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CLASA A VIII-A

Inegalități în mulțimea numerelor reale

Se spune că relația care guvernează cu adevărat matematica este cea de inegalitate, egalitatea fiind un caz special. Cunoaşterea rezultatelor de bază, a inegalităților remarcabile şi a tehnicilor cu aplicabilitate largă este neapărat necesară.

Amintim proprietățile fundamentale ale relației „ $\leq$ ’’ în mulțimea **R**.

1. $a \leq a$, oricare ar fi $a \in \mathbf{R}$
2. dacă $a \leq b$ şi $b \leq a$ atunci $a = b$
3. dacă $a \leq b$ şi $b \leq c$ atunci $a \leq c$
4. dacă $a \leq b$ atunci $a + c \leq b + c$
5. dacă $a \leq b$ şi $c \geq 0$ atunci $ac \leq bc$
6. dacă $a \leq b$ şi $c < 0$ atunci $ac \geq bc$ şi $\frac{a}{c} \geq \frac{b}{c}$
7. dacă $a \leq b$ şi $c \leq d$ atunci $a + c \leq b + d$
8. dacă $0 \leq a \leq b$ şi $0 \leq c \leq d$ atunci $0 \leq ac \leq bd$
9. dacă $0 \leq a \leq b$ atunci $a^2 \leq b^2$
10. $x^2 \geq 0$, oricare ar fi $x \in \mathbf{R}$
11. dacă $a > 0$, $|x| \leq a$ <=> $x \in [-a, a]$
12. dacă $a > 0$, $|x| \geq a$ <=> $x \in (-\infty, -a] \cup [a, +\infty)$

În această lucrare vom face dese trimiteri la inegalități clasice pe care le vom trece în revistă, vom prezenta inegalități simple şi diferite forme ale lor care pot fi folosite în alte exerciții.

Inegalități remarcabile

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1. dacă $a > 1$, atunci $a^k > a^{k-1}$, oricare ar fi $k \geq 1$

$0 < a < 1$, atunci $a^k < a^{k-1}$, oricare ar fi $k \geq 1$

2. dacă $a \leq b$, atunci $(a^m - b^m)(a^n - b^n) \geq 0$, oricare ar fi $m,n \in$ **N**

3. $a + \frac{1}{a} \geq 2$, oricare ar fi $a > 0$

4. $a + \frac{1}{a} \leq -2$, oricare ar fi $a < 0$

5. $(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$, oricare ar fi $a,b \in (0,+\infty)$

6. $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$, oricare ar fi $a, b, c \in (0,+\infty)$

7. $\frac{1}{2\sqrt{k}} < \frac{1}{\sqrt{k} + \sqrt{k-1}} = \sqrt{k} - \sqrt{k-1}$ si $\frac{1}{2\sqrt{k}} > \frac{1}{\sqrt{k} + \sqrt{k+1}} = \sqrt{k} - \sqrt{k+1}$

8. $\frac{a^2+b^2}{2} \geq \left(\frac{a+b}{2}\right)^2$, oricare ar fi $a, b \in$ **R**

9. $\frac{a^2+b^2}{a+b} \geq \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}$, oricare ar fi $a, b > 0$

10. $a^2 + b^2 + c^2 \geq ab + ac + bc$, oricare ar fi a,b,c $\in$ **R**

11. $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$, oricare ar fi a,b,c $\in$ **R**

12. $\frac{a^2+b^2+c^2}{a+b+C} \geq \frac{1}{3}(a + b + c)$, oricare ar fi a,b,c $\in$ $\mathbf{R}_+^*$

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13. $\sqrt{a+b+c} \geq \frac{\sqrt{3}}{3}(\sqrt{a}+\sqrt{b}+\sqrt{c})$, oricare ar fi a,b,c $\in$ **R**$_+$

14. $n(a_1^2+a_2^2+\cdots+a_n^2) \geq (a_1+a_2+\cdots+a_n)^2$, oricare ar fi $a_i \in$ **R**, $i=\overline{1,n}$

15. $\frac{a^n+b^n}{2} \geq \left(\frac{a+b}{2}\right)^n$, oricare ar fi $n \in$ **N** si $a,b > 0$

16. dacă $0 < \frac{a}{b} < 1$, atunci $\frac{a}{b} < \frac{a+r}{b+r}$, oricare ar fi $r > 0$

dacă $1 < \frac{a}{b}$, atunci $\frac{a}{b} > \frac{a+r}{b+r}$, oricare ar fi $r > 0$

17. $|a \pm b| \leq |a| + |b|$, oricare ar fi $a,b \in$ **R**

18. $||a| - |b|| \leq |a - b|$, oricare ar fi $a,b \in$ **R**

19. $\frac{1}{n^2} = \frac{1}{n \cdot n} \leq \frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n}$

20. dacă $x,y \in$ **Z** şi $x < y$, atunci $x + 1 \leq y$

21. dacă $x_1, x_2, \dots, x_n \geq 0$ şi $x_1+x_2+\cdots+x_n = k$ (constant), atunci produsul $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ este maxim când $x_1 = x_2 = \cdots = x_n = \frac{k}{n}$

22. dacă $x_1 \cdot x_2 \cdot \ldots \cdot x_n = k$, atunci suma $x_1+x_2+\cdots+x_n$ este minimă când $x_1 = x_2 = \cdots = x_n = \sqrt[n]{k}$

23. Inegalitatea mediilor

$$\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} \leq \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} \leq \frac{a_1+a_2+\cdots+a_n}{n}$$

unde $a_i > 0, i = \overline{1,n}$, cu egalitate <=> $a_i = a_j$

O consecință imediată $(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2$

24. Inegalitatea lui Cauchy – Buniakovsky – Schwartz

$({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$,

unde a_i şi b_i $\in$ **R**, $i = \overline{1,n}$ **.**Egalitate dacă şi numai dacă $\frac{a_i}{b_i} = \frac{a_j}{b_j}$, oricare ar fi $i, j = \overline{1,n}$.

25. dacă $n \geq 2$, $a_i > 0, i = \overline{1,n}$ atunci $\min a_i \leq m_{arm} \leq m_g \leq m_a \leq \max a_i$

26. Inegalitatea lui Minkowschi

$\sqrt{(x_1 + y_1)^2 + \cdots + (x_n + y_n)^2} \leq \sqrt{x_1^2 + x_2^2 + \cdots + {x_n}^2} + \sqrt{y_1^2 + y_2^2 + \cdots + {y_n}^2}$,

oricare ar fi $x_i, y_i \in$ **R**, $i = \overline{1,n}$

27. Inegalitatea lui Cebîşev

Dacă $\begin{matrix} a_1 \leq a_2 \leq \cdots \leq a_n \\ b_1 \leq b_2 \leq \cdots \leq b_n \end{matrix}$ atunci $(a_1 b_1 + \cdots + a_n b_n) \geq \frac{1}{n}(a_1 + \cdots + a_n)(b_1 + \cdots + b_n)$

Dacă $\begin{matrix} a_1 \leq a_2 \leq \cdots \leq a_n \\ b_1 \geq b_2 \geq \cdots \geq b_n \end{matrix}$ atunci $a_1 b_1 + \cdots + a_n b_n \leq \frac{1}{n}(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)$,

$a_i, b_i \in$ **R**.

Probleme rezolvate

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1) Sa se demonstreze ca daca $x, y \in$ **N**, $x > 2$ si $y \geq 2$ atunci $x + y < xy$.

Solutie: $x+y < xy$ <=> $\frac{x+y}{xy} < 1$ <=> $\frac{1}{x} + \frac{1}{y} < 1$

Din $x > 2$ si $y \geq 2$ => $\frac{1}{x} + \frac{1}{y} < \frac{1}{2} + \frac{1}{2} = 1$

2) Sa se arate ca $2^n \geq n + 1$, oricare ar fi $n \in \mathbf{N}^*$.

Solutie: Se demonstreaza ca $\frac{m}{m+1} \geq \frac{1}{2}$, $m = \overline{1,n}$ si obtinem

$\frac{1}{2} \geq \frac{1}{2}$; $\frac{2}{3} \geq \frac{1}{2}$; $\frac{3}{4} \geq \frac{1}{2}$; …; $\frac{n}{n+1} \geq \frac{1}{2}$ => $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{n}{n+1} \geq \frac{1}{2^n}$ <=> $\frac{1}{n+1} \geq \frac{1}{2^n}$ <=> $2^n \geq n + 1$

3) Daca $a,b,c \in$ **R**$_+^*$, atunci $\frac{a+b}{a+b+c} < \frac{a}{a+c} + \frac{b}{b+C}$.

Solutie: $\frac{a+b}{a+b+c} = \frac{a}{a+b+c} + \frac{b}{a+b+c} < \frac{a}{a+c} + \frac{b}{b+c}$.

4) Sa se demonstreze ca:

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{3}{4}$$

Solutie:

$$\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}<\frac{1}{2^2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)n}=\frac{1}{4}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+$$
$$\frac{1}{n-1}-\frac{1}{n}=\frac{1}{4}+\frac{1}{2}-\frac{1}{n}<\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$$

5) Sa se demonstreze ca:

$$\frac{1}{2}<\frac{1}{101}+\frac{1}{102}+..+\frac{1}{200}<1$$

Solutie: $\frac{1}{101}+\frac{1}{102}+\cdots+\frac{1}{200}>\frac{1}{200}+\frac{1}{200}+\cdots\frac{1}{200}=\frac{100}{200}=\frac{1}{2}$

$$\frac{1}{101}+\frac{1}{102}+\cdots+\frac{1}{200}<\frac{1}{101}+\frac{1}{101}+\cdots+\frac{1}{101}=\frac{100}{101}<1$$

6) Sa se demonstreze ca :

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ \ldots\cdot\frac{99}{100}<\frac{1}{10}$$

Solutie: daca *0 < a < b*, atunci $\frac{a}{b}<\frac{a+k}{b+k}\ \ ,k>0$

=> $\frac{1}{2}<\frac{2}{3}\quad \frac{3}{4}<\frac{4}{5}\quad \ldots.\quad \frac{99}{100}<\frac{100}{101}$ =>

=> $\frac{1}{2}\cdot\frac{3}{4}\cdot\ \ldots\cdot\frac{99}{100}<\frac{2}{3}\cdot\frac{4}{5}\cdot\ \ldots\cdot\frac{100}{101}\quad \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\ \ldots\cdot\frac{99}{100}\right)$

=> $\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{99}{100}\right)^2 < \frac{1}{101} < \frac{1}{100}$

=> $\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{99}{100} < \frac{1}{10}$

7) $a, b > 0, n \in$ **N*** => $\left(1+\frac{a}{b}\right)^n + \left(1+\frac{b}{a}\right)^n \geq 2^{n+1}$.

Solutie:

$\left(1+\frac{a}{b}\right)^n + \left(1+\frac{b}{a}\right)^n \geq 2\sqrt{\left(1+\frac{a}{b}\right)\left(1+\frac{b}{a}\right)} = 2\sqrt{2+\frac{a}{b}+\frac{b}{a}} \geq 2\sqrt{(2+2)^n} = 2\sqrt{2^{2n}} = 2^{n+1}$

8) Oricare ar fi $a, b \in$ **R** $-\{0\}, \left|\frac{a}{b}+\frac{b}{a}\right| \geq 2$

Solutie: $\left|\frac{a}{b}+\frac{b}{a}\right| \geq 2$ <=> $\frac{a^2+b^2-2|b|\cdot|a|}{|ab|} \geq 0 \Rightarrow \frac{(|a|-|b|)^2}{|ab|} \geq 0$. Egalitate pentru

a = $\pm$ *b.*

9) $f(x) = x^{16} - x^{13} + x^{12} - x^5 + x^2 - x + 1$. Sa se demonstreze ca *f(x) > 0*,

oricare ar fi $x \in$ **R**.

Solutie: pentru $x < 0$, $f(x)$ este suma de termeni pozitivi

pentru $x \in [0,1], f(x) = (1-x) + x^2(1-x^3) + x^{12}(1-x) + x^{16} > 0$,

oricare ar fi $x \in$ **R**.

pentru $x > 1$, $f(x) = x^{13}(x^3-1) + x^5(x^3-1) + x(x-1) + 1$

$\Rightarrow f(x) > 0$, oricare ar fi $x \in$ **R**.

10) $\sqrt{x+1} + \sqrt{2x-3} + \sqrt{50-3x} \le 12$

Solutie: $a + b + c \le \sqrt{3(a^2+b^2+c^2)}$, oricare ar fi $a, b, c \in$ **R**$_+$

$$\sqrt{x+1} + \sqrt{2x-3} + \sqrt{50-3x} \le \sqrt{3(x+1+2x-3+50-3x)} = \sqrt{3 \cdot 48} = 12$$

11) Daca $x,y,z > 0$, atunci $\sqrt{x(y+z)} + \sqrt{y(z+x)} + \sqrt{z(x+y)} \le \sqrt{2}(x+y+z)$.

Solutie: Inegalitatea mediilor

$$\sqrt{2x(y+z)} \le \frac{2x+y+z}{2}$$

$$\sqrt{2y(z+x)} \le \frac{2y+z+x}{2}$$

$$\sqrt{2z(y+x)} \le \frac{2z+x+y}{2}$$

$+$

$$\sqrt{2x(y+z)} + \sqrt{2y(z+x)} + \sqrt{2z(y+x)} \le 2(x+y+z)$$

$$\sqrt{2}\left(\sqrt{x(y+z)} + \sqrt{y(z+x)} + \sqrt{z(y+x)}\right) \le 2(x+y+z)$$

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12) Daca *a,b,c > 0*, atunci $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}$.

Solutie: Notam $b + c = x$, $c + a = y$ si $a + b = z$

$2(a+b+c) = x+y+z$

$a+b+c = \frac{x+y+z}{2}$

$a=\frac{x+y+z}{2}-x$ => $a=\frac{y+z-x}{2}$

$b=\frac{x+y+z}{2}-y$ => $b=\frac{x-y+z}{2}$

$c=\frac{x+y-z}{2}$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{y+z-x}{2x}+\frac{x-y+z}{2y}+\frac{x+y-z}{2z}$$
$$=\frac{1}{2}\left(\frac{y}{x}+\frac{z}{x}-1+\frac{x}{y}-1+\frac{z}{y}+\frac{x}{z}+\frac{y}{z}-1\right)$$
$$=\frac{1}{2}\left(\frac{y}{x}+\frac{x}{y}+\frac{z}{x}+\frac{x}{z}+\frac{y}{z}+\frac{z}{y}-3\right)\geq\frac{1}{2}(6-3)=\frac{3}{2}$$

13) $\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\geq\frac{x}{z}+\frac{y}{x}+\frac{z}{y}$,oricare ar fi $x,y,z>0$.

Folosim inegalitatea $a^2+b^{2+}c^2\geq ab+ac+bc$

14) $\sqrt{(a+c)^2+b^2}+\sqrt{(a-c)^2+b^2}\geq 2\sqrt{a^2+b^2}$, oricare ar fi $a,b,c\in$ **R**.

Solutie: Folosim inegalitatea Mincowschi

$$\sqrt{(a+c)^2+b^2}+\sqrt{(a-c)^2+b^2} \geq \sqrt{(a+c+a-c)^2+(b+b)^2}=\sqrt{4(a^2+b^2)}$$

15) Aratati ca daca *a,b > 0, a + b = 1*, atunci $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geq \frac{25}{2}$.

Solutie: $\frac{x^2+y^2}{2} \geq \left(\frac{x+y}{2}\right)^2$, pentru $x=a+\frac{1}{a}$, $y=b+\frac{1}{b}$

$$\frac{\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2}{2} \geq \frac{\left(a+\frac{1}{a}+b+\frac{1}{b}\right)^2}{4}=\frac{1}{4}\left(1+\frac{1}{a}+\frac{1}{b}\right)^2=\frac{1}{4}\left(1+\frac{a+b}{ab}\right)^2$$
$$=\frac{1}{4}\left(1+\frac{1}{ab}\right)^2$$

Dar $ab \leq \left(\frac{a+b}{2}\right)^2=\frac{1}{4} \quad => \frac{1}{ab} \geq 4 \quad => 1+\frac{1}{ab} \geq 5$

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geq \frac{1}{2}\left(1+\frac{1}{ab}\right)^2 \geq \frac{1}{2}\cdot 5^2=\frac{25}{2}$$

Egalitate pentru $a=b=\frac{1}{2}$

16) Fie *a,b,c > 0* . Sa se demonstreze ca:

$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ac}{a+c} \leq \frac{a+b+c}{2}$. In ce conditii are loc egalitatea?

Solutie: Se aplica inegalitatea $\frac{ab}{a+b} \leq \frac{a+b}{4}$.

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17) Fie $a_1, a_2, \dots, a_n \in$ **R** si $b_1, b_2, \dots, b_n \in (0,+\infty)$, atunci

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n}\geq\frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n}$$ (Inegalitatea lui H. Bergstrom)

Solutie: Din inegalitatea Cauchy – Buniakovski – Schwartz avem:

$$\left(\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n}\right)(b_1+b_2+\cdots+b_n)$$
$$=\left(\left(\frac{a_1}{\sqrt{b_1}}\right)^2+\cdots+\left(\frac{a_n}{\sqrt{b_n}}\right)^2\right)\left(\left(\sqrt{b_1}\right)^2+\cdots+\left(\sqrt{b_n}\right)^2\right)$$
$$\geq\left(\frac{a_1}{\sqrt{b_1}}\cdot\sqrt{b_1}+\cdots+\frac{a_n}{\sqrt{b_n}}\cdot\sqrt{b_n}\right)^2=(a_1+\cdots+a_n)^2$$

18) Daca *a,b,c* sunt numere reale, atunci:

$$\sqrt{(a+c)^2+b^2}+\sqrt{(a-c)^2+b^2}\geq2\sqrt{a^2+b^2}.$$

Solutie: Folosind inegalitatea lui Minkowschi avem:

$$\sqrt{(a+c)^2+b^2}+\sqrt{(a-c)^2+b^2}\geq\sqrt{(a+c+a-c)^2+(b+b)^2}=2\sqrt{a^2+b^2}.$$

19) Fie $a,b,c\in(0,+\infty)$ si *n* numar natural, nenul atunci:

$$\frac{a^{n+1}}{b^n}+\frac{b^{n+1}}{c^n}+\frac{c^{n+1}}{a^n}\geq\frac{a^n}{b^{n-1}}+\frac{b^n}{c^{n-1}}+\frac{c^n}{a^{n-1}}\geq\cdots\geq\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c.$$

Solutie: Din $(a^n - b^n)(a-b) \geq 0$ deducem ca $a^{n+1} + b^{n+1} \geq a^n b + ab^n$, adica

$\frac{a^{n+1}}{b^n} + b \geq \frac{a^n}{b^{n-1}} + a$.

Analog obtinem $\frac{b^{n+1}}{c^n} + c \geq \frac{b^n}{c^{n-1}} + b$ si $\frac{c^{n+1}}{a^n} + a \geq \frac{c^n}{a^{n-1}} + c$.

Adunand aceste inegalitati se obtin inegalitatile in enunt.

20) Daca *a, b, c* sunt numere reale, pozitive,nenule, atunci:

$$\frac{a+b}{a^2+ab+b^2} + \frac{b+c}{b^2+bc+c^2} + \frac{c+a}{c^2+ac+a^2} \leq \frac{2}{3}\left(\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}\right).$$

Solutie: Demonstram ca: $\frac{a+b}{a^2+ab+b^2} \leq \frac{4}{3(a+b)} \leq \frac{2}{3\sqrt{ab}}$ oricare ar fi *a, b* numere reale

$\frac{a+b}{a^2+ab+b^2} \leq \frac{4}{3(a+b)}$ <=> $3(a+b)^2 \leq 4(a^2+ab+b^2)$ <=>

<=> $3a^2 + 6ab + 3b^2 \leq 4a^2 + 4ab + 4b^2$ <=> $2ab \leq a^2 + b^2$

<=> $(a-b)^2 \geq 0$ (Adevarat)

$\frac{4}{3(a+b)} \leq \frac{2}{3\sqrt{ab}}$ <=> $2\sqrt{ab} \leq a + b$ <=> $0 \leq \left(\sqrt{a} - \sqrt{b}\right)^2$ (Adevarat)

Avem:

$\frac{a+b}{a^2+ab+b^2} \leq \frac{2}{3\sqrt{ab}}$

$$\frac{b+c}{b^2+bc+c^2} \leq \frac{2}{3\sqrt{ab}}$$

$$\frac{c+a}{c^2+ac+a^2} \leq \frac{2}{3\sqrt{ab}}$$

Prin adunarea acestor inegalitati obtinem inegalitatea din enunt. Egalitate daca $a = b = c$.

21) Fie *a, b , c* $\in (0,+\infty)$ astfel incat $ab+ac+bc=\frac{1}{3}$ .Sa se arate ca:

$$\frac{a}{a^2-bc+1}+\frac{b}{b^2-ca+1}+\frac{c}{c^2-ab+1} \geq \frac{1}{a+b+c}.$$

Solutie: Din inegalitatea H. Bergstorm avem:

$$\sum \frac{a}{a^2-bc+1} = \sum \frac{a^2}{a^3-abc+a} \geq \frac{(a+b+c)^2}{a^3+b^3+c^3-3abc+a+b+c} =$$

$$= \frac{(a+b+c)^2}{(a+b+c)\left((a+b+c)^2-3(ab+bc+ca)\right)+a+b+c} = \frac{(a+b+c)^2}{(a+b+c)^3} = \frac{1}{a+b+c}$$

22) Fie *a, b, c* $\in (0,+\infty)$ astfel incat $a \geq b + c$. Sa se arate ca

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq\geq 10.$$

Solutie: Fie $E(a,b,c)=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$, atunci :

$$E(a,b,c)=3+a\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{1}{a}(b+c)+\frac{b}{c}+\frac{c}{b}.$$

Din $\frac{1}{b}+\frac{1}{c}\geq\frac{4}{b+c}$ deducem ca:

$$E(a,b,c)\geq 5+\frac{4a}{b+c}+\frac{b+c}{a}$$

Notam $\frac{b+c}{a}=t$ avem $t+\frac{4}{t}\geq 5$, deoarece $t\leq 1$, de unde $E(a,b,c)\geq 10$.

Folosirea inegalitatilor in rezolvarea unor ecuatii, sisteme

1. Rezolvati ecuatia $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\left(2+\sqrt{x^2-1}\right)=4$.

Solutie: $x > 0$

$x^2-1\geq 0 \quad <=> \quad |x|\geq 1 \quad <=> \quad x\in(-\infty,-1]\cup[1,+\infty)$ <=>

<=> $x\in[1,+\infty)$

$\sqrt{x}+\frac{1}{\sqrt{x}}\geq 2$ si $2+\sqrt{x^2-1}\geq 2$, deci $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\left(2+\sqrt{x^2-1}\right)\geq 4$ <=>

<=> $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\left(2+\sqrt{x^2-1}\right)=4$ daca $\sqrt{x}+\frac{1}{\sqrt{x}}=2$ si $2+\sqrt{x^2-1}=2$

=> $x = 1$

$S=\{1\}$

2. Rezolvati ecuatia:

$$x^4+\frac{1}{x^4}=\sqrt{x}+\frac{1}{\sqrt{x}}$$

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Solutie: oricare ar fi $a \in \mathbf{R} - \{0\}, a^2 + \frac{1}{a^2} \geq a + \frac{1}{a}$

<=> $a^4 + 1 \geq a^3 + a$ <=> $a^4 - a^3 - a + 1 \geq 0$ <=> $(a^3 - 1)(a - 1) \geq 0$

<=> $(a-1)^2(a^2 + a + 1) \geq 0$. Egalitate pentru $a = 1$

Pentru oricare ar fi $x > 0$

$x^4 + \frac{1}{x^4} \geq x^2 + \frac{1}{x^2} \geq x + \frac{1}{x} \geq \sqrt{x} + \frac{1}{\sqrt{x}}$ <=> $x^4 + \frac{1}{x^4} = \sqrt{x} + \frac{1}{\sqrt{x}}$ <=> $x = 1$

3. Rezolvati in **R** sistemul:

$$\begin{cases} x^2 + u = yz \\ y^2 - u = xz \\ z^2 = xy \\ x^3 + y^3 + z^3 + u^3 = -3 \end{cases}$$

Din primele trei ecuatii: $x^2 + y^2 + z^2 = xy + xz + yz$ =>

=> $x = y = z$ => $x^2 + u = x^2$ => $u = 0$

$\begin{cases} x^3 + y^3 + z^3 = -3 \\ x = y = z \end{cases}$ => $x = y = z = -1$

4. Fie $x, y \in \mathbf{R}$ astfel incat $x^2 + y^2 = x + y + 1$. Determinati valoarea maxima si minima a numerelor x si y.

Solutie: $4x^2 + 4y^2 = 4x + 4y + 4$ <=> $4x^2 - 4x + 1 + 4y^2 - 4y + 1 = 6$

$(2x-1)^2 + (2y-1)^2 = 6$ => $\begin{cases} (2x-1)^2 \leq 6 \\ (2y-1)^2 \leq 6 \end{cases}$ =>

$|2x-1| \leq \sqrt{6} \quad <=> \begin{cases} -\sqrt{6} \leq 2x-1 \leq \sqrt{6} \\ \frac{1-\sqrt{6}}{2} \leq x \leq \frac{1+\sqrt{6}}{2} \end{cases}$

Analog $\frac{1-\sqrt{6}}{2} \leq y \leq \frac{1+\sqrt{6}}{2}$.

Probleme propuse

1) $2 \leq \frac{x}{[x]} + \frac{[x]}{x} < \frac{5}{2}$, oricare ar fi $x \in [1,+\infty)$

unde cu $[x]$ s-a notat partea intreaga a numarului x

2) a) $\frac{1}{2} < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < 1$

b) $1 < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} < 2$, oricare ar fi $n \geq 2$

c) $\sqrt{2} + \sqrt{6} + \cdots + \sqrt{n(n+1)} < \frac{n(n+2)}{2}$, oricare ar fi *n* numar natural, $n \neq 0$

3) $x^4 - x + \frac{1}{2} > 0$,oricare ar fi *x* nr real

4) $\frac{\sqrt{6}}{5} + \frac{\sqrt{10}}{9} + \cdots + \frac{\sqrt{2n(2n+1)}}{4n+1} < \frac{n}{2}$

5) $\sqrt{x^2+y^2} + \sqrt{y^2+z^2} + \sqrt{z^2+x^2} \geq \sqrt{2}(x+y+z)$, oricare ar fi *x,y,z* nr reale

6) $x^4 + x^3 + x^2 + x + 1 > 0$, oricare ar fi *x* nr real

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7) Daca $a,b,c > 0$, astfel incat $a + b + c = 2\sqrt{abc}$, atunci $a^2 + b^2 + c^2 \geq 4\sqrt{abc}$.

8) Daca $a,b,c > 0$ cu $a + b + c = 1$, atunci $a^2 + b^2 + c^2 < 1$.

9) Sa se arate ca $\frac{1\cdot 2}{3^2} + \frac{2\cdot 3}{5^2} + \frac{3\cdot 4}{7^2} + \cdots + \frac{999\cdot 1000}{1999^2} < 2\cdot 5^3$.

10) Oricare ar fi x,y,z > 0 avem:

$$\frac{x}{x + 2y + 2z} + \frac{y}{y + 2x + 2z} + \frac{z}{z + 2y + 2x} \geq \frac{3}{5}$$

11) $abc(a + b + c) \leq a^4 + b^4 + c^4$, *a,b,c* numere reale

12) Oricare ar fi *x,y,z* numere reale avem:

$$3x^2 + 3y^2 + 3z^2 + 12 \geq 4\sqrt{3}(x + y + z)$$

13) Oricare ar fi *a,b,c* numere reale, pozitive, nenule avem:

$$\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} \geq \frac{1}{3}(a + b + c)^2$$

14) $\frac{\sqrt{x}}{(x+y)(x+z)} + \frac{\sqrt{y}}{(y+x)(y+z)} + \frac{\sqrt{z}}{(z+y)(z+x)} \leq \frac{3}{4}\frac{1}{\sqrt{xyz}}$, *x,y,z* numere reale, pozitive, nenule

15) Demonstrati ca daca $x, y, z \in (0, +\infty)$, atunci :

$$\frac{1}{x+2y+3z} + \frac{1}{y+2z+3x} + \frac{1}{z+2x+3y} \geq \frac{3}{2(x+y+z)}$$

16) Aflati numerele reale *x* cu proprietatea ca |x+1| + |x-1| =2