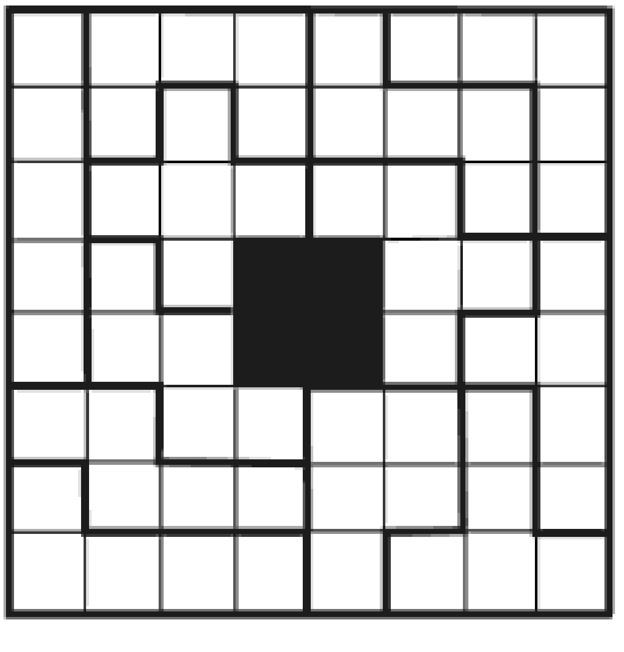


Figure 5.21. Board without central square

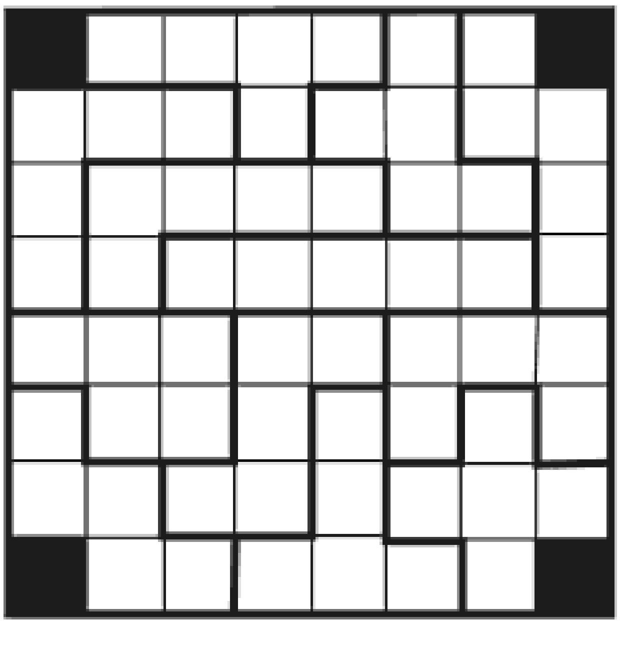


Figure 5.22. Board without corner squares

5.6. Let us recall that the chessboard is black and white colored; see Figure 5.23. Since the cut opposite squares are both black, there are 32 white squares and 30 black squares. One domino, placed either horizontally or vertically, always covers one white and one black square. Therefore, no tiling exists. Notice that the coloring principle can be usefully applied to resolve a variety of packing problems; see Klarner [114].

A similar approach based on the coloring argument deals with the numbers 0 and 1; the number 0 is written in each white square and 1 in each black square. The total sum of these numbers is 30. On the other hand, the domino-sum for each domino is 1, making the total domino-sum equal to $31 \times 1 = 31$, not 30. This proves that the tiling is impossible.

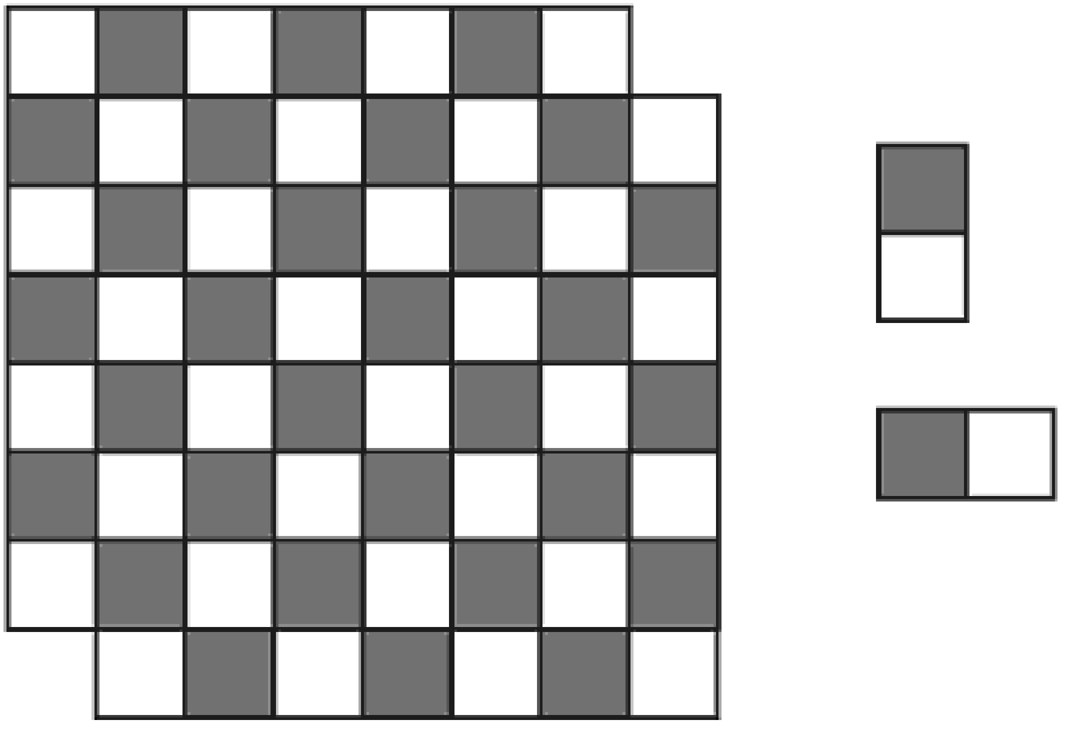


FIGURE 5.23. Tiling the truncated chessboard

5.7. Recall that the L-tile is one of the tetraminoes, the well-known pieces from the famous computer game "Tetris" proposed by the Russians A. Pajitnov and V. Gerasimov. The posed tiling problem can be solved effectively using the number-coloring argument presented in the solution of Problem 5.6.

First we write 1s and 5s in the small cells of the large square, as shown in Figure 5.24. Notice that each L-tetramino covers either three 1s and one 5 or three 5s and one 1. Therefore, the sum on a tile is either 8 or 16, in other words, a multiple of 8. Now we calculate the total sum S of the numbers on the large square: $S = 5 \times 10 + 5 \times 50 = 300$. Since 300 is not divisible by 8, we conclude that the tiling is impossible.

1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5

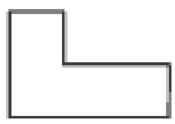


FIGURE 5.24. Tiling by L-tetraminoes

5.9. The required solution is shown in Figure 5.25.

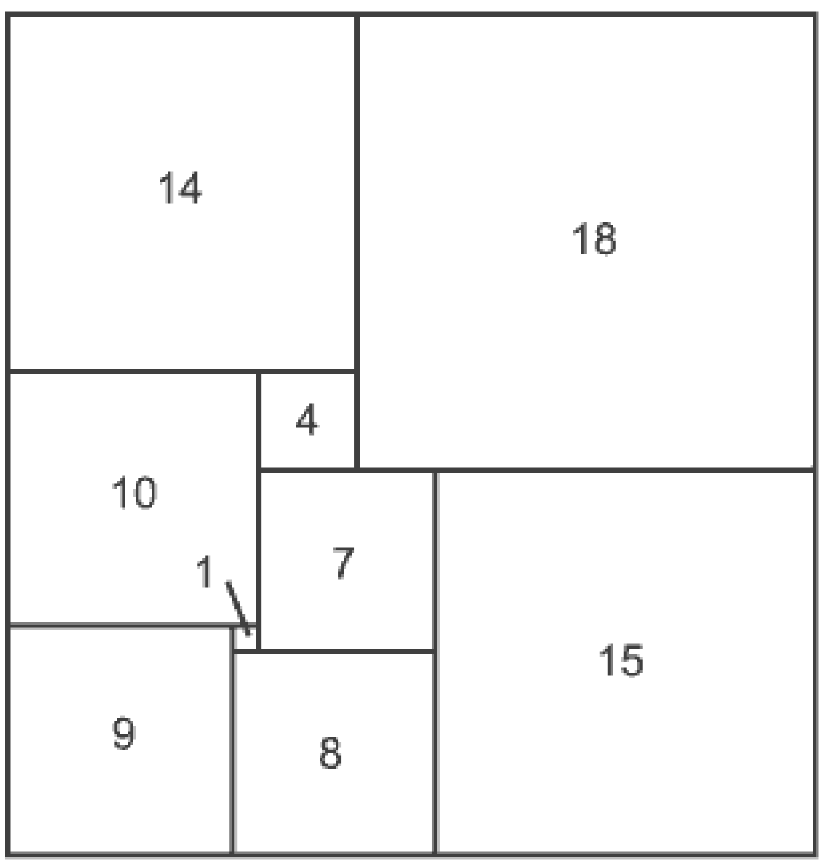


FIGURE 5.25. Rectangle composed of 9 squares

5.12. We will represent the solution in the xyz coordinate system, but taking strips instead points, as shown in Figure 5.18. For example, the unit cube in the origin corner is represented by $(1, 1, 2)$, while the smallest blocks $1 \times 1 \times 3$, placed in the starting position (see Figure 5.18) have the "coordinates" $(1, 1, 345)$, $(2, 234, 2)$, and $(345, 5, 1)$. The number of digits from the set $\{1, 2, 3, 4, 5\}$ gives the dimension of a block (that is, the number of unit cubes) in the corresponding direction. For example, $(345, 5, 1)$ tell us

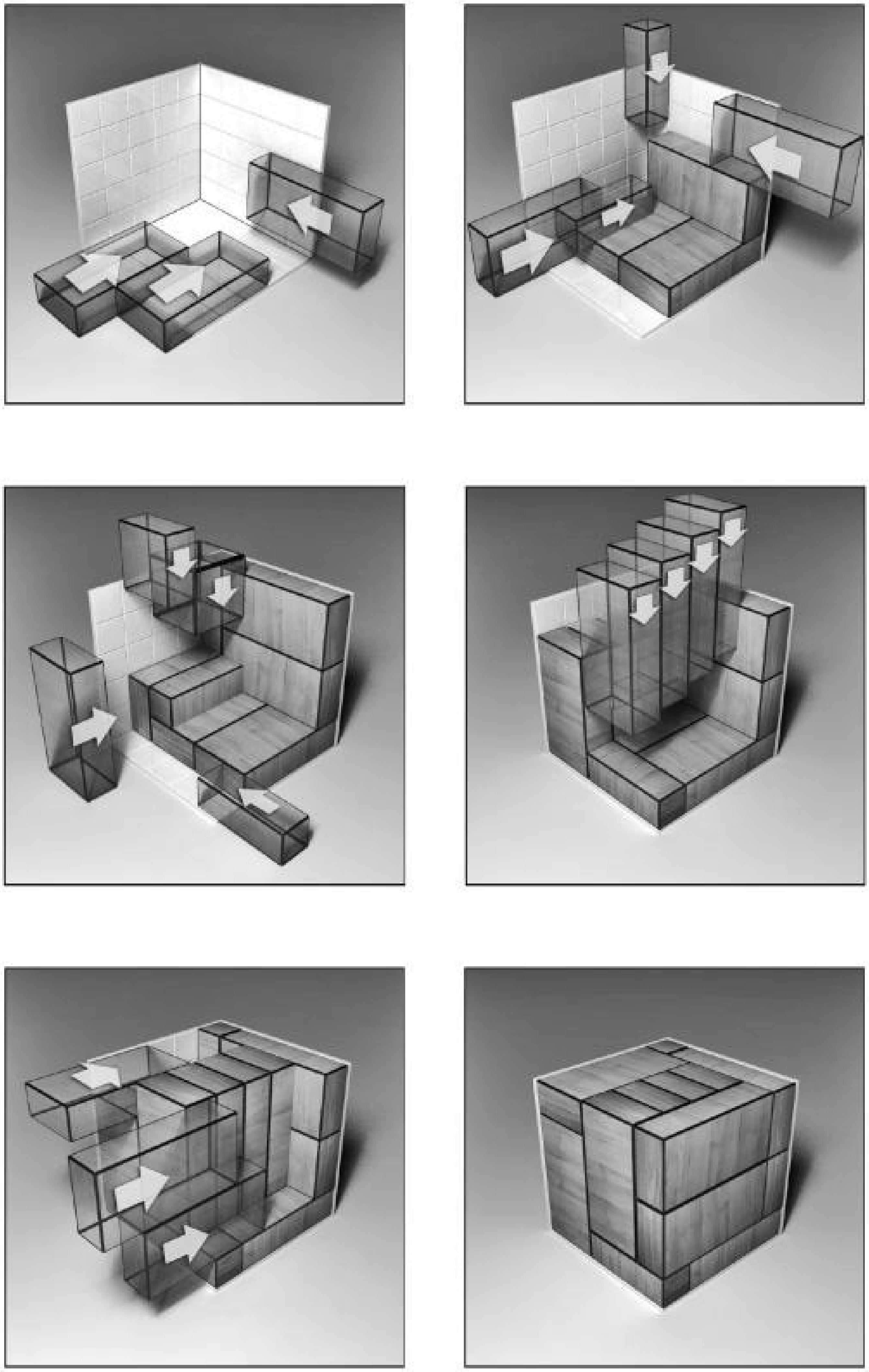


FIGURE 5.26. Conway's $5 \times 5 \times 5$ cube puzzle—solution

that the dimension of the placed block is $3 \times 1 \times 1$ and it occupies the strips 3, 4 and 5 along the x-axis, the strip 5 along the y-axis and the strip 1 along the z-axis. Using this notation, we give one solution below.

1–3	$(23, 1234, 1)$	$(45, 1234, 1)$	$(2345, 1, 23)$
4–6	$(2345, 1, 45)$	$(1, 1234, 12)$	$(1, 1, 345)$
7–9	$(2, 234, 2)$	$(12, 23, 34)$	$(12, 4, 34)$
10–12	$(12, 5, 1234)$	$(345, 5, 1)$	$(34, 2, 2345)$
13–15	$(34, 3, 2345)$	$(34, 4, 2345)$	$(34, 5, 2345)$
16–18	$(5, 2345, 23)$	$(5, 2345, 45)$	$(12, 2345, 5)$

The corresponding illustrations which show the packing of the $5 \times 5 \times 5$ cube are shown in Figure 5.26.

5.13. As in the case of Conway's $5 \times 5 \times 5$ cube, the solution is not hard to find once we come to the correct initial structure. The cube $3 \times 3 \times 3$ has 9 layers (cross sections), three from each perspective, one of them being shown in Figure 5.27 left in checkerboard-coloring. The blocks

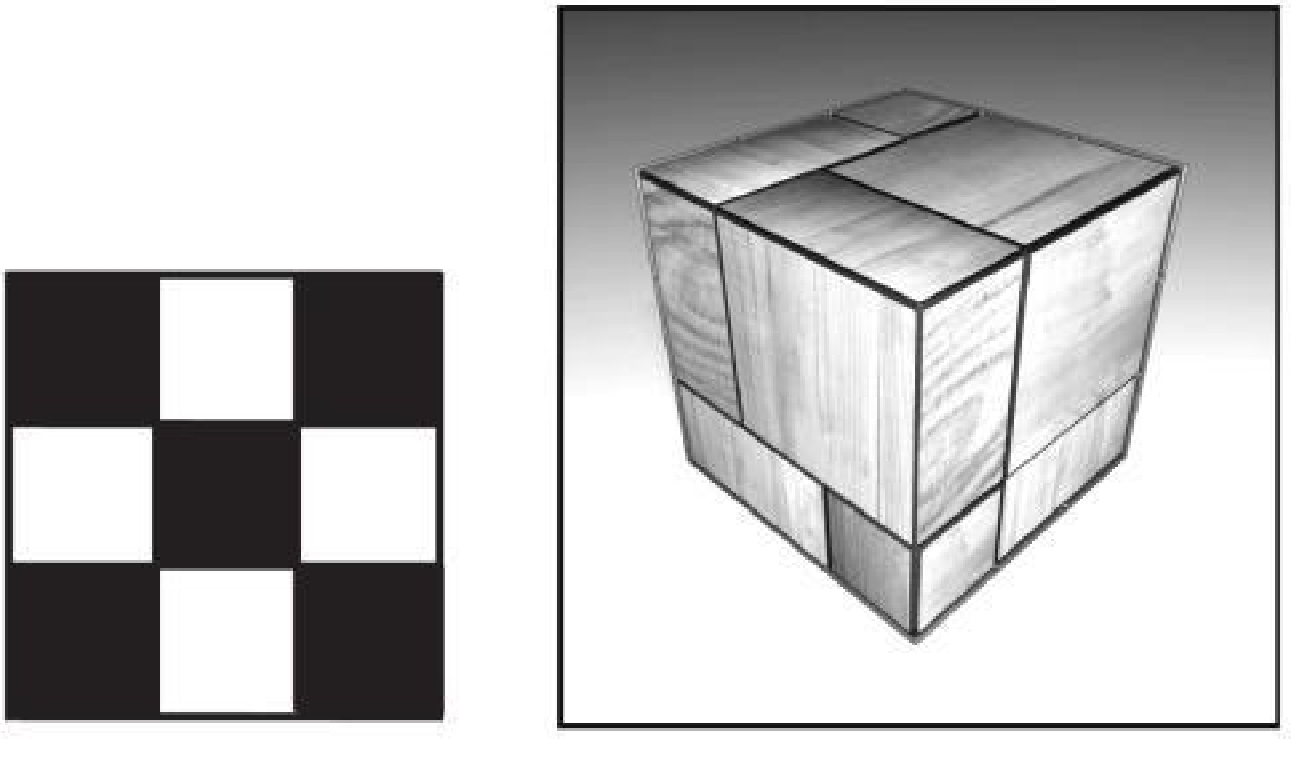


FIGURE 5.27. Conway-like $3 \times 3 \times 3$ cube puzzle—solution

$1 \times 2 \times 2$ always occupy an even number of cells in each layer, leaving 1 or 3 cells for the unit blocks $1 \times 1 \times 1$. It is immediately clear that all 3 unit blocks must not be placed on a single layer because the remaining layers could not be covered completely with the $1 \times 2 \times 2$ blocks. Therefore, we conclude that each of the nine layers must have in it one and only one of the $1 \times 2 \times 2$ block.

Furthermore, we observe that a unit cube must be at the center or at one of the corners (black cells). Indeed, placing a unit cube at one of the white cells leaves 3 white cells and 5 black cells for the arrangement of the $1\times2\times2$ blocks. But this is impossible because there are equal numbers of remaining black and white cells. The only way to fulfill the mentioned conditions is to place 3 blocks $1 \times 1 \times 1$ along a space diagonal. Using the notation from the solution of Problem 5.12, one initial position of the unit cubes can be represented as $(1,1,3)$, $(2,2,2)$, $(3,3,1)$. Obviously, these cubes can also be placed along other space diagonals.

One solution which starts with the mentioned initial position of the unit cubes is given below.

$$\begin{array}{llllll}
1\text{–}5 & (12,1,12) & (12,23,1) & (3,12,12) & (3,3,1) & (2,2,2)\\
6\text{–}9 & (1,23,23) & (23,12,3) & (23,3,23) & (1,1,3) &
\end{array}$$

The illustration of the completed $3\times3\times3$ cube is shown in Figure 5.27 right.

Chapter 6 PHYSICS

In physics, you don't have to go around
making trouble for yourself—nature does it for you.
Frank Wilczek

Among the oldest puzzles are those that rely on elementary physical principles or involve physical objects. Everyone is familiar with Zeno's paradox on the race of Achilles and a tortoise. Aristotle describes in his *Physics* that the quickest runner can never overtake the slowest, since the pursuer must first reach the point from where the leader started, so that the one who started first must always hold a lead. Surprisingly, some philosophers believe that a correct explanation of this paradox has not been given yet.

From the beginning of science, mathematicians and physicists have had close collaboration. Many mathematical theories have arisen in solving physics problems. On the other hand, the development of some new theories in physics was impossible without profound mathematics (for example, Einstein's theory of relativity and quantum mechanics). However, sometimes mathematicians and physicists do not get along. The famous physicist Ernest Rutherford once said: *"All science is either physics or stamp collecting."* His contemporary, the great mathematician David Hilbert, retaliated him: *"Physics is too hard for physicists,"* implying that the necessary mathematics was generally beyond their reach. There is a wildly spread joke on Internet sites: *"An engineer thinks that his equations are an approximation to reality. A physicist thinks reality is an approximation to his equations. A mathematician doesn't care."* And one more: *"A mathematician believes nothing until it is proven, a physicist believes everything until it is proven wrong."*

Many physics-math puzzles are concerned with motion, traced distances, commuter problems, fluid properties, questions of balance, time machines, physical principles and phenomena. Puzzles that belong only to physics and contain very little or nothing of mathematics are not considered in this book. They can be found in many books devoted exclusively to the topics of physics. In this chapter we use physics terminology freely, assuming that the reader possesses some basic high school knowledge of physics.

The first puzzle of this chapter is the classical problem on the gold crown of King Hiero, attributed to Archimedes. Its solution is based on Archimedes' discovery, the first law of hydrostatics. You probably know about the legend of Archimedes who, excited after this discovery about the displacement of water in his bath, ran naked through the street crying *"Eureka!"* A motion problem by Nicole Oresme is given to demonstrate an elegant solution using a geometrical method instead of summing infinite series. It was a fine achievement in the fourteenth century. Summing an infinite series is also mentioned in connection with the ability of John von Neumann to quickly operate with complicated and long expressions by heart. The main attention in this chapter is devoted to the famous problem on the lion and the man in a circular arena, proposed by Richard Rado in 1932 and studied by numerous mathematicians, including Littlewood and Besicovitch. In this problem you again see some infinite series.

*

* *

Archimedes (280 B.C.–220 B.C.) (→ p. 299)

The gold crown of King Hiero

The following frequently told story of King Hiero's[1] gold crown is often connected to Archimedes (see, *e.g.*, [100]):

To celebrate his victories, King Hiero ordered a crown of pure gold. When the crown was finished, the information came that the goldsmith had withheld a certain amount of gold replacing it with silver. The king, unable to find a way of detecting the theft, referred the matter to Archimedes. While Archimedes was considering this problem, he happened to go bathing. As he entered the bathing pool, it occurred to him that the amount of water flowing outside the pool was equal to the volume of his body that was immersed. Archimedes realized that he could apply this fact to the problem at hand. Elated, he rose from the pool and forgetting to clothe himself, he ran home naked shouting loudly: "Eureka, eureka! (I have found it!)*."*

Archimedes' discovery was, in fact, the first law of hydrostatics, given as Proposition 7 in his first book *On Floating Bodies*:

A body immersed in fluid is lighter than its true weight by the weight of the fluid displaced.

[1] Hiero, or Hieron, the king of Syracuse, third century B.C., Archimedes' relative (according to some historians).

Let us return to the problem of the king's crown and solve the following task.

Problem 6.1. *Find the ratio of gold to silver in Hieros' crown using Archimedes' law of hydrostatics.*

Suppose that a crown of weight w is composed of unknown weights w_1 and w_2 of gold and silver, respectively. To determine the ratio of gold to silver in the crown, first weigh the crown in water and let F be the loss of weight. This amount can be determined by weighing the water displaced. Next take a weight w of pure gold and let F_1 be its weight loss in water. It follows that the weight of water displaced by a weight w_1 of gold is $\frac{w_1}{w}F_1$. Similarly, if the weight of water displaced by the weight w of pure silver is F_2, the weight of water displaced by a weight w_2 of silver is $\frac{w_2}{w}F_2$. Therefore,

$$F = \frac{w_1}{w}F_1 + \frac{w_2}{w}F_2.$$

Substituting $w = w_1 + w_2$ in the last relation, we find the ratio of gold to silver,

$$\frac{w_1}{w_2} = \frac{F - F_2}{F_1 - F}.$$

Nicole Oresme (1320–1382) (→ p. 301)

The career of French mathematician Nicole Oresme carried him from a college professorship to a bishopric. He was a mathematician, physicist, astronomer, philosopher, musicologist, theologian, and finally Bishop of Lisieux. Oresme was probably the most eminent and influential philosopher of the fourteenth century. Today the widely known apology, *"I indeed know nothing except that I know that I know nothing,"* is attributed to Oresme.

Considering the problem of motion, and in particular the quantitative representation of velocity and acceleration, Oresme had in essence developed the idea of representing the functional relationship between velocity and time by a curve; see Clagett [36]. Giovanni di Cosali gave an earlier graph of motion, however, it lacked sufficient clarity and impact.

The length of traveled trip

Oresme's geometric technique appeared some 250 years before the work of Galileo Galilei (1564–1642) in this field. Oresme studied the above-mentioned subject in an abstract sense only; this becomes evident in the following problem of velocities that increase without bound.

Problem 6.2. *The velocity of an object is taken to be* 1 *unit during the first half of the time interval* AB, 2 *units in the next quarter,* 3 *units in the next eighth,* 4 *in the next sixteenth, and so on, to the infinity. Calculate the total distance traveled.*

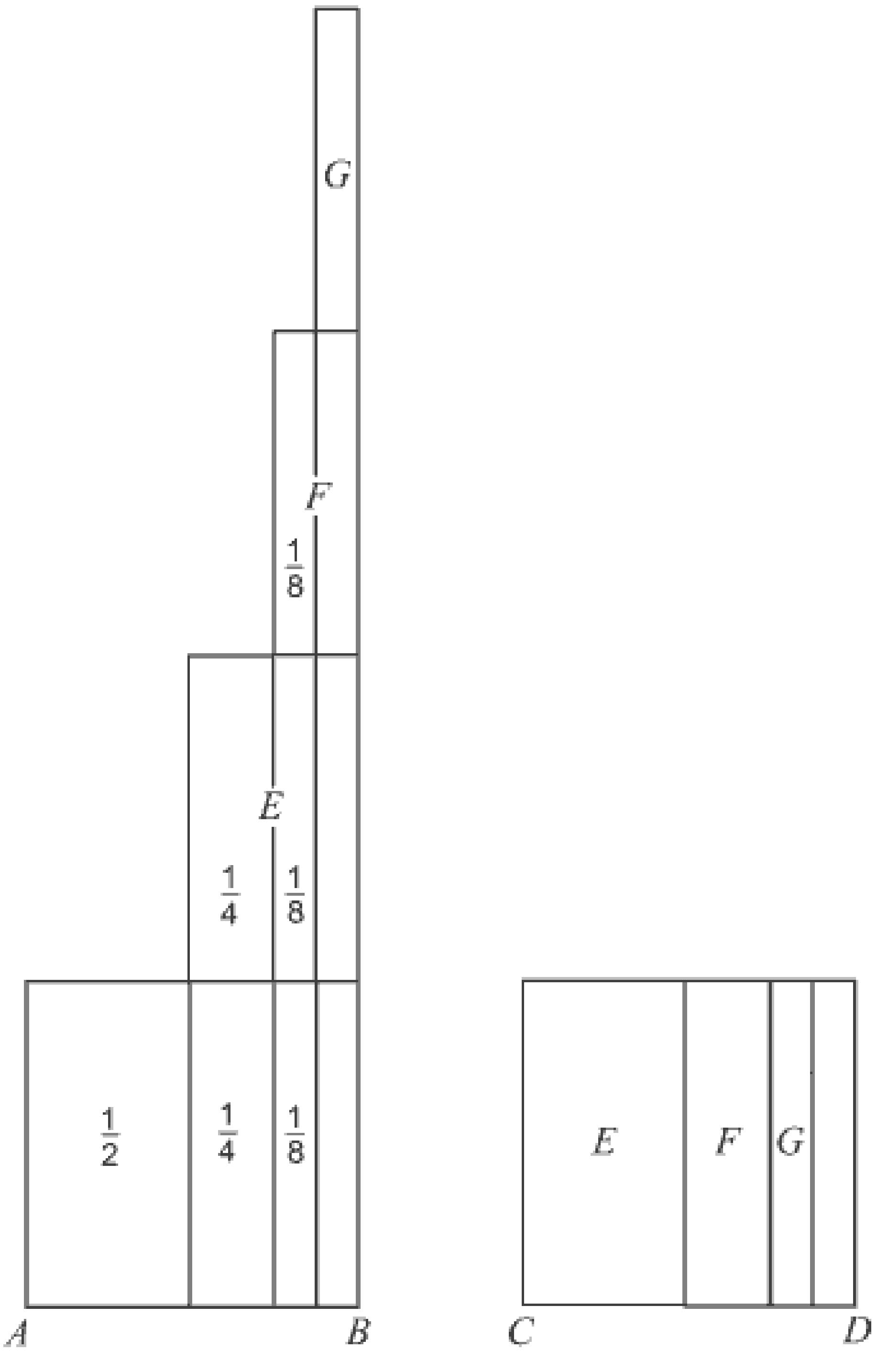


FIGURE 6.1. Geometric method of summing the infinite series

Solution. The sum of the infinite series given below yields the distance traveled:

$$S = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \cdots + \frac{1}{2^n}\cdot n + \cdots \,. \tag{6.1}$$

Oresme found that this sum is equal to 2, using an elegant geometrical method. He drew a square of base CD equal to AB $(= 1)$ and divided it "to infinity into parts continually proportional according to the ratio 2 to 1" (Figure 6.1). In other words, E represents half of the square, F one quarter,

G one eighth, and so on. The rectangle E is placed over the right half of the square on AB, F atop the new configuration over its right quarter, G atop the right eighth, and so on. It is evident that the total area of the new configuration, which represents the total distance traveled, is not only equal to the sum of the infinite series but also equal to the sum of the areas of the two original squares.

Let us pause for a moment and leave to the reader the pleasant work of finding Oresme's sum.

Problem 6.3.* *Find the infinite sum* (6.1) *in an elementary way. No derivatives, please!*

Figure 6.1 and the summation by the "packaging" method resembles very much a subtle problem of Leo Moser, a professor of the University of Alberta (Canada). Consider the squares with the sides $1/2,\ 1/3,\ 1/4, \ldots$. These numbers form the so-called *harmonic series*

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

whose sum is infinite. The proof may be found in almost every textbook on series or calculus. However, the total area of these squares is finite. This was discovered in 1746 by Euler who found that[2]

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2 + \cdots = \frac{\pi^2}{6} - 1 \approx 0.644934.$$

Having in mind this result, Leo Moser posed the following problem:

Problem 6.4.* *Can the infinite number of squares with sides* $1/2,\ 1/3,\ 1/4, \ldots$ *be fitted without overlap into a unit square?*

Edouard Lucas (1842–1891) (→ p. 307)

The French mathematician Edouard Lucas is best known for his results in number theory: he stated a method of testing primality which, after refinement by D. H. Lehmer in 1930, has remained in use up to the present day under the name Lucas–Lehmer test. Lucas particularly studied the sequence defined by $F_n = F_{n-1} + F_{n-2}$, arising from Fibonacci's rabbit problem (page 12), and named it the Fibonacci sequence. Some of the

[2]According to C. B. Boyer [26], Euler's mentor Johan Bernoulli had this result four years before Euler.

brain teasers that we have presented in this book were taken from Lucas' four-volume work on recreational mathematics *Récréations Mathématiques* (1882–94).

Meeting of ships

Problem 6.5. *Daily at noon a ship departs from Le Havre bound for New York and conversely, another ship leaves New York bound for Le Havre. The crossing lasts 7 days and 7 nights. During the passage to New York, how many Le Havre-bound ships will the New York-bound vessel meet, with today as its date of departure?*

You should be careful, this is a bit tricky. A quick answer "seven", forgetting about the ships already en route, is incorrect. A convincing solution is shown graphically in the diagram.

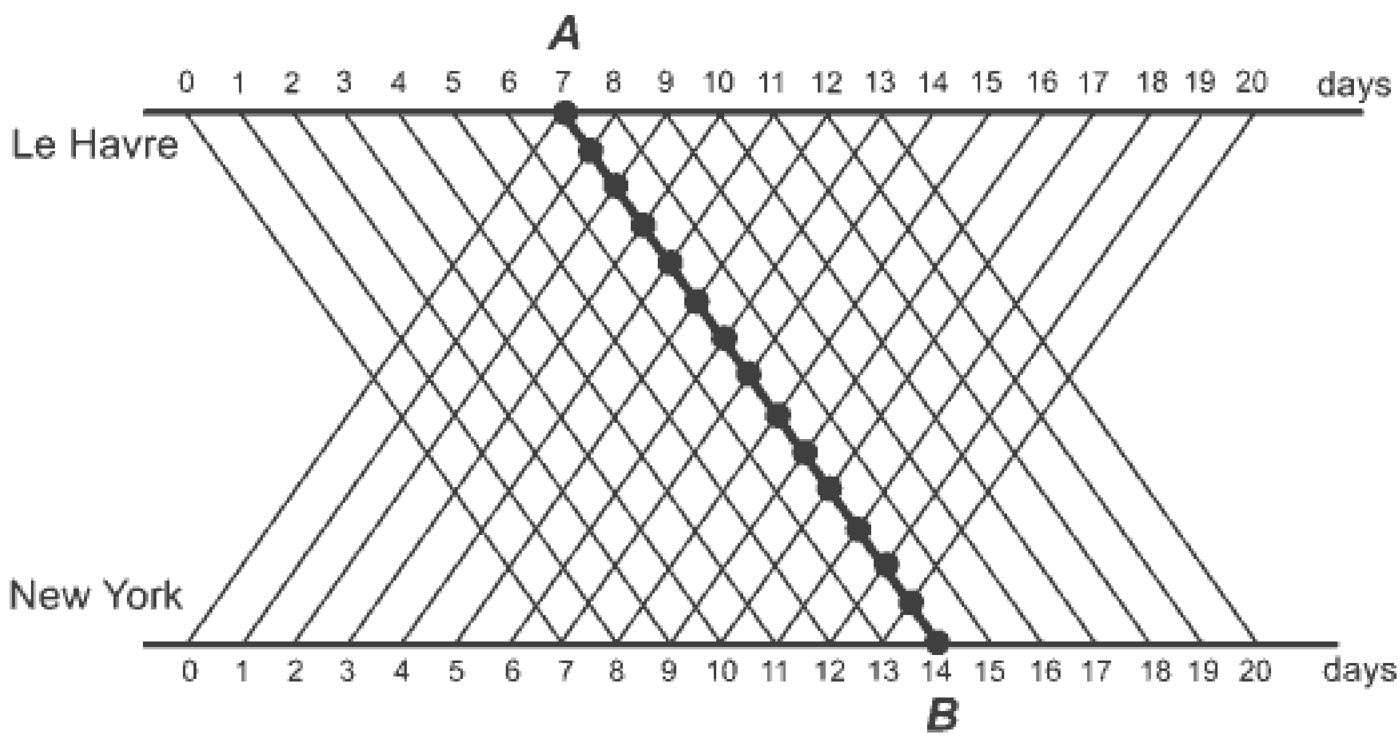


FIGURE 6.2. Diagram of ships' crossings

At the moment of departure from Le Havre (point A in Figure 6.2), 8 ships are en route to Le Havre. In fact, one of them is just entering Le Havre's harbor, and another ship has just left the New York harbor. The ship starting from A will meet all of these ships. In addition, during the course of the Le Havre ship's seven-day crossing, 7 ships leave New York, the last of them at the very moment that it enters the harbor. Therefore, the total number of meetings equals 15. It is clear from the diagram that the ship, whose trip is shown by the line segment AB, meets 13 other ships at the sea, plus two ships at their moments of departure (Le Havre harbor) and arrival (New York harbor), which makes 15 in total.

John von Neumann (1903–1957) (→ p. 309)

John von Neumann is certainly one of the greatest twentieth-century mathematicians, making several remarkable achievements to the field of mathematics. Many consider von Neumann the founder of mathematical game theory and a contributor in the development of high-speed computing machines and cellular automata. He gave the great contribution in designing the EDVAC, one of the first electronic computers that was ever built. Like many scientists, during and after World War II von Neumann also worked on the key problems of pure and applied mathematics and physics which made possible the development of the hydrogen bomb. Unlike many of his colleagues, who focused only on their jobs, von Neumann lived a rather unusual lifestyle. He and his wife Klara enjoyed an active social life, giving memorable parties and banquets in their Princeton home.

John von Neumann
1903–1957

From an early age, John von Neumann manifested incredible powers of memory to which many stories attest. At the age of six, he was exchanging jokes in classical Greek with his father and dividing two eight-digit numbers in his head. The Neumann family sometimes entertained guests with demonstrations of Johnny's ability to memorize pages from phone book. A guest would select a page and column at random from the phone book. Young Johnny, János in Hungarian, would read the column over a few times and after that he could answer any question put to him as to whom a certain number belonged, or recite the names, addresses, and numbers in order. He also loved history and, since he remembered everything he once read, he became an expert on many issues: Byzantine history, the details of the trial of Joan of Arc, minute features of the battles of the American Civil War.

The eminent mathematician Paul Halmos, von Neumann's friend, cited in [98] that another famous mathematician George Pólya once said:[3] *"Johnny was the only student I was ever afraid of. If in the course of a lecture I stated an unsolved problem, the chances were he'd come to me as soon as the lecture was over, with the complete solution in a few scribbles on a slip of paper."* Halmos also mentions that von Neumann's poor driving (in)ability was legendary. Von Neumann himself reported one of his numerous accidents

[3] It is rather interesting that von Neumann, Halmos and Pólya, were all born in Budapest, Hungary.

as: *"I was proceeding down the road. The trees on the right were passing me in orderly fashion at* 60 *miles per hour. Suddenly one of them stepped in my path. Boom!"*

A girl and a bird

This old problem can be found in many books of recreational mathematics in different variants. Here we present a simple one.

Problem 6.6. *A girl stands* 500 *feet from a wall with a bird perched upon on her head. They both start to move towards the wall in a straight line. The girl walks at a rate of* 5 *feet per second and the bird flies at* 15 *feet per second. Upon reaching the wall, the bird immediately reverses direction and returns to the girl's head where it again reverses direction. The bird's to-and-from flight pattern continues until the girl reaches the wall. How far did the bird travel?*

Many people try to solve this classic problem the hard way. They sum the lengths of the bird's path between the girl and the wall during her walk. These paths become shorter and shorter, and such an approach leads to the *summing of an infinite series*, which is complicated. But, the solution is very simple and we could call it the "trick" solution. It is sufficient to note that the girl walks for 100 seconds before hitting the wall. Thus the bird flies for 100 seconds and travels

$$(15 \text{ feet per second}) \times (100 \text{ seconds}) = 1{,}500 \text{ feet}.$$

There is a legend that when his high school teacher gave this problem to John von Neumann, he solved it quickly. When the teacher commented, *"Ah, you saw the trick,"* Neumann replied, *"What trick? It was an easy series."*

We present the "easy series" for the flight time which might be the one von Neumann used.

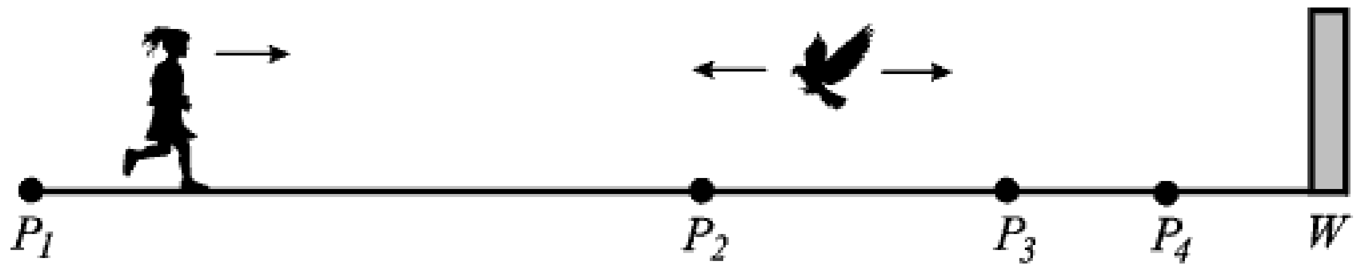


Figure 6.3. The girl and the bird

Let P_1 be their initial position and W the turnaround point at the wall as shown in Figure 6.3. Then P_1 is the point where the bird is first on the girl's head. Let P_i $(i = 1, 2, \dots)$ be the point corresponding to the ith contact of bird and head and t_i the time elapsed between contact i and contact $i+1$. Let D be the distance from P_1 to W, v_B the bird's speed and v_G the girl's speed $(v_B > v_G)$.

In time t_1 the bird goes from P_1 to W and back to P_2. Also in time t_1 the girl goes from P_1 to P_2 and we have

$$t_1 v_B + t_1 v_G = 2|P_1W| = 2D. \tag{6.2}$$

Similarly, in time t_2 the total distance they travelled is twice the distance from P_2 to W, giving

$$t_2 v_B + t_2 v_G = 2(|P_1W| - |P_1P_2|) = 2(D - v_G t_1) = t_1 v_B - t_1 v_G, \tag{6.3}$$

where (6.2) is used to obtain the last equality in (6.3). During time t_3 we have

$$\begin{aligned} t_3 v_B + t_3 v_G &= 2(|P_1W| - |P_1P_2| - |P_2P_3|) \\ &= 2(D - v_G t_1 - v_G t_2) = t_2 v_B - t_2 v_G, \end{aligned} \tag{6.4}$$

where (6.3) is used to obtain the last equality in (6.4). From (6.4) we can conclude that in general we have

$$t_i = \frac{v_B - v_G}{v_B + v_G} t_{i-1} = K t_{i-1} = \cdots = K^{i-1} t_1,$$

where $K = (v_B - v_G)/(v_B + v_G)$. Since $K < 1$, the total time T is given by

$$\begin{aligned} T &= \lim_{n\to\infty} \sum_{i=1}^{n} t_i = \lim_{n\to\infty} t_1 \sum_{i=0}^{n-1} K^i = \frac{2D}{v_B + v_G} \cdot \frac{1}{1-K} \\ &= \frac{2D}{v_B + v_G} \cdot \frac{1}{1 - (v_B - v_G)/(v_B + v_G)} = \frac{D}{v_G}. \end{aligned}$$

Then the total distance traveled by the bird is $v_B T = v_B D / v_G$, as the "trick" solution predicts.

One will find it useful to represent the paths of the girl and the bird by plotting a space-time graph as we see in Figure 6.4. The slopes of the displayed segment lines are proportional to the velocities v_G and v_B of the girl and the bird, respectively. The points P_1, P_2, $P_3, \dots$ represent the points of contacts of the girl and the bird. Of course, we cannot finish drawing the bird's path to W because the zigzags are infinite.

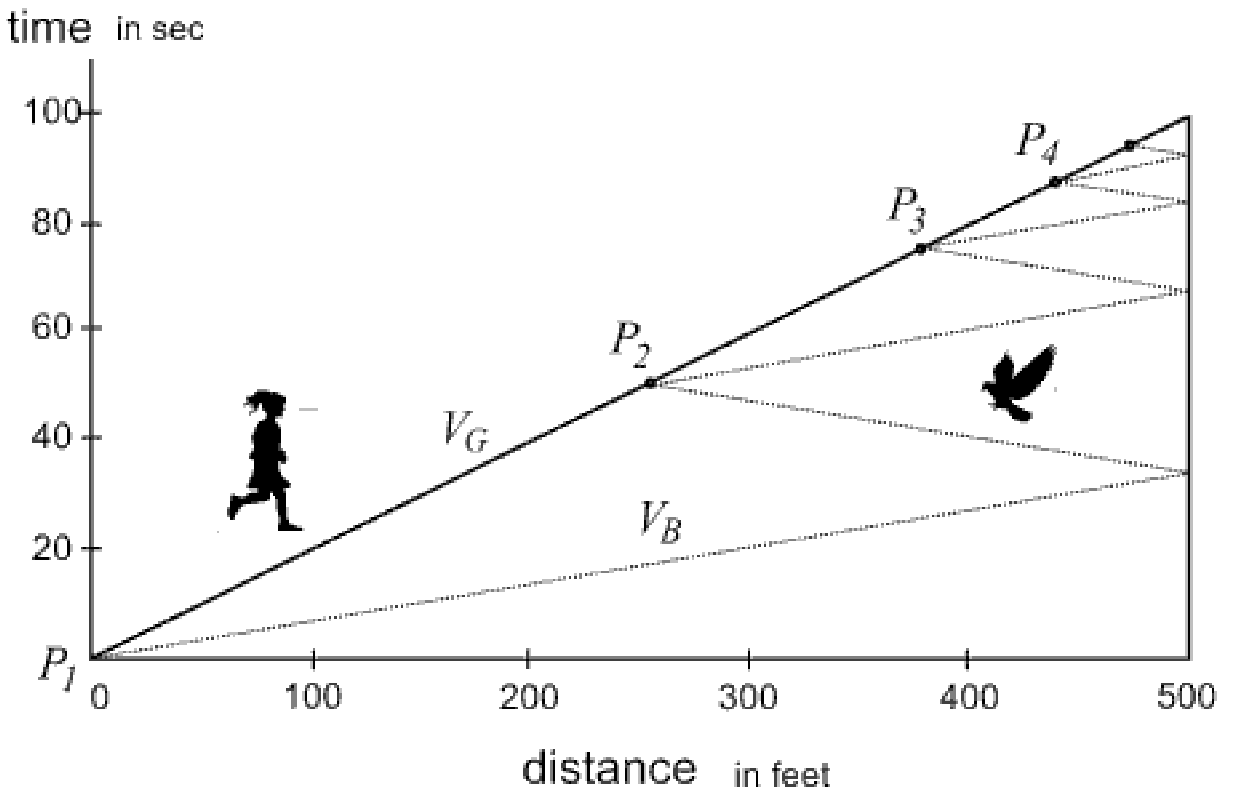


FIGURE 6.4. Space-time graph for the problem of the girl and the bird

Let us consider another more complicated version of this problem in which an infinite series, requiring much tedious time and effort, appears.

Problem 6.7.* *Two cars, setting out from points* A *and* B 140 *miles apart, move toward each other on the same road, until they collide at* C. *Their speeds are* 30 *miles and* 40 *miles per hour. At the very instant they start, a bird takes flight from point* A *heading straight toward the car that has left point* B. *As soon as the bird reaches the other car, it turns and changes direction. The bird flies back and forth in this way at a speed of* 50 *miles per hour until the two cars meet. How long is the bird's path?*

In Chapters 7 and 9 we will encounter the "river-crossing problems". The following problem comes in two flavors, a river-crossing and a bird's path flavor.

Problem 6.8.* *Four rower-mathematicians wish to cross the river by means of a boat that can only hold two men. The rower* R_1 *needs* 1 *minute to cross the river alone, and the rowers* R_2, R_3 *and* R_4 *need* 2, 6 *and* 9 *minutes, respectively. Being mathematicians, the rowers have planned an optimal strategy for crossing the river. They start to row towards the opposite river bank in a straight line. At the very moment they begin to row, a swan starts to swim over the river in a straight line with the speed of* 60 *feet per minute. The swan reaches the opposite river bank at the exact moment when the rowers complete their transfer. How wide is this river?*

John E. Littlewood (1885–1977) (→ p. 308)

Abram Besicovitch (1891–1970) (→ p. 308)

Richard Rado (1906–1989) (→ p. 309)

The lion and the man

The problem about a lion and a man that Richard Rado proposed in 1932 belongs to the area of recreational mathematics that has attracted a great deal of attention of numerous mathematicians, including John E. Littlewood and Abram Besicovitch. The problem reads thus:

Problem 6.9. *A man finds himself in a circular arena with a lion. Both man and lion can move throughout the entire arena at the same maximum speed. Without leaving the arena, can the man pursue a course of motion ensuring that the lion will never catch him? The problem assumes the unlimited strength of both man and lion.*

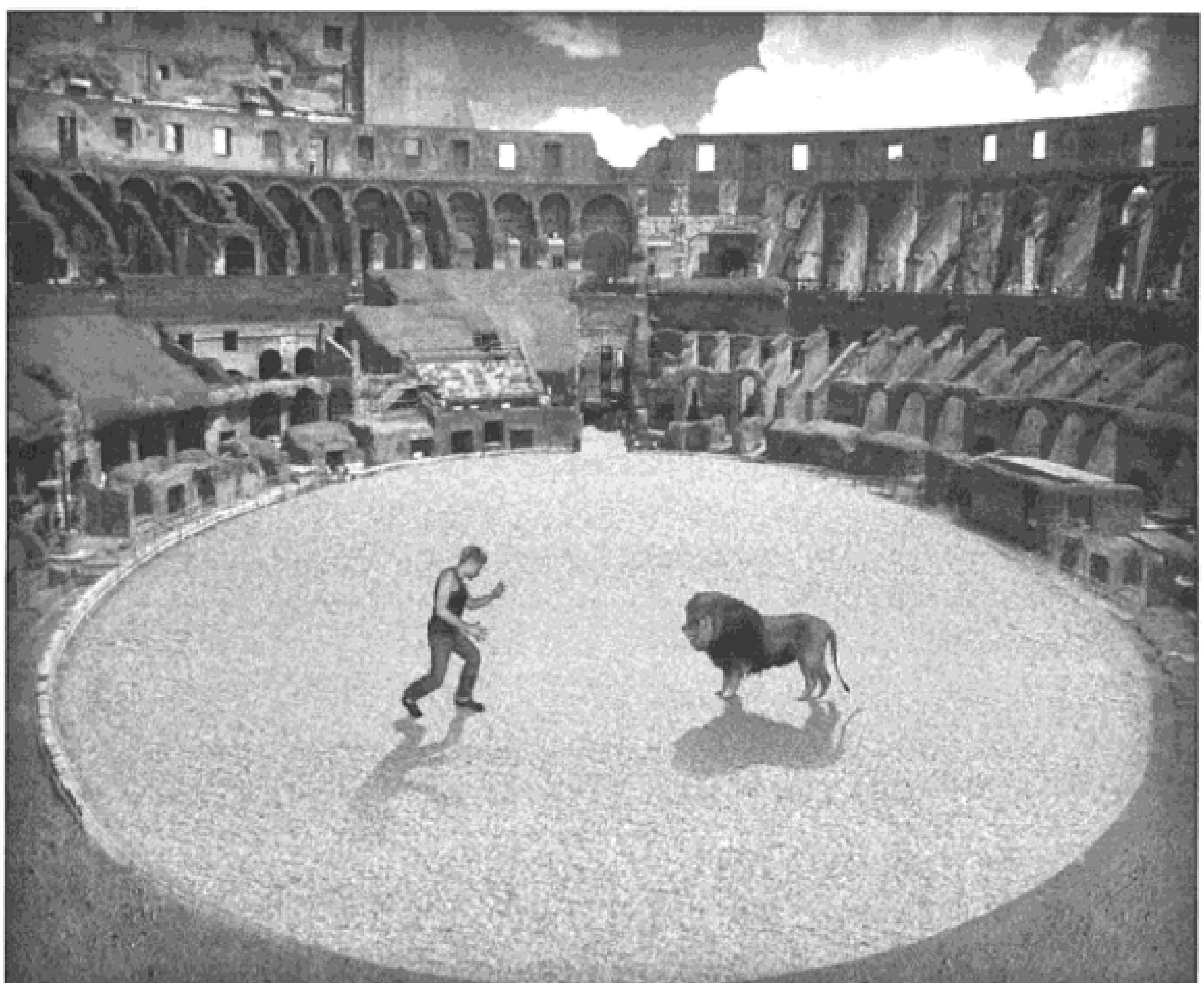


FIGURE 6.5. A man and a lion in a circular arena

For the next 25 years prevailing opinion, substantiated "irrefutable" proofs, held that the man could not escape from the lion. A quarter century later Abram S. Besicovitch, a professor at Cambridge University, proved that these assumptions were wrong. *A Mathematician's Miscellany* (London 1957), an intriguing book by the prominent English mathematician John E. Littlewood (1885–1977), published Besicovitch's proof of a strategy enabling the man to avoid contact with the lion. Evidently this problem interested Littlewood, for he discussed it in his book. More about this problem can also be found in the paper *How the lion tamer was saved* by Richard Rado (*Ontario Secondary School Mathematics Bulletin*, October 1972) (reprinted later in *Mathematical Spectrum* [145]) and in the joint paper [144] by Peter A. Rado and Richard Rado. Here we present a solution of this very interesting and challenging problem, based on the material given in the references above.

For the sake of simplicity, we will assume that both lion and man are mathematical points and that their velocities do not exceed the *maximal speed* v. Let r be the radius of the circle, C its center, M_0 and L_0 the starting positions of the man and the lion, d_0 the starting distance between them ($|M_0L_0| = d_0$) and s_0 the starting distance of the man from the center of the circle ($|M_0C| = s_0 < r$); see Figure 6.6(a). If he wishes to save himself, the man should apply the following strategy.

In the first time interval t_1 the man moves at speed v perpendicularly to the direction M_0L_0 as long as he traverses the distance $(r - s_0)/2$ always choosing the direction that keeps him closer to the center. In the case when M_0L_0 crosses the center, both directions are equal so that the man can choose either of them. Hence the first phase of the man's escape will last

$$t_1 = \frac{r - s_0}{2v}.$$

After that period of time, the distance of the man (who came to the point M_1) from the center of the circle is limited from above by the inequality

$$s_1 \le \sqrt{s_0^2 + \left(\frac{r - s_0}{2}\right)^2},$$

while the distance between the man and the lion (which came to the point L_1) is given by

$$d_1 \ge \sqrt{d_0^2 + \left(\frac{r - s_0}{2}\right)^2} - \frac{r - s_0}{2} > 0$$

(see Figure 6.6(b)). Therefore, it holds that

$$s_1^2 \le s_0^2 + \frac{1}{4}(r - s_0)^2.$$

In the second time interval the man runs at speed v perpendicularly to the direction M_1L_1 until he covers the distance $(r-s_0)/3$ again choosing the direction that keeps him closer to the center C. So, the second time interval is

$$t_2 = \frac{r - s_0}{3v}.$$

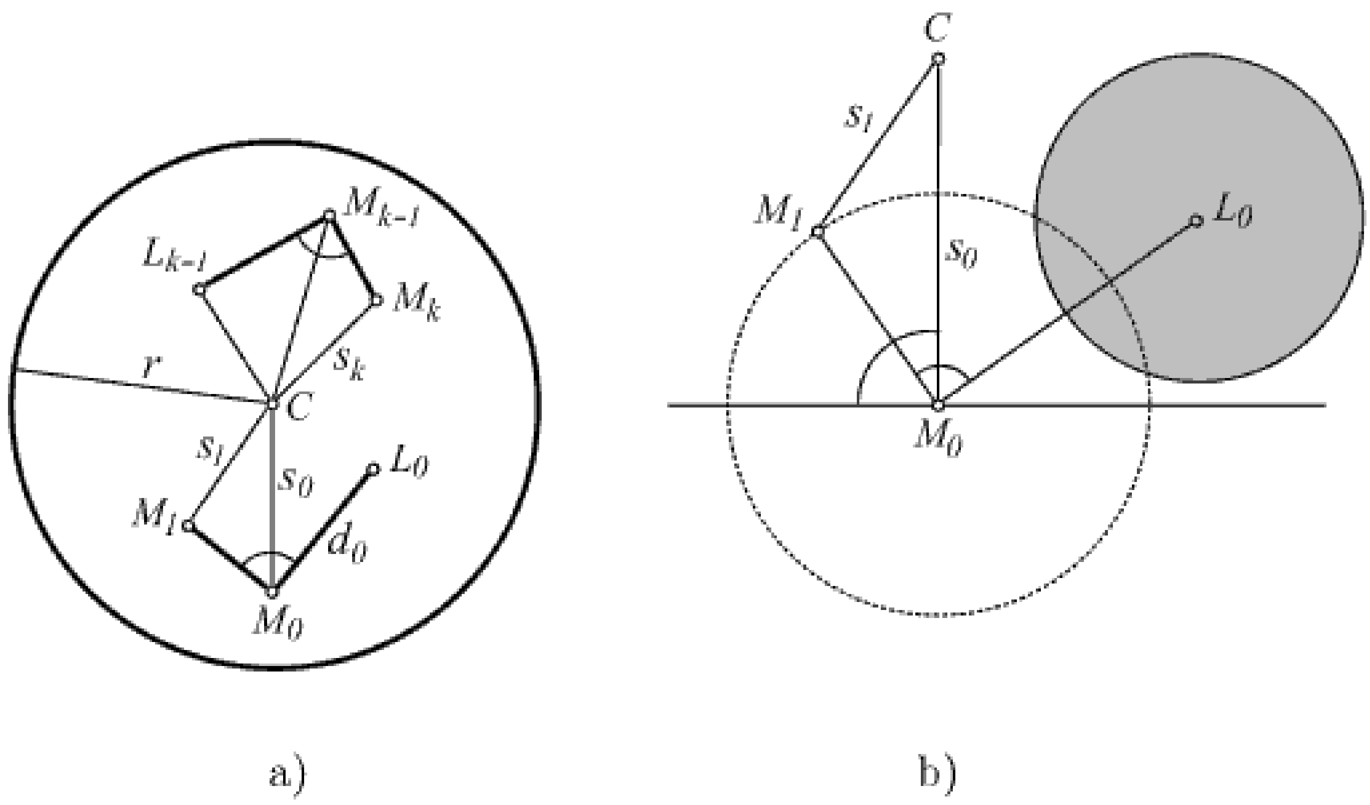


FIGURE 6.6. The paths of a man and a lion

After time $t_1 + t_2$ the distance of the man (who came to point M_2) from the center of the circle is bounded by

$$s_2 \le \sqrt{s_1^2 + \left(\frac{r - s_0}{3}\right)^2}.$$

The upper bound of the distance between the man and the lion (which came to point L_2) is given by

$$d_2 \ge \sqrt{d_1^2 + \left(\frac{r - s_0}{3}\right)^2} - \frac{r - s_0}{3} > 0.$$

From the inequalities for s_1 and s_2 we find

$$s_2^2 \leq s_1^2 + \frac{1}{9}(r - s_0)^2 \leq s_0^2 + \frac{1}{4}(r - s_0)^2 + \frac{1}{9}(r - s_0)^2.$$

The procedure continues in the same way. In the kth time interval t_k the man will run at speed v perpendicularly to the direction $M_{k-1}L_{k-1}$ as long as he crosses the path of the length $(r - s_0)/(k + 1)$, in the direction that enables him to remain closer to the arena's center. The corresponding time interval for this motion is

$$t_k = \frac{r - s_0}{(k + 1)v}.$$

After time $t_1 + t_2 + \cdots + t_k$ the distance of the man (who came to point M_k) from the center of the circle will be

$$s_k \leq \sqrt{s_{k-1}^2 + \left(\frac{r - s_0}{k + 1}\right)^2},$$

and from the lion (which came to point L_k)

$$d_k \geq \sqrt{d_{k-1}^2 + \left(\frac{r - s_0}{k + 1}\right)^2} - \frac{r - s_0}{k + 1} > 0.$$

According to the boundaries for $s_{k-1},\ s_{k-2}, ..., s_1$, we obtain

$$s_k^2 \leq s_0^2 + \frac{1}{2^2}(r - s_0)^2 + \frac{1}{3^2}(r - s_0)^2 + \cdots + \frac{1}{(k + 1)^2}(r - s_0)^2.$$

Let us now prove that the man, following the strategy outlined here, will never be caught by the lion. The proof is based on the following simple assertions:

If $a_n = 1/n$ *for* $n = 2, 3, \ldots$, *then*

(i) $$a_2 + a_3 + a_4 + \cdots = \infty;$$

(ii) $$a_2^2 + a_3^2 + \cdots + a_k^2 < 1 - \frac{1}{k + 1} \quad (k > 1).$$

Let us prove these assertions. Since for every $k = 1, 2, \ldots$ we have

$$a_{k+1} + a_{k+2} + \cdots + a_{2k} \geq k \cdot a_{2k} = \frac{1}{2},$$

we conclude that the infinite series $a_2 + a_3 + a_4 + \cdots$ contains infinitely many mutually disjunctive sequences of consecutive terms such that the sum of each sequence is at least $1/2$. Evidently, this causes the divergence of the series $a_2 + a_3 + a_4 + \cdots$, that is, $a_2 + a_3 + a_4 + \cdots + a_n \to \infty$ when $n \to \infty$.

To prove assertion (ii) we note that

$$\frac{1}{(m+1)^2} < \frac{1}{m(m+1)} = \frac{1}{m} - \frac{1}{m+1} \quad (m = 1, 2, \ldots)$$

so that

$$\begin{aligned} a_2^2 + a_3^2 + \cdots a_k^2 &\le \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} \\ &= \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{2}{3}\Big) + \cdots + \Big(\frac{1}{k} - \frac{1}{k+1}\Big) \\ &= 1 - \frac{1}{k+1}. \end{aligned}$$

We will prove that the man's described motion is possible, that is, the man will not need to leave the arena at any moment. According to assertion (ii) we get for every $k = 1, 2, \ldots$,

$$\begin{aligned} s_k^2 &\le s_0^2 + (r - s_0)^2 \left(\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(k+1)^2}\right) \\ &< s_0^2 + (r - s_0)^2 \left(1 - \frac{1}{k+1}\right) \\ &< s_0^2 + (r - s_0)^2 \\ &< \Big[s_0 + (r - s_0)\Big]^2 = r^2. \end{aligned}$$

Hence, for each k it holds that $s_k < r$, that is, the man always remains inside the circle.

From the inequality $d_k > 0$, which holds for every k, we conclude that the lion will not catch the man if he follows the described strategy of motion. Well, presuming that he is able to run, keep an eye on the lion and make sophisticated calculations simultaneously all the time.

Finally, it is still necessary to prove that this kind of motion can last an infinitely long time. Thus, we have to prove that the sum of the time intervals $t_1 + t_2 + t_3 \cdots$ is infinite. This sum equals

$$\begin{aligned} t_1 + t_2 + t_3 \cdots &= (r - s_0)\Big(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\Big) \\ &= (r - s_0)(a_2 + a_3 + a_4 + \cdots), \end{aligned}$$

and it is infinite according to assertion (i).

From the presented proof we see that the divergence of the harmonic series plays a key role: the described motion of the man can last an infinitely long time. How fast does this series approach "lazy eight", that is, ∞? The following example is illustrative.

A mathematician organizes a lottery which will bring an infinite amount of money to the holder of the winning lottery ticket. After some time all lottery tickets have been sold. When the winning ticket is drawn and the lucky winner came to take the prize, the organizer explains how the prize will be paid: 1 dollar immediately, 1/2 a dollar the next week, 1/3 of a dollar the week after that, and so on. The harmonic series $1+1/2+1/3+\cdots$ has an infinitely large sum, but this sum increases so slowly that the winner would obtain about 8.44 dollars after 50 years (2,600 weeks).

In this chapter we have combined physics (motion) and geometry. The next bonus problem also combines motion and elementary geometry and may be useful in some real-life situations.

Problem 6.10.* *A large avalanche in the Alps traps an unhappy mole. When the avalanche stops, it turns out that the poor mole has been buried somewhere inside a snowball with an ellipsoidal shape* (*the ellipsoid is the shape of a rugby ball, or a more or less flattened ball, if you do not know what an ellipsoid is*) *with a volume of* 500 *cubic meters. The mole can dig a hole through the snow advancing at one meter per minute but he only has the strength and breath for* 24 *minutes. Can the mole reach the surface of the snowball and save his life?*

Answers to Problems

6.3. The sum of multiplied powers is no reason for panic. Let

$$S_n = \sum_{k=0}^{n} kx^k = 1 \cdot x + 2 \cdot x^2 + \cdots + n \cdot x^n.$$

The key idea to solving this is to split the above sum and form an equation in S_n,

$$\begin{aligned} S_n + (n+1)x^{n+1} &= \sum_{k=0}^{n}(k+1)x^{k+1} = \sum_{k=0}^{n} kx^{k+1} + \sum_{k=0}^{n} x^{k+1} \\ &= xS_n + \sum_{k=0}^{n} x^{k+1}. \end{aligned} \tag{6.5}$$

The other sum is a geometric progression. You have learned in high school (we hope) that this sum is

$$\sum_{k=0}^{n} x^{k+1} = \frac{x - x^{n+2}}{1 - x} \quad (x \neq 1).$$

Returning to (6.5) and solving the equation

$$S_n + (n+1)x^{n+1} = xS_n + \frac{x - x^{n+2}}{1 - x}$$

in S_n, we obtain

$$S_n = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}. \tag{6.6}$$

Assume now that $|x| < 1$, and let $n \to \infty$. Then $x^{n+1} \to 0$ and $x^{n+2} \to 0$ and from (6.6) it follows that

$$1 \cdot x + 2 \cdot x^2 + \cdots + n \cdot x^n + \cdots = \lim_{n\to\infty} S_n = \frac{x}{(1-x)^2}.$$

Here $\lim_{n\to\infty} S_n$ denotes a limit value of the sum S_n when the number of addends n is infinitely large. In particular, taking $x = 1/2$, we find the required Oresme's sum

$$S = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \cdots + \frac{1}{2^n} \cdot n + \cdots = \frac{1/2}{(1-1/2)^2} = 2.$$

6.4. The required fitting of "harmonic squares" is possible, as shown in Figure 6.7. The unit square is divided into strips of width $1/2^k$ ($k = 1, 2, 3 \ldots$). Since the sum of the widths of strips is $\pi^2/6 - 1 \approx 0.644$ (see page 155) and the sum of sides of the squares in the kth strip is

$$\frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}-1} < 2^k \cdot \frac{1}{2^k} = 1, \ (k = 1, 2, \cdots),$$

an infinite number of small harmonic squares can be packed inside the unit square. As mentioned in [73], the composer of this problem Leo Moser and his colleague J. W. Moon, both from the University of Alberta (Canada), showed that these squares can be fitted into a square of side no less than 5/6 (= the total width of sides of the squares in the first strip).

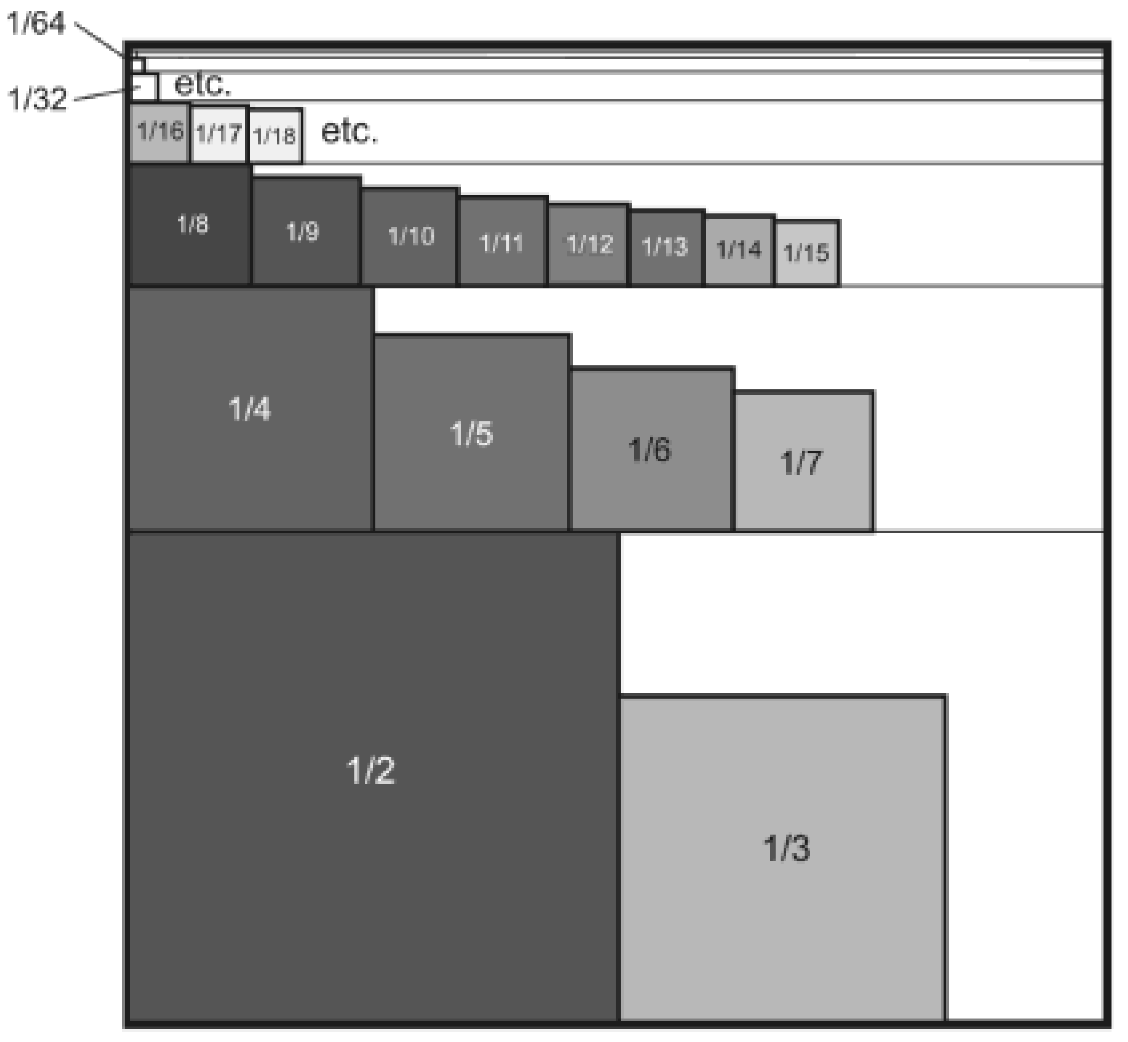


FIGURE 6.7. Packing of harmonic squares into the unit square

6.7. Despite a bit more complicated conditions, the solution does not require much effort and can be found without pen and paper; it does not take a genius to calculate that the cars will meet in exactly two hours. Since the bird flies for two hours, its path must be $2 \cdot 50 = 100$ miles. There is no need to sum an infinite series, which is here considerably more complicated compared to the series appearing in the problem of "a girl and a bird."

6.8. First, notice that a joined pair of rowers will cross the river faster than each of them individually. To find the speed of any pair of rowers, let us assume that an energy E is needed for one transfer. If F_k stands for the power of the rower R_k necessary for one crossing which lasts t_k minutes $(k = 1, 2, 3, 4)$, then

$$E = F_1 t_1 = F_2 t_2 = F_3 t_3 = F_4 t_4, \tag{6.7}$$

where $t_1 = 1,\ t_2 = 2,\ t_3 = 6,\ t_4 = 9$ (expressed in minutes). From (6.7) it follows that $F_k = E/t_k$. The time t_{ij} for crossing the river by any two rowers R_i and R_j is equal to

$$t_{ij} = \frac{E}{F_i + F_j} = \frac{E}{E/t_i + E/t_j} = \frac{t_i t_j}{t_i + t_j}.$$

Let $\rightarrow R$ and $\leftarrow R$ denote forward-trip and backward-trip of a rower R over the river. The optimal strategy of crossing the river is as follows:

	starting bank	rower(s)	crossing-time (in minutes)
—	$R_1R_2R_3R_4$	———	———
1.	R_2R_3	$\rightarrow R_1R_4$	$t_{14} = (t_1t_4)/(t_1 + t_4) = 9/10$
2.	$R_1R_2R_3$	$\leftarrow R_1$	$t_1 = 1$
3.	R_1	$\rightarrow R_2R_3$	$t_{23} = (t_2t_3)/(t_2 + t_3) = 3/2$
4.	R_1R_2	$\leftarrow R_2$	$t_2 = 2$
5.	———	$\rightarrow R_1R_2$	$t_{12} = 2/3$

From the above scheme we find that the total time of all transfers of the rowers is

$$t = t_{14} + t_1 + t_{23} + t_2 + t_{12} = \frac{9}{10} + 1 + \frac{3}{2} + 2 + \frac{2}{3} = \frac{91}{15} \text{ minutes.}$$

The width of the river is equal to the distance traveled by the swan, that is,

$$(60 \text{ feet per minute}) \times (91/15 \text{ minutes}) = 364 \text{ feet.}$$

6.10. Let the point A mark the location of the mole captured in the snowball. In order for its to dig out of the snowball, the mole must first dig a tunnel 8 meters long in a straight line up to point B (see Figure 6.8). Then it must make a right angle turn and go the next 8 meters straight ahead to point C. Finally, it must make another right angle turn in reference to the plane determined by the straight lines AB and BC, continuing in that direction 8 meters to point D.

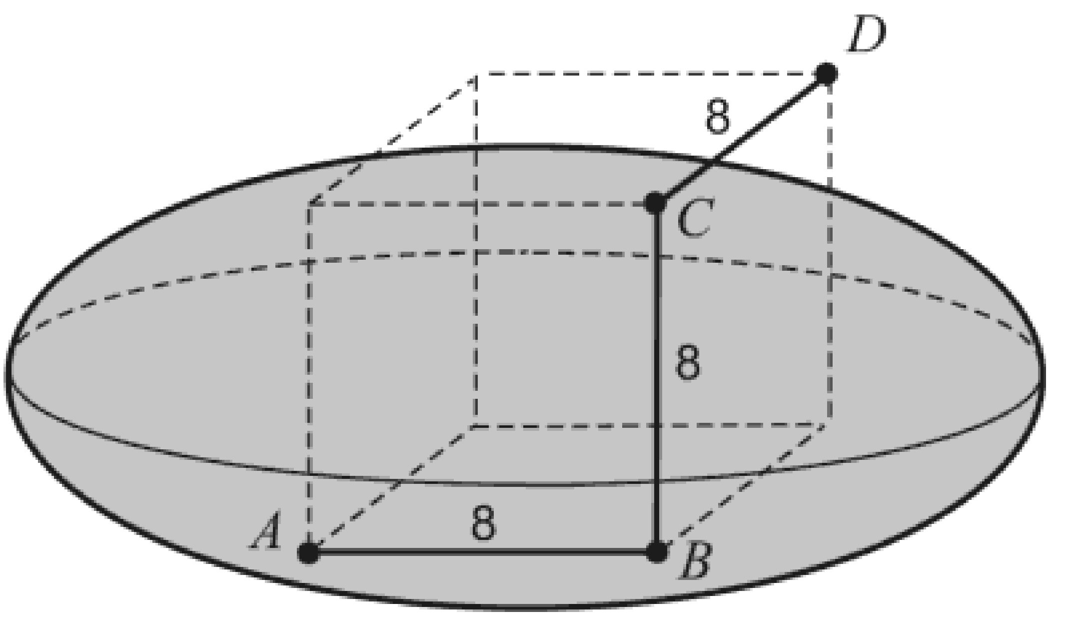


FIGURE 6.8. The mole's rescuing from the snowball

It is not terribly hard to prove that at least one of the points B, C, and D lies outside the snowball. Indeed, if all four points A, B, C, D would lie inside the snowball, then all interior points of the cube constructed from the perpendicular segments AB, BC, and CD would belong to the interior of the snowball. However, the volume of such a cube is $8 \times 8 \times 8 = 512\,m^3$, which would mean that the volume of the snowball is greater than $512\,m^3$. This would contradict the terms of the posed task, which consequently proves that at least one of the points B, C, and D lies outside the snowball.

Chapter 7 COMBINATORICS

Most popular mathematics puzzles and games are essentially problems in combinatorics.

Anders Björner and Richard Stanley
'A Combinatorial Miscellany'(2004)

Combinatorics is a branch of discrete mathematics that studies arrangements of objects of finite sets, satisfying specified criteria. In particular, it is concerned with "counting" the objects in those sets (enumerative combinatorics) and with deciding whether certain "optimal" objects exist (extremal combinatorics). The renowned authors Fuchs and Tabachnikov have said that combinatorial problems look like this: *"Given such and such a number of such and such things, in how many ways can we do such and such a thing?"*

Some examples of combinatorial questions are the following: How many nonempty subset does a set of 64 elements have? What is the minimum number of moves required to transfer a Tower of Hanoi puzzle consisting of 64 disks (see Problem 7.15 in this chapter)? The problem of the chessboard-grains of wheat asks for the total number of grains on the ordinary 8×8 chessboard if one grain of wheat is placed on the first square, two on the second, four on the third, and so on in geometric progression, the number of grains being doubled for each successive square until the final sixty-fourth square of the chessboard. The answer to these questions is the same: $2^{64} - 1$. The solution of these structurally different problems uses the same method of combinatorics.

Recent progress in combinatorics has been initiated by applications to other disciplines, especially in computer science. Modern combinatorics deals with problems of existence and construction of combinatorial objects with given properties, as well as with optimization problems.

A great number of combinatorics has arisen from games and puzzles. In how many ways can you place 8 rooks on the 8×8 chessboard (or, more generally, n rooks on an $n \times n$ board) so that no rook can be attacked by another? How many poker hands are there (choose 5 from 52)? How many ways of placing k balls in n boxes are possible? The last task is of a higher school level, but it might be important when, for example, we consider the distribution of k electrons among n shells in the physics of elementary particles.

Many combinatorial problems can be understood by a large audience because extensive prerequisites are not necessary. To solve most problems in this chapter, you will often need only patience, persistence, imagination and intuition. Once J. L. Synge said: *"The mind is at its best when at play."*

You will find in this chapter a diverse set of famous entertaining problems such as the ring puzzle, Eulerian squares, the Josephus problem, Cayley's counting problem, the Tower of Hanoi puzzle, the river-crossing problems, Kirkman's schoolgirls problem and the three-planting problem of Sylvester. Accordingly, giants such as Euler, Cayley, Sylvester, Steiner, Knuth and Cardano appear as the main players in this chapter.

*

* *

Mahāvira (*ca.* 800–*ca.* 870) (→ p. 300)

Combination with flavors

According to Katz [113, p. 228], the first recorded statements of combinatorial rules (although without any proofs or justification) appeared in India. In the ninth century, the Indian mathematician Mahāvira gave (without proof) an explicit algorithm for calculating the number of combinations. His rule, translated into the modern formula, can be written as

$$C_k^n = \frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k} = \binom{n}{k},$$

where C_k^n is the number of different ways of choosing k ingredients (objects) among n ingredients (objects). The following task from Mahāvira's book *Ganita Sara Samgraha* (about 850 A.D.) concerns this subject.

Problem 7.1. *In how many ways can different numbers of flavors be used in combination together, being selected from the astringent, the bitter, the sour, the pungent, and the saline, together with the sweet taste?*

Obviously, choosing k flavors (among n) is the same as choosing $n-k$ flavors which are *not* included. This is in accordance with the well-known binomial relation

$$\binom{n}{k} = \binom{n}{n-k}.$$

Consequently, we conclude that there are:

$\binom{6}{0} = 1$ way of choosing none of the flavors,

$\binom{6}{1} = 6$ choices of a single flavor,

$\binom{6}{2} = (6 \cdot 5)/2 = 15$ ways of choosing a pair of flavors,

$\binom{6}{3} = (6 \cdot 5 \cdot 4)/(3 \cdot 2) = 20$ ways of choosing three flavors,

$\binom{6}{4} = \binom{6}{2} = 15$ choices of four flavors,

$\binom{6}{5} = \binom{6}{1} = 6$ ways of choosing five flavors,

$\binom{6}{6} = \binom{6}{0} = 1$ way to choose all flavors.

Summing the above answers we obtain that the total number of all different combinations of flavors is 64.

The posed task can be solved using another approach. Let $\overline{f_1 f_2 \cdots f_6}$ be a 6-digit number where $f_k = 0$ ($k \in \{1, \ldots, 6\}$) if the flavor k is rejected and $f_k = 1$ if the flavor k is accepted. We start with

$$000000, \quad 000001, \quad 000010, \quad 000011,$$

and finish with

$$111100, \quad 111101, \quad 111110, \quad 111111.$$

How many different 6-digit numbers exist? The answer is $V_2^6 = 2^6 = 64$ (variation with repetition of 6 elements of the second class).

Claude Gaspar Bachet (1581–1638) (→ p. 302)

Married couples cross the river

The medieval mind delighted in a certain type of puzzle involving the river-crossing in which specific conditions and/or restrictions prevailed. Some references (*e.g.*, [133], [150], [153], [186]) attribute the earliest problems of this sort to the eighth-century English theologian and scholar, Alcuin of York and the eminent Italian mathematician Niccolo Tartaglia (1550–1557). In addition to the next problem and its variants, we present another three problems on pages 240–243.

Problem 7.2. *Three beautiful ladies and their husbands come to a river bank while traveling. They must cross the river by means of a boat that cannot carry more than two people at a time. The problem is further complicated by the jealousy of the husbands. To avoid any scandal, they agree that no woman should cross the river unless in the company of her husband. How can* *the party cross the river in the fewest number of crossings while respecting the stated conditions and also assuming that the women can row?*

Solution. Eleven passages are required and although there are several ways of crossing the river, this number cannot be reduced. Let capital letters A, B, C stand for the husbands, and lower case letters a, b, c stand for their wives, where the same letters correspond to each married couple. The diagram below outlines one possible solution to the problem. The numbered statements record each of the successive crossings, noting each individual's location, whether side X, the point of departure, in the boat, or side Y, the destination point. The arrows $\rightarrow$ and $\leftarrow$ in the list below mean departure and return, respectively. According to D. Wells [186], Alcuin of York put forth the solution consisting of eleven crossings.

	starting bank	rower(s)	arrival bank
—	$ABCabc$	———	———
1.	$BCbc$	$\rightarrow$ Aa	Aa
2.	$ABCbc$	$\leftarrow$ A	a
3.	ABC	$\rightarrow$ bc	abc
4.	$ABCa$	$\leftarrow$ a	bc
5.	Aa	$\rightarrow$ BC	$BCbc$
6.	$ABab$	$\leftarrow$ Bb	Cc
7.	ab	$\rightarrow$ AB	$ABCc$
8.	abc	$\leftarrow$ c	ABC
9.	c	$\rightarrow$ ab	$ABCab$
10.	Cc	$\leftarrow$ C	$ABab$
11.	———	$\rightarrow$ Cc	$ABCabc$

Below, however, D. Wells gives a shorter solution [186, p. 203] that requires only nine crossings in which the letter combinations represent the rower(s):

$$\underset{\longrightarrow}{Aa} \quad \underset{\longleftarrow}{A} \quad \underset{\longrightarrow}{bc} \quad \underset{\longleftarrow}{a} \quad \underset{\longrightarrow}{Aa} \quad \underset{\longleftarrow}{\mathrm{A}} \quad \underset{\longrightarrow}{BC} \quad \underset{\longleftarrow}{\mathrm{a}} \quad \underset{\longrightarrow}{Aa}$$

Yet we can accept this solution only under special conditions of which the text of the task makes no mention. Namely, in the seventh crossing husbands B and C reach the arrival bank where wives a, b and c are situated. Then B and C land on the bank while wife a rows back in the boat. Therefore, husbands B and C pass wife a momentarily, thus violating the condition which forbids the meeting of persons from a "forbidden group". If this brief meeting is overlooked, the nine-stage solution could then be accepted. We note that Rouse Ball and Coxeter [150, p. 118] discussed the eleven-crossing solution.

A great many variations of this type of river-crossing problem with n married couples have been widely spread throughout the literature of the time; see, *e.g.*, [56], [107], [118], [125], [150]. For $n > 3$ the problem can be solved only if there is an island in the middle of the river; its proof may be found in [107].

Problem 7.3. *Solve the "Married couples cross the river" problem with four married couples (the husbands are again jealous), this time with an island in the middle of the river. There is a boat that holds not more than two people and all passengers can row.*

Solution. The entire operation, described by Ignjat'ev [107], requires not less than 17 passages. Let A, B, C, D stand for the husbands and a, b, c, d stand for their wives beginning with the entire party gathered together on the same side of the river. The scheme presented below clearly indicates the combinations obtained for each stage of the crossing, *i.e.*, the composition of the rowing party, and the individuals standing on either side of the river.

At the beginning, all travelers are on the starting bank, which is denoted by $ABCDabcd$ (the names of all passengers). The crossing is carried out according to the following scheme (current states on both banks and the island clearly indicate the river-crossing and rower(s) in the boat):

	starting bank	island	arrival bank
—	$ABCDabcd$	—	—
1.	$ABCDcd$	—	ab
2.	$ABCDbcd$	—	a
3.	$ABCd$	bc	a
4.	$ABCDcd$	b	a
5.	$CDcd$	b	ABa
6.	$BCDcd$	b	Aa
7.	BCD	bcd	Aa
8.	$BCDd$	bc	Aa
9.	Dd	bc	$ABCa$
10.	Dd	abc	ABC
11.	Dd	b	$ABCac$
12.	BDd	b	$ACac$
13.	d	b	$ABCDac$
14.	d	bc	$ABCDa$
15.	d	—	$ABCDabc$
16.	cd	—	$ABCDab$
17.	—	—	$ABCDabcd$

There are several ways to achieve the crossing in 17 passages, but the above solution accomplishes the crossing operation in the fewest trips and the fewest number of passages back and forth.

M. G. Tarry has complicated the problem under consideration by assuming that the wives are unable to row (see [118]). He also proposed a further complication by suggesting that one of the husbands is a bigamist traveling with both of his wives. A similar variant is given in Steven Kranz's excellent book *Techniques of Problem Solving* [119]: *"A group consists of two men, each with two wives, want to cross a river in a boat that holds two people. The jealous bigamists agree that no woman should be located either in the boat or on the river banks unless in the company of her husband."*

Josephus problem

A well-known medieval task consists of arranging men in a circle so that when every kth man is removed, the remainder shall be a certain specified man. This problem appeared for the first time in Ambrose of Milan's book *De Bello Judaico* (bk. iii, chpts. 16–18). Ambrose (*ca.* 370), who wrote this work under the name of Hegeisippus, stated the problem thus:

Problem 7.4. *Vespasian, a successful military commander and later Roman Emperor, and the army under his command, were charged with quelling the Jewish revolt against Roman Domination. Vespasian and his men captured the famous Jewish historian Flavius Josephus and forty other men in a cellar. The Romans decided to kill all the prisoners but two using a selection method that requires the arrangement of all prisoners in a circle and then killing every third man until only two were left. Josephus, not keen to die, quickly calculated where he and his close friend should stand in the murderous circle, thus saving his life and that of his friend. Which two places in the circle did Josephus choose?*

The answer to this question of life or death reads: Josephus placed himself in the 16th place and a close friend in the 31st place (or opposite). See the end of this essay.

A tenth-century European manuscript and the *Ta'hbula* of Rabbi ben Ezra (*ca.* 1140) both make mention of the "Josephus problem". It is interesting that this problem reached the Far East, appearing in Japanese books. Chuquet (1484) and later such eminent writers as Cardano (1539), Buteo (1560), Ramus (1569) and Bachet (1624) gave great prominence to this problem; see D. E. Smith [167, Vol. II, p. 541].

Smith [167, Vol. II], Rouse Ball and Coxeter [150, pp. 32–36], and Skiena [164, pp. 34–35] give numerous details of this problem. Here we present

Bachet's variant of the Josephus problem which appears as problem xxiii in his classic book *Problèmes Plaisants et Délectables*, Lyons 1624.[1]

Problem 7.5.* *Fifteen sailors and fifteen smugglers were sailing on a ship that encountered a storm. The captain could save his ship only if half of the party abandon the ship. He arranged the thirty men in a circle, and every ninth man, reckoning from a given point on the circle, was lowered into a lifeboat. How did the captain arrange the men in such a way that all fifteen of the sailors were saved?*

P. G. Tait[2] has considered a general case when n men are arranged in a circle and every kth man is removed beginning anywhere and going around until only r are left.

Henry E. Dudeney (*Tit-Bits*, London, October 1905), the great English composer of mathematical diversions suggested the following modification to the original problem.[3]

Problem 7.6.* *Let five sailors and five burglars be arranged around a circle thus, B S B S S B S B S B. Suppose that if beginning with the* $\boldsymbol{a}$*th man, every* $\boldsymbol{h}$*th man is selected, all the burglars will be picked out for punishment; but if beginning with the* $\boldsymbol{b}$*th man, every* $\boldsymbol{k}$*th man is selected, all the sailors will be picked out for punishment. The problem is to find* $a,\ b,\ h,\ k$.

Muramatsu's text dated 1663 gives a Japanese version of the "Josephus problem" that reads thus:[4]

Problem 7.7.* *Many years ago there lived a prosperous farmer who had been married twice. From his first marriage, he had fifteen children; from his second marriage, he also had fifteen. His second wife very much wanted her favorite son to inherit the entire property. One day she suggested to her husband that they arrange all* 30 *children in a circle, and designating the first child in the circle as number one, thereafter they would count out every tenth child until only one remained, who would then be named as the heir. The husband agreed to this request, and the wife then arranged all the children in a circle. This counting process eliminated* 14 *of her fifteen stepchildren at once. Feeling quite confident of a successful outcome, the wife then suggested that they reverse the order of the counting. Once again, the husband agreed,*

---

[1] In fact, we give a more human variant to avoid possible criticism, both political and religious, since Bachet's and Buteo's versions involved clashing Turks and Christians.

[2] *Collected Scientific Papers*, Vol. II, Cambridge, 1900, pp. 432–435.

[3] The occupations given for the people in the problem have been changed for the same reasons given in an earlier footnote.

[4] The adapted version of the text quoted in Cajori [32, p. 79].

and the counting proceeded in the reverse order. This time, however, the counting resulted in the unexpected exclusion of each and every one of the second wife's children, and the remaining stepchild consequently inherited the property.

The analysis of possible arrangements of the initial position of the 30 children is left to the reader.

R. L. Graham, D. E. Knuth and O. Patashnik considered the following variation of the Josephus problem in their fascinating book *Concrete Mathematics* [88].

Problem 7.8. *n people numbered 1 to n are disposed around a circle. If one starts with person 1 and eliminates every second remaining person, determine the survivor's number, $J(n)$* (J stands for Josephus).

The mentioned authors have derived in [88] a simple but remarkable recurrence relation that defines $J(n)$ for arbitrary n.[5] In what follows we give a short adaptation of their treatment of the Josephus problem. Let us distinguish between the even and odd case of n. If there are $2n$ people at the start, then after the first trip around the circle only the odd numbers are left (Figure 7.1(a)) and 3 will be the next to eliminate. This is just like the starting position, but without n eliminated persons and a new numeration where each person's number is doubled and decreased by 1. Therefore,

$$J(2n) = 2J(n) - 1, \quad \text{for} \quad n \geq 1. \tag{7.1}$$

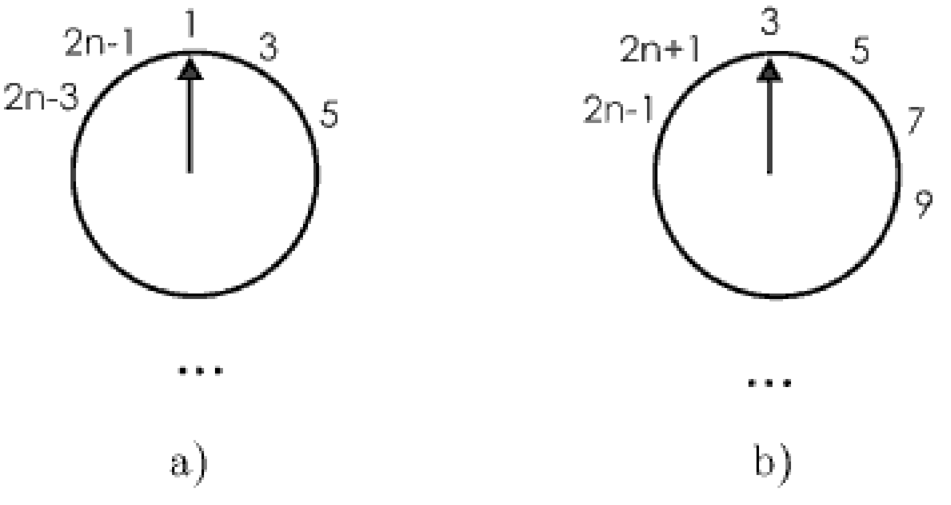


FIGURE 7.1.

Now let us assume that we have $2n + 1$ people originally. After the first pass around the circle, the persons numbered $2, 4, 6, \ldots, 2n$ and 1 (in this

[5] The authors noted, in their characteristic style: *"Josephus and the Jewish-Roman war have led us to some interesting general recurrences."*

order) are executed, leaving again persons with odd numbers, excepting number 1 (see Figure 7.1(b)). This resembles the original situation with n people, but in this case their numbers are doubled and increased by 1. Thus

$$J(2n+1) = 2J(n)+1, \quad \text{for} \ \ n \geq 1. \tag{7.2}$$

Combining (7.1) and (7.2), and taking into account the obvious case $J(1) = 1$, we obtain a recurrence relation that defines J in all cases:

$$\begin{aligned} &J(1) = 1; \\ &J(2n) = 2J(n)-1, \quad \text{for} \ \ n \geq 1; \\ &J(2n+1) = 2J(n)+1, \quad \text{for} \ \ n \geq 1. \end{aligned} \tag{7.3}$$

To find a general explicit formula for $J(n)$, we first form Table 7.1 of small values using (7.3).

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

TABLE 7.1.

We have grouped the entries of $J(n)$ for n from 2^m to $2^{m+1}-1$ $(m = 0, 1, 2, 3)$, indicated by the vertical lines in the table. We observe from the table that $J(n)$ is always 1 at the beginning of a group (for $n = 2^m$) and it increases by 2 within a group. Representing n in the form $n = 2^m + k$, where 2^m is the largest power of 2 not exceeding n, the solution to the recurrence (7.3) seems to be

$$J(2^m + k) = 2k + 1, \quad \text{for} \ \ m \geq 0 \ \ \text{and} \ \ 0 \leq k < 2^m. \tag{7.4}$$

We note that the remainder $k = n - 2^m$ satisfies $0 \leq k < 2^{m+1} - 2^m$ if $2^m \leq n < 2^{m+1}$.

We will prove the formula (7.4) by induction. For $m = 0$ it must be $k = 0$ (following the above bounds). Then from (7.4) we have $J(1) = 1$, which is true. Let $m > 0$ and $2^m + k = 2n + 1$, then k is odd. Assume that for some $m > 0$ the following is valid:

$$J(2^m + k) = 2k + 1.$$

According to this hypothesis and (7.3), we find

$$J(2^m + k) = 2J(2^{m-1} + (k-1)/2) + 1 = 2(2(k-1)/2 + 1) + 1 = 2k + 1,$$

which completes the proof by induction. In a similar way we derive the proof in the even case, when $2^m + k = 2n$. Therefore, formula (7.4) is stated.

To demonstrate solution (7.4), let us calculate $J(101)$. Since $101 = 2^6 + 37$ (that is, $k = 37$), we have $J(101) = 2 \cdot 37 + 1 = 75$.

In [88, p. 11] the authors gave the following interesting solution to $J(n)$ using representations in the binary system.

If the binary expansion of n is

$$n = (b_m b_{m-1} \cdots b_1 b_0)_2,$$

that is, $n = b_m \cdot 2^m + b_{m-1} \cdot 2^{m-1} + \cdots + b_1 \cdot 2 + b_0$, then

$$J(n) = (b_{m-1} b_{m-2} \cdots b_1 b_0 b_m)_2.$$

Therefore, $J(n)$ is obtained from n by a one-bit cyclic shift to the left. For example, if $n = 101 = (1100101)_2$, then

$$J(101) = J\big((1100101)_2\big) = (1001011)_2,$$

which gives

$$J(101) = 64 + 8 + 2 + 1 = 75.$$

We end this essay on the Josephus problem with the remark that the *Mathematica* package *Combinatorica* can simulate the "Josephus process" by the command `InversePermutation[Josephus[n,m]]`. The outcome is the ordered list of men who are consecutively eliminated from the circle consisting of n men, when every mth man is eliminated. For example, $n = 41$ and $n = 3$ in the original Josephus problem (Problem 7.4). The command `InversePermutation[Josephus[41, 3]]` gives the ordered list

```
3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39,
1, 5, 10, 14, 19, 23, 28, 32, 37, 41, 7, 13, 20, 26,
34, 40, 8, 17, 29, 38, 11, 25, 2, 22, 4, 35, 16, 31
```

The last two numbers (in bold) represent beneficial places for Josephus and his friend.

Gerolamo Cardano (1501–1576) (→ p. 301)

The multi-talented Italian mathematician, physician, astronomer and gambler Gerolamo Cardano was not only a controversial figure, but also a man of remarkable contrasts. Furthermore, Cardano was one of the most extraordinary characters in the history of mathematics. At the height of his fame he had a reputation as the world's leading scientist. His turbulent life was marked by great successes and by great misfortunes. Tragedy struck Cardano's family several times; first in 1560, his elder son Giambattista was executed for murdering his wife. Cardano's younger son Aldo was a gambler who associated with individuals of dubious character. He gambled away all his possessions as well as a considerable sum of his father's money. He even broke into his father's house to steal cash and jewelry. Cardano was forced to report his son to the authorities and Aldo was banished from Bologna.

Rings puzzle

Figure 7.2 shows a very familiar toy known as *Chinese rings*. The French call this puzzle La Baguenodier and the English call it Tiring Irons. According to Steinhaus, this device was originally used by French peasants to lock chests. Although very old, this toy is sold even today in toy shops all over the world. Despite its name, no one has ever proven its Chinese origin until today. S. Culin[6] suspects that Chinese general Hung Ming (A.D. 181–234) made this toy puzzle to amuse his wife while he waged his wars. Gerolamo Cardano was apparently the first to describe this puzzle in 1550 in his *De Subtilitate*, bk. xv, paragraph 2, ed. Sponius, vol. iii. However, this puzzle was mentioned in passing in chapter 7 of *Hongloumeng* (The Dream of the Red Chamber), a famous novel in 1791. It was also considered by John Wallis in his book, *Algebra*, Latin edition, 1693, *Opera*, vol. II, chap. cxi.

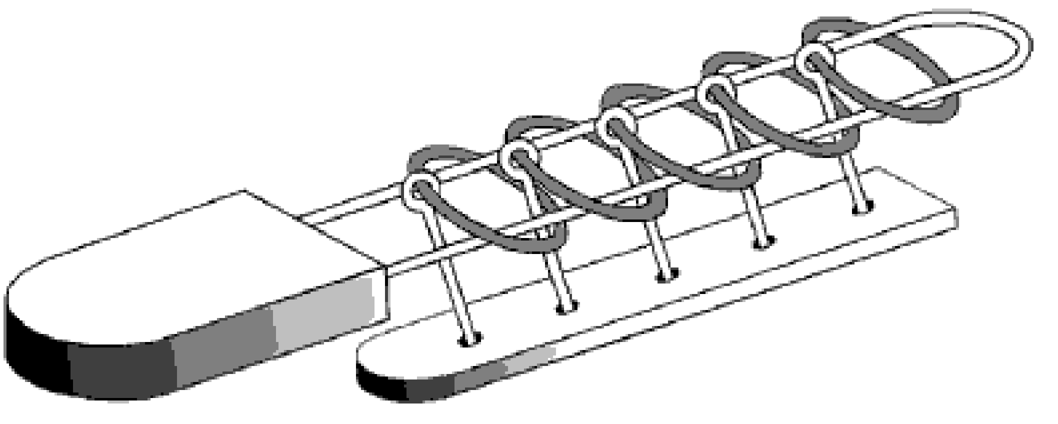


FIGURE 7.2. Chinese rings

The Chinese rings puzzle consists of a number of rings hung upon a long wire loop. Each ring is connected loosely by a post to a platform below the loop. Each of the connecting posts is linked to the corresponding ring to prevent removal of a ring from the loop. The ring can slide along the loop to its end A and can be taken off or put on the loop in such a way that any

[6] S. Culin, *Games of the Orient: Korea, China, Japan*, Charles E. Tuttle, Rutland 1965.

other ring can be taken off or put on only when the one next to it towards A is on, and all the rest towards A are off the loop. The order of the rings cannot be changed. The aim is to remove all the rings in the minimal number of moves.

The solution to the Chinese rings puzzle is similar to that of the Tower of Hanoi puzzle (see page 196), in that they both require a reversal of the procedure, in other words, putting the rings back on the loop. This recursion property provides an obvious link between these two puzzles. Moreover, in both cases the use of binary numbers leads to ingenious solutions, thus making these puzzles nearly identical.

W. W. Rouse Ball and H. S. M. Coxeter in [150] describe a procedure to find the total number of steps necessary to remove all of the rings. The minimal number of moves is either $\frac{1}{3}(2^{n+1}-1)$ if n is odd, or $\frac{1}{3}(2^{n+1}-2)$ if n is even. These numbers can be expressed by the recurrence relation

$$A_0 = 1,\ A_2 = 2, \quad A_n = A_{n-1} + 2A_{n-2} + 1.$$

Interestingly enough, Cardano and Wallis did not find the optimal solution of the rings puzzle. An elegant solution was given by the French mathematician Louis A. Gros in 1872 in his treatise *Théorie du Baguenodier* (Lyons, 1872). His approach to the solving procedure anticipated the so-called "Gray code", named after Frank Gray, an engineer who worked on an error-correcting technique during the 1930s at AT&T Bell Laboratories.

Mathematics literature abounds with details of the rings puzzle; see, for example, the works of Afriat [1], Berlekamp, Conway and Guy [17, Vol. 2, Ch. 19], Dewdney [50], Gardner [81], Rouse Ball and Coxeter [150], D. Singmaster [163] and Skiena [164].

Here we will give only the solution of the five-rings puzzle based on binary numbers and Gray-code numbers, following the mentioned Dewdney's article [50]. The five-rings puzzle requires $\frac{1}{3}(2^6-1) = 21$ moves. Each ring position will be represented by a Gray five-digit binary sequence of 0's and 1's, where 1 stands for a ring on the loop and 0 stands for a ring off the loop. The last digit of a Gray number relates to the position of the first ring (nearest to the end of the loop), and the first digit is related to the fifth ring. For example, we give several ring positions:

1 1 1 1 1	(all rings on)
1 0 1 1 1	(fourth ring off)
1 1 0 1 0	(first and third rings off)
0 0 0 1 1	(first and second rings on)
0 0 0 0 0	(no rings on)

	Binary code	Gray code		Binary code	Gray code
0	0 0 0 0 0	0 0 0 0 0	11	0 1 0 1 1	0 1 1 1 0
1	0 0 0 0 1	0 0 0 0 1	12	0 1 1 0 0	0 1 0 1 0
2	0 0 0 1 0	0 0 0 1 1	13	0 1 1 0 1	0 1 0 1 1
3	0 0 0 1 1	0 0 0 1 0	14	0 1 1 1 0	0 1 0 0 1
4	0 0 1 0 0	0 0 1 1 0	15	0 1 1 1 1	0 1 0 0 0
5	0 0 1 0 1	0 0 1 1 1	16	1 0 0 0 0	1 1 0 0 0
6	0 0 1 1 0	0 0 1 0 1	17	1 0 0 0 1	1 1 0 0 1
7	0 0 1 1 1	0 0 1 0 0	18	1 0 0 1 0	1 1 0 1 1
8	0 1 0 0 0	0 1 1 0 0	19	1 0 0 1 1	1 1 0 1 0
9	0 1 0 0 1	0 1 1 0 1	20	1 0 1 0 0	1 1 1 1 0
10	0 1 0 1 0	0 1 1 1 1	21	1 0 1 0 1	1 1 1 1 1

Table 7.2. Binary numbers and Gray-code numbers

Table 7.2 gives decimal numbers from 0 to 21 in the binary representation. For these binary numbers the associated Gray-code numbers are generated and displayed in the second column. Each five-digit Gray-code number is obtained from its corresponding binary number by the following rule: Reckoning binary numbers from left to right, the first Gray-code digit is always the same as the first binary digit. Afterwards, each Gray digit is 1 if the corresponding binary digit differs from its predecessor; otherwise it is 0. Surprisingly, there is an incredible coincidence between Gray-code numbers and the solution of the rings puzzle: in reverse order (from 21 to 0 in five-rings case) the Gray-code numbers show only the successive ring positions in the required solution.

Before going any further, let us allow a digression for a while to mention the Gray-codes and coding theory in general. Coding theory is a branch of mathematics and computer science dealing with the error-correcting process. The main goal of coding theory is to send messages reliably when transmitting data across noisy channels. For example, it is of great interest to receive relatively good images sent from distant space: from Mars, Venus or even deeper space. Do you believe that coding theory plays an important role when your CD player encounters troubles arising from a badly scratched compact disc? For more details, adapted for a wide circle of readers, see the book [13] of Ehrhard Behrends.

Let us go back to the Gray-codes and the solution of the ring puzzle. According to Table 7.2 we can form a diagram (displayed in Figure 7.3) indicating the steps necessary to remove the first three rings from the set of five rings. The associated Gray-code numbers show the successive positions of the rings, represented by small circles.

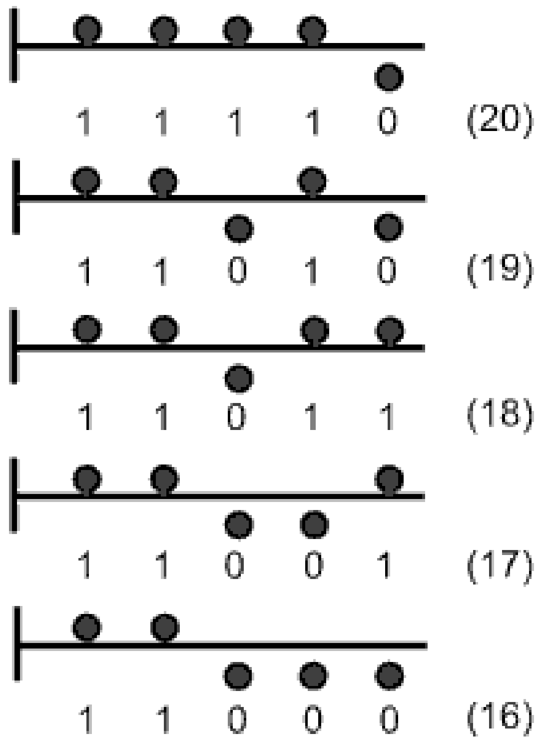


FIGURE 7.3. Five rings and the corresponding Gray-code numbers

Nicolaus II Bernoulli (1687–1759) (→ p. 304)

Leonhard Euler (1707–1783) (→ p. 305)

The problem of the misaddressed letters

The following problem was first considered by Nicolaus II Bernoulli, the nephew of the two great mathematicians Jacob and Johann Bernoulli. Euler later solved this problem independently of Bernoulli.

Problem 7.9(a). *Assume that k distinct objects are displaced in positions 1 through k. In how many different ways can they be reordered so that no object occupies its original place?*

Often found in literature[7] this problem is treated as a curious problem of combinatorics. *The problem of the misaddressed letters* states the problem in a slightly more concrete form.

[7] C. P. R. de Montmort, *Essai d'analyse sur les jeux de hasard*, Paris 1713; J. L. Coolidge, *An Introduction to Mathematical Probability*, Oxford 1925; C. V. Durrel, A. Robson, *Advanced Algebra*, London 1937; A. C. Aitken, *Determinants and Matrices*, Edinburgh 1956; L. Comtet [39]; H. Dörrie [54]; S. Fisk [64]; S. G. Kranz [119]; W. W. Rouse Ball and H. S. M. Coxeter [150].

Problem 7.9(b). *An individual has written k letters to each of k different friends, and addressed the k corresponding envelopes. How many different ways are there to place every letter into a wrong envelope?*

Solution. Let $a_1, \dots, a_k$ denote the objects and let $P_1, \dots, P_k$ be their corresponding positions. If the object a_i lands in P_j we will write $a_i \parallel P_j$, if a_i is not in P_j we write $a_i \nparallel P_j$. Let the required number of misplacements be designated as $M(k)$.

We will distinguish between two different cases: (i) $a_1 \parallel P_2$ and $a_2 \parallel P_1$, while the objects $a_3, \dots, a_n$ are distributed among $P_3, \dots, P_n$ but so that $a_i \nparallel P_i$ $(i = 3, \dots, k)$; (ii) $a_1 \parallel P_2$ but $a_2 \nparallel P_1$.

Case (i): The objects $a_3, a_4, \dots, a_k$ are distributed among $P_3, P_4, \dots, P_k$ so that $a_i \nparallel P_i$ holds for every $i \in \{3, 4, \dots, k\}$. The number of possible ways for these arrangements is, obviously, $M(k-2)$.

Case (ii): This case is equivalent to the following situation: we wish to distribute $a_2, a_3, a_4, \dots, a_k$ among $P_1, P_3, P_4, \dots, P_k$ but in such a way that $a_2 \nparallel P_1, a_3 \nparallel P_3$, and so forth. Therefore, the number is $M(k-1)$.

The number of allowable rearrangements in which a_1 ends up in P_2 is $M(k-2) + M(k-1)$. We can repeat a similar analysis to determine the number of allowable arrangements in which $a_1 \parallel P_3, a_1 \parallel P_4, \dots, a_1 \parallel P_k$. This number will be the same: $M(k-2) + M(k-1)$. Therefore, the total number $M(k)$ of all possible cases is

$$M(k) = (n-1)\big[M(k-2) + M(k-1)\big].$$

The last recurrence relation can be rewritten as

$$M(k) - k \cdot M(k-1) = -\big[M(k-1) - (k-1) \cdot M(k-2)\big]. \qquad (7.5)$$

Let us set $N_i = M(i) - i \cdot M(i-1)$. Then (7.5) becomes

$$N_i = -N_{i-1}.$$

We form this relation for $i = 3, 4, \dots, k$ and obtain

$$N_3 = -N_2, \quad N_4 = -N_3, \quad N_5 = -N_4, \ \dots \ , N_k = -N_{k-1}.$$

Simple substitution yields

$$N_k = -N_{k-1} = (-1)^2 N_{k-2} = (-1)^3 N_{k-3} = \cdots = (-1)^{k-2} N_2,$$

that is,

$$M(k) - k \cdot M(k-1) = (-1)^{k-2}\big[M(2) - M(1)\big].$$

Since

$$M(1) = 0, \quad M(2) = 1, \quad (-1)^{k-2} = (-1)^k,$$

from the last relation we find that

$$M(k) - k \cdot M(k-1) = (-1)^k.$$

After dividing both sides by $k!$ one obtains

$$\frac{M(k)}{k!} - \frac{M(k-1)}{(k-1)!} = \frac{(-1)^k}{k!}. \tag{7.6}$$

Using (7.6) and applying telescoping summation, we find

$$\begin{aligned}\sum_{r=2}^{k}\Big(\frac{M(r)}{r!} - \frac{M(r-1)}{(r-1)!}\Big) &= \frac{M(k)}{k!} - \frac{M(1)}{1!} \\ &= \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \cdots + \frac{(-1)^k}{k!}.\end{aligned}$$

Hence, taking into account that $M(1) = 0$, we obtain

$$\frac{M(k)}{k!} = \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \cdots + \frac{(-1)^k}{k!},$$

leading to the final result of

$$M(k) = k!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{(-1)^k}{k!}\right).$$

For example, the first few entries are

$$M(3) = 2,\ M(4) = 9,\ M(5) = 44,\ M(6) = 265,\ M(7) = 1854.$$

Leonhard Euler (1707–1783) ($\rightarrow$ p. 305)

Eulerian squares

Euler was interested in the topic of magic squares, today known as Latin squares and Graeco-Latin squares (or the Eulerian squares). A *Latin square* of order n consists of n distinct symbols, arranged in the form of a square scheme in such a way that each symbol occurs once in every row and once in every column. In other words, every row and every column is a permutation of n symbols. In his article *Recherches sur une nouvelle espace de quarres magiques*, Euler wrote about this kind of magic squares. Today these squares are called Latin squares after Euler's use of ordinary Latin letters as symbols that should be arranged.

b	a	d	c
d	c	b	a
c	d	a	b
a	b	c	d

a)

γ	δ	α	β
β	α	δ	γ
δ	γ	β	α
α	β	γ	δ

b)

$b\gamma$	$a\delta$	$d\alpha$	$c\beta$
$d\beta$	$c\alpha$	$b\delta$	$a\gamma$
$c\delta$	$d\gamma$	$a\beta$	$b\alpha$
$a\alpha$	$b\beta$	$c\gamma$	$d\delta$

c)

FIGURE 7.4. Eulerian squares

Figure 7.4(a) shows a fourth order Latin square where four Latin letters a, b, c, and d are arranged in the described manner. Figure 7.4(b) also represents a different Latin square with the four corresponding Greek letters. A superposition of these two squares gives a square scheme of pairs in which each Latin letter combines once and only once with each Greek letter; see Figure 7.4(c). A square obtained by such a combination is called an *Eulerian square* or a *Graeco-Latin square*. The name comes after Euler's use of Latin letters for one square and Greek letters for the other square.

Two composite squares are said to be *orthogonal squares*. Let us note that Graeco-Latin squares are now widely used in designing biology, medical, sociology and even marketing experiments. For more details about Graeco-Latin squares see, for instance, the books [18], [74, Ch. 14] and [150, Ch. 10].

Let us pause for a moment to offer a nice problem given in Watkins' book *Across the Board* [181], mentioned several times in this book. This problem concerns a 6×6 Latin square with a flavor of crossword puzzles. The pleasure is yours!

Problem 7.10.* *A* 6×6 *chessboard is divided into* 6 *regions, each of them consisting of* 6 *squares.* 6 *letters are placed on this mini chessboard as shown in Figure* 7.5 *The task is to complete a* 6×6 *Latin square by filling the remaining squares using the letters A, B, C, D, E, F and requiring that each of the 6 regions contains all* 6 *letters.*

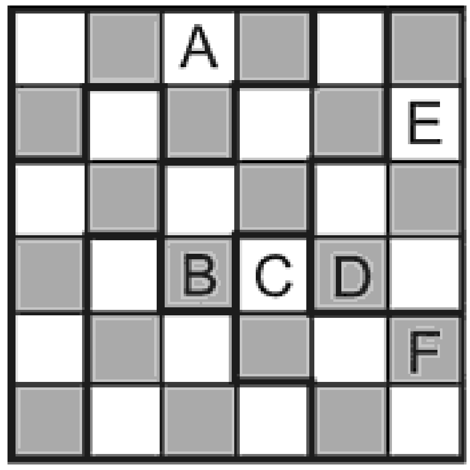


FIGURE 7.5. A 6×6 Latin square-crossword puzzle

The Graeco-Latin squares of orders 3, 4, and 5 were already known in Euler's time, but Euler wondered about the order 6. In 1782, a year before he died, having considered the question, Euler formulated a problem familiarly known as *Euler's officers problem*.

Problem 7.11. *Can one arrange* 36 *officers, each holding* 6 *different ranks, belonging to* 6 *separate regiments, to form a* 6×6 *square so that each row and column contains exactly one officer of each rank and from each regiment?*

Euler demonstrated that the problem of n^2 officers, which is the same as the problem of constructing a Graeco-Latin square of order n, can always be solved if n is odd, or if n is divisible by 4. Furthermore, he stated that Graeco-Latin squares of the order 6, 10 and 14, and in general all squares of the order $n = 4k + 2$ cannot be constructed. This became famous as *Euler's conjecture*.

In 1901, 118 years later, Gaston Tarry[8], the French mathematician, proved Euler's conjecture for the particular case $n = 6$. According to Tarry's study, the required arrangement of officers is not possible. After Tarry, several mathematicians even published "proofs" that the conjecture was true, but later the proofs were found to contain flaws.

Being restricted to pencil-and-paper methods, the work of Tarry and his assistants demanded tedious and exhausting efforts. Because of this, the subsequent cases in particular had to wait until the computer era. Thus 177 years later, Euler's conjecture was disproved in 1959, when R. C. Bose and S. S. Shrikhande of the University of North Carolina constructed a Graeco-Latin square of order 22 by using a modified Kirkman system [23]

[8]G. Tarry, *Le problème de 36 officiers*, Comptes Rendu de l'Association Française pour l'Avancement de Science Naturel, Vol. 2 (1901), pp. 170–203.

and next E.T. Parker, an employee of Remington Rand Univac, a division of the Sperry Rand Corporation, found a square of order 10 [136]. The methods of these contributors grew increasingly more refined; it was ultimately established that Euler's conjecture is wrong for all values of $n = 4k + 2$, where n is greater than 6; see [24] and [25]. Parker's Graeco-Latin square of order 10 is shown in Figure 7.6, where Latin and Greek letters are replaced by digits from 0 to 9.

00	47	18	76	29	93	85	34	61	52
86	11	57	28	79	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

FIGURE 7.6. Parker's Graeco-Latin square of order 10

We end this essay on Eulerian squares with the remark that J. Arkin, P. Smith and E. G. Straus [5] extended Euler's officers problem into a three-dimensional cube and demonstrated the existence of a solution in three dimensions! Page 83 of the paper [5] lists the solution. Theirs was a major accomplishment, especially considering that, nine years ago, prevailing opinion doubted that such a construction was possible [179].

Thomas P. Kirkman (1806–1895) (→ p. 306)

Jacob Steiner (1796–1863) (→ p. 306)

James J. Sylvester (1814–1897) (→ p. 307)

Arthur Cayley (1821–1895) (→ p. 307)

Kirkman's schoolgirls problem

In 1847 Thomas Kirkman, an English vicar who was also an expert in group theory and combinatorics and a fellow of the Royal Society, posed the following problem.[9]

[9]T. Kirkman, *Cambridge and Dublin Mathematical Journal*, Vol. 2 (1847), 191–204.

Problem 7.12. *At a girls' school, it is the daily custom for the pupils to take a walk arranged in five rows of three girls walking side by side. Can the headmistress devise a schedule according to which no two girls walk beside each other more than once for seven consecutive days?*

Kirkman's problem and others similar to it belong to the area of combinatorics called block-design theory; they were studied intensively in the nineteenth century, but mainly as problems of recreational mathematics. It was later discovered that such problems are closely related to topics as diverse as statistics, error-correcting codes, higher-dimensional geometry, Hadamard matrices, and projective geometry.

A generalization of Kirkman's problem leads to *Steiner triple systems*, after Jacob Steiner.[10] A Steiner triple system S_n, if it exists, is an arrangement of n objects in triples, such that any pair of objects is in exactly one triple. There are $\frac{1}{2}n(n-1)$ pairs and $\frac{1}{6}n(n-1)$ triples in Steiner triple system; hence, n has to be congruent to 1 or 3 modulo 6, that is, $n = 7, 9, 13, 15, \dots$. E. H. Moore proved in 1893 that this condition is also sufficient.

In a generalized Kirkman problem the number of days required for the walks is $\frac{1}{2}(n-1)$. The above numbers will be integers only if n is an odd multiple of 3, that is, of the form $3(2k+1) = 6m+3$. Thus, the sequence of possible values is 3, 9, 15, 21 and so on.

The case $n = 3$ is trivial, one day a trio of girls simply goes for a walk. The case of $n = 9$ schoolgirls in four days has a unique basic solution given in the scheme below:

123	147	159	168
456	258	267	249
789	369	348	357

Since 1922, Kirkman's original problem is known to have 80 solutions for the case $n = 15$, only seven of which are the basic solutions. There are many methods for solving Kirkman's problem. Here we present a geometrical solution with the aid of rotating concentric disks, described by Martin Gardner in *Scientific American* (May, 1980). Note that the same method was presented in the book [18, pp. 260–263] for $n = 9$.

Draw a circle and write digits 1 through 14 equally spaced around it. A cardboard disk of the same size is fastened to the circle with a pin through both centers. Label the center of the disk by 15. Draw on the disk a diameter (10,15,3) and five noncongruent triangles (1,2,15), (3,7,10), (4,5,13), (6,9,11),

[10] J. Steiner, *Journal für die reine und angewandte Mathematik*, Vol. 45 (1853), 181–182.

(8,12,14), as shown in Figure 7.7. This starting position actually gives the first day's arrangement.

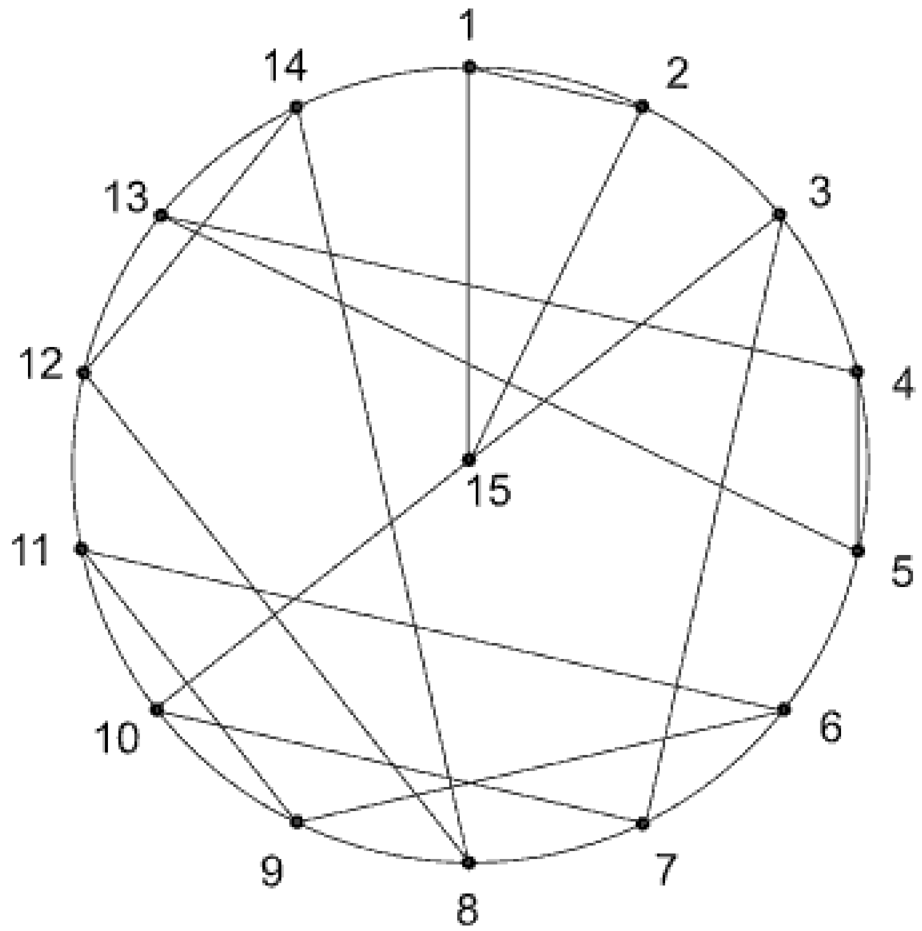


FIGURE 7.7. Geometrical solution to Kirkman's schoolgirls problem

To obtain the arrangements for the remaining six days, rotate the circle in either direction, in increments of two numbers at a time, to six different positions. This means that a point, say number 1, should coincide with the numbers 3, 5, 7, 9, 11 and 13 (or in the reverse order) on the fixed circle. In each of these six positions the vertices of five triangles on the rotating circle mark the numbers on the fixed circle, giving five new triples. Yes, it works! Indeed, a handy rule. Below we give the complete solution for the design shown in Figure 7.7:

Monday:	(1,2,15)	(3,7,10)	(4,5,13)	(6,9,11)	(8,12,14)
Tuesday:	(3,4,15)	(5,9,12)	(1,6,7)	(8,11,13)	(2,10,14)
Wednesday:	(5,6,15)	(7,11,14)	(3,8,9)	(1,10,13)	(2,4,12)
Thursday:	(7,8,15)	(2,9,13)	(5,10,11)	(1,3,12)	(4,6,14)
Friday:	(9,10,15)	(1,4,11)	(7,12,13)	(3,5,14)	(2,6,8)
Saturday:	(11,12,15)	(3,6,13)	(1,9,14)	(2,5,7)	(4,8,10)
Sunday:	(13,14,15)	(1,5,8)	(2,3,11)	(4,7,9)	(6,10,12)

According to James J. Sylvester, who was also interested in this problem (see below), one of the most interesting solutions of Kirkman's problem comes from B. Pierce.[11] Pierce's solution can be found in H. Dörrie's book [54]. MacMillan's 1949 edition of Rouse Ball and Coxeter's famous book, *Mathematical Recreations and Essays*, gave one solution for every case when n is less than 100.

Although the form $6m+3$ for n is necessary for the solution of the general form of Kirkman's problem, it is not sufficient. In the second half of the nineteenth century many papers were written on this subject, giving only solutions for particular values of n. A general solution for all n (of the form $6m+3$) was given in 1970 when D. K. Ray-Chaudhuri and Richard M. Wilson of Ohio State University proved that the answer is **yes.** However, the number of solutions remains unknown, and it was found only for small values of n. The number of the Steiner triple systems S_n increases very rapidly; for example, there are more than $2 \cdot 10^{15}$ non-isomorphic solutions for $n = 31$.

Soon after Thomas Kirkman postulated his schoolgirls problem in 1847, Arthur Cayley wrote one of the first papers on this subject titled, *On the triadic arrangements of seven and fifteen things* and published in *Philos. Mag.*, No. 3 (1850). His friend James J. Sylvester also considered Kirkman's problem and he noticed that there are $\binom{15}{3} = 455$ ways of forming a triple of the 15 schoolgirls. Since $455 = 13 \cdot 35$, and 35 (=7 days $\times$ 5 triples) is the number of triples appearing in the solution to Kirkman's problem, the following question arises:

Problem 7.13. *Can one partition* 455 *different triples into* 13 *different Steiner triple systems that each satisfy the conditions of Kirkman's problem?*

In a general case of n girls, the total number of different triples is

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

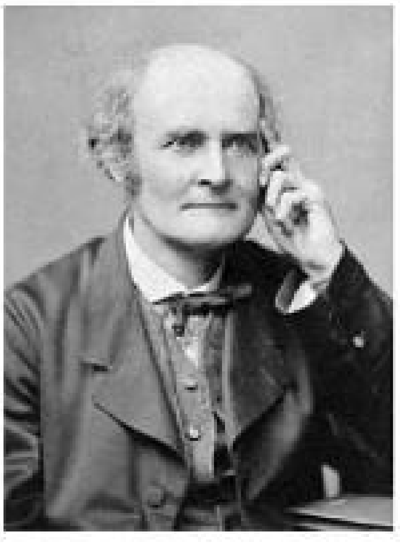
Since the Steiner triple system S_n contains $\frac{1}{6}n(n-1)$ triples, $n-2$ disjoint S_n's will exhaust all of the triples. To generalize Sylvester's problem, we have to arrange the set of all triples into $n-2$ disjoint S_n's, each one of them "parted" into $\frac{1}{2}n(n-1)$ "days" as before.

Sylvester's problem is very difficult and the answer was not found until 1974 when R. H. F. Denniston of Leicester University formulated a solution aided by computer (*Discrete Mathematics*, No. 9 (1974)). Kirkman, incidentally, mistakenly claimed to have solved Sylvester's problem for $n = 15$.

[11] *Cyclic solutions of the schoolgirl puzzle*, The Astronomical Journal, vol. VI, 1859–1861.

Arthur Cayley (1821–1895) (→ p. 307)

By today's standards, the career of one of the greatest nineteenth-century mathematicians was unusual although not uncommon for his times. Before Arthur Cayley became a professor of pure mathematics at Cambridge in 1863, he spent 14 years as a highly competent lawyer working together with his close friend and fellow lawyer, James Joseph Sylvester (1814–1897) who was also a great mathematician. During their working day at Lincoln's Inn Court, they took every opportunity to discuss questions of profound mathematical importance. In the time he worked as a lawyer, Cayley published about 250 mathematical papers from a total life-time output of over 900 papers and notes.



Arthur Cayley
1821–1895

Counting problem

A connected graph that does not contain any cycle is called a *tree*, see Appendix C. The trees with 2, 3 and 4 nodes (vertices) are given in Figure 7.8. Counting the number of different trees with a fixed number of nodes while still taking into account positions of nodes, Cayley derived[12] an interesting recursion relation in 1859. To avoid some elements of graph theory, unfamiliar to many readers, we will present equivalent real-life problems from Mendelson's paper [128]. The solution of these problems is expressed by Cayley's recursion relation.

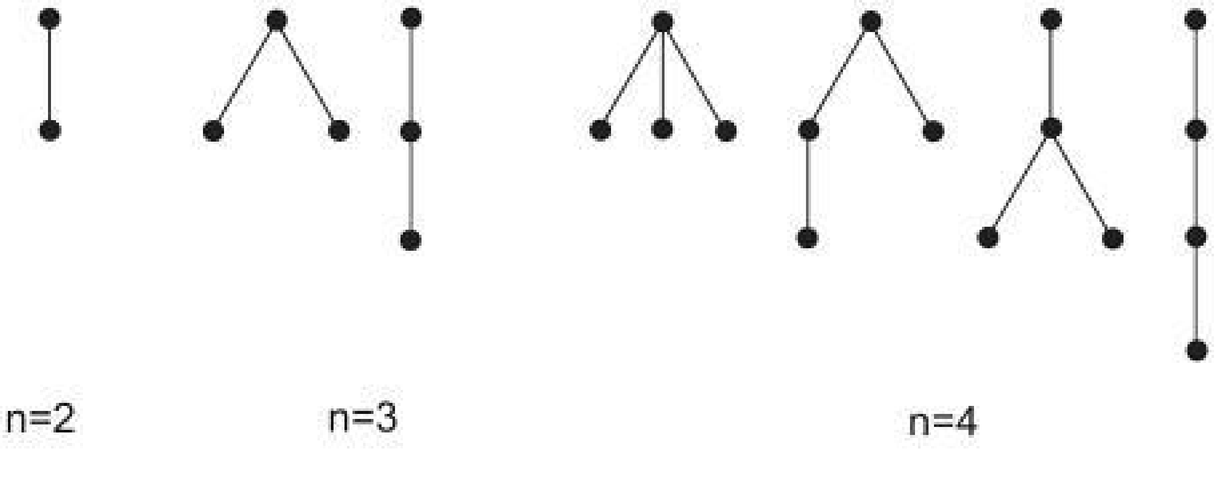


FIGURE 7.8. Trees with 2, 3 and 4 nodes

[12]A. Cayley, *On the analytical forms called trees, second part*, Philos. Mag. 18 (1859), 374–378.

Problem 7.14(a). *n runners compete in a race in which any number of runners may tie for an arbitrary number of places. Find the number of possible outcomes.*

An interesting equivalent form of the above problem thus reads:

Problem 7.14(b). *An election ballot consists of n candidates, some equally favored. Assuming that any number of candidates may obtain the same number of votes, find the number of possible outcomes.*

Let J_n be the number of all possible outcomes with n runners/candidates. It is easy to calculate the first few values of J_n:

$$J_0 = 1;$$
$$J_1 = 1;$$
$$J_2 = 3 : \text{(either (A,B), (BA), or a tie (AB));}$$
$$J_3 = 13 \quad \text{(see the diagram below and Figure 7.9)}$$

1	A	A	B	B	C	C	ABC	AB	AC	BC	A	B	C
2	B	C	A	C	A	B	—	C	B	A	BC	AC	AB
3	C	B	C	A	B	A	—	—	—	—	—	—	—

The tree-graphs with 3 nodes that correspond to all possible outcomes with 3 runners/candidates are presented in Figure 7.9.

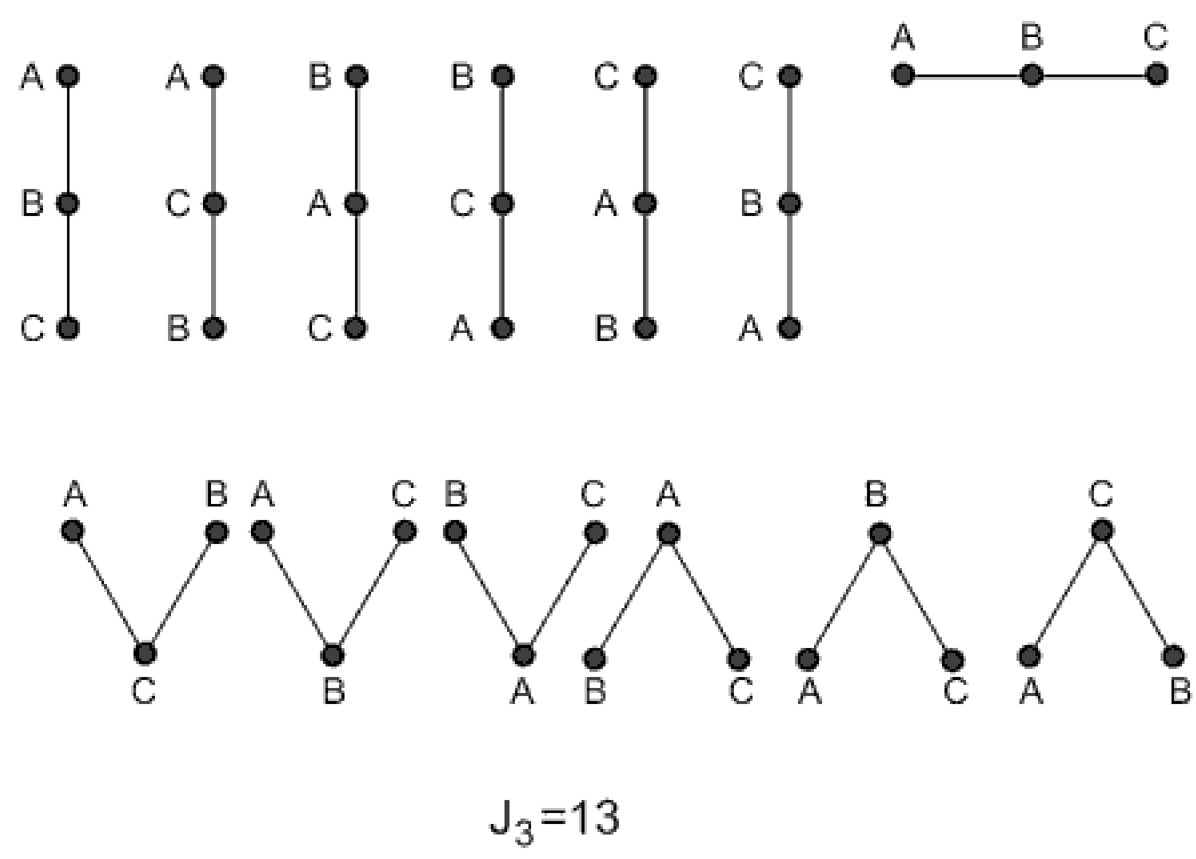


FIGURE 7.9. The number of all possible outcomes with 3 runners/candidates

We give the derivation of the recursion relation for J_n as presented in [128] by S. Mendelson in the next paragraph.

Let us assume that there are $n+1$ runners. If the number of runners who do not finish first is j, then those j runners can finish in second, third, ... place in J_j ways. The number of choices of these j runners from $n+1$ runners is $\binom{n+1}{j}$, so that the number of possible outcomes is $\binom{n+1}{j}\cdot J_j$. Since j can be any number between 0 and n, the value of J_{n+1} is

$$\binom{n+1}{0}J_0 + \binom{n+1}{1}J_1 + \cdots + \binom{n+1}{n}J_n.$$

Thus, we obtain the recursion relation

$$\begin{aligned} J_0 &= 1, \\ J_{n+1} &= \sum_{j=0}^{n}\binom{n+1}{j}J_j. \end{aligned} \tag{7.7}$$

For example, from (7.7) we calculate

$$J_4 = \sum_{j=0}^{3}\binom{4}{j}J_j = J_0 + 4J_1 + 6J2 + 4J_3 = 1 + 4 + 18 + 52 + 52 = 75.$$

As mentioned above, Cayley was the first to derive (7.7), although in an entirely different context.

The table given below lists all values of J_n for $1 \le n \le 10$.

1	2	3	4	5	6	7	8	9	10
1	3	13	75	541	4,683	47,293	545,835	7,087,261	102,247,563

TABLE 7.3.

Using some elements of combinatorics S. Mendelson [128] derived the following expression for J_n in the closed form:

$$J_n = \sum_{k=1}^{n}\sum_{p=1}^{k}(-1)^{k-p}\binom{k}{p}p^n \quad (n = 1, 2, \ldots). \tag{7.8}$$

For the purpose of demonstration, we again compute J_4 but now using formula (7.8):

$$\begin{aligned} J_4 &= 1 + (-2 + 2^4) + (3 - 3\cdot 2^4 + 3^4) + (-4 + 6\cdot 2^4 - 4\cdot 3^4 + 4^4) \\ &= 1 + 14 + 36 + 24 = 75. \end{aligned}$$

Edouard Lucas (1842–1891) (→ p. 307)

The Tower of Hanoi

The ingenious puzzle known as the *Tower of Hanoi* was launched in 1883 by M. Claus, an anagram in fact, of its inventor, Edouard Lucas[13].

Problem 7.15. *The tower puzzle consists of three vertical pegs set into a board, and a number of disks graded in size, eight disks in the case of Lucas' toy, as we see in Figure* 7.10. *These disks are initially stacked on one of the pegs so that the largest rests at the bottom of the stack, the next largest in size atop it, and so on, ending with the smallest disk placed at the top. A player can shift the disks from one peg to another one at a time, however, no disk may rest upon a disk smaller than itself. The task is to transfer the tower of disks from the peg upon which the disks initially rest to one of the other pegs. How does one accomplish this transfer in the minimum number of moves?*

Denote the required minimal number of moves with h_n (h stands for "Hanoi"). It is evident that $h_1 = 1$ and $h_2 = 3$. The first three moves are displayed in Figure 7.10.

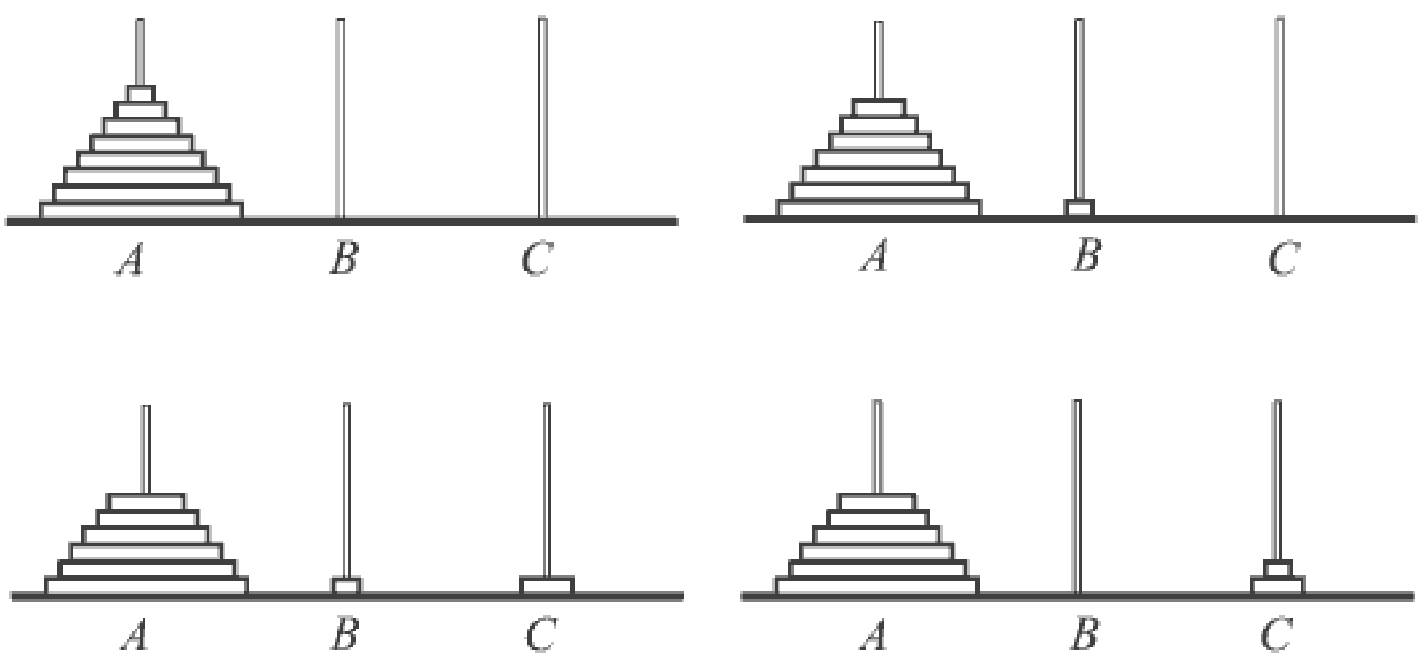


FIGURE 7.10. Tower of Hanoi–first moves

In order to transfer the largest disk from peg A to peg B, we must first construct a tower composed of the remaining $n-1$ disks on peg C, using

[13]N. Claus [=Edouard Lucas], *La tour d'Hanoi, jeu de calcul*, Sci. Nature 1 (1884), 127–128; also E. Lucas, *Récréations Mathmatiques*, Gauthier-Villars 1882–94, reprinted by Blanchard, Paris 1960.

in this process the peg B (see Figure 7.11). The minimal number of moves necessary for this transfer is h_{n-1}. After that, one move is needed for the transfer of the largest disk to peg B and at least h_{n-1} moves to transfer $n-1$ disks from peg C to peg B by using peg A. Therefore, the required number is given by the recurrence relation

$$h_n = 2h_{n-1} + 1, \quad n \geq 2, \quad h_1 = 1. \tag{7.9}$$

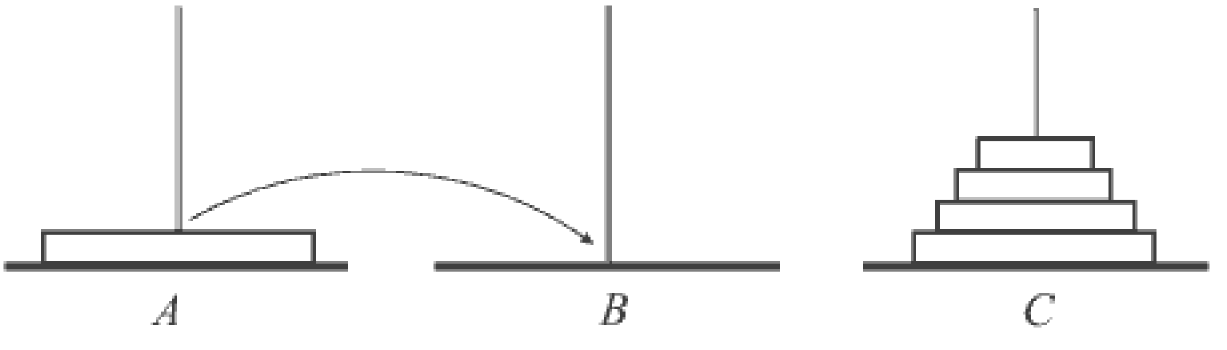


FIGURE 7.11.

For $n,\ n-1, \ldots, 3,\ 2$ the relation (7.9) gives

$$\begin{aligned} h_n &= 2h_{n-1} + 1 \\ h_{n-1} &= 2h_{n-2} + 1 \\ &\vdots \\ h_3 &= 2h_2 + 1 \\ h_2 &= 2h_1 + 1. \end{aligned}$$

Multiplying the above relations by $1,\ 2,\ 2^2, \ldots, 2^{n-2}$, respectively, and summing the left and right sides of multiplied relations (do you remember telescoping summation from page 100), after cancelling the same terms one obtains

$$h_n = 1 + 2 + 2^2 + \cdots 2^{n-2} + 2^{n-1} = 2^n - 1.$$

Therefore, $2^n - 1$ is the minimum number of moves required to effect the complete transfer of n disks. In the case of Lucas' 1883 toy with $n = 8$ disks, this number is $2^8 - 1 = 255$. In fact, $2^n - 1$ is really the minimum number of moves if the described algorithm is applied, but we cannot yet claim "mission finished". This question is discussed later.

A year after Lucas had launched his toy, Henri de Parville told an interesting tale of the origin of the *Tower of Hanoi* in *La Nature* (Paris, 1884, pp. 285–286). Rouse Ball and Coxeter's book, *Mathematical Recreations and Essays* [150], retells this story; we present an adapted version below.

In the Temple of Benares, beneath the dome purportedly marking the very center of the world, there lies a brass plate with three diamond needles affixed to it. Each needle measures one cubit in height, and the size of a bee in thickness. At the same time God created the world, he also created the Tower of Brahma and placed sixty-four pure golden disks upon one of the needles, the largest disk resting upon the brass plate, with each disk progressively smaller in size until reaching the top. In accordance with the unchanging and unchangeable laws of Brahma, the temple priests must transfer the disks from one of these diamond needles to another never ceasing, day or night. The attending priest may not move more than one disk at a time and in placing these disks on the needle, he must respect the order so that no smaller disk rests below a larger one. Once the transfer of the sixty-four disks from the needle on which God first placed them to one of the other needles has been completed, all will turn to dust, tower, temple, and Brahmins together, and the earth itself will vanish in a thunderbolt.

According to the derived general formula, the number of separate transfers of golden disks of the Tower of Brahma in Benares is

$$2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615.$$

Assuming that the priests can transfer one disk per second, the end of the world will occur in about 585 billion years! But wait for a moment. Incidentally, $2^{64} - 1$ is the total number of grains of wheat on the ordinary 8×8 chessboard if one grain is placed on the first square, two on the second, and so on in geometric progression, as mentioned in the introduction of this chapter. Indeed, $1 + 2 + 2^2 + \cdots + 2^{63} = 2^{64} - 1$.

Although many references state $2^n - 1$ as the minimum number of moves, most do so without offering any deeper analysis. However, D. Wood in his paper [189] says: *"What is usually proved is that the number of moves required by the recursive algorithm for n disks is $2^n - 1$. This is **not** a proof that **no** other algorithm exists which takes fewer moves."* In the continuation of his paper, D. Wood proves that $2^n - 1$ is really the minimum number of moves, independently of a moving procedure. Two important corollaries are:

1) The typical recursive solution (shown above) for the Tower of Hanoi problem is optimal in the number of moves.

2) The Tower of Hanoi problem is an exponential time problem.

Wood's paper [189] also contains an analysis of the solution from a computational complexity point of view, a transfer-program generator written in *PASCAL*, some open and some new problems concerning the Tower of Hanoi problem and, finally, an extensive list of references on the subject.

The Tower of Hanoi and, in turn methods for solving it, have generated an abundant literature; for example, [17, Vol. 2], [18], [19], [28], [29], [50], [69], [70, pp. 55–62], [88], [99], [103], [149], as well as the mentioned works by Lucas [125], and Rouse Ball and Coxeter [150]. The use of a special type of graph (called H_n graph for n-disk Hanoi) and of graph theory applications to study the Tower of Hanoi and solve it form the basis of the papers [60], [157], [192]. Curiously enough, the corresponding H_n graph, associated with the solution of n-disk Hanoi, looks more and more like the Sierpiński gasket as n becomes larger and larger; see [171].

A curious reader may wonder whether there is a solution for a tower with four or more pegs. There are algorithms for transferring disks on four pegs but the optimality of these procedures (that is, the minimum number of moves) has not been proved yet. It is solely conjectured that this number is given recursively by

$$f_n = f_{n-1} + 2^x, \quad f_1 = 1,$$

where

$$x = \left\lfloor \frac{\sqrt{8n-7}-1}{2} \right\rfloor.$$

Recall that $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Interchanging the checkers (I)

The following problem by Lucas appearing in his 1883 book *Récréations mathématiques*[14] may also be found in the book [150, pp. 124–125].

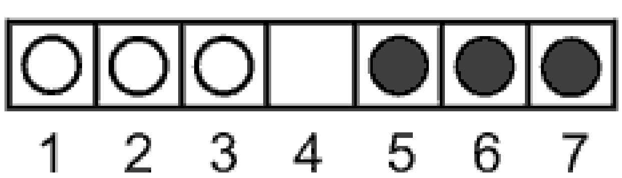


FIGURE 7.12. Interchanging the checkers

Problem 7.16.* *Seven squares, denoted by numbers from 1 to 7, are joined in a row as in Figure 7.12. Three white checkers are placed at one end, and three black checkers at the other end. The middle square 4 is vacant. The aim is to translate all white checkers to the places occupied initially by the black checkers and vice versa, moving white checkers from left to right, and black checkers from right to left. One move consists of either one transposition of a checker to the adjacent unoccupied square or*

[14] Paris 1883, Vol. II, part 5, pp. 141–143.

the jump over a checker of opposite color on the unoccupied square beyond it. The checkers may move in any order, regardless of their color. The minimum number of moves is required.

Another of Lucas' checker-interchanging problems is a little more complicated and may be found in Wells' book [186].

Interchanging the checkers (II)

Problem 7.17.* *Figure 7.13 shows four checkers A, B, C, D occupying the shaded squares. The task is to exchange C for D and A for B. The checkers can move one or more squares in any direction (including rectangular turning), but without skipping over any other checkers.*

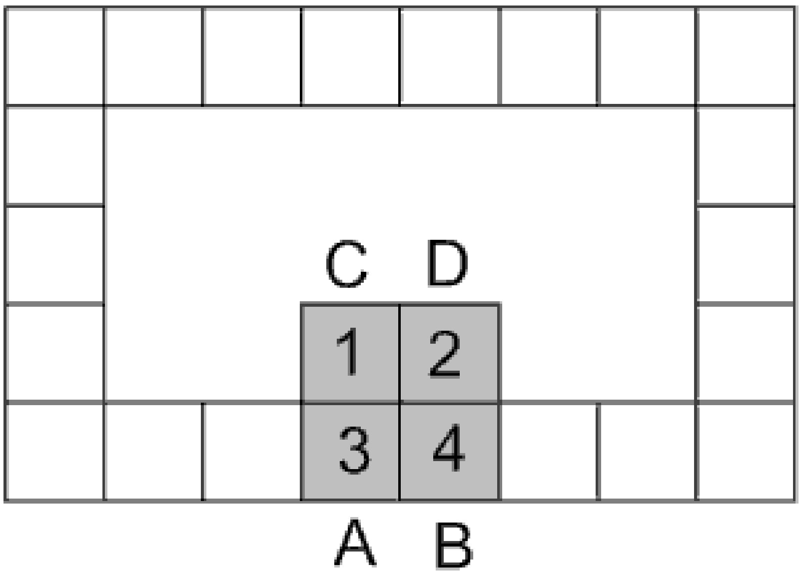


FIGURE 7.13. Interchanging the checkers

Shunting problem

Problem 7.18.* *This problem requires train A to overtake train B, employing a spur track only long enough to hold half of train B; see Figure 7.14. How can the engineers maneuver the trains to reverse their positions?*

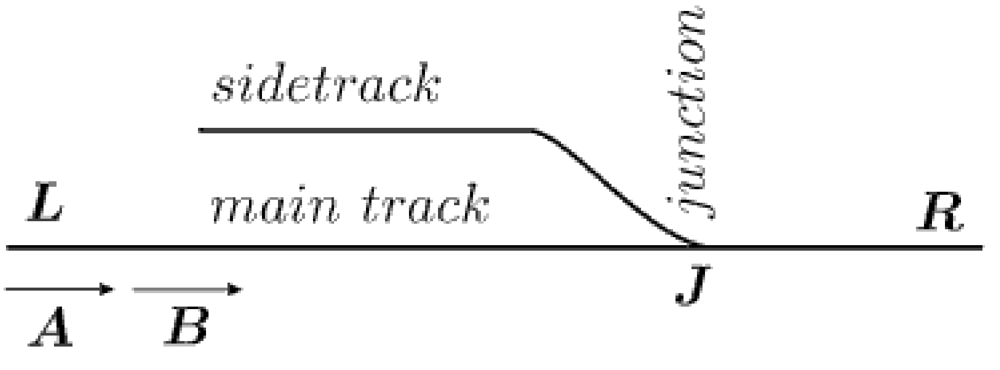


FIGURE 7.14. Reversal of train positions

Problem of married couples (problème des ménages)

The next problem by Lucas appeared in his book [125] in 1891 and reads:

Problem 7.19. *Determine all possible arrangements of n married couples in such a manner that the seating results in the men and women placed alternately about a round table so that none of the husbands is ever seated next to his own wife.*

For more than a century, this classic problem of combinatorial analysis has attracted the attention of numerous authors. For more details see, *e.g.*, Dörrie [54], Dutka [59], Kaplansky and Riordan [111] and Riordan [147].

Solving this challenging problem, also known as *the problème des ménages*, requires some elements of the theory of discordant permutations. The Frenchmen M. Laisant[15] and M. C. Moreau and H. M. Taylor, an Englishman, have all arrived at solutions for the married couples problem[16] using recurrence relation (7.10), given below, without the expression of an explicit formula. Due to lack of space, we cannot give the solution here. Instead, we borrow the final result from H. Dörrie's book [54] in which he presented the complete solution.

The number B_n of possible arrangements of n married couples satisfying the condition of the problem is $B_n = 2n!A_n$, where A_n is calculated from the recurrence formula

$$(n-1)A_{n+1} = (n^2-1)A_n + (n+1)A_{n-1} + 4(-1)^n. \tag{7.10}$$

The derivation of this relation may be found in [54]. The question of relation (7.10)'s priority was discussed in [111]. In his book *Théorie des Nombres* (1891), Lucas gives (7.10) attributing it to Laisant, and independently to Moreau. Nonetheless, Cayley and Muir noted relation (7.10) thirteen years earlier, a fact unknown to Lucas and others after him.

It is not difficult to find $A_3 = 1, A_4 = 2$ directly. Then the above recurrence relation commences, giving

$$\begin{aligned} &A_5 = 13,\ A_6 = 80,\ A_7 = 579,\ A_8 = 4{,}738,\\ &A_9 = 43{,}387,\ A_{10} = 439{,}792,\ \text{etc.}, \end{aligned}$$

and B_n is easy to calculate.

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[15] *Sur deux problèmes de permutations*, Bulletin de la Société Mathématique de France, T. 19 (1890–91), 105–108.

[16] *A problem on arrangement*, Messenger of Mathematics, Vol. 32 (May 1902–April 1903), 60–63.

The recurrence relation (7.10) is a nice contribution, but it gives only indirect information. The quest for a neat closed form of A_n ended successfully when J. Touchard [177] gave the explicit solution of Lucas' problem in the form

$$B_n = 2n!A_n, \quad A_n = \sum_{k=0}^{n}(-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)! \tag{7.11}$$

However, Touchard did not offer a proof. Proofs of (7.11) were given by A. Kaplansky [110] in 1943 and J. Riordan [146] in 1944.

With the assumption that n wives were seated in assigned positions, leaving one vacant place between each pair, then the desired number is exactly A_n. In 1981, D. S. Jones and P. G. Moore [108] considered this variant of Lucas' problem and found the explicit solution in the form

$$A_n = \sum_{k=0}^{n}(-1)^k \frac{(2n-k-1)!n\pi^{1/2}}{k!(n-k-1/2)!2^{2n-2k-1}}. \tag{7.12}$$

The equivalence of formulae (7.11) and (7.12) was shown in [59].

G. E. Thomas [174] reported the genesis and short history of the *problème des ménages*, pointing to the problem's connections to a number of diverse areas. J. Dutka writes in [59] that P. G. Tait (1831–1901), a well-known nineteenth-century physicist and a professor of natural philosophy at Edinburgh, investigated the theory of vortex atoms in the 1870s and stated a knot-problem which is equivalent to Lucas' problem with initially assigned places for wives (or husbands). Thomas Muir (see [85]) formulated Tait's problem as a question concerning the number of terms in the expansion of a particular kind of determinant and obtained the recurrence relation (7.10) sometimes referred to as Laisant's relation. Other related problems were considered in [111], [126, Vol. 1], [147, Ch. 8], [178], [191].

James Joseph Sylvester (1814–1897) (→ p. 307)

The tree planting problem

There are many puzzles based on the arrangement of n points in a plane, but the oldest and most popular is known as the "tree planting problem", or the "orchard-planting problem." Stated simply, it reads as follows:

Problem 7.20. *How can n points in a plane be arranged in rows, each containing exactly k points, to produce a maximum number of rows?*

The puzzle's name comes from an earlier puzzle concerning a farmer who wishes to plant a certain number of trees in an orchard so that the pattern of trees will have r straight rows of exactly k trees in each row. The puzzle becomes difficult when one stipulates a maximum number of rows.

J. J. Sylvester devoted much of his time to this question working continually on the general problem from the late 1860s until his death in 1897 (see H. T. Croft *et al.* [44]). Although this puzzle has been around for many years, the general problem of determining the largest number of rows $r(n,k)$, given n and k, has yet to be solved. The solution was found for only some particular cases. For example, Sylvester found the lower bound given by

$$r(n,3) \geq \left\lfloor \frac{1}{6}(n-1)(n-2) \right\rfloor,$$

where, as a reminder, $\lfloor x \rfloor$ is the greatest integer less than or equal to x. However, this is an inferior limit; compare with values given in Table 7.4.

A wealth of details about the tree planting problem may be found in [30] and [77]. S. Ruberg, in [152], records an interesting approach to this problem using the projection procedure.

When k is 2, the problem is trivial: every pair of n points forms a row of two which means that the maximum number of rows is $\binom{n}{2} = \frac{n(n-1)}{2}$. When k is 3, the problem not only becomes more intriguing but it also relates to such mathematical topics as balanced-block designs, Kirkman–Steiner triples, finite geometries, Weierstrass elliptic functions, cubic curves, projective planes, error-correcting codes and many other significant aspects of mathematics.

Assuming that all n points lie in a finite plane, the maximum solutions for three-in-a row plants, for $n = 3$ to 12, are given in the table below.

n	3	4	5	6	7	8	9	10	11	12
$r(n,3)$	1	1	2	4	6	7	10	12	16	19

TABLE 7.4.

Henry Ernest Dudeney (1857–1931), the great English puzzle expert to whom we have made mention, furnished the 11-point pattern with 16 rows shown in Figure 7.15; see [56]. Dudeney presented this solution as a military puzzle. *"On a World War I battlefield,* 16 *Russian soldiers surround* 11 *Turkish soldiers. Each Russian soldier fires once and each bullet passes through exactly three Turkish heads. How many Turkish soldiers remain alive?"* According to Figure 7.15 the answer is: **Nobody.** The Russian soldiers' positions are situated in the continuation of each line segment.

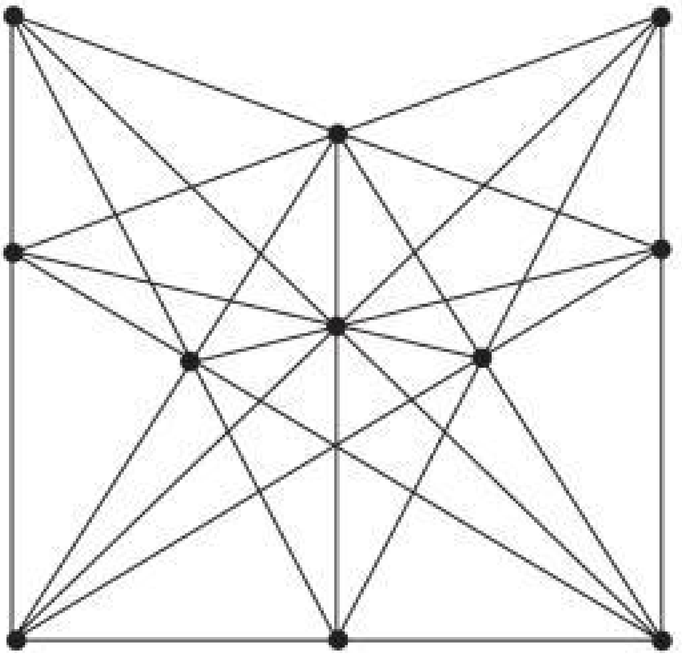


FIGURE 7.15. Eleven points in 16 rows of three

In the case $k = 4$ the problem becomes more difficult. More details can be found in [77]. Here we present an interesting example dealing with $n = 16$ and $k = 4$, discussed by Henry E. Dudeney.[17] The best-known result (15 rows) of that time is shown in Figure 7.16.[18] As reported by Martin Gardner in [77], this pentagonal pattern resembles the blossom of the flower *Hoya carnosa* (Figure 7.16 right), a member of the milkweed family and one of 100 species of *Hoya* that are native to Eastern Asia and Australia.

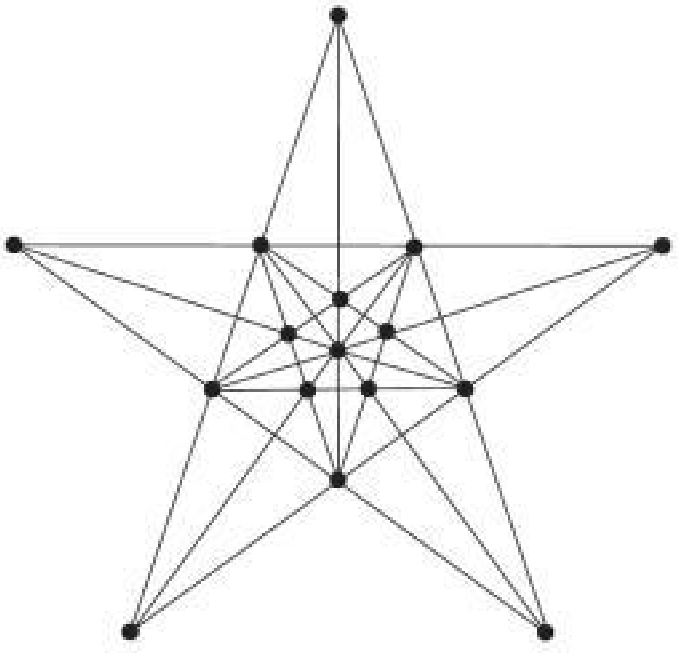


FIGURE 7.16. 16 points in 15 rows of four

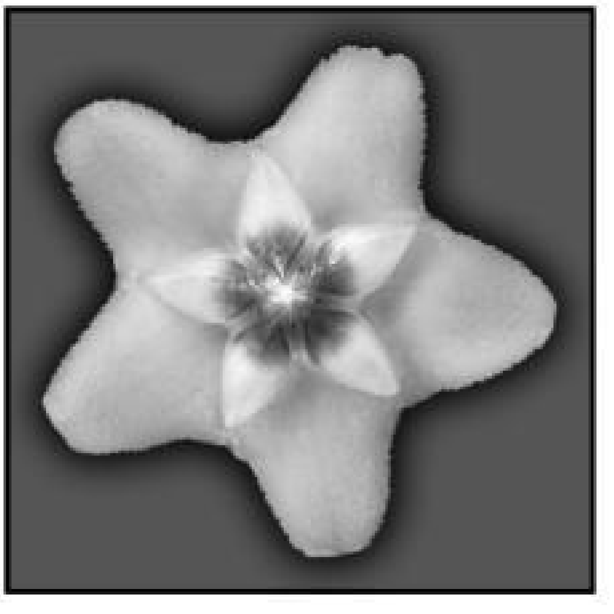


FIGURE 7.17. Hoya carnosa: the flower pattern

[17]Problem 21 in his *The Canterbury Puzzles and Other Curious Problems*, London, 1907.

[18]Today we only know that the maximum number of rows for $k = 4$ and $n = 16$ is at least 15.

After thinking about the tree planting problem for many years, Sylvester posed in 1893 a new related problem (without a proof), today known as Sylvester's line problem:

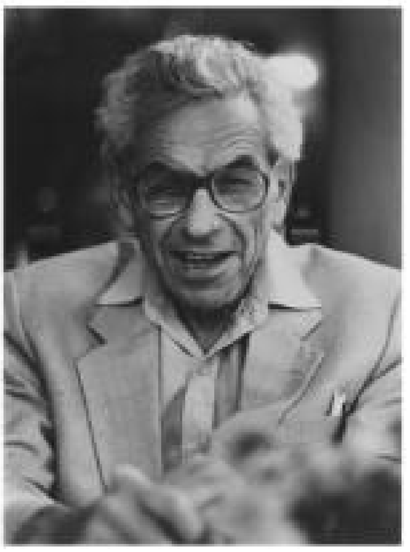
Problem 7.21. *Is it possible to arrange any finite number of points so that a straight line through every set of two points shall always pass through a third, unless they all lie in the same right line?*

Unfortunately, Sylvester did not live to see an answer to his question. T. Grünwald[19] (1944) was the first who correctly solved the problem: the answer is **no.** It turned out that the answer also follows from a result of E. Melchior (1940) applying Euler's polyhedral formula.

Paul Erdös (1913–1996) (→ p. 310)

The great Hungarian mathematician Paul Erdös was one of the twentieth century's most prolific and eccentric mathematicians. He wrote or co-authored 1,475 academic papers collaborating with 485 co-authors, more people than any mathematician in history. Erdös was really crazy about mathematics. He was forever occupied with various kinds of mathematical problems, including many curious and amusing tasks; one of them is presented below.

Erdös, a confirmed bachelor, spent most of his life crisscrossing the world in search of the beauty and the ultimate truth in mathematics. Paul Hoffman published *The Man Who Loved Only Numbers* (Hyperion, New York 2000), an inventive and captivating biography of Paul Erdös. We learn that Erdös had his own particular language, for example:



Paul Erdös
1913–1996

boss = woman
slave = man
captured = married
liberated = divorced
recaptured = remarried
epsilon = child
to exist = to do math
to die = to stop doing math
trivial being = someone who does not do math
noise = music
my brain is open = I am ready to do math

[19]T. Grünwald, *Solution to Problem* 4065, Amer. Math. Montly 51 (1944), 169–171.

We present a problem belonging to combinatorial geometry that attracted the attention of Paul Erdös.

Different distances

Problem 4.22. *Given the 7×7 checkerboard, can you put 7 counters so that all distances between pairs are different? It is assumed that distances are measured on a straight line joining the centers of cells.*

The posed request is less innocent than it might appear at first glance. This is one of those problems with a simple and clear formulation but with a difficult solution. Erdös enjoyed solving such problems. For him, they were like perfume: the packaging is at least as important as the contents.

The general problem of the $n \times n$ board with n counters has had a tempestuous history; see Gardner [82]. It was proved, with significant help from computers, that the 7×7 board is the largest for which there is a solution. Paul Erdös and Richard Guy [82, p. 137] gave the solution for the checkerboard 7×7 (shown in Figure 7.18) and proved that no solutions are possible for $n > 7$. Where there is a will, there is a way.

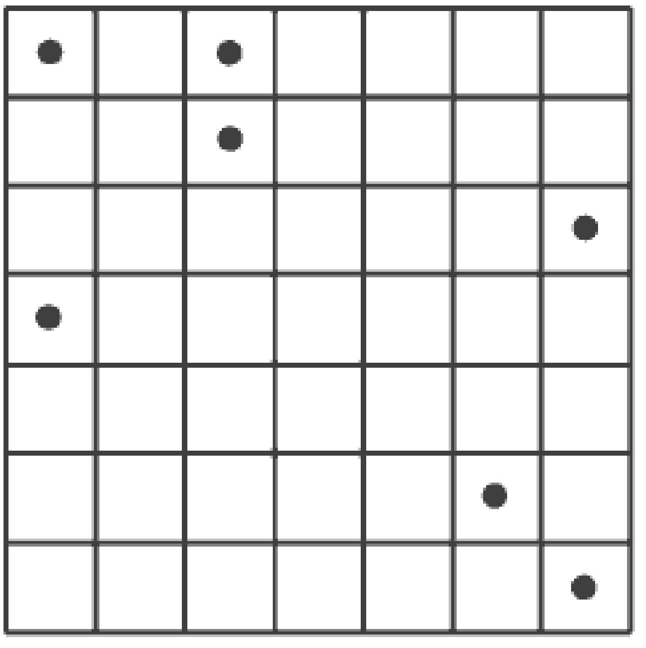


FIGURE 7.18. Different distances on the 7×7 checkerboard

Answers to Problems

7.5. The company must be arranged in a circle as shown in Figure 7.19, where the small white disks represent sailors and the small black disks represent smugglers. The command

```
InversePermutation[Josephus[30,9]]
```

in the programming package *Mathematica* gives the complete list in order of execution (take the first fifteen numbers); see Figure 7.19:

`9, 18, 27, 6, 16, 26, 7, 19, 30, 12, 24, 8, 22, 5, 23`

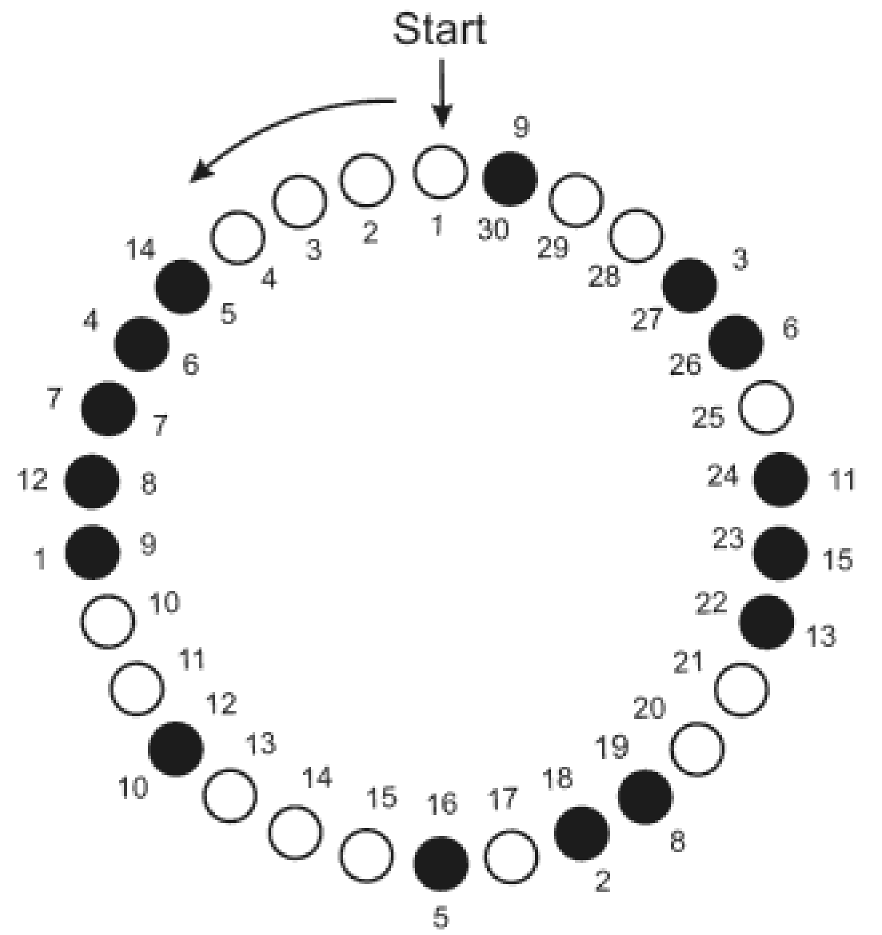


FIGURE 7.19. Disposition of smugglers and sailors

7.6. A solution is $a = 1,\ h = 11,\ b = 9,\ k = 29.$

7.10. The unique solution is shown in Figure 7.20.

D	F	A	E	C	B
B	D	C	F	A	E
A	B	E	D	F	C
F	E	B	C	D	A
E	C	D	A	B	F
C	A	F	B	E	D

FIGURE 7.20. A 6×6 Latin square—solution

7.16. The solution requires 15 moves that can be recorded by the starting and the destination square (see Figure 7.12):

3-4 5-3 6-5 4-6 2-4 1-2 3-1 5-3 7-5 6-7 4-6 2-4 3-2 5-3 4-5

7.17. The first step consists of moving checker B one square to the right. Next, move A around the circuit in the clockwise direction to the square directly right of checker B; see Figure 7.13. To interchange C and D, move them both around the shaded cells 1,2,3,4 thus: C : 1–3–4, D : 2–1, C : 4–2. Last, shift checkers $B - A$ to the left to occupy squares 3 and 4.

7.18. Referring to Figure 7.14, we will describe the maneuvering of the trains in five stages. The arrows $\rightarrow$ and $\leftarrow$ indicate, respectively, the motion of trains forward (left to right) and backward (right to left). The common point and track sections are denoted by the letters J (junction), L (track left from the junction), R (track right from the junction) and S (spur track).

1) **B**: $\rightarrow L$–J–R; $\leftarrow R$–J–S, uncouples its rear half on the spur track; the front half of **B**: $\rightarrow J$–R;

2) **A**: $\rightarrow L$–J–R, joins the rear half of **B**; $\leftarrow R$–J–L;

3) the front half of **B**: $\leftarrow R$–J–S;

4) **A** uncouples the rear half of **B** and proceeds on its way to L–R;

5) the front half of **B** leaves the spur track, joins its rear half and proceeds.

Chapter 8 PROBABILITY

The most important questions of life are,
for the most part, really only problems of probability.
Pierre-Simon de Laplace

Probability is expectation founded upon partial knowledge.
George Boole

The excitement that a gambler feels
when making a bet is equal to the amount
he might win times the probability of winning it.
Blaise Pascal

The great French mathematician Pierre-Simon de Laplace wrote: *"It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge."* He thought about the probability and you will see at the beginning of this chapter that Pascal and Fermat were those fellows who conducted the mentioned treatise of the game of two gamblers who played for a stake. The results of their fruitful discussions led to the foundation of the probability theory, a very important branch of mathematics, which is also very useful in many scientific disciplines and many other human activities.

Probability deals with the estimation of a chance that some event will happen, or figuring out how often this event can occur under given conditions. The probability of the occurrence of an event can be expressed as a fraction or a decimal from 0 (impossible event) to 1 (certain event). It is substantial to many human activities and experimental researches when quantitative analysis of large sets of data is necessary. However, we will consider neither such analysis nor the analysis of random phenomena; we will not even get involved in the pitfalls of defining probability. In this chapter you will find only elementary problems of great mathematicians that need only the knowledge of the basic elements of combinatorics.

"Perhaps the greatest paradox of all is that there are paradoxes in mathematics," said Edward Kasner. Probability is full of surprising results and paradoxes, more than any other branch of mathematics. This is another

paradox. Indeed, thanks to our sense of risks and many aspects of chance, and direct life experience, we should have a good estimation of results. In spite of that, probability is swarming with paradoxes and unexpected results. Would you take the bet that at least two people among soccer players of two teams plus the referee (23 persons in total) have the same birthday? *"No, the chance is extremely small,"* you would have probably said. However, probability theory gives the unexpected result of even 51%. If we know that in a two-child family one child is a boy, what is the probability that both children are boys? A most frequent but incorrect answer is 1/2, a good analysis tells us that the probability is 1/3. Another paradox is given as Problem 8.4 in this chapter.

Charles S. Pierce once said: *"Probability is the only one branch of mathematics in which good writers frequently get results which are entirely erroneous."* Leibniz is said to have thought that the probability of getting 12 with a pair of dice is the same as of getting 11. Similarly, D'Alembert, the great eighteenth-century French mathematician, failed to notice that tossing a coin three times gives the same results as tossing three coins at once.

The following example is instructive: When a statistician passed the airport security check, the security employees discovered a bomb in his bag. He explained: *"Statistics show that the probability of a bomb being on an airplane is* $1/1000$. *However, the chance that there are two bombs on one plane is* $1/1000 \times 1/1000 = 1/1{,}000{,}000$, *which is a very small probability and I can accept that risk. So, I bring my own bomb along to feel much safer."*

How to avoid traps and wrong conclusions? In any probability problem, it is very important to analyze given conditions and understand them well, and identify all different outcomes that could occur.

Except the already mentioned problem of the division of stakes between gamblers discussed by Pascal and Fermat, this chapter contains the gambler's ruin problem which attracted wide attention of eminent mathematicians, including Pascal, Fermat and Huygens. Huygens was the author of the first printed work on probability; so we selected a problem on a gambling game with dice from his collection. It was impossible to omit a very popular probability problem with misaddressed letters, related to the work of Euler and Nicolaus II Bernoulli (see Chapter 7). The Petersburg paradox is given as a good example of an unexpected result that fascinated and confused many mathematicians including the brothers Daniel and Nicolaus III Bernoulli, Cramer, D'Alembert, Poisson, Bertrand, Laplace and others. Finally, Banach's matchbox problem will test how skillfully you can swim in combinatorial waters.

Pierre de Fermat (1601–1665) (→ p. 303)

Blaise Pascal (1623–1662) (→ p. 303)

Pierre de Fermat
1601–1665

Pierre de Fermat, considered one of the leading mathematicians of his time, was a lawyer who spent much of his time doing mathematics. Although he published almost nothing, he communicated all of his work through his extensive correspondence with many of the leading mathematicians of his day. He stated the basic properties of optics, now known to us as Fermat's principle: light always follows the shortest possible path. Fermat made major contributions in several branches of mathematics, but he is best remembered today for his work in number theory, in particular for Fermat's Last Theorem, certainly one of the best-known problems in the history of mathematics.[1]

The French genius Blaise Pascal showed a phenomenal ability in mathematics from an early age; at the age of twelve he made the independent discovery of most of the theorems from Euclid's first book; when he was fourteen he participated in the gatherings of a group of French mathematicians that in 1666 became the French Academy; in 1654, he wrote a brief essay on conic sections. By the age of twenty-two,[2] he invented the first calculating machine which he called *pascaline*.

After experiencing a religious ecstasy on the night of November 23, 1654, Pascal abandoned science and mathematics for theology. From time to time, he would return to mathematics and other sciences. He even wrote one philosophical book in the form of letters to a ficticious provincial gentleman. One of these letters contained the well-known Pascal's apology: *"I would have written a shorter letter, but I did not have the time."* Pascal turned away from worldly concerns and died at the age of 39, after suffering greatly due to an illness of long duration.

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[1] *No positive integers x, y, z, n exist such that $x^n + y^n = z^n$ for $n > 2$.* Fermat's conjecture is probably the best-known problem in the history of mathematics. After eight years of extensive work, the English mathematician Andrew Wiles (1954–), proved this conjecture and published his proof in the paper *Modular elliptic curves and Fermat's Last Theorem*, Annals of Mathematics, Vol. 141, No. 3, May 1995. Wiles' proof contains extraordinary ideas from various highly specialized branches of mathematics, among them, elliptic curves, algebraic geometry, modular forms, and p-adic numbers.

[2] According to some sources, by the age of eighteen or nineteen.

Blaise Pascal
1623–1662

It is perhaps worth mentioning that Descartes, Fermat and Pascal represented a mighty French trio of mathematicians in the first half of the seventeenth century. However, Descartes' arrogance and massive ego often spoiled the harmony of this trio. After Fermat's criticism of Descartes' work, Descartes began to ignore Fermat and endeavored to smear his reputation whenever possible. Pascal suffered similarly, although he had done nothing to deserve this treatment. Commenting on Pascal's study of atmospheric pressure and a vacuum, Descartes noted sarcastically that Pascal *"has too much vacuum in his head."*

The Problem of the points

The origins of probability theory came about as a result of a gamblers' quarrel in 1654, when Antoine Gombaud, the chevalier de Méré a professional gambler, more or less, and a friend of Pascal, asked him to solve an insignificant gambling problem concerning the division of stakes. Pascal and Fermat carried out intense discussions on this problem, often called the *problem of the points*. Through their correspondence they both arrived at a new theory based on ideas that laid the foundations for the theory of probability.

Gerolamo Cardano, mentioned elsewhere on page 180, and Luca Pacioli had earlier considered the division of the stakes problem posed by de Méré to Pascal although they both arrived at incorrect solutions. The *problem of the points*, sometimes named as the *division problem*, thus reads:

Problem 8.1. *Two supposedly equally-skilled gamblers play a game for the stakes which go to the one who first wins a fixed number of points given in advance. However, the game is interrupted due to some reason. The question is how to divide the stakes knowing the scores of the gamblers at the time of interruption and the number of points required to win the game.*

Fermat studied a particular case in which gambler A needs 2 points to win, and gambler B needs 3 points. We present his solution in the following paragraph.

Evidently, not more than four trials are needed to decide the game. Let a indicate a trial where A wins, and b a trial where B wins. There are 16 combinations of the two letters a and b taken 4 at a time, as shown in Table 8.1.

Among 16 possible cases, 11 are favorable for A (cases 1–11 where a appears 2 or more times), and 5 are favorable for B (cases 12–16 where b appears 3 or more times). Therefore, the probability of winning is 11/16 for gambler A, and 5/16 for gambler B. Fermat concluded that the stakes should be divided proportionally to the probabilities of winning, thus, in the ratio 11:5.

1	a	a	a	a
2	a	a	a	b
3	a	a	b	a
4	a	b	a	a
5	b	a	a	a
6	a	a	b	b
7	a	b	a	b
8	b	a	a	b
9	a	b	b	a
10	b	a	b	a
11	b	b	a	a
12	b	b	b	a
13	b	b	a	b
14	b	a	b	b
15	a	b	b	b
16	b	b	b	b

TABLE 8.1. Fermat–Pascal's problem of the points

However, Pascal and Fermat considered the problem of points in a more general form (see [92]). In his work *Usage du Triangle Arithmétique pour Déterminer les Partis Qu'on Doit Faire Entre Deux Joueurs en Plusieurs Parties*, Pascal solved the problem of points using recursion relations and the arithmetic triangle. He always assumed that the two players have an equal chance of winning in a single game.

Let n be in advance a fixed number of points that must be reached in the play, and let $e(k, m)$ denote gambler A's share of the total stake (or A's expectation), if the play is interrupted when A lacks k games and B lacks m games to win. Then the play will be over in *at most* $k+m-1$ further games. Using modern notation, we may write Pascal's procedure as

$$e(0,n)=1 \quad \text{and} \quad e(n,n)=\frac{1}{2}, \quad n=1,2,\dots,$$
$$e(k,m)=\frac{1}{2}\big[e(k-1,m)+e(k,m-1)\big], \quad k,m=1,2,\dots .$$

To find the explicit expression for $e(k,m)$, Pascal used his results about the arithmetic triangle. This recursive method became very popular later in the eighteenth century; among many of its applications in solving difficult problems, de Moivre, Lagrange and Laplace used it to develop general methods for the solution of difference equations. Pascal's solution may be found in [92, p. 58]. We give the final expression for $e(k,m)$ derived by Pascal:

$$e(k,m)=\frac{1}{2^{k+m-1}}\sum_{i=0}^{m-1}\binom{k+m-1}{i}. \tag{8.1}$$

Fermat's approach to solving the problem of points was different; he used what is called today *a waiting-time argument.* Although he did not give a general result explicitly for $e(k,m)$, his argumentation presented on some particular cases clearly leads to the expression

$$e(k,m)=\sum_{i=0}^{m-1}\binom{k-1+i}{k-1}\left(\frac{1}{2}\right)^{k+i}. \tag{8.2}$$

Note that (8.1) and (8.2) give identical results.

For example, for a particular case given previously when $k=2$ and $m=3$, the formula (8.1) (say) gives

$$e(2,3)=\frac{1}{2^4}\left(\binom{4}{0}+\binom{4}{1}+\binom{4}{2}\right)=\frac{11}{16},$$

which is the probability of gambler A to win.

Johann Bernoulli, de Montmort, and de Moivre employed the methods developed by Pascal and Fermat to solve the problem of points for players with different probabilities of winning a single game. More details on this subject can be found in Hald's book *A History of Probability and Statistics and Their Applications before 1750* [92, §14.1].

Christiaan Huygens (1629–1695) (→ p. 303)

Christiaan Huygens
1629–1695

Certainly one of the greatest seventeenth-century scientists, the Dutchman Christiaan Huygens acquired a far-reaching international reputation working in physics, mathematics and astronomy. He was a member of the London Royal Society and frequent visitors of the French Academy of Sciences and other scientific centers in Europe: indeed as he put it himself, *"The world is my country, science is my religion."* Huygens greatly improved the telescope using lens-shaping techniques, which enabled him to discover Saturn's largest moon Titan and to give the first accurate description of the rings of Saturn.

Huygens communicated his discovery of Saturn's rings in a booklet in the form of an anagram:[3]

aaaaaaa cccc d eeeee g h iiiiiii lll mm nnnnnnnnn oooo pp q rr s ttttt uuuuu

Ordering the letters in the proper way results in the following declaration:

Annulo cingitur tenui, plano, nusquam cohaerente, ad eclipticam inclinato.

(Enclosed by thin, flat rings, without support, inclined toward ecliptic.)

It should be noted that the letters of Huygens' anagram can be arranged in about 10^6 ways. Nonetheless, such anagrams were not always sufficient to protect secrets. When Huygens discovered Saturn's moon Titan, he composed a similar anagram that the English mathematician and theologian John Wallis deciphered.

Gambling game with dice

The first printed work on the theory of probability was Huygens' little tract *De ratiociniis in ludo aleae* (On reasoning in games of dice), prompted by the correspondence of the Frenchmen and published in 1657. Here is one example solved by Huygens.

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[3]An anagram creates a new word or phrase that consists of the same letters as an existing one but ordered differently. Some sixteenth- and seventeenth-century scientists often recorded their results or discoveries in the form of anagrams to avoid their publishing until they had checked all of the details. They also communicated with other scientists by using such anagrams.

Problem 8.2. *Two gamblers* ***A*** *and* ***B*** *play a game throwing two ordinary dice.* ***A*** *wins if he obtains the sum* 6 *before* ***B*** *obtains* 7, *and* ***B*** *wins if he obtains the sum* 7 *before* ***A*** *obtains* 6. *Which of the players has a better chance of winning if player* ***A*** *starts the game?*

Let A and B stand for the events denoting the appearances of the sums 6 (***A*** wins) and 7 (***B*** wins). As usual, the opposite events are denoted by $\bar{A}$ and $\bar{B}$. If two dice are thrown, then there are 36 possibilities: $(1,1),(1,2),\dots,(1,6),(2,1),\dots,(6,6)$. There are five favorable odds $(1,5),(2,4),(3,3),(4,2),(5,1)$ for the sum 6, and six favorable odds $(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$ for the sum 7. Then the probabilities of the described events are

$$P(A)=\frac{5}{36},\quad P(\bar{A})=1-P(A)=\frac{31}{36},$$
$$P(B)=\frac{6}{36}=\frac{1}{6},\quad P(\bar{B})=1-P(B)=\frac{30}{36}=\frac{5}{6}.$$

Gambler ***A*** will win if the following global event happens:

$$A+\bar{A}\bar{B}A+\bar{A}\bar{B}\bar{A}\bar{B}A+\bar{A}\bar{B}\bar{A}\bar{B}\bar{A}\bar{B}A+\cdots.$$

Particular events that appear as addends in the above sum are mutually exclusive, so that the probability of the global event is equal to the sum of the probabilities of particular events. Furthermore, the events appearing in the above products are mutually independent so that the probability of the product of events is equal to the product of probabilities of particular events. If p_A denotes the probability of ***A***'s win, then

$$\begin{aligned}p_A =&P(A)+P(\bar{A})P(\bar{B})P(A)+P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(A)\\&+P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(A)+\cdots\\=&\frac{5}{36}+\frac{31}{36}\cdot\frac{5}{6}\cdot\frac{5}{36}+\left(\frac{31}{36}\cdot\frac{5}{6}\right)^2\cdot\frac{5}{36}+\left(\frac{31}{36}\cdot\frac{5}{6}\right)^3\cdot\frac{5}{36}+\cdots\\=&\frac{5}{36}\left(1+x+x^2+x^3+\cdots\right)=\frac{5}{36}\cdot\frac{1}{1-x}=\frac{30}{61},\end{aligned}$$

where we put $x=\dfrac{31}{36}\cdot\dfrac{5}{6}=\dfrac{155}{216}$.

Gambler ***B*** wins if the following global event

$$\bar{A}B+\bar{A}\bar{B}\bar{A}B+\bar{A}\bar{B}\bar{A}\bar{B}\bar{A}B+\bar{A}\bar{B}\bar{A}\bar{B}\bar{A}\bar{B}\bar{A}B+\cdots$$

happens. Hence, in a way similar to the above, we calculate the probability p_B of $\boldsymbol{B}$'s win:

$$\begin{aligned} p_B =& P(\bar{A})P(B) + P(\bar{A})P(\bar{B})P(\bar{A})P(B) \\ &+ P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(\bar{A})P(B) \\ &+ P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(\bar{A})P(B) + \cdots \\ =& \frac{31}{36}\cdot\frac{1}{6} + \frac{31}{36}\cdot\frac{5}{6}\cdot\frac{31}{36}\cdot\frac{1}{6} + \frac{31}{36}\cdot\frac{1}{6}\cdot\left(\frac{31}{36}\cdot\frac{5}{6}\right)^2 \\ &+ \frac{31}{36}\cdot\frac{1}{6}\cdot\left(\frac{31}{36}\cdot\frac{5}{6}\right)^3 + \cdots \\ =& \frac{31}{36}\cdot\frac{1}{6}\left(1 + x + x^2 + x^3 + \cdots\right) = \frac{31}{36}\cdot\frac{1}{6}\cdot\frac{1}{1-x} = \frac{31}{61}. \end{aligned}$$

According to the obtained probabilities p_A and p_B we conclude that player $\boldsymbol{B}$ has a slightly greater chance of winning than his rival $\boldsymbol{A}$. By the way, it must be $p_A + p_B = 1$, which is true.

Blaise Pascal (1623–1662) (→ p. 303)

Pierre de Fermat (1601–1665) (→ p. 303)

Christiaan Huygens (1629–1695) (→ p. 303)

Gambler's ruin problem

This is a light introduction to the Gambler's ruin problem that attracted the attention of Pascal, Fermat and Huygens. The contributions of these originators of the theory of probability have been presented in the previous essays.

Two players A and B play a game tossing a fair coin, in other words, the probability of winning is 1/2 for each player. They each have a finite amount of dollars, say n_A and n_B. After each tossing of the coin, the loser gives one dollar to the winner. The game is over when one of the players holds all of the dollars in the game ($n_A + n_B$ in total). The respective chances P_A and P_B of gaining all the money for the players A and B are

$$P_A = \frac{n_A}{n_A + n_B}, \quad P_B = \frac{n_B}{n_A + n_B}.$$

As expected, the player possessing less money has the greater chance of going bankrupt. The long run success of casinos is indeed, in part at least, due to this simple principle since presumably casinos have much more money than their clients—gamblers. This game (or for some, a profit making enterprise), in which a gambler is ruined, has been known for more than 350 years and it is a hot topic even today. More details and a more complicated discussion is given in what follows.

Problem 8.3(a). *$\boldsymbol{A}$ and $\boldsymbol{B}$ each take* 12 *turns playing with* 3 *dice while observing the following rules: if* 11 *is thrown, $\boldsymbol{A}$ cedes one toss to $\boldsymbol{B}$; if* 14 *is thrown, $\boldsymbol{B}$ cedes one toss to $\boldsymbol{A}$. The winner is the player who first obtains the specified number of turns. Determine the probability of winning for players $\boldsymbol{A}$ and $\boldsymbol{B}$.*

Known as the gambler's ruin problem, this problem was posed by Pascal to Fermat, and through Carcavi to Huygens in a letter dated 28 September 1656 that includes Pascal and Fermat's answers (see [92, p. 76]). Huygens gave the following answer in a note from 1676 (contained in *Ouvres*, Vol. 14, pp. 151–155): $\boldsymbol{A}$'s chance compared to $\boldsymbol{B}$'s is 244,140,625 : 282,429,536,481.

The general formulation of the gambler's ruin problem is as follows:

Problem 8.3(b). *At the beginning of the play gamblers $\boldsymbol{A}$ and $\boldsymbol{B}$ have m and n chips, respectively. Let their probabilities of winning in each trial be p for $\boldsymbol{A}$, and $q = 1-p$ for $\boldsymbol{B}$. After each trial the winner gets a chip from the loser, and the play continues until one of the players is ruined. What is the probability of $\boldsymbol{A}$ being ruined?*

Struyck first gave a complete proof based on difference equations in 1716, using Jacob Bernoulli's recursion formula given in *Ars Conjectandi*. In modern notation, Struyck's solution may be expressed as follows:

Let $e(x)$ denote the expectation of player $\boldsymbol{A}$ when he has x chips, $x = 1, 2, \dots, m+n-1$, $e(0) = 0$, $e(m+n) = 1$. Bernoulli's recursion formula leads to the difference equation

$$\begin{aligned} &e(x) = pe(x+1) + qe(x-1), \ x = 1, 2, \dots, m+n-1, \\ &e(0) = 0, \quad e(m+n) = 1, \end{aligned}$$

which may be written as

$$pe(x+1) = (p+q)e(x) - qe(x-1),$$

wherefrom

$$e(x+1) - e(x) = \frac{q}{p}\big[e(x) - e(x-1)\big].$$

A successive application of the last relation gives

$$e(x+1) - e(x) = \left(\frac{q}{p}\right)^x e(1).$$

Hence, by summing we get,

$$e(m) = \sum_{x=0}^{m-1} \big[e(x+1) - e(x)\big] = \frac{\big[1-(q/p)^m\big]e(1)}{1-(q/p)}. \tag{8.3}$$

In an analogous way we find

$$e(m+n) = \sum_{x=0}^{m+n-1} \big[e(x+1) - e(x)\big] = \frac{\big[1-(q/p)^{m+n}\big]e(1)}{1-(q/p)}.$$

Taking $e(m+1) = 1$ in the last relation we determine

$$e(1) = \frac{1-(q/p)}{1-(q/p)^{m+n}}.$$

Returning with this entry in (8.3), Struyck obtained the explicit solution

$$P_A = e(m) = \frac{1-(q/p)^m}{1-(q/p)^{m+n}}. \tag{8.4}$$

By the same argumentation one gets a similar expression for P_B:

$$P_B = e(n) = \frac{1-(p/q)^n}{1-(p/q)^{m+n}}. \tag{8.5}$$

From (8.4) and (8.5) we obtain the ratio

$$\frac{P_A}{P_B} = \frac{p^n q^m - p^{m+n}}{q^{m+n} - p^n q^m} \quad \text{for} \quad m \neq n. \tag{8.6}$$

Jacob Bernoulli and Abraham de Moivre obtained the same result. De Moivre's solution, which is based on a completely different method, can be found in [92, pp. 203–204].

In a special case when $m = n$, we set $t = q/p$ and obtain from (8.6)

$$\begin{aligned}\frac{P_A}{P_B} &= \frac{p^m}{q^m} \cdot \frac{(q/p)^m - 1}{(q/p)^n - 1} = \frac{1}{t^m} \cdot \frac{(t-1)(t^{m-1} + \cdots + t + 1)}{(t-1)(t^{n-1} + \cdots + t + 1)} \\ &\to \frac{1}{t^m} = \frac{p^m}{q^m} \quad \text{when} \quad m \to n.\end{aligned}$$

Therefore, the expectations ratio of $\boldsymbol{A}$ and $\boldsymbol{B}$ is as $p^m : q^m$, which may also be shown by induction. In Huygens' problem the gamblers started with $m = n = 12$ chips and, according to the stated rules of play, the number of chances to win a point is 15 for $\boldsymbol{A}$ and 27 for $\boldsymbol{B}$ among 42 possible outcomes; thus, $p = 15/42 = 5/14$ and $q = 27/42 = 9/14$. Hence

$$\frac{P_A}{P_B} = \frac{(5/14)^{12}}{(9/14)^{12}} = \left(\frac{5}{9}\right)^{12} = \frac{244{,}140{,}625}{282{,}429{,}536{,}481},$$

which is in agreement with Huygens' answer.

Nicolaus III Bernoulli and Pierre de Montmort also considered the Gambler's ruin problem. The problem's history and Huygens' solution of a special case posed by Pascal may be found in the book [92].

Nicolaus III Bernoulli (1695–1726) (→ p. 305)

Daniel Bernoulli (1700–1782) (→ p. 305)

Gabriel Cramer (1704–1752) (→ p. 305)

and others

The Petersburg paradox

Many eighteenth- and nineteenth-century mathematicians were fascinated by a problem known as the Petersburg paradox. This problem, related to mathematical expectations in tossing a coin, appeared in the *Commentarii* of the Petersburg's Academy[4].

Problem 8.4. *Two players A and B play a game in which they toss a coin until it lands heads down. If this happens on the first throw, player A pays player B one crown. Otherwise, player A tosses again. If heads appear on the second throw, player A pays two crowns; if on the third throw, four crowns; and so on, doubling each time. Thus, if the coin does not land heads down until the nth throw, player B then receives 2^{n-1} crowns. How much should player B pay player A for the privilege of playing this game?*

There is a probability of 1/2 that player B will receive one crown, a probability of 1/4 that he will receive two crowns, and so on. Hence the total

[4]See E. Kamke, *Einführung in die Wahrscheinlichkeitstheorie*, Leipzig, 1932, pp. 82–89.

number of crowns that he may reasonably expect to receive (= mathematical expectation) is

$$\tfrac{1}{2}\cdot 1+\tfrac{1}{4}\cdot 2+\cdots+\left(\tfrac{1}{2}\right)^n 2^{n-1}+\cdots=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\ ,$$

that is, it is infinite! It is almost unbelievable, but true. The result comes like a bolt from the blue.

This unexpected result attracted the attention of many mathematicians including the brothers Daniel and Nicolaus III Bernoulli and a large group of French mathematicians. One of their contemporaries, Jean-le-Rond D'Alembert (1717–1783), also an influential mathematician, wrote to Joseph Louis Lagrange (1736–1813), another famous French mathematician: *"Your memoir on games makes me very eager that you should give us a solution of the Petersburg problem, which seems to me insoluble on the basis of known principle."*

On the one hand, French mathematicians Nicolas Condorcet (1743–1794) and Siméon Poisson (1781–1840) thought that player A entered into an engagement which he could not keep and that the game contained a contradiction. Another French mathematician Joseph Bertand (1822–1900) asserted that the theory, and the result given above, were quite correct and that only the conditions of the game favored player B, leading to the unexpected result. According to him, if the number of throws is limited, then the chances are different. For example, for a hundred games player B's stake is about 15 crowns and he now runs a greater risk of losing. The conditions of the game are still to his advantage as a result of the possibility of large profits, although the probabilities are slight. B's stake depends on the number of games that A is obliged to play. If this number is n, Bertrand calculated that B's stake is $\log n/(2\log 2)$. Bertrand's extended commentary may be found in Kraitchik's book [118].

Some mathematicians offered a solution based on the practical reasoning that since a player's fortune is necessarily finite, the sum, accordingly, cannot be unlimited. The Comte de Buffon (1707–1788) even carried out empirical tests to find an average amount. Daniel Bernoulli sought to resolve the problem through his principle of *moral expectation,* in accordance with which he replaced the amounts

$$1,\ 2,\ 2^2,\ 2^3,\ldots \quad \text{by} \quad 1^{1/2},\ 2^{1/4},\ 4^{1/8},\ 8^{1/16},\ldots\ .$$

Actually, he considered that the worth of a fortune depends not only on the number of crowns it accrues, but also on the satisfaction that it can give. According to his approach, 100 million crowns added to an already-acquired fortune of 100 million, is not sufficient to double the original fortune. Employing his own principle of moral expectation, D. Bernoulli derived the following calculation (see Kraitchik [118]): *"If a given fortune x is increased by an amount dx, the worth of the increase is dx/x. Hence if my fortune increases from an amount a to an amount b, I have gained an advantage which can be measured by*

$$\int_a^b \frac{dx}{x} = \log_e b - \log_e a = \log_e \frac{b}{a} \text{ ."}$$

Some references[5] attribute the *Petersburg paradox* to Nicolaus II Bernoulli (1687–1759), a nephew of Jacob and Johann Bernoulli. The French scientist Pierre-Simon de Laplace (1749–1827) and the aforementioned French mathematicians also studied this problem. In addition, I. Todhunter,[6] E. Kamke,[7] M. Kraitchik [118], Rouse Ball and H. S. M. Coxeter [150] all discussed details of the Petersburg paradox.

Among the various modifications of the Petersburg problem in order to get an finite answer, Gabriel Cramer advanced one of the more satisfactory ones in about 1730.[8] Cramer assumed that player A's wealth was limited.

Problem 8.5.* (Cramer) *Player A's wealth was limited to 2^{24} = 16,777,216 crowns. How much should player B pay player A for the privilege of playing the game described in Problem 8.4?*

Nicolaus II Bernoulli (1687–1759) (→ p. 304)

Leonhard Euler (1707–1783) (→ p. 305)

The probability problem with the misaddressed letters

Many textbooks consider the probability theory task that we present below as closely resembling the Bernoulli–Euler problem presented previously in Chapter 7.

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[5] See, *e.g.*, A. N. Bogol'bov, *Mathematics, Mechanics – Biographical Dictionary* (in Russian), Kiev 1983, p. 44.

[6] *A History of the Mathematical Theory of Probability*, London 1865.

[7] *Einführung in die Wahrscheinlichkeitstheorie*, Leipzig, 1932.

[8] I. Todhunter, *A History of the Mathematical Theory of Probability*, London, 1865, p. 221.

Problem 8.6. *A girl writes k letters to k friends, and addresses the k corresponding envelopes. With her eyes closed, she randomly stuffs one letter into each envelope. What is the probability that just one envelope contains the right letter, and the other $k-1$ each contain the wrong letter?*

The problem's solution corresponds exactly to the solution of the misaddressed letters problem studied in the previous chapter, page 184. The number of ways of getting one letter into the right envelope and the other $k-1$ letters each into the wrong envelope is $k \cdot M(k-1)$, where the number of allowable rearrangements $M(r)$ is defined in the mentioned problem on page 184. The number of all possibilities is equal to the number of various arrangements of k objects, which makes

$$k! = 1 \cdot 2 \cdot 3 \cdots k.$$

Therefore, the required probability is

$$P_k = \frac{k \cdot M(k-1)}{k!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{(-1)^{k-1}}{(k-1)!}. \tag{8.7}$$

Knowing the Taylor development of the function e^x,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

we find (for $x = -1$)

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots.$$

Comparing the last expression with P_k we conclude that for relatively large k we have $P_k \approx e^{-1} = 0.36787944\ldots$.

The following variation of the problem with the misaddressed letters has an unexpected solution.

Problem 8.7.* *A girl writes k letters to k friends, and addresses the k corresponding envelopes. Then she randomly stuffs the letters in the envelopes. What is the probability that at least one envelope contains the right letter? Find the greatest k such that the probability is greater than $1/2$.*

Another variation, proposed by Karapetoff [112] in 1947, is actually a generalization of Problem 8.6.

Problem 8.8.* *A girl writes k letters and addresses the k corresponding envelopes. Then she randomly puts one letter into each envelope. What is the probability that exactly m envelopes contain the right letters, and the other $k-m$ each contains the wrong letter?*

Stephen Banach (1892–1945) (→ p. 308)

Matchbox problem

The famous Polish mathematician Stephen Banach was the founder of modern functional analysis and a great contributor to several branches of mathematics (the Han–Banach theorem, the Banach–Steinhaus theorem, Banach space, Banach algebra, the Banach fixed-point theorem). Banach's following "definition" of a mathematician is often quoted:

"A mathematician is a person who can find analogies between theorems, a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is the one who can see analogies between analogies."

Stephen Banach posed and solved this problem.

Problem 8.9. *A man carries two boxes of matches in his pocket. Every time he wants to light a match, he selects one box or the other at random. After some time the man discovers that one of the boxes is empty. What is the probability that there will be exactly k matches at that moment in the other box if each box originally contained n $(\geq k)$ matches?*

Solution. Let M_k stand for the event that "there is exactly k matches in one of the boxes in the moment when the man observed that the other box is empty." The problem is equivalent to the problem of determining the probability that at the moment when the man attempts to remove the $(n+1)$st match from the box A, exactly $n-k$ matches are taken from the box B. Considering the problem in this form, we see that we can increase the number of matches in the boxes. Moreover, we will assume that each box contains an infinite number of matches.

Now we can consider the experiment of choosing $n+(n-k)=2n-k$ matches, selecting the box from which each match is taken at random. Since the $(2n-k+1)$st match is assumed to come from A, its selection is not part of the experiment. There are 2^{2n-k} equally probable outcomes of this experiment. The favorable outcomes are those in which n matches are chosen from A and the remaining $n-k$ from B. This can be done in $\binom{2n-k}{n}$ ways, and so the desired probability is

$$p(M_k) = P_k = \frac{1}{2^{2n-k}}\binom{2n-k}{n}. \tag{8.8}$$

The following question naturally arises:

Problem 8.10.* *What is the most likely number of matches remaining in the box at the exact moment when the other box is empty? In other words, determine* $\max P_k$ $(n \geq k \geq 1)$.

Here is another related problem of the combinatorial type.

Problem 8.11.* *Using the solution to Banach's matchbox problem, find the following sum:*

$$S_n = \binom{2n}{n} + 2\binom{2n-1}{n} + 2^2\binom{2n-2}{n} + \cdots + 2^n\binom{n}{n}.$$

Answers to Problems

8.5. There is a probability of $1/2^n$ that player B will receive 2^{n-1} crowns at the nth throw only as long as $n < 25$; thereafter he will receive merely 2^{24} crowns. Since

$$\sum_{n=1}^{24}\left(\frac{1}{2}\right)^n \cdot 2^{n-1} + \sum_{n=25}^{\infty}\left(\frac{1}{2}\right)^n 2^{24} = 12 + 1 = 13,$$

the player's expectation is 13 crowns, a reasonable amount.

8.7. Let p_k be the probability that at least one letter (among k) will go to the right address. Assume that the letters and envelopes are numbered from 1 to k. Let A_i be the event that the ith letter goes into the right envelope i. Using the formula for the product of mutually independent events we obtain

$$P(A_{i_1}A_{i_2}\cdots A_{i_r}) = \frac{1}{k}\cdot\frac{1}{k-1}\cdots\frac{1}{k-r+1}. \tag{8.9}$$

Now we will apply the formula for the sum of events (the derivation of it is beyond the scope of this book):

$$p_k = P\left(\sum_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<\lambda} P(A_iA_jA_\lambda) + \cdots + (-1)^k P(A_1A_2\cdots A_k). \tag{8.10}$$

Plugging (8.9) into (8.10) gives the required probability

$$p_k = 1 - \binom{k}{2}\frac{1}{k}\cdot\frac{1}{k-1} + \binom{k}{3}\frac{1}{k}\cdot\frac{1}{k-1}\cdot\frac{1}{k-2} + \cdots + (-1)^{k-1}\binom{k}{k}$$
$$= 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{k-1}\frac{1}{k!}.$$

It does not hurt to find that $p_2 = 0.5$ and $p_k > 0.5$ for all $k \geq 3$ with

$$\lim_{k\to\infty} p_k = 1 - e^{-1} \approx 0.632121.$$

The probability that the considered event will *not* happen, that is, none of the letters will be put into the right envelope, is given by

$$\bar{p}_k = 1 - p_k = \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^k\frac{1}{k!} = \sum_{i=0}^{k}(-1)^i\frac{1}{i!}. \tag{8.11}$$

Warning: The last expression differs from (8.7). One more remark. The result (8.11) finds its application in solving Problem 8.8.

8.8. Assume that exactly m (among k) letters have been stuffed into the correct envelopes. The probability that each of them is contained in the right envelope is

$$p_1 = \frac{1}{k}\cdot\frac{1}{k-1}\cdots\frac{1}{k-m+1};$$

see (8.9).

According to (8.11), the probability that none of the remaining $k-m$ letters is put into the right envelope is

$$p_2 = \sum_{i=0}^{k-m}(-1)^i\frac{1}{i!}.$$

Therefore, the probability that exactly m letters are put into the right envelopes is given by

$$p_3 = p_1 p_2 = \frac{(k-m)!}{k!}\sum_{i=0}^{k-m}\frac{(-1)^i}{i!}. \tag{8.12}$$

Since there are $\binom{k}{m}$ ways of choosing m letters among k letters with the probability p_3, using (8.12) it follows that the required probability is

$$p = \binom{k}{m} p_3 = \frac{1}{m!}\sum_{i=0}^{k-m}\frac{(-1)^i}{i!}.$$

In particular, for $m = 1$ we obtain the solution (8.7) of Problem 8.6.

8.10. We will prove that the sequence of probability $\{P_k\}$ given by (8.8) is monotonically decreasing, that is, $P_{k+1} < P_k,\ k \geq 1$. Starting from the obvious inequality

$$2(n-k) < 2n-k, \ (n \geq k \geq 1),$$

after simple manipulation we obtain

$$\frac{2(2n-k-1)!}{(n-k-1)!n!} < \frac{(2n-k)!}{(n-k)!n!},$$

that is,

$$P_{k+1} = \frac{1}{2^{2n-k-1}}\binom{2n-k-1}{n} < \frac{1}{2^{2n-k}}\binom{2n-k}{n} = P_k$$

and the proof is completed. Some mathematicians would emphasize the end of the proof using w^5, as a witty abbreviation of *"which was what was wanted."* Actually, this notation is never found in academic publications and Halmos' □ or Q.E.D. (*quod erat demonstrandum*—that which was to be proved) are used instead.

The proof of the inequality $P_{k+1} < P_k$ $(k \geq 1)$ simply means that $\max P_k = P_1$, that is, when the man discovers that one of the boxes is empty, the other box will contain most probably **one** match.

8.11. The sum S_n is somewhat difficult to evaluate by standard methods. For this reason, we will use the solution (8.8) of Banach's problem. The events $M_0, M_1, \dots, M_n$ are disjunct and make a complete set of events (see the solution of Problem 8.9), that is,

$$M_0 + M_1 + \cdots + M_n \text{ is a certain event with probability of } 1.$$

Therefore,

$$p(M_0 + M_1 + \cdots + M_n) = \sum_{k=0}^{n} p(M_k) = \sum_{k=0}^{n} P_k = 1.$$

Hence, using (8.8), we obtain

$$\frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\binom{2n-1}{n} + \frac{1}{2^{2n-2}}\binom{2n-2}{n} + \cdots + \frac{1}{2^n}\binom{n}{n} = 1.$$

Multiplying the above relation with 2^{2n}, we obtain

$$S_n = \binom{2n}{n} + 2\binom{2n-1}{n} + 2^2\binom{2n-2}{n} + \cdots + 2^n\binom{n}{n} = 2^{2n}.$$

Chapter 9 GRAPHS

Computers allow us to solve very large problems concerning graphs, while graph theory helps advance computer science.
Narsingh Deo

The full potential and usefulness of graph theory is only beginning to be realized.
Gary Chartrand

In the introductory Chapter 1 we mentioned that some entertaining problems and puzzles were the starting points for important mathematical ideas and results. This chapter begins with a very representative example, Euler's solution of the problem of Königsberg's seven bridges in 1736. It is regarded as the beginning of graph theory, a new and fundamental topic in discrete mathematics and computer science. It is maybe of interest to note that the term "graph" was introduced by Sylvester in a paper published 1878 in the prestigious scientific journal *Nature*, many years after Euler's pioneering work. Watch out though: this term should not be confused with "graphs of functions".

A graph is a set of points (*graph vertices* or *nodes*) and lines (*graph edges*) linking the subset of these points. Graph theory is the study of graphs. Appendix C gives some basic facts and definitions of graph theory. Nowadays, graph theory is extremely valuable not only in the field of mathematics, but also in other scientific disciplines such as computer science, electrical engineering, network analysis, chemistry, physics, biology, operations research, social sciences, to name only some. Remember that the four-color problem, one of the most famous mathematical problems ever, was solved using graph coloring.

In addition to Euler's problem of Königsberg's bridges and its variants, you will find in this chapter the study of various kinds of paths in graphs: Eulerian paths, Eulerian cycles, Hamiltonian paths and Hamiltonian cycles. We use graphs to study Hamilton's puzzle game called Icosian. This game consists in the determination of a path passing each dodecahedron's vertices once and only once, known as Hamiltonian path (or cycle, if it is closed). We recall that the famous knight's re-entrant route on chessboards, described in Chapter 10, is a Hamiltonian cycle. Further, we show that a graph can be useful to model and resolve many common situations, such as diagram-tracing puzzles, river-crossing problems, fluid-measuring puzzles and chess problems. Besides Euler and Hamilton, the renowned scientists Tait, Poinsot, Poisson, Listing and Erdös appear as the authors and solvers of the problems presented in this chapter.

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Leonhard Euler (1707–1783) (→ p. 305)

The problem of Königsberg's bridges

The Prussian town of Königsberg, the present-day Russian city of Kaliningrad, is situated on both banks of the river Pregel (Figure 9.1). There are two islands in the river and seven bridges connecting the islands and river banks, as shown in Figure 9.2. The inhabitants of Königsberg have long amused themselves with this intriguing question:

Problem 9.1. *Can one stroll around the city crossing each bridge exactly once?*

In spite of all efforts, no one has ever succeeded, either in accomplishing this feat or in proving the possibility of doing so.

FIGURE 9.1. Old Königsberg and its seven bridges

Leonhard Euler solved the problem in 1736[1] by proving the impossibility of the Königsberg bridges walk. His proof is usually regarded as the beginning of a new and fruitful mathematical discipline, *graph theory*.

The Königsberg bridges problem bears a resemblance to the following problem (see Figure 9.3, for demonstration):

Draw a set of line segments to join the points located in the plane. Can this diagram be traversed without lifting the pencil from the paper by tracing over each point exactly once?

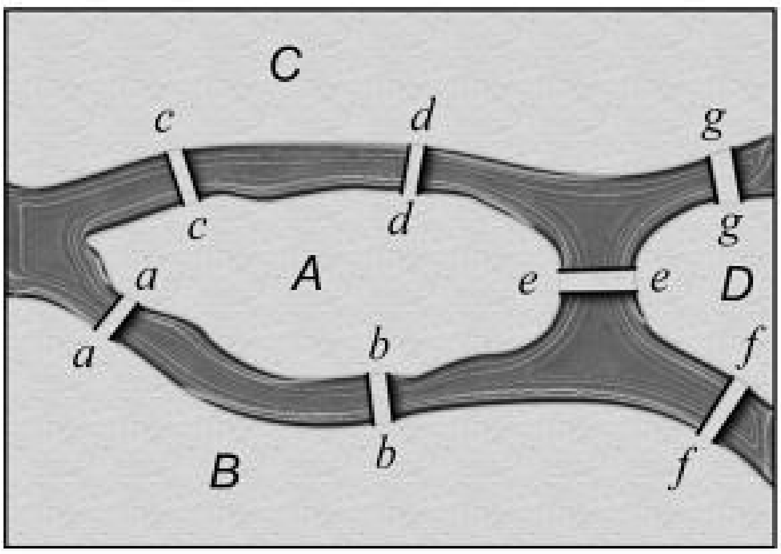

FIGURE 9.2 Königsberg's bridges

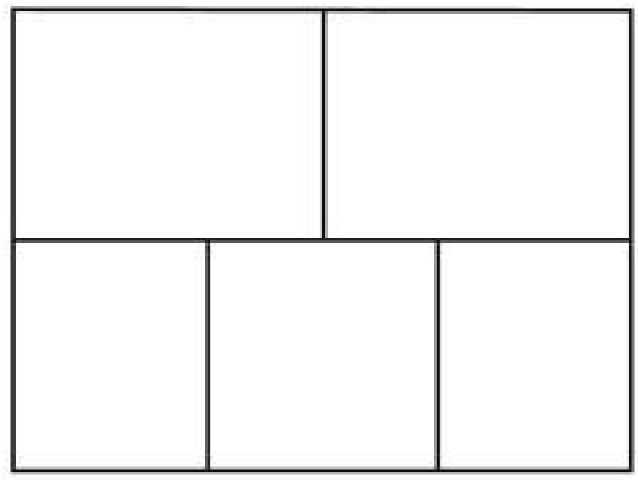

FIGURE 9.3. Drawing of the figure

Euler's solution of the Königsberg bridges problem and that of traversing the graph "by one move" (that is, "without lifting the pencil") is based on the following assertion:

Euler's theorem. *A* (*simple or multiple*) *graph can be traversed without lifting the pencil, while tracing each edge exactly once, if and only if it has not more than two odd vertices.*

Graphs that can be traversed under the required conditions are called *Eulerian graphs* and the corresponding path is called *Euler's path*. Thus, Euler's path exists if and only if there are 0 or at most 2 odd vertices. If the number of odd vertices is 0 (thus, all vertices are even), then Euler's path can be drawn starting from any vertex and ending at the same vertex. In the case when there are 2 odd vertices Euler's path starts from one of these odd vertices and ends at the other.

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[1] *Solutio problematis ad Geometriam situs pertinentis*, Commentarii Academiae Scientiarum Petropolitanae for 1736, Petersburg, 1741, Vol. 8, pp. 128–140.

The graph given in Figure 9.3 has 8 odd vertices and, therefore, cannot be traversed without lifting the pencil. Figure 9.4 shows "Königsberg's multigraph" where the islands and the banks are represented by vertices, and the bridges by edges. This graph has four vertices of odd degree and, according to Euler's theorem, it is not an Eulerian graph. Consequently, one cannot achieve the proposed tour across Königsberg's seven bridges.

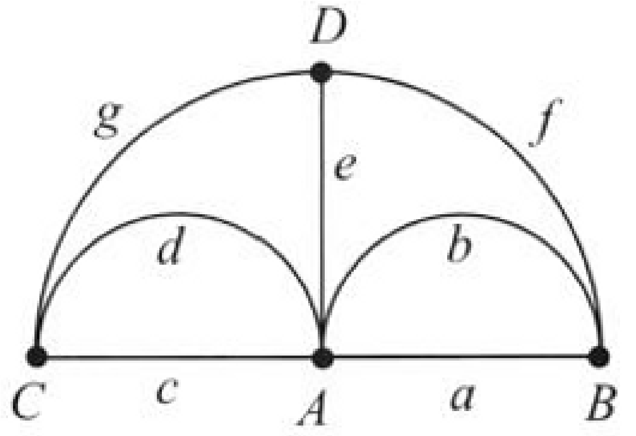

FIGURE 9.4. Graph of Königsberg's bridges

Another more complicated problem related to the application of Euler's theorem bears a resemblance to the Köningsberg bridges puzzle. Adapted from Euler's paper,[2] it requires the crossing of fifteen bridges:

Problem 9.2.* *Fifteen bridges connect four islands A, B, C and D to each side of the river, as well as linking the islands themselves as shown in Figure 9.5. Can one take a stroll in which each bridge is crossed once and only once?*

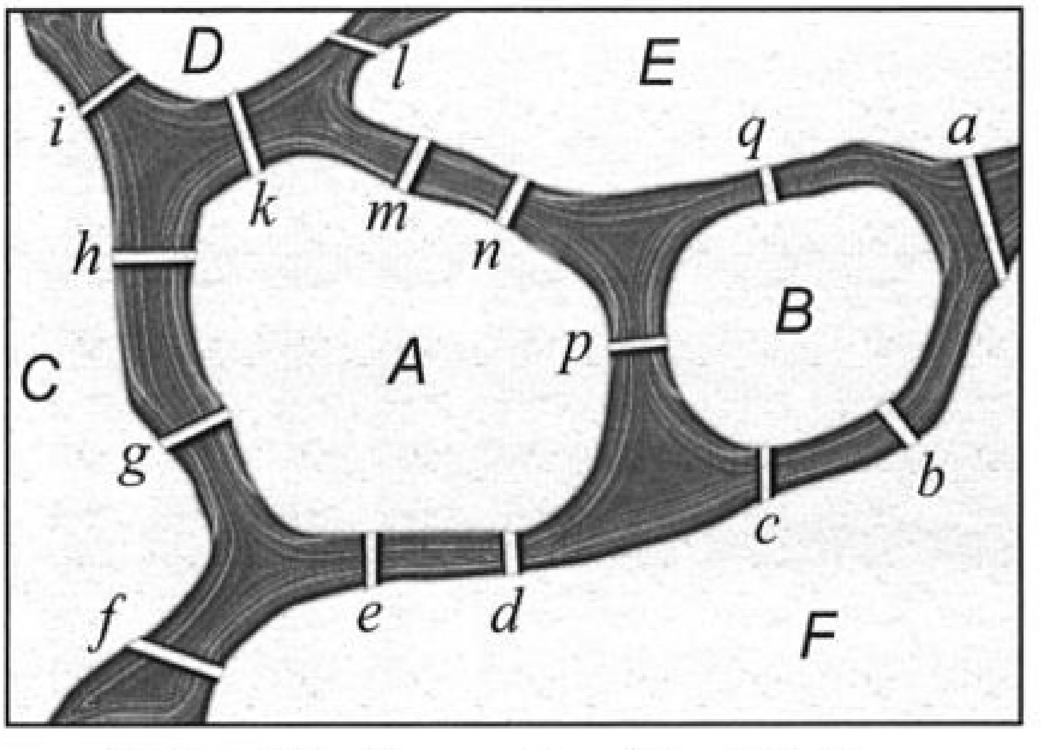

FIGURE 9.5. The crossing of the 15 bridges

[2] According to R. J. Wilson, J. J. Watkins [188, p. 124].

After Euler's theorem, it is easy to be smart. We encourage the reader to find the required path across the fifteen bridges using Euler's theorem and the graph method.

Tait's article[3] *Listing's Topologie*, which appeared in the January 1884 edition of the *Philosophical Magazine* (London), published another problem, also connected with Euler's path.

Problem 9.3.* *Without lifting pencil from paper, can one draw the figure shown in Figure* 9.6 *while tracing over the edges exactly once?*

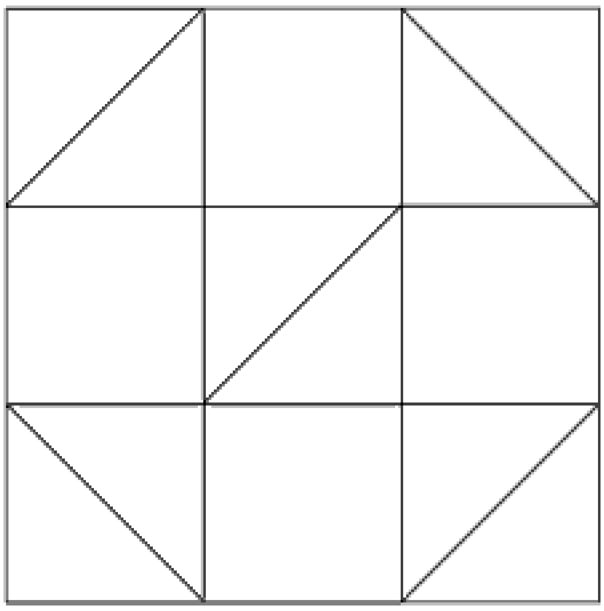

FIGURE 9.6. Tait's net

William R. Hamilton (1805–1865) (→ p. 306)

Several notions in mathematics and mechanics bear the name of William Rowan Hamilton who, with mathematicians James J. Sylvester and Arthur Cayley, whom we will discuss later, gained considerable prominence in the mathematical world of nineteenth-century Britain.

Hamilton, whose education had been guided by his linguist uncle, mastered foreign languages as a child. A child prodigy, Hamilton could read Greek and Hebrew at five as well as reading and writing Latin, French and Italian; at the age of ten he was studying Arabic and Sanskrit; at twelve he had a working knowledge of all these languages, not to mention Syriac, Persian, Hindustani, and Malay. A turning point came in Hamilton's life at the age of 12 when he met the American mental prodigy Zerah Colburn. Colburn could perform amazing mental arithmetical feats and Hamilton put

[3] Peter G. Tait (1831–1901), an outstanding Scottish physicist and mathematician.

his arithmetical abilities to the test in competing with Colburn. Perhaps losing to Colburn sparked Hamilton's interest in mathematics.

Hamilton made very important contributions to various branches of mathematics, but also to optics and mechanics. However, he regarded that his discovery of quaternions in 1843 was his greatest achievement and he devoted the two last decades of his life to working on this subject. However, it has turned out that the theory of quaternions has not found wide application. Exhausted by hard work and disappointed in his private life, Hamilton often sought solace in drinking. After his death, his working room was found in a chaotic condition, full of heaps of unfinished manuscripts and the remains of food. In his book *Men of Mathematics*, E. T. Bell [15] subtitled his chapter on Hamilton, *"An Irish tragedy."*

William R. Hamilton
1805–1865

Hamilton's game on a dodecahedron

As noted above, the history of graph theory started from a recreational mathematics problem. A century later, in a manner similar to Euler's graphs, a game provided the inspiration for yet another type of graph. In 1856 William R. Hamilton proposed an interesting graph problem and three years later turned it into a puzzle game called *Icosian* on a dodecahedron.[4]

	vertices	edges	faces
Tetrahedron	4	6	4 △
Octahedron	6	12	8 △
Cube	8	12	6 □
Icosahedron	12	30	20 △
Dodecahedron	20	30	12 ⬠

TABLE 9.1. Platonic solids

Before going any further, let us digress for a moment to recall the basic features of Platonic solids (also called regular polyhedrons) that may be useful later. A Platonic solid is a convex polyhedron whose faces are congruent

[4]Actually, Hamilton sold the idea of the icosian game to J. Jacques and Son, makers of fine sets, for £25, but it turned out to be a bad bargain—for the dealer.

convex regular polygons. There are only five convex regular polyhedrons whose number of vertices, edges and faces is given in Table 9.1. From this table we see that the dodecahedron is one of the five regular polyhedrons, which has 12 faces, 20 vertices and 30 edges (Figure 9.7). All the faces of the dodecahedron are regular pentagons, three of them joining at each vertex.

A. Beck, M. N. Bleicher and D. W. Crowe pointed in the book *Excursion into Mathematics* [11] that the regular polyhedrons resemble to a significant extent the skeletons of the minute marine animals called *radiolaria*. An example of a dodecahedron and the skeleton of a radiolaria are shown in Figures 9.7 and 9.8.

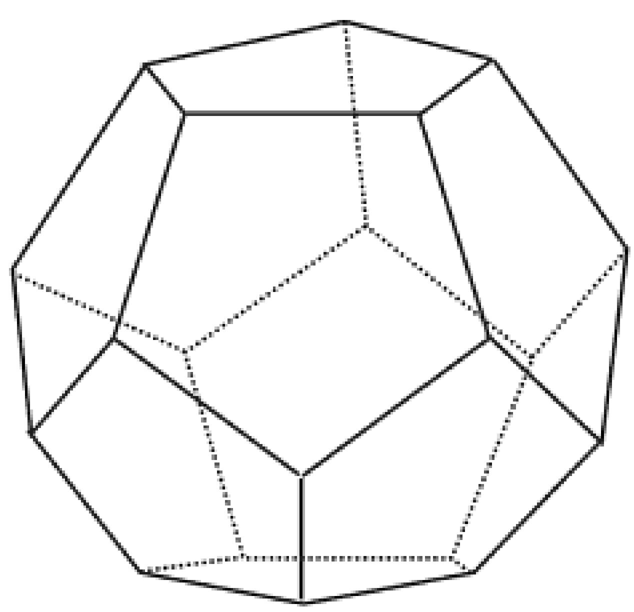

FIGURE 9.7. Dodecahedron

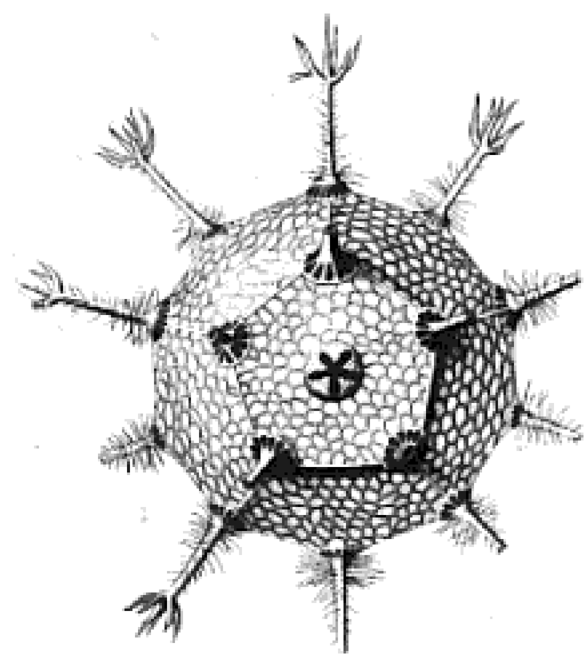

FIGURE 9.8. Radiolaria—dodecahedron

Back to the icosian puzzle. Here is Hamilton's task:

Problem 9.4. *Map a path passing through each of the dodecahedron's vertices once and only once.*

Notice that an edition of Hamiltonian's board game on a dodecahedron named *A Voyage Round the World*, treated the vertices of the dodecahedron as exotic locations all over the world. A traveler must visit each of these locations once and only once.

In solving Hamilton's task, for the sake of clarity and convenience, rather than considering a dodecahedron we shall alter its orientation to that of its stereographic projection, in essence, a dodecahedron's skeleton in planar graph form as shown in Figure 9.9.

Paths as that required in Hamiltonian games are called *Hamiltonian paths* after Hamilton, and graphs with this property are called *Hamilton graphs.*

A cyclic Hamiltonian path which finishes at the starting point is called a *Hamiltonian cycle.*[5] Hamiltonian cycles have several important applications in traffic optimization problems and communication networks.

One Hamiltonian cycle along the skeleton of dodecahedron is represented on Figure 9.9 by the thick line segments coinciding with the letter sequence

$$A-B-C-D-E-F-G-H-I-J-K-L-M-N-O-P-Q-R-S-T-A$$

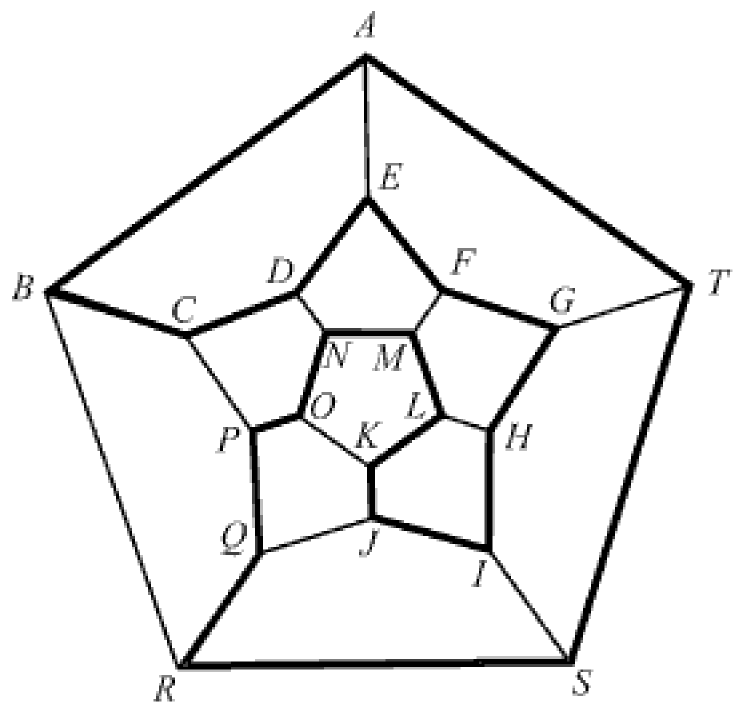

FIGURE 9.9. Hamiltonian cycle

Problem 9.5.* *A variation on Hamilton's game involves determining Hamiltonian cycles given five initial letters. Starting with the initial letters, say, $NMFGH$, the reader should complete a Hamiltonian cycle.*

It is easy to find a Hamiltonian cycle on the 3-cube, as shown in Figure 9.10. Introducing the xyz coordinate system, the presented route can be expressed by a sequence of coordinates (x, y, z); see Figure 9.10 right. As noted in [11], there is a remarkable similarity between the Hamiltonian cycle on the 3-cube and a very familiar Tower of Hanoi puzzle (considered in Chapter 7, pages 196–199). Indeed, let x, y, z stand for the smallest, medium and largest disk of the Tower of Hanoi having three disks, and let (x, y, z)

[5]According to N. L. Briggs, E. K. Lloyd and R. J. Wilson, questions of priority arise concerning the works of W. R. Hamilton and T. P. Kirkman. Briggs, Lloyd and Wilson remarked in their book [27] that Hamilton was concerned with one special case, whereas Kirkman concurrently studied a more general problem: spanning cycles on general convex polyhedra. Nevertheless, the spanning cycles of a graph are now known as Hamiltonian cycles, not as Kirkman cycles.

denote a disk-position. Assuming that these coordinates take the values 0 or 1, the change of one (and only one) coordinate from 0 to 1, or opposite, means that the corresponding disk has been moved. For example, from the two consecutive disk-positions 110 and 010 one concludes that x-disk (the smallest one) has been moved. As we can see from Figure 9.10 right, the solution in 7 moves of the Tower of Hanoi with three disks is given by the same sequence as the vertices of a Hamiltonian cycle on the 3-cube. What is really surprising is that a general assertion is valid (see [11]):

Theorem 9.1. *The simplest solution of the Tower of Hanoi with n disks, consisting of $2^n - 1$ moves, coincides with a Hamiltonian cycle on the n-cube.*

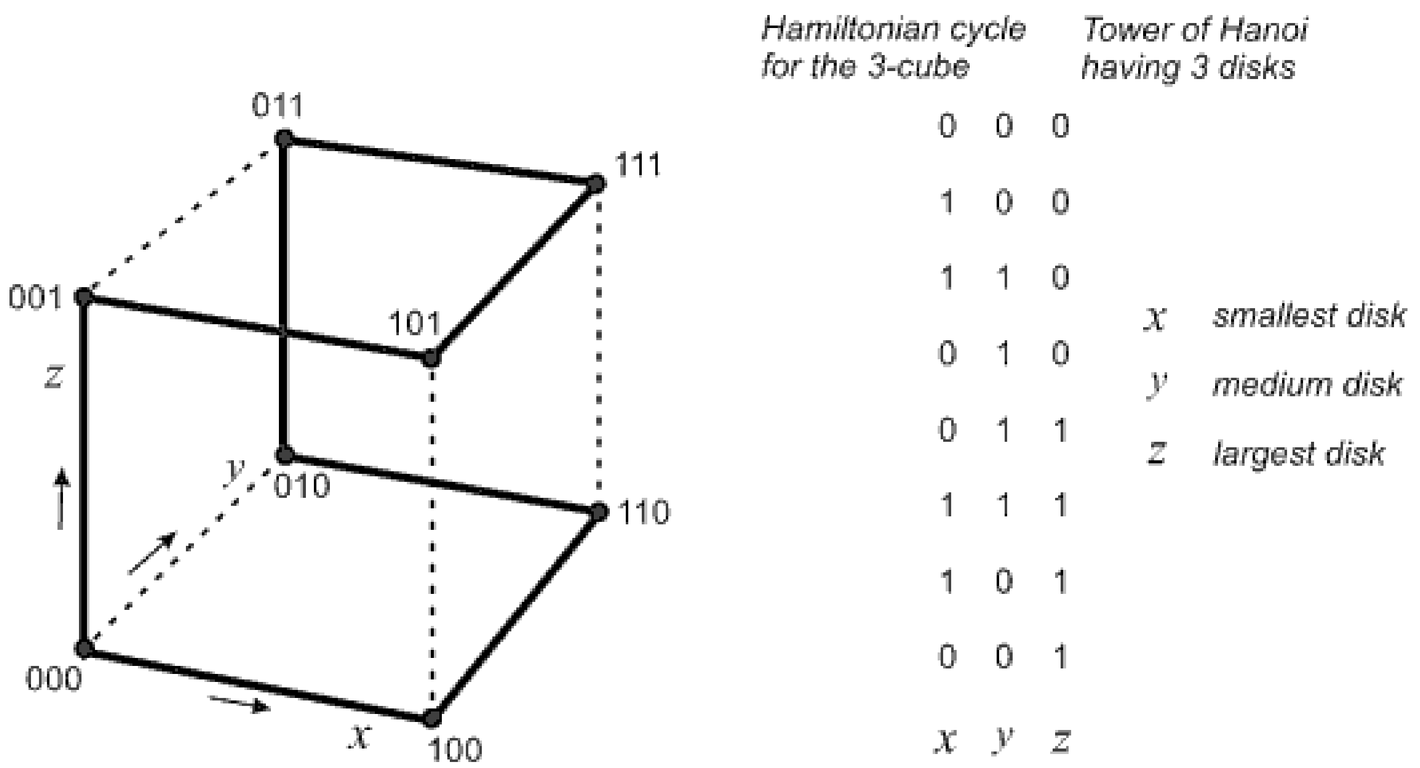

FIGURE 9.10. Hamiltonian cycle on the 3-cube and the Tower of Hanoi solution

Hamilton's original game uses a dodecahedron. The following natural question arises:

Problem 9.6.* *Is it possible to find Hamiltonian cycles on all the Platonic solids, presented in Table* 9.1*?*

Here is another task connected with a dodecahedron whose solution can be found in an easy way using graph theory (see Exercises in [11, p. 18]):

Problem 9.7.* *Can you color the faces of a regular dodecahedron with three colors in such a way that no two neighboring faces have the same color?*

As we have seen, Euler's theorem provides a method allowing for simple characterization of Euler's graphs. However, a similar criterion for Hamilton's graphs has not yet been stated and it thus remains one of the most challenging unsolved problems in graph theory. Graphs that might contain Hamiltonian paths exhibit no particular known features to distinguish them; furthermore, they hold that even in the event that a Hamiltonian path exists, no simple algorithm could be used to determine its existence. Instead, they maintain, proving the existence of a Hamiltonian path would involve considering graphs on an individual case-by-case basis, and even then such a discovery would very likely result from perceptive guesswork and calculated experimentation (see Averbach and Chein [6, Ch. 6]).

We owe the reader an explanation of why modern very powerful computers cannot find Hamiltonian paths (if they exist at all) within a reasonable amount of time. Of course, we restrict ourselves to a finite set of objects that must be traversed for which there are finitely many paths. How large is the number of these paths? Let us assume that twenty cities are given and each of them must be visited exactly once. With a little help of elementary combinatorics we find that there are

$$20 \cdot 19 \cdot 18 \cdots 3 \cdot 2 \cdot 1 = 2{,}432{,}902{,}008{,}176{,}640{,}000$$

possible trips. If we assume that a very fast computer may find one billion (10^9) trips in one second, then it will need 77 years to examine all trips. Quite unacceptable for real-life problems! The problem becomes considerably more complicated if the distances between cities should be taken into account and we want to find the shortest Hamiltonian path. In this way we come to the famous (unsolved, of course) *traveling salesman problem* (see [13] for a popular exposition of this problem).

A number of results have been established concerning sufficient conditions for Hamilton graphs, and here we include two of the most important sufficient conditions formulated by G. A. Dirac (1952) and O. Ore (1960), using the notion of the node degree $\deg(\cdot)$ (see Appendix C).

Dirac's theorem. *Let G be a simple graph with n ($n \geq 3$) vertices. If $\deg(v) \geq n/2$ for each vertex v belonging to G, then G is a Hamilton graph.*

Ore's theorem. *Let G be a simple graph with n ($n \geq 3$) vertices. If $\deg(v) + \deg(w) \geq n$ for each pair of nonadjacent vertices v and w belonging to G, then G is a Hamilton graph.*

Let us note that Dirac's theorem is a special case of Ore's theorem. The proof of Ore's theorem can be found, *e.g.*, in [188]. The following problem nicely illustrates the application of Dirac's theorem.

Problem 9.8. *King Arthur gathered his $2n$ knights of the Round Table to prepare for an important council. Each knight has at most $n-1$ enemies among the knights. Can the knights be seated at the Round Table so that each of them has two friends for neighbors?*

We solve this problem by constructing a graph with $2n$ vertices, where each vertex represents one knight. Two vertices are connected if and only if the corresponding knights are friends. As stated by the conditions given, each knight has at least n friends. This means that the degree of each vertex is at least n. Since $\deg(G) = n = 2n/2$, according to Dirac's theorem, it follows that graph G has a Hamiltonian cycle. This cycle gives the required arrangement of the knights.

To illustrate the point, we present a particular case of 8 knights (that is, $n = 4$). We designate the knights by the capital letters A, B, C, D, E, F, G, H. Each knight has exactly 3 enemies given in the parentheses below.

$\boldsymbol{A}\ (B, E, F)$, $\boldsymbol{B}\ (A, D, G)$, $\boldsymbol{C}\ (F, G, H)$, $\boldsymbol{D}\ (B, E, H)$
$\boldsymbol{E}\ (A, D, G)$, $\boldsymbol{F}\ (A, C, H)$, $\boldsymbol{G}\ (B, C, E)$, $\boldsymbol{H}\ (C, D, F)$

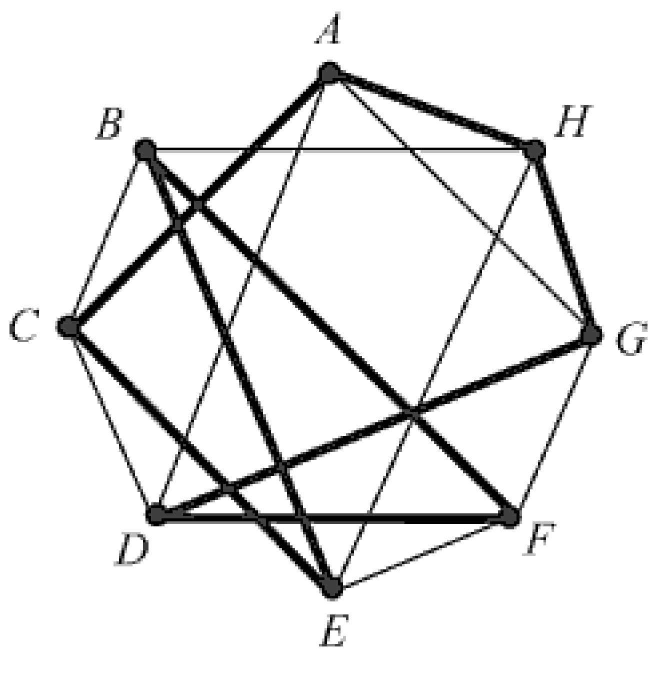

FIGURE 9.11. King Arthur's graph

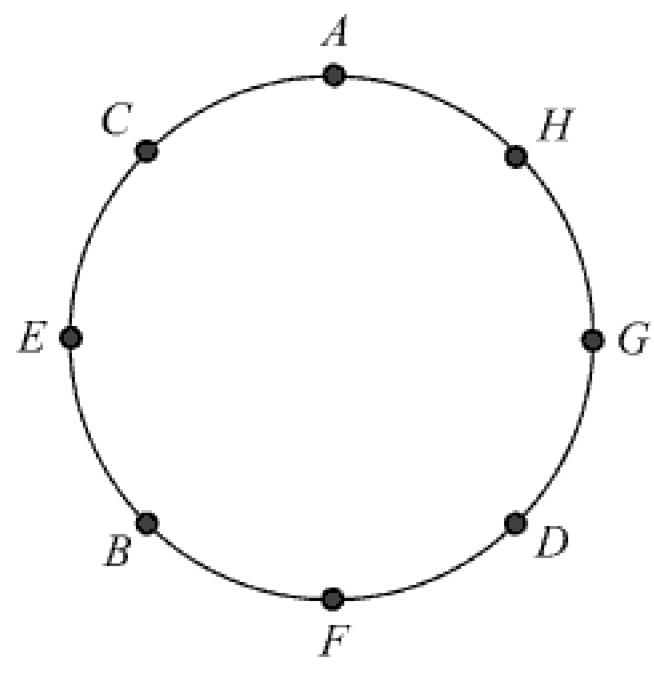

FIGURE 9.12. Arrangement of knights

According to our "lists of enemies" we can construct the corresponding graph given in Figure 9.11. Then we find a Hamiltonian cycle, marked by the thick line starting from A. Taking the vertices along this Hamiltonian cycle and keeping the ordering of vertices, we obtain the arrangement of knights shown in Figure 9.12.

Let us note that B. Averbach and O. Chein in [6, Pog. 6] considered a version of the problem of arranging King Arthur's knights. The famous *knight's tour problem*, considered by Euler, de Moivre, de Montmort, Vandermonde and other outstanding mathematicians, is an earlier example of a problem which can be expressed in terms of Hamiltonian cycles. We discuss this problem on pages 258–264.

Alcuin of York (735–804) (→ p. 299)

A man, a wolf, a goat and a cabbage

As previously discussed, river-crossing problems appear as a recurrent theme in recreational mathematics. One familiar twist on these well-known river-crossing problems involves a man, a wolf, a goat, and a cabbage. This problem dates back at least to the eighth century when it appeared for the first time in a booklet, very likely written by Alcuin of York.

Problem 9.9. *A man wishes to ferry a wolf, a goat, and a cabbage across a river in a boat that can carry only the man and one of the others at a time. He cannot leave the goat alone with the wolf nor leave goat alone with the cabbage on either bank. How will he safely manage to carry all of them across the river in the fewest crossings?*

Using digraphs (for the definition of digraph, see Appendix C), we can solve this kind of problem elegantly as shown by R. Freley, K. L. Cooke and P. Detrick in their paper [66] (see also Chapter 7 of the book [42] by K. L. Cooke, R. E. Bellman and J. A. Lockett). M. Gardner also made these puzzles the subject of his column *Mathematical Games* in the *Scientific American*, No. 3 (1980), and many other journals and books of recreational mathematics. We shall apply some elements of graph theory (see Appendix C) to solve Alcuin's classic problem.

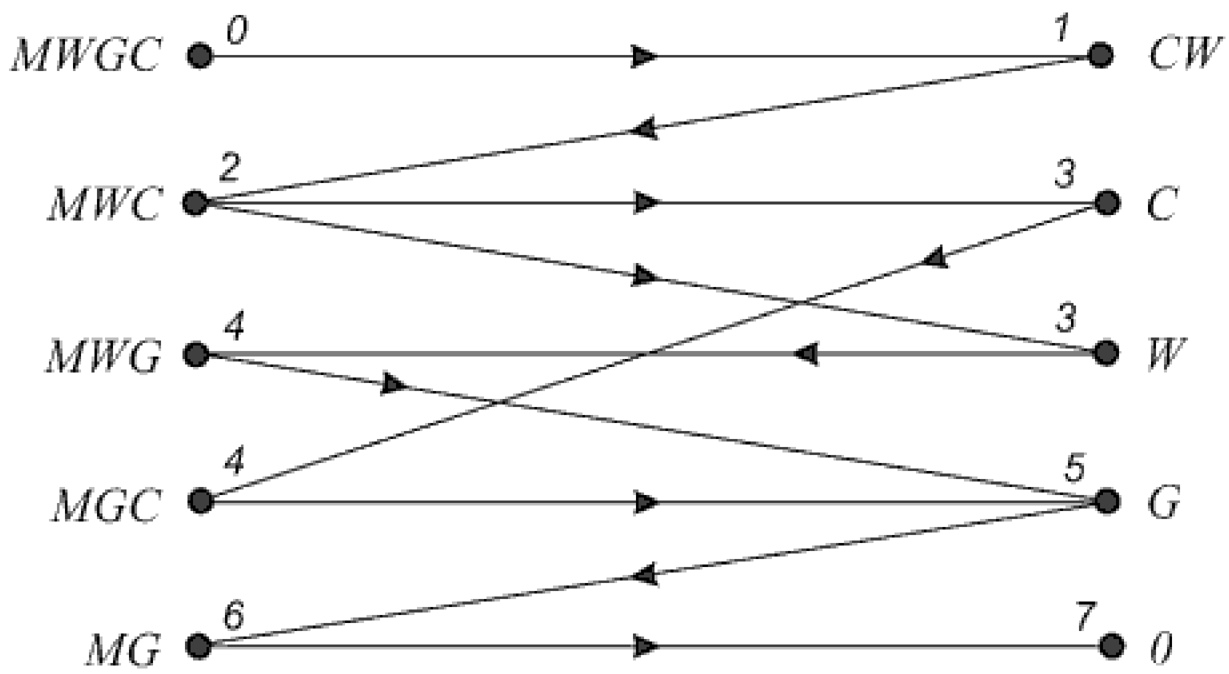

FIGURE 9.13. Crossing the river—a graph

Let M, W, G, and C stand for the man, the wolf, the goat and the cabbage, respectively. According to the puzzle's conditions the following sets of symbols denote the permissible states on the starting bank: $MWGC$, MWG, MWC, MGC, MG, CW, W, G, C, 0. The symbol 0 refers to the state once the river crossing has been accomplished.

Figure 9.13 shows the graph of all possible transits among the accepted states. Now we can simply reduce the solution of the puzzle to the determination of the shortest path between the vertex $MWGC$ (the initial state)

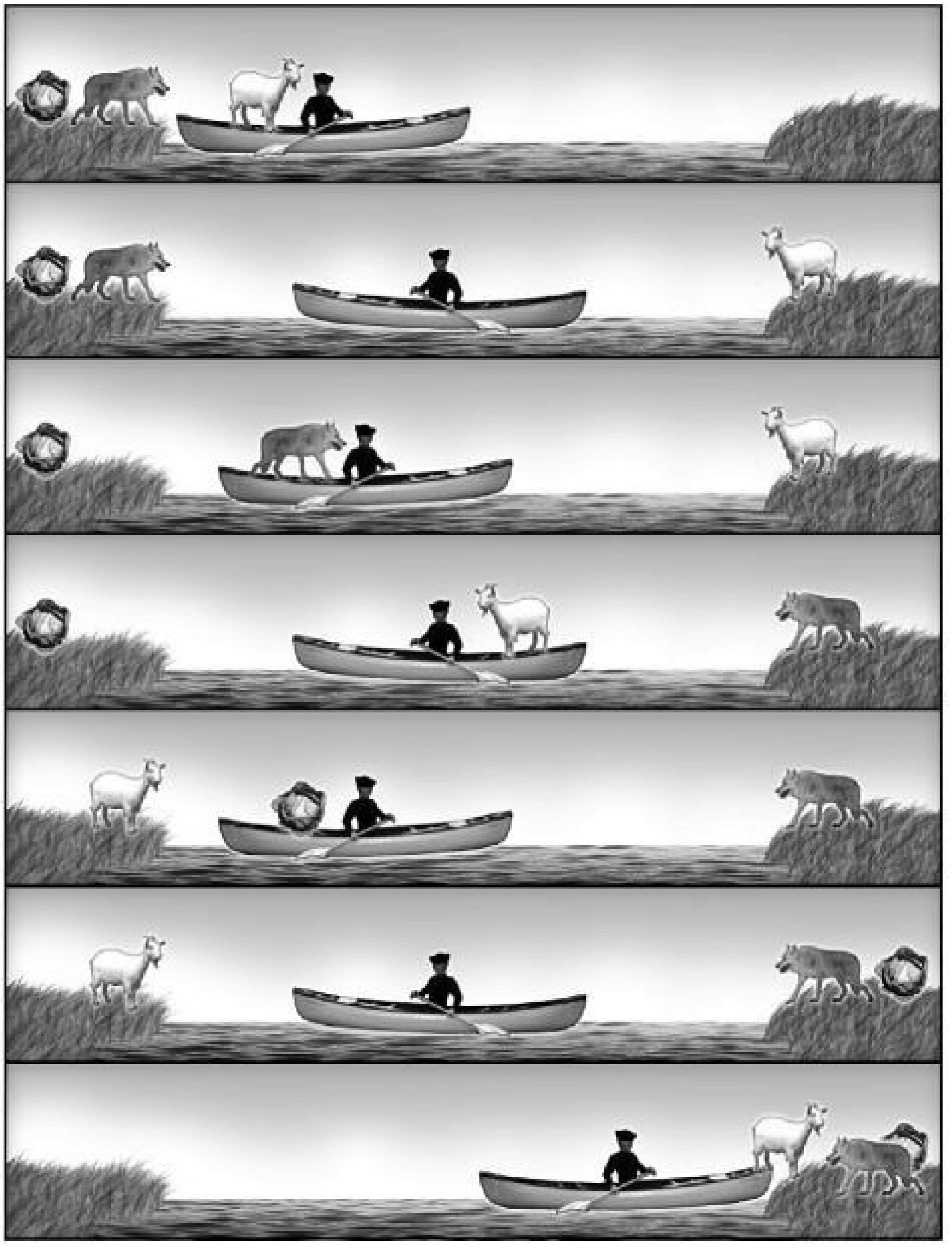

FIGURE 9.14. River-crossing scheme

and the vertex 0 (the final state). There are two minimal solutions, each requiring seven transits, recorded by each progressive state of transfer:

I $MWGC$, CW, MWC, W, MWG, G, MG, 0;
II $MWGC$, CW, MWC, C, MGC, G, MG, 0.

Due to the possibilities of taking a fourth and a fifth step, two paths appear. The difference between two adjacent states indicates what is in the boat with the man during the trip across the river. As a result, we can easily record the above solutions by successive lists of passengers in the boat:

I $MG - M - MC - MG - MW - M - MG$;
II $MG - M - MW - MG - MC - M - MG$.

We give a graphical illustration of the second solution II in Figure 9.14 based on the problem's solution given in an interesting graphical interpretation by B. Kordemski [117].

Distinctive variants of Alcuin's task occur in Africa: in Algeria the objects are a jackal, a goat, and a bundle of hay; in Liberia a man's company are a cheetah, a fowl, and some rice, while in Zanzibar, a man must ferry a leopard, a goat, and some leaves across a river (see Katz's book *A History of Mathematics* [113, p. 339]).

A stout family crosses the river

This puzzle, belonging to the river-crossing category, appeared for the first time as Problem XIX in Alcuin's work. We give the problem as it is found in [186], after its translation from Latin.

Problem 9.10. *A man and a woman who each weigh as much as a loaded cart must cross a river with their two children, each of whom, in turn, weighs the same, and whose total weight together equals that of a loaded cart. They find a boat that can only hold a single cartload. Make the transfer, if you can, without sinking the boat!*

Solution. Alcuin correctly found that nine passages are necessary. Let the letters F, M, s stand for father, mother and sons, respectively. Here is one solution recorded by the name(s) of traveler(s) where each crossing is described by 1) the names of persons on the starting (left) bank, 2) name(s) of rower(s) in the boat and 3) names of persons on the arrival (right) bank after crossing. The arrows $\rightarrow$ and $\leftarrow$ denote departure and return, respectively. Let us note that F can be replaced by M, and vice versa, since their weights are equal.

	starting bank	rower(s)	arrival bank
—	$FMss$	———	———
1.	FM	$\rightarrow\ ss$	ss
2.	FMs	$\leftarrow\ s$	s
3.	Fs	$\rightarrow\ M$	Ms
4.	Fss	$\leftarrow\ s$	M
5.	F	$\rightarrow\ ss$	Mss
6.	Fs	$\leftarrow\ s$	Ms
7.	s	$\rightarrow\ F$	FMs
8.	ss	$\leftarrow\ s$	FM
9.	———	$\rightarrow\ ss$	$FMss$

The posed problem may be effectively solved using digraphs. All possible states on the starting bank are given in Figure 9.15. The minimal solution is represented by the tick lines with arrows.

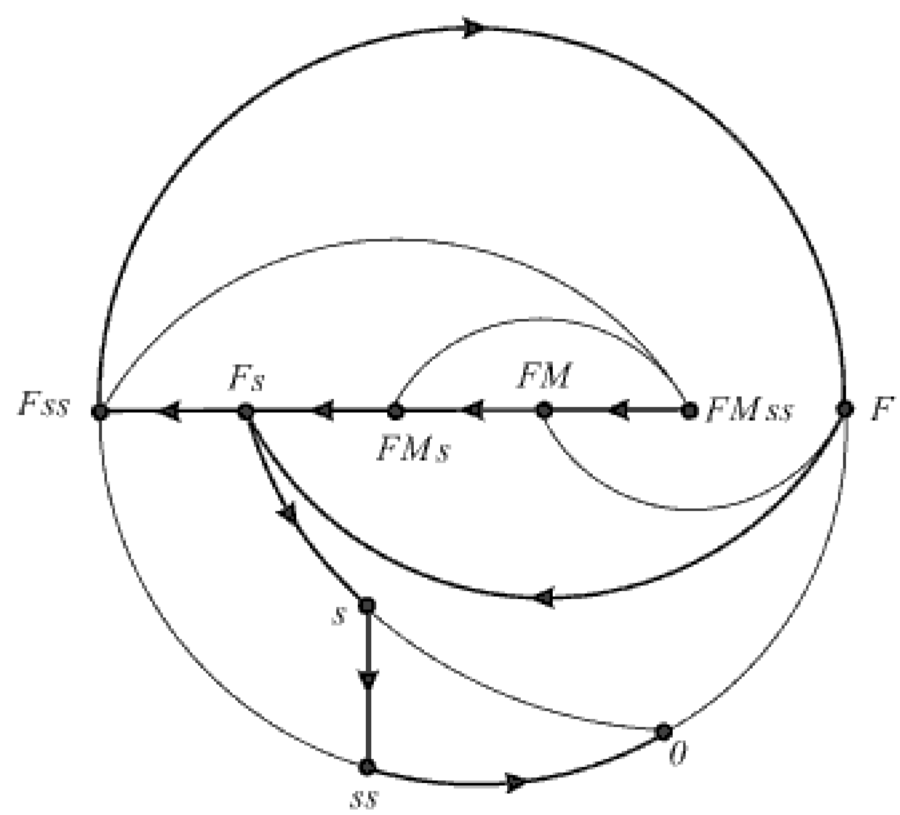

FIGURE 9.15. A stout family crosses the river—a digraph solution

We leave to the reader the solution of the well-known river-crossing problem that can be found in many books of recreational mathematics.

Problem 9.11.* *Three explorers and three helpful natives—who are incidentally also cannibals—must cross the river by means of a small rowing boat that can hold at most two passengers at a time. Naturally, the explorers* *must not allow to be outnumbered by the cannibals on either river bank. How can these six people safely cross the river in the fewest number of crossings?*

Can you solve this problem if only one explorer and only one cannibal can row?

Paul Erdös (1913–1996) (→ p. 310)

The famous mathematician Paul Erdös enjoyed posing and solving challenging and beautiful problems that are simple to understand, especially those belonging to number theory, combinatorics and graph theory. One of his problems is included in Chapter 7. Here is another dealing with graphs.

Seven towns and one-way roads

Problem 9.12. *There are 7 towns in a country, each of them connected to the other by a two-way road. Can one reconstruct all of these roads as one-way roads so that for any two specified towns it is always possible to reach each town in one step from some third town?*

By drawing a digraph with 7 vertices, we can employ arrows whose orientations indicate the particular direction of one-way roads and arrayed in such a way so that for any specified pair of towns, there is a third town from which you can drive directly to the other two. The solution is shown in Figure 9.16.

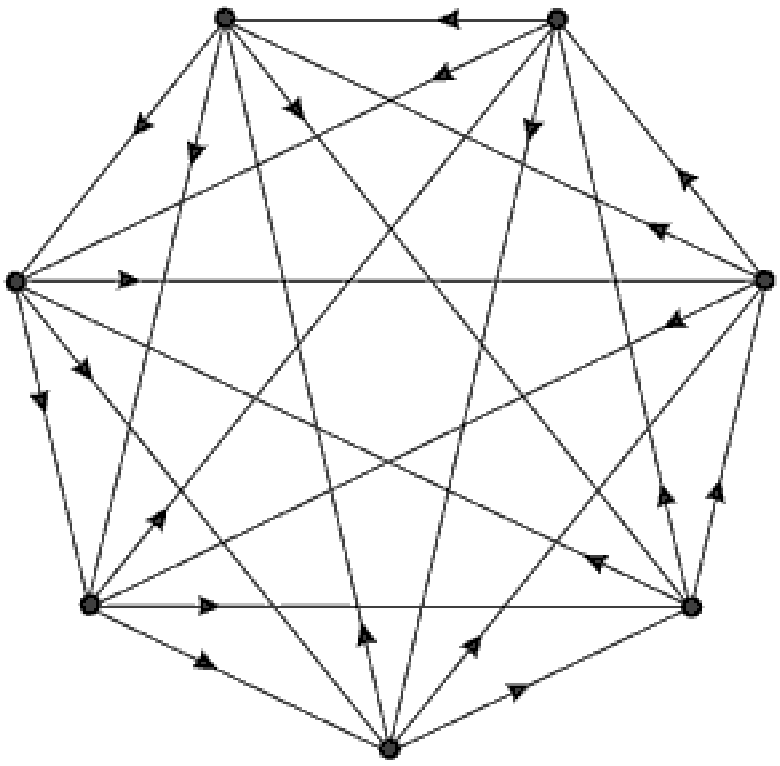

Figure 9.16. Graphing the one-way road problem

Louis Poinsot (1777–1859) (→ p. 306)

The French mechanist and mathematician Louis Poinsot, a member of the Académie des Sciences, is best known for his contribution in geometry, statics and mechanics. He played a leading role in mathematical research and education in eighteenth-century France.

Poinsot's diagram-tracing puzzle

Problem 9.13. *Each of n points, disposed on a circumference, is connected by straight lines with each of the remaining $n-1$ points. Can a diagram constructed in this way be traversed in one continuous stroke without covering any part more than once?*

In 1809, Poinsot showed that the diagram consisting of n interconnected points can be traversed under the stated conditions only if n is odd, but not if n is even. In graph theory terminology, this is equivalent to saying that the complete graph K_n is Eulerian only for an odd value of n. Poinsot also gave a method for finding an Eulerian path when n is odd. The five diagrams in Figure 9.17 illustrate Poinsot's statement.

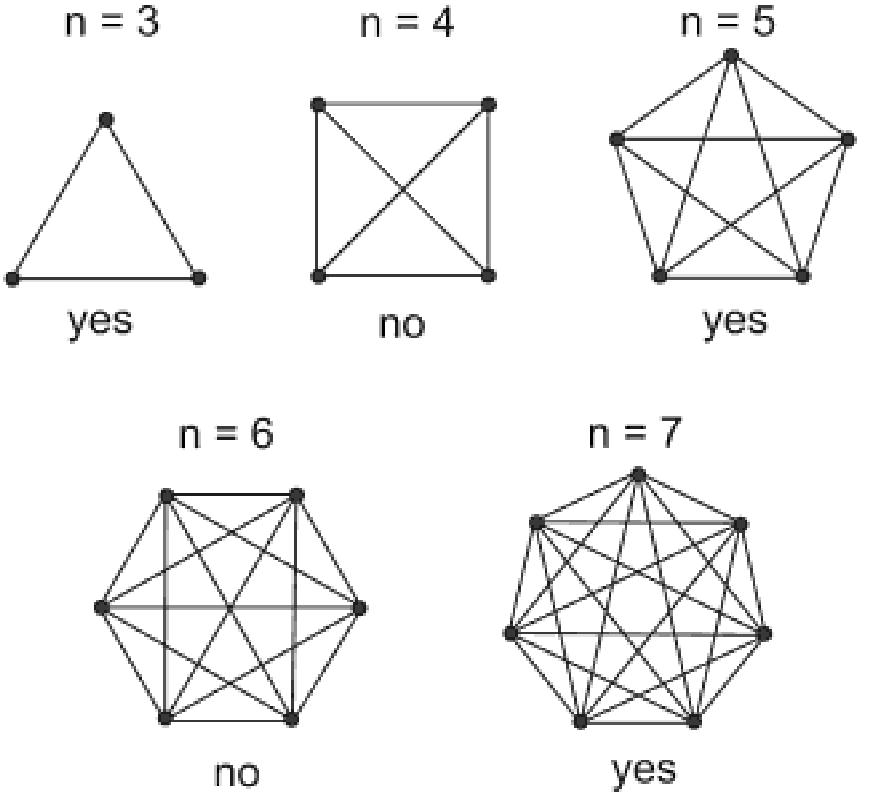

FIGURE 9.17. The complete graphs K_3 to K_7

In the following paragraph, we describe an algorithm for tracing the complete graph K_n (n is odd).

Let us label the vertices by $0, 1, \dots, n-1$ (n is odd) in the clockwise direction, and let $m = 0, 1, \dots, n-4$ be the counter.

1° Set $m = 0$;

2° Start from an arbitrary vertex r ($\in \{0, 1, \ldots, n-1\}$) and trace the edges in the clockwise direction skipping m vertices in each move until the starting vertex r is again reached; then go to 3°;

3° $m := m + 1$;

4° If $m < n - 3$, go to step 2°, otherwise STOP—the diagram is traced.

For example, in the case of the diagram K_7, Figure 9.17, the Eulerian trial is as follows (taking $r = 0$):

$$\underbrace{01, 12, 23, 34, 45, 56, 60}_{m=0} \quad \underbrace{02, 24, 46, 61, 13, 35, 50}_{m=1} \quad \underbrace{03, 36, 62, 25, 51, 14, 40}_{m=2}$$

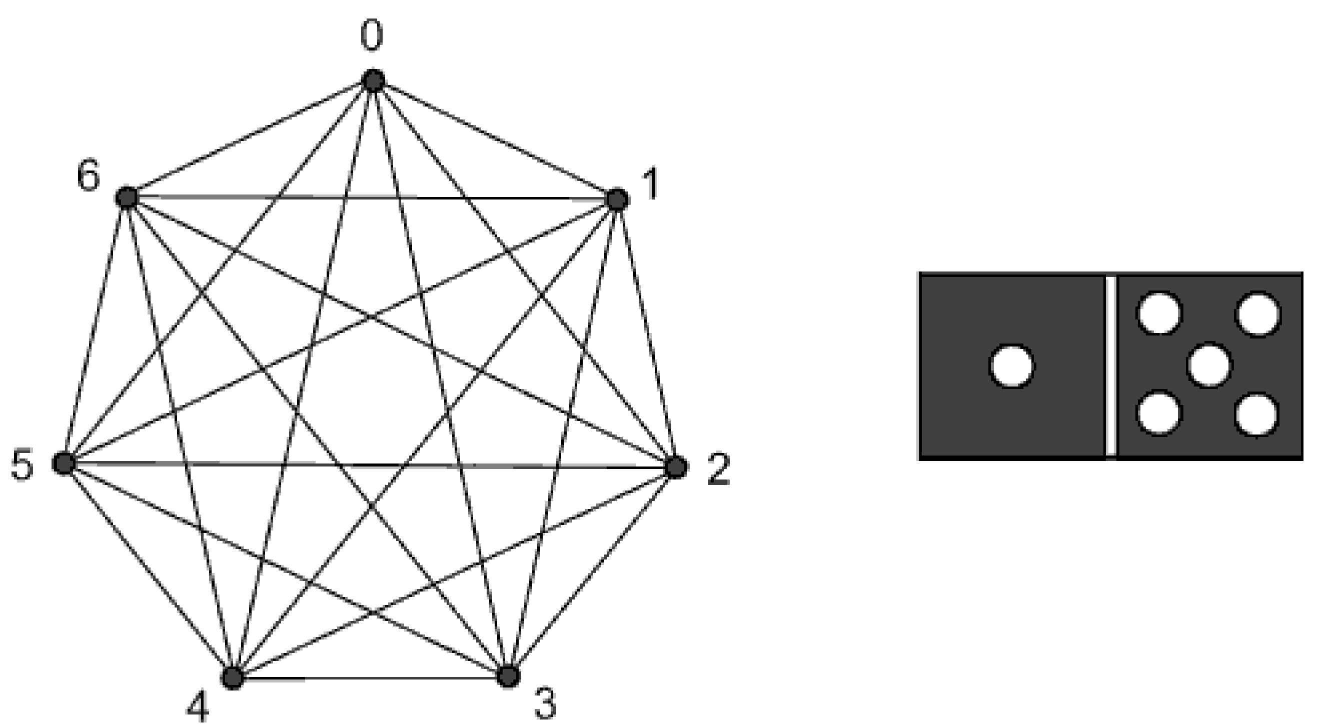

FIGURE 9.18. Eulerian path on the complete graph K_7

An interesting connection exists in the case of the complete graph K_7 (Figure 9.18 left) between the usual set of dominoes and the described Eulerian path; see [188, p. 129]. Let us regard each of the edges of K_7 as a domino; for example, the edge 1–5 corresponds to the domino displayed on the right-hand side of Figure 9.18.

As one may observe, the Eulerian path above corresponds to an arrangement of a set of dominoes including all but doubles 0–0, 1–1, ..., 6–6, in a continuous sequence. Once the basic sequence is found, the doubles can be inserted in the appropriate places. In this way we show that a complete game of dominoes is *possible*. Figure 9.19 displays the ring of dominoes corresponding to the above Eulerian path.

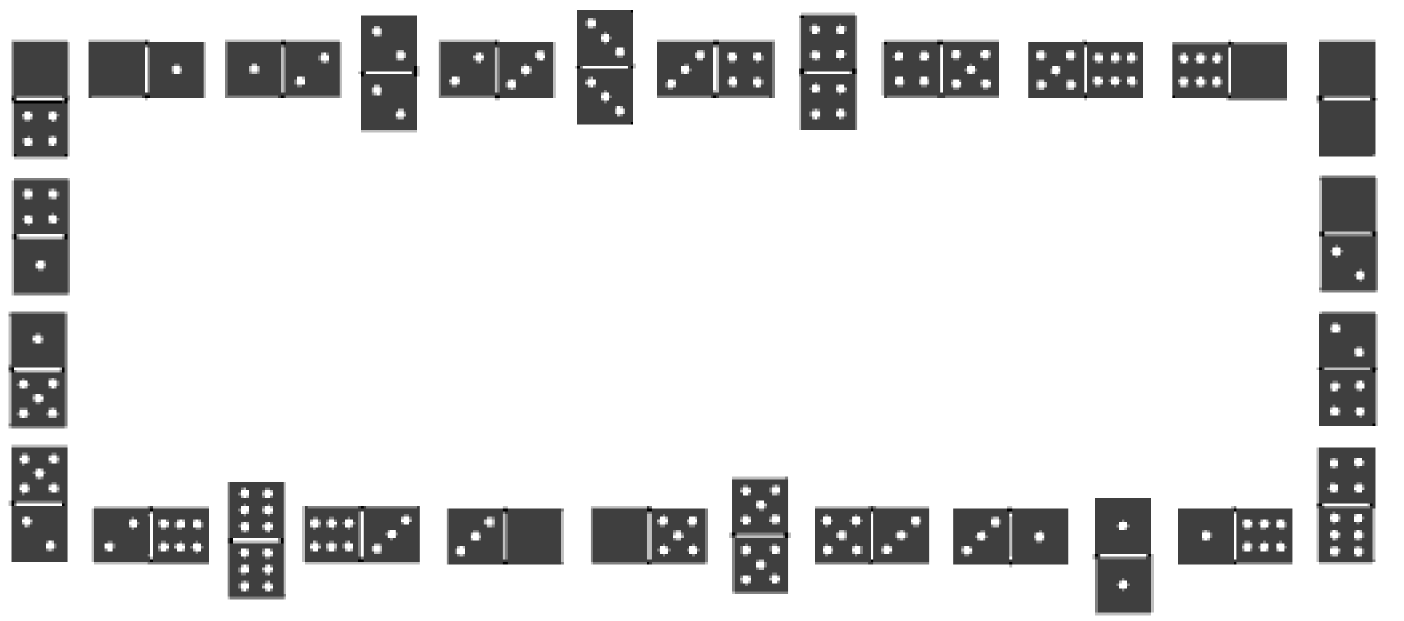

FIGURE 9.19. The ring of dominoes corresponds to the Eulerian trail

Gaston Tarry, the French mathematician, considered a more general case (see [150, pp. 249–253]). He stated the connection between a complete graph K_n (n is odd) and a set of dominoes running up to double-$(n-1)$, and determined the number of ways in which this set of dominoes can be arranged. For example, for $n = 7$ he found that the number of possible arrangements in a line of the usual set of 28 dominoes (from 0−0 to 6−6) is 7,959,229,931,520.

Problem 9.14.* *Let us consider the complete graph K_n. What is the maximum number of regions obtained by straight lines that connect the nodes of this graph?*

Siméon Poisson (1781–1840) (→ p. 306)

Milk puzzle

Fluid-measuring puzzles require one to measure a certain quantity of fluid using no other measuring device than that of given containers having a precise and exact capacity. When the great French mathematician Poisson was a boy, he encountered the following puzzle belonging to the fluid-measuring category of puzzles.

Problem 9.15. *A milkman has a container of milk with the capacity of* 12 *liters. He must deliver* 6 *liters of milk to a customer who possesses an* 8*-liter and a* 5*-liter container. How can he pour exactly* 6 *liters of milk from* *his container into the customer's larger container, and keep* 6 *liters in his container? While pouring, he may use all three containers.*

The story goes that young Poisson took so much delight in this puzzle[6] that he decided to make mathematics his life's vocation. Indeed, as the prominent mathematician Guglielmo Libri said of him, Poisson's only passion was mathematics: he lived and died for it. Poisson himself once said: *"Life is good for only two things, discovering mathematics and teaching mathematics."*

Table 9.2 presents the solution to the milk-decanting puzzle in terms of the minimal number of pourings.

12 l	12	4	4	9	9	1	1	6
8 l	0	8	3	3	0	8	6	6
5 l	0	0	5	0	3	3	5	0

TABLE 9.2. Solution to Poisson's milk puzzle

We note that this puzzle can also be solved effectively by the use of graphs as described by O. Ore in [135]. In the following problem, one very similar to Poisson's milk puzzle, we shall illustrate such an approach.

Problem 9.16. *Two containers, A and B, have holding capacities of* 3 *and* 5 *gallons, respectively. Drawing water from a lake or a pond, how does one pour exactly* 4 *gallons of water into the larger container using only these two containers?*

Solution. We can describe each state of water in the two containers A and B, by denoting a and b the quantities of water contained in A and B, respectively. In this manner every possible distribution of water is represented by an ordered pair of integers (a, b). At the beginning we have $a = b = 0$, which means that one starts with the state (0,0). The final state is given by the pair $(a, 4)$, where $a \in \{0, 3\}$ is the quantity of water in a smaller container at the moment that the measurement is completed.

It is easy to see that there are $6 \times 4 = 24$ possible states that can be represented by the vertices of a graph. It is convenient to draw this graph in the two-dimensional coordinate system as shown in Figure 9.20. Now we connect all integer pairs (a, b) by edges whenever it is possible to move from one vertex to another by pouring water between the containers or by taking

[6]This problem appeared for the first time in *Triparty en la science des nomvres*, a work of Nicolas Chuquet published in 1484. Chuquet (died 1487) was a French physician and also the best French mathematician of his time.

water from the lake. For example, in the first step it is possible to move from (0,0) to (0,5) or (3,0).

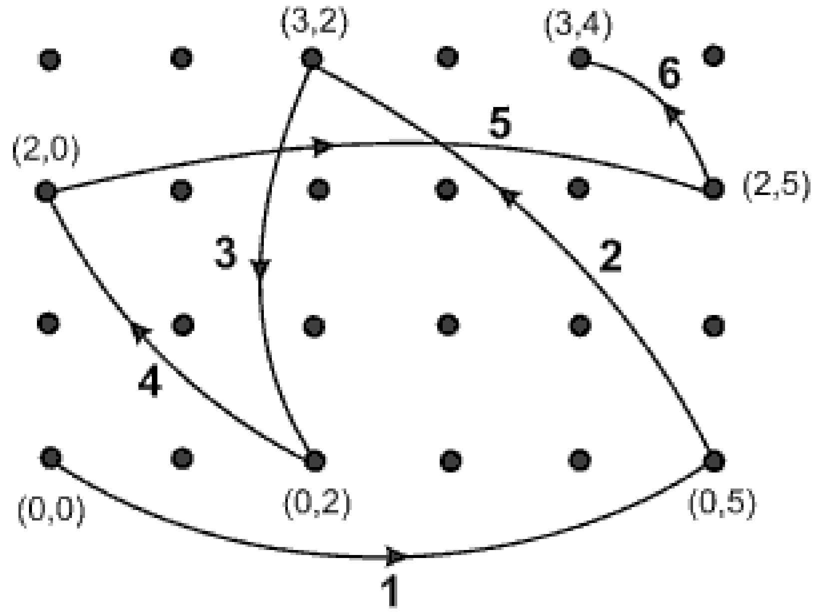

FIGURE 9.20. Graph solution for the pouring problem

The transition from (a, b) to $(a, 0)$ means that the larger container B is completely emptied. Continuing in this manner we arrive at the required target state $(3, 4)$ in 6 steps. All movements are represented by the edges indicated in Figure 9.20. We note that only those vertices and edges involved in the solution are denoted.

Following the edges of the graph, the solution can be presented by the table given below.

A 3g	0	0	3	0	2	2	3
B 5g	0	5	2	2	0	5	4

TABLE 9.3.

Johann B. Listing (1808–1882) (→ p. 307)

Listing's diagram-tracing puzzle

In 1847, a versatile German scientist J. B. Listing who worked in mathematics, geodesy, terrestrial magnetism, meteorology and other disciplines, wrote an important treatise entitled, *Vorstudien zur Topologie* (Introductory Studies in Topology). This work is often considered to have introduced a new branch of mathematics: topology. Among many fundamental topics, this book includes a discussion of diagram-tracing puzzles. Listing remarked that the diagram shown in Figure 9.21 can be drawn without lifting pencil from paper, and tracing over the edges exactly once starting at one end and finishing at the other. Euler's path exists since there are only two points which correspond to vertices of odd degree. We hope that the reader will show great patience and presence of mind when confronted by this huge network.

Problem 9.17.* *Is it possible to reproduce the diagram shown in Figure* 9.21 *drawing it in one continuous line while tracing over the edges once and only once?*

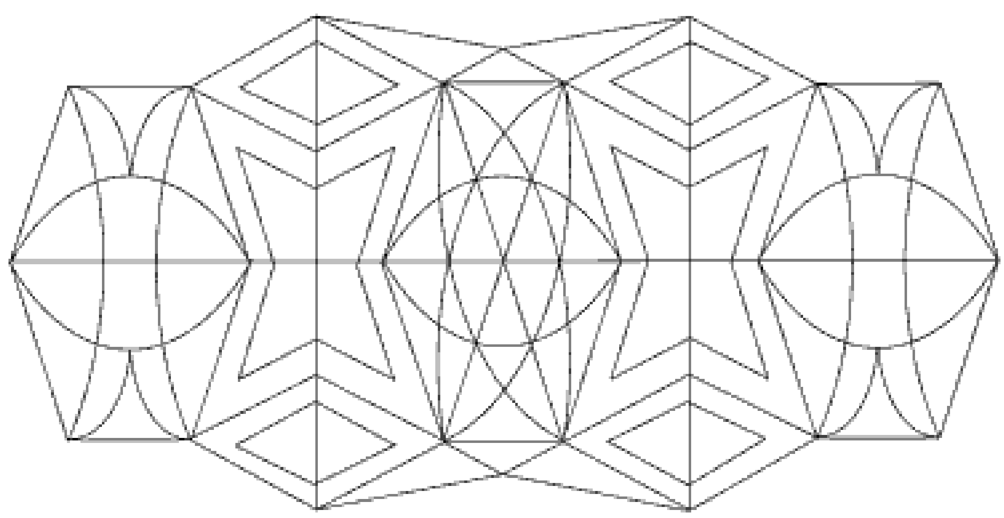

FIGURE 9.21. Listing's diagram-tracing puzzle

Answers to Problems

9.2. To solve this problem let the upper-case letters represent the islands (A,B,C,D) and the shore (E,F) by vertices of a graph. The lower-case letters represent the bridges by edges of a graph. When we look at Figure 9.22, we see that all vertices of the graph are even. Euler's theorem demonstrates that such a route exists in which one may cross all fifteen bridges starting from any point and ending the stroll at the same starting point.

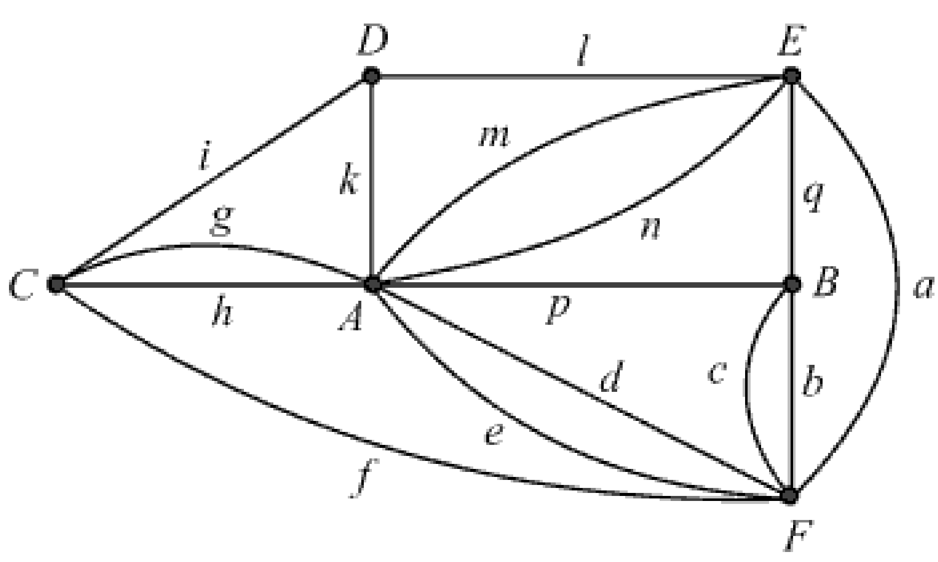

FIGURE 9.22. Graph of 15 bridges

We can carry out the crossing in this ordering

$E\ a\ F\ b\ B\ c\ F\ d\ A\ e\ F\ f\ C\ g\ A\ h\ C\ i\ D\ k\ A\ m\ E\ n\ A\ p\ B\ q\ E\ l\ D\ j\ E$

9.3. An Euler path is given by the sequence of nodes $\{1, 2, \ldots, 22\}$, marked in Figure 9.23.

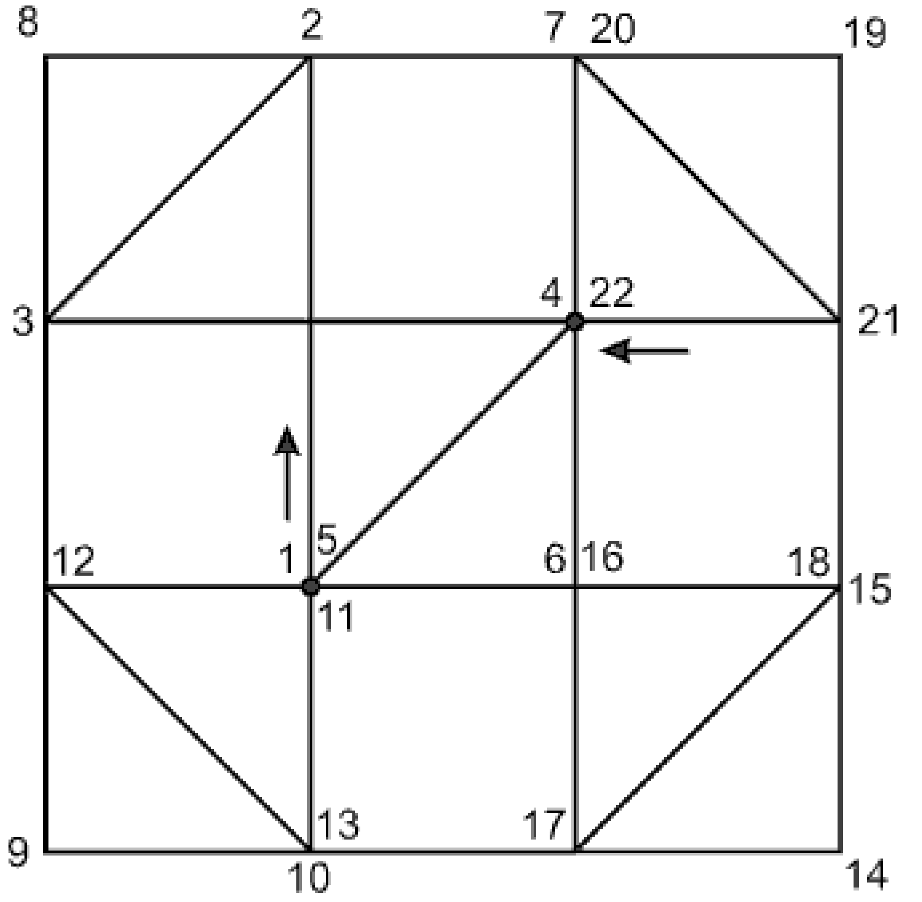

Figure 9.23. Tracing Tait's net

9.5. The completion of a Hamiltonian cycle is possible in exactly two ways:

$$N - M - F - G - H \mid$$
$$L - K - J - I - S - T - A - E - D - C - B - R - Q - P - O,$$

$$N - M - F - G - H \mid$$
$$L - K - O - P - C - B - R - Q - J - I - S - T - A - E - D.$$

9.6. Yes, it is possible. We have already seen Hamiltonian cycles on the dodecahedron and the 3-cube, the remaining three cycles are illustrated in Figure 9.24 (see [69]). As in the case of the two mentioned regular polyhedrons, we have considered stereographic projections—skeletons of a tetrahedron, an octahedron and an icosahedron.

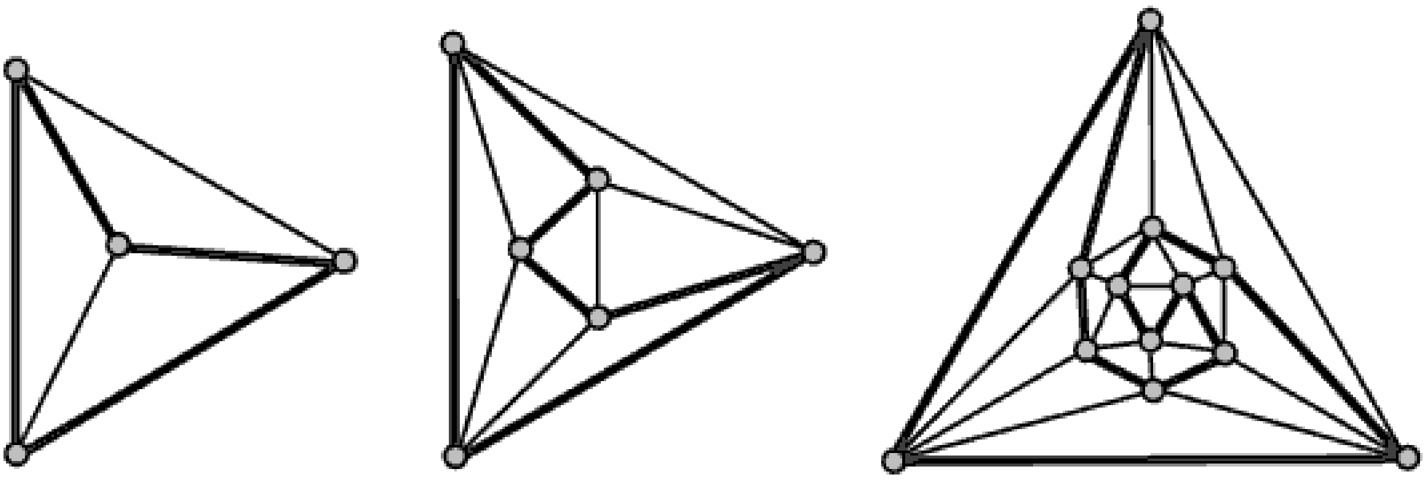

FIGURE 9.24. Hamiltonian cycles on tetrahedron, octahedron and icosahedron

9.7. Let us represent each of 12 faces of a dodecahedron by the nodes of a graph shown in Figure 9.25. The numbering is obviously irrelevant due to symmetry, but the connecting lines must be correctly drawn.

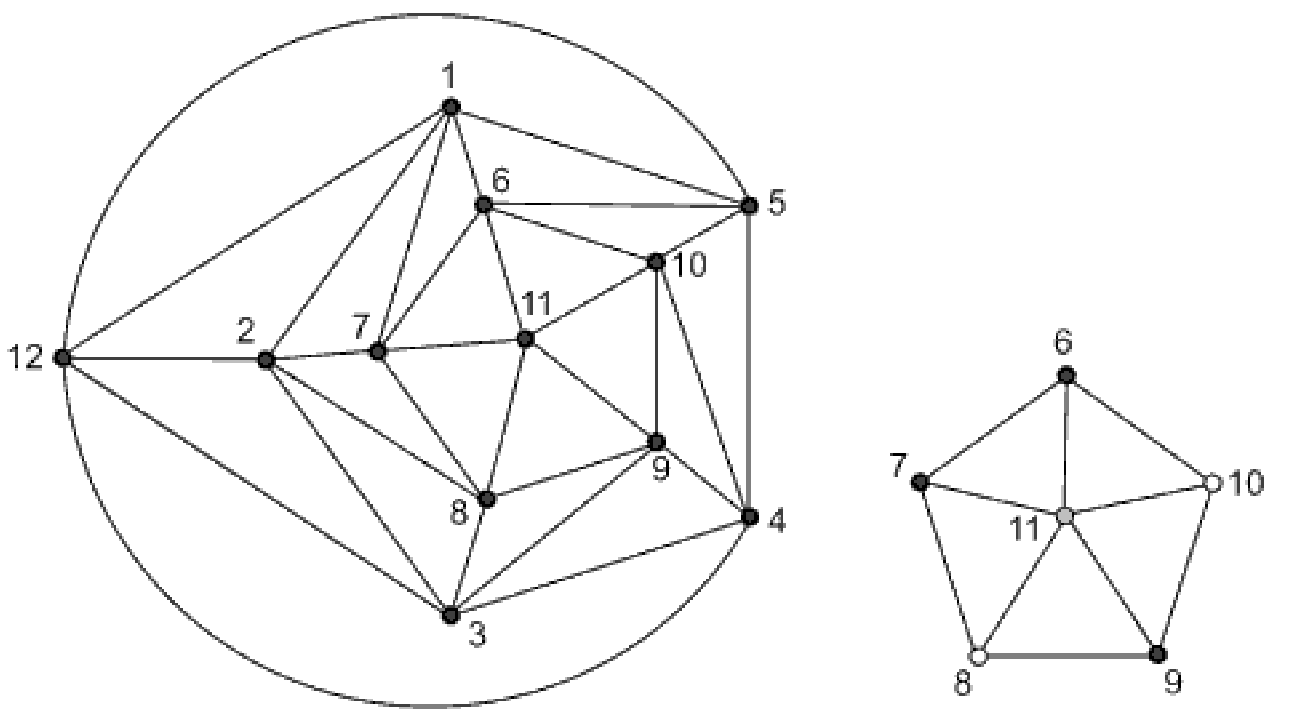

FIGURE 9.25. Coloring the faces of a dodecahedron

It is sufficient to consider a separate subgraph as shown in Figure 9.25 right. Suppose that the central node 11 is colored in one color. Then the surrounding (outer) nodes 6, 7, 8, 9 and 10 must be colored *alternatively* by two other colors. However, this is impossible because the number of outer nodes is odd. Therefore, the faces of a regular dodecahedron cannot be colored with three colors so that two neighboring faces are a different color.

9.11. The crossing requires not less than 11 passages. The reader may come to the optimal operation by trial and error, but we give an elegant solution using the digraphs. Since we have already used the graphs for solving crossing puzzles in this chapter, only an outline of the digraph-solution will be given. More details can be found in [66] and [80].

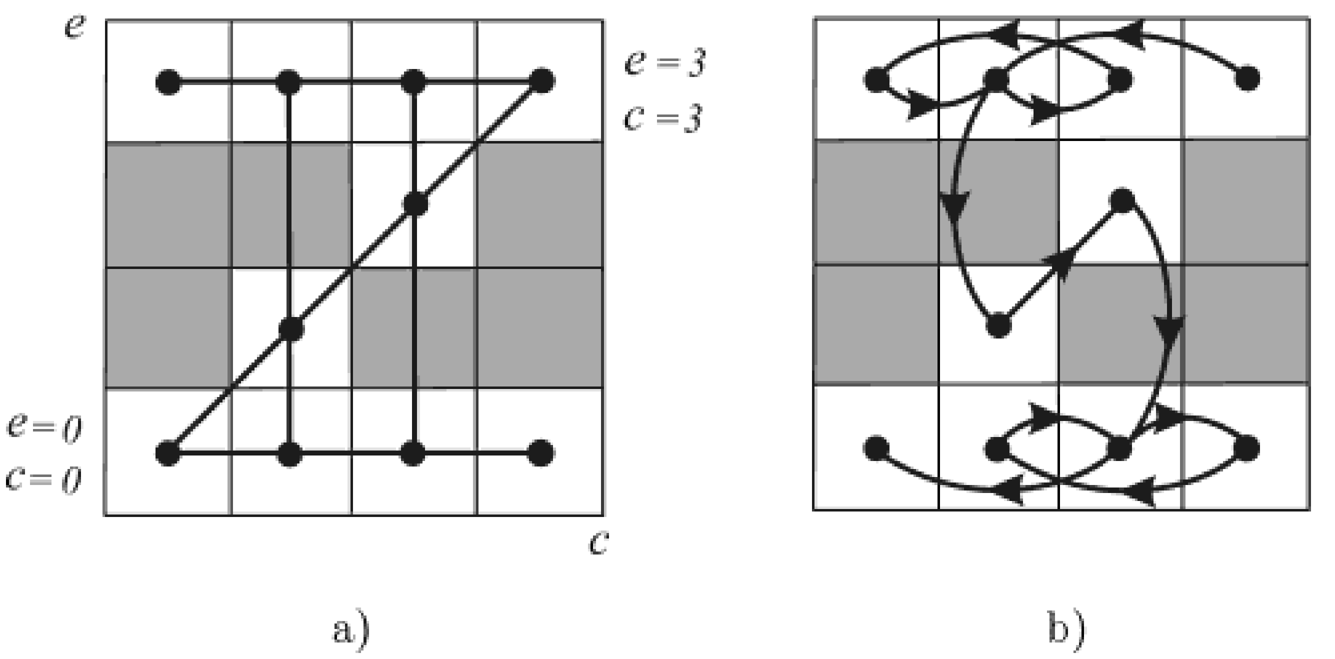

FIGURE 9.26. A graphical solution of the explorer-cannibal puzzle

Let e and c denote the number of explorers and cannibals, respectively. As in the previous graph-solutions, we will consider all possible states on the starting bank. Since $e, c \in \{0, 1, 2, 3\}$, there are 16 possible states represented in the matrix-like form, Figure 9.26(a). Six shaded cells denote the forbidden states where the cannibals outnumber the explorers. The remaining acceptable 10 states are marked by points that are connected by lines to show all possible transfers. In solving this problem we ought to choose the route among these lines which provides: 1) at most two passengers in the rowing boat; 2) the safety of explorers on either bank; 3) the passage from the (starting) state ($e = 3, c = 3$) to the (final) state ($e = 0, c = 0$); 4) the minimum number of crossings.

One of the four solutions in 11 moves is shown in Figure 9.26(b). Referring to Figure 9.26(b), and using the notation $\rightarrow$ $(rower(s))$ and $\leftarrow$ $(rower(s))$ to denote the direction of crossing and passenger(s) in the boat, the solution can be recorded as follows:

1. $\rightarrow$ (c, c) 2. $\leftarrow$ (c) 3. $\rightarrow$ (c, c) 4. $\leftarrow$ (c) 5. $\rightarrow$ (e, e) 6. $\leftarrow$ (c, e)
7. $\rightarrow$ (e, e) 8. $\leftarrow$ (c) 9. $\rightarrow$ (c, c) 10. $\leftarrow$ (c) 11. $\rightarrow$ (c, c).

In a variant of the explorer-cannibal puzzle in which only one explorer (let us distinguish him with a capital E) and only one cannibal (capital C) can row, the fewest crossings consist of 13 transfers. Here is one solution:

1. $\to$ (C, c) 2. $\leftarrow$ (C) 3. $\to$ (C, c) 4. $\leftarrow$ (C) 5. $\to$ (E, e)
6. $\leftarrow$ (c, E) 7. $\to$ (C, E) 8. $\leftarrow$ (c, E) 9. $\to$ (E, e) 10. $\leftarrow$ (C)
11. $\to$ (C, c) 12. $\leftarrow$ (C) 13. $\to$ (C, c).

Is this solution unique?

9.14. The maximum number of regions R_n is given by the formula

$$R_n = \binom{n}{4} + \binom{n-1}{2} = \frac{n^4 - 6n^3 + 23n^2 - 43n + 24}{24}.$$

This result can be obtained using various methods. One of them follows directly from the following problem posed by Leo Moser (see Gardner [82, p. 559]): *n spots are placed along a circle's circumstance. The circle is divided into regions by connecting every pair of points by a straight line* (*see Figure* 9.27 *for* $n = 5$). *What is the maximum number of regions?* The answer is given by the formula

$$M_n = n + \binom{n}{4} + \binom{n-1}{2}$$

(M stands for Moser). This is a real-life formula since it gives the maximum number of slices of a pizza that can be produced by n knight's straight cuts.

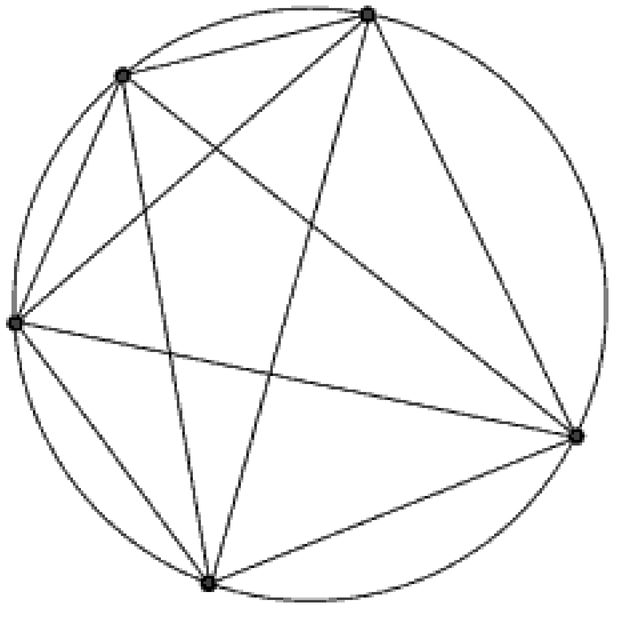

FIGURE 9.27. Moser's point problem for $n = 5$

From Figure 9.27 we notice that the diagram obtained by deleting the circumstance is actually the complete graph K_n. It is obvious from this figure that

$$R_n = M_n - n = \binom{n}{4} + \binom{n-1}{2}.$$

9.17. One of numerous solutions is shown in Figure 9.28. For the sake of clarity, the "questionable crossroads" are marked by small black circles. For symmetry, only a part is shown; this path continues from point 73 toward the endpoint **E** in a symmetrical way in reference to point 41 (white circles) going back to the starting point **S**.

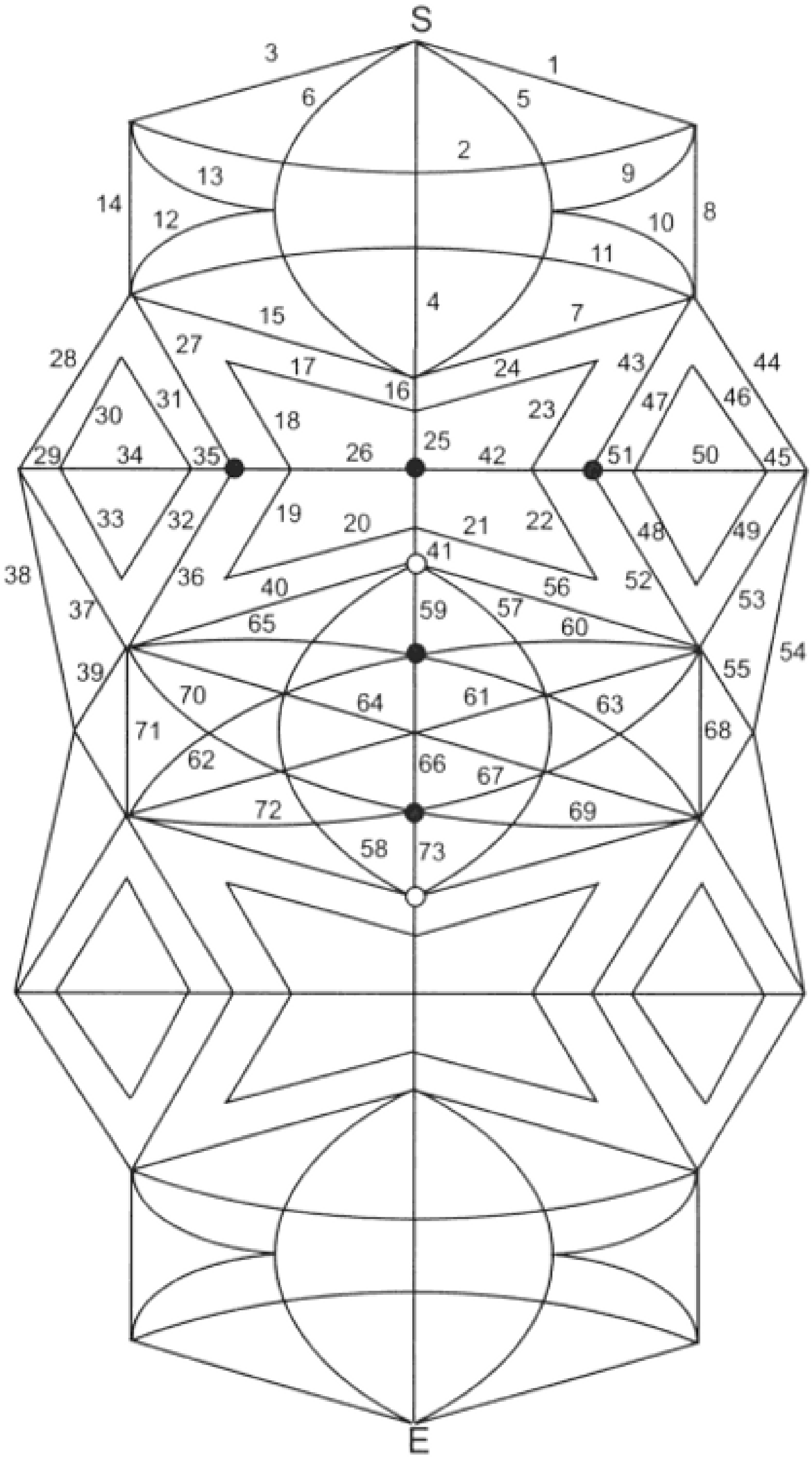

FIGURE 9.28. Tracing Listing's diagram

Chapter 10 CHESS

In many cases, mathematics as well as chess,
is an escape from reality.
Stanislaw Ulam

Chess is the gymnasium of the mind.
Blaise Pascal

The chessboard is the world,
the pieces are the phenomena of the Universe,
the rules of the game are what we call the laws of Nature.
Thomas Huxley

Puzzles concern the chessboards (of various dimensions and different shapes) and chess pieces have always lent themselves to mathematical recreations. Over the last five centuries so many problems of this kind have arisen. Find a re-entrant path on the chessboard that consists of moving a knight so that it moves successively to each square once and only once and finish its tour on the starting square. How to place 8 queens on the 8×8 chessboard so that no queen can be attacked by another? For many years I have been interested in these types of chess-math problems and, in 1997, I wrote the book titled *Mathematics and Chess* (Dover Publications) [138] as a collection of such problems. Some of them are presented in this chapter.

Mathematics, the queen of the sciences, and chess, the queen of games, share an axiomatical approach and an abstract way of reasoning in solving problems. The logic of the rules of play, the chessboard's geometry, and the concepts "right" and "wrong" are reminiscent of mathematics. Some mathematical problems can be solved in an elegant manner using some elements of chess. Chess problems and chess-math puzzles can ultimately improve analytical reasoning and problem solving skills.

In its nature, as well as in the very structure of the game, chess resembles several branches of mathematics. Solutions of numerous problems and puzzles on a chessboard are connected and based on mathematical facts from graph theory, topology, number theory, arithmetic, combinatorial analysis, geometry, matrix theory, and other topics. In 1913, Ernst Zermelo used these connections as a basis to develop his theory of game strategies, which is considered as one of the forerunners of game theory.

The most important mathematical challenge of chess has been how to develop algorithms that can play chess. Today, computer engineers, programmers and chess enthusiasts design chess-playing machines and computer programs that can defeat the world's best chess players. Recall that, in 1997, IBM's computer Deep Blue beat Garry Kasparov, the world champion of that time.

Many great mathematicians were interested in chess problems: Euler, Gauss, Vandermonde, de Moivre, Legendre. On the other hand, several world-class chess players have made contributions to mathematics, before all, Emanuel Lasker. One of the best English contemporary grandmasters and twice world champion in chess problem solving, John Nunn, received his Ph.D. in mathematics from Oxford University at the age of twenty-three.

The aim of this chapter is to present amusing puzzles and tasks that contain both mathematical and chess properties. We have mainly focused on those problems posed and/or solved by great mathematicians. The reader will see some examples of knight's re-entrant tours (or "knight's circles") found by Euler, de Moivre and Vandermond. We have presented a variant of knight's chessboard (uncrossed) tour, solved by the outstanding computer scientist Donald Knuth using a computer program. You will also find the famous eight queens problem, that caught Gauss' interest. An amusing chessboard problem on non-attacking rooks was solved by Euler.

None of the problems and puzzles exceed a high school level of difficulty; advanced mathematics is excluded. In addition, we presume that the reader is familiar with chess rules.

*

* *

Abraham de Moivre (1667–1754) (→ p. 304)

Pierre de Montmort (1678–1733) (→ p. 304)

Alexandre Vandermonde (1735–1796) (→ p. 305)

Leonhard Euler (1707–1783) (→ p. 305)

Knight's re-entrant route

Among all re-entrant paths on a chessboard, the knight's tour is doubtless the most interesting and familiar to many readers.

Problem 10.1. *Find a re-entrant route on a standard* 8×8 *chessboard which consists of moving a knight so that it moves successively to each square once and only once and finishes its tour on the starting square.*

Closed knight's tours are often called "knight's circles". This remarkable and very popular problem was formulated in the sixth century in India [181]. It was mentioned in the Arab, *Mansubas*[1], of the ninth century A.D. There are well-known examples of the knight's circle in the Hamid I Mansubas (Istanbul Library) and the Al-Hakim Mansubas (Ryland Library, Manchester). This task has delighted people for centuries and continues to do so to this day. In his beautiful book *Across the Board: The Mathematics of Chessboard Problems* [181] J. J. Watkins describes his unforgettable experience at a Broadway theater when he was watching the famous sleight-of-hand artist Ricky Jay performing blindfolded a knights tour on stage.

The knight's circle also interested such great mathematicians as Euler, Vandermonde, Legendre, de Moivre, de Montmort, and others. De Montmort and de Moivre provided some of the earliest solutions at the beginning of the eighteenth century. Their method is applied to the standard 8×8 chessboard divided into an inner square consisting of 16 cells surrounded by

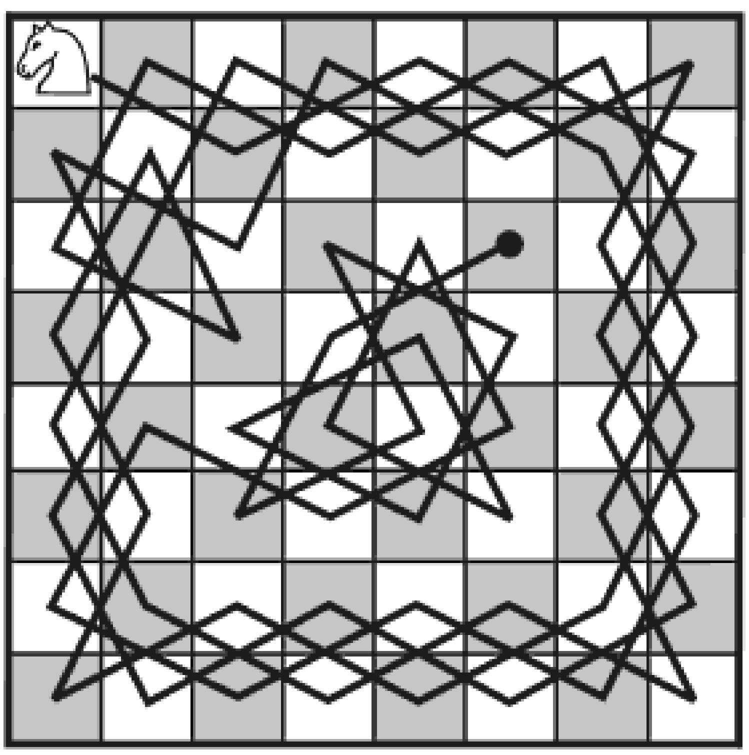

FIGURE 10.1. Knight's tour—de Moivre's solution

an outer ring of cells two deep. If the knight starts from a cell in the outer ring, it always moves along this ring filling it up and continuing into an inner ring cell only when absolutely necessary. The knight's tour, shown in

[1]The Mansubas, a type of book, collected and recorded the games, as well as remarkably interesting positions, accomplished by well-known chess players.

Figure 10.1, was composed by de Moivre (which he sent to Brook Taylor). Surprisingly, the first 23 moves are in the outer two rows. Although it passes all 64 cells, the displayed route is not a re-entrant route.

Even though L. Bertrand of Geneva initiated the analysis, according to *Mémoires de Berlin* for 1759, Euler made the first serious mathematical analysis of this subject. In his letter to the mathematician Goldbach (April 26, 1757), Euler gave a solution to the knight's re-entrant path shown in Figure 10.2.

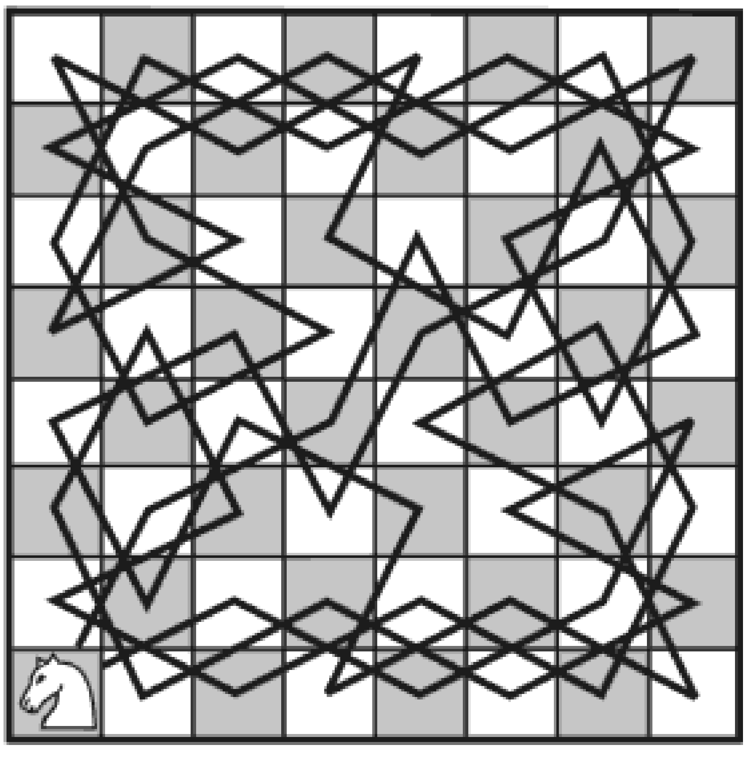

FIGURE 10.2. Euler's knight's circle solution

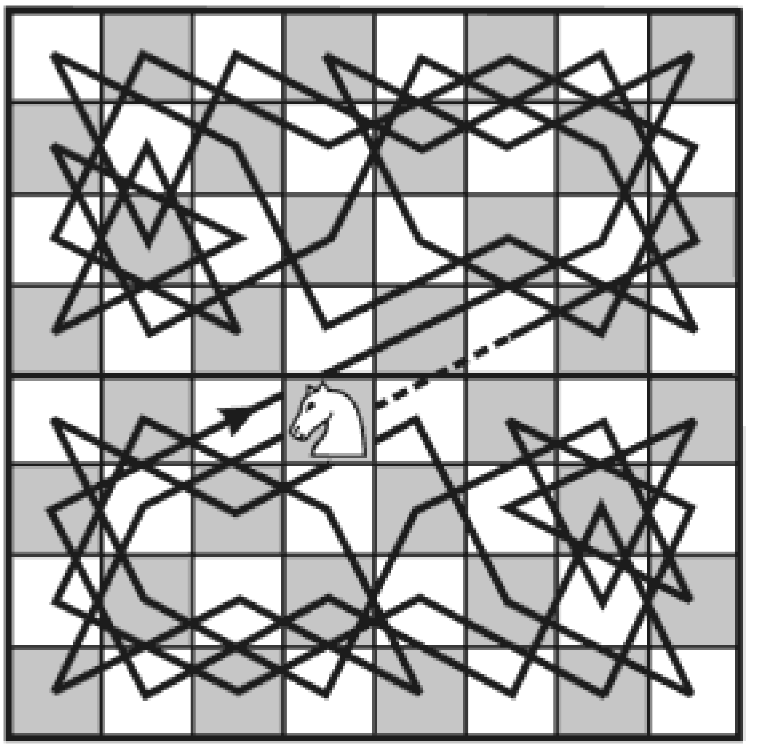

FIGURE 10.3. Euler's half-board solution

Euler's method consists of a knight's random movement over the board as long as it is possible, taking care that this route leaves the least possible number of untraversed cells. The next step is to interpolate these untraversed cells into various parts of the circuit to make the re-entrant route. Details on this method may be found in the books, *Mathematical Recreations and Essays* by Rouse Ball and Coxeter [150], *Across the Board* by J. J. Watkins [181] and *In the Czardom of Puzzles* (in Russian) [107] by E. I. Ignat'ev, the great Russian popularizer of mathematics. Figure 10.3 shows an example of Euler's modified method where the first 32 moves are restricted to the lower half of the board, then the same tour is repeated in a symmetric fashion for the upper half of the board.

Vandermonde's approach to solving the knight's re-entrant route uses fractions of the form x/y, where x and y are the coordinates of a traversed cell.[2]

[2] *L'Historie de l'Académie des Sciences* for 1771, Paris 1774, pp. 566–574.

For example, 1/1 is the lower left corner square (a1) and 8/8 is the upper right corner square (h8). The values of x and y are limited by the dimensions of the chessboard and the rules of the knight's moves. Vandermonde's basic idea consists of covering the board with two or more independent paths taken at random. In the next step these paths are connected. Vandermonde has described a re-entrant route by the following fractions (coordinates):

5/5, 4/3, 2/4, 4/5, 5/3, 7/4, 8/2, 6/1, 7/3, 8/1, 6/2, 8/3, 7/1, 5/2, 6/4, 8/5,
7/7, 5/8, 6/6, 5/4, 4/6, 2/5, 1/7, 3/8, 2/6, 1/8, 3/7, 1/6, 2/8, 4/7, 3/5, 1/4,
2/2, 4/1, 3/3, 1/2, 3/1, 2/3, 1/1, 3/2, 1/3, 2/1, 4/2, 3/4, 1/5, 2/7, 4/8, 3/6,
4/4, 5/6, 7/5, 8/7, 6/8, 7/6, 8/8, 6/7, 8/6, 7/8, 5/7, 6/5, 8/4, 7/2, 5/1, 6/3.

The usual chess notation corresponding to the above fraction notation would be e5, d3, b4, d5, e3, and so on.

An extensive literature exists on the knight's re-entrant tour.[3] In 1823, H. C. Warnsdorff[4] provided one of its most elegant solutions. His method is very efficient, not only for the standard chessboard but also for a general $n \times n$ board as well.

Recalling Problem 9.4 we immediately conclude that the knight's circles are in fact Hamiltonian cycles. There are 13,267,364,410,532 closed knight's tours, calculated in 2000 by Wegener [183]. The same number was previously claimed by Brendan McKay in 1997.[5] One of the ways to find a knight's

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[3]For instance, P. Volpicelli, *Atti della Reale Accademia dei Lincei* (Rome, 1872); C.F. de Jaenisch, *Applications de l'Analyse mathématique au Jeu des Echecks*, 3 vols. (Petrograd, 1862–63); A. van der Linde, *Geschichte und Literatur des Schachspiels*, vol. 2 (Berlin, 1874); M. Kraitchik, *La Mathématique des Jeux* (Brussels, 1930); W. W. Rouse Ball, *Mathematical Recreations and Essays*, rev. ed. (Macmillan, New York 1960); E. Gik, *Mathematics on the Chessboard* (in Russian, Nauka, Moscow 1976); E. I. Ignat'ev, *In the Czardom of Puzzles* (in Russian, Nauka, Moscow 1979); D'Hooge, *Les Secrets du cavalier* (Bruxelles-Paris, 1962); M. Petković, *Mathematics and Chess*, Dover Publications, Mineola (1997); I. Wegener, *Branching Programs and Binary Decision Diagrams*, SIAM, Philadelphia (2000); J. J. Watkins, *Across the Board: The Mathematics of Chessboard Problems*, Princeton University Press, Princeton and Oxford (2004); N. D. Elkies, R. P. Stanley, *The mathematical knight*, Mathematical Intellegencer 22 (2003), 22–34; A. Conrad, T. Hindrichs, H. Morsy, I. Wegener, *Solution of the knight's Hamiltonian path problem on chessboards*, Discrete Applied Mathematics 50 (1994), 125–134.

[4]*Des Rösselsrpunges einfachste und allgemeinste Lösung*, Schalkalden 1823.

[5]A powerful computer, finding tours at a speed of 1 million tours per second, will have to run for more than 153 days and nights to reach the number of tours reported by McKay and Wegener.

tour is the application of *backtracking* algorithms[6], but this kind of search is very slow so that even very powerful computers need considerable time. Another algorithm developed by A. Conrad *et al.* [40] is much faster and finds the knight's re-entrant tours on the $n \times n$ board for $n \geq 5$.

An extensive study of the possibility of the knight's re-entrant routes on a general $m \times n$ chessboard can be found in [181]. A definitive solution was given by Allen Schwenk [156] in 1991, summarized in the form of the following theorem.

Theorem 10.1 (Schwenk). *An $m \times n$ chessboard ($m \leq n$) has a knight's tour unless one or more of the following three conditions hold:*

(i) *m and n are both odd;*

(ii) *$m = 1, 2,$ or 4; or*

(iii) *$m = 3$ and $n = 4,\ 6,$ or 8.*

One more remark. If a knight's closed tour exists, then it is obvious that any square on the considered chessboard can be taken as the starting point.

It is a high time for the reader to get busy and try to find the solution to the following problem.

Problem 10.2.* *Prove the impossibility of knight's tours for $4 \times n$ boards.*

The previous problem tell us that a knight's tour on a 4×4 board is impossible. The question of existence of such a tour for the three-dimensional $4 \times 4 \times 4$ board, consisting of four horizontal 4×4 boards which lie one over the other, is left to the reader.

Problem 10.3.* *Find a knight's re-entrant tour on a three-dimensional $4 \times 4 \times 4$ board.*

Many composers of the knight's circles have constructed re-entrant paths of various unusual and intriguing shapes while also incorporating certain esthetic elements or other features. Among them, magic squares using a knight's tour (not necessarily closed) have attracted the most attention. J. J. Watkins calls the quest for such magic squares the *Holy Grail.* The Russian chess master and officer de Jaenisch (1813–1872) composed many notable problems concerning the knight's circles. Here is one of them [138, Problem 3.5], just connected with magic squares.

[6] A backtracking algorithm searches for a partial candidate to the solution step by step and eliminates this candidate when the algorithm reaches an impasse, then backing up a number of steps to try again with another partial candidate—the knight's path in this particular case.

Problem 10.4. *Let the successive squares that form the knight's re-entrant path be denoted by the numbers from* 1 *to* 64 *consecutively,* 1 *being the starting square and* 64 *being the square from which the knight plays its last move, connecting the squares* 64 *and* 1. *Can you find a knight's re-entrant path such that the associated numbers in each row and each column add up to* 260?

The first question from the reader could be: *Must the sum be just* 260? The answer is very simple. The total sum of all traversed squares of the chessboard is

$$1 + 2 + \cdots + 64 = \frac{64 \cdot 65}{2} = 2{,}080,$$

and 2,080 divided by 8 gives 260. It is rather difficult to find magic or "semi-magic squares" ("semi-" because the sums over diagonals are not taken into account), so we recommend Problem 10.4 only to those readers who are well-versed in the subject. One more remark. De Jaenisch was not the first who constructed the semi-magic squares. The first semi-magic knight's tour, shown in Figure 10.4, was composed in 1848 by William Beverley, a British landscape painter and designer of theatrical effects.

There are 280 distinct arithmetical semi-magic tours (not necessarily closed). Taking into account that each of these semi-magic tours can be oriented by rotation and reflection in eight different ways, a total number of semi-magic squares is 2,240 (= 280 × 8).

Only a few knight's *re-entrant* paths possess the required "magic" properties. One of them, constructed by de Jaenisch, is given in Figure 10.5.

1	30	47	52	5	28	43	54
48	51	2	29	44	53	6	27
31	46	49	4	25	8	55	42
50	3	32	45	56	41	26	7
33	62	15	20	9	24	39	58
16	19	34	61	40	57	10	23
63	14	17	36	21	12	59	38
18	35	64	13	60	37	22	11

FIGURE 10.4. Beverley's tour

63	22	15	40	1	42	59	18
14	39	64	21	60	17	2	43
37	62	23	16	41	4	19	58
24	13	38	61	20	57	44	3
11	36	25	52	29	46	5	56
26	51	12	33	8	55	30	45
35	10	49	28	53	32	47	6
50	27	34	9	48	7	54	31

FIGURE 10.5. De Jaenisch's tour

The question of existence of a proper magic square (in which the sums over the two main diagonals are also equal to 260) on the standard 8 × 8 chessboard has remained open for many years. However, in August 2003, Guenter Stertenbrink announced that an exhaustive search of all possibilities using a computer program had led to the conclusion that no such knight's tour exists (see Watkins [181]).

Our discussion would be incomplete without addressing the natural question of whether a magic knight's tour exists on a board of any dimension $n \times n$. Where there is magic, there is hope. Indeed, it has been proved recently that such magic tours do exist on boards of size 16×16, 20×20, 24×24, 32×32, 48×48, and 64×64 (see [181]).

The following problem concerns a knight's tour which is closed, but in another sense. Namely, we define a *closed knight's route* as a closed path consisting of knight's moves which do not intersect and do not necessarily traverse all squares. For example, such a closed route is shown in Figure 10.13.

Problem 10.5.* *Prove that the area enclosed by a closed knight's route is an integral multiple of the area of a square of the $n \times n$ ($n \geq 4$) chessboard.*

Hint: Exploit the well-known Pick's theorem which reads: *Let A be the area of a non-self-intersecting polygon P whose vertices are points of a lattice. Assume that the lattice is composed of elementary parallelograms with the area S. Let B denote the number of points of the lattice on the polygon edges and I the number of points of the lattice inside P. Then*

$$A = \Big(I + \tfrac{1}{2}B - 1\Big)S. \tag{10.1}$$

Many generalizations of the knight's tour problem have been proposed which involve alteration of the size and shape of the board or even modifying the knight's standard move; see Kraitchik [118]. Instead of using the perpendicular components 2 and 1 of the knight's move, written as the pair (2,1), Kraitchik considers the (m, n)-move.

A Persian manuscript, a translation of which can be found in Duncan Forbes' *History of Chess* (London, 1880), explains the complete rules of fourteenth-century Persian chess. A piece called the "camel", used in Persian chess and named the "cook" by Solomon Golomb, is actually a modified knight that moves three instead of two squares along a row or a file, then one square at right angles which may be written as (3,1). Obviously, this piece can move on the 32 black squares of the standard 8×8 chessboard without leaving the black squares. Golomb posed the following task.

Problem 10.6.* *Is there a camel's tour over all 32 black squares of the chessboard in such a way that each square is traversed once and only once?*

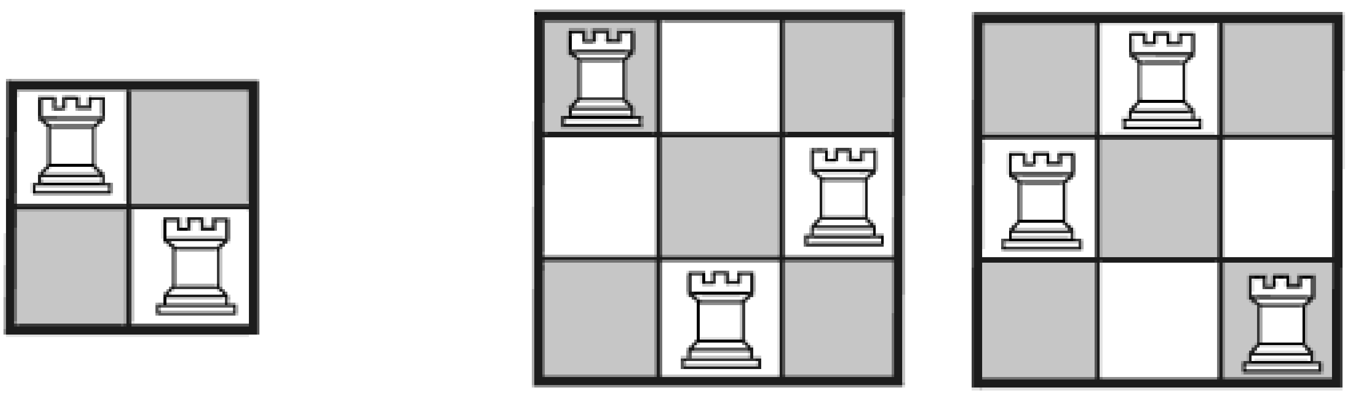
Leonhard Euler (1707–1783) (→ p. 305)

Non-attacking rooks

Apart from the knight's re-entrant tours on the chessboard, shown on pages 258–264, another amusing chessboard problem caught Leonhard Euler's interest.

Problem 10.7. *Let Q_n $(n \geq 2)$ be the number of arrangements of n rooks that can be placed on an $n \times n$ chessboard so that no rook attacks any other and no rook lies on the squares of the main diagonal. One assumes that the main diagonal travels from the lower left corner square $(1, 1)$ to the upper right corner square (n, n). The task is to find Q_n for an arbitrary n.*

The required positions of rooks for $n = 2$ and $n = 3$, for example, are shown in Figure 10.6 giving $Q_2 = 1$ and $Q_3 = 2$.



$n = 2,\ Q_2 = 1$ $\qquad$ $n = 3,\ Q_3 = 2$

FIGURE 10.6. Non-attacking rooks outside the main diagonal

The above-mentioned problem is, in essence, the same one as that referred to as the Bernoulli–Euler *problem of misaddressed letters* appearing on page 184. Naturally, the same formula provides solutions to both problems. As our problem involves the placement of rooks on a chessboard, we will express the solution of the *problem of non-attacking rooks* in the context of the chessboard.

According to the task's conditions, every row and every column contain one and only one rook. For an arbitrary square (i, j), belonging to the ith row and jth column, we set the square (j, i) symmetrical to the square (i, j) with respect to the main diagonal.

The rook can occupy $n-1$ squares in the first column (all except the square belonging to the main diagonal). Assume that the rook in the first column is placed on the square $(r, 1)$, $r \in \{2, \ldots, n\}$. Depending on the arrangement of the rooks in the remaining $n-1$ columns, we can distinguish two groups of positions with non-attacking rooks: if the symmetrical square $(1, r)$ (related to the rook on the square $(r, 1)$) is not occupied by a rook, we will say that the considered position is of the first kind, otherwise, it is of the second kind. For example, the position on the left in Figure 10.7 (where $n = 4$ and $r = 2$) is of the first kind, while the position on the right is of the second kind.

FIGURE 10.7. Positions of the first kind (left) and second kind (right)

Let us now determine the number of the first kind positions. If we remove the rth row from the board and substitute it by the first row, and then remove the first column, a new $(n-1) \times (n-1)$ chessboard is obtained. Each arrangement of rooks on the new chessboard satisfies the conditions of the problem. The opposite claim is also valid: for each arrangement of rooks on the new chessboard satisfying the conditions of the problem, the unique position of the first kind can be found. Hence, the number of the first kind positions is exactly Q_{n-1}.

To determine the number of second kind positions, let us remove the first column and the rth row, and also the rth column and the first row from the $n \times n$ chessboard (regarding only positions of the second kind). If we join the remaining rows and columns without altering their order, a new $(n-2) \times (n-2)$ chessboard is formed. It is easy to check that the arrangements of rooks on such $(n-2) \times (n-2)$ chessboards satisfy the conditions of the posed problem. Therefore, it follows that there are Q_{n-2} positions of the second kind.

After consideration of the above, we conclude that there are $Q_{n-1} + Q_{n-2}$ positions of non-attacking rooks on the $n \times n$ chessboard, satisfying the problem's conditions and corresponding to the fixed position of the rook—the square $(r, 1)$—in the first column. Since r can take $n - 1$ values $(= 2, 3, \ldots, n)$, one obtains

$$Q_n = (n-1)(Q_{n-1} + Q_{n-2}). \tag{10.2}$$

The above recurrence relation derived by Euler is a difference equation of the second order. It can be reduced to a difference equation of the first order in the following manner. Starting from (10.2) we find

$$\begin{aligned} Q_n - nQ_{n-1} &= (n-1)(Q_{n-1} + Q_{n-2}) - nQ_{n-1} \\ &= -(Q_{n-1} - (n-1)Q_{n-2}). \end{aligned}$$

Using successively the last relation we obtain

$$\begin{aligned} Q_n - nQ_{n-1} &= -(Q_{n-1} - (n-1)Q_{n-2}) \\ &= (-1)^2(Q_{n-2} - (n-2)Q_{n-3}) \\ &\;\vdots \\ &= (-1)^{n-3}(Q_3 - 3Q_2). \end{aligned}$$

Since $Q_2 = 1$ and $Q_3 = 2$ (see Figure 10.6), one obtains the difference equation of the first order

$$Q_n - nQ_{n-1} = (-1)^n. \tag{10.3}$$

To find the general formula for Q_n, we apply (10.3) backwards and obtain

$$\begin{aligned} Q_n &= nQ_{n-1} + (-1)^n = n\big((n-1)Q_{n-2} + (-1)^{n-1}\big) + (-1)^n \\ &= n(n-1)Q_{n-2} + n(-1)^{n-1} + (-1)^n \\ &= n(n-1)\big((n-2)Q_{n-3} + (-1)^{n-2}\big) + n(-1)^{n-1} + (-1)^n \\ &= n(n-1)(n-2)Q_{n-3} + n(n-1)(-1)^{n-2} \\ &\quad + n(-1)^{n-1} + (-1)^n \\ &\;\vdots \\ &= n(n-1)(n-2)\cdots 3\cdot Q_2 + n(n-1)\cdots 4\cdot(-1)^3 + \cdots \\ &\quad + n(-1)^{n-1} + (-1)^n \\ &= n!\Big(\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\Big), \end{aligned}$$

that is,

$$Q_n = n!\sum_{k=2}^{n} \frac{(-1)^k}{k!} \quad (n \geq 2).$$

The last formula gives

$$Q_2 = 1, \quad Q_3 = 2, \quad Q_4 = 9, \quad Q_5 = 44, \quad Q_6 = 265, \quad \text{etc.}$$

Carl Friedrich Gauss (1777–1855) (→ p. 305)

Carl Friedrich Gauss indisputably merits a place among such illustrious mathematicians as Archimedes and Newton. Sometimes known as "the Prince of mathematicians," Gauss is regarded as one of the most influential mathematicians in history. He made a remarkable contribution to many fields of mathematics and science (see short biography on page 305).

As a ten-year old schoolboy, Gauss was already exhibiting his formidable mathematical talents as the following story recounts. One day Gauss' teacher Mr. Büttner, who had a reputation for setting difficult problems, set his pupils to the task of finding the sum of the arithmetic progression $1 + 2 + \cdots + 100$.[7] The lazy teacher assumed that this problem would occupy the class for the entire hour since the pupils knew nothing about arithmetical progression and the general sum formula. Almost immediately, however, gifted young Gauss placed his slate on the table. When the astonished teacher finally looked at the results, he saw the correct answer, 5,050, with no further calculation. The ten-year-old boy had mentally computed the sum by arranging the addends in 50 groups $(1+100)$, $(2+99), \dots, (50, 51)$, each of them with the sum 101, and multiplying this sum by 50 in his head to obtain the required sum $101 \cdot 50 = 5{,}050$. Impressed by his young student, Büttner arranged for his assistant Martin Bartels (1769–1836), who later became a mathematics professor in Russia, to tutor Gauss.

Carl Friedrich Gauss
1777–1855

Like Isaac Newton, Gauss was never a prolific writer. Being an ardent perfectionist, he refused to publish his works which he did not consider complete, presumably fearing criticism and controversy. His delayed publication of results, like the delays of Newton, led to many high profile controversies and disputes.

[7]Some authors claim that the teacher gave the arithmetic progression $81{,}297 + 81{,}495 + \cdots + 100{,}899$ with the difference 198. It does not matter!

Gauss' short dairy of only 19 pages, found after his death and published in 1901 by the renowned German mathematician Felix Klein, is regarded as one of the most valuable mathematical documents ever. In it he included 146 of his discoveries, written in a very concise form, without any traces of derivation or proofs. For example, he jotted down in his dairy *"Heureka! $num = \triangle + \triangle + \triangle$,"* a coded form of his discovery that every positive integer is representable as a sum of at most three triangular numbers.

Many details about the work and life of Gauss can be found in G. W. Dunnington's book, *Carl Friedrich Gauss, Titan of Science* [58]. Here is a short list of monuments, objects and other things named in honour of Gauss:

– The CGS unit for magnetic induction was named *Gauss* in his honour,
– Asteroid 1001 *Gaussia*,
– The *Gauss* crater on the Moon,
– The ship *Gauss*, used in the Gauss expedition to the Antarctic,
– *Gaussberg*, an extinct volcano on the Antarctic,
– The *Gauss* Tower, an observation tower in Dransfeld, Germany,
– *Degaussing* is the process of decreasing or eliminating an unwanted magnetic field (say, from TV screens or computer monitors).

The eight queens problem

One of the most famous problems connected with a chessboard and chess pieces is undoubtedly the *eight queens problem*. Although there are claims that the problem was known earlier, in 1848 Max Bezzel put forward this problem in the chess journal *Deutsche Schachzeitung* of Berlin:

Problem 10.8. *How does one place eight queens on an 8×8 chessboard, or, for general purposes, n queens on an $n \times n$ board, so that no queen is attacked by another. In addition, determine the number of such positions.*

Before we consider this problem, let us note that although puzzles involving non-attacking queens and similar chess-piece puzzles may be intriguing in their right, more importantly, they have applications in industrial mathematics; in maximum cliques from graph theory, and in integer programming (see, *e.g.*, [65]).

The eight queens problem was posed again by Franz Nauck in the more widely read, *Illustrirte Zeitung*, of Leipzig in its issue of June 1, 1850. Four weeks later Nauck presented 60 different solutions. In the September issue he corrected himself and gave 92 solutions but he did not offer a proof that there are not more. In 1910 G. Bennett[8] concluded that there are only 12

[8]G. Bennett, *The eight queens problem*, Messenger of Mathematics, 39 (1910), 19.

distinctly different solutions to the queens problem, that is, solutions that could not be obtained one from another by rotations for 90°, 180° and 270°, and mirror images; T. Gosset later proved this in 1914.[9]

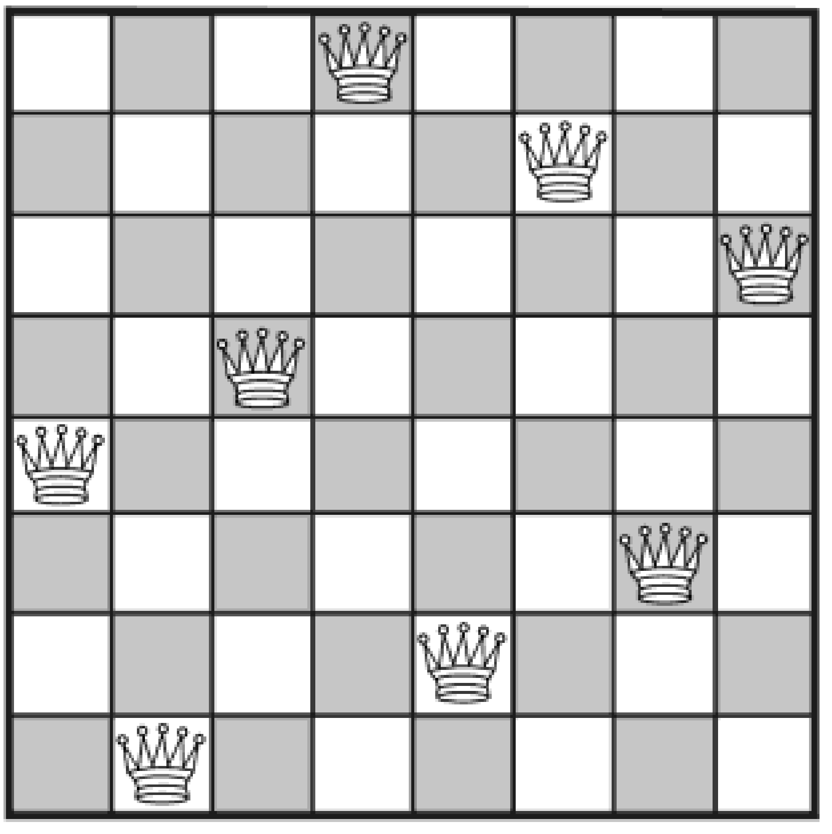


Figure 10.8. The 8-queens problem; one fundamental solution 41582736

Each position of the non-attacking queens on the 8×8 board can be indicated by an array of 8 numbers $k_1 k_2 \cdots k_8$. The solution $k_1 k_2 \cdots k_8$ means that one queen is on the k_1th square of the first column, one on the k_2th square of the second column, and so on. Therefore, twelve fundamental solutions can be represented as follows:

41582736	41586372	42586137
42736815	42736851	42751863
42857136	42861357	46152837
46827135	47526138	48157263

Each of the twelve basic solutions can be rotated and reflected to yield 7 other patterns (except the solution 10, which gives only 3 other patterns because of its symmetry). Therefore, counting reflections and rotations as different, there are 92 solutions altogether. One fundamental solution given by the first sequence 41582736 is shown in Figure 10.8.

Gauss himself also found great interest in the eight queens problem reading *Illustrirte Zeitung*. In September of 1850 he concluded that there were

[9]T. Gosset, *The eight queens problem,* Messenger of Mathematics, 44 (1914), 48.

76 solutions. Only a week later, Gauss wrote to his astronomer friend H. C. Schumacher that four of his 76 solutions are false, leaving 72 as the number of true solutions. In addition, Gauss noted that there might be more, remembering that Franz Nauck did not prove his assertion that there are exactly 92 solutions. One can imagine that Gauss did not find all the solutions on the first attempt, presumably because at that time, he lacked the systematic and strongly supported methods necessary for solving problems of this kind. More details about Gauss and the eight queens problem can be found in [34] and [65].

Considering that the method of solving the eight queens problem via trail and error was inelegant, Gauss turned this problem into an arithmetical problem; see [34] and [86]. We have seen that each solution can be represented as a permutation of the numbers 1 through 8. Such a representation automatically confirms that there is exactly one queen in each row and each column. It was necessary to check in an easy way if any two queens occupy the same diagonal and Gauss devised such a method. We will illustrate his method with the permutation 41582736 that represents the eight-queens solution shown in Figure 10.8.

Let us form the following sums:

$$\begin{array}{ccccccccc} & 4 & 1 & 5 & 8 & 2 & 7 & 3 & 6 \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ & - & - & - & - & - & - & - & - \\ \Sigma & 5 & 3 & 8 & 12 & 7 & 13 & 10 & 14 \end{array}$$

and

$$\begin{array}{ccccccccc} & 4 & 1 & 5 & 8 & 2 & 7 & 3 & 6 \\ & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ & - & - & - & - & - & - & - & - \\ \Sigma & 12 & 8 & 11 & 13 & 6 & 10 & 5 & 7 \end{array}$$

In both cases the eight sums are distinct natural numbers, which means that no two queens lie on the same negative diagonal \ (the sums above) and no two queens lie on the same positive diagonal / (the sums below). According to these outcomes, Gauss concluded that the queens with positions represented by the permutation 41582736 are non-attacking.

In 1874 J. W. Glaisher[10] proposed expanding the eight queens problem to the *n-queens problem*, that is, solving the queens' puzzle for the general $n \times n$ chessboard. He attempted to solve it using determinants. It was suspected

[10] J. W. Glaisher, *On the problem of eight queens*, Philosophical Magazine, Sec. 4, 48 (1874), 457.

that exactly n non-attacking queens could be placed on an $n \times n$ chessboard, but it was not until 1901 that Wilhelm Ahrens [2] could provide a positive answer. Other interesting proofs can be found in [104], [181] and[193]. In their paper [104] Hoffman, Loessi and Moore reduced the n-queens task to the problem of finding a maximum internally stable set of a symmetric graph, the vertices of which correspond to the n^2 square elements of an $n \times n$ matrix.

Considering the more general problem of the $n \times n$ chessboard, first we verify that there is no solution if $n < 4$ (except the trivial case of one queen on the 1×1 square). Fundamental solutions for $4 \leq n \leq 7$ are as follows:

$$
\begin{aligned}
n = 4: &\ \ 3142, \\
n = 5: &\ \ 14253, 25314, \\
n = 6: &\ \ 246135, \\
n = 7: &\ \ 1357246, 3572461, 5724613, 4613572, 3162574, 2574136.
\end{aligned}
$$

The number of fundamental solutions $F(n)$ and the number of all solutions $S(n)$, including those obtained by rotations and reflections, are listed below for $n = 1, \ldots, 12$. A general formula for the number of solutions $S(n)$ when n is arbitrary has not been found yet.

n	1	2	3	4	5	6	7	8	9	10	11	12
$F(n)$	1	–	–	1	2	1	6	12	46	92	341	1,784
$S(n)$	1	–	–	2	10	4	40	92	342	724	2,680	14,200

Table 10.1. The number of solutions to the $n \times n$ queens problem

Some interesting relations between magic squares and the n-queens problem have been considered by Demirörs, Rafraf and Tanik in [48]. The authors have introduced a procedure for obtaining the arrangements of n non-attacking queens starting from magic squares of order n not divisible by 2 and 3.

The following two problems are more complicated modern variants of the eight queens problem and we leave them to the reader. In solving these problems, it is advisable to use a computer program.

In his *Mathematical Games* column, M. Gardner [79] presented a version of the n-queens problem with constraints. In this problem a queen may attack other queen directly (as in ordinary chess game) or by reflection from either the first or the $(n+1)$-st horizontal virtual line. To put the reader at ease, we shall offer the special case $n = 8$.

Problem 10.9.* *Place* 8 *chess queens on the* 8×8 *board with a reflection strip in such a way that no two queens attack each other either directly or by reflection.*

A superqueen (known also as "Amazon") is a chess piece that moves like a queen as well as a knight. This very powerful chess piece was known in some variants of chess in the Middle Ages. Obviously, the n-superqueen problem is an extension of the n-queen problem in which new restrictions should be taken into account. So it is not strange that the n-superqueen problem has no solution for $n < 10$.

Problem 10.10.* *Place* 10 *superqueens on the* 10×10 *chessboard so that no superqueen can attack any other.*

There is just one fundamental solution for the case $n = 10$. Can you find this solution?

A variation of the chess that would be worth mentioning is one in which the game is played on a cylindrical board. The pieces in so-called cylindrical chess are arranged as on an ordinary chessboard, and they move following the same rules. But the board is in a cylindrical form because its vertical edges are joined ("vertical cylindrical chess") so that the verticals a and h are juxtaposed. Also, it is possible to join the horizontal edges of the board ("horizontal cylindrical chess") so that the first and the eighth horizontal are connected.

We have already seen that the eight queens problem on the standard 8×8 chessboard has 92 solutions. The following problem on a cylindrical chessboard was considered by the outstanding chess journalist and chessmaster Edvard Gik [84].

Problem 10.11.* *Solve the problem of non-attacking queens on a cylindrical chessboard that is formed of an* 8×8 *chessboard.*

Donald Knuth (1938–) (→ p. 310)

The longest uncrossed knight's tour

On pages 258–262 we previously considered a knight's tour over a chessboard such that all 64 squares are traversed once and only once. The difficult problem presented below imposes certain restrictions on the knight's tour:

Problem 10.12. *Find the largest uncrossed knight's tour on a chessboard.*

Apparently T. R. Dawson once posed this problem, but L. D. Yardbrough launched the same problem again in the July 1968 *Journal of Recreational Mathematics.*

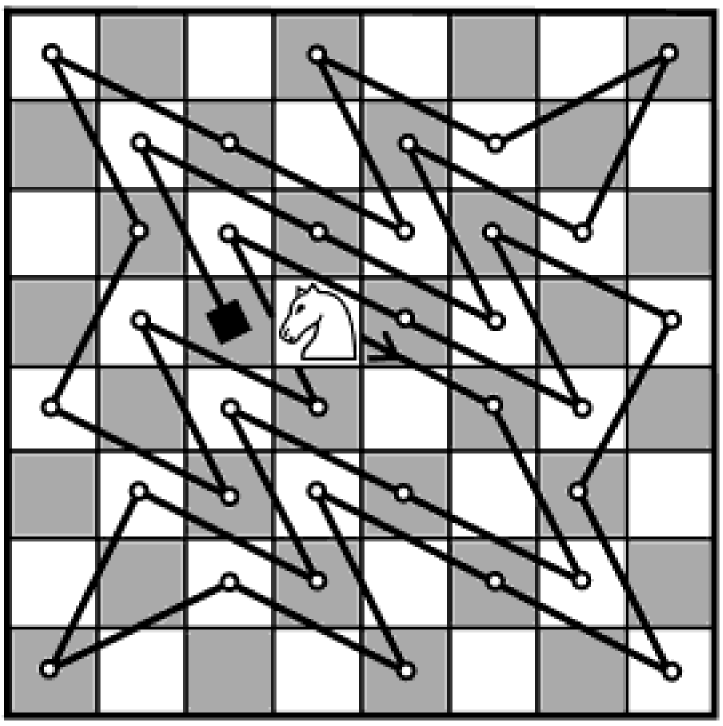


FIGURE 10.9. Knuth's solution for the longest uncrossed knight's tour

Donald E. Knuth wrote a "backtrack" computer program to find four fundamental solutions for the knight's tour. To find these tours, the computer examined 3,137,317,289 cases. One of these solutions is shown in Figure 10.9 (see, *e.g.*, the book [138, p. 61]).

Guarini's knight-switching problem

We end this chapter with Guarini's classic knight-switching problem from 1512, mentioned in Chapter 1. A number of mathematicians have considered problems of this type, in modern times most frequently in connection with planar graphs. No matter how unexpected it sounds, a kind of "graph approach" was known to al-Adli (*ca.* 840 A.D.) who considered in his work on chess a simple circuit that corresponds to the knight-move network on a 3×3 board.

Problem 10.13. *The task is to interchange two white knights on the top corner squares of a 3×3 chessboard and two black knights on the bottom corner squares. The white knights should move into the places occupied initially by the black knights—and vice versa—in the minimum number of moves. The knights may be moved in any order, regardless of their color. Naturally, no two of them can occupy the same square.*

Solution. This puzzle belongs to the class of problems that can be solved in an elegant manner using the theory of planar graphs. Possibly this problem could find its place in Chapter 9 on graphs, but we regard that it is an unimportant dilemma.

The squares of the chessboard represent *nodes* of a graph, and the possible moves of the pieces between the corresponding squares (the nodes of the graph) are interpreted as the *connecting lines* of the graph. The corresponding graph for the board and the initial positions of the knights are shown in Figure 10.10(a).

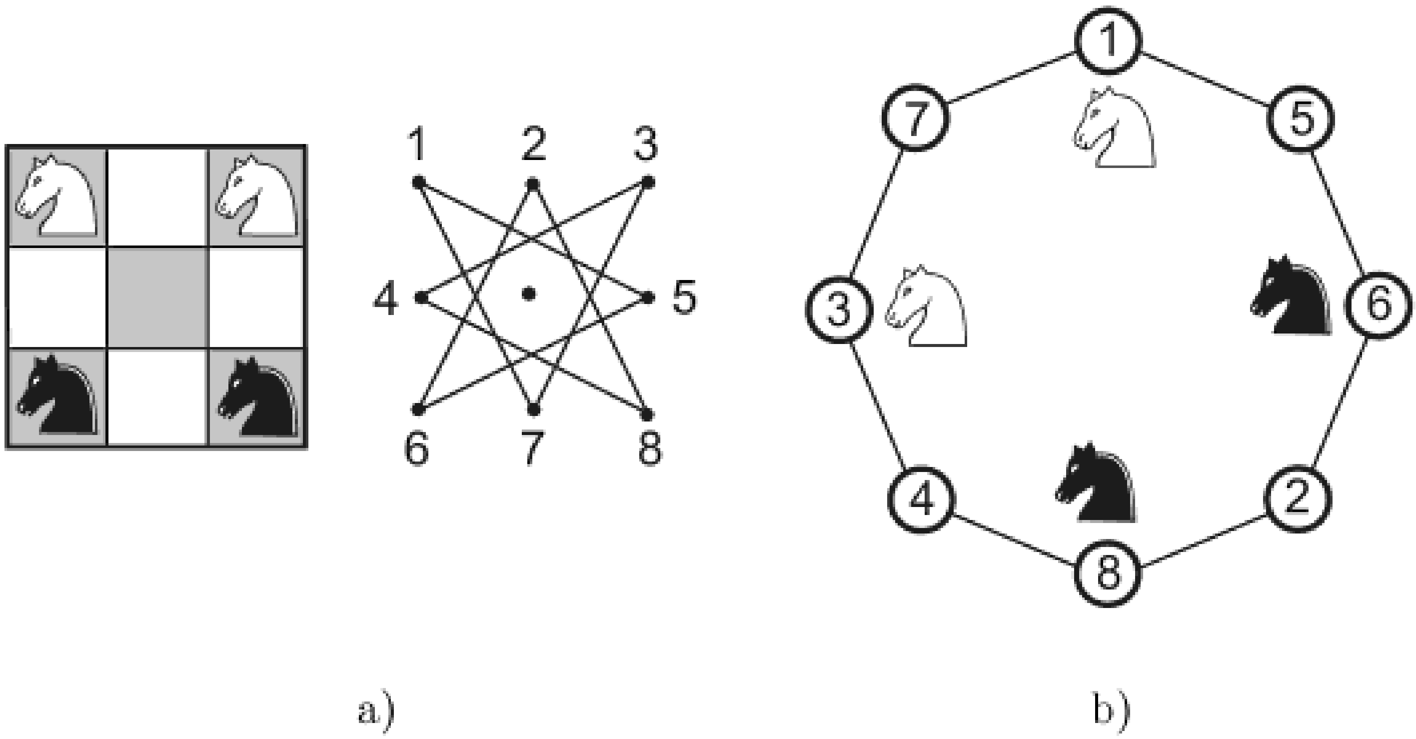


FIGURE 10.10. a) Graph to Guarini's problem b) Equivalent simplified graph

The initial positions of the knights are indicated and all possible moves of the knights between the squares (the nodes of the graph) are marked by lines. Using Dudeney's famous "method of unraveling a graph,"[11] starting from any node, the graph 10.10(a) can be "unfolded" to the equivalent graph 10.10(b), which is much clearer and more convenient for the analysis. Obviously, the topological structure and the connectedness are preserved. To find the solution it is necessary to write down the moves (and reproduce them on the 3×3 board according to some correspondence), moving the knights along the circumference of the graph until they exchange places. The minimum number of moves is 16 although the solution is not unique (because the movement of the knights along the graph is not unique). Here is one solution:

[11]This "method" was described in detail by E. Gik in the journal *Nauka i Zhizn* 12 (1976); see also M. Gardner, *Mathematical Puzzles and Diversions* (New York: Penguin, 1965).

1–5 6–2 3–7 8–4 5–6 2–8 7–1 4–3
1–5 8–4 6–2 3–7 5–6 7–1 2–8 4–3.

A similar problem also involves two white and two black chess knights and requires their interchange in the fewest possible moves.

Problem 10.14.* *Two white and two black knights are placed on a board of an unusual form, as shown in Figure* 10.11. *The goal is to exchange the white and black knights in the minimum number of moves.*

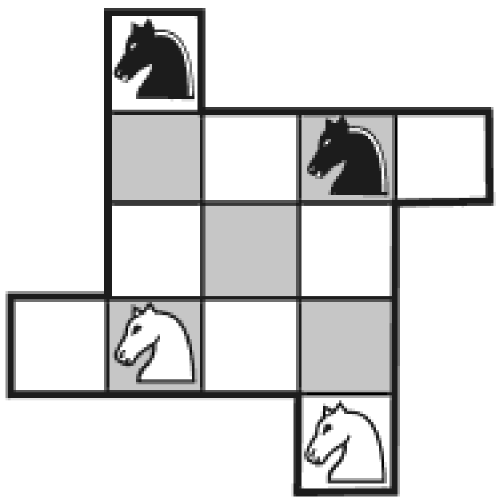


FIGURE 10.11. Knight-switching problem

Answers to Problems

10.2. Suppose that the required knight's re-entrant route exists. We assume that this board is colored alternately white and black (in a chess order). The upper and lower row will be called the *outer* lines (O), and the two remaining rows the *middle* lines (M). Since a knight, starting from any *outer* square, can land only on a *middle* square, it follows that among $4n$ moves that should make the route, $2n$ moves must be played from the *outer* to the *middle* squares. Therefore, there remain exactly $2n$ moves that have to be realized from the *middle* to the *outer* squares.

Since any square of the closed knight's tour can be the starting square, without loss of generality, we can assume that we start from a "middle" square. The described tour gives an alternate sequence

$$M(\text{start}) - O - M - O - \cdots - M - O - M(\text{finish}), \qquad (10.4)$$

ending at the starting square. We emphasize that a knight can't dare visit two middle squares in a row anywhere along the tour because of the following. Assume that we start with this double move $M-M$ (which is always possible because these moves belong to the circuit), then we will have the sequence

$M - M - (2n-1) \times (O - M)$. In this case we have $2n+1$ M moves and $2n-1$ O moves, thus each different from $2n$. Note further that the double move $O - O$ in the parenthesis in the last sequence is impossible because a knight cannot jump from the outside line to the outside line.

On the other hand, the same knight's tour alternates between white and black squares, say,

$$\textit{black} - \textit{white} - \textit{black} - \textit{white} - \cdots - \textit{black} - \textit{white} - \textit{black} \tag{10.5}$$

(or opposite). Comparing the sequences (10.4) and (10.5) we note that all squares of the outer lines are in one color and all squares of the middle lines are in the other color. But this is a *contradiction* since the board is colored alternately. Thus, the required path is impossible.

10.3. One solution is displayed in Figure 10.12. The three-dimensional 4×4×4 board is represented by the four horizontal 4×4 boards, which lie one over the other; the lowest board is indicated by I, the highest by IV. The knight's moves are denoted by the numbers from 1 (starting square) to 64 (ending square). The knight can make a re-entrant tour because the squares 64 and 1 are connected by the knight's moves.

57	30	47	36
48	33	58	31
29	60	35	46
34	45	32	59

I

42	37	56	51
55	52	43	40
38	41	50	53
49	54	39	44

II

27	62	15	2
14		26	63
61	28	3	16
4	13	64	25

III

10	7	22	17
21	18	9	6
8	11	20	23
19	24	5	12

IV

Figure 10.12. Knight's re-entrant path on the $4 \times 4 \times 4$ board

10.5. Let S be the area of a square of the $n \times n$ chessboard. Considering formula (10.1) in Pick's theorem, it is sufficient to prove that the number of boundary points B is even. Since the knight's tour alternates between white (w) and black squares (b), in the case of any closed tour (the starting square coincides with the ending square) it is easy to observe that the number of traversed squares must be even. Indeed, the sequence $b\,(\text{start}) - w - b - w - \cdots - b - w - b\,(\text{finish})$, associated to the closed knight's path, always has an even number of moves (= traversed squares); see Figure 10.13. Since the number of squares belonging to the required closed knight's path is equal to the number of boundary points B, the proof is completed.

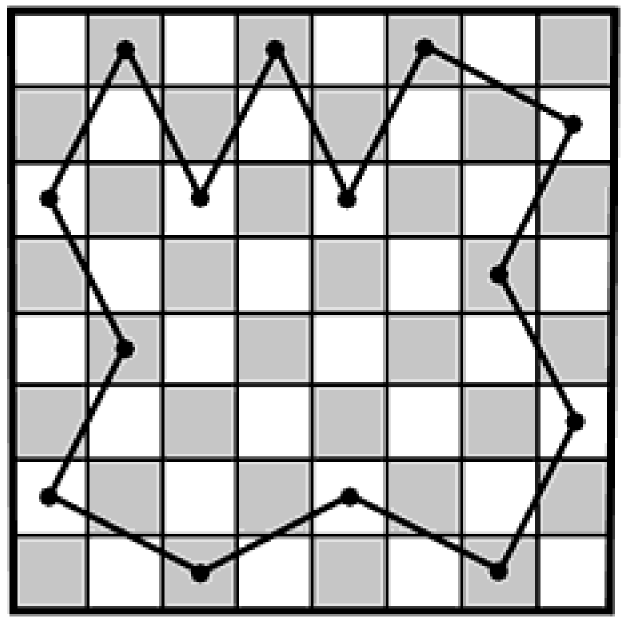


FIGURE 10.13. Area of a simply closed lattice polygon

10.6. As mentioned by M. Gardner in *Scientific American* 7 (1967), S. Golomb solved the problem of a camel's tour by using a transformation of the chessboard suggested by his colleague Lloyd R. Welch and shown in Figure 10.14: the chessboard is covered by a jagged-edged board consisting of 32 cells, each of them corresponding to a black square. It is easy to observe that the camel's moves over black squares of the chessboard are playable on the jagged board and turn into knight's moves on the jagged board. Therefore, a camel's tour on the chessboard is equivalent to a knight's tour on the jagged board. One simple solution is

1–14–2–5–10–23–17–29–26–32–20–8–19–22–9–21–18–30–27–15–3–6–11–24–12–7–4–16–28– 31–25–13.

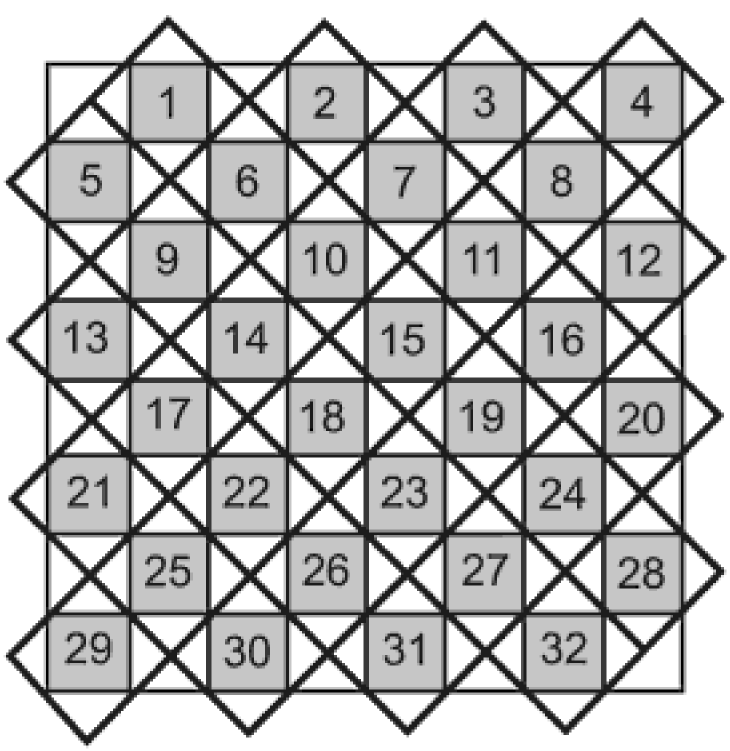


FIGURE 10.14. Solution of camel's tour by transformation

10.9. If you have not succeeded in solving the given problem, see the following solutions found by Y. Kusaka [120]. Using a computer program and backtracking algorithm he established that there are only 10 solutions in this eight queens problem with constraints (we recall that this number is 92 for the ordinary case; see Table 10.1 for $n = 8$):

25741863	27581463	36418572	36814752	36824175
37286415	42736815	51468273	51863724	57142863

10.10. The author of this book provided in his book *Mathematics and Chess* [138] a computer program in the computer language C that can find all possible solutions: the fundamental one and similar ones obtained by the rotations of the board and by the reflections in the mirror. The program runs for arbitrary n and solves the standard n-queens problem as well as the n-superqueens problem. We emphasize that the running time increases very quickly if n increases.

The fundamental solution is shown in Figure 10.15, which can be denoted as the permutation (3,6,9,1,4,7,10,2,5,8). As before, such denotation means that one superqueen is on the third square of the first column, one on the sixth square of the second column, and so on.

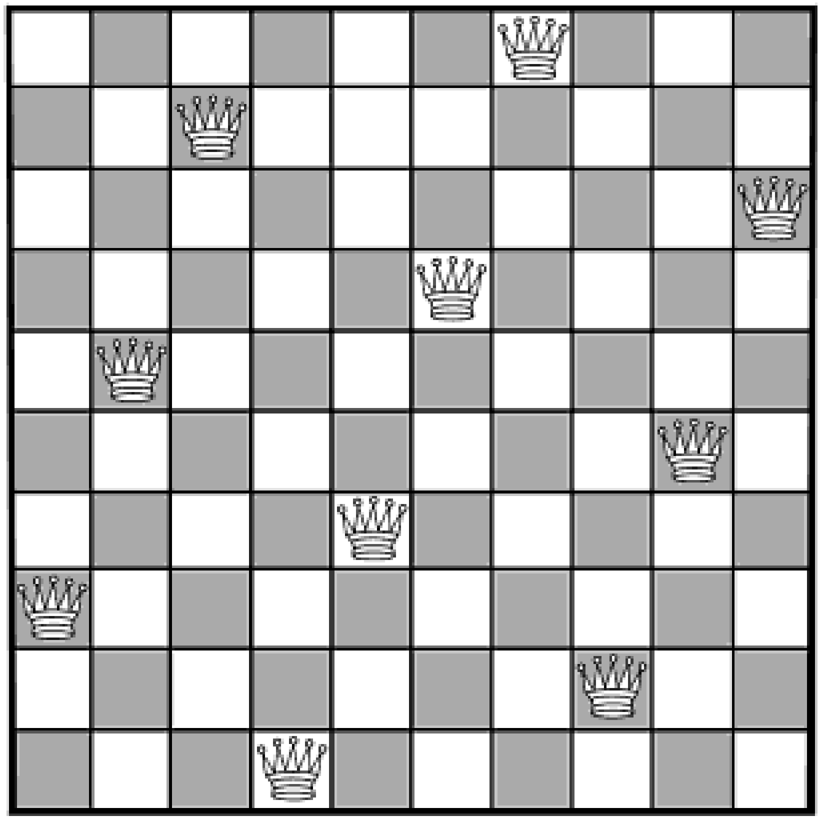


FIGURE 10.15. The fundamental solution of the superqueens problem for $n = 10$

The three remaining solutions (found by the computer) arise from the fundamental solution, and they can be expressed as follows:

(7,3,10,6,2,9,5,1,8,4) (4,8,1,5,9,2,6,10,3,7) (8,5,2,10,7,4,1,9,6,3)

10.11. There is no solution of the eight queens problem on the cylindrical chessboard of the order 8. We follow the ingenious proof given by E. Gik [84].

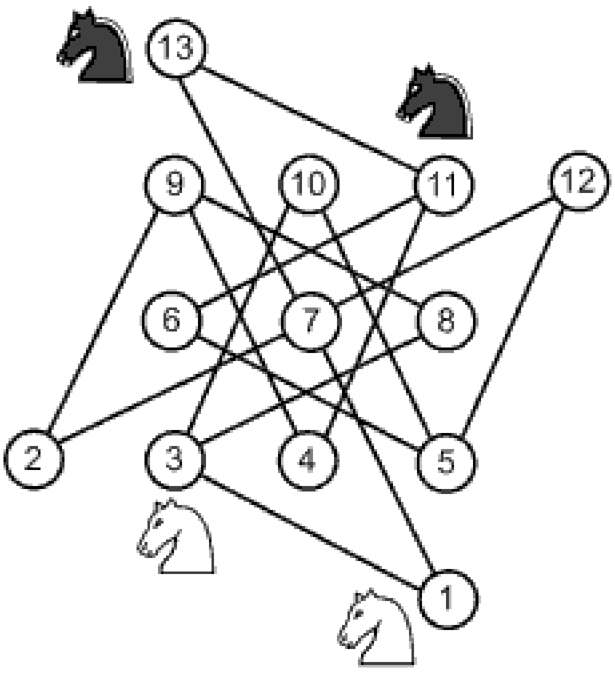
Let us consider an ordinary chessboard, imagining that its vertical edges are joined ("vertical cylindrical chess"). Let us write in each of the squares three digits (i, j, k), where $i, j, k \in \{1, \ldots, 8\}$ present column, row, and diagonal (respectively) of the traversing square (Figure 10.16). Assume that there is a replacement of 8 non-attacking queens and let $(i_1, j_1, k_1), \ldots, (i_8, j_8, k_8)$ be the ordered triples that represent 8 occupied squares. Then the numbers $i_1, \ldots, i_8$ are distinct and belong to the set $\{1, \ldots, 8\}$; therefore, $\sum i_m = 1 + \cdots + 8 = 36$. The same holds for the numbers from the sets $\{j_1, \ldots, j_8\}$ and $\{k_1, \ldots, k_8\}$.

187	286	385	484	583	682	781	888
178	277	376	475	574	673	772	871
161	268	367	466	565	664	763	862
152	251	358	457	556	655	754	853
143	242	341	448	547	646	745	844
134	233	332	431	538	637	736	835
125	224	323	422	521	628	727	826
116	215	314	413	512	611	718	817

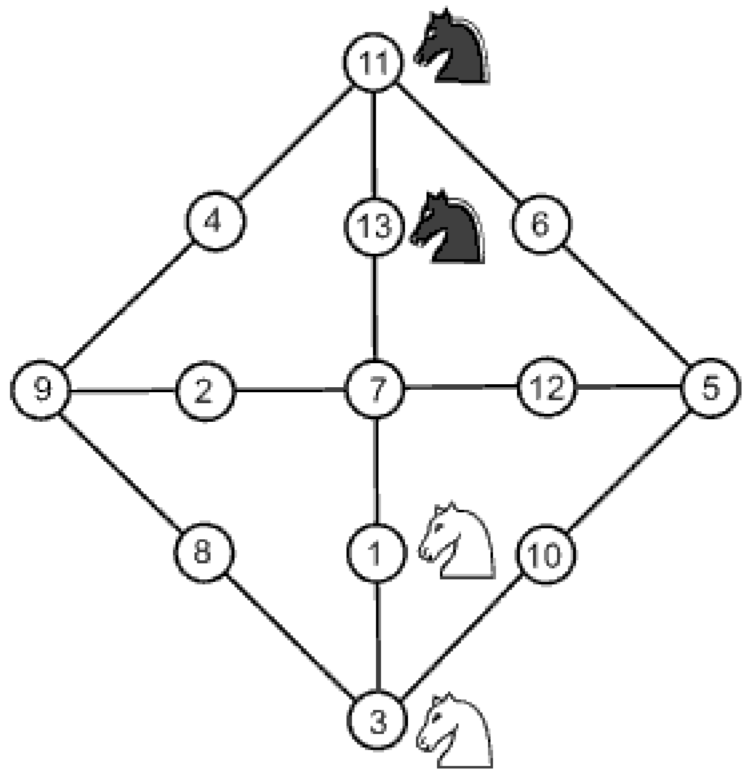
FIGURE 10.16. Gik's solution

We see that the sum $(i_1 + \cdots + i_8) + (j_1 + \cdots + j_8) + (k_1 + \cdots + k_n)$ of all 24 digits written in the squares occupied by the queens is equal to $(1 + \cdots + 8) \times 3 = 108$. Since the sum $i_\nu + j_\nu + k_\nu$ of the digits on each of the squares is divided by 8 (see Figure 10.16), it follows that the sum of the mentioned 24 digits must be divisible by 8. But 108 is not divisible by 8—a contradiction, and the proof is completed, we are home free.

10.14. Although the chessboard has an unusual form, the knight-switching problem is effectively solved using graphs, as in the case of Guarini's problem 10.13. The corresponding graph for the board and the knight's moves is shown in Figure 10.17(a), and may be reduced to the equivalent (but much simpler) graph 10.17(b).



a)



b)

FIGURE 10.17. A graph of possible moves and a simplified graph

The symmetry of the simplified graph and the alternative paths of the knights along the graph permit a number of different solutions, but the minimum number of 14 moves cannot be decreased. Here is one solution:

13–7–2 11–4–9–8 1–7–13–11 3–1–7–13 8–3 2–7–1

Chapter 11 MISCELLANY

In this chapter the reader will find some interesting problems and puzzles of Alcuin of York, Abu'l-Wafa, Fibonacci, Bachet, Huygens, Newton and Euler. These problems have not been classified into previous chapters mainly for two reasons: either they are similar to the presented problems or they do not clearly belong to the subjects considered in the previous chapters. We leave it to the readers to find the solutions.

*

* *

Problems from Alcuin of York[1]

Problem 11.1.* *Three sons must equally divide and share thirty flasks and their contents among themselves. How will they accomplish this given that of the thirty flasks, ten are full, ten half-empty, and ten entirely empty?*

Problem 11.2.* *A rabbit pursued by a dog has a head start of* 150 *feet. For every* 7 *feet that the rabbit jumps, the dog bounds* 9 *feet. In how many leaps will the dog overtake the rabbit?*

Problem 11.3.* *In his will, a dying man stipulates that his wife, who is expecting a child, shall inherit* $\frac{1}{4}$ *of the property if she gives birth to a son, and the son shall inherit the other* $\frac{3}{4}$*. However, if his widow gives birth to a daughter, she will receive* $\frac{5}{12}$ *of the property, and the daughter,* $\frac{7}{12}$*. In the event that both a son and a daughter are born, how will the property be divided?*

Problems from Abu'l-Wafa[2]

Problem 11.4.* *Construct an equilateral triangle embedded in a given square so that one of its vertices is at a corner of the square and the other two lie on the opposite sides of the square.*

---

[1] These problems of various origin are contained in Alcuin's collection of problems for "quickening of the mind".

[2] The solutions of these five problems can be found in [186].

Problem 11.5.* *Dissect two regular hexagons of different sizes into seven pieces and then assemble one, larger, regular hexagon from the seven available pieces.*

Problem 11.6.* *Construct the perpendicular to the segment AB at the endpoint A using only a straightedge and a fixed-opening compass, without extending the segments beyond A.*

Problem 11.7.* *Divide a given line segment into any given number of equal parts using only a straightedge and fixed-opening compass.*

Problem 11.8.* *Using only a straightedge and a compass with a fixed opening equal to the radius of a given circle, construct a regular pentagon with vertices on this circle.*

Amusing problems from Fibonacci

Among the many problems that Fibonacci includes in the third section of *Liber abacci*, here are six presented in the form as in [61, Ch. 8]:

Problem 11.9.* *A lion trapped in a pit 50 feet deep tries to climb out of it. Each day he climbs up $\frac{1}{7}$ of a foot, but each night slips back $\frac{1}{9}$ of a foot. How many days will it take the lion to reach the top of the pit?*[3]

Problem 11.10.* *Two men each possess a certain amount of money. The first says to the second, "If you give me nine denarii, we will both have the same amount." The second man replies to the first, "If you give me nine denarii, I will have ten times as much as you." How much money does each man have?*

Problem 11.11.* *A hound whose speed increases arithmetically chases a hare whose speed also increases arithmetically; how far do they run before the hound catches the hare?*

Problem 11.12.* *If partridges cost 3 coins each, pigeons 3 coins each, and sparrows cost 1 coin for 2, how many birds of each kind will a merchant have if he buys 30 birds for 30 coins?*

[3]Fibonacci, by the way, gave a false solution; see [113, p. 308]. He started from 63 as a number divisible by both 7 and 9 and found that in 63 days the lion would climb up 9 feet and fall down 7. Hence, the lion advances 2 feet every day and, by proportionality, he calculated that the lion would take $(50 : 2) \times 63 = 1575$ days to climb the 50 feet to reach the top of the pit. The correct answer is 1572 days; actually, the lion will be only $\frac{8}{63}$ of a foot from the top at the end of 1571 days, so that he will reach the top on the next day.

Problem 11.13.* *Four men each have a certain sum of money. Together, the first, second, and third have* 27 *denarii; the second, third, and fourth men have* 31 *denarii among them; the third, fourth, and first have* 34*; finally, the fourth, first and second men have* 37*. How much money does each man have?*

Problem 11.14.* *A man bequeathed one bezant and a seventh of his estate to his eldest son. To his next son he left two bezants and again, a seventh of what was left of the estate. Next, from the new remainder, he left three bezants and a seventh of what remained to his third son. He continued in this way, giving each son one bezant more than the previous son and a seventh of what remained. As a result of this distribution, the last son received all that was left and all the sons shared equally. How many sons were there and how large was the man's estate?*

Problems from Bachet

Problem 11.15.* *A person randomly chooses an hour, say m, and then points to it on a watch displaying some other hour, say n. Beginning with the randomly-chosen hour, and moving in the counterclockwise direction, if the person counts each successive numeral on the watch as $m, m+1$, etc., until he reaches $n+12$, then the last numeral he points to will be the hour he originally chose at random. Prove this.*

Problem 11.16.* *Within a group of people, an individual secretly chooses a number less than* 60*, and announces the remainders, for example, a, b, c, when the chosen number is divided by* 3*, by* 4*, and by* 5*. Prove that the number originally chosen equals the remainder obtained when $40a+45b+36c$ is divided by* 60.

Huygens' probability problems

Problem 11.17.* *A and B play a game tossing two dice; A wins if* 7 *is thrown; B wins if* 10 *is thrown; the game is a draw if any other number is thrown. Find the chances of winning for players A and B.*

Problem 11.18.* *An urn holds* 4 *white balls and* 8 *black balls for a total of* 12*. Three players, A, B, C, each blindfolded, draw a ball, A first, next B, and then C. The player who wins is the first one to draw a white ball. If each black ball is replaced after a player has drawn it, find the ratio of the three players' chances.*

Problem 11.19.* *A certain pack of cards contains* 40 *cards,* 10 *from each suit. A wagers B that in drawing four cards, he will draw one card from each suit. What amounts can be fairly wagered for each player?*

Problems from Newton

Problem 11.20.* *One morning two couriers A and B, separated by a distance of* 59 *miles, set out to meet each other. While A has completed* 7 *miles in* 2 *hours, B travels* 8 *miles in* 3 *hours; B, however, started his journey* 1 *hour later than A. How far a distance must A still travel before meeting B?*

Problem 11.21.* *A certain scribe takes* 8 *days to copy* 15 *sheets. How many scribes, capable of producing the same amount, will be needed to copy* 405 *sheets in* 9 *days?*

Problem 11.22.* *Among three workmen, every three weeks A finishes a given job once; every eight weeks, B completes the job three times, and C finishes the same job five times every twelve weeks. If the three workmen undertake to complete the job together, how long will it take them?*

Problem 11.23.*[4] *A number of unbiased (fair) dice are rolled simultaneously. Determine which of the following events is most likely:*

1) *The appearance of at least one six when* 6 *dice are rolled;*
2) *The appearance of at least two sixes when* 12 *dice are rolled;*
3) *The appearance of at least three sixes when* 18 *dice are rolled.*

Problems from Euler

Problem 11.24.* *Twenty men and women have dinner at a tavern. Each man's share of the bill is* 8 *crowns, each woman's share* 7 *crowns; the entire bill amounts to* 145 *crowns. Of the twenty diners, how many are men and how many are women?*

Problem 11.25.* *A horse dealer buys a horse for a certain number of crowns, and then sells it again for* 119 *crowns. This amount includes his*

[4]This problem was posed by Samuel Pepys (1633–1703) in a letter to Isaac Newton; see E. D. Schell's paper, *Samuel Pepys, Isaac Newton and Probability*, The American Statician 14 (1960), 27–30. Pepys was an English naval administrator, a member of Parliament and a fellow of the Royal Society, and wrote a famous diary.

profit, which was as much per cent as the horse cost him. What is the initial purchase price of the horse?

Problem 11.26.* *Three brothers buy a vineyard for* 100 *crowns. The youngest says that he could purchase the vineyard on his own if the second brother gives him half of the money he has; the second brother says that if the eldest would give him only a third of his money, he could pay for the vineyard by himself; lastly, the eldest asks for only a fourth of the youngest brother's money to purchase the vineyard himself. How much money does each brother have?*

Problem 11.27.* *Three gamblers play together; in the first game, player one loses to the two other players a sum of money equal to the sum that each of the other two players possesses. In the next game, the second gambler loses to each of the other two as much money as they have already. In the third game, the first and second gamblers each gain from the third gambler as much money as they had before. At that moment they stop their play to discover that they all possess the equal sum of* 24 *crowns each. How much money does each gambler possess when they first begin to play?*

APPENDIX A

Method of continued fractions for solving Pell's equation

In this book Pell's equation appears in several tasks studied by great mathematicians. To clarify the presentation of solutions of these tasks, we give in the sequel an appendix which presents a procedure for solving Pell's equation

$$x^2 - Ny^2 = 1,$$

where N is a natural number which is not a perfect square. For more details see, *e.g.*, Davenport's book [45]. Since the described method uses the convergents of a continued fraction of $\sqrt{N}$, let us recall first a basic fact from the theory of continued fractions.

The expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}}$$

is called a *continued fraction* and can be written in the short form

$$\left[a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \cdots\right]. \tag{A.1}$$

In particular, if $b_1 = b_2 = \cdots = b_n = \cdots = 1$, then the following simpler notation is used:

$$\left[a_0; a_1, a_2, a_3, \ldots\right].$$

Let us consider a sequence of continued fractions which is obtained from (A.1) taking a finite number of its terms, and set

$$c_k = \frac{P_k}{Q_k} = \left[a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots \frac{b_k}{a_k}\right].$$

The ratio c_k is called the *k-convergent* and the limit

$$c = \lim_{k\to\infty} c_k,$$

if it exists, is the value of the continued fraction (A.1).

Taking

$$P_0 = a_0, \quad Q_0 = 1, \quad P_{-1} = 0, \quad Q_{-1} = 0,$$

by induction one can prove the following recurrence relations for P_k and Q_k:

$$\begin{aligned} P_k &= a_k P_{k-1} + b_k P_{k-1}, \\ Q_k &= a_k Q_{k-1} + b_k Q_{k-2}. \end{aligned} \tag{A.2}$$

These relations are known as Euler's formulae. Let us note that P_k and Q_k are two solutions of the difference equation

$$y_k - a_k y_{k-1} - b_k y_{k-2} = 0. \tag{A.3}$$

Let $\lfloor n \rfloor$ denote the integer part of a number n. If q_0 is the integer part of $\sqrt{N}$ (that is, $q_0 = \lfloor \sqrt{N} \rfloor$), then it can be shown (see [45]) that the continued fraction for $\sqrt{N} + q_0$ is purely periodic and has the form

$$\sqrt{N} + q_0 = \Big[2q_0; q_1, q_2, \dots, q_n, 2q_0, q_1, q_2, \dots, 2q_0, \dots\Big].$$

Hence, the continued fraction for $\sqrt{N}$ is of the form

$$\sqrt{N} = \Big[q_0; \overline{q_1, q_2, \dots, q_n, 2q_0}\Big],$$

where the overline points to the period. The period begins immediately after the first term q_0. In addition, the following is valid:

$$q_n = q_1, \; q_{n-1} = q_2, \dots .$$

Hence, we have

$$\sqrt{N} = \big[q_0; \overline{q_1, q_2, \dots, q_2, q_1, 2q_0}\big].$$

The symmetric part may or may not have a central term.

The development of $\sqrt{N}$ into a continued fraction is easy to realize by implementing the following simple procedure:

1° $k = 0, \quad A_k = \sqrt{N}, \quad q_k = \lfloor A_k \rfloor$;

2° $k := k + 1, \quad A_k = 1/(A_{k-1} - q_{k-1}), \quad q_k = \lfloor A_k \rfloor$;

3° If $q_k = 2q_0$, then STOP; otherwise, go to step 2°.

For example, using the above procedure, we find that

$$\sqrt{19} = \Big[4; \overline{2, 1, 3, 1, 2, 8}\Big]$$

with the central term 3, and

$$\sqrt{29} = \left[5; \overline{2, 1, 1, 2, 10}\right]$$

without a central term.

Now we may present a method for solving Pell's equation. Let A_n/B_n be the convergent coming before the term $2q_0$ in the continued fraction for $\sqrt{N}$, that is,

$$\frac{A_n}{B_n} = \left[q_0; q_1, q_2, \dots, q_n\right].$$

It is not difficult to prove that A_n and B_n satisfy the relation

$$A_n^2 - NB_n^2 = (-1)^{n-1}.$$

Hence, if n is odd, $x = A_n$ and $y = B_n$ are solutions of Pell's equation. If n is even, we continue searching for the convergent until the end of the next period A_{2n+1}/B_{2n+1},

$$\frac{A_{2n+1}}{B_{2n+1}} = \left[q_0; q_1, q_2, \dots, q_n, 2q_0, q_1, q_2, \dots, q_n\right].$$

In this case we have

$$A_{2n+1}^2 - NB_{2n+1}^2 = (-1)^{2n} = 1$$

and we take $x = A_{2n+1}$ and $y = B_{2n+1}$ for the solutions of Pell's equation.

We recall that the numerator and denominator of the convergents A_n/B_n and A_{2n+1}/B_{2n+1} can be found using Euler's recurrence relations

$$\begin{aligned} A_m &= q_m A_{m-1} + A_{m-2}, \\ B_m &= q_m B_{m-1} + B_{m-2}, \end{aligned} \qquad (m = 1, 2, \dots)$$

starting with $A_0 = q_0,\ B_0 = 1,\ A_{-1} = 0,\ B_{-1} = 0$ (see (A.2)). Below we give a program written in the *Mathematica* for finding the least solutions of Pell's equation $x^2 - ny^2 = 1$; n can take arbitrary values (odd and even) from the set of natural numbers that are not perfect squares.

```
Pel[n ] :=
Module[{k = 1, A = Sqrt[n], a, vek, ok, kk, m, i,
a1, b1, a2, b2, a3, b3},
a = IntegerPart[A]; vek[k] = a; ok = True;
```

```
While[ok, k++; A = 1/(A - a); a = IntegerPart[A];
vek[k] = a; ok = vek[k] < 2 vek[1] ];
kk = k - 1; m = k - 2 IntegerPart[k/2];
If[m < 1, Do[vek[k + i] = vek[1 + i], i, k - 2] ];
kk = (k - 1) (2 - m); a1 = vek[1]; b1 = 1;
a2 = a1 vek[2] + 1; b2 = vek[2];
Do[a3 = vek[i]*a2 + a1; b3 = vek[i]*b2 + b1;
a1 = a2; b1 = b2; a2 = a3; b2 = b3, {i, 3, kk} ];
{a2, b2} ]
```

Solving Pell's equation $x^2 - ny^2 = 1$ is executed by calling `Pel[n ]` for a specific value of n.

We note that from one solution of Pell's equation $x^2 - Dy^2 = \pm 1$, an infinite number of solutions may be found. If p and q are the least values satisfying the equation $x^2 - Dy^2 = 1$, then

$$x^2 - Dy^2 = (p^2 - Dq^2)^n = 1,$$

which leads to the factorization

$$(x + y\sqrt{D})(x - y\sqrt{D}) = (p + q\sqrt{D})^n (p - q\sqrt{D})^n.$$

Equating the factors with the same sign, we obtain

$$x + y\sqrt{D} = (p + q\sqrt{D})^n,$$
$$x - y\sqrt{D} = (p - q\sqrt{D})^n.$$

Solving for x and y we get the general formulae:

$$x = \frac{(p + q\sqrt{D})^n + (p - q\sqrt{D})^n}{2}, \tag{A.4}$$

$$y = \frac{(p + q\sqrt{D})^n - (p - q\sqrt{D})^n}{2\sqrt{D}}. \tag{A.5}$$

Taking $n = 1, 2, 3, \dots$ we obtain an infinite number of solutions.

Basic facts from the theory of difference equations (see Appendix D) indicate that formulae (A.4) and (A.5) present a general solution of a difference equation (with different initial conditions), whose characteristic equation has the roots

$$r_1 = p + q\sqrt{D}, \quad r_2 = p - q\sqrt{D}.$$

Therefore, this equation reads $(r - r_1)(r - r_2) = 0$, that is,

$$r^2 - 2pr + p^2 - q^2 D = 0.$$

Hence, the difference equation, associated with Pell's equation $x^2 - Dy^2 = 1$, has the form

$$y_{n+2} - 2py_{n+1} + (p^2 - q^2 D)y_n = 0. \tag{A.6}$$

Formula (A.4) is obtained for the initial values $y_0 = 1,\ y_1 = p$, while (A.5) arises for $y_0 = 0,\ y_1 = q$.

APPENDIX B

Geometrical inversion

Below we give a review of the basic properties of inversion necessary for solving the arbelos problem (page 68). For more details see, *e.g.*, Coxeter's book *Introduction to Geometry* [43, Ch. 6].

Given a fixed circle with center I and radius ρ (Figure B.1(a)), we define the *inverse* of any point A (distinct from I) as the point A' on the straight line IA whose distance from I satisfies the equation

$$|IA| \cdot |IA'| = \rho^2.$$

The circle centered at I (the *center of inversion*) and with radius ρ is called the *circle of inversion*.

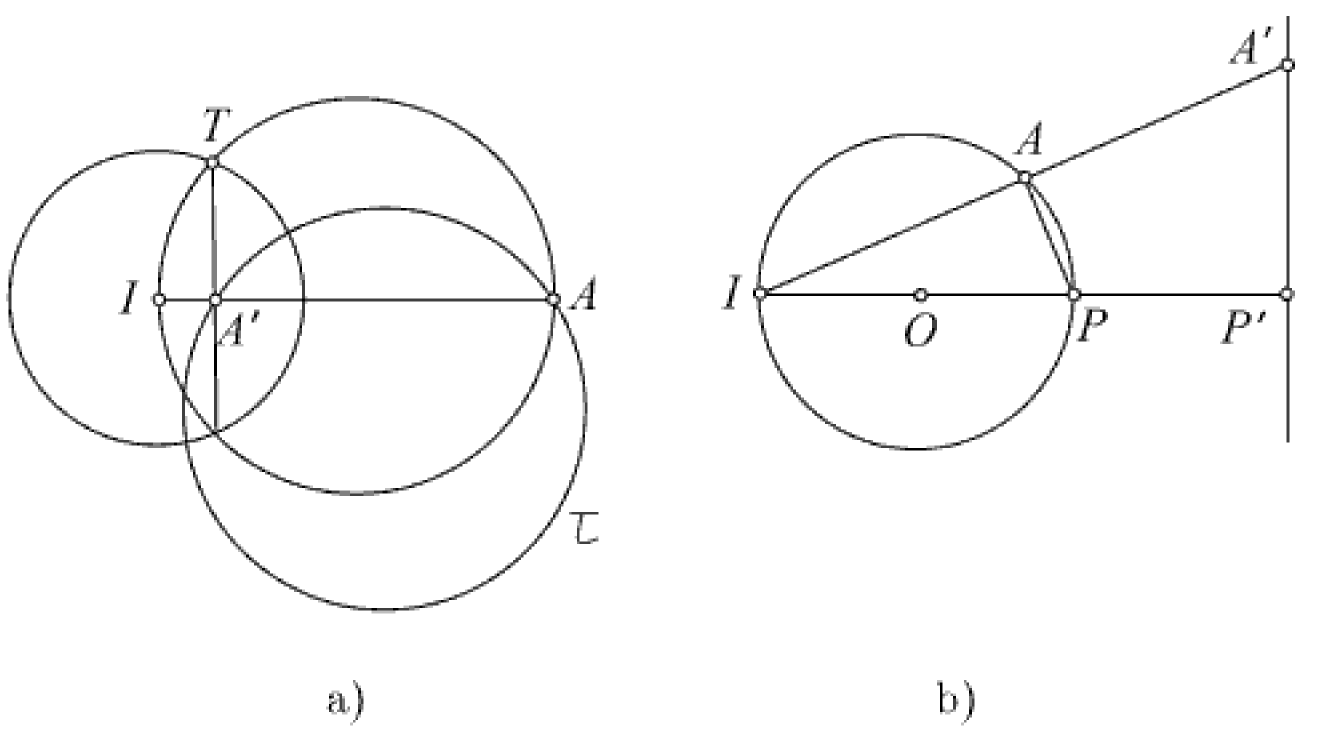


FIGURE B.1. Inverse of circles and straight lines

To find the inverse of a given point A outside the circle of inversion, let us construct the semicircle on IA as diameter. If T is the point of intersection of this semicircle and the circle of inversion, then the desired point A' is the foot of the perpendicular from T to IA (see Figure B.1(a)). Indeed, since the right triangles $\triangle ITA$ and $\triangle IA'T$ are similar and $|IT| = \rho$, we obtain

$$\frac{|IA'|}{|IT|} = \frac{|IT|}{|IA|}, \quad \text{whence} \quad |IA'| \cdot |IA| = |IT|^2 = \rho^2.$$

It follows from the definition that the inverse of A' is A itself.

From the definition of inversion, the following properties follow:

(1) The straight line (circle) through the pair of associated points A and A' inverts into itself and represents a *fixed* line (circle) of inversion (for example, the straight line IA and the circle τ in Figure B.1(a)).

(2) Every circle orthogonal to the circle of inversion inverts into itself.

(3) All points belonging to the circle of inversion are fixed and invariable. Accordingly, the circle of inversion inverts into itself.

(4) Any straight line not passing through the center of inversion I inverts into a circle through I, and vice versa; a circle through I inverts into a straight line not passing through I. For example, the circle centered at O that passes through the center of inversion I inverts into the straight line $A'P'$ as in Figure B.1(b).

(5) The inverse of any circle not passing through the center of inversion I is another circle that also does not pass through I.

(6) Inversion preserves angles between curves, although it reverses the sense of the angles, and hence it preserves tangency of curves.

APPENDIX C

Some basic facts from graph theory

In graph theory a graph is defined as any set of points, some of which are connected by line segments. The points are called *vertices* or *nodes* of the graph (*e.g.*, the points A, B, C, D, E in Figure C.1(a)), and the lines are called its *edges*. The line segment that joins two adjacent vertices is *incidental* for these two vertices. A graph that has no self-loops, (that is, lines joining a point to itself, for instance, λ and τ as in Figure C.1(b) and C.1(d)) and no multiple edges of two or more lines connecting the same pair of points, is a *simple graph* (Figure C.1(a)), otherwise it is a *multigraph* (Figure C.1(b)).

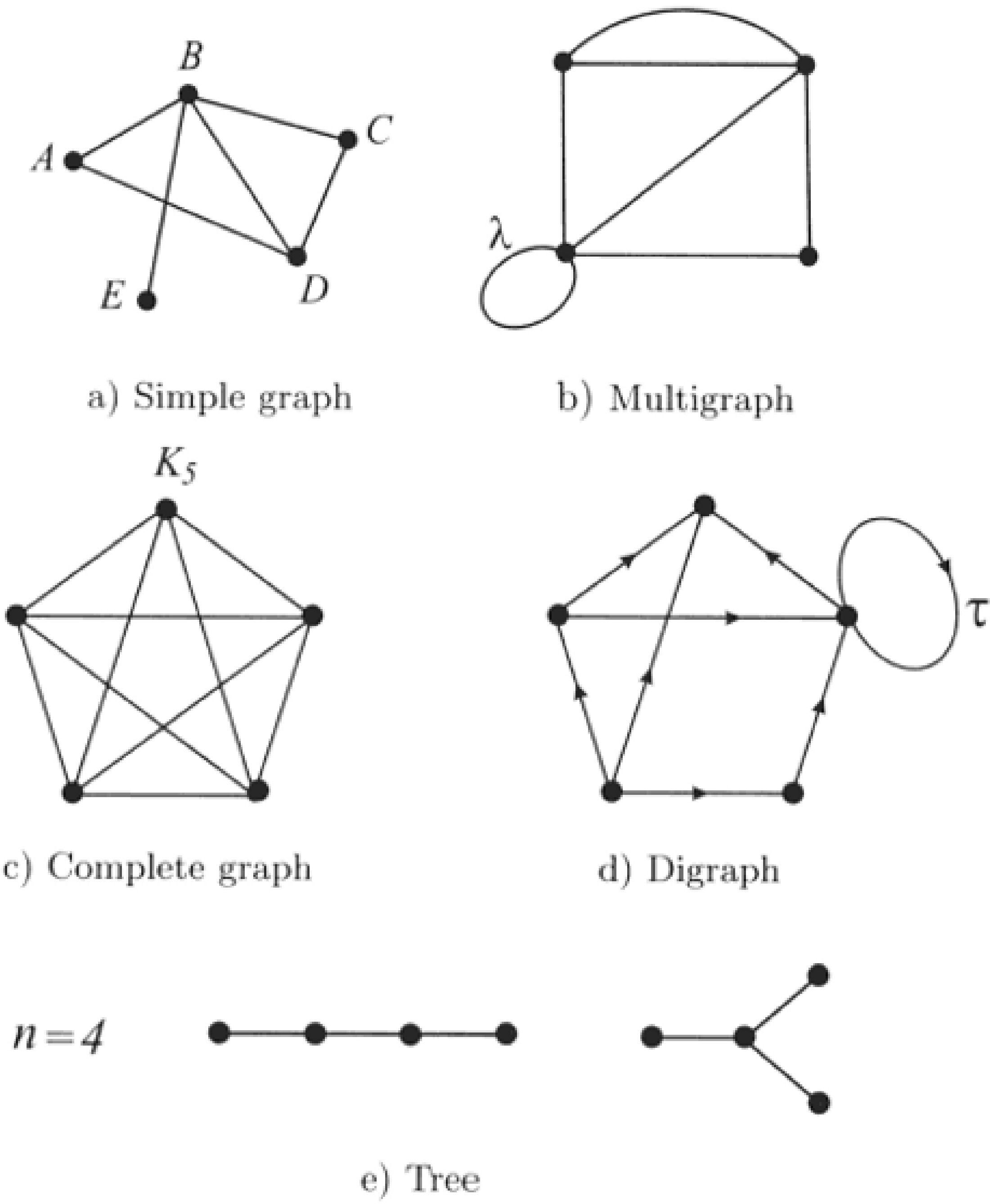


FIGURE C.1. Various types of graphs

The *degree* of a vertex is the number of edges that are connected to this vertex, counting self-loop and parallel lines. The degree of a vertex, say a, is denoted by $\deg(a)$. For example, the degrees of the vertices in Figure C.1(a) are $\deg(A) = 2$, $\deg(B) = 4$, $\deg(C) = 2$, $\deg(D) = 3$, $\deg(E) = 1$. A graph is *regular* if all its vertices have the same degree (as the one shown in Figure C.1(c)). A vertex of a graph having an odd degree is called an *odd* vertex. A vertex having an even degree is called an *even* vertex. Since the sum of several integers is even if and only if the number of odd addends is even, it follows that the number of odd vertices in any graph must be even.

A graph is called *connected* if any two of its vertices are connected by a path (a sequence of edges, each of which starts at the endpoint of the previous one). A closed path, a path whose starting and ending vertices coincide, is called a *cycle*. A graph consisting of n vertices is *complete* (denoted by K_n) if each vertex is connected with all $n-1$ remaining vertices, that is, if any two vertices are adjacent. Figure C.1(c) shows a complete graph K_5.

A connected graph that does not contain any cycle is called *tree*. The tree contains exactly $n-1$ edges. The two trees for $n=4$ are displayed in Figure C.1(e).

If an arrowhead is added to each edge of a graph, determining a direction for each line that orders its endpoints, the graph becomes a *directed graph*, or briefly *digraph*. Figure C.1(d) shows a digraph.

APPENDIX D

Linear difference equations with constant coefficients

A *linear difference equation of order n with constant coefficients* is a recurrence relation having the form

$$a_0 y_{n+k} + a_1 y_{n+k-1} + \cdots + a_n y_k = f(k), \tag{D.1}$$

where $a_0 \neq 0$ and $a_0, a_1, \ldots, a_n$ are all real constants. If $f(k) = 0$, then we have the *homogenous linear difference equation*

$$a_0 y_{n+k} + a_1 y_{n+k-1} + \cdots + a_n y_k = 0. \tag{D.2}$$

Assume that $y_k = r^k$ is a solution of (D.2). Then, upon substitution we obtain

$$a_0 r^{n+k} + a_1 r^{n+k-1} + \cdots + a_n r^k = 0$$

or

$$r^k (a_0 r^n + a_1 r^{n-1} + \cdots + a_n) = 0.$$

Hence it follows that r^k is a solution of (D.2) if r is a solution of

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_n = 0, \tag{D.3}$$

which is called the *characteristic equation*. This equation has n roots $r_1, r_2, \ldots, r_n$ which may or may not be different. Also, some conjugate complex numbers may occur among these roots.

We restrict our attention to the case when $r_1, \ldots, r_n$ are all real and distinct roots of the equation (D.3). Then one can prove that the general solution of the difference equation (D.2) is

$$y_k = c_1 r_1^k + c_2 r_2^k + \cdots + c_n r_n^k, \tag{D.4}$$

where $c_1, c_2, \dots, c_n$ are real constants. These constants can be determined if the initial conditions of the form $y_0 = A_0,\ y_1 = A_1, \dots, y_{n-1} = A_{n-1}$ are known. Then we put $k = 0, 1, \dots, n-1$ in (D.4) and obtain a system of linear equations

$$\begin{aligned}
&c_1 + c_2 + \cdots + c_n = A_0\\
&c_1 r_1 + c_2 r_2 + \cdots + c_n r_n = A_1\\
&\cdots\\
&c_1 r_1^{n-1} + c_2 r_2^{n-1} + \cdots + c_n r_n^{n-1} = A_{n-1}
\end{aligned}$$

whose solutions are the constants $c_1, c_2, \dots, c_n$.

As an example, let us consider the linear homogenous difference equation of the second order,

$$y_{n+2} - y_{n+1} - y_n = 0, \tag{D.5}$$

with the initial condition $y_1 = 1,\ y_2 = 1$. From the above recurrence relation it follows that any term of the sequence $y_3,\ y_4,\ y_5, \dots$ is equal to the sum of the two previous terms. Substituting $y_k = r^k$ in (D.5) we get

$$r^{n+2} - r^{n+1} - r^n = 0.$$

After dividing with r^n one obtains the *characteristic equation*

$$r^2 - r - 1 = 0$$

of the difference equation (D.5) with real solutions

$$r_1 = \frac{1}{2}(1 + \sqrt{5}) \quad\text{and}\quad r_2 = \frac{1}{2}(1 - \sqrt{5}).$$

Now the general solution of (D.5) is given by

$$y_k = A r_1^k + B r_2^k = A\Big(\frac{1+\sqrt{5}}{2}\Big)^k + B\Big(\frac{1-\sqrt{5}}{2}\Big)^k,$$

where A and B are constants which can be determined from the initial conditions $y_1 = 1,\ y_2 = 1$. These conditions lead to the system of two linear equations in A and B with solutions $A = 1/\sqrt{5},\ B = -1/\sqrt{5}$ giving the explicit formula

$$y_k = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^k - \Big(\frac{1-\sqrt{5}}{2}\Big)^k\Big], \tag{D.6}$$

for any term from the sequence $\{y_k\}$. Let us recall that the difference equation (D.5) defines the well-known Fibonacci sequence $1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13, \dots$. The kth term of this sequence can be obtained directly from the explicit formula (D.6).

BIOGRAPHIES – A CHRONOLOGICAL ORDER

Archimedes. Born *ca.* 287 B.C., Syracuse, Sicily, died *ca.* 212 B.C., Syracuse. Considered one of the greatest mathematicians of all times, Archimedes received his education in Alexandria. His remarkable achievements in pure and applied mathematics, physics, and mechanics include the method of exhaustion (the early form of integration), geometrical solution to the cubic equation, the quadrature of the parabola, and the famous principle named after him. Legend has it that a Roman soldier attacked Archimedes with his sword while the latter was immersed in solving a geometrical problem, illustrating the classic example of brute force and ignorance triumphing over intellect and nobility of spirit.

Heron of Alexandria. Born *ca.* 10 A.D., Alexandria, died *ca.* 75 A.D. Some historians, however, suggest that the dates *ca.* 65 A.D.–*ca.* 125 A.D. more closely match his lifetime. Best known for the pneumatical device commonly known as Heron's fountain and the formula $S = \sqrt{s(s-a)(s-b)(s-c)}$ for the area of a triangle, Heron invented a simple form of the steam engine and wrote on pneumatics, dioptrics, mechanics, geometry and mensuration. He also derived an iterative method $x_{n+1} = x_n + a/x_n$ for approximating the square root of a number a, although the Babylonians already knew of this method some 2000 years before.

Diophantus of Alexandria. Born *ca.* 200–died *ca.* 284. One of ancient Greece's most celebrated mathematicians, Diophantus introduced improved algebraic notation and worked on numerical solutions of determinate and indeterminate equations. His most important work, *Arithmetica*, collects about 130 problems from this field. Indeterminate algebraic problems, where one must find only the rational solutions, have become known as *diophantine problems*.

Brahmagupta. Born *ca.* 598–died *ca.* 670. A prominent seventh-century Hindu mathematician, Brahmagupta wrote his major work on astronomy, *Brahma-sphuta-sidd'hānta* (Correct astronomical system of Brahma) in 628. The book contained 21 chapters, including two chapters dealing with mathematics. Brahmagupta succeeded in solving the indeterminate linear equation $ax+by=c$ in integers and also solving some special cases of the so-called Pell equation $y^2 = Ax^2+1$, using methods rediscovered several centuries later.

Alcuin of York. Born 735, York, died 804, Tours, France. The headmaster at York, one of the most important centers of learning in Europe, Alcuin later became the abbot at Tours. He wrote elementary texts on arithmetic, geometry, and astronomy. Known as an historian, Alcuin also collected puzzles and mathematical recreation problems.

Mahāvira. Born *ca.* 800, Mysore, India, died *ca.* 870. He wrote the only known book, *Ganita Sara Sangraha*, to update the work of Brahmagupta, the seventh-century Indian mathematician, providing significant simplifications and explanations in greater detail of Brahmagupta's text. Mahāvira examined methods of squaring numbers, operations with fractions and discussed integer solutions of first degree indeterminate equations. He was the first to give an explicit algorithm for calculating the number of combinations.

Tābit ibn Qorra. Born 826, Harran, Turkey, died 901, Baghdad. The Arabian mathematician, astronomer, physician and philosopher ibn Qorra moved to Baghdad in about 870 where he became a great scholar. He translated Euclid's *Elements* into Arabian and wrote a commentary on this famous work. He solved algebraic equations and studied number theory and trigonometry problems.

Abu'l-Wafa. Born 940–died 998. A famous Persian astronomer, algebraist and trigonometer who investigated the moon's orbit and wrote, *Theory of the Moon*; he translated one of the last great classics, the *Arithmetica* by Diophantus, from Greek. However, Abu'l-Wafa is best known for the first use of the *tan* function as well as introducing the *sec* and *cosec* (reciprocals of *cos* and *sin*). Abu'l-Wafa derived a new method of calculating *sin* tables. His trigonometric tables are accurate to 8 decimal places!

Ibn al-Haytham. Born 965, Basra, Iraq, died 1039, Cairo, Egypt. Although born in Basra, **Alhazen,** as he is known in Europe, spent most of his life in Egypt. One of Islam's most illustrious scientists, his opus magnus, *Kitaaz* (Optics) fills seven books. After its translation into Latin, it greatly influenced European thought for several centuries thereafter. Alhazen solved numerous problems related to a variety of reflecting surfaces successfully using elementary and advanced geometry of the Greeks.

Bhāskara. Born 1114–died 1185. A prominent twelfth-century Hindu mathematician and astronomer, Baskara spent most of his life at the astronomical observatory in Ujjain where he wrote, *Siddhāntaśiromani*, a major work on astronomy. Bhāskara also wrote mathematical works on arithmetic, *Lilāvati* and on algebra, *Vijaganita*. These books contain a number of problems dealing with determinate and indeterminate linear and quadratic equations, arithmetic and geometric progressions, Pythagorean triads, and other problems.

Leonardo Pisano (Fibonacci). Born 1170, Pisa?, Italy, died 1250. Although born in Italy, Fibonacci, known also as Leonardo de Pisano or Leonardo of Pisa, received his education in North Africa. The Middle Ages' greatest and most productive mathematician, after traveling extensively, he wrote *Liber Abaci* (The Book of the Abacus), in which he treats arithmetic and elementary algebra in 1202. This book played an important role in the introduction of the Hindu-Arabic place-valued decimal system and Arabic numerals into Europe.

Yang Hui. Born *ca.* 1238–died *ca.* 1298. Hui, who lived in south China under the Song dynasty, wrote two books *Xiangjie Jiushang Suanfa* (A Detailed Analysis of Arithmetical Rules in Nine Sections) (1261) and *Yang Hui Suanfa* (Yang Hui's Methods of Computation) (1275). He made contributions mainly in the form of reports on the work of an eleventh-century Chinese mathematician, Jia Xian, who gave a method for the calculation of square and cubic roots to higher roots using what the West knows as *Pascal's triangle*. Hui also wrote on multiplication and division as well as mathematical education.

Nicolas Oresme. Born 1323?, Normandy, France, died 1382, Lisieux. A French cleric, scholar, and one of the greatest mathematicians of the fourteenth century, Oresme was a professor in the Collège de Navarre of the University of Paris (1355), dean of Rouen (1361) and bishop of Lisieux (1377). He wrote five mathematical works and translated Aristotle's *De Coelo et Mundo*. The first known use of fractional exponents appears in his tract *Algorismus Proportionum*. Oresme's *Tractatus de Uniformitate*, which may have influenced Descartes, anticipated coordinate geometry by using two coordinates to locate points.

Johann Müller. Born June 6, 1436, Königsberg, died July, 1476, Rome. Better known as **Regiomontanus**, the Latin translation of his birthplace, Königsberg (king's mountain). An influential mathematician, his greatest work, *Detriangulis Omnimodis* (On Triangles of Every Kind) contained important results devoted to plane and spherical trigonometry. Regiomontanus composed an extensive table of sines with the interval of $1'$. He translated Ptolemy's, *Almagest*, as well as works of Apollonius, Heron, and Archimedes. His death was said to have occurred under mysterious circumstances

Niccolo Fontana Tartaglia. Born 1500, Brescia, Italy, died December 13, 1557, Venice. One of Italy's greatest mathematicians, as a boy he survived a sabre attack from which he suffered facial wounds. The attack resulted in permanent speech difficulties, and gave rise to the nickname "Tartaglia," or "the stammerer." Today, Tartaglia is best remembered for his formula to solve cubic equations, the Cardano–Tartaglia formula. In addition to being the first to describe new ballistic methods in artillery science, including the first firing tables, Tartaglia also published editions of Euclid and Archimedes.

Gerolamo Cardano. Born September 24, 1501, Pavia, Italy, died September 21, 1576, Rome. Immensely talented and versatile, Gerolamo Cardano worked as a physician, mathematician, physicist, astronomer, astrologer, a professor at the University of Bologna. His most important mathematical work *Ars Magna* (The Great Art, 1545), one of the most influential books in mathematics of his century, contains the formula providing the solution of the cubic equation, today known as the Cardano–Tartaglia formula.

Robert Recorde. Born 1510, Tenby, Wales, died 1558, London. A highly influential writer of textbooks in sixteenth-century England, Recorde studied medicine at Oxford and Cambridge and taught private classes in mathematics at both institutions. After leaving Cambridge, Recorde served as a physician to Edward VI and Queen Mary. He wrote at least five books (one of them, *The Ground of Artes*, had more than 28 editions) on mathematics, astronomy and medicine.

His book *The Whetstone of Whitte*, published in 1557, was the first major English resource on algebra.

François Viète. Born 1540, Fontelay-le-Comte, France, died December 13, 1603, Paris. Educated as a lawyer, Viète achieved prominence in a diplomatic career, serving as a parliamentary councilor and as the king's confidante. Viète, also a gifted mathematician, made valuable and significant contributions in geometry, arithmetic, algebra, and trigonometry. He introduced the first systematic algebraic notation in his book, *In Artem Analyticam Isagoge* (1591), using symbols $+$ (plus) and $-$ (minus), and letters as symbols for quantities, both known and unknown.

Johannes Kepler. Born December 27, 1571, Weil der Stadt, Germany, died November 15, 1639, Regensburg. The great German scientist Johannes Kepler is chiefly known in the field of astronomy, although he made important contributions to mathematics and optics. Kepler made significant advances in the use of infinitesimals in geometry and astronomy, and did important work on polyhedra (1619). He also studied the problem of filling spaces with regular polyhedra and spheres. Kepler's monumental discovery that the planets move around the sun in elliptical orbits with the sun as their focus, as well as his formulation of the three mathematical laws of planetary motion, helped enormously to advance scientific thought.

Claude-Gaspar Bachet, Sieur de Méziriac. Born October 9, 1581, Bourg-en-Bresse, France, died February 26, 1638. A mathematician, philosopher, theologian, and poet, Bachet made initial steps in number theory even before Fermat. Bachet authored such classic books on mathematical recreations such as *Problèms Plaisants et Délectables* (1612, 1624). He also achieved renown for his Latin translation of Diophantus' Greek text, *Arithmetica* (1621).

Renè Descartes. Born March 31, 1596, La Haye (since renamed Descartes), France, died February 11, 1650, Stockholm, Sweden. Although an insightful philosopher, Descartes' achievements in mathematics, especially his masterpiece *La Géométrie* in which he unified algebra and geometry, have assured his place in history. By thus unifying them, he created a new mathematical discipline, *analytical geometry*, one that represented a turning point and an extremely powerful point of departure to further the advancement of mathematics and natural sciences. Many mathematical terms testify to his influence: Cartesian product, Cartesian coordinates, Cartesian coordinate system.

Bonaventura Cavalieri. Born 1598, Milan, Italy, died November 30, 1647, Bologna. An influential seventeenth-century mathematician, Cavalieri was a disciple of Galileo and a professor of mathematics at the University of Bologna from 1629 until his death. He has largely gained recognition for introducing Italy to logarithms as a computational tool. Cavalieri wrote on mathematics, optics, astronomy, and astrology, however, he also laid the groundwork for integral calculus. In his *Geometria indivisiblis continuorum nova* (1635), Cavalieri elaborated his *principle of indivisibles*, a kind of crude calculus and used it in the computation of areas and volumes.

Pierre de Fermat. Born August 17, 1601, Beaumont-de-Lomagne, France, died January 12, 1665, Castres. A lawyer by training and vocation, Fermat made great discoveries in mathematics; he was a pioneer in the development of differential calculus, number theory, and, with Pascal, probability theory. His method for finding the extremes of a function represents his most important contribution. Fermat's conjecture that no integral values of x, y, z can be found to satisfy the equation $x^n + y^n = z^n$ if n is an integer greater than 2, is probably the best-known problem in the entire history of mathematics.

Evangelista Torricelli. Born October 15, 1608, Faenza, Italy, died October 25, 1647, Florence, Italy. Torricelli collaborated with Galileo and became his successor as court mathematician to the Grand Duke of Tuscany. Most famous for his discoveries in physics, *e.g.*, the invention of the barometer, acceleration due to gravity, the motion of fluids, and the theory of projectiles, Torricelli also took great interest in mathematics. Early on, he made use of infinitesimal methods (the tangent method), squared the cycloid and found the length of an arc of the logarithmic spiral.

Blaise Pascal. Born June 19, 1623, Clermont-Ferrand, France, died August 19, 1662, Paris. As a young man Pascal produced important theorems in projective geometry, and by the age of 22 he had invented the first calculating machine. Pascal laid the foundations for the theory of probability in his correspondence with Fermat. Through his investigations of the action of fluids subjected to air pressure, he gained himself a reputation as a physicist. Pascal also worked on the triangular arrangement of the coefficients of the powers of a binomial (Pascal's Triangle). Although he was one of the most talented mathematicians ever, at the age of twenty-five he suddenly abandoned scientific work to devote his life to the study of philosophy and religion.

Christiaan Huygens. Born April 14, 1629, The Hague, Netherlands, died July 8, 1695, The Hague. Chiefly recognized as the inventor of the pendulum clock (1656), the law of falling bodies and the wave theory of light, Huygens was one of the world greatest physicists. He also developed an international reputation in mechanics and astronomy for his detection of the first moon of Saturn in 1655. In mathematics he introduced the notion of evolutes and involutes; wrote on the logarithmic curve and probability; proved that the cycloid is a tautochronous curve and made significant contributions in the application of mathematics to physics. Huygens was a member of the London Royal Society.

Seki Shinsuke Kowa, or **Takakazu**. Born 1642, Fujioka, Japan, died October 24, 1708. Seki Kowa is rightly celebrated not only as the greatest Japanese mathematician of the seventeenth century, but as one of Japan's greatest mathematicians ever. He systematized and improved already-known methods such as the Chinese method of solving higher order equations and the early Chinese use of determinants in solving simultaneous equations, thus anticipating Leibniz's discovery. Seki Kowa's reputation as a great teacher won him numerous pupils. Aside from mathematics Seki Kowa demonstrated a keen ability and ingenuity in mechanics. Excelling in the affairs of life in general, he became a Shogun samurai serving as master of ceremonies to Shogun Koshu.

Isaac Newton. Born January 4, 1643, Woolsthorpe, England, died March 31, 1727, London. Some consider Newton to be the greatest scientist who ever lived. He made revolutionary advances in mathematics, physics, mechanics, optics, and astronomy. Published in 1687, his *Philosophiae Naturalis Principia Mathematica*, in which he stated the laws of motion and the law of gravitation is probably the most monumental work in the whole history of science. Newton originated differential and integral calculus, several years before Leibniz made his independent discovery of it. From 1703 until his death Newton was the president of the London Royal Society.

David Gregory. Born June 3, 1659, Aberdeen, Scotland, died October 10, 1708, Maidenhead, England. A nephew of the eminent mathematician James Gregory, David Gregory started his studies at the University of Aberdeen when he was 12 years old! At the age of 24, he was appointed professor at the University of Edinburgh, where he lectured on mathematics, mechanics, and hydrostatics. There, he was the first to teach modern Newtonian theories. Receiving support from Newton, David Gregory was elected professor at Oxford in 1681 and in the same year became a fellow of the Royal Society. He is best known for his experiments with telescopes and work on series and optics.

Abraham de Moivre. Born May 26, 1667, Vitry-le-François, France, died November 27, 1754, London. De Moivre spent most of his life in England where he worked mainly on trigonometry, probability, analytic geometry, and the theory of annuities. The well-known relationship for complex numbers $(\cos x + i\,\sin x)^n = \cos nx + i\,\sin nx$ bears his name. Despite his eminence in the scientific community, he did not succeed in obtaining a chair of mathematics since his foreign birth put him at a disadvantage. However, he was admitted to membership in the London Royal Society and into the academies of Paris and Berlin.

Pierre-Rémond de Montmort. Born October 27, 1678, Paris, died October 7, 1719, Paris. Montmort lived most of his life on his estate, the Château de Montmort, to which he often invited eminent European mathematicians. He wrote on the theory of probability, combinatorial problems, and infinite series. In 1708 de Montmort published an important work on probability, *Essay d'analyse sur les jeux de hazard*. He is remembered for his extensive correspondence with many prominent mathematicians. De Montmort was a member of the London Royal Society and the Paris Académie des Sciences.

Nicolaus II Bernoulli. Born 1687, Basel, Switzerland, died 1759, Basel. The nephew of Jacob and Johann Bernoulli, Nicolaus worked on geometry and differential equations as the appointee to Galileo's chair at Padua from 1717 to 1722. After teaching at Padua, Nicolaus II served as a professor of logic and later as a professor of law at the University of Basel. He made significant contributions in the study of orthogonal trajectories, differential equations, integral calculus, and probability theory. Besides editing Jacob Bernoulli's complete works, Nicolaus II Bernoulli was elected a member of the Berlin Academy in 1713, a fellow of the Royal Society of London in 1714, and a member of the Academy of Bologna in 1724.

Nicolaus III Bernoulli. Born February 6, 1695, Basel, died July 31, 1726, St. Petersburg. One of three sons of the outstanding mathematician Johann Bernoulli, Nicolaus III studied law and became a professor of law at Bern. In 1725 he and his younger brother Daniel traveled to St. Petersburg where they accepted positions as mathematics professors. Nicolaus worked on the geometry of curves, differential equations, mechanics and probability, but a promising career was cut short by his death at age 31.

Daniel Bernoulli. Born February 8, 1700, Groningen, Netherlands, died March 17, 1782, Basel, Switzerland. Daniel's father, the famous mathematician Johann Bernoulli held the chair of mathematics at the University of Basel. Daniel himself became a professor the prestigious Academy of Sciences in St. Petersburg and later in Basel. He was also a member of the London Royal Society, and the academies of Petersburg, Berlin and Paris. D. Bernoulli is regarded as the founder of mathematical physics. He made important contributions to hydrodynamics (Bernoulli's principle), vibrating systems, the kinetic theory of gases, magnetism, etc. In mathematics, he worked on differential equations, the theory of probability, series and other topics. Daniel Bernoulli won the Grand Prize of the Paris Académie of Sciences 10 times for topics in astronomy and nautical sciences.

Gabriel Cramer. Born July 31, 1704, Geneva, died January 4, 1752, Bagnols-sur-Cze, France. A professor of mathematics and physics at Geneva, Cramer worked on geometry, algebraic curves, analysis and the history of mathematics. Cramer is best known for his work on determinants and their use in solving linear systems of equations (Cramer's rule).

Leonhard Euler. Born April 15, 1707, Basel, Switzerland, died September 18, 1783, St. Petersburg. One of the most outstanding mathematicians of all time, Leonhard Euler wrote close to 900 scientific papers on algebra, differential equations, power series, special functions, differential geometry, number theory, rational mechanics, calculus of variations, music, optics, hydrodynamics, and astronomy. He produced almost half of all his work even after becoming nearly blind. Euler was a member of the Petersburg Academy of Science and the Berlin Academy of Science; as a testament to his achievements, several algebraic expressions were named after him.

Alexandre Théophile Vandermonde. Born February 28, 1735, Paris, died January 1, 1796, Paris. Music was the first love of the French mathematician Vandermonde, and he did not begin his work in mathematics until he was 35 years old. He contributed to the theory of equations and the general theory of determinants. Vandermonde also devoted time to the mathematical solution of the knight's tour problem, and was a member of the Académie des Sciences at Paris.

Carl Friedrich Gauss. Born April 30, 1777, Brunswick, Germany, died February 23, 1855, Göttingen, Germany. Gauss figures among Archimedes and Newton as one of the greatest mathematicians of all time. He spent almost forty years as the director of the Göttingen Observatory. Gauss worked on problems in astronomy, geodesy, electricity, celestial mechanics, and in almost all of the leading topics in the field of mathematics: number theory, complex numbers, the theory of surfaces, congruences, least squares, hyperbolic geometry, etc. He was one of the first to consider the question of non-Euclidean geometry.

Louis Poinsot. Born January 3, 1777, Paris, France, died December 5, 1859, Paris. Together with Monge, the French mathematician and mechanist Louis Poinsot was one of the leading French mathematicians in the field of geometry during the eighteenth century. He invented geometric mechanics, which investigates the system of forces acting on a rigid body. Poinsot also made important contributions in statics, dynamics, number theory (diophantine equations), and to the theory of polyhedra. In 1816, by the age of thirty-nine, he had been elected to the Académie des Sciences.

Siméon Poisson. Born June 27, 1781, Pithiviers, France, died April 25, 1840, Sceaux (near Paris). Poisson published over 300 mathematical works covering a variety of applications from electricity, elasticity, and magnetism, to astronomy. His most important papers treated definite integrals and his own advances in Fourier series. Poisson also contributed to the theory of probability (the Poisson distribution), differential equations, surfaces, the calculus of variations, and algebraic equations. His name is associated with such wide-ranging scientific branches as elasticity (Poisson's ratio); potential theory (Poisson's equation); electricity (Poisson's constant) and mathematics (Poisson's integral).

Jakob Steiner. Born March 18, 1796, Utzenstorf, Switzerland, died April 1, 1863, Bern. Although the Swiss mathematician Steiner did not learn to read and write until the age of fourteen, he later became a professor of mathematics at the University of Berlin in 1834, a post he held until his death. Steiner, regarded as the greatest geometer of modern times, wrote a series of prestigious papers on projective geometry and the theory of curves and surfaces of the second degree.

William Rowan Hamilton. Born August 4, 1805, Dublin, Ireland, died September 2, 1865, Dublin. A child prodigy who knew 13 foreign languages by the time he was thirteen, Hamilton created a new algebra in 1843 by introducing quaternions, an extension of complex numbers to three dimensions. He devoted 22 years to the study of quaternions, and furthermore, obtained noteworthy results in optics, mechanics, calculus of variations, geometry, algebra, differential equations. Several notions in mathematics and mechanics bear his name. Hamilton was a member of many academies of sciences and scientific associations.

Thomas Kirkman. Born March 31, 1806, Bolton, England, died February 4, 1895, Bowdon. Kirkman served vicar to the Parish of Southworth in Lancashire for 52 years, while also dedicating much effort to mathematics. Although he did not take up mathematics until the age of 40, he became an expert in group theory and combinatorics, working on knots. Kirkman gained reconition for his work on Steiner systems and a related topic, the *fifteen schoolgirls problem*. As a result of his work on the enumeration of polyhedra, he was named a fellow of the Royal Society in 1857.

Johann Benedict Listing. Born July 25, 1808, Frankfurt am Main, Germany, died December 24, 1882, Göttingen. A German mathematician and physicist who helped to found a new branch of mathematics: topology. He also made an independent discovery of the properties of the Möbius band contemporaneously with Möbius. Listing made important observations in meteorology, terrestrial magnetism, geodesy, and spectroscopy. He introduced such new terms as topology, entropic phenomena, nodal points, and one micron. Listing was a member of the Göttingen Academy and the Royal Society of Edinburgh.

James Joseph Sylvester. Born September 3, 1814, London, died on March 15, 1897, London. Together with W. R. Hamilton and Arthur Cayley, Sylvester was one of Britains most prominent nineteenth-century mathematicians. He was a professor at Johns Hopkins University in Baltimore, Maryland, from 1877 to 1883, and at Oxford from 1883 to 1893. Sylvester helped to further the progress of mathematics in America by founding the *American Journal of Mathematics* in 1878. He performed important work on matrix theory, invariants, theoretical and applied cinematics, mathematical physics and higher algebra. Sylvester was the second president of the London Mathematical Society (after de Morgan).

Arthur Cayley. Born August 16, 1821, Richmond, Surrey, England, died January 26, 1895, Cambridge. Cayley spent 14 years as a lawyer devoting his leisure hours to mathematics until 1863 when he was appointed professor at Cambridge. He published over 900 papers covering nearly every aspect of modern mathematics. Cayley developed the theory of algebraic invariance, and worked on problems of elliptic functions and non-Euclidean geometry. His development of n-dimensional geometry has been applied in physics to the study of space-time continuum, while his work on matrices served as the foundation for quantum mechanics.

Edouard Lucas. Born April 4, 1842, Amiens, France, died October 3, 1891, Paris. He worked at the Paris Observatory and as a professor of mathematics in Paris. He is best known for his results in number theory; in particular, he studied the Fibonacci sequence and the sequence associated with it and named for him, the Lucas sequence. Lucas also devised the methods of testing primality that essentially remain those in use today. His four volume work on recreational mathematics, *Recréations Mathḿatiques* (1882–94), attained status as a classic in its field. While attending a banquet, Lucas was struck on the cheek by a piece of glass when a plate was dropped. As the result of this bizarre accident, he died of erysipelas a few days later.

Ferdinand Georg Frobenius. Born October 26, 1849, Charlottenburg, a suburb of Berlin, Germany, died August 3, 1917, Berlin. He received his doctorate in 1870 supervised by Weierstrass at the University of Berlin. Frobenius was a professor at the Eidgenössische Polytechnikum (now ETH) in Zurich between 1875 and 1892 and then he was appointed professor at the University of Berlin. He made remarkable contributions to differential equations, group theory (particularly in the representation theory of groups), number theory and the theory of positive and non-negative matrices (Peron–Frobenius theorem). Frobenius was the first who gave general proof of the famous Cayley–Hamilton theorem (1878). He was elected to the Prussian Academy of Sciences in 1892.

Frederick Soddy. Born September 2, 1877, Eastbourne, England, died September 22, 1956, Brighton. The British physicist and chemist was a professor at the universities of Aberdeen and Oxford. In 1921, he received the Nobel Prize in chemistry for his discovery of isotopes. The *Soddy–Fajans–Russel law* was named in recognition of his research on radioactive decay. Soddy devoted his leisure time to mathematics and poetry.

John E. Littlewood. Born June 9, 1885, Rochester, England, died September 6, 1977, Cambridge. The English mathematician J. E. Littlewood was a mathematics professor at Trinity College, Cambridge, and was a member of the Royal Society. He made distinguished contributions to function theory, nonlinear differential equations, the theory of series, inequalities, the Riemann zeta function, summability, number theory, Tauberian theory, etc. He also gained recognition for his collaboration with Godfrey H. Hardy, another famous English mathematician.

George Pólya. Born December 13, 1887, Budapest, died September 7, 1985, Palo Alto, CA. Pólya received his doctorate in mathematics from the University of Budapest in 1912 and worked at the University of Zurich from 1914 until 1940 when he left for America. After working at Brown University for two years, he took an appointment at Stanford until his retirement. Pólya worked in probability (theorem of random walks), analysis, number theory, geometry, astronomy, combinatorics (enumeration theorem), mathematical physics, and other matters. He jointly published a famous monograph *Inequalities* with Hardy and Littlewood in 1934. Pólya is widely known for his contributions to mathematical teaching. The first edition of his book *How to Solve It*, published in 1945, sold over one million copies.

Srinivasa Ramanujan. Born December 22, 1887, Erode, Tamil Nadu state, India, died April 26, 1920, Kumbakonam. The story of this Indian mathematician who died at a very young age makes for some extremely compelling reading. A self-taught mathematical genius, Ramanujan demonstrated an uncanny and amazing ability for intuitive reasoning and stating fascinating number relations. The outstanding British number theorist G. H. Hardy observed his work and brought him to England to study at Cambridge University. Hardy and Ramanujan co-authored seven remarkable mathematical papers. Ramanujan made important contributions to the analytic theory of numbers, elliptic integrals, hypergeometric series, continued fractions, and infinite series. He was elected a fellow of the Royal Society in 1918.

Abram Besicovitch. Born January 24, 1891, Berdyansk, Russia, died November 2, 1970, Cambridge, England. He studied and worked in St. Petersburg until the mid-1920s when he escaped from Russia and made his way to Copenhagen. He later worked at Trinity College, Cambridge, where he spent over 40 years of his life. He made important contributions to periodic functions, the classical theory of real functions, fractal geometry, measure theory, etc. Besicovitch was elected a fellow of the Royal Society in 1934.

Stephen Banach. Born March 30, 1892, Kraków, Poland, died August 31, 1945, Lvov, Ukraine. The Polish mathematician Stefan Banach attended school in Kraków, but he received his doctorate in the Ukrainian city of Lvov, where he lectured at the Institute of Technology and at the University of Lvov. Banach founded modern functional analysis and made major contributions to the theory of topological vector spaces, measure theory, integration, and orthogonal series. Banach literally left his signature on mathematics with theorems and concepts such as Banach space, Banach algebra, the Hahn–Banach theorem, the Banach–Steinhaus theorem, the Banach fixed-point theorem, and the Banach–Tarski paradox. Banach's most important work is the *Théorie des Opérations Linéaires* (1932).

Paul Dirac. Born August 8, 1902, Bristol, England, died October 20, 1984, Tallahassee, FL. The English physicist and mathematician Paul Dirac played a huge role in the creation of quantum mechanics and quantum electrodynamics; in the words of Silvan Schweber, he was "one of the principal architects of quantum field theory". Dirac was appointed Lucasian professor of mathematics at the University of Cambridge in 1932, a post he held for 37 years. He began his research in the field of quantum theory in 1925, and five years later he published *The principles of quantum mechanics*, for which he was awarded the Nobel Prize for physics in 1933. He was made a fellow of the Royal Society in 1930.

John von Neumann. Born December 28, 1903, Budapest, Hungary, died February 8, 1957, Washington, D. C. Von Neumann, regarded as one of the twentieth-century's most illustrious mathematicians began his scientific work in Budapest, moved on to Berlin and Hamburg, and from 1930, continued his career in the United States. He contributed substantially to set theory, quantum physics, functional analysis, operator theory, logic, meteorology, probability, among other things. He laid the foundations for mathematical game theory and applied it to economics. Von Neumann also played a part in developing high-speed computing machines.

Richard Rado. Born April 28, 1906, Berlin, died December 23, 1989, Henley-on-Thames, England. Rado studied at the University of Berlin where he completed his doctoral dissertation in 1933. When the Nazis came to power in 1933, Rado, being Jewish, could not secure a teaching position, and so left Germany with his family for England. Rado held various appointments as professor of mathematics at Sheffield, Cambridge, London, and Reading. He carried out important work in combinatorics, convergence of sequences and series. In addition, Rado studied inequalities, geometry, and measure theory. In the field of graph theory, he worked on infinite graphs and hypergraphs.

Harold Scott MacDonald Coxeter. Born February 9, 1907, London, England, died March 31, 2003, Toronto, always known as Donald, from the third name MacDonald. He received his graduate diploma and doctorate (1931) at the University of Cambridge. Coxeter became a professor of mathematics at the University of Toronto (1936), a post he held until his death. He is best known for his work in geometry. He made significant contribution in the theory of polytopes, non-Euclidean geometry, combinatorics and group theory (Coxeter groups). Coxeter wrote several very influential books and revised and updated Rouse Ball's *Mathematical Recreations and Essays* (1938). He received nine honorary doctorates and was a fellow of the Royal Society of London and a fellow of the Royal Society of Canada.

Paul Erdös. Born March 26, 1913, Budapest, died September 20, 1996, Warsaw, Poland. The Hungarian mathematician Paul Erdös studied at the University of Budapest, and received his doctorate there in 1934. He continued his career in England, the United States, and Israel. A giant among twentieth-century mathematicians, Erdös contributed significantly to number theory, combinatorial analysis, and discrete mathematics. He loved to pose and solve problems that were beautiful, simple to understand, yet very difficult to solve. Erdös wrote prolifically and published some 1475 papers.

Roger Penrose. Born August 8, 1931, Colchester, Essex, England. Roger Penrose, a mathematical physicist, cosmologist and philosopher, is one of British most prominent scientists. He received his doctorate in mathematics from the University of Cambridge in 1957. In 1973 Penrose was appointed Rouse Ball Professor of Mathematics at the University of Oxford, a post he held for 25 years. Endeavoring to unite relativity and quantum theory, he invented the twistor theory in 1967. Penrose is best known for his works on general relativity, quantum mechanics and cosmology, but also for his very popular books on science. Penrose has been awarded many honorary degrees from eminent universities and prizes for his contributions to science. He was elected a fellow of the Royal Society of London (1972) and a Foreign Associate of the United States National Academy of Sciences (1998). In 1994 he was knighted for services to science.

John Horton Conway. Born December 26, 1937, Liverpool, England. J. H. Conway received his doctorate in 1964 at the University of Cambridge. He was a professor of mathematics at Cambridge until 1986, when he was appointed the John von Neumann Chair of Mathematics at Princeton. Conway made distinguished contributions to he theory of finite groups (Conway groups), knot theory, number theory (he proved Waring's conjecture that every positive integer could be represented as the sum of 37 fifth powers), combinatorial game theory, quadratic forms, coding theory and geometry (studying the symmetries of crystal lattices). In March 1981 Conway was elected a fellow of the Royal Society of London. He is widely known for his contributions and inventions to recreational mathematics, primarily the discovery of the cellular automata called the Game of Life.

Donald Knuth. Born January 10, 1938, Milwaukee, WI. Knuth, a professor at Stanford University, has achieved international renown as an incredible computer scientist. He has written more than 150 papers dealing with software, compilers, programming languages, construction and analysis of algorithms, mathematical modelling, combinatorial geometry, and many other subjects. He authored the three-volume monumental work *The Art of Computer Programming* (1968–1973) and invented a revolutionary typesetting program for technical material named $\TeX$. In 1974 Professor Knuth won the Turing Prize, computer science's highest achievement; he holds more than 30 honorary doctorates from eminent universities throughout the world.

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NAME INDEX

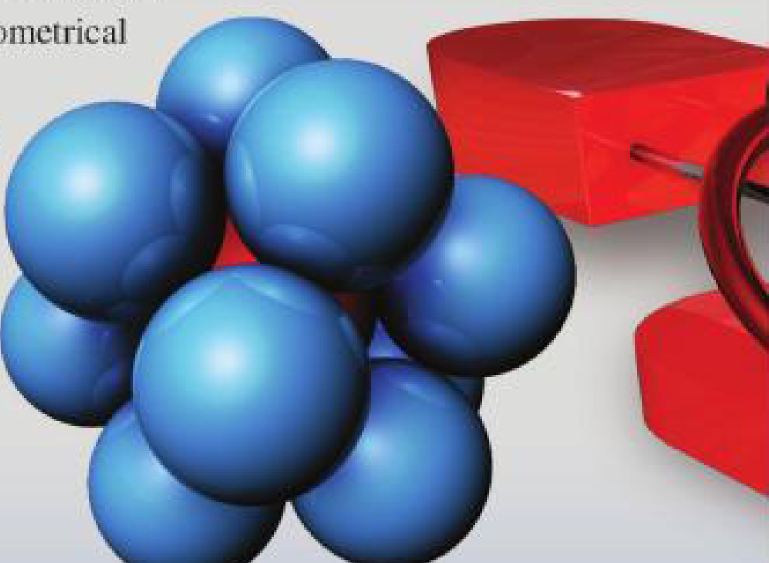
The names of mathematicians related to the considered recreational problems are boldfaced. The first appearance of any problem is marked by the page in boldface. Italic numbers point to the page where the corresponding short biography is situated.

This entertaining book presents a collection of 180 famous mathematical puzzles and intriguing elementary problems that great mathematicians have posed, discussed, and/or solved. The selected problems do not require advanced mathematics, making this book accessible to a variety of readers.

Mathematical recreations offer a rich playground for both amateur and professional mathematicians. Believing that creative stimuli and aesthetic considerations are closely related, great mathematicians from ancient times to the present have always taken an interest in puzzles and diversions. The goal of this book is to show that famous mathematicians have all communicated brilliant ideas, methodological approaches, and absolute genius in mathematical thoughts by using recreational mathematics as a framework. Concise biographies of many mathematicians mentioned in the text are also included.

The majority of the mathematical problems presented in this book originated in number theory, graph theory, optimization, and probability. Others are based on combinatorial and chess problems, while still others are geometrical and arithmetical puzzles.

This book is intended to be both entertaining as well as an introduction to various intriguing mathematical topics and ideas. Certainly, many stories and famous puzzles can be very useful to prepare classroom lectures, to inspire and amuse students, and to instill affection for mathematics.



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