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MATHEMATICAL CONSTANTS

Steven R. Finch

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Mathematical Constants

Famous mathematical constants include the ratio of circular circumference to diameter, $\pi = 3.14\ldots$, and the natural logarithmic base, $e = 2.178\ldots$. Students and professionals usually can name at most a few others, but there are many more buried in the literature and awaiting discovery.

How do such constants arise, and why are they important? Here Steven Finch provides 136 essays, each devoted to a mathematical constant or a class of constants, from the well known to the highly exotic. Topics covered include the statistics of continued fractions, chaos in nonlinear systems, prime numbers, sum-free sets, isoperimetric problems, approximation theory, self-avoiding walks and the Ising model (from statistical physics), binary and digital search trees (from theoretical computer science), the Prouhet–Thue–Morse sequence, complex analysis, geometric probability, and the traveling salesman problem. This book will be helpful both to readers seeking information about a specific constant and to readers who desire a panoramic view of all constants coming from a particular field, for example, combinatorial enumeration or geometric optimization. Unsolved problems appear virtually everywhere as well. This is an outstanding scholarly attempt to bring together all significant mathematical constants in one place.

Steven R. Finch studied at Oberlin College and the University of Illinois in Urbana-Champaign, and held positions at TASC, MIT Lincoln Laboratory, and MathSoft Inc. He is presently a freelance mathematician in the Boston area. He is also a composer and has released a CD entitled "An Apple Gathering" devoted to his vocal and choral music.

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For Nancy Armstrong, the one constant

Contents

Preface

All numbers are not created equal. The fact that certain constants appear at all and then echo throughout mathematics, in seemingly independent ways, is a source of fascination. Formulas involving φ, e, π, or γ understandably fill a considerable portion of this book.

There are also many constants whose purposes are more specialized. Often such exotic quantities have been buried in the literature, known only to the experts of a narrow field, and invisible to the wider public. In some cases, the constants are easily computable; in other cases, they may be known to only one decimal digit of precision (or worse, none at all). Even rigorous proofs of existence might be unavailable.

My belief is that these latter constants are not as isolated as they may seem. The associated branches of research (unlike those involving φ, e, π, or γ) might simply require more time to develop the languages, functions, symmetries, etc., to express the constants more naturally. That is, if we work and listen hard enough, the echoes will become audible.

An elaborate taxonomy of mathematical constants has not yet been achieved; hence the organization of this book (by discipline) is necessarily subjective. A table of decimal approximations at the end gives an alternative organizational strategy (if ascending numerical order is helpful). The emphasis for me is not on the decimal expansions, but rather on the mathematical origins of the constants and their interrelationships. In short, the stories, not the table, tie the book together.

Material about well-known constants appears early and carefully, for the sake of readers without much mathematical background. Deeper into the text, however, I necessarily become more terse. My intended audience is advanced undergraduates and beyond (so I may assume readers have had calculus, matrix theory, differential equations, probability, some abstract algebra, and analysis). My aim is always to be clear and complete, to motivate why a particular constant or idea is important, and to indicate exactly where in the literature one should look for rigorous proofs and further elaboration.

I have incorporated Richard Guy's use of the ampersand (&) to denote joint work. For example, phrases like "...follows from the work of Hardy & Ramanujan and

Rademacher" are unambiguous when presented as here. The notation [3, 7] means references 3 and 7, whereas [3.7] refers to Section 3.7 of this book. The presence of a comma or decimal point is clearly crucial.

Many people have speculated on the role of the Internet in education and research. I have no question about the longstanding impact of the Web as a whole, but I remain skeptical that any specific Web address I might give here will exist in a mere five years. Of all mathematical Web sites available today, I expect that at least the following three will survive the passage of time:

- the ArXiv preprint server at Los Alamos National Laboratory (the meaning of a pointer to "math.CA/9910045" or to "solv-int/9801008" should be apparent to all ArXiv visitors),
- MathSciNet, established by the American Mathematical Society (subscribers to this service will be acquainted with *Mathematical Reviews* and the meaning of "MR 3,270e," "MR 33 #3320," or "MR 87h:51043"), and
- the On-Line Encyclopedia of Integer Sequences, created by Neil Sloane (a sequence identifier such as "A000688" will likewise suffice),

but not many more will outlive us. Even those that persist will be moved to various new locations and the old addresses will eventually fail. I have therefore chosen not to include Web URLs in this book. When I cite a Web site (e.g., "Numbers, Constants and Computation," "Prime Pages," "MathPages," "Plouffe's Tables," or "Geometry Junkyard"), the reference will be by name only.

A project of this magnitude cannot possibly be the work of one person. These pages are filled with innumerable acts of kindness by friends. To express my appreciation to all would considerably lengthen this preface; hence I will not attempt this. Special thanks are due to Philippe Flajolet, my mentor, who provided valuable encouragement from the very beginning. I am grateful to Victor Adamchik, Christian Bower, Anthony Guttmann, Joe Keane, Pieter Moree, Gerhard Niklasch, Simon Plouffe, Pascal Sebah, Craig Tracy, John Wetzel, and Paul Zimmermann. I am also indebted to MathSoft Inc.,* the Algorithms Group at INRIA, and CECM at Simon Fraser University for providing Web sites for my online research notes – my window to the world! – and to Cambridge University Press for undertaking this publishing venture with me.

Comments, corrections, and suggestions from readers are always welcome. Please send electronic mail to *Steven.Finch@inria.fr*. Thank you.

Notation

$\lfloor x \rfloor$	*floor function:* largest integer $\leq x$
$\lceil x \rceil$	*ceiling function:* smallest integer $\geq x$
$\{x\}$	*fractional part:* $x - \lfloor x \rfloor$
$\ln x$	*natural logarithm:* $\log_e x$
$\binom{n}{k}$	*binomial coefficient:* $\dfrac{n!}{k!(n-k)!}$
$b_0 + \dfrac{a_1\vert}{\vert b_1} + \dfrac{a_2\vert}{\vert b_2} + \dfrac{a_3\vert}{\vert b_3} + \cdots$	*continued fraction:* $b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$
$f(x) = O(g(x))$	*big O:* $\vert f(x)/g(x)\vert$ is bounded from above as $x \to x_0$
$f(x) = o(g(x))$	*little o:* $f(x)/g(x) \to 0$ as $x \to x_0$
$f(x) \sim g(x)$	*asymptotic equivalence:* $f(x)/g(x) \to 1$ as $x \to x_0$
$\sum_p$	summation over all prime numbers $p = 2, 3, 5, 7, 11, \ldots$ (only when the letter p is used)
$\prod_p$	same as $\sum_p$, with addition replaced by multiplication
$f(x)^n$	*power*: $(f(x))^n$, where n is an integer
$f^n(x)$	*iterate*: $\underbrace{f(f(\cdots f}_{n \text{ times}}(x)\ldots))$ where $n \geq 0$ is an integer

1

Well-Known Constants

1.1 Pythagoras' Constant, $\sqrt{2}$

The diagonal of a unit square has length $\sqrt{2} = 1.4142135623\ldots$. A theory, proposed by the Pythagorean school of philosophy, maintained that all geometric magnitudes could be expressed by rational numbers. The sides of a square were expected to be commensurable with its diagonals, in the sense that certain integer multiples of one would be equivalent to integer multiples of the other. This theory was shattered by the discovery that $\sqrt{2}$ is irrational [1–4].

Here are two proofs of the irrationality of $\sqrt{2}$, the first based on divisibility properties of the integers and the second using well ordering.

- If $\sqrt{2}$ were rational, then the equation $p^2 = 2q^2$ would be solvable in integers p and q, which are assumed to be in lowest terms. Since p^2 is even, p itself must be even and so has the form $p = 2r$. This leads to $2q^2 = 4r^2$ and thus q must also be even. But this contradicts the assumption that p and q were in lowest terms.
- If $\sqrt{2}$ were rational, then there would be a least positive integer s such that $s\sqrt{2}$ is an integer. Since $1 < 2$, it follows that $1 < \sqrt{2}$ and thus $t = s \cdot (\sqrt{2} - 1)$ is a positive integer. Also $t\sqrt{2} = s \cdot (\sqrt{2} - 1)\sqrt{2} = 2s - s\sqrt{2}$ is an integer and clearly $t < s$. But this contradicts the assumption that s was the smallest such integer.

Newton's method for solving equations gives rise to the following first-order recurrence, which is very fast and often implemented:

$$x_0 = 1, \qquad x_k = \frac{x_{k-1}}{2} + \frac{1}{x_{k-1}} \quad \text{for } k \geq 1, \qquad \lim_{k\to\infty} x_k = \sqrt{2}.$$

Another first-order recurrence [5] yields the reciprocal of $\sqrt{2}$:

$$y_0 = \frac{1}{2}, \qquad y_k = y_{k-1}\left(\frac{3}{2} - y_{k-1}^2\right) \quad \text{for } k \geq 1, \qquad \lim_{k\to\infty} y_k = \frac{1}{\sqrt{2}}.$$

The binomial series, also due to Newton, provides two interesting summations [6]:

$$1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^{2n}(2n-1)}\binom{2n}{n}=1+\frac{1}{2}-\frac{1}{2\cdot 4}+\frac{1\cdot 3}{2\cdot 4\cdot 6}-+\cdots=\sqrt{2},$$

$$1+\sum_{n=1}^{\infty}\frac{(-1)^{n}}{2^{2n}}\binom{2n}{n}=1-\frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+-\cdots=\frac{1}{\sqrt{2}}.$$

The latter is extended in [1.5.4]. We mention two beautiful infinite products [5, 7, 8]

$$\prod_{n=1}^{\infty}\left(1+\frac{(-1)^{n+1}}{2n-1}\right)=\left(1+\tfrac{1}{1}\right)\left(1-\tfrac{1}{3}\right)\left(1+\tfrac{1}{5}\right)\left(1-\tfrac{1}{7}\right)\cdots=\sqrt{2},$$

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{4(2n-1)^2}\right)=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{9\cdot 11}{10\cdot 10}\cdot\frac{13\cdot 15}{14\cdot 14}\cdots=\frac{1}{\sqrt{2}}$$

and the regular continued fraction [9]

$$2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}=2+\frac{1|}{|2}+\frac{1|}{|2}+\frac{1|}{|2}+\cdots=1+\sqrt{2}=(-1+\sqrt{2})^{-1},$$

which is related to **Pell's sequence**

$$a_0=0, \qquad a_1=1, \qquad a_n=2a_{n-1}+a_{n-2} \quad \text{for } n\geq 2$$

via the limiting formula

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1+\sqrt{2}.$$

This is completely analogous to the famous connection between the Golden mean φ and Fibonacci's sequence [1.2]. See also Figure 1.1.

Viète's remarkable product for Archimedes' constant π [1.4.2] involves only the number 2 and repeated square-root extractions. Another expression connecting π and radicals appears in [1.4.5].

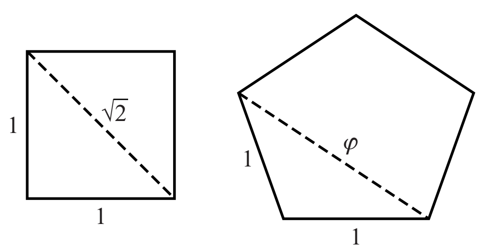

Figure 1.1. The diagonal of a regular unit pentagon, connecting any two nonadjacent corners, has length given by the Golden mean φ (rather than by Pythagoras' constant).

We return finally to irrationality issues: There obviously exist rationals x and y such that x^y is irrational (just take $x = 2$ and $y = 1/2$). Do there exist *irrationals* x and y such that x^y is *rational*? The answer to this is very striking. Let

$$z = \sqrt{2}^{\sqrt{2}}.$$

If z is rational, then take $x = y = \sqrt{2}$. If z is irrational, then take $x = z$ and $y = \sqrt{2}$, and clearly $x^y = 2$. Thus we have answered the question ("yes") without addressing the actual arithmetical nature of z. In fact, z is transcendental by the Gel'fond–Schneider theorem [10], proved in 1934, and hence is irrational. There are many unsolved problems in this area of mathematics; for example, we do not know whether

$$\sqrt{2}^z = \sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$$

is irrational (let alone transcendental).

1.1.1 Generalized Continued Fractions

It is well known that any quadratic irrational possesses a periodic regular continued fraction expansion and vice versa. Comparatively few people have examined the generalized continued fraction [11–17]

$$w(p,q) = q + \cfrac{p + \cfrac{1}{q + \cfrac{p+\cdots}{q+\cdots}}}{q + \cfrac{p + \cfrac{1+\cdots}{q+\cdots}}{q + \cfrac{p+\cdots}{q+\cdots}}},$$

which exhibits a fractal-like construction. Each *new* term in a particular generation (i.e., in a partial convergent) is replaced according to the rules

$$p \to p + \frac{1}{q}, \qquad q \to q + \frac{p}{q}$$

in the next generation. Clearly

$$w = q + \frac{p + \frac{1}{w}}{w}; \qquad \text{that is,} \quad w^3 - qw^2 - pw - 1 = 0.$$

In the special case $p = q = 3$, the higher-order continued fraction converges to $(-1 + \sqrt[3]{2})^{-1}$. It is conjectured that regular continued fractions for cubic irrationals behave like those for almost all real numbers [18–21], and no patterns are evident. The ordinary replacement rule

$$r \to r + \frac{1}{r}$$

is sufficient for the study of quadratic irrationals, but requires extension for broader classes of algebraic numbers.

Two alternative representations of $\sqrt[3]{2}$ are as follows [22]:

$$\sqrt[3]{2} = 1 + \cfrac{1}{3 + \cfrac{3}{a} + \cfrac{1}{b}}, \qquad \text{where} \quad a = 3 + \frac{3}{a} + \frac{1}{b}, \qquad b = 12 + \frac{10}{a} + \frac{3}{b}$$

and [23]

$$\sqrt[3]{2} = 1 + \frac{1|}{|3} + \frac{2|}{|2} + \frac{4|}{|9} + \frac{5|}{|2} + \frac{7|}{|15} + \frac{8|}{|2} + \frac{10|}{|21} + \frac{11|}{|2} + \cdots.$$

Other usages of the phrase "generalized continued fractions" include those in [24], with application to simultaneous Diophantine approximation, and in [25], with a geometric interpretation involving the boundaries of convex hulls.

1.1.2 Radical Denestings

We mention two striking radical denestings due to Ramanujan:

$$\sqrt[3]{\sqrt[3]{2} - 1} = \sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{4}{9}}, \qquad \sqrt[2]{\sqrt[3]{5} - \sqrt[3]{4}} = \tfrac{1}{3}\left(\sqrt[3]{2} + \sqrt[3]{20} - \sqrt[3]{25}\right).$$

Such simplifications are an important part of computer algebra systems [26].

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1.2 The Golden Mean, φ

Consider a line segment:

What is the most "pleasing" division of this line segment into two parts? Some people might say at the halfway point:

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Others might say at the one-quarter or three-quarters point. The "correct answer" is, however, none of these, and is supposedly found in Western art from the ancient Greeks onward (aestheticians speak of it as the principle of "dynamic symmetry"):

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If the right-hand portion is of length $v = 1$, then the left-hand portion is of length $u = 1.618\ldots$. A line segment partitioned as such is said to be divided in Golden or Divine section. What is the justification for endowing this particular division with such elevated status? The length u, as drawn, is to the whole length $u + v$, as the length v is to u:

$$\frac{u}{u+v} = \frac{v}{u}.$$

Letting $\varphi = u/v$, solve for φ via the observation that

$$1 + \frac{1}{\varphi} = 1 + \frac{v}{u} = \frac{u+v}{u} = \frac{u}{v} = \varphi.$$

The positive root of the resulting quadratic equation $\varphi^2 - \varphi - 1 = 0$ is

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots,$$

which is called the **Golden mean** or **Divine proportion** [1,2].

The constant φ is intricately related to **Fibonacci's sequence**

$$f_0 = 0, \qquad f_1 = 1, \qquad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2.$$

This sequence models (in a naive way) the growth of a rabbit population. Rabbits are assumed to start having bunnies once a month after they are two months old; they always give birth to twins (one male bunny and one female bunny), they never die, and they never stop propagating. The number of rabbit pairs after n months is f_n.

What can φ possibly have in common with $\{f_n\}$? This is one of the most remarkable ideas in all of mathematics. The partial convergents leading up to the regular continued fraction representation of φ,

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = 1 + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + \cdots,$$

are all ratios of successive Fibonacci numbers; hence

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \varphi.$$

This result is also true for arbitrary sequences satisfying the same recursion $f_n = f_{n-1} + f_{n-2}$, assuming that the initial terms f_0 and f_1 are distinct [3,4].

The rich geometric connection between the Golden mean and Fibonacci's sequence is seen in Figure 1.2. Starting with a single Golden rectangle (of length φ and width 1), there is a natural sequence of nested Golden rectangles obtained by removing the leftmost square from the first rectangle, the topmost square from the second rectangle, etc. The length and width of the n^{th} Golden rectangle can be written as linear expressions $a + b\varphi$, where the coefficients a and b are always Fibonacci numbers. These Golden rectangles can be inscribed in a logarithmic spiral as pictured. Assume that the lower left corner of the first rectangle is the origin of an xy-coordinate system.

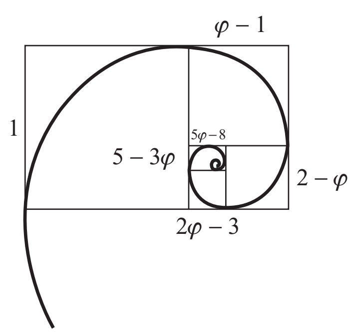

Figure 1.2. The Golden spiral circumscribes the sequence of Golden rectangles.

The accumulation point for the spiral can be proved to be $(\frac{1}{5}(1+3\varphi), \frac{1}{5}(3-\varphi))$. Such logarithmic spirals are "equiangular" in the sense that every line through (x_∞, y_∞) cuts across the spiral at a constant angle ξ. In this way, logarithmic spirals generalize ordinary circles (for which $\xi = 90^\circ$). The logarithmic spiral pictured gives rise to the constant angle $\xi = \operatorname{arccot}(\frac{2}{\pi}\ln(\varphi)) = 72.968\ldots^\circ$. Logarithmic spirals are evidently found throughout nature; for example, the shell of a chambered nautilus, the tusks of an elephant, and patterns in sunflowers and pine cones [4–6].

Another geometric application of the Golden mean arises when inscribing a regular pentagon within a given circle by ruler and compass. This is related to the fact that

$$2\cos\left(\frac{\pi}{5}\right) = \varphi, \qquad 2\sin\left(\frac{\pi}{5}\right) = \sqrt{3-\varphi}.$$

The Golden mean, just as it has a simple regular continued fraction expansion, also has a simple radical expansion [7]

$$\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}.$$

The manner in which this expansion converges to φ is discussed in [1.2.1]. Like Pythagoras' constant [1.1], the Golden mean is irrational and simple proofs are given in [8,9].

Here is a series [10] involving φ:

$$\begin{aligned}\frac{2\sqrt{5}}{5}\ln(\varphi) &= \left(1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}\right)\\ &\quad+\left(\frac{1}{11}-\frac{1}{12}-\frac{1}{13}+\frac{1}{14}\right)+\cdots,\end{aligned}$$

which reminds us of certain series connected with Archimedes' constant [1.4.1]. A direct expression for φ as a sum can be obtained from the Taylor series for the square root function, expanded about 4. The Fibonacci numbers appear in yet another representation [11] of φ:

$$4-\varphi = \sum_{n=0}^{\infty}\frac{1}{f_{2^n}} = \frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_4}+\frac{1}{f_8}+\cdots.$$

Among many other possible formulas involving φ, we mention the four Rogers–Ramanujan continued fractions

$$\begin{aligned}
\frac{1}{\alpha-\varphi}\exp\left(-\frac{2\pi}{5}\right) &= 1+\frac{e^{-2\pi}|}{|1}+\frac{e^{-4\pi}|}{|1}+\frac{e^{-6\pi}|}{|1}+\frac{e^{-8\pi}|}{|1}+\cdots,\\
\frac{1}{\beta-\varphi}\exp\left(-\frac{2\pi}{\sqrt{5}}\right) &= 1+\frac{e^{-2\pi\sqrt{5}}|}{|1}+\frac{e^{-4\pi\sqrt{5}}|}{|1}+\frac{e^{-6\pi\sqrt{5}}|}{|1}+\frac{e^{-8\pi\sqrt{5}}|}{|1}+\cdots,\\
\frac{1}{\kappa-(\varphi-1)}\exp\left(-\frac{\pi}{5}\right) &= 1-\frac{e^{-\pi}|}{|1}+\frac{e^{-2\pi}|}{|1}-\frac{e^{-3\pi}|}{|1}+\frac{e^{-4\pi}|}{|1}-+\cdots,\\
\frac{1}{\lambda-(\varphi-1)}\exp\left(-\frac{\pi}{\sqrt{5}}\right) &= 1-\frac{e^{-\pi\sqrt{5}}|}{|1}+\frac{e^{-2\pi\sqrt{5}}|}{|1}-\frac{e^{-3\pi\sqrt{5}}|}{|1}+\frac{e^{-4\pi\sqrt{5}}|}{|1}-+\cdots,
\end{aligned}$$

where

$$\alpha = \left(\varphi\sqrt{5}\right)^{\frac{1}{2}}, \qquad \alpha' = \frac{1}{\sqrt{5}}\left((\varphi-1)\sqrt{5}\right)^{\frac{5}{2}}, \qquad \beta = \frac{\sqrt{5}}{1+\sqrt[5]{\alpha'}-1},$$

$$\kappa = \left((\varphi-1)\sqrt{5}\right)^{\frac{1}{2}}, \qquad \kappa' = \frac{1}{\sqrt{5}}\left(\varphi\sqrt{5}\right)^{\frac{5}{2}}, \qquad \lambda = \frac{\sqrt{5}}{1+\sqrt[5]{\kappa'}-1}.$$

The fourth evaluation is due to Ramanathan [9, 12–16].

1.2.1 Analysis of a Radical Expansion

The radical expansion [1.2] for φ can be rewritten as a sequence $\{\varphi_n\}$:

$$\varphi_1 = 1, \qquad \varphi_n = \sqrt{1+\varphi_{n-1}} \quad \text{for } n \geq 2.$$

Paris [17] proved that the rate in which φ_n approaches the limit φ is given by

$$\varphi - \varphi_n \sim \frac{2C}{(2\varphi)^n} \quad \text{as } n \to \infty,$$

where $C = 1.0986419643\ldots$ is a new constant. Here is an exact characterization of C. Let $F(x)$ be the analytic solution of the functional equation

$$F(x) = 2\varphi F(\varphi - \sqrt{\varphi^2 - x}), \qquad |x| < \varphi^2,$$

subject to the initial conditions $F(0) = 0$ and $F'(0) = 1$. Then $C = \varphi F(1/\varphi)$. A power-series technique can be used to evaluate C numerically from these formulas. It is simpler, however, to use the following product:

$$C = \prod_{n=2}^{\infty} \frac{2\varphi}{\varphi + \varphi_n},$$

which is stable and converges quickly [18].

Another interesting constant is defined via the radical expression [7, 19]

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\sqrt{5+\cdots}}}}} = 1.7579327566\ldots,$$

but no expression of this in terms of other constants is known.

1.2.2 Cubic Variations of the Golden Mean

Perrin's sequence is defined by

$$g_0 = 3, \qquad g_1 = 0, \qquad g_2 = 2, \qquad g_n = g_{n-2} + g_{n-3} \quad \text{for } n \geq 3$$

and has the property that $n > 1$ divides g_n if n is prime [20, 21]. The limit of ratios of successive Perrin numbers

$$\psi = \lim_{n\to\infty} \frac{g_{n+1}}{g_n}$$

satisfies $\psi^3 - \psi - 1 = 0$ and is given by

$$\psi = \left(\frac{1}{2} + \frac{\sqrt{69}}{18}\right)^{\frac{1}{3}} + \frac{1}{3}\left(\frac{1}{2} + \frac{\sqrt{69}}{18}\right)^{-\frac{1}{3}} = \frac{2\sqrt{3}}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\right)\right)$$
$$= 1.3247179572\ldots.$$

This also has the radical expansion

$$\psi = \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \cdots}}}}}.$$

An amusing account of ψ is given in [20], where it is referred to as the Plastic constant (to contrast against the Golden constant). See also [2.30].

The so-called **Tribonacci sequence** [22, 23]

$$h_0 = 0, \qquad h_1 = 0, \qquad h_2 = 1, \qquad h_n = h_{n-1} + h_{n-2} + h_{n-3} \quad \text{for } n \geq 3$$

has an analogous limiting ratio

$$\chi = \left(\frac{19}{27} + \frac{\sqrt{33}}{9}\right)^{\frac{1}{3}} + \frac{4}{9}\left(\frac{19}{27} + \frac{\sqrt{33}}{9}\right)^{-\frac{1}{3}} + \frac{1}{3} = \frac{4}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{19}{8}\right)\right) + \frac{1}{3}$$
$$= 1.8392867552\ldots,$$

that is, the real solution of $\chi^3 - \chi^2 - \chi - 1 = 0$. See [1.2.3]. Consider also the **four-numbers game**: Start with a 4-vector (a, b, c, d) of nonnegative real numbers and determine the cyclic absolute differences $(|b - a|, |c - b|, |d - c|, |a - d|)$. Iterate indefinitely. Under most circumstances (e.g., if a, b, c, d are each positive integers), the process terminates with the zero 4-vector after only a finite number of steps. Is this always true? No. It is known [24] that $v = (1, \chi, \chi^2, \chi^3)$ is a counterexample, as well as any positive scalar multiple of v, or linear combination with the 4-vector $(1, 1, 1, 1)$. Also, $w = (\chi^3, \chi^2 + \chi, \chi^2, 0)$ is a counterexample, as well as any positive scalar multiple of w, or linear combination with the 4-vector $(1, 1, 1, 1)$. These encompass all the possible exceptions. Note that, starting with w, one obtains v after one step.

1.2.3 Generalized Continued Fractions

Recall from [1.1.1] that generalized continued fractions are constructed via the replacement rule

$$p \to p + \frac{1}{q}, \qquad q \to q + \frac{p}{q}$$

applied to each new term in a particular generation. In particular, if $p = q = 1$, the partial convergents are equal to ratios of successive terms of the Tribonacci sequence, and hence converge to χ. By way of contrast, the replacement rule [25, 26]

$$r \to r + \cfrac{1}{r + \cfrac{1}{r}}$$

is associated with a root of $x^3 - rx^2 - r = 0$. If $r = 1$, the limiting value is

$$\left(\frac{29}{54} + \frac{\sqrt{93}}{18}\right)^{\frac{1}{3}} + \frac{1}{9}\left(\frac{29}{54} + \frac{\sqrt{93}}{18}\right)^{-\frac{1}{3}} + \frac{1}{3} = \frac{2}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{29}{2}\right)\right) + \frac{1}{3}$$
$$= 1.4655712318\ldots.$$

Other higher-order analogs of the Golden mean are offered in [27–29].

1.2.4 Random Fibonacci Sequences

Consider the sequence of random variables

$$x_0 = 1, \qquad x_1 = 1, \qquad x_n = \pm x_{n-1} \pm x_{n-2} \quad \text{for } n \geq 2,$$

where the signs are equiprobable and independent. Viswanath [30–32] proved the surprising result that

$$\lim_{n\to\infty} \sqrt[n]{|x_n|} = 1.13198824\ldots$$

with probability 1. Embree & Trefethen [33] proved that generalized random linear recurrences of the form

$$x_n = x_{n-1} \pm \beta x_{n-2}$$

decay exponentially with probability 1 if $0 < \beta < 0.70258\ldots$ and grow exponentially with probability 1 if $\beta > 0.70258\ldots.$

1.2.5 Fibonacci Factorials

We mention the asymptotic result $\prod_{k=1}^{n} f_k \sim c \cdot \varphi^{n(n+1)/2} \cdot 5^{-n/2}$ as $n \to \infty$, where [34, 35]

$$c = \prod_{n=1}^{\infty}\left(1 - \frac{(-1)^n}{\varphi^{2n}}\right) = 1.2267420107\ldots.$$

See the related expression in [5.14].

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1.3 The Natural Logarithmic Base, e

It is not known who first determined

$$\lim_{x \to 0}(1+x)^{\frac{1}{x}} = e = 2.7182818284\ldots.$$

We see in this limit the outcome of a fierce tug-of-war. On the one side, the exponent explodes to infinity. On the other side, $1+x$ rushes toward the multiplicative identity 1. It is interesting that the additive equivalent of this limit

$$\lim_{x \to 0} x \cdot \frac{1}{x} = 1$$

is trivial. A geometric characterization of e is as follows: e is the unique positive root x of the equation

$$\int_1^x \frac{1}{u} du = 1,$$

which is responsible for e being employed as the natural logarithmic base. In words, e is the unique positive number exceeding 1 for which the planar region bounded by the curves $v = 1/u$, $v = 0$, $u = 1$, and $u = e$ has unit area.

The definition of e implies that

$$\frac{d}{dx}(c \cdot e^x) = c \cdot e^x$$

and, further, that any solution of the first-order differential equation

$$\frac{dy}{dx} = y(x)$$

must be of this form. Applications include problems in population growth and radioactive decay. Solutions of the second-order differential equation

$$\frac{d^2y}{dx^2} = y(x)$$

are necessarily of the form $y(x) = a \cdot e^x + b \cdot e^{-x}$. The special case $y(x) = \cosh(x)$ (i.e., $a = b = 1/2$) is called a catenary and is the shape assumed by a certain uniform flexible cable hanging under its own weight. Moreover, if one revolves part of a catenary around the x-axis, the resulting surface area is smaller than that of any other curve with the same endpoints [1,2].

The series

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots$$

is rapidly convergent – ordinary summation of the terms as listed is very quick for all practical purposes – so it may be surprising to learn that a more efficient means for computing the n^{th} partial sum is possible [3,4]. Define two functions $p(a, b)$ and

$q(a, b)$ recursively as follows:

$$\begin{pmatrix} p(a,b) \\ q(a,b) \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 \\ b \end{pmatrix} & \text{if } b = a+1, \\ \begin{pmatrix} p(a,m)q(m,b) + p(m,b) \\ q(a,m)q(m,b) \end{pmatrix} & \text{otherwise, where} \\ & m = \left\lfloor \frac{a+b}{2} \right\rfloor. \end{cases}$$

Then it is not difficult to show that $1 + p(0, n)/q(0, n)$ gives the desired partial sum. Such a **binary splitting** approach to computing e has fewer single-digit arithmetic operations (i.e., reduced bit complexity) than the usual approach. Accelerated methods like this grew out of [5–7]. When coupled with FFT-based integer multiplication, this algorithm is asymptotically as fast as any known.

The factorial series gives the following matching problem solution [8]. Let $P(n)$ denote the probability that a randomly chosen one-to-one function $f : \{1, 2, 3, \ldots, n\} \to \{1, 2, 3, \ldots, n\}$ has at least one fixed point; that is, at least one integer k for which $f(k) = k$, $1 \le k \le n$. Then

$$\lim_{n\to\infty} P(n) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} = 1 - \frac{1}{e} = 0.6321205588\ldots.$$

See Figure 1.3; a generalization appears in [5.4]. Also, let $X_1, X_2, X_3, \ldots$ be independent random variables, each uniformly distributed on the interval [0, 1]. Define an integer N by

$$N = \min\left\{ n : \sum_{k=1}^{n} X_k > 1 \right\};$$

then the expected value $\mathrm{E}(N) = e$. In the language of stochastics, a renewal process with uniform interarrival times X_k has a mean renewal count involving the natural logarithmic base [9].

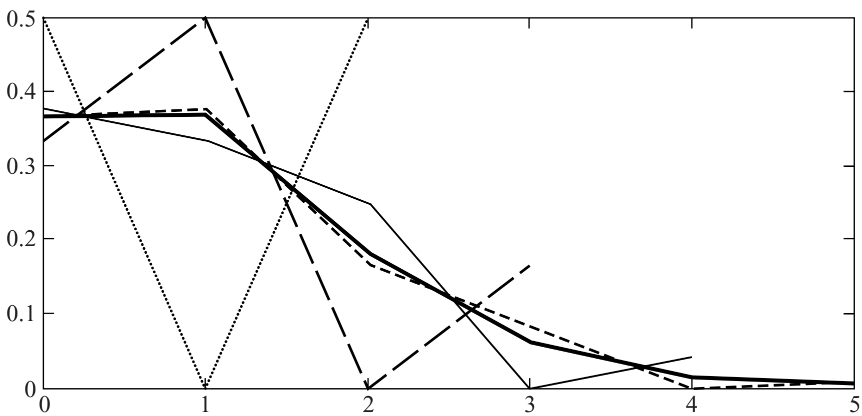

Figure 1.3. Distribution of the number of fixed points of a random permutation f on n symbols, tending to Poisson(1) as $n \to \infty$.

Break a stick of length r into m equal parts [10]. The integer m such that the product of the lengths of the parts is maximized is $\lfloor r/e \rfloor$ or $\lfloor r/e \rfloor + 1$. See [5.15] for information on a related application known as the secretary problem.

There are several Wallis-like infinite products [4, 11]

$$e = \frac{2}{1} \cdot \left(\frac{4}{3}\right)^{\frac{1}{2}} \cdot \left(\frac{6 \cdot 8}{5 \cdot 7}\right)^{\frac{1}{4}} \cdot \left(\frac{10 \cdot 12 \cdot 14 \cdot 16}{9 \cdot 11 \cdot 13 \cdot 15}\right)^{\frac{1}{8}} \cdots,$$

$$\frac{e}{2} = \left(\frac{2}{1}\right)^{\frac{1}{2}} \cdot \left(\frac{2 \cdot 4}{3 \cdot 3}\right)^{\frac{1}{4}} \cdot \left(\frac{4 \cdot 6 \cdot 6 \cdot 8}{5 \cdot 5 \cdot 7 \cdot 7}\right)^{\frac{1}{8}} \cdots$$

and continued fractions [1.3.2] as well as the following fascinating connection to prime number theory [12]. If we define

$$n? = \prod_{\substack{p \le n \\ p \text{ prime}}} p, \qquad \text{then} \quad \lim_{n \to \infty} (n?)^{\frac{1}{n}} = e,$$

which is a consequence of the Prime Number Theorem. Equally fascinating is the fact that

$$\lim_{n \to \infty} \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$$

by Stirling's formula; thus the growth of $n!$ exceeds that of $n?$ by an order of magnitude. We also have [13–15]

$$\lim_{n \to \infty} (n!)^{\frac{1}{n}} - ((n-1)!)^{\frac{1}{n-1}} = \frac{1}{e}, \qquad \lim_{n \to \infty} \prod_{k=1}^{n} (n^2 + k)(n^2 - k)^{-1} = e.$$

The irrationality of e was proved by Euler and its transcendence by Hermite; that is, the natural logarithmic base e cannot be a zero of a polynomial with integer coefficients [4, 16–18].

An unusual procedure for calculating e, known as the spigot algorithm, was first publicized in [19]. Here the intrigue lies not in the speed of the algorithm (it is slow) but in other characteristics: It is entirely based on integer arithmetic, for example.

Some people call e *Euler's constant*, but the same phrase is so often used to refer to the Euler–Mascheroni constant γ that confusion would be inevitable. Napier came very close to discovering e in 1614; consequently, some people call e *Napier's constant* [1].

1.3.1 Analysis of a Limit

The Maclaurin series

$$\frac{1}{e}(1+x)^{\frac{1}{x}} = 1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \frac{2447}{5760}x^4 - \frac{959}{2304}x^5 + O(x^6)$$

describes more fully what happens in the limiting definition of e; for example,

$$\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} = -\frac{1}{2}e, \qquad \lim_{x\to 0} \frac{\dfrac{(1+x)^{\frac{1}{x}} - e}{x} + \dfrac{1}{2}e}{x} = \frac{11}{24}e.$$

Quicker convergence is obtained by the formulas [20–24]:

$$\lim_{x\to 0} \left(\frac{2+x}{2-x}\right)^{\frac{1}{x}} = e, \qquad \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}} = e.$$

To illustrate, the first terms in the corresponding asymptotic expansions are $1 + x^2/12$ and $1 + 1/(24n^2)$. Further improvements are possible.

1.3.2 Continued Fractions

The regular continued fraction for e,

$$e = 2 + \frac{1|}{|1} + \frac{1|}{|2} + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|4} + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|6} + \frac{1|}{|1} + \cdots,$$

is (after suitable transformation) one of a family of continued fractions [25–28]:

$$\coth\left(\frac{1}{m}\right) = \frac{e^{2/m}+1}{e^{2/m}-1} = m + \frac{1|}{|3m} + \frac{1|}{|5m} + \frac{1|}{|7m} + \frac{1|}{|9m} + \cdots,$$

where m is any positive integer. Davison [29] obtained an algorithm for computing quotients of $\coth(3/2)$ and $\coth(2)$, for example, but no patterns can be found. Other continued fractions include [1,26,30,31]

$$e - 1 = 1 + \frac{2|}{|2} + \frac{3|}{|3} + \frac{4|}{|4} + \frac{5|}{|5} + \cdots, \qquad \frac{1}{e-2} = 1 + \frac{1|}{|2} + \frac{2|}{|3} + \frac{3|}{|4} + \frac{4|}{|5} + \cdots,$$

and still more can be found in [32,33].

1.3.3 The Logarithm of Two

Finally, let us say a few words [34] about the closely related constant $\ln(2)$,

$$\ln(2) = \int_0^1 \frac{1}{1+t}dt = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = 0.6931471805\ldots,$$

which has limiting expressions similar to that for e:

$$\lim_{x\to 0} \frac{2^x - 1}{x} = \ln(2) = \lim_{x\to 0} \frac{2^x - 2^{-x}}{2x}.$$

Well-known summations include the Maclaurin series for $\ln(1+x)$ evaluated at $x = 1$ and $x = -1/2$,

$$\ln(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \sum_{k=1}^{\infty} \frac{1}{k2^k}.$$

A binary digit extraction algorithm can be based on the series

$$\ln(2) = \sum_{k=1}^{\infty} \left(\frac{1}{8k+8} + \frac{1}{4k+2} \right) \frac{1}{4^k},$$

which enables us to calculate the d^{th} bit of ln(2) without being forced to calculate all the preceding $d - 1$ bits. See also [2.1], [6.2], and [7.2].

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1.4 Archimedes' Constant, π

Any brief treatment of π, the most famous of the transcendental constants, is necessarily incomplete [1–5]. Its innumerable appearances throughout mathematics stagger the mind.

The area enclosed by a circle of radius 1 is

$$A = \pi = 4\int_0^1 \sqrt{1-x^2}\,dx = \lim_{n\to\infty} \frac{4}{n^2}\sum_{k=0}^{n} \sqrt{n^2-k^2} = 3.1415926535\ldots$$

while its circumference is

$$C = 2\pi = 4\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = 4\int_0^1 \sqrt{1+\left(\frac{d}{dx}\sqrt{1-x^2}\right)^2}\,dx.$$

The formula for A is based on the definition of area in terms of a Riemann integral, that is, a limit of Riemann sums. The formula for C uses the definition of arclength, given a continuously differentiable curve. How is it that the same mysterious π appears in both formulas? A simple integration by parts provides the answer, with no trigonometry required [6].

In the third century B.C., Archimedes considered inscribed and circumscribed regular polygons of 96 sides and deduced that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. The recursion

$$a_0 = 2\sqrt{3}, \qquad b_0 = 3,$$
$$a_{n+1} = \frac{2a_nb_n}{a_n+b_n}, \qquad b_{n+1} = \sqrt{a_{n+1}b_n} \qquad \text{for } n \ge 0$$

(often called the Borchardt–Pfaff algorithm) essentially gives Archimedes' estimate on the fourth iteration [7–11]. It is only linearly convergent (meaning that the number of iterations is roughly proportional to the number of correct digits). It resembles the arithmetic-geometric-mean (AGM) recursion discussed with regard to Gauss' lemniscate constant [6.1].

The utility of π is not restricted to planar geometry. The volume enclosed by a sphere of radius 1 in n-dimensional Euclidean space is

$$V = \begin{cases} \dfrac{\pi^k}{k!} & \text{if } n = 2k, \\ 2^{2k+1}\dfrac{k!}{(2k+1)!}\pi^k & \text{if } n = 2k+1, \end{cases}$$

while its surface area is

$$S = \begin{cases} \dfrac{2\pi^k}{(k-1)!} & \text{if } n = 2k, \\ 2^{2k+1}\dfrac{k!}{(2k)!}\pi^k & \text{if } n = 2k+1. \end{cases}$$

These formulas are often expressed in terms of the gamma function, which we discuss in [1.5.4]. The planar case (a circle) corresponds to $n = 2$.

Another connection between geometry and π arises in Buffon's needle problem [1, 12–15]. Suppose a needle of length 1 is thrown at random on a plane marked by parallel lines of distance 1 apart. What is the probability that the needle will intersect one of the lines? The answer is $2/\pi = 0.6366197723\ldots$.

Here is a completely different probabilistic interpretation [16, 17] of π. Suppose two integers are chosen at random. What is the probability that they are coprime, that is, have no common factor exceeding 1? The answer is $6/\pi^2 = 0.6079271018\ldots$ (in the limit over large intervals). Equivalently, let $R(N)$ be the number of *distinct* rational numbers a/b with integers a, b satisfying $0 < a, b < N$. The total number of ordered pairs (a, b) is N^2, but $R(N)$ is strictly less than this since many fractions are not in lowest terms. More precisely, by preceding statements, $R(N) \sim 6N^2/\pi^2$.

Among the most famous limits in mathematics is Stirling's formula [18]:

$$\lim_{n\to\infty} \frac{n!}{e^{-n}n^{n+1/2}} = \sqrt{2\pi} = 2.5066282746\ldots.$$

Archimedes' constant has many other representations too, some of which are given later. It was proved to be irrational by Lambert and transcendental by Lindemann [2, 16, 19]. The first truly attractive formula for computing decimal digits of π was found by Machin [1, 13]:

$$\begin{aligned} \frac{\pi}{4} &= 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) \\ &= 4\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\cdot 5^{2k+1}} - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\cdot 239^{2k+1}}. \end{aligned}$$

The advantage of this formula is that the second term converges very rapidly and the first is nice for decimal arithmetic. In 1706, Machin became the first individual to correctly compute 100 digits of π.

We skip over many years of history and discuss one other significant algorithm due to Salamin and Brent [2, 20–23]. Define a recursion by

$$\begin{aligned} &a_0 = 1, & &b_0 = 1/\sqrt{2}, & &c_0 = 1/2, & &s_0 = 1/2, \\ &a_{n+1} = \frac{a_n + b_n}{2}, & &b_{n+1} = \sqrt{a_n b_n}, & &c_{n+1} = \left(\frac{c_n}{4a_{n+1}}\right)^2, & &s_{n+1} = s_n - 2^{n+1} c_{n+1} \end{aligned}$$

for $n \geq 0$. Then the ratio $2a_n^2/s_n$ converges quadratically to π (meaning that each iteration approximately doubles the number of correct digits). Even faster cubic and quartic algorithms were obtained by Borwein & Borwein [2, 22, 24, 25]; these draw upon Ramanujan's work on modular equations. These are each a far cry computationally from Archimedes' approach. Using techniques like these, Kanada computed close to a trillion digits of π.

There is a spigot algorithm for calculating π just as for e [26]. Far more important, however, is the digit-extraction algorithm discovered by Bailey, Borwein & Plouffe [27–29] based on the formula

$$\begin{aligned} \pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \\ \times \left(\frac{4+8r}{8k+1} - \frac{8r}{8k+2} - \frac{4r}{8k+3} - \frac{2+8r}{8k+4} - \frac{1+2r}{8k+5} - \frac{1+2r}{8k+6} + \frac{r}{8k+7} \right) \end{aligned}$$

(for $r = 0$) and requiring virtually no memory. (The extension to complex $r \neq 0$ is due to Adamchik & Wagon [30, 31].) A consequence of this breakthrough is that we now know the quadrillionth digit in the binary expansion for π, thanks largely to Bellard and Percival. An analogous base-3 formula was found by Broadhurst [32].

Some people call π *Ludolph's constant* after the mathematician Ludolph van Ceulen who devoted most of his life to computing π to 35 decimal places.

The formulas in this essay have a qualitatively different character than those for the natural logarithmic base e. Wimp [33] elaborated on this: What he called "e-mathematics" is linear, explicit, and easily capable of abstraction, whereas "π-mathematics" is nonlinear, mysterious, and generalized usually with difficulty. Cloitre [34], however, gave formulas suggesting a certain symmetry between e and π: If $u_1 = v_1 = 0$, $u_2 = v_2 = 1$ and

$$u_{n+2} = u_{n+1} + \frac{u_n}{n}, \quad v_{n+2} = \frac{v_{n+1}}{n} + v_n, \quad n \geq 0,$$

then $\lim_{n\to\infty} n/u_n = e$ whereas $\lim_{n\to\infty} 2n/v_n^2 = \pi$.

1.4.1 Infinite Series

Over five hundred years ago, the Indian mathematician Madhava discovered the formula [35–38]

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + - \cdots,$$

which was independently found by Gregory [39] and Leibniz [40]. This infinite series is conditionally convergent; hence its terms may be rearranged to produce a series that has any desired sum or even diverges to $+\infty$ or $-\infty$. The same is also true for the alternating harmonic series [1.3.3]. For example, we have

$$\frac{1}{4}\ln(2) + \frac{\pi}{4} = 1 + \frac{1}{5} - \frac{1}{3} + \frac{1}{9} + \frac{1}{13} - \frac{1}{7} + \frac{1}{17} + \frac{1}{21} - \frac{1}{11} + + - \cdots$$

(two positive terms for each negative term). Generalization is possible.

Changing the pattern of plus and minus signs in the Gregory–Leibniz series, for example, gives [41,42]

$$\frac{\pi}{4}\sqrt{2} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + + - - \cdots$$

or extracting a subseries gives [43]

$$\frac{\pi}{8}(1+\sqrt{2}) = 1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - \frac{1}{31} + - \cdots.$$

We defer discussion of Euler's famous series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}$$

until [1.6] and [1.7]. Among many other series of his, there is [1,44]

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)\binom{2n}{n}} = 1 + \frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \cdots.$$

We note that [2,45]

$$\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\binom{2n}{n}} = 2\ln(\varphi)^2$$

and wonder in what other ways π and the Golden mean φ [1.2] can be so intricately linked.

Ramanujan [23,24,46] and Chudnovsky & Chudnovsky [47–50] discovered series at the foundation of some of the fastest known algorithms for computing π.

1.4.2 Infinite Products

Viète [51] gave the first known analytical expression for π:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \cdots,$$

which he obtained by considering a limit of areas of Archimedean polygons, and Wallis [52] derived the formula

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots = \lim_{n\to\infty} \frac{2^{4n}}{(2n+1)\binom{2n}{n}^2}.$$

These products are, in fact, children of the same parent [53]. We might prove their truth in many different ways [54]. One line of reasoning involves what some regard as the definition of sine and cosine. The following infinite polynomial factorizations hold [55]:

$$\sin(x) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right), \qquad \cos(x) = \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right).$$

The sine and cosine functions form the basis for trigonometry and the study of periodic phenomena in mathematics. Applications include the undamped simple oscillations of a mechanical or electrical system, the orbital motion of planets around the sun, and much more [56]. It is well known that

$$\frac{d^2}{dx^2}(a \cdot \sin(x) + b \cdot \cos(x)) + (a \cdot \sin(x) + b \cdot \cos(x)) = 0$$

and, further, that any solution of the second-order differential equation

$$\frac{d^2y}{dx^2} + y(x) = 0$$

must be of this form. The constant π plays the same role in determining sine and cosine as the natural logarithmic base e plays in determining the exponential function. That these two processes are interrelated is captured by Euler's formula $e^{i\pi} + 1 = 0$, where i is the imaginary unit.

Famous products relating π and prime numbers appear in [1.6] and [1.7], as a consequence of the theory of the zeta function. One such product, due to Euler, is [57]

$$\frac{\pi}{2} = \prod_{p \text{ odd}} \frac{p}{p + (-1)^{(p-1)/2}} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdots,$$

where the numerators are the odd primes and the denominators are the closest integers of the form $4n+2$. See also [2.1]. A different appearance of π in number theory is the asymptotic expression

$$p(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

due to Hardy & Ramanujan [58], where $p(n)$ is the number of unrestricted partitions of the positive integer n (order being immaterial). Hardy & Ramanujan [58] and Rademacher [59] proved an *exact* analytical formula for $p(n)$ [60, 61], which is too far afield for us to discuss here.

1.4.3 Definite Integrals

The most famous integrals include [62, 63]

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \qquad \text{(Gaussian probability density integral)},$$

$$\int_0^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2} \qquad \text{(limiting value of arctangent)},$$

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\pi\sqrt{2}}{4} \qquad \text{(Fresnel integrals)},$$

$$\int_0^{\frac{\pi}{2}} \ln(\sin(x))dx = \int_0^{\frac{\pi}{2}} \ln(\cos(x))dx = -\frac{\pi}{2}\ln(2),$$

$$\int_0^1 \sqrt{\ln\left(\frac{1}{x}\right)}dx = \frac{\sqrt{\pi}}{2}.$$

It is curious that

$$\int_0^\infty \frac{\cos(x)}{1+x^2}dx = \frac{\pi}{2e}, \qquad \int_0^\infty \frac{x\sin(x)}{1+x^2}dx = \frac{\pi}{2e}$$

have simple expressions, but interchanging $\cos(x)$ and $\sin(x)$ give complicated results. See [6.2] for details.

Also, consider the following sequence:

$$s_n = \int_0^\infty \left(\frac{\sin(x)}{x}\right)^n dx, \qquad n = 1, 2, 3, \ldots.$$

The first several values are $s_1 = s_2 = \pi/2$, $s_3 = 3\pi/8$, $s_4 = \pi/3$, $s_5 = 115\pi/384$, and $s_6 = 11\pi/40$. An exact formula for s_n, valid for all n, is found in [64].

1.4.4 Continued Fractions

Starting with Wallis's formula, Brouncker [1, 2, 52] discovered the continued fraction

$$1+\frac{4}{\pi}=2+\frac{1^2|}{|2}+\frac{3^2|}{|2}+\frac{5^2|}{|2}+\frac{7^2|}{|2}+\frac{9^2|}{|2}+\cdots,$$

which was subsequently proved by Euler [41]. It is fascinating to compare this with other related expansions, for example [65–67],

$$\frac{4}{\pi}=1+\frac{1^2|}{|3}+\frac{2^2|}{|5}+\frac{3^2|}{|7}+\frac{4^2|}{|9}+\frac{5^2|}{|11}+\cdots,$$

$$\frac{6}{\pi^2-6}=1+\frac{1^2|}{|1}+\frac{1\cdot 2|}{|1}+\frac{2^2|}{|1}+\frac{2\cdot 3|}{|1}+\frac{3^2|}{|1}+\frac{3\cdot 4|}{|1}+\frac{4^2|}{|1}+\cdots,$$

$$\frac{2}{\pi-2}=1+\frac{1\cdot 2|}{|1}+\frac{2\cdot 3|}{|1}+\frac{3\cdot 4|}{|1}+\frac{4\cdot 5|}{|1}+\frac{5\cdot 6|}{|1}+\frac{6\cdot 7|}{|1}+\cdots,$$

$$\frac{12}{\pi^2}=1+\frac{1^4|}{|3}+\frac{2^4|}{|5}+\frac{3^4|}{|7}+\frac{4^4|}{|9}+\frac{5^4|}{|11}+\cdots,$$

$$\pi+3=6+\frac{1^2|}{|6}+\frac{3^2|}{|6}+\frac{5^2|}{|6}+\frac{7^2|}{|6}+\frac{9^2|}{|6}+\cdots.$$

1.4.5 Infinite Radical

Let S_n denote the length of a side of a regular polygon of 2^{n+1} sides inscribed in a unit circle. Clearly $S_1=\sqrt{2}$ and, more generally, $S_n=2\sin(\pi/2^{n+1})$. Hence, by the half-angle formula,

$$S_n=\sqrt{2-\sqrt{4-S_{n-1}^2}}.$$

(A purely geometric argument for this recursion is given in [68, 69].) The circumference of the 2^{n+1}-gon is $2^{n+1}S_n$ and tends to 2π as $n\to\infty$. Therefore

$$\pi=\lim_{n\to\infty}2^nS_n=\lim_{n\to\infty}2^n\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}}},$$

where the right-hand side has n square roots.

Although attractive, this radical expression for π is numerically sound only for a few iterations. It is a classic illustration of the loss of floating-point precision that occurs when subtracting two nearly equal quantities. There are many ways to approximate π: This is not one of them!

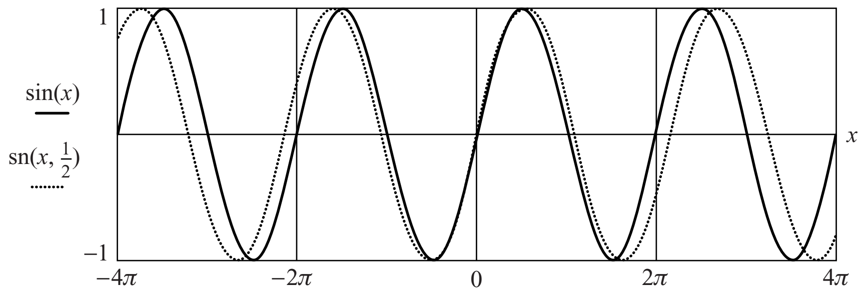

Figure 1.4. The circular function $\sin(x)$ has period $2\pi \approx 6.28$, while the elliptic function $\mathrm{sn}(x, 1/2)$ has (real) period $4K(1/2) \approx 6.74$.

1.4.6 Elliptic Functions

Consider an ellipse with semimajor axis length 1 and semiminor axis length $0 < r \le 1$. The area enclosed by the ellipse is πr while its circumference is $4E\left(\sqrt{1-r^2}\right)$, where

$$K(x) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-x^2\sin(\theta)^2}}\,d\theta = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-x^2t^2)}}\,dt,$$

$$E(x) = \int_0^{\frac{\pi}{2}} \sqrt{1-x^2\sin(\theta)^2}\,d\theta = \int_0^1 \sqrt{\frac{1-x^2t^2}{1-t^2}}\,dt$$

are **complete elliptic integrals** of the first and second kind. (One's first encounter with $K(x)$ is often with regard to computing the period of a physical pendulum [56].) The analog of the sine function is the **Jacobi elliptic function** $\mathrm{sn}(x, y)$, defined by

$$x = \int_0^{\mathrm{sn}(x,y)} \frac{1}{\sqrt{(1-t^2)(1-y^2t^2)}}\,dt \quad \text{for } 0 \le y \le 1.$$

See Figure 1.4. Clearly we have $\mathrm{sn}(x, 0) = \sin(x)$ for $-\pi/2 \le x \le \pi/2$ and $\mathrm{sn}(x, 1) = \tanh(x)$. An assortment of extended trigonometric identities involving sn and its counterparts cn and dn can be proved. For fixed $0 < y < 1$, the function $\mathrm{sn}(x, y)$ can be analytically continued over the whole complex plane to a doubly periodic meromorphic function. Just as $\sin(z) = \sin(z + 2\pi)$ for all complex z, we have $\mathrm{sn}(z) = \mathrm{sn}\left(z + 4K(y) + 2iK\left(\sqrt{1-y^2}\right)\right)$. Hence the constants $K(y)$ and $K\left(\sqrt{1-y^2}\right)$ play roles for elliptic functions analogous to the role π plays for circular functions [2, 70].

1.4.7 Unexpected Appearances

A fascinating number-theoretic function $f(n)$ is described in [71–77]. Take any positive integer n, round it up to the nearest multiple of $n-1$, then round this result up to the nearest multiple of $n-2$, and then (more generally) round the k^{th} result up to the nearest multiple of $n-k-1$. Stop when $k=n-1$ and let $f(n)$ be the final value. For example, $f(10)=34$ since

$$10 \to 18 \to 24 \to 28 \to 30 \to 30 \to 32 \to 33 \to 34 \to 34.$$

The ratio $n^2/f(n)$ approaches π as n increases without bound. In the same spirit, Matiyasevich & Guy [78] obtained

$$\pi = \lim_{m\to\infty} \sqrt{\frac{6 \cdot \ln(f_1 \cdot f_2 \cdot f_3 \cdots f_m)}{\ln(\operatorname{lcm}(f_1, f_2, f_3, \ldots, f_m))}},$$

where $f_1, f_2, f_3, \ldots$ is Fibonacci's sequence [1.2] and lcm denotes least common multiple. It turns out that Fibonacci's sequence may be replaced by many other second-order, linear recurring sequences without changing the limiting value π.

In [1.4.1] and [1.4.2], we saw expressions resembling $\binom{2n}{n}/(n+1)$. These are known as **Catalan numbers** and are important in combinatorics, for example, when enumerating strictly binary trees with $2n+1$ vertices. The average height h_n of such trees satisfies

$$\lim_{n\to\infty} \frac{h_n}{\sqrt{n}} = 2\sqrt{\pi}$$

by a theorem of Flajolet & Odlyzko [79, 80] (we introduce the language of trees in [5.6]). This is yet another unexpected appearance of the constant π.

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1.5 Euler–Mascheroni Constant, γ

The Euler–Mascheroni constant, γ, is defined by the limit [1–8]

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right) = 0.5772156649\ldots.$$

In words, γ measures the amount by which the partial sums of the harmonic series (the simplest divergent series) differ from the logarithmic function (its approximating integral). It is an important constant, shadowed only by π and e in significance. It appears naturally whenever estimates of $\sum_{k=1}^{n} 1/k$ are required. For example, let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with continuous distribution function. Define R_n to be the number of **upper records** in the sequence [9–12], that is, the count of times that $X_k > \max\{X_1, X_2, \ldots, X_{k-1}\}$. By convention, X_1 is included. The random variable R_n has expectation $\mathrm{E}(R_n)$ satisfying $\lim_{n\to\infty}(\mathrm{E}(R_n) - \ln(n)) = \gamma$. As another example, let the set $C = \{1, 2, \ldots, n\}$ of **coupons** be sampled repeatedly with replacement [13–15], and let S_n denote the number of trials needed to collect all of C. Then $\lim_{n\to\infty}((\mathrm{E}(S_n) - n\ln(n))/n) = \gamma$.

There are certain applications, however, where γ appears quite mysteriously. Suppose we wish to factor a random permutation π on n symbols into disjoint cycles. For example, the permutation π on $\{0, 1, 2, \ldots, 8\}$ defined by $\pi(x) = 2x \bmod 9$ has cycle structure $\pi = (0)(124875)(36)$. What is the probability that no two cycles of π possess the same length, as $n \to \infty$? The answer to the question is $e^{-\gamma} = 0.5614594835\ldots$. More about random permutations is found in [5.4]. Suppose instead that we wish to factor a random integer polynomial $F(x)$ of degree n, modulo a prime p. What is the probability that no two irreducible factors of $F(x)$ possess the same degree, as $p \to \infty$ and $n \to \infty$? The same answer $e^{-\gamma}$ applies [16–21], but proving this is complicated by the double limit.

Euler's constant appears frequently in number theory, for example, in connection with the Euler totient function [2.7]. Here are more applications. If $d(n)$ denotes the number of distinct divisors of n, then the average value of the divisor function satisfies [22–24]

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(k) - \ln(n)\right) = 2\gamma - 1 = 0.1544313298\ldots.$$

We discuss this again in [2.10]. A surprising result, due to de la Vallée Poussin [25–28], is

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left\{\frac{n}{k}\right\} = 1 - \gamma = 0.4227843351\ldots,$$

where $\{x\}$ denotes the fractional part of x. In words, if a large integer n is divided by each integer $1 \le k \le n$, then the average fraction by which the quotient n/k falls short of the next integer is not $1/2$, but γ! One can also restrict n to being all terms of an arithmetic sequence, or even to being all terms of the sequence of primes, and obtain the same mean value. Also, let $M(n)$ denote the number of primes p, not exceeding n, for which $2^p - 1$ is prime. It has been suggested [29–32] that $M(n) \to \infty$ at approximately the same rate as $\ln(n)$ and, moreover, $\lim_{n\to\infty} M(n)/\ln(n) = e^{\gamma}/\ln(2) = 2.5695443449\ldots$. The empirical data supporting this claim is quite thin: There are only 39 known Mersenne primes [33]. Other number-theoretic applications include [34–37].

Calculating Euler's constant has not attracted the same public intrigue as calculating π, but it has still inspired the dedication of a few. The evaluation of γ is difficult and only several hundred million digits are known. For π, we have the Borweins' *quartically convergent* algorithm: Each successive iteration approximately quadruples the number of correct digits. By contrast, for γ, not even a quadratically convergent algorithm is known [38–40].

The definition of γ converges too slowly to be numerically useful. This fact is illustrated by the following inequality [41,42]:

$$\frac{1}{2(n+1)} < \sum_{k=1}^{n}\frac{1}{k} - \ln(n) - \gamma < \frac{1}{2n},$$

which serves as a double-edged sword. On the one hand, if we wish K digits of accuracy (after truncation), then $n \ge 10^{K+1}$ suffices in the summation. On the other hand, $n < 10^K$ will not be large enough. Some alternative estimates and inequalities were reported in [43,44]. The best-known technique, called Euler–Maclaurin summation, gives an improved family of estimates, including

$$\gamma = \sum_{k=1}^{n} \frac{1}{k} - \ln(n) - \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{1}{240n^8} + \frac{1}{132n^{10}} - \frac{691}{32760n^{12}} + O\left(\frac{1}{n^{14}}\right).$$

Euler correctly obtained γ to 15 digits using $n = 10$ in this formula [45–48]. Fast algorithms like Karatsuba's FEE method [49, 50] and Brent's binary splitting method [51] were essential in the latest computations [52–55]. Papanikolaou calculated the first 475006 partial quotients in the regular continued fraction expansion for γ (using results in [56]) and deduced that if γ is a rational number, then its denominator must exceed 10^{244663}. This is compelling evidence that Euler's constant is not rational. A *proof* of irrationality (let alone transcendence) is still beyond our reach [57]. See two invalid attempts in [58, 59].

Here are two other unanswered questions, the first related to the harmonic series and the second similar to the coupon collector problem. Given a positive integer k, let n_k be the unique integer n satisfying $\sum_{j=1}^{n-1} 1/j < k < \sum_{j=1}^{n} 1/j$. Is n_k equal to the integer nearest $e^{k-\gamma}$ always [60–65]? Suppose instead we are given a binary sequence B, generated by independent fair coin tosses, and a positive integer n. What is the waiting time T_n for all 2^n possible different patterns of length n to occur (as subwords of B)? It might be conjectured (on the basis of [66, 67]) that the mean waiting time satisfies $\lim_{n\to\infty}((\mathrm{E}(T_n) - 2^n n \ln(2))/2^n) = \gamma$, but this remains open. However, the minimum possible waiting time is only $2^n + n - 1$, as a consequence of known results concerning what are called de Bruijn sequences [68].

1.5.1 Series and Products

The following series is a trivial restatement of the definition of γ:

$$\gamma = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right)\right).$$

Other formulas involving γ include two more due to Euler [1],

$$\gamma = \frac{1}{2} \cdot \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) - \frac{1}{3} \cdot \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\right) + \frac{1}{4} \cdot \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right) - + \cdots,$$

$$\gamma = \frac{1}{2} \cdot \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) + \frac{2}{3} \cdot \left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots\right) + \frac{3}{4} \cdot \left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right) + \cdots,$$

one due to Vacca [69–75],

$$\gamma = \frac{1}{2} - \frac{1}{3} + 2 \cdot \left(\frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7}\right) + 3 \cdot \left(\frac{1}{8} - \frac{1}{9} + \cdots - \frac{1}{15}\right) + 4 \cdot \left(\frac{1}{16} - \frac{1}{17} + \cdots - \frac{1}{31}\right) + \cdots,$$

one due to Pólya [26, 76],

$$\gamma = 1 - \left(\frac{1}{2} + \frac{1}{3}\right) + \frac{3}{4} - \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \frac{5}{9} - \left(\frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{15}\right) + \frac{7}{16} - + \cdots,$$

and two due to Mertens [22, 77],

$$e^{\gamma} = \lim_{n\to\infty} \frac{1}{\ln(n)} \cdot \prod_{p\le n} \frac{p}{p-1}, \qquad \frac{6e^{\gamma}}{\pi^2} = \lim_{n\to\infty} \frac{1}{\ln(n)} \cdot \prod_{p\le n} \frac{p+1}{p},$$

where both products are taken over all primes p not exceeding n. Mertens' first formula may be rewritten as [55]

$$\gamma = \lim_{n\to\infty} \left(\sum_{p\le n} \ln\left(\frac{p}{p-1}\right) - \ln(\ln(n)) \right).$$

If, in this series, the expression $\ln(p/(p-1))$ is replaced by its asymptotic equivalent $1/p$, then a different constant arises [2.2]. Other series and products appear in [78–95].

1.5.2 Integrals

There are many integrals that involve Euler's constant, including

$$\int_0^\infty e^{-x}\ln(x)dx = -\gamma, \qquad \int_0^\infty e^{-x^2}\ln(x)dx = -\frac{\sqrt{\pi}}{4}(\gamma + 2\ln(2)),$$

$$\int_0^\infty e^{-x}\ln(x)^2 dx = \frac{\pi^2}{6} + \gamma^2, \qquad \int_0^1 \ln\left(\ln\left(\frac{1}{x}\right)\right) dx = -\gamma,$$

$$\int_0^\infty \frac{e^{-x^a} - e^{-x^b}}{x} dx = \frac{a-b}{ab}\gamma, \qquad \int_0^\infty \frac{x}{1+x^2} \cdot \frac{1}{e^{2\pi x} - 1} dx = \frac{1}{4}(2\gamma - 1),$$

$$\int_0^1 \left(\frac{1}{\ln(x)} + \frac{1}{1-x}\right) dx = \gamma, \qquad \int_0^1 \frac{1}{1+x}\left(\sum_{k=1}^\infty x^{2^k}\right) dx = 1 - \gamma,$$

to mention a few [55, 75, 96, 97]. It is assumed here that the two parameters a and b satisfy $a > 0$ and $b > 0$. If $\{x\}$ denotes the fractional part of x, then [22, 24]

$$\int_1^\infty \frac{\{x\}}{x^2} dx = \int_0^1 \left\{\frac{1}{y}\right\} dy = 1 - \gamma$$

and similar integrals appear in [1.6.5], [1.8], and [2.21]. See also [98–101].

1.5.3 Generalized Euler Constants

Boas [102–104] wondered why the original Euler constant has attracted attention but other types of constants of the form

$$\gamma(m, f) = \lim_{n\to\infty}\left(\sum_{k=m}^{n} f(k) - \int_m^n f(x)dx\right)$$

have been comparatively neglected. The case $f(x) = x^{-q}$, where $0 < q < 1$, gives the constant $\zeta(q) + 1/(1-q)$ involving a zeta function value [1.6] and the case $f(x) = \ln(x)^r/x$, where $r \geq 0$, gives the Stieltjes constant γ_r [2.21]. We give some sample numerical results in Table 1.1. Briggs [105] and Lehmer [106] studied the analog of γ corresponding to the arithmetic progression a, $a + b$, $a + 2b$, $a + 3b$, ...:

$$\gamma_{a,b} = \lim_{n\to\infty}\left(\sum_{\substack{0<k\leq n\\ k\equiv a \bmod b}} \frac{1}{k} - \frac{1}{b}\ln(n)\right).$$

For example, $\gamma_{0,b} = (\gamma - \ln(b))/b$, $\sum_{a=0}^{b-1}\gamma_{a,b} = \gamma$, and

$$\gamma_{1,3} = \frac{1}{3}\gamma + \frac{\sqrt{3}}{18}\pi + \frac{1}{6}\ln(3), \quad \gamma_{1,4} = \frac{1}{4}\gamma + \frac{1}{8}\pi + \frac{1}{4}\ln(2).$$

See also [107, 108]. A two-dimensional version of Euler's constant appears in [7.2] and a (different) n-dimensional lattice sum version is discussed in [1.10].

Table 1.1. *Generalized Euler Constants*

m	$f(x)$	$\gamma(m, f)$
1	$1/x$	$0.5772156649\ldots = \gamma_0$
2	$1/\ln(x)$	$0.8019254372\ldots$
2	$1/(x \cdot \ln(x))$	$0.4281657248\ldots$
1	$1/\sqrt{x}$	$0.5396454911\ldots = \zeta(1/2) + 2$
1	$\ln(x)/x$	$-0.0728158454\ldots = \gamma_1$

1.5.4 Gamma Function

For complex z, the Euler gamma function $\Gamma(z)$ is often defined by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! \cdot n^z}{\prod_{k=0}^{n}(z+k)}$$

and is analytic over the whole complex plane except for simple poles at the nonpositive integers. For real $x > 0$, this simplifies to the integral formula

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds = \int_0^1 \left(\ln\left(\frac{1}{t}\right)\right)^{x-1} dt$$

and, if n is a positive integer, $\Gamma(n) = (n-1)!$ This is the reason we sometimes see the expression

$$\left(-\frac{1}{2}\right)! = \sqrt{\pi} = 1.7724538509\ldots$$

since $\Gamma(1/2)$ transforms, by change of variable, to the well-known Gaussian probability density integral.

The Bohr–Mollerup theorem [109, 110] maintains that $\Gamma(z)$ is the most natural possible extension of the factorial function (among infinitely many possible extensions) to the complex plane.

For what argument values is the gamma function known to be transcendental? Chudnovsky [111–114] showed in 1975 that $\Gamma(1/6)$, $\Gamma(1/4)$, $\Gamma(1/3)$, $\Gamma(2/3)$, $\Gamma(3/4)$, and $\Gamma(5/6)$ are each transcendental and that each is algebraically independent from π. (It is curious [115, 116] that we have known $\Gamma(1/4)^4/\pi$ and $\Gamma(1/3)^2/\pi$ to be transcendental for many more years.) Nesterenko [117–121] proved in 1996 that π, e^π, and $\Gamma(1/4) = 3.6256099082\ldots$ are algebraically independent. The constant $\Gamma(1/4)$ appears in [3.2], [6.1], and [7.2]. Nesterenko also proved that π, $e^{\pi\sqrt{3}}$, and $\Gamma(1/3) = 2.6789385347\ldots$ are algebraically independent. A similarly strong result has not yet been proved for $\Gamma(1/6) = 5.5663160017\ldots$, nor has $\Gamma(1/5) = 4.5908437119\ldots$ even been demonstrated to be irrational. The reflection formula provides that

$$\Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right) = \pi\sqrt{2}, \qquad \Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right) = \tfrac{2}{3}\pi\sqrt{3},$$
$$\Gamma\left(\tfrac{1}{6}\right)\Gamma\left(\tfrac{5}{6}\right) = 2\pi, \quad \Gamma\left(\tfrac{1}{5}\right)\Gamma\left(\tfrac{4}{5}\right) = \tfrac{2}{5}\pi\sqrt{5}\sqrt{2+\varphi},$$

where φ is the Golden mean [1.2]. Furthermore [122, 123],

$$\Gamma\left(\tfrac{1}{4}\right) = 2^{\frac{1}{2}}\pi^{\frac{3}{4}}h_1^{\frac{1}{2}}, \ \ \Gamma\left(\tfrac{1}{3}\right) = 2^{\frac{4}{9}}3^{-\frac{1}{12}}\pi^{\frac{2}{3}}h_3^{\frac{1}{3}}, \ \ \Gamma\left(\tfrac{1}{6}\right) = 2^{\frac{5}{9}}3^{\frac{1}{3}}\pi^{\frac{5}{6}}h_3^{\frac{2}{3}},$$

where

$$h_1 = \tfrac{2}{\pi} K\left(\tfrac{\sqrt{2}}{2}\right) = \left(\sum_{n=-\infty}^{\infty} e^{-n^2\pi}\right)^2 = 1.1803405990\ldots,$$

$$h_3 = \tfrac{2}{\pi} K\left(\tfrac{\sqrt{2}}{4}(\sqrt{3}-1)\right) = \left(\sum_{n=-\infty}^{\infty} e^{-n^2\sqrt{3}\pi}\right)^2 = 1.0174087975\ldots,$$

and $K(x)$ is the complete elliptic integral of the first kind [1.4.6].

When plotting the gamma function $y = \Gamma(x)$, the minimum point in the upper right quadrant has xy-coordinates $(x_{\min}, \Gamma(x_{\min})) = (1.4616321449\ldots, 0.8856031944\ldots)$. If θ is the unique positive root of the equation

$$\frac{d}{dx} \ln(\Gamma(x))\bigg|_{x=\theta} = \ln(\pi)$$

then $d_s = 2\theta = 7.2569464048\ldots$ and $d_s = 2(\theta - 1) = 5.2569464048\ldots$ are the fractional dimensions at which d-dimensional spherical surface area and volume, respectively, are maximized [124].

Several relevant series appear in [125–129]. Two series due to Ramanujan, for example [130–132], are

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n}} \binom{2n}{n}^2 = (2\pi)^{-\frac{3}{2}} \Gamma\left(\tfrac{1}{4}\right)^2, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{6n}} \binom{2n}{n}^3 = \left(\frac{\Gamma\left(\frac{9}{8}\right)}{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{7}{8}\right)}\right)^2,$$

which extend a series mentioned in [1.1]. Two products [96, 133] are

$$\prod_{n=1}^{\infty}\left(1 - \frac{1}{(4n+1)^2}\right) = \frac{4\cdot 6}{5\cdot 5}\cdot\frac{8\cdot 10}{9\cdot 9}\cdot\frac{12\cdot 14}{13\cdot 13}\cdot\frac{16\cdot 18}{17\cdot 17}\cdots = \frac{1}{8\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)^2,$$

$$\prod_{n=1}^{\infty}\left(1 - \frac{1}{(2n+1)^2}\right)^{(-1)^n} = \frac{3^2}{3^2-1}\cdot\frac{5^2-1}{5^2}\cdot\frac{7^2}{7^2-1}\cdot\frac{9^2-1}{9^2}\cdots = \frac{1}{16\pi^2}\Gamma\left(\frac{1}{4}\right)^4.$$

A sample integral, with real parameters $u > 0$ and $v > 0$, is [96, 134, 135]

$$\int_0^{\frac{\pi}{2}} \sin(x)^{u-1}\cos(x)^{v-1}dx = \int_0^1 y^{u-1}(1-y^2)^{\frac{v}{2}-1}dy = \frac{1}{2}\frac{\Gamma(\frac{u}{2})\Gamma(\frac{v}{2})}{\Gamma(\frac{u+v}{2})}.$$

The significance of Euler's constant to Euler's gamma function is best summarized by the formula $\psi(1) = -\gamma$, where [90]

$$\psi(x) = \frac{d}{dx}\ln(\Gamma(x)) = -\gamma - \sum_{n=0}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n+1}\right)$$

is the digamma function. Higher-order derivatives at $x = 1$ involve zeta function values [1.6]. Information on such derivatives (polygamma functions) is found in [134, 136].

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1.6 Apéry's Constant, $\zeta(3)$

Apéry's constant, $\zeta(3)$, is defined to be the value of **Riemann's zeta function**

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad x > 1,$$

when $x = 3$. This designation of $\zeta(3)$ as Apéry's constant is new but well deserved. In 1979, Apéry stunned the mathematical world with a miraculous proof that $\zeta(3) = 1.2020569031\ldots$ is irrational [1–10]. We will return to this after a brief discussion of Riemann's function.

The zeta function can be evaluated exactly [11–14] at positive even integer values of x,

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k} B_{2k}}{2(2k)!},$$

where $\{B_n\}$ denotes the **Bernoulli numbers** [1.6.1]. For example,

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

Clearly $\zeta(1)$ cannot be defined, at least by means of our definition of $\zeta(x)$, since the harmonic series diverges. The zeta function can be analytically continued over the whole complex plane via the functional equation [15–19]:

$$\zeta(1-z) = \frac{2}{(2\pi)^z} \cos\left(\frac{\pi z}{2}\right) \Gamma(z)\zeta(z)$$

with just one singularity, a simple pole, at $z = 1$. Here $\Gamma(z) = (z-1)!$ is the gamma function [1.5.4]. The connection between $\zeta(x)$ and prime number theory is best summarized by the two formulas

$$\zeta(x) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^x}\right)^{-1}, \quad \frac{\zeta(2x)}{\zeta(x)} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^x}\right)^{-1}.$$

If the famous Riemann hypothesis [1.6.2] can someday be proved, more information about the distribution of prime numbers will become available.

A closely associated function is [20–22]

$$\eta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^x}, \quad x > 0,$$

which equals $(1 - 2^{1-x})\zeta(x)$ for $x \neq 1$. For example,

$$\eta(1) = \ln(2), \quad \eta(2) = \frac{\pi^2}{12}, \quad \eta(4) = \frac{7\pi^4}{720}.$$

The constant $\zeta(3)$ has a probabilistic interpretation [23, 24]: Given three random integers, the probability that no factor exceeding 1 divides them all is $1/\zeta(3) = 0.8319073725\ldots$ (in the limit over large intervals). By way of contrast, the probability that the three integers are pairwise coprime is only $0.2867474284\ldots$; see the formulation in [2.5]. If n is a power of 2, define $c(n)$ to be the number of positive integer solutions (i, j, p) with p prime of the equation $n = p + ij$ [25, 26]. Then $\lim_{n\to\infty} c(n)/n = 105\zeta(3)/(2\pi^4)$. Other occurrences of $\zeta(3)$ in number theory are discussed in [2.7] and [27–30]. It also appears in random graph theory with regard to minimum spanning tree lengths [8.5].

A generalization of Apéry's work to $\zeta(2k+1)$ for any $k > 1$ remains, as van der Poorten wrote, "a mystery wrapped in an enigma" [2]. It remains open whether $\zeta(3)$ is transcendental, or even whether $\zeta(3)/\pi^3$ is irrational. Rivoal [31, 32] recently proved that there are infinitely many integers k such that $\zeta(2k+1)$ is irrational, and Zudilin [33, 34] further showed that at least one of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational. This is the most dramatic piece of relevant news since Apéry's irrationality proof of $\zeta(3)$.

1.6.1 Bernoulli Numbers

Define $\{B_n\}$, the Bernoulli numbers, by the generating function [7, 19–22]

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

From this, it follows that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, and $B_{2n+1} = 0$ for $n > 0$.

(There is, unfortunately, an alternative definition of the Bernoulli numbers to confuse matters. Under this alternative definition, the subscripting is somewhat different and all the numbers are positive. One must be careful when reading any paper to establish which definition has been used.)

The Bernoulli numbers also arise in certain other series expansions, such as

$$\tan(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2^{2k}(2^{2k} - 1)B_{2k}}{(2k)!} x^{2k-1}.$$

1.6.2 The Riemann Hypothesis

With Wiles' recent proof of Fermat's Last Theorem now confirmed, the most notorious unsolved problem in mathematics becomes the Riemann hypothesis. This conjecture states that all the zeros of $\zeta(z)$ in the strip $0 \leq \mathrm{Re}(z) \leq 1$ lie on the central line $\mathrm{Re}(z) = 1/2$.

Here is a completely elementary restatement of the Riemann hypothesis [35]. Define a positive square-free integer to be **red** if it is the product of an even number of distinct primes, and **blue** if it is the product of an odd number of distinct primes. Let $R(n)$ be the number of red integers not exceeding n, and let $B(n)$ be the number of blue integers not exceeding n. The Riemann hypothesis is equivalent to the following statement: For any $\varepsilon > 0$, there exists an integer N such that for all $n > N$,

$$|R(n) - B(n)| < n^{\frac{1}{2}+\varepsilon}.$$

This is usually stated in terms of the Möbius mu function [2.2]. It turns out that setting $\varepsilon = 0$ is impossible; what is known as the Mertens hypothesis is false!

Another restatement (among several [36, 37]) is as follows. The Riemann hypothesis is true if and only if [38]

$$\int_0^\infty \int_{\frac{1}{2}}^\infty \frac{1-12y^2}{(1+4y^2)^3} \ln|\zeta(x+iy)|\, dx\, dy = \frac{3-\gamma}{32}\pi,$$

where γ is the Euler–Mascheroni constant [1.5]. It is interesting to compare this conditional equality with formulas we know to be unconditionally true. For example, if Z denotes the set of all zeros ρ in the critical strip, then [39–41]

$$\sum_{\rho \varepsilon Z} \frac{1}{\rho} = \frac{1}{2}\gamma + 1 - \ln(2) - \frac{1}{2}\ln(\pi) = 0.0230957089\ldots.$$

That is, although the zero locations remain a mystery, we know enough about them to exactly compute their reciprocal sum. Care is needed: $\sum_\rho |\rho|^{-1}$ diverges, but $\sum_\rho \rho^{-1}$converges provided that we group together conjugate terms.

One consequence of Riemann's hypothesis (among many [17]) is mentioned in [2.13]. Our knowledge of the distribution of prime numbers will be much deeper if a successful proof is someday found. The essay on the de Bruijn–Newman constant [2.32] has details of a computational approach. A deeper hypothesis, called the Gaussian unitary ensemble hypothesis [2.15.3], governs the vertical spacing distribution between the zeros.

1.6.3 Series

Summing over certain arithmetic progressions gives slight variations [42, 43]:

$$\lambda(3) = \sum_{k=0}^\infty \frac{1}{(2k+1)^3} = \frac{7}{8}\zeta(3), \quad \sum_{k=0}^\infty \frac{1}{(3k+1)^3} = \frac{2\pi^3}{81\sqrt{3}} + \frac{13}{27}\zeta(3),$$

$$\sum_{k=0}^\infty \frac{1}{(4k+1)^3} = \frac{\pi^3}{64} + \frac{7}{16}\zeta(3), \quad \sum_{k=0}^\infty \frac{1}{(6k+1)^3} = \frac{\pi^3}{36\sqrt{3}} + \frac{91}{216}\zeta(3).$$

We will discuss $\lambda(x)$ later in [1.7]. Two formulas involving central binomial sums are [42, 44–47]

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}} = \frac{2}{5}\zeta(3), \quad \sum_{k=1}^{\infty} \frac{30k-11}{(2k-1)k^3\binom{2k}{k}^2} = 4\zeta(3),$$

the former of which has become famous because of Apéry's work.

What is the analog for $\zeta(2n+1)$ of the exact formula for $\zeta(2n)$? No one knows, but series obtained by Grosswald [48–51],

$$\zeta(3) = \frac{7}{180}\pi^3 - 2\sum_{k=1}^{\infty} \frac{1}{k^3\left(e^{2\pi k}-1\right)}, \quad \zeta(7) = \frac{19}{56700}\pi^7 - 2\sum_{k=1}^{\infty} \frac{1}{k^7\left(e^{2\pi k}-1\right)},$$

and by Plouffe [52] and Borwein [26, 53],

$$\zeta(5) = \frac{1}{294}\pi^5 - \frac{72}{35}\sum_{k=1}^{\infty} \frac{1}{k^5\left(e^{2\pi k}-1\right)} - \frac{2}{35}\sum_{k=1}^{\infty} \frac{1}{k^5\left(e^{2\pi k}+1\right)},$$

might be regarded as leading candidates. The formulas were inspired by certain entries in Ramanujan's notebooks [54].

Some multiple series appearing in [55–62] include

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{ij(i+j)} = 2\zeta(3), \quad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^{i-1}}{ij(i+j)} = \frac{5}{8}\zeta(3),$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^{i+j}}{ij(i+j)} = \frac{1}{4}\zeta(3), \quad \sum_{i=2}^{\infty}\sum_{j=1}^{i-1} \frac{1}{i^2 j} = \zeta(3),$$

$$\sum_{i=3}^{\infty}\sum_{j=2}^{i-1}\sum_{k=1}^{j-1} \frac{1}{i^3 j^2 k} = -\frac{29}{6480}\pi^6 + 3\zeta(3)^2,$$

and many more such evaluations (of arbitrary depth) are known [63–75].

If $0 < x < 1$, then the following is true [19]:

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k^x} - \frac{n^{1-x}}{1-x}\right) = \zeta(x) = \left(1-2^{1-x}\right)^{-1}\eta(x) = \frac{-1}{2^{1-x}-1}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^x}.$$

For example, when $x = 1/2$, the limiting value is [76]

$$\begin{aligned}\lim_{n\to\infty}\left(1+\tfrac{1}{\sqrt{2}}+\cdots+\tfrac{1}{\sqrt{n}}-2\sqrt{n}\right) &= -\left(\sqrt{2}+1\right)\left(1-\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{3}}-+\cdots\right)\\ &= -1.4603545088\ldots\end{aligned}$$

as mentioned with regard to Euler's constant [1.5.3]. Recall too from [1.5.1] that

$$\gamma = \sum_{k=2}^{\infty}(-1)^k \frac{\zeta(k)}{k}, \quad 1-\gamma = \sum_{k=2}^{\infty} \frac{\zeta(k)-1}{k}.$$

A notable family of series involving zeta function values is [77, 78]

$$S(n) = \sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+n)2^{2k-1}}, \quad n = 0, 1, 2, \ldots.$$

For example [79–83],

$$S(0) = \ln(\pi) - \ln(2), \quad S(1) = -\ln(2) + 1, \quad S(2) = \frac{7}{2\pi^2}\zeta(3) - \ln(2) + \frac{1}{2},$$

$$S(3) = \frac{9}{2\pi^2}\zeta(3) - \ln(2) + \frac{1}{3}, \quad S(4) = -\frac{93}{2\pi^4}\zeta(5) + \frac{9}{\pi^2}\zeta(3) - \ln(2) + \frac{1}{4}.$$

These can be combined in various ways (via partial fractions) to obtain more rapidly convergent series, for example,

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}} = -\frac{7}{4\pi^2}\zeta(3) + \frac{1}{4}$$

due to Euler [84–89] and

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(k+1)(2k+1)(2k+3)2^{2k}} = \frac{2}{\pi^2}\zeta(3) - \frac{11}{18} + \frac{1}{3}\ln(\pi)$$

due to Wilton [90–92]. Many more series exist [93–102].

Broadhurst [103] determined digit-extraction algorithms for $\zeta(3)$ and $\zeta(5)$ similar to the Bailey–Borwein–Plouffe algorithm for π. The corresponding series for $\zeta(3)$ is

$$\begin{aligned}\zeta(3) = \tfrac{48}{7}\sum_{k=0}^{\infty} \tfrac{1}{2\cdot 16^k}\Big(&\tfrac{1}{(8k+1)^3} - \tfrac{7}{(8k+2)^3} - \tfrac{1}{2(8k+3)^3} + \tfrac{10}{2(8k+4)^3} - \tfrac{1}{2^2(8k+5)^3} - \tfrac{7}{2^2(8k+6)^3} \\ &+ \tfrac{1}{2^3(8k+7)^3}\Big) + \tfrac{32}{7}\sum_{k=0}^{\infty} \tfrac{1}{8\cdot 16^{3k}}\Big(\tfrac{1}{(8k+1)^3} + \tfrac{1}{2(8k+2)^3} - \tfrac{1}{2^3(8k+3)^3} - \tfrac{2}{2^4(8k+4)^3} \\ &- \tfrac{1}{2^6(8k+5)^3} + \tfrac{1}{2^7(8k+6)^3} + \tfrac{1}{2^9(8k+7)^3}\Big).\end{aligned}$$

Amdeberhan, Zeilberger and Wilf [104–106] discovered extremely fast series for computing $\zeta(3)$, which presently is known to several hundred million decimal digits. See also [107–110]. We mention [111–114]

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3(k+1)^3} = 10 - \frac{3}{2}\zeta(3) - 12\ln(2),$$

$$\mathrm{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) + \frac{\pi^2}{12}\ln\left(\frac{1}{2}\right) - \frac{1}{6}\ln\left(\frac{1}{2}\right)^3,$$

$$\mathrm{Li}_3(2-\varphi) = \frac{4}{5}\zeta(3) + \frac{\pi^2}{15}\ln(2-\varphi) - \frac{1}{12}\ln(2-\varphi)^3,$$

where Li_3 denotes the trilogarithm function [1.6.8] and φ denotes the Golden mean [1.2].

Finally, the generating function for $\zeta(4n+3)$ [115, 116]

$$\sum_{n=0}^{\infty} \zeta(4n+3)x^n = \frac{5}{2}\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i^3\binom{2i}{i}} \frac{1}{1-\frac{x}{i^4}} \prod_{j=1}^{i-1} \frac{j^4+4x}{j^4-x}, \quad |x| < 1,$$

includes the Apéry series in the special case $x = 0$. If we differentiate both sides with respect to x and then set $x = 0$, a fast series for $\zeta(7)$ emerges:

$$\zeta(7) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}} + \frac{25}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{m=1}^{k-1}\frac{1}{m^4}$$

and likewise for larger n. No analogous generating function is known for $\zeta(4n+1)$. How can the series [117]

$$\zeta(5) = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}} - \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{m=1}^{k-1}\frac{1}{m^2}$$

be correspondingly extended?

1.6.4 Products

There is a striking family of matrix products due to Gosper [118]. The simplest case is

$$\prod_{k=1}^{\infty}\begin{pmatrix} -\frac{k}{2(2k+1)} & \frac{5}{4k^2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \zeta(3) \\ 0 & 1 \end{pmatrix},$$

which is equivalent to a central binomial sum given earlier. The general case involves $(n+1)\times(n+1)$ upper-triangular matrices, where $n \geq 2$:

$$\prod_{k=1}^{\infty}\begin{pmatrix} -\frac{k}{2(2k+1)} & \frac{1}{2k(2k+1)} & 0 & \cdots & 0 & \frac{1}{k^{2n}} \\ 0 & -\frac{k}{2(2k+1)} & \frac{1}{2k(2k+1)} & \cdots & 0 & \frac{1}{k^{2n-2}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2k(2k+1)} & \frac{1}{k^4} \\ 0 & 0 & 0 & \cdots & -\frac{k}{2(2k+1)} & \frac{5}{4k^2} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \cdots 0 & \zeta(2n+1) \\ 0 \cdots 0 & \zeta(2n-1) \\ \vdots \quad \vdots & \vdots \\ 0 \cdots 0 & \zeta(5) \\ 0 \cdots 0 & \zeta(3) \\ 0 \cdots 0 & 1 \end{pmatrix},$$

where the diagonal and superdiagonal are extended (by repetition) as indicated, the rightmost column contains reciprocals of k^{2m}, and all remaining entries are zero.

1.6.5 Integrals

Riemann's zeta function has an alternative expression [17] for $x > 1$:

$$\zeta(x) = \frac{1}{\Gamma(x)}\int_0^{\infty}\frac{t^{x-1}}{e^t - 1}dt.$$

If $\{t\}$ denotes the fractional part of t, then [18, 19]

$$\int_1^{\infty}\frac{\{t\}}{t^{x+1}}dt = \begin{cases} \dfrac{1}{x-1} - \dfrac{\zeta(x)}{x} & \text{if } 0 < x < 1 \text{ or } x > 1, \\ 1 - \gamma & \text{if } x = 1. \end{cases}$$

For all remaining x the integral is divergent. A quick adjustment is, however, possible over a subinterval:

$$\int_1^{\infty} \frac{\{t\} - \frac{1}{2}}{t^{x+1}} dt = \begin{cases} \dfrac{1}{x-1} - \dfrac{1}{2x} - \dfrac{\zeta(x)}{x} & \text{if } -1 < x < 0, \\ \dfrac{1}{2}\ln(2\pi) - 1 & \text{if } x = 0. \end{cases}$$

Munthe Hjortnaes [119] proved that

$$\zeta(3) = 10 \int_0^{\ln(\varphi)} x^2 \coth(x) dx = 10 \int_0^{\frac{1}{2}} \frac{\operatorname{arcsinh}(y)^2}{y} dy,$$

which, after integration by parts, gives [120]

$$\zeta(3) = -5 \int_0^{2\ln(\varphi)} \theta \ln\left(2 \sinh\left(\frac{\theta}{2}\right)\right) d\theta.$$

Starting with an integral of Euler's [84, 121],

$$4 \int_0^{\pi} \theta \ln\left(\sin\left(\frac{\theta}{2}\right)\right) d\theta = 7\zeta(3) - 2\pi^2 \ln(2),$$

the same reasoning can be applied as before (but in reverse) to obtain [80, 81]

$$-8 \int_0^{1} \frac{\arcsin(y)^2}{y} dy = -8 \int_0^{\frac{\pi}{2}} x^2 \cot(x) dx = 7\zeta(3) - 2\pi^2 \ln(2).$$

1.6.6 Continued Fractions

Stieltjes [122] and Ramanujan [54] discovered the continued fraction expansion

$$\zeta(3) = 1 + \frac{1|}{|2\cdot 2} + \frac{1^3|}{|1} + \frac{1^3|}{|6\cdot 2} + \frac{2^3|}{|1} + \frac{2^3|}{|10\cdot 2} + \frac{3^3|}{|1} + \frac{3^3|}{|14\cdot 2} + \cdots.$$

If we group terms together in a pairwise manner, we obtain

$$\zeta(3) = 1 + \frac{1|}{|5} - \frac{1^6|}{|21} - \frac{2^6|}{|55} - \frac{3^6|}{|119} - \frac{4^6|}{|225} - \frac{5^6|}{|385} - \cdots,$$

where the partial denominators are generated according to the polynomial $2n^3 + 3n^2 + 11n + 5$. The convergence rate of this expansion is not fast enough to demonstrate the irrationality of $\zeta(3)$. Apéry succeeded in accelerating the convergence to

$$\zeta(3) = \frac{6|}{|5} - \frac{1^6|}{|117} - \frac{2^6|}{|535} - \frac{3^6|}{|1463} - \frac{4^6|}{|3105} - \frac{5^6|}{|5665} - \cdots,$$

where the partial denominators are generated according to the polynomial $34n^3 + 51n^2 + 27n + 5$.

1.6.7 Stirling Cycle Numbers

Define $s_{n,m}$ to be the number of permutations of n symbols that have exactly m cycles [123]. The quantity $s_{n,m}$ is called the **Stirling number of the first kind** and satisfies the recurrence

$$s_{n,0} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \geq 1, \end{cases}$$

$$s_{n,m} = (n-1)s_{n-1,m} + s_{n-1,m-1} \quad \text{if } n \geq m \geq 1.$$

For example, $s_{3,1} = 2$ since (123) and (321) are distinct permutations. More generally, $s_{n,1} = (n-1)!$ and $s_{n,2} = (n-1)! \sum_{k=1}^{n-1} 1/k$. Similar complicated formulas involving higher-order harmonic sums apply for $m \geq 3$. Consequently [124],

$$\sum_{n=1}^{\infty} \frac{s_{n,m}}{n!n} = \zeta(m+1)$$

for $m \geq 1$. The case for $m = 2$ follows from one of the earlier multiple series (due to Euler [67]). The asymptotics of $s_{n,m}$ as $n \to \infty$ are found in [125].

1.6.8 Polylogarithms

Before defining the polylogarithm function Li_n, let us ask a question. It is known that

$$(-1)^k k! \zeta(k+1) = \int_0^1 \frac{\ln(x)^k}{1-x} dx, \quad k = 1, 2, 3, \ldots.$$

What happens if the interval of integration is changed from $[0, 1]$ to $[1, 2]$? Ramanujan [42] showed that, if

$$a_k = \int_1^2 \frac{\ln(x)^k}{1-x} dx,$$

then $a_1 = \zeta(2)/2 = \pi^2/6$ and $a_2 = \zeta(3)/4$. We would expect the pattern to persist and for a_k to be a rational multiple of $\zeta(k+1)$ for all $k \geq 1$. This does not appear to be true, however, even for $k = 3$.

Define $\mathrm{Li}_1(x) = -\ln(1-x)$ and [113, 114]

$$\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n} = \int_0^x \frac{\mathrm{Li}_{n-1}(t)}{t} dt \quad \text{for any integer } n \geq 2, \text{ where } |x| \leq 1.$$

Clearly $\mathrm{Li}_n(1) = \zeta(n)$. We mentioned special values, due to Landen, of the trilogarithm Li_3 earlier. Not much is known about the tetralogarithm Li_4, but Levin [126] demonstrated that

$$a_3 = \frac{\pi^4}{15} + \frac{\pi^2 \ln(2)^2}{4} - \frac{\ln(2)^4}{4} - \frac{21 \ln(2)}{4} \zeta(3) - 6 \,\mathrm{Li}_4\left(\frac{1}{2}\right)$$

and more. To fully answer our question, therefore, requires an understanding of the arithmetic nature of $\mathrm{Li}_n(1/2)$. Further details on polylogarithms are found in [127–131].

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1.7 Catalan's Constant, G

Catalan's constant, G, is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159655941\ldots.$$

Our discussion parallels that of Apéry's constant [1.6] and a comparison of the two is worthwhile. Here we work with **Dirichlet's beta function**

$$\beta(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^x}, \quad x > 0$$

(also referred to as Dirichlet's L-series for the nonprincipal character modulo 4) and observe that $G = \beta(2)$.

The beta function can be evaluated exactly [1–3] at positive odd integer values of x:

$$\beta(2k+1) = \frac{(-1)^k E_{2k}}{2(2k)!}\left(\frac{\pi}{2}\right)^{2k+1},$$

where $\{E_n\}$ denote the **Euler numbers** [1.7.1]. For example,

$$\beta(1) = \frac{\pi}{4}, \quad \beta(3) = \frac{\pi^3}{32}, \quad \beta(5) = \frac{5\pi^5}{1536}.$$

Like the zeta function [1.6], $\beta(x)$ can be analytically continued over the whole complex plane via the functional equation [4–6]:

$$\beta(1-z) = \left(\frac{2}{\pi}\right)^z \sin\left(\frac{\pi z}{2}\right)\Gamma(z)\beta(z),$$

where $\Gamma(z) = (z-1)!$ is the gamma function [1.5.4]. Dirichlet's function, unlike Riemann's function, is defined everywhere and has no singularities. Its connection to prime number theory is best summarized by the formula [7]

$$\beta(x) = \prod_{\substack{p \text{ prime} \\ p \equiv 1 \bmod 4}} \left(1 - \frac{1}{p^x}\right)^{-1} \cdot \prod_{\substack{p \text{ prime} \\ p \equiv 3 \bmod 4}} \left(1 + \frac{1}{p^x}\right)^{-1} = \prod_{\substack{p \text{ odd} \\ \text{prime}}} \left(1 - \frac{(-1)^{\frac{p-1}{2}}}{p^x}\right)^{-1},$$

and the rearrangement of factors is justified by absolute convergence. A closely associated function is [8–10]

$$\lambda(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^x} = \left(1 - \frac{1}{2^x}\right)\zeta(x), \quad x > 1,$$

with sample values

$$\lambda(2) = \frac{\pi^2}{8}, \quad \lambda(4) = \frac{\pi^4}{96}, \quad \lambda(6) = \frac{\pi^6}{960}.$$

Unlike Apéry's constant, it is unknown whether G is irrational [11, 12]. We also know nothing about the arithmetic character of G/π^2. In statistical mechanics, G/π arises as part of the exact solution of the dimer problem [5.23]. Schmidt [13] pointed out a curious coincidence:

$$\frac{\pi^2}{12\ln(2)} = \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + - \cdots\right)\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots\right)^{-1},$$

$$\frac{4G}{\pi} = \left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + - \cdots\right)\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots\right)^{-1},$$

where the former expression (Lévy's constant) is important in continued fraction asymptotics [1.8]. A variation of this,

$$\frac{8G}{\pi^2} = \left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + - \cdots\right)\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right)^{-1},$$

occurs as the best coefficient for which a certain conjugate function inequality [7.7] is valid. The constant $2G/(\pi \ln(2))$ also appears as the average root bifurcation ratio of binary trees [5.6].

1.7.1 Euler Numbers

Define $\{E_n\}$, the Euler numbers, by the generating function [1, 8–10]

$$\operatorname{sech}(x) = \frac{2e^x}{e^{2x} + 1} = \sum_{k=0}^{\infty} E_k \frac{x^k}{k!}.$$

It can be shown that all Euler numbers are integers: $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61, \ldots$ and $E_{2n-1} = 0$ for $n > 0$.

(There is, unfortunately, an alternative definition of the Euler numbers to confuse matters. Under this alternative definition, the subscripting is somewhat different and all the numbers are positive. One must be careful when reading any paper to establish which definition has been used.)

The Euler numbers also arise in certain other series expansions, such as

$$\sec(x) = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}}{(2k)!} x^{2k}.$$

1.7.2 Series

Summing over certain arithmetic progressions gives slight variations [14–16]:

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)^2} = \frac{1}{16}\pi^2 + \frac{1}{2}G, \quad \sum_{k=0}^{\infty} \frac{1}{(4k+3)^2} = \frac{1}{16}\pi^2 - \frac{1}{2}G.$$

Four formulas involving central binomial sums are [1, 17–19]

$$\sum_{k=0}^{\infty} \frac{2^{2k}}{(2k+1)^2\binom{2k}{k}} = 2G, \quad \sum_{k=0}^{\infty} \frac{1}{2^{3k}(2k+1)^2}\binom{2k}{k} = \frac{\pi}{4\sqrt{2}}\ln(2) + \frac{1}{\sqrt{2}}G,$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2\binom{2k}{k}} = \frac{8}{3}G - \frac{\pi}{3}\ln(2+\sqrt{3}),$$

$$\sum_{k=0}^{\infty} \frac{2^{4k}}{(k+1)(2k+1)^2\binom{2k}{k}^2} = 2\pi G - \frac{7}{2}\zeta(3).$$

As Berndt [17] remarked, it is interesting that the first of these is reminiscent of the famous Apéry series [1.6.3], yet it was discovered many years earlier. A family of related series is [20, 21, 23]

$$R(n) = \sum_{k=0}^{\infty} \frac{1}{2^{4k}(2k+n)}\binom{2k}{k}^2, \quad n = 0, 1, 2, \ldots,$$

which can be proved to satisfy the recurrence [1, 22, 24]

$$R(0) = 2\ln(2) - \frac{4G}{\pi}, \quad R(1) = \frac{4G}{\pi},$$

$$(n-1)^2 R(n) = (n-2)^2 R(n-2) + \frac{2}{\pi} \quad \text{for } n \geq 2.$$

What is the analog for $\beta(2n)$ of the exact formula for $\beta(2n+1)$? No one knows, but the series obtained by Ramanujan [16, 25],

$$G = \frac{5}{48}\pi^2 - 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2(e^{\pi(2k+1)}-1)} - \frac{1}{4}\sum_{k=1}^{\infty} \frac{\operatorname{sech}(\pi k)}{k^2},$$

might provide a starting point for research.

Some multiple series include [16, 17, 26–28]

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} = G, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sum_{k=1}^{n} \frac{1}{k} = G - \frac{\pi}{2}\ln(2),$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sum_{k=0}^{n-1} \frac{1}{2k+1} = \frac{\pi}{8}\ln(2) - \frac{1}{2}G, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \sum_{k=0}^{n-1} \frac{1}{2k+1} = \pi G - \frac{7}{4}\zeta(3),$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \sum_{k=1}^{n} \frac{1}{k+n} = \pi G - \frac{33}{16}\zeta(3), \quad \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)\binom{2n}{n}} \sum_{k=0}^{n} \frac{1}{2k+1} = 2G.$$

Two series involving zeta function values are [29–31]

$$\sum_{n=1}^{\infty} \frac{n\zeta(2n+1)}{2^{4n}} = 1 - G, \quad \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2^{4n}(2n+1)} = \frac{1}{2} - \frac{1}{4}\ln(2) - \frac{1}{\pi}G.$$

Broadhurst [32–34] determined a digit-extraction algorithm for G via the following series:

$$G = 3\sum_{k=0}^{\infty} \frac{1}{2\cdot 16^k}\left(\frac{1}{(8k+1)^2} - \frac{1}{(8k+2)^2} + \frac{1}{2(8k+3)^2} - \frac{1}{2^2(8k+5)^2} + \frac{1}{2^2(8k+6)^2} - \frac{1}{2^3(8k+7)^2}\right)$$

$$- 2\sum_{k=0}^{\infty} \frac{1}{8\cdot 16^{3k}}\left(\frac{1}{(8k+1)^2} + \frac{1}{2(8k+2)^2} + \frac{1}{2^3(8k+3)^2} - \frac{1}{2^6(8k+5)^2} - \frac{1}{2^7(8k+6)^2} - \frac{1}{2^9(8k+7)^2}\right).$$

1.7.3 Products

As with values of the zeta function at odd integers [1.6.4], Gosper [35] found an infinite matrix product that gives beta function values at even integers. We exhibit the 4×4 case only:

$$\prod_{k=1}^{\infty} \begin{pmatrix} \frac{4k^2}{(4k-1)(4k+1)} & \frac{-1}{(4k-1)(4k+1)} & 0 & \frac{1}{(2k-1)^5} \\ 0 & \frac{4k^2}{(4k-1)(4k+1)} & \frac{-1}{(4k-1)(4k+1)} & \frac{1}{(2k-1)^3} \\ 0 & 0 & \frac{4k^2}{(4k-1)(4k+1)} & \frac{6k-1}{2(2k-1)(4k-1)} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \beta(6) \\ 0 & 0 & 0 & \beta(4) \\ 0 & 0 & 0 & \beta(2) \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The extension to the $(n+1) \times (n+1)$ case and to $\beta(2n)$ follows the same pattern as before.

1.7.4 Integrals

The beta function has an alternative expression [4] for $x > 0$:

$$\beta(x) = \frac{1}{2\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{\cosh(t)} dt.$$

There are many integrals involving Catalan's constant [10, 15, 16, 36, 37], including

$$2\int_0^1 \frac{\arctan(x)}{x} dx = \int_0^{\frac{\pi}{2}} \frac{x}{\sin(x)} dx = 2G, \quad \frac{1}{2}\int_0^1 K(x)dx = \int_0^1 E(x)dx - \frac{1}{2} = G,$$

$$\int_0^1 \frac{\ln(x)}{1+x^2} dx = -\int_1^{\infty} \frac{\ln(x)}{1+x^2} dx = -G,$$

$$\int_0^{\frac{\pi}{4}} \ln(2\cos(x))dx = -\int_0^{\frac{\pi}{4}} \ln(2\sin(x))dx = \frac{1}{2}G,$$

$$4\int_0^1 \frac{\arctan(x)^2}{x}dx = \int_0^{\frac{\pi}{2}} \frac{x^2}{\sin(x)}dx = 2\pi G - \frac{7}{2}\zeta(3),$$

$$\int_0^{\frac{\pi}{2}} \operatorname{arcsinh}(\sin(x))dx = \int_0^{\frac{\pi}{2}} \operatorname{arcsinh}(\cos(x))dx = G,$$

where $K(x)$ and $E(x)$ are complete elliptic integrals [1.4.6]. See also [1.7.6].

1.7.5 Continued Fractions

The following expansions are due to Stieltjes [38], Rogers [39], and Ramanujan [40]:

$$2G = 2 - \frac{1|}{|3} + \frac{2^2|}{|1} + \frac{2^2|}{|3} + \frac{4^2|}{|1} + \frac{4^2|}{|3} + \frac{6^2|}{|1} + \frac{6^2|}{|3} + \cdots,$$

$$2G = 1 + \frac{1|}{|\frac{1}{2}} + \frac{1^2|}{|\frac{1}{2}} + \frac{1\cdot 2|}{|\frac{1}{2}} + \frac{2^2|}{|\frac{1}{2}} + \frac{2\cdot 3|}{|\frac{1}{2}} + \frac{3^2|}{|\frac{1}{2}} + \frac{3\cdot 4|}{|\frac{1}{2}} + \frac{4^2|}{|\frac{1}{2}} + \cdots.$$

1.7.6 Inverse Tangent Integral

Define $\mathrm{Ti}_1(x) = \arctan(x)$ and [41]

$$\begin{aligned}\mathrm{Ti}_n(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n} x^{2k+1} \\ &= \int_0^x \frac{\mathrm{Ti}_{n-1}(s)}{s}ds, \text{ for any integer } n \geq 2, \text{ where } |x| \leq 1.\end{aligned}$$

Clearly $\mathrm{Ti}_n(1) = \beta(n)$. The special case $n = 2$ is called the **inverse tangent integral**. It has alternative expressions

$$\mathrm{Ti}_2(\tan(\theta)) = \frac{1}{2}\int_0^{2\theta} \frac{t}{\sin(t)}dt = \theta\ln(\tan(\theta)) - \int_0^{\theta} \ln(2\sin(t))dt + \int_0^{\theta} \ln(2\cos(t))dt$$

for $0 < \theta < \pi/2$, and sample values [21, 41]

$$\mathrm{Ti}_2(2-\sqrt{3}) = \frac{2}{3}G + \frac{\pi}{12}\ln(2-\sqrt{3}), \quad \mathrm{Ti}_2(2+\sqrt{3}) = \frac{2}{3}G + \frac{5\pi}{12}\ln(2+\sqrt{3}).$$

In the latter formula, we use the integral expression (since the series diverges for $x > 1$, but the integral converges). Very little is known about $\mathrm{Ti}_n(x)$ for $n > 2$.

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1.8 Khintchine–Lévy Constants

Let x be a real number. Expand x (uniquely) as a regular continued fraction:

$$x = q_0 + \frac{1|}{|q_1} + \frac{1|}{|q_2} + \frac{1|}{|q_3} + \cdots,$$

where q_0 is an integer and $q_1, q_2, q_3, \ldots$ are positive integers. Unlike a decimal expansion, the properties of a regular continued fraction do not depend on the choice of base. Hence, to number theorists, terms of a continued fraction are more "natural" to look at than decimal digits.

What can be said about the average behavior of q_k, where $k > 0$ is arbitrary? Consider, for example, the geometric mean

$$M(n, x) = (q_1 q_2 q_3 \cdots q_n)^{\frac{1}{n}}$$

in the limit as $n \to \infty$. One would expect this limiting value to depend on x in some possibly complicated way. Since any sequence of qs determines a unique x, there exist xs for which the qs obey any conceivable condition. To attempt to compute $\lim_{n\to\infty} M(n, x)$ would thus seem to be impossibly difficult.

Here occurs one of the most astonishing facts in mathematics. Khintchine [1–4] proved that

$$\lim_{n\to\infty} M(n,x) = \prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+2)}\right)^{\frac{\ln(k)}{\ln(2)}} = K = e^{0.9878490568\ldots} = 2.6854520010\ldots,$$

a *constant*, for almost all real numbers x. This means that the set of exceptions x to Khintchine's result (e.g., all rationals, quadratic irrationals, and more) is of Lebesgue measure zero. We can be probabilistically certain that a truly randomly selected x will obey Khintchine's law. This is a profound statement about the nature of real numbers. Another proof, drawing upon ergodic theory and due to Ryll-Nardzewski [5], is found in Kac [6].

The infinite product representation of K converges very slowly. Fast numerical procedures for computing K appear in [7–13]. Among several different representations of K are [8, 11, 13, 14]

$$\ln(2)\ln(K) = -\sum_{i=2}^{\infty}\ln\left(1-\frac{1}{i}\right)\ln\left(1+\frac{1}{i}\right) = \sum_{j=2}^{\infty}\frac{(-1)^j(2-2^j)}{j}\zeta'(j),$$

$$\ln(2)\ln(K) = \sum_{k=1}^{\infty}\frac{\zeta(2k)-1}{k}\left(1-\frac{1}{2}+\frac{1}{3}-+\cdots+\frac{1}{2k-1}\right),$$

$$\ln(2)\ln(K) = -\int_0^1 \frac{1}{x(1+x)}\ln\left(\frac{\sin(\pi x)}{\pi x}\right)dx$$

$$= \frac{\pi^2}{12}+\frac{\ln(2)^2}{2}+\int_0^{\pi}\frac{\ln|\theta\cot(\theta)|}{\theta}d\theta,$$

where $\zeta(x)$ denotes the Riemann zeta function [1.6] and $\zeta'(x)$ is its derivative.

Many questions arise. Is K irrational? What well-known irrational numbers are among the meager exceptions to Khintchine's result? Lehmer [7, 15] observed that e is an exception; whether $\sqrt[3]{2}$, π, and K itself (!) are likewise remains unsolved.

Related ideas include the asymptotic behavior of the coprime positive integers P_n and Q_n, where P_n/Q_n is the n^{th} partial convergent of x. That is, P_n/Q_n is the value of the finite regular continued fraction expansion of x up through q_n. Lévy [16, 17] determined that

$$\lim_{n\to\infty} Q_n^{\frac{1}{n}} = e^{\frac{\pi^2}{12\ln(2)}} = e^{1.1865691104\ldots} = 3.2758229187\ldots = \lim_{n\to\infty}\left(\frac{P_n}{x}\right)^{\frac{1}{n}}$$

for almost all real x. Philipp [18, 19] provided improvements to error bounds associated with both Khintchine and Lévy limits. A different perspective is given by [20–22]:

$$-\lim_{n\to\infty}\frac{1}{n}\log_{10}\left|x-\frac{P_n}{Q_n}\right| = \frac{\pi^2}{6\ln(2)\ln(10)} = 1.0306408341\ldots,$$

which indicates that the information in a typical continued fraction term is approximately 1.03 decimal digits (valid for almost all real x). Equivalently, the metric entropy of the continued fraction map $x \mapsto \{1/x\}$ is [23, 24]

$$\lim_{n\to\infty} \frac{Q_{n+1}^2}{Q_n^2} = e^{\frac{\pi^2}{6\ln(2)}} = 10.7310157948\ldots = (0.0931878229\ldots)^{-1},$$

where $\{x\}$ denotes the fractional part of x. That is, an additional term reduces the uncertainty in x by a factor of 10.73. The corresponding entropy for the shift map $x \mapsto \{10x\}$ is 10.

Corless [13, 25] pointed out the interesting contrasting formulas

$$\ln(K) = \int_0^1 \frac{\ln\left\lfloor\frac{1}{x}\right\rfloor}{\ln(2)(1+x)}dx, \quad \frac{\pi^2}{12\ln(2)} = \int_0^1 \frac{\ln\left(\frac{1}{x}\right)}{\ln(2)(1+x)}dx,$$

where $\lfloor x \rfloor$ is the largest integer $\leq x$.

Let us return to the original question: What can be said about the average behavior of the k^{th} partial denominator q_k, $k > 0$? We have examined the situation for only one type of mean value, the geometric mean. A generalization [26] of mean value is

$$M(s,n,x) = \left(\frac{1}{n}\sum_{k=1}^{n} q_k^s\right)^{\frac{1}{s}},$$

which reduces to the harmonic mean, geometric mean, arithmetic mean, and root mean square, respectively, when $s = -1$, 0, 1, and 2. Thus the well-known means fit into a continuous hierarchy of mean values. It is known [3, 27] that, if $s \geq 1$, then $\lim_{n\to\infty} M(s,n,x) = \infty$ for almost all real x. What can be said about the value of $M(s,n,x)$ for $s < 1$, $s \neq 0$? The analog of Khintchine's formula here is

$$\lim_{n\to\infty} M(s,n,x) = \left[\frac{1}{\ln(2)}\sum_{k=1}^{\infty} k^s \ln\left(1 + \frac{1}{k(k+2)}\right)\right]^{\frac{1}{s}} = K_s$$

for almost all real x. It is known [13, 28] that $K_{-1} = 1.7454056624\ldots$, $K_{-2} = 1.4503403284\ldots$, $K_{-3} = 1.3135070786\ldots$, and clearly $K_s = 1 + O(1/s)$ as $s \to -\infty$.

Closely related topics are discussed in [2.17], [2.18], and [2.19].

1.8.1 Alternative Representations

There are alternative ways of representing real numbers, akin to regular continued fractions, that have associated Khintchine–Lévy constants. For example, every real number $0 < x < 1$ can be uniquely expressed in the form

$$\begin{aligned} x &= \frac{1}{a_1+1} + \sum_{n=2}^{\infty}\left(\prod_{k=1}^{n-1}\frac{1}{a_k(a_k+1)}\right)\frac{1}{a_n+1} \\ &= \frac{1}{b_1} + \sum_{n=2}^{\infty}\left(\prod_{k=1}^{n-1}\frac{1}{b_k(b_k+1)}\right)\frac{(-1)^{n-1}}{b_n}, \end{aligned}$$

where $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are positive integers. These are called the **Lüroth** and **alternating Lüroth representations** of x, respectively. The limiting constants are the same whether we use as or bs, and [29–31]

$$\lim_{n\to\infty}(a_1a_2a_3\cdots a_n)^{\frac{1}{n}}=\prod_{k=1}^{\infty}k^{\frac{1}{k(k+1)}}=e^{0.7885305659\ldots}=2.2001610580\ldots=U,$$

$$\lim_{n\to\infty}\left|x-\frac{P_n}{Q_n}\right|^{\frac{1}{n}}=\prod_{k=1}^{\infty}[k(k+1)]^{\frac{-1}{k(k+1)}}=e^{-2.0462774528\ldots}=V,$$

where P_n/Q_n is the n^{th} partial sum. A variation of this [32],

$$\lim_{n\to\infty}((a_1+1)(a_2+1)\cdots(a_n+1))^{\frac{1}{n}}=\prod_{k=1}^{\infty}(k+1)^{\frac{1}{k(k+1)}}=e^{1.2577468869\ldots}=W,$$

also appears in [2.9]. Of course, $UVW=1$ and

$$\ln(U)=-\sum_{i=2}^{\infty}(-1)^i\zeta'(i),\ \ \ln(V)=2\sum_{j=1}^{\infty}\zeta'(2j),\ \ \ln(W)=-\sum_{k=2}^{\infty}\zeta'(k).$$

A second example [22] is the **Bolyai–Rényi representation** of $0<x<1$,

$$x=-1+\sqrt{a_1+\sqrt{a_2+\sqrt{a_3+\cdots}}},$$

where each $a_k\in\{0,1,2\}$. Whereas an exact expression $\pi^2/(6\ln(2))=2.373138\ldots$ arises for the entropy of continued fractions, only a numerical result $1.056313\ldots$ exists for the entropy of radical expansions [33].

A third example [34–41] is the **nearest integer continued fraction** of $-1/2<x<1/2$,

$$x=\frac{1|}{|c_1}+\frac{1|}{|c_2}+\frac{1|}{|c_3}+\cdots,$$

which is generated according to

$$c_1=\left\lfloor\frac{1}{x}+\frac{1}{2}\right\rfloor,\ \ x_1=\frac{1}{x}-c_1,\ \ c_2=\left\lfloor\frac{1}{x_1}+\frac{1}{2}\right\rfloor,\ \ x_2=\frac{1}{x_1}-c_2,\ldots.$$

Some of the cs may be negative. The formulas for the Khintchine–Lévy constants in this case are

$$\lim_{n\to\infty}|c_1c_2\cdots c_n|^{\frac{1}{n}}=\left(\frac{5\varphi+3}{5\varphi+2}\right)^{\frac{\ln(2)}{\ln(\varphi)}}\prod_{k=3}^{\infty}\left(\frac{8(k-1)\varphi+(2k-3)^2+4}{8(k-1)\varphi+(2k-3)^2}\right)^{\frac{\ln(k)}{\ln(\phi)}}$$
$$=e^{1.6964441175\ldots}=5.4545172445\ldots,$$

$$\lim_{n\to\infty}Q_n^{\frac{1}{n}}=e^{\frac{\pi^2}{12\ln(\varphi)}}=e^{1.7091579853\ldots}=5.5243079702\ldots,$$

Table 1.2. *Nonexplicit Constants Recursively Derived from K*

$y = 2.3038421962\ldots$	q_n is the largest possible integer: $\prod_{k=0}^{n} q_k < K^{n+1}$
$y = 3.3038421963\ldots$	q_n is the smallest possible integer: $\prod_{k=0}^{n} q_k > K^{n+1}$
$y = 2.2247514809\ldots$	$\prod_{k=0}^{n} q_k$ is just less than K^{n+1} when n is even, and $\prod_{k=0}^{n} q_k$ is just greater than K^{n+1} when n is odd
$y = 3.4493588902\ldots$	$\prod_{k=0}^{n} q_k$ is just greater than K^{n+1} when n is even, and $\prod_{k=0}^{n} q_k$ is just less than K^{n+1} when n is odd

where P_n/Q_n is the n^{th} partial convergent and φ is the Golden mean [1.2]. Such expansions are also called **centered continued fractions** [42].

1.8.2 Derived Constants

Although we know exceptions x (which all belong to a set of measure zero) to Khintchine's law, we do not know a single explicit y that provably satisfies it. This is remarkable because one would expect y to be easy to find, being so much more plentiful than x. The requirement that y be "explicit" is the difficult part. It means, in particular, that the partial denominators q_n in the regular continued fraction for y should not depend on knowing K to arbitrary precision. Robinson [43] described four nonexplicit constants that are recursively derived from K in a simple manner (see Table 1.2). Bailey, Borwein & Crandall [13] gave other, more sophisticated constructions in which at least the listing $q_0, q_1, q_2, \ldots$ is explicit (although the constant y still is not).

1.8.3 Complex Analog

Schmidt [44–46] introduced what appears to be the most natural approach for generalizing continued fraction theory to the complex field. For example [47–50], the complex analog of Lévy's constant is $\exp(G/\pi)$, where G is Catalan's constant [1.7]. Does Khintchine's constant possess a complex analog?

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1.9 Feigenbaum–Coullet–Tresser Constants

Let $f(x) = ax(1 - x)$, where a is constant. The interval $[0, 1]$ is mapped into itself by f for each value of $a \in [0, 4]$. This family of functions, parametrized by a, is known as the family of **logistic maps** [1–8].

What are the 1-cycles (i.e., fixed points) of f? Solving $x = f(x)$, we obtain

$$x = 0 \quad \text{(which attracts for } a < 1 \text{ and repels for } a > 1\text{)}$$

and

$$x = \frac{a-1}{a} \quad \text{(which attracts for } 1 < a < 3 \text{ and repels for } a > 3).$$

What are the 2-cycles of f? That is, what are the fixed points of the iterate f^2 that are not fixed points of f? Solving $x = f^2(x)$, $x \neq f(x)$, we obtain the 2-cycle

$$x = \frac{a+1 \pm \sqrt{a^2-2a-3}}{2a} \quad \begin{array}{l}\text{(which attracts for } 3 < a < 1+\sqrt{6} \\ \quad\text{and repels for } a > 1+\sqrt{6}).\end{array}$$

For $a > 1+\sqrt{6} = 3.4495\ldots$, an attracting 4-cycle emerges. We can obtain the 4-cycle by numerically solving $x = f^4(x)$, $x \neq f^2(x)$. It can be shown that the 4-cycle attracts for $3.4495\ldots < a < 3.5441\ldots$ and repels for $a > 3.5441\ldots$.

For $a > 3.5441$, an attracting 8-cycle emerges. We can obtain the 8-cycle by numerically solving $x = f^8(x)$, $x \neq f^4(x)$. It can be shown that the 8-cycle attracts for $3.5441\ldots < a < 3.5644\ldots$ and repels for $a > 3.5644\ldots$.

For how long does the sequence of period-doubling bifurcations continue? It is interesting that this behavior stops far short of 4. Letting

$$a_0 = 1, \;\; a_1 = 3, \;\; a_2 = 3.4495\ldots, \;\; a_3 = 3.5441\ldots, \;\; a_4 = 3.5644\ldots,$$

etc. denote the sequence of **bifurcation points** of f, it can be proved that

$$a_\infty = \lim_{n\to\infty} a_n = 3.5699\ldots < 4.$$

This limiting point marks the separation between the "periodic regime" and the "chaotic regime" for this family of quadratic functions. Much research has been aimed at developing a theory of chaos and applying it to the study of physical, chemical, and biological systems. We will focus on only a small aspect of the theory: two "universal" constants associated with the exponential accumulation described earlier. The bifurcation diagram in Figure 1.5 is helpful for defining the following additional symbols. The sequence of **superstable points** of f is

$$\tilde{a}_1 = 1+\sqrt{5} = 3.2360\ldots, \;\; \tilde{a}_2 = 3.4985\ldots, \;\; \tilde{a}_3 = 3.5546\ldots, \;\; \tilde{a}_4 = 3.5666\ldots,$$

where $\tilde{a}_n$ is the least parameter value at which a 2^n-cycle contains the critical element $1/2$. Call this cycle $\tilde{C}(n)$. The sequence of **superstable widths** of f is

$$\tilde{w}_1 = (\sqrt{5}-1)/4 = 0.3090\ldots, \;\; \tilde{w}_2 = 0.1164\ldots, \;\; \tilde{w}_3 = 0.0459\ldots,$$

where $\tilde{w}_n$ is the distance between $1/2$ and the element $f^{2^{n-1}}(1/2) \in \tilde{C}(n)$ nearest to $1/2$. Also, the sequence of **bifurcation widths** of f is

$$w_1 = \sqrt{2}(\sqrt{6}-1)/5 = 0.4099\ldots, \;\; w_2 = 0.1603\ldots, \;\; w_3 = 0.0636\ldots,$$

where w_n is the corresponding cycle distance at a_{n+1}. The superstable variants $\tilde{a}_n$ and $\tilde{w}_n$ are numerically easier to compute than a_n and w_n. Define the two **Feigenbaum–**

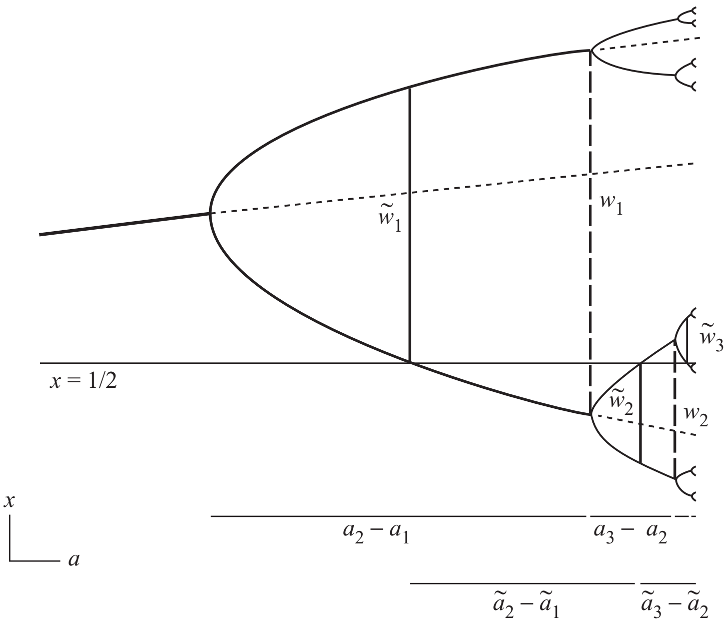

Figure 1.5. Horizontal and vertical characteristics of the bifurcation are quantified by a_n and w_n.

Coullet–Tresser constants to be [9–17]

$$\delta = \lim_{n\to\infty} \frac{a_n - a_{n-1}}{a_{n+1} - a_n} = \lim_{n\to\infty} \frac{\tilde{a}_n - \tilde{a}_{n-1}}{\tilde{a}_{n+1} - \tilde{a}_n} = 4.6692016091\ldots$$

and

$$\alpha = \lim_{n\to\infty} \frac{w_n}{w_{n+1}} = \lim_{n\to\infty} \frac{\tilde{w}_n}{\tilde{w}_{n+1}} = 2.5029078750\ldots = (0.3995352805\ldots)^{-1}.$$

As indicated here, the tildes can be included or excluded without change to the limiting ratios δ and α.

What qualifies these constants to be called "universal"? If we replace the logistic maps f by, for example, $g(x) = b\,\sin(\pi x)$, $0 \le b \le 1$, then interestingly the same constants δ and α occur. Both functions f and g have quadratic maximum points; we extend this condition to obtain generalized Feigenbaum constants [1.9.1]. We mention a two-dimensional example [1.9.2] as well. Rigorous proofs of universality for the one-dimensional, quadratic maximum case were first given by Lanford [18–22] and Campanino & Epstein [23–28]; the former apparently was the first computer-assisted proof of its kind in mathematics.

Does there exist a simpler definition of the Feigenbaum constants? One would like to see a more classical characterization in terms of a limit or an integral that would not require quite so much explanation. The closest thing to this involves a certain functional equation [1.9.3], which in fact appears to provide the most practical algorithm for calculating the constants to high precision [29–37]. We also mention maps on a circle [1.9.4] and a different form of chaos.

The numbers 3.5441 ... and 3.5644 ... mentioned previously are known to be algebraic of degrees 12 and 240, as discussed in [38, 39].

Salamin [40] has speculated that the (unitless) fine structure constant $(137.0359\ldots)^{-1}$ from quantum electrodynamics will, in a better theory than we have today, be related to a Feigenbaum-like constant.

1.9.1 Generalized Feigenbaum Constants

Consider the functions f and g defined earlier. Consider also the function $h(x) = 1 - c|x|^r$ defined on the interval $[-1, 1]$, where $1 < c < 2$ and $r > 1$ are constants. Each function is unimodal, concave, symmetric, and analytic everywhere with the possible exception of h at $x = 0$. Further, each second derivative, evaluated at the maximum point, is strictly negative if $r = 2$. That is, f, g, and h have quadratic maximum points.

In contrast, the order of the maximum of h is cubic if $r = 3$, quartic if $r = 4$, etc. This is an important distinction with regard to the values of the Feigenbaum constants.

Many authors have used the word "universal" to describe δ and α, and this is appropriate if quadratic maximums are all one is concerned about. Vary r, however, and different values of δ and α emerge. Numerical evidence indicates that δ increases with r, and α decreases to a limiting value of 1 [36, 41] (see Table 1.3). In fact, we have [42–48]

$$\lim_{r\to\infty} \delta(r) = 29.576303\ldots, \quad \lim_{r\to\infty} \alpha(r)^{-r} = 0.0333810598\ldots.$$

At the other extreme [15, 31], $\lim_{r\to 1^+} \delta(r) = 2$ whereas $\lim_{r\to 1^+} \alpha(r) = \infty$.

A somewhat different generalization involves period triplings rather than period doublings [1, 16, 29, 30, 49–51]. For the logistic map f, when $3.8284\ldots \le a \le 3.8540\ldots$, a cascade of trifurcations to 3^n-cycles at parameter values $\hat{a}_n$ occur with Feigenbaum constants:

$$\hat{\delta} = \lim_{n\to\infty} \frac{\hat{a}_n - \hat{a}_{n-1}}{\hat{a}_{n+1} - \hat{a}_n} = 55.247\ldots, \quad \hat{\alpha} = \lim_{n\to\infty} \frac{\hat{w}_n}{\hat{w}_{n+1}} = 9.27738\ldots.$$

Three-cycles are of special interest since they guarantee the existence of chaos [2]. We do not know precisely the minimum value of a for which f has points that are not asymptotically periodic. The first 6-cycle appears [2] at 3.6265 ..., and the first odd-cycle appears [1] at 3.6786 ....

The constants 55.247 ... and 9.27738 ... have not been computed to the same precision as the original Feigenbaum constants. Existing theory [27, 28] seems to apply

Table 1.3. *Feigenbaum Constants as Functions of Order r*

r	3	4	5	6
$\delta(r)$	5.9679687038 ...	7.2846862171 ...	8.3494991320 ...	9.2962468327 ...
$\alpha(r)$	1.9276909638 ...	1.6903029714 ...	1.5557712501 ...	1.4677424503 ...

only to period doublings. Our knowledge of period triplings is evidently based more on numerical heuristics than on mathematical rigor at present.

Incidently, the bifurcation points of h, when $r = 2$, are

$$c_2 = \tfrac{5}{4} = 1.25,\ \ c_3 = 1.3680\ldots,\ \ c_4 = 1.3940\ldots,\ \ \ldots,\ \ c_\infty = 1.4011\ldots$$

and are related to a_n via the transformation $c_n = a_n(a_n - 2)/4$. The limit point $c_\infty = 1.4011551890\ldots$ is due to Myrberg [52] but is not universal in any sense. Similarly, we can find the successive superstable width ratios of h, when $r = 2$:

$$\alpha_1 = 3.2185\ldots,\ \ \alpha_2 = 2.6265\ldots,\ \ \alpha_3 = 2.5281\ldots,\ \ \ldots\ \ \alpha_\infty = \alpha = 2.5029\ldots.,$$

in terms of symbols defined earlier: $\alpha_n = \tilde{w}_n(\tilde{a}_{n+1} - 2)\tilde{w}_{n+1}^{-1}(\tilde{a}_n - 2)^{-1}$. Both sequences $\{c_n\}$ and $\{\alpha_n\}$ are needed in [1.9.3].

1.9.2 Quadratic Planar Maps

The quadratic area-preserving (conservative) **Hénon map** [53, 54]

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - a x_n^2 + y_n \\ x_n \end{pmatrix}$$

also leads to a cascade of period doublings, but with Feigenbaum constants $\alpha = 4.0180767046\ldots$, $\beta = 16.3638968792\ldots$ (scaling for two directions), and $\delta = 8.7210972\ldots$ that are larger than those for the one-dimensional case. These are characteristic for a certain subclass of the class of two-dimensional maps with quadratic maxima [50, 55, 56]. There is a different subclass, however, for which the *original* Feigenbaum constant $\delta = 4.6692016091\ldots$ appears: the area-contracting (dissipative) Hénon maps [49, 57, 58]

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - a x_n^2 + y_n \\ b x_n \end{pmatrix}$$

(where the additional parameter b satisfies $|b| < 1$). It appears in higher dimensions too. The extent of the universality of δ is therefore larger than one may have expected!

Like period-tripling constants discussed in [1.9.1], the quantities $4.01808\ldots$, $16.36389\ldots$, and $8.72109\ldots$ have not been computed to the same precision as the original Feigenbaum constants. For two-dimensional conservative maps, Eckmann, Koch & Wittwer [59, 60] proved that these are indeed universal. For N-dimensional dissipative maps, Collet, Eckmann & Koch [61, 62] sketched a proof that the constant $4.66920\ldots$ is likewise universal.

1.9.3 Cvitanovic–Feigenbaum Functional Equation

Let D be an open, connected set in the complex plane containing the interval [0, 1]. Let X be the real Banach space of functions F satisfying $F(0) = 0$ that are complex-analytic on D, continuous on the closure of D, and real on [0, 1], equipped with the supremum norm.

Fix a real number $r > 1$. Let Ω_r be the set of functions $f : [-1, 1] \to (-1, 1]$ of the form $f(x) = 1 + F(|x|^r)$, $F \in X$, with $F'(y) < 0$ for all $y \in [0, 1]$. In words, Ω_r is the set of even, folding self-maps f of the interval $[-1, 1]$ that can be written as power series in $|x|^r$ and satisfy $-1 < f^2(0) < f(0) = 1$. Define also $\Omega_{r,0}$ to be the subset of Ω_r subject to the additional constraint $f^2(0) < 0 < f^4(0) < -f^2(0) < f^3(0) < 1$.

By using the correspondence between f and F, the sets $\Omega_{r,0}$ and Ω_r are naturally identified with nested, open subsets of X. Hence $\Omega_{r,0}$ and Ω_r are Banach manifolds, both based on X. We can thus perform differential calculus on what is called the **period-doubling operator** $T_r : \Omega_{r,0} \to \Omega_r$, obtaining a linear operator $L_r : X \to X$ that best fits T_r in the vicinity of a certain function φ. This will be done shortly and is necessary to rigorously formulate the Feigenbaum constants [15, 27, 63].

Consider the function h defined earlier. Let us make its dependence on the parameter c explicit and write h_c from now on. Clearly $h_c \in \Omega_r$. Recall the sequences $\{c_n\}$ and $\{\alpha_n\}$ defined at the conclusion of [1.9.1] for $r = 2$; analogous sequences can be defined for arbitrary $r > 1$. We are interested in the "universality" of iterates of h_c as the parameter c increases to c_∞ and as the middle portion of the graph is magnified without bound. The remarkable limit

$$\lim_{n\to\infty} (-\alpha_n)^n \cdot h_{c_n}^{2^n}\left(\frac{x}{\alpha_n^n}\right) = \varphi(x)$$

exists [64–67] and satisfies the **Cvitanovic–Feigenbaum functional equation**

$$\varphi(x) = \varphi(1)^{-1} \cdot \varphi(\varphi(\varphi(1) \cdot x)) = T_r[\varphi](x)$$

with $\varphi \in \Omega_{r,0}$. See Figure 1.6 for a nice geometric interpretation. Moreover, the solution φ has been proven to be unique if r is an even integer [68–71]. Extending this uniqueness

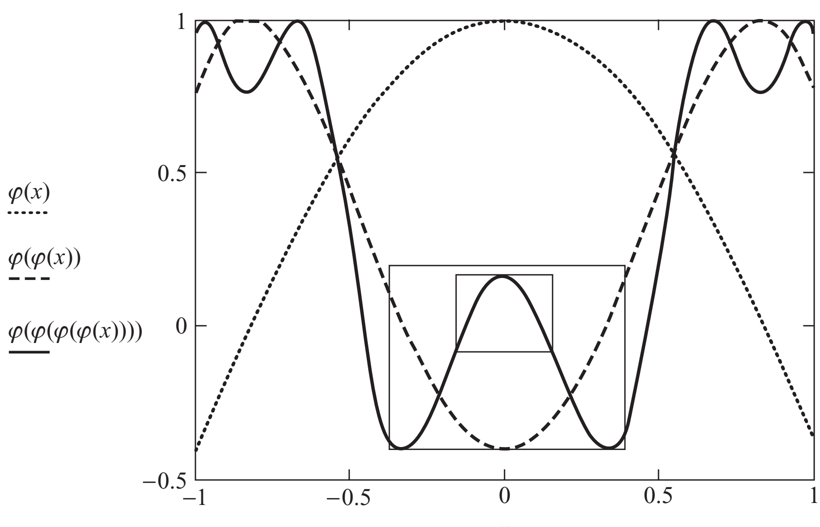

Figure 1.6. Self-similarity of iterates of φ are illustrated inside diminishing rectangular windows: The condition $\varphi(1) < 0$ reverses orientation.

result to arbitrary $r > 1$ is an unsolved challenge [72]. As a consequence, for each r, we have $\alpha(r) = -\varphi(1)^{-1}$.

Consider now the local linearization (Fréchet derivative) of T_r at the fixed point φ:

$$L_r[\psi](x) = \varphi(1)^{-1} \cdot \left\{\varphi'(\varphi(\varphi(1) \cdot x)) \cdot \psi(\varphi(1) \cdot x) + \psi(\varphi(\varphi(1) \cdot x)) \right. \\ \left. + \psi(1) \cdot \left[\varphi'(x) \cdot x - \varphi(x)\right]\right\}.$$

Then, for each r, $\delta(r)$ is the largest eigenvalue associated with L_r and is, in fact, the only eigenvalue that lies outside the unit disk. This is the basis for accurate estimates of $\delta(r)$. Fortunately, only the first two of the three terms in $L_r[\psi](x)$ are needed for computations [27, 36]. Alternatively, $\delta(r) = \lim_{n\to\infty} \sigma_{n+1}/\sigma_n$, where [45–47]

$$\sigma_n = \frac{1}{\xi(1)^n} \sum_{k=1}^{2^n-1} \xi^k(0) \cdot \left(\prod_{j=0}^{k-1} \xi'(\xi^j(0))\right)^{-1}$$

and $\xi(x) = \left|\varphi(x^{1/r})\right|^r$ for $0 \le x \le 1$. This formula is attractive, but unfortunately it is not numerically feasible for high-precision results. More formulas for δ appear in [73–75].

For period tripling [1.9.1], the analog of the Cvitanovic–Feigenbaum equation [29]

$$\varphi(x) = \varphi(1)^{-1} \cdot \varphi(\varphi(\varphi(\varphi(1) \cdot x)))$$

gives an estimate of $\hat{\alpha}$, and a linearization of the right-hand side gives $\hat{\delta}$. For planar maps, a matrix analog applies. Other functional equations will appear shortly.

1.9.4 Golden and Silver Circle Maps

We briefly mention a different example [76–79]:

$$\theta_{n+1} = k_a(\theta_n) = \theta_n + a - \frac{1}{2\pi}\sin(2\pi\theta_n),$$

which can be thought of as a homeomorphic mapping of a circle of circumference 1 onto itself. For any such circle map l, the limit

$$\rho(l) = \lim_{n\to\infty} \frac{l^n(\theta) - \theta}{n}$$

exists and is independent of θ. The quantity $\rho(l)$ is called the **winding** or **rotation number** of l. Our interest here is not in period doubling but rather quasiperiodicity: The subject offers an alternative transition into chaos and is rooted in the tension created under conditions when ρ is irrational.

Let $f_1 = f_2 = 1,\ f_3 = 2, \ldots$ denote the Fibonacci numbers [1.2], and define sequences $\{a_n\}$ and $\{w_n\}$ by [80, 81]

$$k_{a_n}^{f_n}(0) = f_{n-1}, \quad w_n = k_{a_n}^{f_{n-1}}(0) - f_{n-2}.$$

It can be proved that $\rho(k_{a_\infty}) = (1 - \sqrt{5})/2$; hence the family of circle maps k_{a_n} is **golden** and the corresponding Feigenbaum constants are $\alpha = 1.2885745539\ldots$ and $\delta = 2.8336106558\ldots$. Moreover, for all golden circle maps with a single cubic point of inflection, the constants α and δ are universal. If we replace the Fibonacci numbers by Pell numbers [1.1], then $\rho(k_{a_\infty}) = \sqrt{2} - 1$; hence the family of circle maps k_{a_n} is **silver** with $\alpha = 1.5868266790\ldots$ and $\delta = 6.7992251609\ldots$. Similar universality holds for cubic silver circle maps; other irrational winding numbers have been studied too [30]. If, instead, we examine golden circle maps with a single r^{th}-order inflection point, then functions $\alpha(r)$ and $\delta(r)$ emerge, satisfying [47, 80, 82–86]

$$\lim_{r\to\infty} \alpha(r) = 1, \qquad \lim_{r\to\infty} \alpha(r)^r = 3.63600703\ldots,$$
$$\alpha\left(\tfrac{1}{r}\right) = \alpha(r)^r \text{ for all } r > 0, \qquad \lim_{r\to\infty} \delta(r) = 4.121326\ldots.$$

It is conjectured, but not yet proven, that $\delta(1/r) = \delta(r)$ for all r.

As with interval maps, certain functional equations provide the numerical key to precisely computing $\alpha(r)$ and $\delta(r)$ associated with circle maps [81]:

$$\varphi(\theta) = \varphi(1)^{-1} \cdot \varphi(\varphi(\varphi(1)^2 \cdot \theta))$$

for the golden case and

$$\varphi(\theta) = \varphi(1)^{-1} \cdot \varphi(\varphi(1) \cdot \varphi(\varphi(\varphi(1)^2 \cdot \theta)))$$

for the silver case.

McCarthy [87] compared the two famous functional equations

$$\varphi(x) \cdot \varphi(y) = \varphi(x + y), \quad \varphi(\varphi(y)) = s^{-1}\varphi(s\, y).$$

In the former, multiplication is simply a form of addition; in the latter, self-composition is just a rescaling. He invoked the appropriate phrase "twentieth-century exponential function" for a solution of the latter. Research in this area will, however, continue for many more years.

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1.10 Madelung's Constant

Consider the square lattice in the plane with unit charges located at integer lattice points $(i, j) \neq (0, 0)$ and of sign $(-1)^{i+j}$. The electrostatic potential at the origin due to the charge at (i, j) is $(-1)^{i+j}/\sqrt{i^2 + j^2}$. The total electrostatic potential at the origin due to all charges is hence [1]

$$M_2 = \sum_{i,j=-\infty}^{\infty}{}' \frac{(-1)^{i+j}}{\sqrt{i^2 + j^2}},$$

where the prime indicates that we omit (0, 0) from the summation.

How is this infinite lattice sum to be interpreted? This is a delicate issue since the subseries with $i = j$ is divergent, so the alternating character of the full series needs to be carefully exploited [2–7]. We may, nonetheless, work with either expanding circles or with expanding squares and still obtain the same convergent sum [8–15]:

$$M_2 = 4(\sqrt{2} - 1)\zeta\left(\frac{1}{2}\right)\beta\left(\frac{1}{2}\right) = -1.6155426267\ldots,$$

where $\zeta(x)$ is Riemann's zeta function [1.6] and $\beta(x)$ is Dirichlet's beta function [1.7]. The sum M_2 is called Madelung's constant for a two-dimensional NaCl crystal. Rewriting lattice sums in terms of well-known functions as such is essential because convergence rates otherwise are extraordinarily slow.

The three-dimensional analog

$$M_3 = \sum_{i,j,k=-\infty}^{\infty}{}' \frac{(-1)^{i+j+k}}{\sqrt{i^2 + j^2 + k^2}}$$

is trickier because, surprisingly, the expanding-spheres method for summation leads to divergence! This remarkable fact was first noticed by Emersleben [16]. Using expanding cubes instead, we obtain the Benson–Mackenzie formula [17, 18]

$$M_3 = -12\pi \sum_{m,n=1}^{\infty} \operatorname{sech}\left(\frac{\pi}{2}\sqrt{(2m-1)^2 + (2n-1)^2}\right)^2 = -1.7475645946\ldots,$$

which is rapidly convergent. Of many possible reformulations, there is a formula due to Hautot [19]

$$M_3 = \frac{\pi}{2} - \frac{9}{2}\ln(2) + 12 \sum_{m,n=1}^{\infty} (-1)^m \frac{\operatorname{csch}\left(\pi\sqrt{m^2 + n^2}\right)}{\sqrt{m^2 + n^2}},$$

that is not quite as fast but is formally consistent with other lattice sums we discuss later. The quantity M_3 is called Madelung's constant for a three-dimensional NaCl crystal or, more simply, **Madelung's constant**. Note that, in their splendid survey, Glasser &

Zucker [20] called $\pm 2M_3$ the same, so caution should be exercised when reviewing the literature. Other representations of M_3 appear in [21–23].

The four-, six-, and eight-dimensional analogs can also be found [24]:

$$M_4 = \sum_{i,j,k,l=-\infty}^{\infty}{}' \frac{(-1)^{i+j+k+l}}{\sqrt{i^2+j^2+k^2+l^2}} = -8\left(5-3\sqrt{2}\right)\zeta\left(\frac{1}{2}\right)\zeta\left(-\frac{1}{2}\right)$$
$$= -1.8393990840\ldots,$$

$$M_6 = \frac{3}{\pi^2}\left[4\left(\sqrt{2}-1\right)\zeta\left(\frac{1}{2}\right)\beta\left(\frac{5}{2}\right) - \left(4\sqrt{2}-1\right)\zeta\left(\frac{5}{2}\right)\beta\left(\frac{1}{2}\right)\right]$$
$$= -1.9655570390\ldots,$$

$$M_8 = \frac{15}{4\pi^3}\left(8\sqrt{2}-1\right)\zeta\left(\frac{1}{2}\right)\zeta\left(\frac{7}{2}\right) = -2.0524668272\ldots.$$

A general result due to Borwein & Borwein [4] shows that the n-dimensional analog of Madelung's constant is convergent for any $n \geq 1$. Of course, $M_1 = -2\ln(2)$. Rapidly convergent series expressions for $M_5 = -1.9093378156\ldots$ or $M_7 = -2.0124059897\ldots$ seem elusive [25]. It is known, however, that for all n,

$$M_n = \frac{1}{\sqrt{\pi}}\int_0^\infty \left\{\left(\sum_{k=-\infty}^{\infty}(-1)^k e^{-k^2 t}\right)^n - 1\right\}\frac{dt}{\sqrt{t}},$$

from which high-precision numerical computations are possible [26,27]. Using this integral, it can be proved [28] that $M_n \sim -\sqrt{4\ln(n)/\pi}$ as $n \to \infty$.

There are many possible variations on these lattice sums. One could, for example, remove the square root in the denominator and obtain [15,20]

$$N_1 = \sum_{i=-\infty}^{\infty}{}' \frac{(-1)^i}{i^2} = -\frac{\pi^2}{6}, \quad N_2 = \sum_{i,j=-\infty}^{\infty}{}' \frac{(-1)^{i+j}}{i^2+j^2} = -\pi\ln(2),$$

$$N_3 = \sum_{i,j,k=-\infty}^{\infty}{}' \frac{(-1)^{i+j+k}}{i^2+j^2+k^2}$$
$$= \frac{\pi^2}{3} - \pi\ln(2) - \frac{\pi}{\sqrt{2}}\ln\left(2(\sqrt{2}+1)\right) + 8\pi\sum_{m,n=1}^{\infty}(-1)^n\frac{\operatorname{csch}\left(\pi\sqrt{m^2+2n^2}\right)}{\sqrt{m^2+2n^2}}$$
$$= -2.5193561520\ldots,$$

$$N_4 = \sum_{i,j,k,l=-\infty}^{\infty}{}' \frac{(-1)^{i+j+k+l}}{i^2+j^2+k^2+l^2} = -4\ln(2),$$

with asymptotics $N_n \sim -\ln(n)$ determined similarly. One could alternatively perform the summation over a different lattice; for example, a regular hexagonal lattice in the plane rather than the square lattice [2,7], with basis vectors $(1,0)$ and $(1/2, \sqrt{3}/2)$. This yields the expression

$$H_2 = \frac{4}{3}\sum_{i,j=-\infty}^{\infty}{}' \frac{\sin((i+1)\theta)\sin((j+1)\theta) - \sin(i\theta)\sin((j-1)\theta)}{\sqrt{i^2+ij+j^2}},$$

where $\theta = 2\pi/3$, which may be rewritten as

$$\begin{aligned} H_2 &= -3\left(\sqrt{3}-1\right)\zeta\left(\frac{1}{2}\right) \\ &\quad \times \left(1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{10}} - \frac{1}{\sqrt{11}} + - \cdots\right) \\ &= 1.5422197217\ldots. \end{aligned}$$

This is Madelung's constant for the planar hexagonal lattice; the three-dimensional analog H_3 of this perhaps has a chemical significance akin to M_3. If we remove the square root in the denominator as well, then

$$K_2 = \frac{4}{3}\sum_{i,j=-\infty}^{\infty}{}' \frac{\sin((i+1)\theta)\sin((j+1)\theta) - \sin(i\theta)\sin((j-1)\theta)}{i^2+ij+j^2} = \sqrt{3}\pi\ln(3).$$

A lattice sum generalization of the Euler–Mascheroni constant [1.5] appears in [1.10.1]. This, by the way, has no connection with different extensions due to Stieltjes [2.21] or to Masser and Gramain [7.2].

Forrester & Glasser [29] discovered that

$$\sum_{i,j,k=-\infty}^{\infty} \frac{(-1)^{i+j+k}}{\sqrt{\left(i-\frac{1}{6}\right)^2 + \left(j-\frac{1}{6}\right)^2 + \left(k-\frac{1}{6}\right)^2}} = \sqrt{3},$$

which may be as close to an exact evaluation of M_3 as possible (in the sense that no such formula is known at any point closer to the origin). Some variations involving trigonometric functions were explored in [30,31]. There are many more relevant summations available than we can possibly give here [20,32].

1.10.1 Lattice Sums and Euler's Constant

For any integer $p \geq 2$, define

$$\Delta(n,p) = \sum_{i_1,i_2,\ldots,i_p=-n}^{n}{}' \frac{1}{\sqrt{i_1^2+i_2^2+\cdots+i_p^2}} - \int_{x_1,x_2,\ldots,x_p=-n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{dx_1 dx_2 \cdots dx_p}{\sqrt{x_1^2+x_2^2+\cdots+x_p^2}}.$$

The integral converges in spite of the singularity at the origin. In two dimensions, we have [33]

$$\begin{aligned}\Delta(n,2) &= \sum_{i,j=-n}^{n}{}' \frac{1}{\sqrt{i^2+j^2}} - 4\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\left(n+\frac{1}{2}\right)\\ &\to 4\zeta\left(\frac{1}{2}\right)\beta\left(\frac{1}{2}\right)\\ &= \left(\sqrt{2}+1\right)M_2 = -3.9002649200\ldots = \delta_2\end{aligned}$$

as $n \to \infty$. It is interesting that if we define a function

$$f(z) = \sum_{i,j=-\infty}^{\infty}{}' \frac{1}{(i^2+j^2)^z}, \quad \mathrm{Re}(z) > 1,$$

then f can be analytically continued to a function F over the whole complex plane via the formula $F(z) = 4\zeta(z)\beta(z)$ with just one singularity, a simple pole, at $z = 1$. So although the lattice sum $f(1/2) = \infty$, we have $\delta_2 = F(1/2) = -3.90026\ldots$; that is, the integral "plays no role" in the final answer.

In the same way, by starting with the function

$$g(z) = \sum_{i,j,k=-\infty}^{\infty}{}' \frac{1}{(i^2+j^2+k^2)^z}, \quad \mathrm{Re}(z) > \frac{3}{2},$$

g can be analytically continued to a function G that is analytic everywhere except for a simple pole at $z = 3/2$. Unlike the two-dimensional case, however, we here have [33]

$$\begin{aligned}\Delta(n,3) &= \sum_{i,j,k=-n}^{n}{}' \frac{1}{\sqrt{i^2+j^2+k^2}} - 12\left(-\frac{\pi}{6}+\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\right)\left(n+\frac{1}{2}\right)^2\\ &\to G\left(\frac{1}{2}\right)+\frac{\pi}{6}\\ &= -2.3136987039\ldots = \delta_3\end{aligned}$$

as $n \to \infty$; that is, here the integral *does* play a role and a "correction term" $\pi/6$ is needed. A fast expression for evaluating $G(1/2)$ is [20, 34]

$$\begin{aligned}G\left(\frac{1}{2}\right) &= \frac{7\pi}{6} - \frac{19}{2}\ln(2) + 4\sum_{m,n=1}^{\infty}\left[3+3(-1)^m+(-1)^{m+n}\right]\frac{\operatorname{csch}\left(\pi\sqrt{m^2+n^2}\right)}{\sqrt{m^2+n^2}}\\ &= -2.8372974794\ldots,\end{aligned}$$

which bears some similarity to Hautot's formula for M_3.

Now define, for any integer $p \ge 1$,

$$\gamma_p = \lim_{n\to\infty}\left(\sum_{i_1,i_2,\ldots,i_p=1}^{n}\frac{1}{\sqrt{i_1^2+i_2^2+\cdots+i_p^2}} - \int_{x_1,x_2,\ldots,x_p=1}^{n}\frac{dx_1dx_2\cdots dx_p}{\sqrt{x_1^2+x_2^2+\cdots+x_p^2}}\right).$$

Everyone knows that $\gamma_1 = \gamma$ is the Euler–Mascheroni constant [1.5], but comparatively few people know that [35–37]

$$\gamma_2 = \frac{1}{4}\left\{\delta_2 + 2\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - 4\gamma_1\right\} = -0.6709083078\ldots,$$

$$\gamma_3 = \frac{1}{8}\left\{\delta_3 + 3\left[-\frac{\pi}{6} + \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\right] + 12\gamma_2 - 6\gamma_1\right\} = 0.5817480456\ldots.$$

No one has computed the value of γ_p for any $p \geq 4$.

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1.11 Chaitin's Constant

Here is a brief discussion of algorithmic information theory [1–4]. Our perspective is number-theoretic and our treatment is informal: We will not attempt, for example, to define "computer" (Turing machine) here.

A **diophantine equation** involves a polynomial $p(x_1, x_2, \ldots, x_n)$ with integer coefficients. Hilbert's tenth problem asked for a general algorithm that could ascertain whether $p(x_1, x_2, \ldots, x_n) = 0$ has positive integer solutions $x_1, x_2, \ldots, x_n$, given arbitrary p. The work of Matiyasevic, Davis, Putnam, and Robinson [5] culminated in a proof that no such algorithm can exist. In fact, one can find a **universal diophantine equation** $P(N, x_1, x_2, \ldots, x_n) = 0$ such that, by varying the parameter N, the corresponding set D_N of solutions x can be any recursively enumerable set of positive integers. Equivalently, any set of positive integers x that could *possibly* be the output of a deterministic computer program *must* be D_N for some N. The existence of P is connected to Gödel's incompleteness theorem in mathematical logic and Turing's negative solution of the halting problem in computability theory.

Now, define a real number A in terms of its binary expansion $0.A_1A_2A_3\ldots$ as follows:

$$A_N = \begin{cases} 1 & \text{if } D_N \neq \emptyset, \\ 0 & \text{if } D_N = \emptyset. \end{cases}$$

There is no algorithm for deciding, given arbitrary N, whether $A_N = 1$ or 0, so A is an uncomputable real number. Is it possible to say more about A?

There is an interesting interplay between computability and randomness. We say that a real number z is **random** if the first N bits of z cannot be compressed into a program shorter than N bits. It follows that the successive bits of z cannot be distinguished from the result of independent tosses of a fair coin. The thought that randomness might occur in number theory staggers the imagination. No *computable* real number z is random [6, 7]. It turns out that A is not random either! We must look a little harder to find unpredictability in arithmetic.

An **exponential diophantine equation** involves a polynomial $q(x_1, x_2, \ldots, x_n)$ with integer coefficients as before, with the added freedom that there may be certain positive integers c and $1 \le i < j \le n$ for which $x_j = c^{x_i}$, and there may be certain $1 \le i \le j < k \le n$ for which $x_k = x_i^{x_j}$. That is, exponents are allowed to be variables as well. Starting with the work of Jones and Matiyasevic, Chaitin [6, 7] found an exponential diophantine equation $Q(N, x_1, x_2, \ldots, x_n) = 0$ with the following remarkable property. Let E_N denote the set of positive integer solutions x of $Q = 0$ for each N. Define a real number Ω in terms of $0.\Omega_1\Omega_2\Omega_3\ldots$ as follows:

$$\Omega_N = \begin{cases} 1 & \text{if } E_N \text{ is infinite,} \\ 0 & \text{if } E_N \text{ is finite.} \end{cases}$$

Then Ω is not merely uncomputable, but it is random too! So although the equation $P = 0$ gave us uncomputable A, the equation $Q = 0$ gives us random Ω; this provides our first glimpse of genuine uncertainity in mathematics [8–10].

Chaitin explicitly wrote down his equation $Q = 0$, which has 17000 variables and requires 200 pages for printing. The corresponding constant Ω is what we call **Chaitin's constant**. Other choices of the expression Q are possible and thus other random Ω exist. The basis for Chaitin's choice of Q is akin to Gödel numbering - Chaitin's modified LISP implementations make this very concrete - but the details are too elaborate to explain here.

Chaitin's constant is the halting probability of a certain self-delimiting universal computer. A different machine will, as before, usually give a different constant. So whereas Turing's fundamental result is that the halting *problem* is unsolvable, Chaitin's result is that the halting *probability* is random. We have a striking formula [2–4]:

$$\Omega = \sum_{\pi} 2^{-|\pi|},$$

the infinite sum being over all self-delimiting programs π that cause Chaitin's universal computer to eventually halt. Here $|\pi|$ denotes the length of π (thinking of programs as strings of bits).

It turns out that the first several bits of Chaitin's original Ω are known and all are ones thus far. This observation gives rise to some interesting philosophical developments. Assume that ZFC (Zermelo–Fraenkel set theory, coupled with the Axiom of Choice) is arithmetically sound. That is, assume any theorem of arithmetic proved by ZFC is true. Under this condition, there is an explicit finite bound on the number of bits of Ω that

ZFC can determine. Solovay [11, 12] dramatically constructed a worst-case machine U for which ZFC cannot calculate any bits of $\Omega(U)$ at all! Further, ZFC cannot predict more than the initial block of ones for any Chaitin constant Ω; although the k^{th} bit may be a zero in truth, this fact is unprovable in ZFC. As Calude [13] wrote, "As soon as you get a 0, it's all over". Solovay's Ω starts with a zero; hence it is unknowable. More recently, a procedure for computing the first 64 bits of such an Ω was implemented [14] via the construction of a non-Solovay machine V that satisfies $\Omega(V) = \Omega(U)$ but is more manageable than U.

It is also known that the set of computably enumerable, random reals coincides with the set of all halting probabilities Ω of Chaitin universal computers [15–17]. Is it possible to define a "simpler" random Ω whose description would not be so complicated as to strain credibility? The latter theorem states that all such numbers have a Diophantine representation $Q = 0$; whether we can significantly reduce the size of the equation remains an open question.

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2

Constants Associated with Number Theory

2.1 Hardy–Littlewood Constants

The sequence of prime numbers $2, 3, 5, 7, 11, 13, 17, \ldots$ has fascinated mathematicians for centuries. Consider, for example, the counting function

$$P_n = \sum_{p \le n} 1 = \text{the number of primes} \le n,$$

where the sum is over all primes p. We write $P_n(p) = P_n$, and the motivation behind this unusual notation will become clear momentarily. It was not until 1896 that Hadamard and de la Vallée Poussin (building upon the work of many) proved what is known as the **Prime Number Theorem**:

$$P_n(p) \sim \frac{n}{\ln(n)}$$

as $n \to \infty$. For every problem that has been solved in prime number theory, however, there are several that remain unsolved. Two of the most famous problems are the following:

Goldbach's Conjecture. *Every even number > 2 can be expressed as a sum of two primes.*

Twin Prime Conjecture. *There are infinitely many primes p such that $p + 2$ is also prime.*

The latter can be rewritten in the following way:

If $P_n(p, p + 2)$ is the number of twin primes with the lesser of the two $\le n$, then $\lim_{n\to\infty} P_n(p, p + 2) = \infty$.

Striking theoretical progress has been achieved toward proving these conjectures, but insurmountable gaps remain. We focus on certain heuristic formulas, developed by Hardy & Littlewood [1]. These formulas attempt to answer the following question: Putting aside the existence issue, what is the distribution of primes satisfying various additional constraints? In essence, one desires asymptotic distributional formulas analogous to that in the Prime Number Theorem.

Extended Twin Prime Conjecture [2–6].

$$P_n(p, p+2) \sim 2C_{\text{twin}} \frac{n}{\ln(n)^2},$$

where $C_{\text{twin}} = \prod_{p>2} \frac{p(p-2)}{(p-1)^2} = 0.6601618158\ldots = \frac{1}{2}(1.3203236316\ldots).$

Conjectures involving two different kinds of prime triples [2].

$$P_n(p, p+2, p+6) \sim P_n(p, p+4, p+6) \sim D\frac{n}{\ln(n)^3},$$

where $D = \frac{9}{2}\prod_{p>3} \frac{p^2(p-3)}{(p-1)^3} = 2.8582485957\ldots.$

Conjectures involving two different kinds of prime quadruples [2].

$$P_n(p, p+2, p+6, p+8) \sim \frac{1}{2}P_n(p, p+4, p+6, p+10) \sim E\frac{n}{\ln(n)^4},$$

where $E = \frac{27}{2}\prod_{p>3} \frac{p^3(p-4)}{(p-1)^4} = 4.1511808632\ldots.$

Conjecture involving primes of the form m^2+1 [3,4,7–9]. *If Q_n is defined to be the number of primes $p \le n$ satisfying $p = m^2+1$ for some integer m, then*

$$Q_n \sim 2C_{\text{quad}} \frac{\sqrt{n}}{\ln(n)},$$

where $C_{\text{quad}} = \frac{1}{2}\prod_{p>2}\left(1 - \frac{(-1)^{\frac{p-1}{2}}}{p-1}\right) = 0.6864067314\ldots = \frac{1}{2}(1.3728134628\ldots).$

Extended Goldbach Conjecture [3,4,10,11]. *If R_n is defined to be the number of representations of an even integer n as a sum of two primes (order counts), then*

$$R_n \sim 2C_{\text{twin}} \cdot \prod_{\substack{p>2 \\ p|n}} \frac{p-1}{p-2} \cdot \frac{n}{\ln(n)^2},$$

where the product is over all primes p dividing n.

It is intriguing that both the Extended Twin Prime Conjecture and the Extended Goldbach Conjecture involve the same constant C_{twin}. It is often said that the Goldbach conjecture is “conjugate” to the Twin Prime conjecture [12]. We talk about recent progress in estimating Q_n [2.1.1] and in estimating R_n [2.1.2]. Shah & Wilson [13] extensively tested the asymptotic formula for R_n; thus C_{twin} is sometimes called the Shah–Wilson constant [14]. A formula for computing C_{twin} is given in [2.4].

The Hardy–Littlewood constants discussed here all involve infinite products over primes. Other such products occur in our essays on the Landau–Ramanujan constant [2.3], Artin's constant [2.4], the Hafner–Sarnak–McCurley constant [2.5], Bateman–Grosswald constants [2.6.1], Euler totient constants [2.7], and Pell–Stevenhagen constants [2.8].

Riesel [2] discussed prime constellations, which generalize prime triples and quadruples, and demonstrated how one computes the corresponding Hardy–Littlewood constants. He emphasized the remarkable fact that, although we do not know the sequence of primes in its entirety, we can compute Hardy–Littlewood constants to *any decimal accuracy* due to a certain transformation in terms of Riemann's zeta function $\zeta(x)$ [1.6].

There is a cubic analog [2.1.3] of the conjecture for prime values taken by the preceding quadratic polynomial. Incidently, if we perturb the product $2C_{\text{quad}}$ only slightly, we obtain a closed-form expression:

$$\prod_{p>2}\left(1-\frac{(-1)^{\frac{p-1}{2}}}{p}\right)=\frac{4}{\pi}=\frac{1}{\beta(1)},$$

where $\beta(x)$ is Dirichlet's beta function [1.7].

Mertens' well-known formula gives [2.2]

$$\lim_{n\to\infty}\frac{1}{\ln(n)}\prod_{2<p<n}\frac{p}{p-1}=\frac{1}{2}e^{\gamma}=0.8905362089\ldots,$$

where γ is the Euler–Mascheroni constant [1.5]. Here is a less famous result [15–17]:

$$\lim_{n\to\infty}\frac{1}{\ln(n)^2}\prod_{2<p<n}\frac{p}{p-2}=\frac{1}{4C_{\text{twin}}}e^{2\gamma}=1.2013035599\ldots=\frac{1}{0.8324290656\ldots}.$$

Here also is an extension of $C_{\text{twin}}=C_2$ introduced by Hardy & Littlewood [16–20]:

$$C_n=\prod_{p>n}\left(\frac{p}{p-1}\right)^{n-1}\frac{p-n}{p-1}=\prod_{p>n}\left(1-\frac{1}{p}\right)^{-n}\left(1-\frac{n}{p}\right),$$

for which $C_3=0.6351663546\ldots=2D/9$, $C_4=0.3074948787\ldots=2E/27$, $C_5=0.4098748850\ldots$, $C_6=0.1866142973\ldots$, and $C_7=0.3694375103\ldots$.

In a study of Waring's problem, Bateman & Stemmler [21–24] examined the conjecture

$$P_n(p,p^2+p+1)\sim H\frac{n}{\ln(n)^2},$$

where

$$H=\frac{1}{2}\prod_{p}\left(1-\frac{1}{p}\right)^{-2}\left(1-\frac{2+\chi(p)}{p}\right)=1.5217315350\ldots=2\cdot 0.7608657675\ldots$$

and $\chi(p)=-1,0,1$ accordingly as $p\equiv -1,0,1 \bmod 3$, respectively. See also [25–28].

We give two problems vaguely related to Goldbach's conjecture. Let $f(n)$ denote the number of representations of n as the sum of one or more *consecutive* primes.

For example, $f(41) = 3$ since $41 = 11 + 13 + 17 = 2 + 3 + 5 + 7 + 11 + 13$. Moser [29] proved that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n) = \ln(2) = 0.6931471805\ldots.$$

Let $g(n)$ denote the number of integers not exceeding n that can be represented as a sum of a prime and a power of 2. Romani [30] numerically investigated the ratio $g(n)/n$ and concluded that the asymptotic density of such integers is $0.434\ldots$.

2.1.1 Primes Represented by Quadratics

We defined Q_n earlier. Let $\tilde{Q}_n$ be the number of positive integers $k \le n$ having ≤ 2 prime factors and satisfying $k = m^2 + 1$ for some integer m. Hardy & Littlewood's conjecture regarding the limiting behavior of Q_n remains unproven; some supporting numerical work appeared long ago [31,32]. Iwaniec, however, recently demonstrated the asymptotic inequality [4,33]

$$\tilde{Q}_n > \frac{1}{77} \cdot 2C_{\text{quad}} \cdot \frac{\sqrt{n}}{\ln(n)} = 0.0178\ldots \cdot \frac{\sqrt{n}}{\ln(n)},$$

which shows that there are infinitely many **almost primes** of the required form. His results extend to any irreducible quadratic polynomial $am^2 + bm + c$ with $a > 0$ and c odd. A good upper bound on Q_n does not seem to be known.

Shanks [32] mentioned a formula

$$C_{\text{quad}} = \frac{3}{4G}\frac{\zeta(6)}{\zeta(3)} \prod_{p\equiv 1 \bmod 4} \left(1 + \frac{2}{p^3 - 1}\right)\left(1 - \frac{2}{p(p-1)^2}\right),$$

where $G = \beta(2)$ is Catalan's constant [1.7]. He added that more rapid convergence may be obtained by multiplying through by the identity

$$1 = \frac{17}{16}\frac{\zeta(8)}{\zeta(4)\beta(4)} \prod_{p\equiv 1 \bmod 4} \left(1 + \frac{2}{p^4 - 1}\right).$$

2.1.2 Goldbach's Conjecture

Some progress has been made recently in proving Goldbach's conjecture, that is, in turning someone's guess into a theorem. Here are both binary and ternary versions:

Conjecture G. *Every even integer > 2 can be expressed as a sum of two primes.*

Conjecture G′. *Every odd integer > 5 can be expressed as a sum of three primes.*

Note that if G is true, then G′ is true. Here are the corresponding asymptotic versions:

Conjecture AG. *There exists N so large that every even integer $> N$ can be expressed as a sum of two primes.*

Conjecture AG′. *There exists N' so large that every odd integer $> N'$ can be expressed as a sum of three primes.*

The circle method of Hardy & Littlewood [1] led Vinogradov [34] to prove that AG′ is true; moreover, he showed that

$$S_n \sim \prod_p \left(1 + \frac{1}{(p-1)^3}\right) \cdot \prod_{\substack{p>2 \\ p|n}} \left(1 - \frac{1}{p^2 - 3p + 3}\right) \cdot \frac{n^2}{2\ln(n)^3},$$

where S_n is the number of representations of the large odd integer n as a sum of three primes. Observe that this is not a conjecture, but a theorem. Further, Borodzkin [35] showed that **Vinogradov's number** N' could be taken to be $3^{3^{15}} \approx 10^{7000000}$ and Chen & Wang [36, 37] improved this to 10^{7194}. It is not possible with today's technology to check all odd integers up to this threshold and hence deduce G′. But by *assuming* the truth of a generalized Riemann hypothesis, the number N' was reduced to 10^{20} by Zinoviev [38], and Saouter [39] and Deshouillers et al. [40] successfully diminished N' to 5. Therefore G′ is true, subject to the truth of a generalized Riemann hypothesis.

We do not have any analogous conditional proof for AG or for G. Here are two known weakenings of these:

Theorem (Ramaré [41, 42]). *Every even integer can be expressed as a sum of six or fewer primes (in other words,* ***Schnirelmann's number*** *is ≤ 6).*

Theorem (Chen [11, 12, 43, 44]). *Every sufficiently large even integer can be expressed as a sum of a prime and a positive integer having ≤ 2 prime factors.*

In fact, Chen proved the asymptotic inequality

$$\tilde{R}_n > 0.67 \cdot \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \cdot \prod_{\substack{p>2 \\ p|n}} \frac{p-1}{p-2} \cdot \frac{n}{\ln(n)^2},$$

where $\tilde{R}_n$ is the number of corresponding representations. Chen also proved that there are infinitely many primes p such that $p+2$ is an almost prime, a weakening of the twin prime conjecture, and the same coefficient 0.67 appears.

Here are additional details on these results. Kaniecki [45] proved that every odd integer can be expressed as a sum of at most five primes, under the condition that the Riemann hypothesis is true. With a large amount of computation, this will eventually be improved to at most four primes. By way of contrast, Ramaré's result that every even integer is a sum of at most six primes is unconditional (not dependent on the Riemann hypothesis).

Vinogradov's result may be rewritten as

$$\begin{aligned}\liminf_{n\to\infty} \frac{\ln(n)^3}{n^2} S_n &= \frac{1}{2} \prod_p \left(1 + \frac{1}{(p-1)^3}\right) \cdot \prod_{p>2} \left(1 - \frac{1}{p^2 - 3p + 3}\right) = C_{\text{twin}} \\ &= 0.6601618158\ldots,\end{aligned}$$

$$\limsup_{n\to\infty} \frac{\ln(n)^3}{n^2} S_n = \frac{1}{2} \prod_p \left(1 + \frac{1}{(p-1)^3}\right) = 1.1504807723\ldots.$$

That is, although $S(n)$ is asymptotically misbehaved, its growth remains within the same order of magnitude. This cannot be said for Chen's result:

$$\liminf_{n\to\infty} \frac{\ln(n)^2}{n} \tilde{R}_n > 0.67 \cdot C_{\text{twin}} = 0.44,$$

$$\limsup_{n\to\infty} \frac{\ln(n)}{n} \tilde{R}_n > 0.67 \cdot \frac{1}{2} e^{\gamma} = 0.59.$$

Note that the limit superior bound grows at a logarithmic factor faster than the limit inferior bound. We have made use of Mertens' formulas in obtaining these expressions.

Chen's coefficient 0.67 for the Goldbach conjecture [43] was replaced by 0.81 in [46] and by 2 in [11]. His inequality for the twin prime conjecture can likewise be improved; the sharpenings in this case include 1.42 in [47], 1.94 in [48], 2.03 in [49], and 2.1 in [50].

Chen [51], building upon [52–54], proved the upper bound

$$R_n \le 7.8342 \cdot \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \cdot \prod_{\substack{p>2 \\ p|n}} \frac{p-1}{p-2} \cdot \frac{n}{\ln(n)^2}.$$

Pan [55] gave a simpler proof but a weaker result with coefficient 7.9880. Improvements on the corresponding coefficient 7.8342 for twin primes include 7.8156 in [56], 7.5555 in [57], 7.5294 in [58], 7 in [59], 6.9075 in [47], 6.8354 in [50], and 6.8325 in [60]. (A claimed upper bound of 6.26, mentioned in [3] and in the review of [50], was incorrect.)

Most of the sharpenings for twin primes are based on [59], which does *not* apply to the Goldbach conjecture for complicated reasons.

There is also a sense in which the set of possible counterexamples to Goldbach's conjecture must be small [61–66]. The number $\varepsilon(n)$ of positive even integers $\le n$ that are *not* sums of two primes provably satisfies $\varepsilon(n) = o\left(n^{0.914}\right)$ as $n \to \infty$. Of course, we expect $\varepsilon(n) = 1$ for all $n \ge 2$. See also [67–69].

2.1.3 Primes Represented by Cubics

Hardy & Littlewood [1] conjectured that there exist infinitely many primes of the form $m^3 + k$, where the fixed integer k is not a cube. Further, if T_n is defined to be the number of primes $p \le n$ satisfying $p = m^3 + 2$ for some integer m, then

$$\lim_{n\to\infty} \frac{\ln(n)}{\sqrt[3]{n}} T_n = A = \prod_{p\equiv 1 \bmod 6} \frac{p-\alpha(p)}{p-1} = 1.2985395575\ldots,$$

where

$$\alpha(p) = \begin{cases} 3 & \text{if 2 is a cubic residue mod } p \text{ (i.e., if } x^3 \equiv 2 \bmod p \text{ is solvable)}, \\ 0 & \text{otherwise.} \end{cases}$$

Likewise, if U_n is defined to be the number of primes $p \le n$ satisfying $p = m^3 + 3$ for some integer m, then

$$\lim_{n\to\infty} \frac{\ln(n)}{\sqrt[3]{n}} U_n = B = \prod_{p\equiv 1 \bmod 6} \frac{p - \beta(p)}{p-1} = 1.3905439387\ldots,$$

where

$$\beta(p) = \begin{cases} 3 & \text{if 3 is a cubic residue mod } p \text{ (i.e., if } x^3 \equiv 3 \bmod p \text{ is solvable)}, \\ 0 & \text{otherwise.} \end{cases}$$

The constants A and B are known as **Bateman's constants** and were first computed to high precision by Shanks & Lal [3, 22, 70, 71].

Here is an example involving a quartic [72]. If V_n is defined to be the number of primes $p \le n$ satisfying $p = m^4 + 1$ for some integer m, then

$$\lim_{n\to\infty} \frac{\ln(n)}{\sqrt[4]{n}} V_n = 4I = 2.6789638796\ldots,$$

where

$$I = \frac{\pi^2}{16\ln(1+\sqrt{2})} \prod_{p\equiv 1 \bmod 8} \left(1 - \frac{4}{p}\right)\left(\frac{p+1}{p-1}\right)^2 = 0.6697409699\ldots.$$

It seems appropriate to call this **Shanks' constant**. Similar estimates for primes of the form $m^5 + 2$ or $m^5 + 3$ evidently do not appear in the literature.

The Bateman–Horn conjecture [3, 21, 73] extends this theory to polynomials of arbitrary degree. It also applies in circumstances when several such polynomials must simultaneously be prime. For example [74–77], if F_n is defined to be the number of prime pairs of the form $(m-1)^2 + 1$ and $(m+1)^2 + 1$ with the lesser of the two $\le n$, then

$$\lim_{n\to\infty} \frac{\ln(n)^2}{\sqrt{n}} F_n = 4J = 1.9504911124\ldots,$$

where

$$J = \frac{\pi^2}{8} \prod_{p\equiv 1 \bmod 4} \left(1 - \frac{4}{p}\right)\left(\frac{p+1}{p-1}\right)^2 = 0.4876227781\ldots.$$

Note that F_n is also the number of Gaussian twin primes $(m - 1 + i, m + 1 + i)$ situated on the line $x + i$ in the complex plane; hence J might be called the **Gaussian twin prime constant**. (These are *not* all Gaussian twin primes in the plane: On the line $x + 2i$, consider $m = 179984$.)

As another example, if G_n is defined to be the number of prime pairs of the form $(m-1)^4 + 1$ and $(m+1)^4 + 1$ with the lesser of the two $\le n$, then

$$\lim_{n\to\infty} \frac{\ln(n)^2}{\sqrt[4]{n}} G_n = 16K = 12.6753318106\ldots,$$

where

$$K = 2I^2 \prod_{p \equiv 1 \bmod 8} \frac{p(p-8)}{(p-4)^2} = 0.7922082381\ldots.$$

The latter is known as **Lal's constant**. Sebah [77] computed this and many of the constants in this essay.

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2.2 Meissel–Mertens Constants

All of the infinite series discussed here and in [2.14] involve reciprocals of the prime numbers 2, 3, 5, 7, 11, 13, 17, ... The sum of the reciprocals of all primes is divergent and, in fact [1–6],

$$\lim_{n\to\infty}\left(\sum_{p\le n}\frac{1}{p}-\ln(\ln(n))\right)=M=\gamma+\sum_{p}\left[\ln\left(1-\frac{1}{p}\right)+\frac{1}{p}\right]=0.2614972128\ldots,$$

where both sums are over all primes p and where γ is Euler's constant [1.5]. According to [7,8], the definition of M was confirmed to be valid by Meissel in 1866 and independently by Mertens in 1874. The quantity M is sometimes called Kronecker's constant [9] or the prime reciprocal constant [10]. A rapidly convergent series for M is [11–13]

$$M=\gamma+\sum_{k=2}^{\infty}\frac{\mu(k)}{k}\ln(\zeta(k)),$$

where $\zeta(k)$ is the Riemann zeta function [1.6] and $\mu(k)$ is the Möbius mu function

$$\mu(k)=\begin{cases}1 & \text{if } k=1,\\ (-1)^r & \text{if } k \text{ is a product of } r \text{ distinct primes},\\ 0 & \text{if } k \text{ is divisible by a square } >1.\end{cases}$$

If $\omega(n)$ denotes the number of *distinct* prime factors of an arbitrary integer n, then interestingly the average value of $\omega(1)$, $\omega(2)$, ..., $\omega(n)$:

$$\mathrm{E}_n(\omega)=\frac{1}{n}\sum_{k=1}^{n}\omega(k)$$

can be expressed asymptotically via the formula [2,9,14–16]

$$\lim_{n\to\infty}\left(\mathrm{E}_n(\omega)-\ln(\ln(n))\right)=M.$$

A somewhat larger average value for the *total* number, $\Omega(n)$, of prime factors of n (repeated factors counted) is as follows:

$$\begin{aligned}M'&=\lim_{n\to\infty}\left(\mathrm{E}_n(\Omega)-\ln(\ln(n))\right)=M+\sum_{p}\frac{1}{p(p-1)}\\&=\gamma+\sum_{p}\left[\ln\left(1-\frac{1}{p}\right)+\frac{1}{p-1}\right]=\gamma+\sum_{k=2}^{\infty}\frac{\varphi(k)}{k}\ln(\zeta(k))\\&=1.0346538818\ldots,\end{aligned}$$

where $\varphi(k)$ is the Euler totient function [2.7]. A related limit [1,17] is

$$\lim_{n\to\infty}\left(\sum_{p\le n}\frac{\ln(p)}{p}-\ln(n)\right)=-M''=-\gamma-\sum_{p}\frac{\ln(p)}{p(p-1)}=-1.3325822757\ldots,$$

and a fast way to compute M'' uses the series [18]

$$M'' = \gamma + \sum_{k=2}^{\infty} \mu(k) \frac{\zeta'(k)}{\zeta(k)}.$$

Dirichlet's famous theorem states that if a and b are coprime positive integers then there exist infinitely many prime numbers of the form $a + bl$. What can be said about the sum of the reciprocals of all such primes? The limit

$$m_{a,b} = \lim_{n \to \infty} \left(\sum_{\substack{p \le n \\ p \equiv a \bmod b}} \frac{1}{p} - \frac{1}{\varphi(b)} \ln(\ln(n)) \right)$$

can be shown to exist and is finite for each a and b. For example [19–23],

$$m_{1,4} = \ln\left(\frac{\sqrt{\pi}}{4K}\right) + \frac{\gamma}{2} + \sum_{p \equiv 1 \bmod 4} \left[\ln\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right] = -0.2867420562\ldots,$$

$$m_{3,4} = \ln\left(\frac{2K}{\sqrt{\pi}}\right) + \frac{\gamma}{2} + \sum_{p \equiv 3 \bmod 4} \left[\ln\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right] = 0.0482392690\ldots,$$

where K is the Landau–Ramanujan constant [2.3]. Of course, $m_{1,4} + m_{3,4} + 1/2 = M$.

The sum of the squared reciprocals of primes is

$$N = \sum_p \frac{1}{p^2} = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln(\zeta(2k)) = 0.4522474200\ldots,$$

which is connected to the variance of $\omega(1), \omega(2), \ldots, \omega(n)$:

$$\mathrm{Var}_n(\omega) = \mathrm{E}_n(\omega^2) - \mathrm{E}_n(\omega)^2$$

via the formula [9, 14]

$$\lim_{n \to \infty} (\mathrm{Var}_n(\omega) - \ln(\ln(n))) = M - N - \pi^2/6 = -1.8356842740\ldots.$$

Likewise,

$$N' = \sum_p \frac{1}{(p-1)^2} = 1.3750649947\ldots$$

appears in the following:

$$\lim_{n \to \infty} (\mathrm{Var}_n(\Omega) - \ln(\ln(n))) = M' + N' - \pi^2/6 = 0.7647848097\ldots.$$

See [15, 24] for detailed accounts of evaluating N and N' and [25–27] for the asymptotic probability distributions of ω and Ω.

Given a positive integer n, let $D_n = \max\{d : d^2 | n\}$. Define S to be the set of n such that D_n is prime, and define $\tilde{S}$ to be the set of $n \in S$ such that $D_n^3 \nmid n$. The asymptotic densities of S and $\tilde{S}$ are, respectively [28–30],

$$\frac{6}{\pi^2} \sum_p \frac{1}{p^2} = 0.2749334633\ldots, \quad \frac{6}{\pi^2} \sum_p \frac{1}{p(p+1)} = 0.2007557220\ldots.$$

In words, S is the set of integers, each of whose prime factors are simple with exactly one exception; in $\tilde{S}$, the exception must be a prime squared. See related discussions of square-free sets [2.5] and square-full sets [2.6].

Bach [12] estimated the computational complexity of calculating M, as well as Artin's constant C_{Artin} [2.4] and the twin prime constant C_{twin} [2.1].

The alternating series

$$\sum_{k=1}^{\infty}(-1)^k \frac{1}{p_k} = -0.2696063519\ldots,$$

where $p_1 = 2,\ p_2 = 3,\ p_3 = 5, \ldots$, is clearly convergent [31]. This is perhaps not so interesting as the two non-alternating series [32–35]

$$\sum_{k=2}^{\infty} \varepsilon_k \frac{1}{p_k} = 0.3349813253\ldots, \quad \sum_{k=1}^{\infty} \varepsilon'_k \frac{1}{p_k} = 0.6419448385\ldots,$$

where

$$\varepsilon_k = \begin{cases} -1 & \text{if } p_k \equiv 1 \bmod 4, \\ 1 & \text{if } p_k \equiv 3 \bmod 4, \end{cases} \quad \varepsilon'_k = \begin{cases} -1 & \text{if } p_k \equiv 1 \bmod 3, \\ 1 & \text{if } p_k \equiv 2 \bmod 3, \\ 0 & \text{if } p_k \equiv 0 \bmod 3. \end{cases}$$

Of course, the following is also convergent [36]:

$$\sum_{k=2}^{\infty} \varepsilon_k \frac{1}{p_k^2} = 0.0946198928\ldots.$$

Erdös [37, 38] wondered if the same is true for the series $\sum_{k=1}^{\infty}(-1)^k k/p_k$.

Merrifield [39] and Lienard [40] tabulated values of the series $\sum_p p^{-n}$ for $2 \le n \le 167$, as well as M and $\gamma - M = 0.3157184521\ldots$.

2.2.1 Quadratic Residues

Let $f(p)$ denote the smallest positive quadratic nonresidue modulo p, where p is prime. The average value of $f(p)$ is [41, 42]

$$\lim_{n\to\infty} \frac{\sum\limits_{p\le n} f(p)}{\sum\limits_{p\le n} 1} = \lim_{n\to\infty} \frac{\ln(n)}{n} \sum_{p\le n} f(p) = \sum_{k=1}^{\infty} \frac{p_k}{2^k} = 3.6746439660\ldots.$$

More generally, if m is odd, let $f(m)$ denote the least positive integer k for which the Jacobi symbol $(k/m) < 1$, where m is nonsquare, and $f(m) = 0$ if m is square. (If $(k/m) = -1$, for example, then k is a quadratic nonresidue modulo m.) The average value of $f(m)$ is [41, 43, 44]

$$\lim_{n\to\infty} \frac{2}{n} \sum_{\substack{m\le n \\ m \text{ odd}}} f(m) = 1 + \sum_{j=2}^{\infty} \frac{p_j + 1}{2^{j-1}} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i}\right) = 3.1477551485\ldots.$$

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2.3 Landau–Ramanujan Constant

Let $B(x)$ denote the number of positive integers not exceeding x that can be expressed as a sum of two integer squares. Clearly $B(x) \to \infty$ as $x \to \infty$, but the rate at which it does so is quite fascinating!

Landau [1–3] and Ramanujan [4, 5] independently proved that the following limit exists:

$$\lim_{x\to\infty} \frac{\sqrt{\ln(x)}}{x} B(x) = K,$$

where K is the remarkable constant

$$K = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 \bmod 4} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}} = \frac{\pi}{4} \prod_{p \equiv 1 \bmod 4} \left(1 - \frac{1}{p^2}\right)^{\frac{1}{2}}$$

and the two products are restricted to primes p. An empirical confirmation of this limit is found in [6]. Shanks [7, 8] discovered a rapidly convergent expression for K:

$$K = \frac{1}{\sqrt{2}} \prod_{k=1}^{\infty} \left[\left(1 - \frac{1}{2^{2^k}}\right) \frac{\zeta(2^k)}{\beta(2^k)}\right]^{\frac{1}{2^{k+1}}} = 0.7642236535\ldots,$$

where $\zeta(x)$ is the Riemann zeta function [1.6] and $\beta(x)$ is the Dirichlet beta function [1.7]. A stronger conclusion, due to Landau, is that

$$\lim_{x \to \infty} \frac{\ln(x)^{\frac{3}{2}}}{Kx} \left(B(x) - \frac{Kx}{\sqrt{\ln(x)}}\right) = C,$$

where C is given by [7, 9–12]

$$\begin{aligned} C &= \frac{1}{2} + \frac{\ln(2)}{4} - \frac{\gamma}{4} - \frac{\beta'(1)}{4\beta(1)} + \frac{1}{4}\frac{d}{ds} \ln\left(\prod_{p \equiv 3 \bmod 4} \left(1 - \frac{1}{p^{2s}}\right)\right)\Bigg|_{s=1} \\ &= \frac{1}{2}\left(1 - \ln\left(\frac{\pi e^{\gamma}}{2L}\right)\right) - \frac{1}{4}\sum_{k=1}^{\infty}\left(\frac{\zeta'(2^k)}{\zeta(2^k)} - \frac{\beta'(2^k)}{\beta(2^k)} + \frac{\ln(2)}{2^{2^k} - 1}\right) \\ &= 0.5819486593\ldots, \end{aligned}$$

γ is Euler's constant [1.5], and $L = 2.6220575542\ldots$ is Gauss' lemniscate constant [6.1]. These formulas were the basis for several recent high-precision computations by Flajolet & Vardi, Zimmermann, Adamchik, Golden & Gosper, MacLeod, and Hare.

2.3.1 Variations

Here are some variations. Define K_n to be the analog of K when counting positive integers of the form $a^2 + nb^2$. Clearly $K = K_1$. Define C_n likewise. It can be proved that [10, 13–16]

$$K_2 = \frac{1}{\sqrt[4]{2}} \prod_{p \equiv 5 \text{ or } 7 \bmod 8} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}} = 0.8728875581\ldots,$$

$$K_3 = \frac{1}{\sqrt{2\sqrt{3}}} \prod_{p \equiv 2 \bmod 3} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}} = 0.6389094054\ldots,$$

$$K_4 = \tfrac{3}{4}K = 0.5731677401\ldots, \quad C_4 = C = 0.5819486593\ldots.$$

Moree & te Riele [17] recently computed $C_3 = 0.5767761224\ldots$, but no one has yet found the value of C_n for $n = 2$ or $n > 4$. In the case $n = 3$, counting positive integers of the form $a^2 + 3b^2$ is equivalent to counting those of the form $a^2 + ab + b^2$.

Define instead $K_{l,m}$ to be the analog of K when counting positive integers simultaneously of the form $a^2 + b^2$ and $lc + m$, where l and m are coprime. Here, $K_{l,m}$ is simply a rational multiple of K depending on l only [18, 19].

Here are more variations. Let $B_{\text{sqfr}}(x)$ be the number of positive square-free integers not exceeding x that can be expressed as a sum of two squares. Also, let $B_{\text{copr}}(x)$ be the number of positive integers not exceeding x that can be expressed as a sum of two coprime squares. It can be proved that [20–22]

$$\lim_{x\to\infty} \frac{\sqrt{\ln(x)}}{x} B_{\text{sqfr}}(x) = \frac{6K}{\pi^2} = 0.4645922709\ldots,$$

$$\lim_{x\to\infty} \frac{\sqrt{\ln(x)}}{x} B_{\text{copr}}(x) = \frac{3}{8K} = 0.4906940504\ldots.$$

A conclusion from the first limit is that being square-free and being a sum of two squares are asymptotically independent properties. Of course, the two squares must be coprime; otherwise the sum could not be square-free.

Dividing the first expression by the second expression, we obtain that the asymptotic relative density of the first set as a subset of the second set is [22]

$$\lim_{x\to\infty} \frac{B_{\text{sqfr}}(x)}{B_{\text{copr}}(x)} = \frac{16K^2}{\pi^2} = \prod_{p\equiv 1 \bmod 4} \left(1 - \frac{1}{p^2}\right) = 0.9468064072\ldots.$$

This is a large density! On the one hand, if we randomly select two coprime integers, square them, and then add them, the sum is very likely to be square-free. On the other hand, there are infinitely many counterexamples: Consider, for example, the primitive Pythagorean triples [5.2].

Let $B_j(x)$ be the number of positive integers up to x, all of whose prime factors are congruent to j modulo 4, where $j = 1$ or 3. It can be shown that [20, 21, 23, 24]

$$\lim_{x\to\infty} \frac{\sqrt{\ln(x)}}{x} B_1(x) = \frac{1}{4K} = 0.3271293669\ldots,$$

$$\lim_{x\to\infty} \frac{\sqrt{\ln(x)}}{x} B_3(x) = \frac{2K}{\pi} = 0.4865198884\ldots.$$

It is interesting that these are not equal! This is a manifestation of the **Chebyshev effect** described by Rubenstein & Sarnak [25]. See [2.8] for a related discussion.

We mention two limits discovered by Uchiyama [26]:

$$\lim_{x\to\infty} \sqrt{\ln(x)} \prod_{\substack{p\le x \\ p\equiv 1 \bmod 4}} \left(1 - \frac{1}{p}\right) = \frac{4}{\sqrt{\pi}} \exp\left(-\frac{\gamma}{2}\right) K = 1.2923041571\ldots,$$

$$\lim_{x\to\infty} \sqrt{\ln(x)} \prod_{\substack{p\le x \\ p\equiv 3 \bmod 4}} \left(1 - \frac{1}{p}\right) = \frac{\sqrt{\pi}}{2} \exp\left(-\frac{\gamma}{2}\right) \frac{1}{K} = 0.8689277682\ldots,$$

which when multiplied together give Mertens' famous theorem [2.2]. Extensions of these results appear in [27–29]. As corollaries, we have

$$\lim_{x\to\infty} \frac{1}{\sqrt{\ln(x)}} \prod_{\substack{p\le x \\ p\equiv 1 \bmod 4}} \left(1+\frac{1}{p}\right) = \frac{4}{\pi^{\frac{3}{2}}} \exp\left(\frac{\gamma}{2}\right) K = 0.7326498193\ldots,$$

$$\lim_{x\to\infty} \frac{1}{\sqrt{\ln(x)}} \prod_{\substack{p\le x \\ p\equiv 3 \bmod 4}} \left(1+\frac{1}{p}\right) = \frac{1}{\sqrt{\pi}} \exp\left(\frac{\gamma}{2}\right) \frac{1}{K} = 0.9852475810\ldots.$$

Here are formulas that complement the expression for $16K/\pi^2$ earlier:

$$\prod_{p\equiv 3 \bmod 4} \left(1-\frac{1}{p^2}\right) = \frac{1}{2K^2} = 0.8561089817\ldots,$$

$$\prod_{p\equiv 1 \bmod 4} \left(1+\frac{1}{p^2}\right) = \frac{192K^2G}{\pi^4} = 1.0544399448\ldots,$$

$$\prod_{p\equiv 3 \bmod 4} \left(1+\frac{1}{p^2}\right) = \frac{\pi^2}{16K^2G} = 1.1530805616\ldots,$$

where $G = \beta(2)$ denotes Catalan's constant [1.7]. A similar expression emerges when dealing with the following situation. Let $\hat{B}(x)$ be the number of positive square-free integers that belong to the sequence n^2+1 with $1 \le n \le x$. Then [30, 31]

$$\lim_{x\to\infty} \frac{\hat{B}(x)}{x} = \prod_{p\equiv 1 \bmod 4} \left(1-\frac{2}{p^2}\right) = 0.8948412245\ldots.$$

Vast generalizations of this result are described in [32–34].

Let $\tilde{B}(x)$ denote the number of positive integers n not exceeding x for which n^2 *cannot* be expressed as a sum of two *distinct nonzero* squares. Shanks [35, 36] called these **non-hypotenuse numbers**, proved that

$$\tilde{K} = \lim_{x\to\infty} \frac{\sqrt{\ln(x)}}{x} \tilde{B}(x) = \frac{4K}{\pi} = 0.9730397768\ldots,$$

$$\lim_{x\to\infty} \frac{\ln(x)^{\frac{3}{2}}}{\tilde{K}x} \left(\tilde{B}(x) - \frac{\tilde{K}x}{\sqrt{\ln(x)}}\right) = C + \frac{1}{2}\ln\left(\frac{\pi e^{\gamma}}{2L^2}\right) = 0.7047534517\ldots,$$

and also mentioned that a third-order term is known to be positive (but did not compute this).

Let $A(x)$ denote the number of *primes* not exceeding x that can be expressed as a sum of two squares. Since odd primes of the form a^2+b^2 are precisely those that are 1 modulo 4, we have

$$\lim_{x\to\infty} \frac{\ln(x)}{x} A(x) = \frac{1}{2}.$$

Define $U(x)$ to be the number of primes not exceeding x that can be expressed in the form a^2+b^4. Friedlander & Iwaniec [37,38] proved that

$$\lim_{x\to\infty}\frac{\ln(x)}{x^{\frac{3}{4}}}U(x)=\frac{4L}{3\pi}=1.1128357889\ldots.$$

By coincidence, the constant L appeared in the second-order approximation of $B(x)$ as well. Drawing inspiration from this achievement, Heath-Brown [39] recently proved an analogous result for primes of the form a^3+2b^3.

Let $V(x)$ be the number of positive integers not exceeding x that can be expressed in the form a^2+b^4. It turns out that for almost all integers, the required representation is unique; hence a formula in [38] is applicable and

$$\lim_{x\to\infty}x^{-\frac{3}{4}}V(x)=\frac{L}{3}=0.8740191847\ldots.$$

The corresponding asymptotics for positive integers of the form a^3+2b^3 would be good to see. Related material appears in [40,41].

Let $Q(x)$ be the number of positive integers not exceeding x that can be expressed as a sum of three squares. Landau [1] proved that $Q(x)/x\to 5/6$ as $x\to\infty$. The error term $\Delta(x)=Q(x)-5x/6$ is not well behaved asymptotically [42–44], in the sense that

$$0=\liminf_{x\to\infty}\Delta(x)<\limsup_{x\to\infty}\Delta(x)=\frac{1}{3\ln(2)}.$$

The average value of $\Delta(x)$ can be precisely quantified in terms of a periodic, continuous, nowhere-differentiable function. More about such formulation is found in [2.16]. The asymptotics for counts of x of the form $a^3+b^3+c^3$ or $a^4+b^4+c^4+d^4$ remain open [45].

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2.4 Artin's Constant

Fermat's Little Theorem says that if p is a prime and n is an integer not divisible by p, then $n^{p-1} - 1$ is divisible by p.

Consider now the set of all positive integers e such that $n^e - 1$ is divisible by p. If $e = p - 1$ is the smallest such positive integer, then n is called a **primitive root modulo** p.

For example, 6 is a primitive root mod 11 since none of the remainders of $6^1, 6^2, 6^3, \ldots, 6^9$ upon division by 11 are equal to 1; thus $e = 10 = 11 - 1$. However, 6 is not a primitive root mod 19 since $6^9 - 1$ is divisible by 19 and $e = 9 < 19 - 1$.

Here is an alternative, more algebraic phrasing. The set $Z_p = \{0, 1, 2, \ldots, p - 1\}$ with addition and multiplication mod p forms a field. Further, the subset $U_p = \{1, 2, \ldots, p - 1\}$ with multiplication mod p forms a cyclic group. Hence we see that the integer n (more precisely, its residue class mod p) is a primitive root mod p if and only if n is a generator of the group U_p.

Here is another interpretation. Let $p > 5$ be a prime. The decimal expansion of the fraction $1/p$ has maximal period ($= p - 1$) if and only if 10 is a primitive root modulo p. Primes satisfying this condition are also known as **long primes** [1–4].

Artin [5] conjectured in 1927 that if $n \neq -1, 0, 1$ is not an integer square, then the set $S(n)$ of all primes for which n is a primitive root must be infinite. Some remarkable progress toward proving this conjecture is indicated in [6–9]. For example, it is known that at least one of the sets $S(2)$, $S(3)$, or $S(5)$ is infinite.

Suppose additionally that n is not an r^{th} integer power for any $r > 1$. Let n' denote the square-free part of n, equivalently, the divisor of n that is the outcome after all factors of the form d^2 have been eliminated. Artin further conjectured that the density of the set $S(n)$, relative to the primes, exists and equals

$$C_{\text{Artin}} = \prod_p \left(1 - \frac{1}{p(p-1)}\right) = 0.3739558136\ldots$$

independently of the choice of n, if $n' \not\equiv 1 \bmod 4$. A proof of this incredible conjecture is still unknown. For other cases, a rational correction factor is needed – see [2.4.2] – but

Artin's constant remains the central feature of such formulas. Hooley [10, 11] proved that such formulas are valid, subject to the truth of a generalized Riemann hypothesis.

A rapidly convergent expression for Artin's constant is as follows [12–18]. Define **Lucas' sequence** as

$$l_0 = 2, \qquad l_1 = 1, \qquad l_n = l_{n-1} + l_{n-2} \quad \text{for } n \geq 2$$

and observe that $l_n = \varphi^n + (1-\varphi)^n$, where φ is the Golden mean [1.2]. Then

$$\begin{aligned} C_{\text{Artin}} &= \prod_{n\geq 2} \zeta(n)^{-\frac{1}{n}\sum_{k|n} l_k \cdot \mu\left(\frac{n}{k}\right)} \\ &= \zeta(2)^{-1}\zeta(3)^{-1}\zeta(4)^{-1}\zeta(5)^{-2}\zeta(6)^{-2}\zeta(7)^{-4}\zeta(8)^{-5}\zeta(9)^{-8}\cdots, \end{aligned}$$

where $\zeta(n)$ is Riemann's zeta function [1.6] and $\mu(n)$ is Möbius' mu function [2.2]. For comparison's sake, here is the analogous expression for the twin prime constant [2.1]:

$$\begin{aligned} C_{\text{twin}} &= \prod_{n\geq 2} \left[\left(1 - \frac{1}{2^n}\right)\zeta(n)\right]^{-\frac{1}{n}\sum_{k|n} 2^k \cdot \mu\left(\frac{n}{k}\right)} \\ &= \left(\tfrac{3\zeta(2)}{4}\right)^{-1}\left(\tfrac{7\zeta(3)}{8}\right)^{-2}\left(\tfrac{15\zeta(4)}{16}\right)^{-3}\left(\tfrac{31\zeta(5)}{32}\right)^{-6}\left(\tfrac{63\zeta(6)}{64}\right)^{-9}\left(\tfrac{127\zeta(7)}{128}\right)^{-18}\cdots. \end{aligned}$$

We briefly examine two k-dimensional generalizations of Artin's constant, omitting technical details. First, let $S(n_1, n_2, \ldots, n_k)$ denote the set of all primes p for which the integers $n_1, n_2, \ldots, n_k$ are simultaneously primitive roots mod p. Matthews [19, 20] deduced the analog of C_{Artin} corresponding to the density of $S(n_1, n_2, \ldots, n_k)$, relative to the primes [21]:

$$C_{\text{Matthews},k} = \prod_p \left(1 - \frac{p^k - (p-1)^k}{p^k(p-1)}\right) = \begin{cases} 0.1473494003\ldots & \text{if } k = 2, \\ 0.0608216553\ldots & \text{if } k = 3, \\ 0.0261074464\ldots & \text{if } k = 4, \end{cases}$$

which is valid up to a rational correction factor. Second, let N denote the subgroup of the cyclic group U_p generated by the set $\{n_1, n_2, \ldots, n_k\} \subseteq U_p$, and define $\tilde{S}(n_1, n_2, \ldots, n_k)$ to be the set of all primes p for which $N = U_p$. Pappalardi [22, 23] obtained the analog of C_{Artin} corresponding to the density of $\tilde{S}(n_1, n_2, \ldots, n_k)$, relative to the primes [17]:

$$C_{\text{Pappalardi},k} = \prod_p \left(1 - \frac{1}{p^k(p-1)}\right) = \begin{cases} 0.6975013584\ldots & \text{if } k = 2, \\ 0.8565404448\ldots & \text{if } k = 3, \\ 0.9312651841\ldots & \text{if } k = 4, \end{cases}$$

which again is valid up to a rational correction factor. Niklasch & Moree [17] computed $C_{\text{Pappalardi},k}$ and many of the constants in this essay.

In the context of quadratic number fields [24, 25], a suitably extended Artin's conjecture involves $C_{\text{Pappalardi},2}$ as well as the constant

$$\frac{8C_{\text{twin}}}{\pi^2} = \prod_{p>2}\left(1 - \frac{2}{p(p-1)}\right) = 0.5351070126\ldots.$$

A generalization to arbitrary algebraic number fields seems to be an open problem. See [26–28] for a curious variation of C_{Artin} involving Fibonacci primitive roots, and see [29] likewise for pseudoprimes and Carmichael numbers.

We describe an unsolved problem. Define, for any odd prime p, $g(p)$ to be the least positive integer that is a primitive root mod p, and define $G(p)$ to be the least prime that is a primitive root mod p. What are the expected values of $g(p)$ and $G(p)$? Murata [21,30] argued heuristically that $g(p)$ is never very far from

$$1 + C_{\text{Murata}} = 1 + \prod_p \left(1 + \frac{1}{(p-1)^2}\right) = 3.8264199970\ldots$$

for almost all p. This estimate turns out to be too low. Empirical data [21,31,32] suggest that $\mathrm{E}(g(p)) = 4.9264\ldots$ and $\mathrm{E}(G(p)) = 5.9087\ldots$. There is a complicated infinite series for $\mathrm{E}(g(p))$ involving Matthews' constants [21], but it is perhaps computationally infeasible. See [2.7] for another occurrence of C_{Murata}.

2.4.1 Relatives

Here are some related constants from various parts of number theory. Let nonzero integers a and b be multiplicatively independent in the sense that $a^m b^n \neq 1$ except when $m = n = 0$. Let $T(a,b)$ denote the set of all primes p for which $p|(a^k - b)$ for some nonnegative integer k. Assuming a generalized Riemann hypothesis, Stephens [33] proved that the density of $T(a,b)$ relative to the primes is

$$\prod_p \left(1 - \frac{p}{p^3 - 1}\right) = 0.5759599688\ldots$$

up to a rational correction factor. Moree & Stevenhagen [34] extended Stephens' work and offered adjustments to the correction factors. They further proved unconditionally that the density of $T(a,b)$ must be positive. A rapidly convergent expression for Stephens' constant is given in [16,17].

The Feller–Tornier constant [35–37]

$$\frac{1}{2} + \frac{1}{2}\prod_p \left(1 - \frac{2}{p^2}\right) = \frac{1}{2} + \frac{3}{\pi^2}\prod_p \left(1 - \frac{1}{p^2 - 1}\right) = 0.6613170494\ldots$$

is the density of integers that have an even number of powers of primes in their canonical factorization. By *power*, we mean a power higher than the first. Thus $2 \cdot 3^2 \cdot 5^3$ has two powers of primes in it and contributes to the density, whereas $3 \cdot 7 \cdot 19 \cdot 31^2$ has one power of a prime in it and does not contribute to the density.

Consider the set of integer vectors (x_0, x_1, x_2, x_3) satisfying the equation $x_0^3 = x_1 x_2 x_3$ and the constraints $0 < x_j \le X$ for $1 \le j \le 3$ and $\gcd(x_1, x_2, x_3) = 1$. What are the asymptotics of the cardinality, $N(X)$, of this set as $X \to \infty$? Heath-Brown & Moroz [38] proved that

$$\lim_{X\to\infty} \frac{2880 N(X)}{X \ln(X)^6} = \prod_p \left(1 - \frac{1}{p^7}\right)\left(1 + \frac{7}{p} + \frac{1}{p^2}\right) = 0.0013176411\ldots.$$

Counting problems such as these for arbitrary cubic surfaces are very difficult.

Given a positive integer n, let $D_n^2 = n/n'$, the largest square divisor of n. Define Σ to be the set of n such that D_n and n' are coprime. Then Σ has asymptotic density [37]

$$\chi = \prod_p \left(1 - \frac{1}{p^2(p+1)}\right) = 0.8815138397\ldots.$$

Interestingly, the constant χ appears in the following as well.

If d is the fundamental discriminant of an imaginary quadratic field ($d < 0$) and $h(d)$ is the associated class number, then the ratio $2\pi h(d)/\sqrt{-d}$ is equal to χ on average [39,40]. This constant plays a role for real quadratic fields too ($d > 0$). In connection with indefinite binary quadratic forms, Sarnak [41] obtained that the average value of $h(d)$, taken over the thin subset of discriminants $0 < d < D$ of the form $c^2 - 4$, is asymptotically

$$\frac{5\pi^2}{48} \prod_p \left(1 - \frac{1}{p^2} - \frac{2}{p^3}\right) \cdot \frac{\sqrt{D}}{\ln(D)} = 0.7439711933\ldots \cdot \frac{\sqrt{D}}{\ln(D)}$$

as $D \to \infty$. The analogous constants for $0 < d < D$ of the form $c^{2\nu} - 4$, $\nu \geq 2$, do not appear to possess similar formulation.

The $2k^{\text{th}}$ moment (over the critical line) of the Riemann zeta function

$$m_{2k}(T) = \frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2k} dt$$

is known to satisfy $m_2(T) \sim \ln(T)$ and $m_4(T) \sim (1/(2\pi^2)) \ln(T)^4$ as $T \to \infty$. It is conjectured that $m_{2k}(T) \sim \gamma_k \ln(T)^{k^2}$ and further that [42–44]

$$\frac{9!}{42}\gamma_6 = \prod_p \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right),$$

$$\frac{16!}{24024}\gamma_8 = \prod_p \left(1 - \frac{1}{p}\right)^9 \left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right).$$

This analysis can be extended to Dirichlet L-functions. Understanding the behavior of moments such as these could have numerous benefits for number theory.

2.4.2 Correction Factors

We have assumed that $n \neq -1, 0, 1$ is not an r^{th} power for any $r > 1$ and that n' is the square-free part of n. If $n' \equiv 1 \bmod 4$, then the density of the set $S(n)$ relative to the primes is conjectured to be [8,10,14,45,46]

$$\left(1 - \mu(|n'|) \prod_{q|n'} \frac{1}{q^2 - q - 1}\right) \cdot C_{\text{Artin}},$$

where the product is restricted to primes q. For example, if $n' = u$ is prime, then this formula simplifies to

$$\left(1 + \frac{1}{u^2 - u - 1}\right) \cdot C_{\text{Artin}}.$$

If instead $n' = uv$, where $u \equiv 1 \bmod 4$ and $v \equiv 1 \bmod 4$ are both primes, then the formula instead simplifies to

$$\left(1 - \frac{1}{u^2 - u - 1} \frac{1}{v^2 - v - 1}\right) \cdot C_{\text{Artin}}.$$

If n is an r^{th} power, a slightly more elaborate formula applies.

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2.5 Hafner–Sarnak–McCurley Constant

We start with a well-known theorem [1]. The probability that two randomly chosen integers are coprime is $6/\pi^2 = 0.6079271018\ldots$ (in the limit over large intervals). What happens if we replace the integers by integer square matrices? Given two randomly chosen integer $n \times n$ matrices, what is the probability, $\Delta(n)$, that the two corresponding determinants are coprime?

Hafner, Sarnak & McCurley [2] showed that

$$\Delta(n) = \prod_p \left[1 - \left(1 - \prod_{k=1}^{n} \left(1 - p^{-k}\right)\right)^2\right]$$

for each n, where the outermost product is restricted to primes p. It can be proved that

$$\Delta(1) = \frac{6}{\pi^2} > \Delta(2) > \Delta(3) > \ldots > \Delta(n-1) > \Delta(n) > \ldots,$$

and Vardi [3, 4] computed the limiting value

$$\lim_{n\to\infty} \Delta(n) = \prod_p \left[1 - \left(1 - \prod_{k=1}^{\infty} \left(1 - p^{-k}\right)\right)^2\right] = 0.3532363719\ldots.$$

2.5.1 Carefree Couples

It is also well known that $6/\pi^2$ is the probability that a randomly chosen integer x is square-free [1], meaning x is divisible by no square exceeding 1. Schroeder [5] asked the following question: Are the properties of being square-free and coprime statistically independent? The answer is no: There appears to be a positive correlation between the two properties. More precisely, define two randomly chosen integers x and y to be **carefree** [5, 6] if x and y are coprime and x is square-free. The probability that x and y are carefree is somewhat larger than $36/\pi^4 = 0.3695\ldots$ and is exactly equal to

$$P = \frac{6}{\pi^2} \prod_p \left(1 - \frac{1}{p(p+1)}\right) = 0.4282495056\ldots.$$

Moree [7] proved that Schroeder's formula is correct. Further, he defined x and y to be **strongly carefree** when x and y are coprime, and x and y are both square-free. The probability in this case is [8]

$$Q = \frac{6}{\pi^2} \prod_p \left(1 - \frac{2}{p(p+1)}\right) = \frac{36}{\pi^4} \prod_p \left(1 - \frac{1}{(p+1)^2}\right) = 0.2867474284\ldots.$$

Define finally x and y to be **weakly carefree** when x and y are coprime, and x or y is square-free. As a corollary, the probability here is $2P - Q = 0.5697515829\ldots$, using the fact that $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$. Do there exist matrix analogs of these joint probabilities?

The constants P and Q appear elsewhere in number theory [7]. Let $D_n = \max\{d : d^2|n\}$. Define

$$\kappa(n) = \frac{n}{D_n^2}, \quad \text{the } \textbf{square-free part} \text{ of } n,$$

$$K(n) = \prod_{p|n} p, \quad \text{the } \textbf{square-free kernel} \text{ of } n;$$

then [9–11]

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{n=1}^{N} \kappa(n) = \frac{\pi^2}{30} = 0.3289\ldots, \quad \lim_{N\to\infty} \frac{1}{N^2} \sum_{n=1}^{N} K(n) = \frac{\pi^2 P}{12} = 0.3522\ldots$$

(see [2.10] for the average of D_n instead). Let $\omega(n)$ be the number of distinct prime factors of n, as in [2.2]; then [11–13]

$$\lim_{N\to\infty} \frac{1}{N\ln(N)} \sum_{n=1}^{N} 2^{\omega(n)} = \frac{6}{\pi^2} = 0.6079\ldots,$$

$$\lim_{N\to\infty} \frac{1}{N\ln(N)^2} \sum_{n=1}^{N} 3^{\omega(n)} = \frac{Q}{2} = 0.1433\ldots.$$

If $\omega(n)$ is replaced by $\Omega(n)$, the total number of prime factors of n, then alternatively [11, 14, 15]

$$\lim_{N\to\infty} \frac{1}{N\ln(N)^2} \sum_{n=1}^{N} 2^{\Omega(n)} = \frac{1}{8\ln(2)C_{\text{twin}}} = 0.2731707223\ldots,$$

where C_{twin} is the twin prime constant [2.1], which seems to be unrelated to P and Q.

We conclude with a generalization. The probability that k randomly chosen integers are coprime is $1/\zeta(k)$, as suggested in [1.6]. The probability that they are *pairwise* coprime is known to be [5, 7]

$$\prod_p \left(1 - \frac{1}{p}\right)^{k-1} \left(1 + \frac{k-1}{p}\right)$$

for $2 \le k \le 3$, but a proof for $k > 3$ has not yet been found. The expression naturally reduces to $6/\pi^2$ if $k = 2$. More surprisingly, if $k = 3$, it reduces to Q.

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2.6 Niven's Constant

Let m be a positive integer with prime factorization $p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_k^{a_k}$. We assume that each exponent $a_i \geq 1$ and each prime $p_i \neq p_j$ for all $i \neq j$. Define two functions

$$h(m) = \begin{cases} 1 & \text{if } m = 1, \\ \min\{a_1, \ldots, a_k\} & \text{if } m > 1, \end{cases} \quad H(m) = \begin{cases} 1 & \text{if } m = 1, \\ \max\{a_1, \ldots, a_k\} & \text{if } m > 1, \end{cases}$$

that is, the smallest and largest exponents for m. Niven [1,2] proved that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} h(m) = 1$$

and, moreover,

$$\lim_{n\to\infty} \frac{\left(\sum\limits_{m=1}^{n} h(m)\right) - n}{\sqrt{n}} = \frac{\zeta(\frac{3}{2})}{\zeta(3)} = 2.1732543125\ldots,$$

where $\zeta(x)$ denotes Riemann's zeta function [1.6]. He also proved that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} H(m) = C$$

and we call C **Niven's constant**:

$$C = 1 + \sum_{k=2}^{\infty} \left(1 - \frac{1}{\zeta(k)}\right) = 1.7052111401\ldots.$$

Subsequent authors discovered the following extended results [3,4]:

$$\sum_{m=1}^{n} h(m) = n + c_{02}n^{\frac{1}{2}} + (c_{12}+c_{03})n^{\frac{1}{3}} + (c_{13}+c_{04})n^{\frac{1}{4}} + (c_{23}+c_{14}+c_{05})n^{\frac{1}{5}} + O(n^{\frac{1}{6}}),$$

$$\sum_{m=1}^{n} \frac{1}{h(m)} = n - \frac{c_{02}}{2}n^{\frac{1}{2}} - \frac{3c_{12}+c_{03}}{6}n^{\frac{1}{3}} - \frac{2c_{13}+c_{04}}{12}n^{\frac{1}{4}} - \frac{10c_{23}+5c_{14}+3c_{05}}{60}n^{\frac{1}{5}} + O(n^{\frac{1}{6}}),$$

where the coefficients c_{ij} are given in [2.6.1]; additionally, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \frac{1}{H(m)} = \sum_{k=2}^{\infty} \frac{1}{k(k-1)\zeta(k)} = 0.7669444905\ldots.$$

Averages for H are not as well understood asymptotically as averages for h.

The constant $c_{02} = \zeta(3/2)/\zeta(3)$ also occurs when estimating the asymptotic growth of the number of square-full integers [2.6.1], as does $c_{12} = \zeta(2/3)/\zeta(2) = -1.4879506635\ldots$ In contrast, the constant $6/\pi^2$ arises in connection with the square-free integers [2.5].

A generalization of Niven's theorem to the setting of a free abelian normed semigroup appears in [5].

Here is a problem that gives expressions similar to C. First, observe that [6,7]

$$\sum_{l=2}^{\infty}\sum_{n=2}^{\infty} \frac{1}{n^l} = \sum_{l=2}^{\infty}(\zeta(l)-1) = 1, \quad \sum_{p}\sum_{n=2}^{\infty}\frac{1}{n^p} = \sum_{p}(\zeta(p)-1) = 0.8928945714\ldots,$$

where the sum over p is restricted to primes. Both series involve reciprocal nontrivial integer powers with duplication, for example, $2^4 = 4^2$ and $4^3 = 8^2$. Now, let $S = \{4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, \ldots\}$ be the set of nontrivial integer powers *without* duplication. It follows that [8]

$$\sum_{s\in S}\frac{1}{s} = -\sum_{k=2}^{\infty}\mu(k)(\zeta(k)-1) = 0.8744643684\ldots,$$

where $\mu(k)$ is Möbius' mu function [2.2]; we also have [8,9]

$$\sum_{s\in S}\frac{1}{s-1} = 1, \quad \sum_{s\in S}\frac{1}{s+1} = \frac{\pi^2}{3} - \frac{5}{2}.$$

Given an arbitrary integer $c \notin S$, what can be said about $\sum_{s\in S}(s-c)^{-1}$? (By Mihailescu's recent proof of Catalan's conjecture, the only two integers in S that differ by 1 are 8 and 9.) See other expressions in [5.1].

2.6.1 Square-Full and Cube-Full Integers

Let $k \geq 2$ be an integer. A positive integer m is k-**full** (or **powerful of type** k) if $m = 1$ or if, for any prime number p, $p|m$ implies $p^k|m$.

Let $N_k(x)$ denote the number of k-full integers not exceeding x. For the case $k = 2$, Erdös & Szekeres [10] showed that

$$N_2(x) = \frac{\zeta(\frac{3}{2})}{\zeta(3)} x^{\frac{1}{2}} + O\left(x^{\frac{1}{3}}\right)$$

and Bateman & Grosswald [11–13] proved the more accurate result

$$N_2(x) = \frac{\zeta(\frac{3}{2})}{\zeta(3)} x^{\frac{1}{2}} + \frac{\zeta(\frac{2}{3})}{\zeta(2)} x^{\frac{1}{3}} + o\left(x^{\frac{1}{6}}\right).$$

This is essentially as sharp an error estimate as possible without additional knowledge concerning the unsolved Riemann hypothesis. A number of researchers have studied this problem. The current best-known error term [14, 15], assuming Riemann's hypothesis, is $O(x^{1/7+\varepsilon})$ for any $\varepsilon > 0$, and several authors conjecture that $1/7$ can be replaced by $1/10$.

For the case $k = 3$, Bateman & Grosswald [12] and Krätzel [16, 17] demonstrated unconditionally that

$$N_3(x) = c_{03} x^{\frac{1}{3}} + c_{13} x^{\frac{1}{4}} + c_{23} x^{\frac{1}{5}} + o\left(x^{\frac{1}{8}}\right).$$

By assuming Riemann's hypothesis, the error term [15] can be improved to $O(x^{97/804+\varepsilon})$. Formulas for the coefficients c_{ij} include [3, 12, 18–20]

$$c_{0j} = \prod_p \left(1 + \sum_{m=j+1}^{2j-1} p^{-\frac{m}{j}}\right) = \begin{cases} 4.6592661225\ldots & \text{if } j = 3, \\ 9.6694754843\ldots & \text{if } j = 4, \\ 19.4455760839\ldots & \text{if } j = 5, \end{cases}$$

$$\begin{aligned} c_{1j} &= \zeta\left(\tfrac{j}{j+1}\right) \prod_p \left(1 + \sum_{m=j+2}^{2j-1} p^{-\frac{m}{j+1}} - \sum_{m=2j+2}^{3j} p^{-\frac{m}{j+1}}\right) \\ &= \begin{cases} -5.8726188208\ldots & \text{if } j = 3, \\ -16.9787814834\ldots & \text{if } j = 4, \end{cases} \end{aligned}$$

$$c_{23} = \zeta\left(\tfrac{3}{5}\right) \zeta\left(\tfrac{4}{5}\right) \prod_p \left(1 - p^{-\frac{8}{5}} - p^{-\frac{9}{5}} - p^{-\frac{10}{5}} + p^{-\frac{13}{5}} + p^{-\frac{14}{5}}\right) = 1.6824415102\ldots,$$

where all products are restricted to primes p. The decimal approximations for the **Bateman–Grosswald constants** listed here are due to Niklasch & Moree [21] and Sebah [22]. Higher-order coefficients appear in the expansions of $N_k(x)$ for $k \geq 4$.

We observe that the Erdös–Szekeres paper [10] also plays a crucial role in the asymptotics of abelian group enumeration [5.1]. The books by Ivić [23] and Krätzel [24] provide detailed analyses and background. See also [5.4] for discussion of the smallest and largest prime factors of m.

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2.7 Euler Totient Constants

When n is a positive integer, Euler's totient function, $\varphi(n)$, is defined to be the number of positive integers not greater than n and relatively prime to n. For example, if p and q are distinct primes and r and s are positive integers, then

$$\varphi(p^r) = p^{r-1}(p-1),$$
$$\varphi(p^r q^s) = p^{r-1}q^{s-1}(p-1)(q-1).$$

In the language of group theory, $\varphi(n)$ is the number of generators in a cyclic group of order n. Landau [1–4] showed that

$$\limsup_{n\to\infty} \frac{\varphi(n)}{n} = 1$$

but

$$\liminf_{n\to\infty} \frac{\varphi(n)\ln(\ln(n))}{n} = e^{-\gamma} = 0.5614594835\ldots,$$

where γ is the Euler–Mascheroni constant [1.5].

The average behavior of $\varphi(n)$ over all positive integers has been of interest to many authors. Walfisz [5, 6], building on the work of Dirichlet and Mertens [2], proved that

$$\sum_{n=1}^{N} \varphi(n) = \frac{3N^2}{\pi^2} + O\left(N \ln(N)^{\frac{2}{3}} \ln(\ln(N))^{\frac{4}{3}}\right)$$

as $N \to \infty$, which is the sharpest such asymptotic formula known. (A claim in [7] that the exponent 4/3 could be replaced by $1 + \varepsilon$, for any $\varepsilon > 0$, is incorrect [8].) It is also known [9, 10] that the error term is not $o(N \ln(\ln(\ln(N))))$.

Interesting constants emerge if we consider instead the series of reciprocals of $\varphi(n)$. Landau [11–13] proved that

$$\sum_{n=1}^{N} \frac{1}{\varphi(n)} = A \cdot (\ln(N) + B) + O\left(\frac{\ln(N)}{N}\right),$$

where

$$A = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = \frac{315}{2\pi^4}\zeta(3) = 1.9435964368\ldots,$$

$$B = \gamma - \sum_p \frac{\ln(p)}{p^2 - p + 1} = \gamma - 0.6083817178\ldots = \frac{-0.0605742294\ldots}{A},$$

and $\zeta(x)$ is Riemann's zeta function [1.6]. Sums and products over p are restricted to primes. The sum within B has inspired several accurate computations by Jameson [14], Moree [15] and Sebah [16]. Landau's error term $O(\ln(N)/N)$ was improved to $O(\ln(N)^{2/3}/N)$ by Sitaramachandrarao [17, 18].

Define $K(x)$ to be the number of all positive integers n that satisfy $\varphi(n) \le x$. It is known [19–22] that the following distributional result is true:

$$K(x) = Ax + O\left(x \exp\left(-c\sqrt{\ln(x)\ln(\ln(x))}\right)\right)$$

for any $0 < c < 1/\sqrt{2}$. Other relevant formulas are [18, 23, 24]

$$\sum_{n=1}^{N} \frac{\varphi(n)}{n} = \frac{6N}{\pi^2} + O\left(\ln(N)^{\frac{2}{3}} \ln(\ln(N))^{\frac{4}{3}}\right),$$

$$\sum_{n=1}^{N} \frac{n}{\varphi(n)} = AN - \frac{1}{2}\ln(N) - \frac{1}{2}C + O\left(\ln(N)^{\frac{2}{3}}\right),$$

$$\sum_{n=1}^{N} \frac{1}{n\varphi(n)} = D - \frac{A}{N} + O\left(\frac{\ln(N)}{N^2}\right),$$

where

$$C = \ln(2\pi) + \gamma + \sum_p \frac{\ln(p)}{p(p-1)} = \ln(2\pi) + 1.3325822757\ldots = 3.1704593421\ldots,$$

which occurred in [2.2], and

$$D = \frac{\pi^2}{6} \prod_p \left(1 + \frac{1}{p^2(p-1)}\right) = 2.2038565964\ldots,$$

which came from a sharpening by Moree [24] of estimates in [25]. See [26] for numerical evaluations of such prime products. The constant A occurs in [27, 28] as the asymptotic mean of a certain prime divisor function and elsewhere too [29]. The constant D also occurs in a certain Hardy–Littlewood conjecture proved by Chowla [30].

We note the following alternative representation of A:

$$A = \prod_p \frac{1 - p^{-6}}{(1 - p^{-2})(1 - p^{-3})} = \prod_p \left(1 + \frac{1}{p(p-1)}\right),$$

which bears a striking resemblance to Artin's constant [2.4]. The only distinction is that an addition is replaced by a subtraction. Curiously, Artin's constant and Murata's constant [2.4] arise explicitly in the following asymptotic results [31, 32]:

$$\lim_{N\to\infty} \frac{\ln(N)}{N} \sum_{p\le N} \frac{\varphi(p-1)}{p-1} = C_{\text{Artin}} = 0.3739558136\ldots,$$

$$\lim_{N\to\infty} \frac{\ln(N)}{N} \sum_{p\le N} \frac{p-1}{\varphi(p-1)} = C_{\text{Murata}} = 2.8264199970\ldots.$$

Let $L(x)$ denote the number of all positive integers n not exceeding x for which n and $\varphi(n)$ are relatively prime. Erdös [33, 34] proved that

$$\lim_{n\to\infty} \frac{L(n)\ln(\ln(\ln(n)))}{n} = e^{-\gamma},$$

another interesting occurrence of the Euler–Mascheroni constant.

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2.8 Pell–Stevenhagen Constants

If an integer $d > 1$ is not a square, then the **Pell equation**

$$x^2 - dy^2 = 1$$

has a solution in integers (in fact, infinitely many). This fact was known long ago [1–5]. We are here concerned with a more difficult question. What can be said about the set D of integers $d > 1$ for which the **negative Pell equation**

$$x^2 - dy^2 = -1$$

has a solution in integers? Only recently has progress been made in answering this.

First, define the **Pell constant**

$$P = 1 - \prod_{\substack{j \geq 1 \\ j \text{ odd}}} \left(1 - \frac{1}{2^j}\right) = 0.5805775582\ldots,$$

which is needed in the following. The constant P is provably irrational [6] but only conjectured to be transcendental. Define also a function

$$\psi(p) = \frac{2 + \left(1 + 2^{1-v_p}\right) p}{2(p+1)},$$

where v_p is the number of factors of 2 occurring in $p - 1$.

For any set S of positive integers, let $f_S(n)$ denote the number of elements in S not exceeding n. Stevenhagen [6–8] developed several conjectures regarding the distribution of D. He hypothesized that the counting function $f_D(n)$ satisfies the following [7]:

$$\lim_{n\to\infty} \frac{\sqrt{\ln(n)}}{n} f_D(n) = \frac{3P}{2\pi} \prod_{p \equiv 1 \bmod 4} \left(1 + \frac{\psi(p)}{p^2 - 1}\right)\left(1 - \frac{1}{p^2}\right)^{\frac{1}{2}} = 0.28136\ldots,$$

where the product is restricted to primes p.

Let U be the set of positive integers not divisible by 4, and let V be the set of positive integers not divisible by any prime congruent to 3 module 4. Clearly D is a subset of $U \cap V$, and $U \cap V$ is the set of positive integers that can be written as a sum of two coprime squares. By the conjectured limit mentioned here and by a coprimality result given in [2.3.1] due to Rieger [9], the density of D inside $U \cap V$ is [7]

$$\lim_{n\to\infty} \frac{f_D(n)}{f_{U\cap V}(n)} = P \prod_{p \equiv 1 \bmod 4} \left(1 + \frac{\psi(p)}{p^2 - 1}\right)\left(1 - \frac{1}{p^2}\right) = 0.57339\ldots.$$

Here is another conjecture. Let W be the set of square-free integers, that is, integers that are divisible by no square exceeding 1. Stevenhagen [6] hypothesized that

$$\lim_{n\to\infty} \frac{\sqrt{\ln(n)}}{n} f_{D\cap W}(n) = \frac{6}{\pi^2} PK = 0.2697318462\ldots,$$

where K is the Landau–Ramanujan constant [2.3]. Clearly $V \cap W$ is the set of positive square-free integers that can be written as a sum of two (coprime) squares. By the second conjectured limit and by a square-free result given in [2.3.1] due to Moree [10], the density of $D \cap W$ inside $V \cap W$ is [8]

$$\lim_{n\to\infty} \frac{f_{D\cap W}(n)}{f_{V\cap W}(n)} = P = 0.5805775582\ldots.$$

A fascinating connection to continued fractions is as follows [7]: An integer $d > 1$ is in D if and only if $\sqrt{d}$ is irrational and has a regular continued fraction expansion with *odd* period length.

A constant Q similar to P here appears in [5.14]; however, exponents in Q are not constrained to be odd integers.

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2.9 Alladi–Grinstead Constant

Let n be a positive integer. The well-known formula

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$$

is only one of many available ways to decompose $n!$ as a product of n positive integer factors. Let us agree to disallow 1 as a factor and to further restrict each of the n factors to be a prime power:

$$p_k^{b_k}, \text{ each } p_k \text{ is prime and } b_k \geq 1,\ k = 1, 2, \ldots, n.$$

(Thus the previously stated natural decomposition of $n!$ is inadmissible.) Let us also write the factors in nondecreasing order from left to right. If $n = 9$, for example, all of the admissible decompositions are

$$\begin{aligned}
9! &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2^2 \cdot 5 \cdot 7 \cdot 3^4 \\
&= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 2^3 \cdot 3^3 \\
&= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 7 \cdot 2^3 \cdot 3^2 \cdot 3^2 \\
&= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 2^2 \cdot 2^2 \cdot 5 \cdot 7 \cdot 3^3 \\
&= 2 \cdot 2 \cdot 2 \cdot 2^2 \cdot 2^2 \cdot 5 \cdot 7 \cdot 3^2 \cdot 3^2 \\
&= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 3^2 \cdot 2^4 \\
&= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 2^2 \cdot 5 \cdot 7 \cdot 2^3 \cdot 3^2 \\
&= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 2^5 \\
&= 2 \cdot 3 \cdot 3 \cdot 2^2 \cdot 2^2 \cdot 2^2 \cdot 5 \cdot 7 \cdot 3^2 \\
&= 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 2^2 \cdot 5 \cdot 7 \cdot 2^4 \\
&= 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 2^3 \cdot 2^3 \\
&= 3 \cdot 3 \cdot 3 \cdot 3 \cdot 2^2 \cdot 2^2 \cdot 5 \cdot 7 \cdot 2^3.
\end{aligned}$$

Note that eleven of the leftmost factors are 2 and one is 3. The maximum leftmost factor, considering all admissible decompositions of 9! into 9 prime powers, is therefore 3. We define

$$\alpha(9) = \frac{\ln(3)}{\ln(9)}.$$

In the same way, for arbitrary n, one determines the maximum leftmost factor p^b over all admissible decompositions of $n!$ into n prime powers and defines

$$\alpha(n) = \frac{\ln(p^b)}{\ln(n)}.$$

Clearly $\alpha(n) < 1$ for each n. What can be said about $\alpha(n)$ for large n?

Alladi & Grinstead [1,2] determined that the limit of $\alpha(n)$ as $n \to \infty$ exists and

$$\lim_{n\to\infty} \alpha(n) = e^{c-1} = 0.8093940205\ldots,$$

where

$$\begin{aligned}
c &= -\sum_{k=2}^{\infty} \frac{1}{k} \ln\left(1 - \frac{1}{k}\right) = \sum_{j=2}^{\infty} \frac{\zeta(j) - 1}{j-1} = 0.7885305659\ldots \\
&= -\ln(0.4545121805\ldots)
\end{aligned}$$

and $\zeta(x)$ is Riemann's zeta function [1.6].

How strongly does Alladi & Grinstead's result depend on decomposing $n!$ and not some other function $f(n)$? It is assumed that f provides sufficiently many small and varied prime factors for each n. See [3] for a related unsolved problem.

Let $d(m)$ denote the number of positive integer divisors of m. What can be said about $d(n!)$? Erdös et al. [4] proved that

$$\lim_{n\to\infty} \frac{\ln(\ln(n!))^2}{\ln(n!)} \ln(d(n!)) = C,$$

where

$$C = \sum_{k=2}^{\infty} \frac{1}{k(k-1)} \ln(k) = -\sum_{j=2}^{\infty} \zeta'(j) = 1.2577468869\ldots$$

as mentioned in [1.8]. The similarity between c and C is quite interesting.

Here are four related infinite products [5, 6]:

$$\prod_{n\geq 2}\left(1+\frac{1}{n}\right)^{\frac{1}{n}} = 1.7587436279\ldots, \quad \prod_{n\geq 2}\left(1-\frac{1}{n}\right)^{\frac{1}{n}} = 0.4545121805\ldots,$$
$$\prod_{p}\left(1+\frac{1}{p}\right)^{\frac{1}{p}} = 1.4681911223\ldots, \quad \prod_{p}\left(1-\frac{1}{p}\right)^{\frac{1}{p}} = 0.5598656169\ldots,$$

the latter two of which are restricted to primes p. The second product is e^{-c}, and the fourth appears in [7, 8]. A related problem, regarding the asymptotics of the smallest and largest prime factors of n, is discussed in [5.4].

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2.10 Sierpinski's Constant

In his 1908 dissertation, Sierpinski [1] studied certain series involving the function $r(n)$, defined to be the number of representations of the positive integer n as a sum of two squares, counting order and sign. For example, $r(1) = 4$, $r(p) = 0$ for primes $p \equiv 3 \bmod 4$, and $r(q) = 8$ for primes $q \equiv 1 \bmod 4$.

Certain results about $r(n)$ are not difficult to see; for example [2–4],

$$\sum_{k=1}^{n} r(k) = \pi n + O\left(n^{\frac{1}{2}}\right)$$

as $n \to \infty$. More details on this estimate are in [2.10.1]. Sierpinski's series include [1,5,6]

$$\sum_{k=1}^{n} \frac{r(k)}{k} = \pi\left(\ln(n) + S\right) + O\left(n^{-\frac{1}{2}}\right),$$

$$\sum_{k=1}^{n} r(k^2) = \frac{4}{\pi}\left(\ln(n) + \hat{S}\right) n + O\left(n^{\frac{2}{3}}\right),$$

$$\sum_{k=1}^{n} r(k)^2 = 4\left(\ln(n) + \tilde{S}\right) n + O\left(n^{\frac{3}{4}} \ln(n)\right),$$

where the constants $\hat{S}$ and $\tilde{S}$ are defined in terms of S as

$$\hat{S} = \gamma + S - \frac{12}{\pi^2}\zeta'(2) + \frac{\ln(2)}{3} - 1, \quad \tilde{S} = 2S - \frac{12}{\pi^2}\zeta'(2) + \frac{\ln(2)}{3} - 1,$$

where γ is the Euler–Mascheroni constant [1.5] and $\zeta(x)$ is Riemann's zeta function [1.6]. See [2.15] and [2.18] for other occurrences of $\zeta'(2)$.

The constant S, which we call **Sierpinski's constant**, thus plays a role in the summation of all three series. It can be defined as

$$S = \gamma + \frac{\beta'(1)}{\beta(1)} = \ln\left(\frac{\pi^2 e^{2\gamma}}{2L^2}\right) = \ln\left(\frac{4\pi^3 e^{2\gamma}}{\Gamma\left(\frac{1}{4}\right)^4}\right) = \frac{2.5849817595\ldots}{\pi},$$

where $\beta(x)$ is Dirichlet's beta function [1.7], $L = 2.6220575542\ldots$ is Gauss' lemniscate constant [6.1], and $\Gamma(x)$ is the Euler gamma function [1.5.4]. It also appears in our essays on the Landau–Ramanujan constant [2.3] and the Masser–Gramain constant [7.2]. Sierpinski, in fact, defined S as a limit:

$$S = \frac{1}{\pi} \lim_{z \to 1}\left(F(z) - \frac{\pi}{z-1}\right),$$

and the function $F(z) = 4\zeta(z)\beta(z)$ is central to our discussion of lattice sums [1.10.1]. Other formulas for S include a definite integral representation:

$$S = 2\gamma + \frac{4}{\pi}\int_{0}^{\infty} \frac{e^{-x}\ln(x)}{1 + e^{-2x}} dx.$$

Clearly this is a meeting place for many ideas, all coming together at once.

2.10.1 Circle and Divisor Problems

More precisely [7–12], the sum of the first n values of r provably satisfies

$$\sum_{k=1}^{n} r(k) = \pi n + O\left(n^{\frac{23}{73}} \ln(n)^{\frac{315}{146}}\right),$$

and it is conjectured that

$$\sum_{k=1}^{n} r(k) = \pi n + O\left(n^{\frac{1}{4}+\varepsilon}\right)$$

for all $\varepsilon > 0$. The problem of estimating the error term is known as the **circle problem** since this is the same as counting the number of integer ordered pairs falling within the disk of radius $\sqrt{n}$ centered at the origin.

Here is a related problem, known as the **divisor problem**, mentioned briefly in [1.5]. If $d(n)$ is the number of distinct divisors of n, then

$$\sum_{k=1}^{n} d(k) = n\ln(n) + (2\gamma - 1)n + O\left(n^{\frac{23}{73}}\ln(n)^{\frac{461}{146}}\right)$$

is the best-known estimate of the sum of the first n values of d. Again, the conjectured exponent is $1/4 + \varepsilon$, but this remains unproven. The analog of Sierpinski's third series, for example, is [13–15]

$$\sum_{k=1}^{n} d(k)^2 = \left(A\ln(n)^3 + B\ln(n)^2 + C\ln(n) + D\right)n + O\left(n^{\frac{1}{2}+\varepsilon}\right),$$

where

$$A = \frac{1}{\pi^2}, \quad B = \frac{12\gamma - 3}{\pi^2} - \frac{36}{\pi^4}\zeta'(2),$$

and the constants C and D have more complicated expressions. The analog of Sierpinski's first series is [16]

$$\sum_{k=1}^{n} \frac{d(k)}{k} = \frac{1}{2}\ln(n)^2 + 2\gamma\ln(n) + (\gamma^2 - 2\gamma_1) + O(n^{-\frac{1}{2}}),$$

where $\gamma_1 = -0.0728158454\ldots$ is the first Stieltjes constant [2.21].

In a variation of $d(n)$, we might restrict attention to divisors of n that are square-free [17]. Likewise, for $r(n)$, we might count only representations $n = u^2 + v^2$ for which u, v are coprime, or examine differences rather than sums. Here is another variation: Define $r_m(n)$ to be the number of representations $n = |u|^m + |v|^m$, where u, v are arbitrary integers. It is known that, if $m \geq 3$, then [12, 18, 19]

$$\sum_{k=1}^{n} r_m(k) = \frac{2\Gamma\left(\frac{1}{m}\right)^2}{m\Gamma\left(\frac{2}{m}\right)} n^{\frac{2}{m}} + O\left(n^{\frac{1}{m}(1-\frac{1}{m})}\right)$$

and, further, the error term may be replaced by

$$2^{3-\frac{1}{m}}\pi^{-1-\frac{1}{m}}m^{\frac{1}{m}}\Gamma\left(1+\tfrac{1}{m}\right)\cdot\sum_{k=1}^{\infty} k^{-1-\frac{1}{m}}\sin\left(2\pi k n^{\frac{1}{m}} - \tfrac{\pi}{2m}\right)\cdot n^{\frac{1}{m}(1-\frac{1}{m})}$$
$$+O\left(n^{\frac{46}{73m}}\ln(n)^{\frac{315}{146}}\right).$$

A full asymptotic analysis of such circle or divisor sums will be exceedingly difficult and cannot be expected soon.

In a related 1908 paper, Sierpinski [20–22] discovered the following fact. Let $D_n = \max\{d : d^2|n\}$; that is, D_n^2 is the largest square divisor of n. Then

$$\frac{1}{n}\sum_{k=1}^{n} D_k = \frac{3}{\pi^2}\ln(n) + \frac{9\gamma}{\pi^2} - \frac{36}{\pi^4}\zeta'(2) + o(1)$$

as $n \to \infty$. By way of contrast, the average square-free part of n appears in [2.5].

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2.11 Abundant Numbers Density Constant

If n is a positive integer, let $\sigma(n)$ denote the sum of all positive divisors of n. Then n is said to be **perfect** if $\sigma(n) = 2n$, **deficient** if $\sigma(n) < 2n$, and **abundant** if $\sigma(n) > 2n$.

The smallest examples of perfect numbers are 6 and 28. If the Mersenne number $2^{m+1} - 1$ is prime, then $2^m(2^{m+1} - 1)$ is perfect. Here are two famous unanswered questions [1]. Do there exist infinitely many even perfect numbers? Does there exist an odd perfect number? (According to [2], a counterexample cannot be less than 10^{300}.)

For positive real x, define the density function

$$A(x) = \lim_{n\to\infty} \frac{|\{n : \sigma(n) \geq x\, n\}|}{n}.$$

Behrend [3,4], Davenport [5], and Chowla [6] independently proved that $A(x)$ exists and is continuous for all x. Erdös [7,8] gave a proof requiring only elementary considerations. Clearly $A(x) = 1$ for $x \leq 1$, and $A(x) \to 0$ as $x \to \infty$. Refining Behrend's technique, Wall [9,10] obtained the following bounds on the **abundant numbers density constant**:

$$0.2441 < A(2) < 0.2909,$$

and Deléglise [11] improved this to

$$|A(2) - 0.2477| < 0.0003.$$

Further, it can be demonstrated [12] that $A(x)$ is differentiable everywhere except on a set of Lebesgue measure zero, and

$$\int_0^\infty x^{s-1} A(x)dx = \frac{1}{s}\prod_p \left[\left(1 - \frac{1}{p}\right)^{-s+1} \sum_{k=0}^\infty \frac{1}{p^k}\left(1 - \frac{1}{p^{k+1}}\right)^s\right]$$

for complex s satisfying $\mathrm{Re}(s) > 1$. The product is over all primes p. An inversion of this identity (Mellin transform) is theoretically possible but not yet numerically feasible [11].

As an aside, define an **exponential divisor** d of $n = p_1^{a_1} \cdots p_r^{a_r}$ to be a divisor of the form $d = p_1^{b_1} \cdots p_r^{b_r}$, where $b_j | a_j$ for each j. Let $\sigma^{(e)}(n)$ denote the sum of all exponential divisors of n, with the convention $\sigma^{(e)}(1) = 1$. Then [13–16]

$$\lim_{N\to\infty} \frac{1}{N^2}\sum_{n=1}^N \sigma(n) = \frac{\pi^2}{12}, \quad \lim_{N\to\infty} \frac{1}{N^2}\sum_{n=1}^N \sigma^{(e)}(n) = B,$$

where

$$\begin{aligned} B &= \frac{1}{2}\prod_p \left[1 + \frac{1}{p(p^2-1)} - \frac{1}{p^2-1} + \left(1 - \frac{1}{p}\right)\sum_{k=2}^\infty \frac{p^k}{p^{2k}-1}\right] \\ &= 0.5682854937\ldots. \end{aligned}$$

A study of the corresponding density function $A^{(e)}(x)$ was begun in [17].

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2.12 Linnik's Constant

We first discuss prime values of a specific sequence. Dirichlet's theorem states that any arithmetic progression $\{an + b : n \geq 0\}$, for which $a \geq 1$ and $b \geq 1$ are coprime, must contain infinitely many primes. This raises a natural question: How large is the first such prime $p(a, b)$?

Define $p(a)$ to be the maximum of $p(a, b)$ over all b satisfying $1 \leq b < a$, $\gcd(a, b) = 1$ and let

$$K = \sup_{a \geq 2} \frac{\ln(p(a))}{\ln(a)}, \quad L = \lim_{a \to \infty} \frac{\ln(p(a))}{\ln(a)}.$$

That is, K is the infimum of κ satisfying $p(a) < a^{\kappa}$ for all $a \geq 2$, and L is the infimum of λ satisfying $p(a) < a^{\lambda}$ for all sufficiently large a. Much research [1,2] has been devoted to evaluating K and L, as well as to determining other forms of upper and lower bounds on $p(a, b)$.

Clearly $K > 1.82$ (witness the case $p(5) = 19$). Schinzel & Sierpinski [3] and Kanold [4,5] conjectured that $K \leq 2$. If true, this would imply that there exists a prime somewhere in the following list:

$$b, a+b, 2a+b, \ldots, (a-1)a+b$$

if $\gcd(a, b) = 1$. Such a statement is beyond the reach of present-day mathematics. Schinzel & Sierpinski that confessed they did not know what the fate of their hypothesis (among several) might be. Ribenboim [2] wondered if such hypotheses might be undecidable within the framework of Peano axiomatic arithmetic.

Linnik [6,7] proved that L exists and is finite. Clearly $L \le K$. If we assume a generalized Riemann hypothesis, it is known that [8–10]

$$p(a) = O(\varphi(a)^2 \ln(a)^2),$$

which would imply that $L \le 2$. Here $\varphi(x)$ denotes the Euler totient function [2.7]. The search for an unconditional upper bound for **Linnik's constant** L has occupied many researchers [11–13]. A culmination of this work is Heath-Brown's proof [14] that $L \le 5.5$.

Partial evidence for $L \le 2$ includes the following. For any fixed positive integers b and k, Bombieri, Friedlander & Iwaniec [15] proved that

$$p(a, b) < \frac{a^2}{\ln(a)^k}$$

for every a outside a set of density zero, as observed by Granville [16,17]. We may therefore infer $L \le 2$ for *almost all* integers a.

Chowla [18] believed that $L = 1$. Subsequent authors [19–23] conjectured that

$$p(a) = O(\varphi(a) \ln(a)^2),$$

which would imply that $L = 1$. An earlier theorem of Elliott & Halberstam [24] provides partial support for this new estimate.

We now turn attention to prime solutions of a specific equation. Liu & Tsang [25–28], among others, investigated existence issues of prime solutions p, q, r of the linear equation $ap + bq + cr = d$, where a, b, c are nonzero integers and where it is further assumed that $a + b + c - d$ is even and that $\gcd(a, b, c)$, $\gcd(d, a, b)$, $\gcd(d, a, c)$, $\gcd(d, b, c)$ are each 1. (Note that, if we were to allow $c = 0$, then the case $a = b = 1$ would be equivalent to Goldbach's conjecture and the case $a = 1, b = -1, d = 2$ would be equivalent to the twin prime conjecture.)

There are two cases, depending on whether a, b, c are all positive or not. We discuss only one case here: Suppose a, b, c are not all of the same sign. Then there exists a constant μ with the property that the equation $ap + bq + cr = d$ must have a solution in primes p, q, r satisfying

$$\max(p, q, r) \le 3|d| + (\max(3, |a|, |b|, |c|))^{\mu}.$$

This result is a generalization of Linnik's original theorem.

The infimum M of all such μ is known as **Baker's constant** [29] and it can be proved that $L \le M$. The best-known upper bound [30,31] for M is 45 (unconditional) and 4 (assuming a generalized Riemann hypothesis). Liu & Tsang, like Chowla, conjectured that $M = 1$.

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2.13 Mills' Constant

Mills [1] demonstrated the surprising existence of a positive constant C such that the expression $\left\lfloor C^{3^N} \right\rfloor$ yields only prime numbers for all positive integers N. (Recall that $\lfloor x \rfloor$ denotes the largest integer not exceeding x.) The proof is based on a difficult theorem in prime number theory due to Hoheisel [2] and refined by Ingham [3]: If $p < p'$ are consecutive primes, then given $\varepsilon > 0$,

$$p' - p < p^{(5/8)+\varepsilon}$$

for sufficiently large p. This inequality is used to define the following recursive sequence. Let $q_0 = 2$ and q_{n+1} be the least prime exceeding q_n^3 for each $n \geq 0$. For example [4, 5], $q_1 = 11$, $q_2 = 1361$, and $q_3 = 2521008887$. The Hoheisel–Ingham theorem implies that

$$q_n^3 < q_{n+1} < q_{n+1} + 1 < q_n^3 + q_n^{(15/8)+3\varepsilon} + 1 < (q_n + 1)^3$$

for large n; hence

$$q_n^{3^{-n}} < q_{n+1}^{3^{-(n+1)}} < (q_{n+1} + 1)^{3^{-(n+1)}} < (q_n + 1)^{3^{-n}}.$$

We deduce that $C = \lim_{n\to\infty} q_n^{3^{-n}}$ exists, which yields the desired prime-representing result. For the particular sequence selected here [4, 6, 7], it is easily computed that $C = 1.3063778838\ldots$.

A different choice of starting value q_0 or variation in the exponent 3 will provide a different value of C. There are infinitely many such quantities C; that is, Mills' constant $1.3063778838\ldots$ is not the unique value of C to give only prime numbers. A generalization of Mills' theorem (to arbitrary sequences of positive integers obeying a growth restriction) is an exercise in [8].

Another constant, $c = 1.9287800\ldots$, appears in Wright [9] as part of an alternative prime-representing function:

$$\left\lfloor 2^{2^{\cdot^{\cdot^{2^c}}}} \right\rfloor,$$

the iterated exponential with N 2s and c at the top. Unlike Mills' example, this example does not require a deep theorem to work. All that is needed is the fact that $p' < 2p$, which is known as Bertrand's postulate.

Several authors [6, 7, 10] wisely pointed out that formulas like that of Mills are not very useful. One would need to know C correctly to many places to compute only a few primes. To make matters worse, there does not seem to be any way of estimating C *except* via the primes $q_1, q_2, q_3, \ldots$ (i.e., the reasoning becomes circular). The only manner in which Mills' formula could be useful is if an *exact* value for C were to somehow become available, which no one has conjectured might ever happen.

Nevertheless, the sheer existence of C is striking. It is not known whether C must necessarily be irrational. A similar constant, $1.6222705028\ldots$, due to Odlyzko & Wilf, arises in [2.30]. See [11] for a related problem concerning expressions of the form $\lfloor C^N \rfloor$.

Huxley [12], among others, succeeded in replacing the exponent $5/8$ by $7/12$. Recent work in sharpening the Hoheisel–Ingham theorem includes [13–16]. The best result known to date is

$$p' - p = O(p^{0.525}).$$

Assuming the Riemann hypothesis to be true, Cramér [17, 18] proved that

$$p' - p = O(\sqrt{p}\ln(p)),$$

which would be a dramatic improvement if the unproved assertion someday falls to analysis. He subsequently conjectured that [19]

$$p' - p = O(\ln(p)^2)$$

and, moreover,

$$\limsup_{p\to\infty} \frac{p' - p}{\ln(p)^2} = 1.$$

Granville [20, 21], building upon the work of Maier [22], revised this conjecture as follows:

$$\limsup_{p\to\infty} \frac{p' - p}{\ln(p)^2} \geq 2e^{-\gamma} = 1.122\ldots,$$

where γ is Euler's constant [1.5]. It has been known for a long time [23] that

$$\limsup_{p\to\infty} \frac{p' - p}{\ln(p)} = \infty;$$

thus Cramér's bound $\ln(p)^2$ cannot be replaced by $\ln(p)$. However, we have [24–26]

$$\liminf_{p\to\infty} \frac{p' - p}{\ln(p)} \leq 0.248.$$

Is further improvement possible? If the twin prime conjecture is true [2.1], then the limit infimum is clearly 0.

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2.14 Brun's Constant

Brun's constant is defined to be the sum of the reciprocals of all twin primes [1,2]:

$$B_2 = \left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\left(\frac{1}{29}+\frac{1}{31}\right)+\cdots.$$

Note that the prime 5 is taken twice (some authors do not do this). If this series were divergent, then a proof of the twin prime conjecture [2.1] would follow immediately. Brun proved, however, that the series is convergent and thus B_2 is finite [3–8]. His result demonstrates the scarcity of twin primes relative to all primes (whose reciprocal sum is divergent [2.2]), but it does not shed any light on whether the number of twin primes is finite or infinite.

Selmer [9], Fröberg [10], Bohman [11], Shanks & Wrench [12], Brent [13,14], Nicely [15–18], Sebah [19], and others successively improved numerical estimates of B_2. The most recent calculations give

$$B_2 = 1.9021605831\ldots$$

using large datasets of twin primes and assuming the truth of the extended twin prime conjecture [2.1]. Let us elaborate on the latter issue. Under Hardy & Littlewood's hypothesis, the raw summation of twin prime reciprocals converges very slowly:

$$\sum_{\substack{\text{twin}\\ p\le n}} \frac{1}{p} - B_2 = O\left(\frac{1}{\ln(n)}\right),$$

but the following extrapolation helps to accelerate the process [10,12,15]:

$$\left(\sum_{\substack{\text{twin}\\ p\le n}} \frac{1}{p} + \frac{4C_{\text{twin}}}{\ln(n)}\right) - B_2 = O\left(\frac{1}{\sqrt{n}\ln(n)}\right),$$

where $C_{\text{twin}} = 0.6601618158\ldots$ is the twin prime constant. Higher order extrapolations exist but do not present practical advantages as yet. In the midst of his computations, Nicely [15] uncovered the infamous Intel Pentium error.

We discuss three relevant variations. Let A_3 denote the reciprocal sum of prime 3-tuples of the form $(p, p+2, p+6)$, A_3' the reciprocal sum of prime 3-tuples of the form $(p, p+4, p+6)$, and A_4 the reciprocal sum of prime 4-tuples of the form $(p, p+2, p+6, p+8)$. Nicely [2,20] calculated

$$A_3 = 1.0978510391\ldots,\quad A_3' = 0.8371132125\ldots,\quad A_4 = 0.8705883800\ldots.$$

Define B_h, where $h \geq 2$ is an even integer, to be the reciprocal sum of primes separated by h, and define $\tilde{B}_h$ to be the reciprocal sum of *consecutive* primes separated by h. Segal proved that B_h is finite for all h [5,21,22]; thus $\tilde{B}_h$ is finite as well. Clearly $B_2 = \tilde{B}_2$ and

$$B_4 = \left(\frac{1}{3} + \frac{1}{7}\right) + \left(\frac{1}{7} + \frac{1}{11}\right) + \left(\frac{1}{13} + \frac{1}{17}\right) + \left(\frac{1}{19} + \frac{1}{23}\right) + \cdots = \tilde{B}_4 + \frac{10}{21},$$

but highly precise computations of B_h or $\tilde{B}_h$, $h \geq 4$, have not yet been performed. Wolf [23] speculated that, for $h \geq 6$,

$$\tilde{B}_h = \frac{4C_{\text{twin}}}{h} \prod_{\substack{p|h \\ p>2}} \frac{p-1}{p-2}$$

on the basis of a small dataset. Even if his conjecture is eventually shown to be false, it should inspire more attempts to relate such generalized Brun's constants to other constants found in number theory.

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2.15 Glaisher–Kinkelin Constant

Stirling's formula [1]

$$\lim_{n\to\infty} \frac{n!}{e^{-n}n^{n+\frac{1}{2}}} = \sqrt{2\pi}$$

provides a well-known estimate for large factorials. If we replace $n! = \Gamma(n+1)$ by different expressions, for example,

$$K(n+1) = \prod_{m=1}^{n} m^m \text{ or } G(n+1) = \frac{(n!)^n}{K(n+1)} = \prod_{m=1}^{n-1} m!$$

then the approximation takes different forms. Kinkelin [2], Jeffery [3], and Glaisher [4–6] demonstrated that

$$\lim_{n\to\infty} \frac{K(n+1)}{e^{-\frac{1}{4}n^2} n^{\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{12}}} = A \text{ and } \lim_{n\to\infty} \frac{G(n+1)}{e^{-\frac{3}{4}n^2}(2\pi)^{\frac{1}{2}n} n^{\frac{1}{2}n^2-\frac{1}{12}}} = \frac{e^{\frac{1}{12}}}{A}.$$

The constant A, which plays the same role in these approximations as $\sqrt{2\pi}$ plays in Stirling's formula, has the following closed-form expression:

$$A = \exp\left(\frac{1}{12} - \zeta'(-1)\right) = \exp\left(\frac{-\zeta'(2)}{2\pi^2} + \frac{\ln(2\pi)+\gamma}{12}\right) = 1.2824271291\ldots,$$

where $\zeta'(x)$ is the derivative of the Riemann zeta function [1.6] and γ is the Euler–Mascheroni constant [1.5]. See [2.10] and [2.18] for other occurrences of $\zeta'(2)$.

Many beautiful formulas involving A exist, including two infinite products [6]:

$$1^{\frac{1}{1}} \cdot 2^{\frac{1}{4}} \cdot 3^{\frac{1}{9}} \cdot 4^{\frac{1}{16}} \cdot 5^{\frac{1}{25}} \cdots = \left(\frac{A^{12}}{2\pi e^{\gamma}}\right)^{\frac{\pi^2}{6}},$$

$$1^{\frac{1}{1}} \cdot 3^{\frac{1}{9}} \cdot 5^{\frac{1}{25}} \cdot 7^{\frac{1}{49}} \cdot 9^{\frac{1}{81}} \cdots = \left(\frac{A^{36}}{2^4\pi^3 e^{3\gamma}}\right)^{\frac{\pi^2}{24}},$$

and two definite integrals [4, 7]:

$$\int_0^\infty \frac{x\ln(x)}{e^{2\pi x}-1}dx = \frac{1}{24} - \frac{1}{2}\ln(A),$$

$$\int_0^{1/2} \ln(\Gamma(x+1))dx = -\frac{1}{2} - \frac{7}{24}\ln(2) + \frac{1}{4}\ln(\pi) + \frac{3}{2}\ln(A).$$

More formulas are found in [8–12].

A generalization of the latter integral,

$$\int_0^x \ln(\Gamma(t+1))dt = \frac{1}{2}\ln(2\pi)x - \frac{1}{2}x(x+1) + x\ln(\Gamma(x+1)) - \ln(G(x+1)),$$

was obtained by Alexeiewsky [13], Hölder [14], and Barnes [15–17] using an analytic extension of $G(n+1)$. Just as the gamma function extends the factorial function $\Gamma(n+1)$ to the complex z-plane, the **Barnes G-function**

$$G(z+1) = (2\pi)^{\frac{1}{2}z} e^{-\frac{1}{2}z(z+1)-\frac{\gamma}{2}z^2} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^n e^{-z+\frac{1}{2n}z^2}$$

extends $G(n+1)$. Just as the gamma function assumes a special value at $z = 1/2$:

$$\Gamma\left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right)! = \sqrt{\pi},$$

the Barnes function satisfies

$$G\left(\frac{1}{2}\right) = 2^{\frac{1}{24}} e^{\frac{1}{8}} \pi^{-\frac{1}{4}} A^{-\frac{3}{2}}.$$

A similar, natural extension of Kinkelin's function via $K(z+1) = \Gamma(z+1)^z/G(z+1)$ has been comparatively neglected by researchers in favor of G. Here is a sample application. Define

$$D(x) = \lim_{n\to\infty} \prod_{k=1}^{2n+1} \left(1+\frac{x}{k}\right)^{(-1)^{k+1}k} = \exp(x)\cdot \lim_{n\to\infty} \prod_{k=1}^{2n} \left(1+\frac{x}{k}\right)^{(-1)^{k+1}k}.$$

Melzak [18] proved that $D(2) = \pi e/2$. Borwein & Dykshoorn [19] extended this result to

$$D(x) = \left(\frac{\Gamma(\frac{x}{2}+\frac{1}{2})}{\Gamma(\frac{x}{2})}\right)^x \left(\frac{K(\frac{x}{2})K(\frac{1}{2})}{K(\frac{x}{2}+\frac{1}{2})}\right)^2 \exp(-\tfrac{x}{2}).$$

where $x > 0$. As a special case, $D(1) = A^6/(2^{\frac{1}{6}}\pi^{\frac{1}{2}})$.

Apart from infrequent whispers [20–27], the Glaisher–Kinkelin constant seemed largely forgotten until recently. Vignéras [28], Voros [29], Sarnak [30], Vardi [31], and others revived interest in the Barnes G-function because of its connection to certain spectral functions in mathematical physics and differential geometry. There is also a connection with random matrix theory and the spacing of zeta function zeros [32–34]. See [2.15.3] and [5.22] as well. Thus generalizations of the formulas here for $\Gamma(1/2)$ and $G(1/2)$ possess a significance unanticipated by their original discoverers.

2.15.1 Generalized Glaisher Constants

Bendersky [35, 36] studied the product $1^{1^k} \cdot 2^{2^k} \cdot 3^{3^k} \cdot 4^{4^k} \cdots n^{n^k}$, which is $n!$ for $k = 0$ and $K(n+1)$ for $k = 1$. More precisely, he examined the logarithm of the product and determined the value of the limit

$$\ln(A_k) = \lim_{n\to\infty}\left(\sum_{m=1}^{n} m^k \ln(m) - p_k(n)\right),$$

where

$$p_k(n) = \left(\tfrac{n^{k+1}}{k+1} + \tfrac{n^k}{2} + \tfrac{B_{k+1}}{k+1}\right)\ln(n) - \tfrac{n^{k+1}}{(k+1)^2}$$
$$+\, k!\sum_{j=1}^{k-1} \tfrac{B_{j+1}}{(j+1)!}\tfrac{n^{k-j}}{(k-j)!}\left(\ln(n) + \sum_{i=1}^{j}\tfrac{1}{k-i+1}\right)$$

and B_n is the n^{th} Bernoulli number [1.6.1]. Clearly $A_0 = \sqrt{2\pi}$ and $A_1 = A$. Choudhury [37] and Adamchik [38] obtained the following exact expression for all $k \geq 0$:

$$A_k = \exp\left(\frac{B_{k+1}}{k+1}\sum_{j=1}^{k}\frac{1}{j} - \zeta'(-k)\right) = \begin{cases} 1.0309167521\ldots & \text{if } k = 2,\\ 0.9795555269\ldots & \text{if } k = 3,\\ 0.9920479745\ldots & \text{if } k = 4,\\ 1.0096803872\ldots & \text{if } k = 5. \end{cases}$$

Zeta derivatives at negative integers can be transformed: If $n > 0$, then [12, 39]

$$\zeta'(-2n) = (-1)^n \frac{(2n)!}{2(2\pi)^{2n}}\zeta(2n+1),$$

$$\zeta'(-2n+1) = \frac{1}{2n}\left[(-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\zeta'(2n) + \left(\sum_{j=1}^{2n-1}\frac{1}{j} - \ln(2\pi) - \gamma\right)B_{2n}\right].$$

It follows that $\ln(A_2) = \zeta(3)/(4\pi^2)$ and $\ln(A_3) = 3\zeta'(4)/(4\pi^4) - (\ln(2\pi)+\gamma)/120$.

2.15.2 Multiple Barnes Functions

Barnes [40] defined a sequence of functions $\{G_n(z)\}$ on the complex plane satisfying

$$G_0(z) = z, \qquad G_n(1) = 1, \qquad G_{n+1}(z+1) = \frac{G_{n+1}(z)}{G_n(z)} \quad \text{for } n \geq 0.$$

The sequence is unique, by an argument akin to the Bohr–Mollerup theorem [41], if it is further assumed that

$$(-1)^n \frac{d^{n+1}}{dx^{n+1}}\ln(G_n(x)) \geq 0 \quad \text{for } x > 0.$$

Clearly $G_1(z) = 1/\Gamma(z)$ and $G_2(z) = G(z)$. Properties of $\{G_n(z)\}$ are given in [31, 42, 43]. Of special interest are the values of $G_n(1/2)$. Adamchik [42] determined the simplest known formula for these:

$$\ln\left(G_n\left(\tfrac{1}{2}\right)\right) = \tfrac{1}{(n-1)!}\left[-\tfrac{\ln(\pi)}{2^n}\prod_{k=2}^{n}(2k-3) + \sum_{m=1}^{n}\left(\ln(2)\tfrac{B_{m+1}}{m+1} + (2^{m+1}-1)\zeta'(-m)\right)\tfrac{q_{m,n}}{2^m}\right],$$

where $q_{m,n}$ is the coefficient of x^m in the expansion of the polynomial $2^{1-n}\prod_{j=1}^{n-1}(2x+2j-1)$. We may hence write

$$\ln\left(G_3\left(\frac{1}{2}\right)\right) = \frac{1}{8} + \frac{1}{24}\ln(2) - \frac{3}{16}\ln(\pi) - \frac{3}{2}\ln(A_1) - \frac{7}{8}\ln(A_2),$$

$$\ln\left(G_4\left(\frac{1}{2}\right)\right) = \frac{265}{2304} + \frac{229}{5760}\ln(2) - \frac{5}{32}\ln(\pi) - \frac{23}{16}\ln(A_1) - \frac{21}{16}\ln(A_2) - \frac{5}{16}\ln(A_3)$$

in terms of the generalized Glaisher constants A_k.

2.15.3 GUE Hypothesis

Assume that the Riemann hypothesis [1.6.2] is true. Let

$$\gamma_1 = 14.1347251417\ldots \le \gamma_2 = 21.0220396387\ldots \le \gamma_3 = 25.0108575801\ldots \le \gamma_4 \le \gamma_5 \le \ldots$$

denote the imaginary parts of the nontrivial zeros of $\zeta(z)$ in the upper half-plane. If $N(T)$ denotes the number of such zeros with imaginary part $< T$, then the Riemann–von Mongoldt formula [44] gives

$$N(T) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) + O(\ln(T))$$

as $T \to \infty$, and hence

$$\gamma_n \sim \frac{2\pi n}{\ln(n)}$$

as $n \to \infty$. The mean spacing between γ_n and γ_{n+1} tends to zero as $n \to \infty$, so it is useful to renormalize (or "unfold") the consecutive differences to be

$$\delta_n = \frac{\gamma_{n+1} - \gamma_n}{2\pi}\ln\left(\frac{\gamma_n}{2\pi}\right),$$

and thus δ_n has mean value 1.

What can be said about the probability distribution of δ_n? That is, what density function $p(s)$ satisfies

$$\lim_{N\to\infty}\frac{1}{N}\left|\{n : 1 \le n \le N,\ \alpha \le \delta_n \le \beta\}\right| = \int_\alpha^\beta p(s)ds$$

for all $0 < \alpha < \beta$?

Here is a fascinating conjectured answer. A random Hermitian $N \times N$ matrix X is said to belong to the **Gaussian unitary ensemble** (GUE) if its (real) diagonal elements x_{jj} and (complex) upper triangular elements $x_{jk} = u_{jk} + iv_{jk}$ are independently chosen from zero-mean Gaussian distributions with $\mathrm{Var}(x_{jj}) = 2$ for $1 \le j \le N$ and

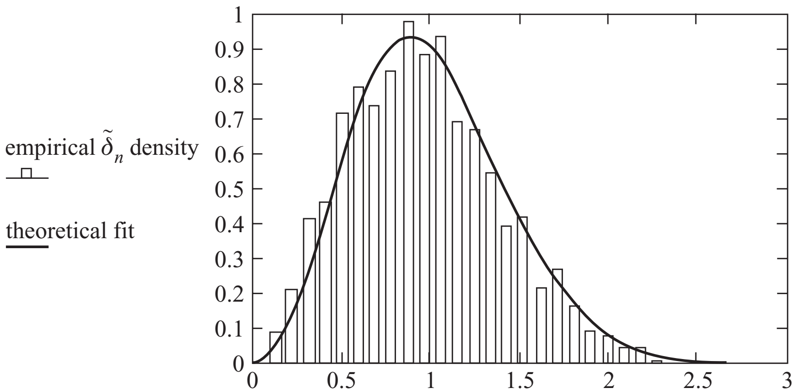

Figure 2.1. In a small simulation, the eigenvalues of fifty 120×120 random GUE matrices were generated. The resulting histogram plot of $\tilde{\delta}_n$ compares well against $p(s)$.

$\mathrm{Var}(u_{jk}) = \mathrm{Var}(v_{jk}) = 1$ for $1 \le j < k \le N$. Let $\lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \le \lambda_N$ denote the (real) eigenvalues of X and consider the normalized spacings

$$\tilde{\delta}_n = \frac{\lambda_{n+1} - \lambda_n}{4\pi}\sqrt{8N - \lambda_n^2}, \quad n \approx \frac{N}{2}.$$

With this choice of scaling, $\tilde{\delta}_n$ has mean value 1. The probability density of $\tilde{\delta}_n$, in the limit as $N \to \infty$, tends to what is called the **Gaudin density** $p(s)$. Inspired by some theoretical work by Montgomery [45], Odlyzko [46–50] experimentally determined that the distributions for δ_n and $\tilde{\delta}_n$ are very close. The **GUE hypothesis** (or **Montgomery–Odlyzko law**) is the astonishing conjecture that the two distributions are identical. See Figure 2.1.

Furthermore, there are extensive results concerning the function $p(s)$. Define

$$E(s) = \exp\left(\int_0^{\pi s} \frac{\sigma(t)}{t} dt\right),$$

where $\sigma(t)$ satisfies the Painlevé V differential equation (in "sigma form")

$$(t \cdot \sigma'')^2 + 4(t \cdot \sigma' - \sigma)[t \cdot \sigma' - \sigma + (\sigma')^2] = 0$$

with boundary conditions

$$\sigma(t) \sim -\frac{t}{\pi} - \left(\frac{t}{\pi}\right)^2 \text{ as } t \to 0^+, \quad \sigma(t) \sim -\left(\frac{t}{2}\right)^2 - \frac{1}{4} \text{ as } t \to \infty;$$

then $p(s) = d^2E/ds^2$.

The Painlevé representation [51–56] above allows straightforward numerical calculation of $p(s)$, although historically a Fredholm determinant representation [49, 57, 58] for $E(s)$ came earlier. (Incidently, Painlevé II arises in our discussion of the longest increasing subsequence problem [5.20], and Painlevé III arises in connection with the Ising model [5.22].)

Using $p(s)$, one could compute the median, mode, and variance of $\tilde{\delta}_n$, as well as higher moments.

Here is an interesting problem having to do with the tail of the Gaudin distribution [59,60]. The function $E(s)$ can be interpreted as the probability that the interval $[0, s]$ contains no (scaled) eigenvalues. If the specific interval $[0, s]$ is replaced by an arbitrary interval of length s, then the probability remains the same. We know that [49,61]

$$E(s) \sim 1 - s + \frac{\pi^2 s^4}{360} \text{ as } s \to 0^+, \quad E(s) \sim C \cdot (\pi s)^{-\frac{1}{4}} \exp\left(-\tfrac{1}{8}(\pi s)^2\right) \text{ as } s \to \infty,$$

where C is a constant. Dyson [49,62] nonrigorously identified

$$C = 2^{\frac{1}{3}} e^{3\zeta'(-1)} = 2^{\frac{1}{4}} e^{2B}$$

using a result of Widom [63], where

$$B = \frac{1}{24}\ln(2) + \frac{3}{2}\zeta'(-1) = -0.2192505830\ldots.$$

This, in turn, is related to Glaisher's constant A via the formula [22]

$$e^{2B} = 2^{\frac{1}{12}} e^{\frac{1}{4}} A^{-3}.$$

It is curious that a complete asymptotic expansion for $E(s)$ is now known [60,64–66], all rigorously obtained except for the factor C! Similar phenomena were reported in [67–70] in connection with certain associated problems.

There is another way of looking at the GUE hypothesis. Let us return to the normalized differences δ_n of consecutive zeta function zeros and define

$$\Delta_{nk} = \sum_{j=0}^{k} \delta_{n+j}.$$

Earlier, k was constrained to be 0. If now $k \geq 0$ is allowed to vary, what is the "distribution" of Δ_{nk}? Montgomery [45] conjectured that the following simple formula is true:

$$\lim_{N\to\infty} \frac{1}{N} \left|\{(n,k) : 1 \leq n \leq N,\ k \geq 0,\ \alpha \leq \Delta_{nk} \leq \beta\}\right| = \int_{\alpha}^{\beta} \left[1 - \left(\frac{\sin(\pi r)}{\pi r}\right)^2\right] dr.$$

In other words, $1 - (\sin(\pi r)/(\pi r))^2$ is the **pair correlation function** of zeros of the zeta function, as predicted by Montgomery's partial results. Incredibly, it has been proved that GUE eigenvalues possess the same pair correlation function. Odlyzko [46–49] again has accumulated extensive numerical evidence supporting this conjecture. The implications of the pair correlation conjecture for prime number theory were explored in [71]. Hejhal [72] studied a three-dimensional analog, known as the triple correlation conjecture; higher level correlations were examined in [73].

Careful readers will note the restriction $n \approx N/2$ in the preceding definition of $\tilde{\delta}_n$. In our small simulation, we took only the middle third of the eigenvalues, sampling what is known as the "bulk" of the spectrum. If we sampled instead the "edges" of the spectrum, a different density emerges [69,70]. The sine kernel in the Fredholm determinant for the "bulk" is replaced by the Airy kernel for the "edges."

Rudnick & Sarnak [73,74] and Katz & Sarnak [75,76] generalized the GUE hypothesis to a wider, more abstract setting. They gave proofs in certain important special cases, but not in the original case discussed here.

There is interest in the limit superior and limit inferior of δ_n, which are conjectured to be ∞ and 0, respectively [77–80].

A huge amount of research has been conducted in the area of random matrices (with no symmetry assumed) and the related subject of random polynomials. We mention only one sample result. Let $q(x)$ be a random polynomial of degree n, with real coefficients independently chosen from a standard Gaussian distribution. Let z_n denote the expected number of real zeros of $q(x)$. Kac [81,82] proved that

$$\lim_{n\to\infty} \frac{z_n}{\ln(n)} = \frac{2}{\pi},$$

and it is known that [82–88]

$$\lim_{n\to\infty} z_n - \frac{2}{\pi}\ln(n) = c,$$

where

$$\begin{aligned} c &= \frac{2}{\pi}\left[\ln(2) + \int_0^\infty \left(\sqrt{x^{-2} - 4e^{-2x}(1-e^{-2x})^{-2}} - (x+1)^{-1}\right) dx\right] \\ &= 0.6257358072\ldots. \end{aligned}$$

More terms of the asymptotic expansion are known; see [82,87] for an overview.

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2.16 Stolarsky–Harborth Constant

Given a positive integer k, let $b(k)$ denote the number of ones in the binary expansion of k. Glaisher [1–6] showed that the number of odd binomial coefficients of the form $\binom{k}{j}$, $0 \le j \le k$, is $2^{b(k)}$. As a consequence, the number of odd elements in the first n rows of Pascal's triangle is

$$f(n) = \sum_{k=0}^{n-1} 2^{b(k)}$$

and satisfies the recurrence

$$f(0) = 0, \quad f(1) = 1, \quad f(n) = \begin{cases} 3f(m) & \text{if } n = 2m \\ 2f(m) + f(m+1) & \text{if } n = 2m+1 \end{cases} \quad \text{for } n \ge 2.$$

The question is: Can a simple approximation for $f(n)$ be found? The answer is yes. Let $\theta = \ln(3)/\ln(2) = 1.5849625007\ldots$, the fractal dimension [7, 8] of Pascal's triangle modulo 2. It turns out that n^θ is a reasonable approximation for $f(n)$. It also turns out that $f(n)$ is not well behaved asymptotically. Stolarsky [9] and Harborth [10] determined that

$$0.812556 < \lambda = \liminf_{n\to\infty} \frac{f(n)}{n^\theta} < 0.812557 < \limsup_{n\to\infty} \frac{f(n)}{n^\theta} = 1,$$

and we call $\lambda = 0.8125565590\ldots$ the **Stolarsky–Harborth constant**.

Here is a generalization. Let p be a prime and $f_p(n)$ be the number of elements in the first n rows of Pascal's triangle that are not divisible by p. Define

$$\theta_p = \frac{\ln\left(\frac{p(p+1)}{2}\right)}{\ln(p)}$$

and note that $\lim\limits_{n\to\infty} \theta_p = 2$. Of course, $f_2(n) = f(n)$ and $\theta_2 = \theta$. It is known that [11–14]

$$\lambda_p = \liminf_{n\to\infty} \frac{f_p(n)}{n^{\theta_p}} < \limsup_{n\to\infty} \frac{f_p(n)}{n^{\theta_p}} = 1,$$

$$\lambda_3 = \left(\tfrac{3}{2}\right)^{1-\theta_3} = 0.7742\ldots, \quad \lim_{p\to\infty} \lambda_p = \tfrac{1}{2}$$

and further conjectured that

$$\lambda_5 = \left(\tfrac{3}{2}\right)^{1-\theta_5} = 0.7582\ldots, \quad \lambda_7 = \left(\tfrac{3}{2}\right)^{1-\theta_7} = 0.7491\ldots,$$
$$\lambda_{11} = \tfrac{59}{44}\left(\tfrac{22}{31}\right)^{\theta_{11}} = 0.7364\ldots.$$

Curiously, no such exact formula for $\lambda_2 = \lambda$ has been found. A broader generalization involves multinomial coefficients [15–17].

2.16.1 Digital Sums

The expression $f_2(n)$ is an **exponential sum of digital sums**. Another example is

$$m_p(n) = \sum_{k=0}^{n-1} (-1)^{b(p\,k)},$$

which, in the case $p = 3$, quantifies an empirical observation that multiples of 3 prefer to have an even number of 1-digits. We will first discuss, however, a **power sum of digital sums**:

$$s_q(n) = \sum_{k=0}^{n-1} b(k)^q$$

and set $q = 1$ for the sake of concreteness.

Trollope [18] and Delange [19], building upon [20–26], proved that

$$s_1(n) = \frac{1}{2\ln(2)} n \ln(n) + n\, S\left(\frac{\ln(n)}{\ln(2)}\right)$$

exactly, where $S(x)$ is a certain continuous nowhere-differentiable function of period 1,

$$-0.2075\ldots = \frac{\ln(3)}{2\ln(2)} - 1 = \inf_x S(x) < \sup_x S(x) = 0,$$

and the Fourier coefficients of $S(x)$ are all known. See Figure 2.2. The mean value of $S(x)$ is [19,27]

$$\int_0^1 S(x)dx = \frac{1}{2\ln(2)}(\ln(2\pi) - 1) - \frac{3}{4} = -0.1455\ldots.$$

Extensions of this remarkable result to arbitrary q appear in [28–36].

Let $\omega = \theta/2$ and $\varepsilon(n) = (-1)^{b(3n-1)}$ if n is odd, 0 otherwise. Newman [37–39] proved that $m_3(n) > 0$ always and is $O(n^{\omega})$. Coquet [40] strengthened this to

$$m_3(n) = n^{\omega}\, M\left(\frac{\ln(n)}{2\ln(2)}\right) + \frac{1}{3}\varepsilon(n),$$

where $M(x)$ is a continuous nowhere-differentiable function of period 1,

$$1.1547\ldots = \frac{2\sqrt{3}}{3} = \inf_x M(x) < \sup_x M(x) = \frac{55}{3}\left(\frac{3}{65}\right)^{\omega} = 1.6019\ldots$$

and, again, the Fourier coefficients of $M(x)$ are all known. The mean value [27] of $M(x)$ is 1.4092203477... but has a complicated integral expression. Extensions of this result to $p = 5$ and 17 appear in [41–43]. The pattern in $\{(-1)^{b(k)}\}$ follows the well-known Prouhet–Thue–Morse sequence [6.8], and associated sums of subsequences of the form $\{(-1)^{b(pk+r)}\}$ are discussed in [44–46].

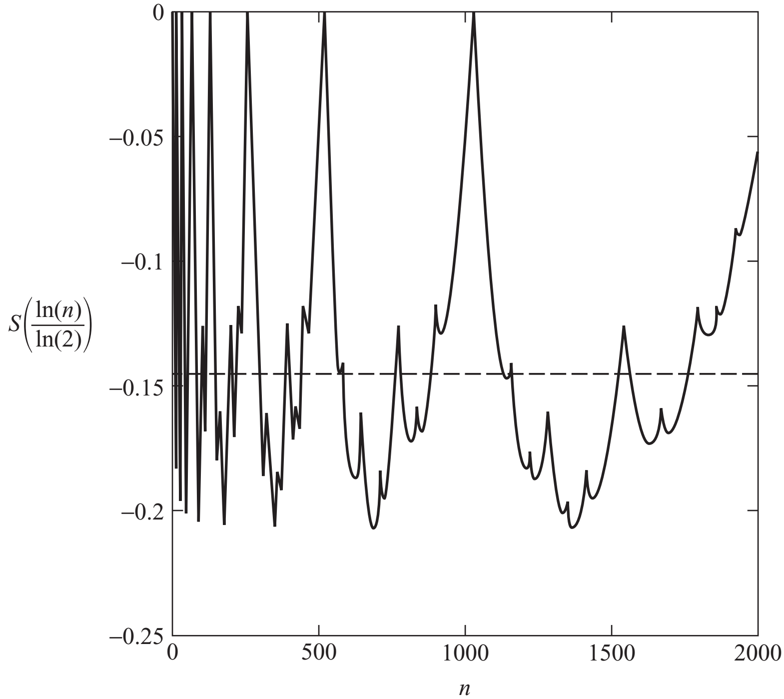

Figure 2.2. The Trollope–Delange function is pictured, as well as its mean value.

We return to binomial coefficients. Stein [47] proved that

$$f_2(n) = n^{\theta}\, F\left(\frac{\ln(n)}{\ln(2)}\right),$$

where $F(x)$ is a continuous function of period 1; by way of contrast, $F(x)$ is differentiable almost everywhere, but is nowhere monotonic [48]. This fact, however, does not appear to give any insight concerning an exact formula for $\lambda_2 = \inf_x F(x)$. The Fourier coefficients of $F(x)$ are all known, and the mean value [27] of $F(x)$ is $0.8636049963\ldots$. Again, the underlying integral is complicated.

This material plays a role in the analysis of algorithms, for example, in approximating the register function for binary trees [49], and in studying mergesort [50], maxima finding [51], and other divide-and-conquer recurrences [52, 53].

2.16.2 Ulam 1-Additive Sequences

There is an unexpected connection between digital sums and **Ulam 1-additive sequences** [54]. Let $u < v$ be positive integers. The 1-additive sequence with base u, v is the infinite sequence $(u, v) = a_1, a_2, a_3, \ldots$ with $a_1 = u$, $a_2 = v$ and a_n is the least integer exceeding a_{n-1} and possessing a unique representation $a_n = a_i + a_j$, $i < j$,

$n \geq 3$. Ulam's archetypal sequence

$$(1, 2) = 1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, \ldots$$

remains a mystery. No pattern in its successive differences has ever been observed. Ulam conjectured that the density of $(1, 2)$, relative to the positive integers, is 0. No one has yet found a proof of this.

Substantially more is known about the cases $(2, v)$, where v is odd, and $(4, v)$, where v additionally is congruent to 1 modulo 4. Cassaigne & Finch [55] proved that the successive differences of the Ulam 1-additive sequence $(4, v)$ are eventually periodic and that the density of $(4, v)$ is

$$d(v) = \frac{1}{2(v+1)} \sum_{k=0}^{(v-1)/2} 2^{-b(k)}.$$

It can be shown that $d(v) \to 0$ as $v \to \infty$. The techniques giving rise to the Stolarsky–Harborth constant λ can be modified to give the following more precise asymptotic estimate of the density:

$$\frac{1}{4} = \liminf_{\substack{v \to \infty \\ v \equiv 1 \bmod 4}} \left(\frac{v}{2}\right)^{2-\theta} d(v) < 0.272190 < \limsup_{\substack{v \to \infty \\ v \equiv 1 \bmod 4}} \left(\frac{v}{2}\right)^{2-\theta} d(v) < 0.272191.$$

A certain family of ternary quadratic recurrences and its periodicity properties play a crucial role in the proof in [55]. It is natural to ask how far this circle of ideas and techniques can be extended.

2.16.3 Alternating Bit Sets

If n is a positive integer satisfying $2^{k-1} \leq n < 2^k$, clearly the binary expansion of n has k bits. Define an **alternating bit set** in n to be a subset of the k bit positions of n with the following properties [6, 56–58]:

- The bits of n that lie in these positions are alternatively 1s and 0s.
- The leftmost (most significant) of these is a 1.
- The rightmost (least significant) of these is a 0.

Let $c(n)$ be the cardinality of all alternating bit sets of n. For example, $c(26) = 8$ since 26 is 11010 in binary and hence all alternating bit sets of 26 are

$$\{\}, \{5, 3\}, \{5, 1\}, \{4, 3\}, \{4, 1\}, \{2, 1\}, \{5, 3, 2, 1\}, \text{ and } \{4, 3, 2, 1\}.$$

Although $c(n)$ is not a digital sum like $b(n)$, it has similarly interesting combinatorial properties: $c(n)$ is the number of ways of writing n as a sum of powers of 2, with each power used at most twice. It satisfies the recurrence

$$c(0) = 1, \qquad c(n) = \begin{cases} c(m) + c(m-1) & \text{if } n = 2m \\ c(m) & \text{if } n = 2m+1 \end{cases} \qquad \text{for } n \geq 1.$$

It is also linked to the Fibonacci sequence in subtle ways and one can prove that [57]

$$0.9588 < \limsup_{n\to\infty} \frac{c(n)}{n^{\ln(\varphi)/\ln(2)}} < 1.1709,$$

where φ is the Golden mean [1.2]. What is the exact value of this limit supremum? Is there a reason to doubt that its exact value is 1?

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2.17 Gauss–Kuzmin–Wirsing Constant

Let x_0 be a random number drawn uniformly from the interval $(0, 1)$. Write x_0 (uniquely) as a regular continued fraction

$$x_0 = 0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots,$$

where each a_k is a positive integer, and define for all $n > 0$,

$$x_n = 0 + \frac{1|}{|a_{n+1}} + \frac{1|}{|a_{n+2}} + \frac{1|}{|a_{n+3}} + \cdots.$$

For each n, x_n is also a number in $(0, 1)$ since $x_n = \{1/x_{n-1}\}$, where $\{y\}$ denotes the fractional part of y.

In 1812, Gauss examined the distribution function [1]

$$F_n(x) = \text{probability that } x_n \le x$$

and believed that he possessed a proof of a remarkable limiting result:

$$\lim_{n\to\infty} F_n(x) = \frac{\ln(1+x)}{\ln(2)}, \quad 0 \le x \le 1.$$

The first published proof is due to Kuzmin [2], with subsequent improvements in error bounds by Lévy [3] and Szüsz [4]. Wirsing [5] went farther and gave a proof that

$$\lim_{n\to\infty} \frac{F_n(x) - \frac{\ln(1+x)}{\ln(2)}}{(-c)^n} = \Psi(x),$$

where $c = 0.3036630028\ldots$ and Ψ is an analytic function satisfying $\Psi(0) = \Psi(1) = 0$. A graph in [6] suggests that Ψ is convex and $-0.1 < \Psi(x) < 0$ for $0 < x < 1$. The constant c is apparently unrelated to more familiar constants and is computed as an eigenvalue of a certain infinite-dimensional linear operator [2.17.1], with $\Psi(x)$ as the corresponding eigenfunction. The key to this analysis is the identity

$$F_{n+1}(x) = T[F_n](x) = \sum_{k=1}^{\infty}\left[F_n\left(\frac{1}{k}\right) - F_n\left(\frac{1}{k+x}\right)\right].$$

Babenko & Jurev [7–9] went even farther in establishing that a certain eigenvalue/eigenfunction expansion,

$$F_n(x) - \frac{\ln(1+x)}{\ln(2)} = \sum_{k=2}^{\infty} \lambda_k^n \cdot \Psi_k(x), \quad 1 = \lambda_1 > |\lambda_2| \ge |\lambda_3| \ge \ldots,$$

is valid for all x and all $n > 0$. Building upon the work of others [1,5,6,10,11], Sebah [12] computed the Gauss–Kuzmin–Wirsing constant c to 100 digits, as well as the eigenvalues λ_k for $3 \le k \le 50$.

Some related paths of research are indicated in [13–19], but these are too far afield for us to discuss.

2.17.1 Ruelle–Mayer Operators

The operators examined here first arose in dynamical systems [20,21]. Let Δ denote the open disk of radius $3/2$ with center at 1, and let $s > 1$. Let X denote the Banach space of functions f that are analytic on Δ and continuous on the closure of Δ, equipped with the supremum norm. Define a linear operator $G_s : X \to X$ by the formula [10,11,22,23]

$$G_s[f](z) = \sum_{k=1}^{\infty} \frac{1}{(k+z)^s} f\left(\frac{1}{k+z}\right), \quad z \in \Delta.$$

We will examine only the case $s = 2$ here; the case $s = 4$ is needed in [2.19].

Note that the derivative $T[F]'(x) = G_2[f](x)$, where $F' = f$, hence an understanding of G_2 carries over to T. The first six eigenvalues [1,6,10–12] of G_2 after $\lambda_1 = 1$ are

$$\lambda_2 = -0.3036630028\ldots, \ \lambda_3 = 0.1008845092\ldots, \quad \lambda_4 = -0.0354961590\ldots,$$
$$\lambda_5 = 0.0128437903\ldots, \ \lambda_6 = -0.0047177775\ldots, \ \lambda_7 = 0.0017486751\ldots.$$

On the one hand, it might be conjectured that

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = -1 - \varphi = -2.6180339887\ldots,$$

where φ is the Golden mean [1.2]. On the other hand, it has been proved that the trace of G_2 is exactly given by [11]

$$\tau_1 = \frac{1}{2} - \frac{1}{2\sqrt{5}} + \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\binom{2k}{k}(\zeta(2k)-1) = 0.7711255236\ldots,$$

where $\tau_n = \sum_{j=1}^{\infty} \lambda_j^n$. The connection between G_s and zeta function values [1.6] is not surprising: Look at G_s applied to $f(z) = z^r$; then consider Maclaurin expansions of arbitrary functions f and the linearity of G_s.

Other interesting trace formulas include the following. Let [24, 25]

$$\xi_n = 0 + \frac{1|}{|n} + \frac{1|}{|n} + \frac{1|}{|n} + \cdots, \quad n = 1, 2, 3, \ldots.$$

Then

$$\tau_1 = \int_0^\infty \frac{J_1(2u)}{e^u - 1} du = \sum_{n=1}^{\infty} \frac{1}{1 + \xi_n^{-2}},$$

where

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!}\left(\frac{x}{2}\right)^{2k+1}$$

is the Bessel function of first order. In the same way, if

$$\xi_{m,n} = 0 + \frac{1|}{|m} + \frac{1|}{|n} + \frac{1|}{|m} + \frac{1|}{|n} + \cdots$$

then

$$\tau_2 = \int_0^\infty\int_0^\infty \frac{J_1(2\sqrt{uv})^2}{(e^u - 1)(e^v - 1)} du\, dv = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(\xi_{m,n}\xi_{n,m})^{-2} - 1} = 1.1038396536\ldots.$$

Generalization of these is possible.

It can be proved that the dominant eigenvalue $\lambda_1(s)$ of G_s (of largest modulus) is positive and unique, that the function $s \to \lambda_1(s)$ is analytic and strictly decreasing, and that [26]

$$\lim_{s\to 1^+}(s-1)\lambda_1(s) = 1, \quad \lambda_1(2) = 1, \quad \lim_{s\to\infty} \frac{1}{s}\ln(\lambda_1(s)) = -\ln(\varphi).$$

A simple argument [22] shows that $\lambda_1'(2) = -\pi^2/(12\ln(2))$ is Lévy's constant [1.8]. Later, we will see that both $\lambda_1'(2)$ and $\lambda_1''(2)$ arise in connection with determining precisely the efficiency of the Euclidean algorithm [2.18]. Likewise, $\lambda_1(4)$ occurs in the analysis of certain comparison and sorting algorithms [2.19]. It is known that all eigenvalues $\lambda_j(s)$ are real, but questions of sign and uniqueness remain open for $j > 1$.

Here is an alternative definition of $\lambda_1(s)$. For any k-dimensional vector $w = (w_1, \ldots, w_k)$ of positive integers, let $\langle w \rangle$ denote the denominator of the continued fraction

$$0 + \frac{1|}{|w_1} + \frac{1|}{|w_2} + \frac{1|}{|w_3} + \cdots + \frac{1|}{|w_k}$$

and let $W(k)$ be the set of all such vectors. Then

$$\lambda_1(s) = \lim_{k\to\infty} \left(\sum_{w \in W(k)} \langle w \rangle^{-s} \right)^{\frac{1}{k}}$$

is true for all $s > 1$. This is the reason $\lambda_1(s)$ is often called a pseudo-zeta function associated with continued fractions.

2.17.2 Asymptotic Normality

We initially studied the denominator $Q_n(x)$ of the n^{th} continued fraction convergent to x in [1.8]. With the machinery introduced in the previous section, more can now be said.

If x is drawn uniformly from (0, 1), then the mean and variance of $\ln(Q_n(x))$ satisfy [22,26]

$$\mathrm{E}(\ln(Q_n(x))) = An + B + O(c^n), \quad \mathrm{Var}(\ln(Q_n(x))) = Cn + D + O(c^n),$$

where $c = -\lambda_2(2) = 0.3036630028\ldots$, $A = -\lambda_1'(2) = 1.1865691104\ldots$, and [2.18]

$$C = \lambda_1''(2) - \lambda_1'(2)^2 = 0.8621470373\ldots = (0.9285187329\ldots)^2.$$

The constants B and D await numerical evaluation. Further, the distribution of $\ln(Q_n(x))$ is asymptotically normal:

$$\lim_{n\to\infty} \mathrm{P}\left(\frac{\ln(Q_n(x)) - An}{\sqrt{Cn}} \le y \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left(-\frac{t^2}{2} \right) dt.$$

This is the first of several appearances of the Central Limit Theorem in this book.

2.17.3 Bounded Partial Denominators

A consequence of the Gauss–Kuzmin density is that almost all real numbers have unbounded partial denominators a_k. What does the set of all real numbers with only 1s and 2s for partial denominators "look like"? It is known [27–31] that this set has Hausdorff dimension between 0.53128049 and 0.53128051. Further discussion of this parameter is deferred until [8.20].

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2.18 Porter–Hensley Constants

Given two nonnegative integers m and n, let $L(m, n)$ denote the number of division steps required to compute $\gcd(m, n)$ by the classical Euclidean algorithm. By definition, if $m \geq n$, then

$$L(m, n) = \begin{cases} 1 + L(n, m \bmod n) & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \end{cases}$$

and if $m < n$, then $L(m, n) = 1 + L(n, m)$. Equivalently, $L(m, n)$ is the length of the regular continued fraction representation of m/n. We are interested in determining precisely the efficiency of the Euclidean algorithm and will do so by examining three types of random variables:

$$X_n = L(m, n), \quad \text{where } 0 \leq m < n \text{ is chosen at random,}$$

$$Y_n = L(m, n), \quad \text{where } 0 \leq m < n \text{ is chosen at random and } m \text{ is coprime to } n,$$

$$Z_N = L(m, n), \quad \text{where both } 1 \leq m \leq N \text{ and } 1 \leq n \leq N \text{ are chosen at random.}$$

Of these three, the expected value of Y_n is best behaved and was the first to succumb to analysis. It is interesting to follow the progress in understanding these average values. In his first edition, Knuth [1] observed that, empirically, $\mathrm{E}(Y_n) \sim 0.843 \ln(n) + 1.47$ and gave compelling reasons for

$$\mathrm{E}(Y_n) \sim \frac{12 \ln(2)}{\pi^2} \ln(n) + 1.47, \quad \mathrm{E}(Z_N) \sim \frac{12 \ln(2)}{\pi^2} \ln(N) + 0.06,$$

where the coefficient of $\ln(n)$ is Lévy's constant [1.8]. He decried the gaping theoretical holes in proving these asymptotics, however, and wrote, "The world's most famous algorithm deserves a complete analysis!"

By the second edition [2], remarkable progress had been achieved by Heilbronn [3], Dixon [4, 5], and Porter [6]. For any $\varepsilon > 0$, the following asymptotic formula is true:

$$\mathrm{E}(Y_n) \sim \frac{12 \ln(2)}{\pi^2} \ln(n) + C + O\left(n^{-\frac{1}{6}+\varepsilon}\right),$$

and **Porter's constant** C is defined by

$$C = \frac{6\ln(2)}{\pi^2}\left(3\ln(2) + 4\gamma - \frac{24}{\pi^2}\zeta'(2) - 2\right) - \frac{1}{2} = 1.4670780794\ldots,$$

where γ is the Euler–Mascheroni constant [1.5],

$$\zeta'(2) = \frac{d}{dx}\zeta(x)\bigg|_{x=2} = -\sum_{k=2}^{\infty}\frac{\ln(k)}{k^2} = -0.9375482543\ldots,$$

and $\zeta(x)$ is the Riemann zeta function [1.6]. This expression for C was discovered by Wrench [7], who also computed $\zeta'(2)$, and hence C, to 120 decimal places [8]. See [2.10] for more occurrences of $\zeta'(2)$.

What can be said of the other two average values? Norton [9] proved that, for any $\varepsilon > 0$,

$$\mathrm{E}(Z_N) \sim \frac{12\ln(2)}{\pi^2}\ln(N) + B + O\left(N^{-\frac{1}{6}+\varepsilon}\right),$$

where

$$B = \frac{12\ln(2)}{\pi^2}\left(-\frac{1}{2} + \frac{6}{\pi^2}\zeta'(2)\right) + C - \frac{1}{2} = 0.0653514259\ldots.$$

The asymptotic expression for $\mathrm{E}(X_n)$ is similar to that for $\mathrm{E}(Y_n)$ minus a correction term [2,9] based on the divisors of n:

$$\mathrm{E}(X_n) \sim \frac{12\ln(2)}{\pi^2}\left(\ln(n) - \sum_{d|n}\frac{\Lambda(d)}{d}\right) + C + \frac{1}{n}\sum_{d|n}\varphi(d)\cdot O\left(d^{-\frac{1}{6}+\varepsilon}\right),$$

where φ is Euler's totient function [2.7] and Λ is von Mangoldt's function:

$$\Lambda(d) = \begin{cases} \ln(p) & \text{if } d = p^r \text{ for } p \text{ prime and } r \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

In the midst of the proof in [9], Norton mentioned the Glaisher–Kinkelin constant A, which we discuss in [2.15]. Porter's constant C can be written in terms of A as

$$C = \frac{6\ln(2)}{\pi^2}(48\ln(A) - 4\ln(\pi) - \ln(2) - 2) - \frac{1}{2}$$

Knuth [7] mentioned a long-forgotten paper [10] containing $(1 - 2B)/4 = 0.2173242870\ldots$ and proposed that C instead be called the Lochs–Porter constant.

It is far more difficult to compute the corresponding variance of $L(m, n)$. Let us focus only on Z_N. Hensley [11] proved that

$$\mathrm{Var}(Z_N) = H\ln(N) + o(\ln(N)),$$

where

$$H = -\frac{\lambda_1''(2) - \lambda_1'(2)^2}{\pi^6\lambda_1'(2)^3} = 0.0005367882\ldots = (0.0231686908\ldots)^2$$

and $\lambda_1'(2)$ and $\lambda_1''(2)$ are precisely as described in [2.17.1]. Numerical work by Flajolet & Vallée [12] yielded the estimate $4\lambda_1''(2) = 9.0803731646\ldots$ needed to evaluate H. Furthermore, the distribution of Z_N is asymptotically normal:

$$\lim_{N\to\infty} \mathrm{P}\left(\frac{Z_N - \frac{12\ln(2)}{\pi^2}\ln(N)}{\sqrt{H\ln(N)}} \le w\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w} \exp\left(-\frac{t^2}{2}\right) dt.$$

A recent paper [13] contains several Porter-like constants in connection with the problem of sorting several real numbers via their continued fraction representations.

2.18.1 Binary Euclidean Algorithm

Assume m and n are positive odd integers. Let $e(m,n)$ be the largest integer such that $2^{e(m,n)}$ divides $m-n$. The number of subtraction steps required to compute $\gcd(m,n)$ by the binary Euclidean algorithm [14] is

$$K(m,n) = \begin{cases} 1 + K\left(\dfrac{m-n}{2^{e(m,n)}}, n\right) & \text{if } m > n, \\ 0 & \text{if } m = n, \\ K(n,m) & \text{if } m < n. \end{cases}$$

Define the random variable

$$W_N = K(m,n), \quad \text{where odd } 0 < m \le N \text{ and } 0 < n \le N \text{ are chosen at random.}$$

Computing the expected value of W_N is much more complicated than for Z_N. As in [2.17.1], study of a linear operator on function spaces [15, 16]

$$V_s[f](z) = \sum_{k\ge 1} \sum_{\substack{1\le j<2^k \\ \text{odd}}} \frac{1}{(j+2^k z)^s} f\left(\frac{1}{j+2^k z}\right),$$

is needed. For $s = 2$, let ψ denote the unique fixed point of V_s (up to scaling) and define a constant

$$\kappa = \frac{2}{\pi^2 \psi(1)} \sum_{\substack{r\ge 1 \\ \text{odd}}} 2^{-\left\lfloor \frac{\ln(r)}{\ln(2)} \right\rfloor} \int_0^{\frac{1}{r}} \psi(x)dx;$$

then $\mathrm{E}(W_N) \sim \kappa \ln(N)$. Further, if a certain conjecture by Vallée is true [15, 16], then some heuristic formulas due to Brent [17–19] are applicable and

$$\kappa = 1.0185012157\ldots = \ln(2)^{-1} \cdot 0.7059712461\ldots.$$

A direct computation, based on the exact definition of κ, has yet to be carried out.

Other performance parameters [15, 16] and alternative algorithms [17] have been studied, giving more constants. There is a continued fraction interpretation of these results. A general framework for investigating Euclidean-like algorithms [20, 21] provides analyses of methods for evaluating the Jacobi symbol from number theory [22].

Even more constants emerge if we examine average bit complexity rather than arithmetical operation counts [23, 24]. Many related questions remain unanswered.

2.18.2 Worst-Case Analysis

It is known [14, 25, 26] that the maximum value of Z_N occurs when m and n are consecutive Fibonacci numbers f_k and f_{k+1}, and k is the largest integer with $f_{k+1} \le N$. Therefore

$$\max(Z_N) = k \sim \frac{1}{\ln(\varphi)} \ln(N) = 2.0780869212\ldots \cdot \ln(N),$$

where φ is the Golden mean [1.2]. In contrast [14],

$$\max(W_N) \sim \frac{1}{\ln(2)} \ln(N) = 1.4426950408\ldots \cdot \ln(N),$$

and this occurs when m and n are of the form $2^{k-1} - 1$ and $2^{k-1} + 1$.

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2.19 Vallée's Constant

Let x and y be random numbers drawn uniformly and independently from the interval $(0, 1)$. To **compare** x and y is to determine which of the following is true: $x < y$ or $x > y$. There is an obvious algorithm for comparing x and y: Search for where the decimal or binary expansions of x and y first disagree. In base b, the number L of iterations of this algorithm has mean value

$$\mathrm{E}(L) = \frac{b}{b-1}$$

and a probability distribution given by

$$p_n = \mathrm{P}(L \geq n+1) = b^{-n}, \quad n = 0, 1, 2 \ldots.$$

Clearly

$$\lim_{n\to\infty} p_n^{\frac{1}{n}} = \frac{1}{b}$$

is a simply a way of expressing the (asymptotic) rate at which digits in the two base-b expansions coincide.

Here is a less obvious algorithm, proposed in [1], for comparing x and y. Write x and y (uniquely) as regular continued fractions:

$$x = 0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots, \quad y = 0 + \frac{1|}{|b_1} + \frac{1|}{|b_2} + \frac{1|}{|b_3} + \cdots,$$

where each a_j and b_j is a positive integer and search for where $a_k \neq b_k$ first occurs. If k is even, then $x < y$ if and only if $a_k < b_k$. If k is odd, then $x < y$ if and only if $a_k > b_k$. (There are other necessary provisions if x or y are rational, i.e., where a_j or b_j might be 0, which we do not discuss.)

The analysis of this algorithm is much more difficult and uses techniques and ideas discussed in [2.17.1]. Daudé, Flajolet & Vallée [2–5] proved that the mean number of iterations is

$$\begin{aligned} \mathrm{E}(L) &= \frac{3}{4} + \frac{180}{\pi^4} \sum_{i=1}^{\infty} \sum_{j=i+1}^{2i} \frac{1}{i^2 j^2} = \frac{17}{4} + \frac{360}{\pi^4} \sum_{i=1}^{\infty} \sum_{j=1}^{i} \frac{(-1)^i}{i^2 j^2} \\ &= 17 - \frac{60}{\pi^4} \left[24 \operatorname{Li}_4\left(\frac{1}{2}\right) - \pi^2 \ln(2)^2 + 21\zeta(3)\ln(2) + \ln(2)^4 \right] \\ &= 1.3511315744\ldots, \end{aligned}$$

where $\mathrm{Li}_4(z)$ is the tetralogarithm function [1.6.8] and $\zeta(3)$ is Apéry's constant [1.6]. This closed-form evaluation draws upon work in [6–8]. We also have

$$p_1 = \sum_{i=1}^{\infty} \frac{1}{i^2(i+1)^2} = \frac{\pi^2}{3} - 3 = 0.2898681336\ldots,$$

$$\begin{aligned} p_2 &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(ij+1)^2(ij+i+1)^2} = 0.0484808014\ldots \\ &= -5 + \frac{2\pi^2}{3} - 2\zeta(3) + 2\sum_{n=0}^{\infty} (-1)^n (n+1)\zeta(n+4)\left[\zeta(n+2) - 1\right], \end{aligned}$$

$$p_3 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(ijk+i+k)^2(ijk+ij+i+k+1)^2} = 0.0102781647\ldots,$$

but unlike earlier, a nice compact formula for p_n is not known. The elaborate recurrence giving rise to p_n appears later [2.19.1]. It can be deduced that [2–5]

$$v = \lim_{n\to\infty} p_n^{\frac{1}{n}} = 0.1994588183\ldots$$

using the fact that this is the largest eigenvalue of the linear operator G_4 defined in [2.17.1]. As with G_2, the eigenvalues of G_4 are real and seem to alternate in sign (the next one is $-0.0757395140\ldots$). A similar argument applies in the analysis of the Gaussian algorithm for finding a short basis of a lattice in two-dimensional space, given an initially skew basis. Vallée's constant v also appears in connection with the problem of sorting $n > 2$ real numbers via their continued fraction representations [9].

If, when comparing x and y, we instead use centered continued fractions, then the number $\hat{L}$ of iterations satisfy [2, 5]

$$\mathrm{E}(\hat{L}) = \frac{360}{\pi^4} \sum_{i=1}^{\infty} \sum_{j=\lceil \varphi i \rceil}^{\lfloor (\varphi+1)i \rfloor} \frac{1}{i^2 j^2} = 1.0892214740\ldots,$$

$$\hat{v} = \lim_{n\to\infty} \hat{p}_n^{\frac{1}{n}} = 0.0773853773\ldots,$$

where φ is the Golden mean [1.2]. Since $1/v = 5.01\ldots$ and $1/\hat{v} = 12.92\ldots$, it follows that continued fractions behave roughly like base-5 and base-13 representations in this respect. Not much is known about the corresponding operator $\hat{G}_s$ and its spectrum. Flajolet & Vallée [5] also numerically computed values of the mock zeta function

$$\zeta_\theta(z) = \sum_{k=1}^{\infty} \frac{1}{\lfloor k\theta \rfloor^z}, \quad \mathrm{Re}(z) > 1, \quad \theta > 1,$$

where $\theta > 1$ is irrational. For example, $\zeta_\varphi(2) = 1.2910603681\ldots$.

2.19.1 Continuant Polynomials

Define functions recursively by the rule [3]

$$f_k(x_1, x_2, \ldots, x_k) = x_k f_{k-1}(x_1, x_2, \ldots, x_{k-1}) + f_{k-2}(x_1, x_2, \ldots, x_{k-2}),$$
$$k = 2, 3, 4, \ldots,$$

where

$$f_0 = 1, \quad f_1(x_1) = x_1.$$

These are called **continuant polynomials** and can also be defined by taking the sum of monomials obtained from $x_1 x_2 \cdots x_k$ by crossing out in all possible ways pairs of adjacent variables $x_j x_{j+1}$. For example,

$$f_2(x_1, x_2) = x_1x_2 + 1, \quad f_3(x_1, x_2, x_3) = x_1x_2x_3 + x_1 + x_3,$$
$$f_4(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4 + x_1x_2 + x_1x_4 + x_3x_4 + 1.$$

The probability of interest to us is

$$p_k = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{f_k^2 (f_k + f_{k-1})^2}.$$

Each p_k can be expressed in terms of complicated series involving Riemann zeta function values and thus falls in the class of polynomial-time computable constants [5].

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2.20 Erdös' Reciprocal Sum Constants

2.20.1 A-Sequences

An infinite sequence of positive integers $1 \le a_1 < a_2 < a_3 < \ldots$ is called an A-**sequence** if no a_k is the sum of two or more distinct earlier terms of the sequence [1]. For example, the sequence of nonnegative powers of 2 is an A-sequence. Erdös [2] proved that

$$S(A) = \sup_{A\text{-sequences}} \sum_{k=1}^{\infty} \frac{1}{a_k} < 103$$

and thus the largest reciprocal sum must be *finite* in particular. Levine & O'Sullivan [3,4] proved that any A-sequence must satisfy what we call the χ**-inequality**:

$$(j+1)a_j + a_i \ge (j+1)i$$

for all i and j, and consequently $S(A) < 3.9998$. In the other direction, Abbott [5] and Zhang [6] gave specific examples that demonstrate that $S(A) > 2.0649$. These are the best-known bounds on $S(A)$ so far.

The χ-inequality is itself interesting. Levine & O'Sullivan [3,7] defined a specific integer sequence by the greedy algorithm: $\chi_1 = 1$ and

$$\chi_i = \max_{1 \le j \le i-1} (j+1)(i - \chi_j)$$

for $i > 1$, that is, 1, 2, 4, 6, 9, 12, 15, 18, 21, 24, 28, 32, 36, 40, 45, 50, 55, 60, 65, ... They conjectured that

$$S(A) \leq \sum_{k=1}^{\infty} \frac{1}{\chi_k} = 3.01\ldots$$

and further that $\{\chi_k\}$ dominates the reciprocal sum of any other integer sequence satisfying the χ-inequality. Finch [8–10] wondered if this latter conjecture still holds for arbitrary (not necessarily integer) real sequences.

The authors of [3–5] used the phrase "sum-free sequence" to refer to A-sequences, which is unfortunate terminology since the word "sum-free" usually refers to an entirely different class of sequences [2.25]. We have adopted the phrase "A-sequence" from Guy [1]. See also [2.28] concerning sets with distinct subset sums.

2.20.2 B_2-Sequences

An infinite sequence of positive integers $1 \leq b_1 < b_2 < b_3 < \ldots$ is called a **B_2-sequence** (or **Sidon sequence**) if all pairwise sums $b_i + b_j$, $i \leq j$, are distinct [1]. For example, the greedy algorithm gives the Mian–Chowla [7, 11] sequence 1, 2, 4, 8, 13, 21, 31, 45, 66, 81, 97, 123, 148, 182, 204, 252, 290, ... , which is known to have reciprocal sum [12] between 2.158435 and 2.158677. Zhang [13] proved that

$$S(B_2) = \sup_{B_2\text{-sequences}} \sum_{k=1}^{\infty} \frac{1}{b_k} > 2.1597$$

and thus is larger than the Mian–Chowla sum. An observation by Levine [1, 13] shows that $S(B_2)$ is necessarily finite; in fact, it is < 2.374. More recent work [12, 14] gives the improved bounds $2.16086 < S(B_2) < 2.247327$.

Erdös & Turán [15–17] asked if a finite B_2-sequence of positive integers $b_1 < b_2 < \ldots < b_m$ with $b_m \leq n$ must satisfy $m \leq n^{1/2} + C$ for some constant C. Lindström [18] demonstrated that $m < n^{1/2} + n^{1/4} + 1$. Zhang [19] computed that if such a C exists, it must be > 10.27. Lindström [20] improved the lower bound for C to 13.71. In a more recent paper [21], he concluded that C probably does not exist and conjectured that $m \leq n^{1/2} + o(n^{1/4})$.

2.20.3 Nonaveraging Sequences

An infinite sequence of positive integers $1 \leq c_1 < c_2 < c_3 < \ldots$ is said to be **nonaveraging** if it contains no three terms in arithmetic progression. Equivalently, $c + d \neq 2e$ for any three distinct terms c, d, e of the sequence [1]. For example, the greedy algorithm gives the Szekeres [7, 22] sequence 1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41, 82, 83, ... ; that is, n is in the sequence if and only if the ternary expansion of $n - 1$ contains only 0s and 1s. This is known to have reciprocal sum between 3.00793 and 3.00794. Wróblewski [23], building upon [24, 25], constructed a special nonaveraging sequence to demonstrate that

$$S(C) = \sup_{\substack{\text{nonaveraging} \\ \text{sequences}}} \sum_{k=1}^{\infty} \frac{1}{c_k} > 3.00849.$$

A proof that $S(C)$ is necessarily finite is not known; the best lower bound [26] for c_k is only $O(k\sqrt{\ln(k)/\ln(\ln(k))})$.

Some related studies of the density of $\{c_k\} \cap [1, n]$, constructed greedily with alternative formation rules or different initial values, appear in [27–31]. Under certain conditions, as n increases, the density oscillates with peaks and valleys (rather than falling smoothly) in roughly geometric progression. The ratio between two consecutive peaks seems, as $N \to \infty$, to approach a limit. This phenomenon deserves to be better understood.

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2.21 Stieltjes Constants

The Riemann zeta function $\zeta(z)$, as defined in [1.6], has a Laurent expansion in a neighborhood of its simple pole at $z = 1$:

$$\zeta(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\gamma_n (z-1)^n.$$

The coefficients γ_n can be proved to satisfy [1–9]

$$\gamma_n = \lim_{m\to\infty}\left(\sum_{k=1}^{m}\frac{\ln(k)^n}{k} - \frac{\ln(m)^{n+1}}{n+1}\right) = \begin{cases} 0.5772156649\ldots & \text{if } n = 0, \\ -0.0728158454\ldots & \text{if } n = 1, \\ -0.0096903631\ldots & \text{if } n = 2, \\ 0.0020538344\ldots & \text{if } n = 3, \\ 0.0023253700\ldots & \text{if } n = 4, \\ 0.0007933238\ldots & \text{if } n = 5, \end{cases}$$

and, in particular, $\gamma_0 = \gamma$, the Euler–Mascheroni constant [1.5].

Here is a sample application to number theory. Define a positive integer N to be **jagged** if its largest prime factor is $> \sqrt{N}$, and let $j(N)$ be the number of such integers not exceeding N. The first several jagged numbers are $2, 3, 5, 6, 7, 10, 11, 13, 14, \ldots$ and, asymptotically [10, 11],

$$j(N) = \ln(2)N - (1-\gamma_0)\frac{N}{\ln(N)} - (1-\gamma_0-\gamma_1)\frac{N}{\ln(N)^2} + O\left(\frac{N}{\ln(N)^3}\right),$$

where $1 - \gamma_0 = 0.4227843351\ldots$ and $1 - \gamma_0 - \gamma_1 = 0.4956001805\ldots$. See the related discussion of smooth numbers in [5.4]. Other occurrences of γ_n include [12–17].

The signs of the Stieltjes constants γ_n follow a seemingly random pattern. Briggs [18] proved that infinitely many γ_n are positive and infinitely many are negative. Mitrovic [19] extended this result by demonstrating that each of the inequalities

$$\gamma_{2n} < 0, \quad \gamma_{2n} > 0, \quad \gamma_{2n-1} < 0, \quad \gamma_{2n-1} > 0$$

must hold for infinitely many n. In an elaborate analysis, Matsuoka [20,21] proved that, for any $\varepsilon > 0$, there exist infinitely many integers n for which all of γ_n, γ_{n+1}, $\gamma_{n+2}, \ldots, \gamma_{n+\lfloor(2-\varepsilon)\ln(n)\rfloor}$ have the same sign, and there exist only finitely many integers n for which all of $\gamma_n, \gamma_{n+1}, \gamma_{n+2}, \ldots, \gamma_{n+\lfloor(2+\varepsilon)\ln(n)\rfloor}$ have the same sign. Also, if

$$f(n) = |\{0 \le k \le n : \gamma_n > 0\}|, \quad g(n) = |\{0 \le k \le n : \gamma_n < 0\}|$$

then $f(n) = n/2 + o(n)$ and $g(n) = n/2 + o(n)$.

The first few Stieltjes constants γ_n are close to 0, but this is deceptive. In fact, their magnitudes seem to $\to \infty$ as $n \to \infty$, although a proof is not known. Upper bounds for $|\gamma_n|$ were successively obtained by several authors [18,22–26], culminating in

$$|\gamma_n| \le \frac{(3 + (-1)^n)(2n)!}{n^{n+1}(2\pi)^n}.$$

The last word again belongs to Matsuoka [20,21], who proved that the lower bound

$$\exp(n \ln(\ln(n)) - \varepsilon n) < |\gamma_n|$$

holds for infinitely many n, while the upper bound

$$|\gamma_n| \le \frac{1}{10000} \exp(n \ln(\ln(n)))$$

holds for all $n \ge 10$.

We mentioned in [1.5] the following formula due to Vacca:

$$\gamma_0 = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left\lfloor \frac{\ln(k)}{\ln(2)} \right\rfloor.$$

Hardy [27] gave an analog for γ_1:

$$\gamma_1 = \sum_{j=1}^{\infty} \frac{(-1)^j \ln(j)}{j} \left\lfloor \frac{\ln(j)}{\ln(2)} \right\rfloor - \frac{\ln(2)}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left\lfloor \frac{\ln(2k)}{\ln(2)} \right\rfloor \left\lfloor \frac{\ln(k)}{\ln(2)} \right\rfloor,$$

and Kluyver [28] presented more such series for higher-order constants. Also, if $\{x\}$ denotes the fractional part of x, then [29]

$$\int_1^\infty \frac{\{x\}}{x^2} dx = 1 - \gamma_0, \quad \int_1^\infty \int_x^\infty \frac{\{y\}}{xy^2} dy dx = 1 - \gamma_0 - \gamma_1.$$

Additional formulas for γ_n appear in [7,8,30–32].

We now discuss certain associated constants. An alternating series variant,

$$\tau_n = \sum_{k=1}^{\infty}(-1)^k \frac{\ln(k)^n}{k}$$
$$= \begin{cases} -\ln(2) = -0.6931471805\ldots & \text{if } n=0, \\ -\frac{1}{2}\ln(2)^2 + \gamma_0 \ln(2) = 0.1598689037\ldots & \text{if } n=1, \\ -\frac{1}{3}\ln(2)^3 + \gamma_0 \ln(2)^2 + 2\gamma_1 \ln(2) = 0.0653725925\ldots & \text{if } n=2, \end{cases}$$

can be related to the Stieltjes constants via the formulas [1,4,8,26]

$$\tau_n = -\tfrac{\ln(2)^{n+1}}{n+1} + \sum_{k=0}^{n-1}\binom{n}{k}\ln(2)^{n-k}\gamma_k, \quad \gamma_n = \tfrac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}B_{n+1-k}\ln(2)^{n-k}\tau_k,$$

where B_j is the j^{th} Bernoulli number [1.6.1]. Consider also the Laurent expansion for $\zeta(z)$ at the origin (rather than at unity):

$$\zeta(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\delta_n z^n.$$

Sitaramachandrarao [33] proved that [3,34]

$$\delta_n = \lim_{m\to\infty}\left(\sum_{k=1}^{m}\ln(k)^n - \int_1^m \ln(x)^n dx - \frac{1}{2}\ln(m)^n\right) = (-1)^n(\zeta^{(n)}(0) + n!)$$
$$= \begin{cases} \frac{1}{2} = 0.5 & \text{if } n=0, \\ \frac{1}{2}\ln(2\pi) - 1 = -0.0810614667\ldots & \text{if } n=1, \\ -\frac{\pi^2}{24} - \frac{1}{2}\ln(2\pi)^2 + \frac{\gamma_0^2}{2} + \gamma_1 + 2 = -0.0063564559\ldots & \text{if } n=2, \end{cases}$$

and these, in turn, were helpful to Lehmer [35] in approximating sums of the form [7,26]

$$\sigma_n = \sum_{\rho}\frac{1}{\rho^n} = \begin{cases} -\frac{1}{2}\ln(4\pi) + \frac{\gamma_0}{2} + 1 = 0.0230957089\ldots & \text{if } n=1, \\ -\frac{\pi^2}{8} + \gamma_0^2 + 2\gamma_1 + 1 = -0.0461543172\ldots & \text{if } n=2, \\ -\frac{7\zeta(3)}{8} + \gamma_0^3 + 3\gamma_0\gamma_1 + \frac{3\gamma_2}{2} + 1 = -0.0001111582\ldots & \text{if } n=3, \end{cases}$$

where each sum is over all nontrivial zeros ρ of $\zeta(z)$. The constant σ_1 also appears in [1.6] and [2.32]. Keiper [36] and Kreminski [37] vastly extended Lehmer's computations.

The analog of γ_n corresponding to the arithmetic progression a, $a+b$, $a+2b$, $a+3b$, … was studied by Knopfmacher [38], Kanemitsu [39], and Dilcher [40]:

$$\gamma_{n,a,b} = \lim_{m\to\infty}\left(\sum_{\substack{0<k\le m \\ k\equiv a \bmod b}}\frac{\ln(k)^n}{k} - \frac{1}{b}\frac{\ln(m)^{n+1}}{n+1}\right).$$

For example, $\sum_{a=0}^{b-1} \gamma_{n,a,b} = \gamma_n$ and

$$\gamma_{n,0,2} = \frac{1}{2}\left[\sum_{j=0}^{n} \binom{n}{j} \gamma_{n-j} \ln(2)^j - \frac{\ln(2)^{n+1}}{n+1}\right], \quad \gamma_{1,0,3} = \frac{1}{3}\left[\gamma_1 + \gamma_0 \ln(3) - \frac{\ln(3)^2}{2}\right],$$

$$\gamma_{1,1,3} = \frac{1}{6}\left[2\gamma_1 - \gamma_0 \ln(3) + \frac{\ln(3)^2}{2} - \left(\frac{\gamma_0 + \ln(2\pi)}{3} - \ln\left[\Gamma(\tfrac{1}{3})^2 \frac{\sqrt{3}}{2\pi}\right]\right) \pi\sqrt{3}\right].$$

Different extensions of γ_n are found in [23, 26, 41–46].

The reader should be warned that some authors define the Stieltjes constants to be $(-1)^n \gamma_n / n!$ rather than γ_n, so care is needed when reviewing the literature.

2.21.1 Generalized Gamma Functions

For complex z, the generalized gamma function $\Gamma_n(z)$ is defined by [47, 48]

$$\Gamma_n(z) = \lim_{m\to\infty} \frac{\exp\left(\dfrac{\ln(m)^{n+1}}{n+1} z\right) \displaystyle\prod_{k=1}^{m} \exp\left(\dfrac{\ln(k)^{n+1}}{n+1}\right)}{\displaystyle\prod_{k=0}^{m} \exp\left(\dfrac{\ln(k+z)^{n+1}}{n+1}\right)}$$

and is analytic over the complex plane slit along the negative x-axis. Clearly $\Gamma_0(z) = \Gamma(z)$ and $\Gamma_n(z)$ satisfies

$$\Gamma_n(1) = 1, \quad \Gamma_n(z+1) = \exp\left(\frac{\ln(z)^{n+1}}{n+1}\right) \Gamma_n(z).$$

The connection between $\Gamma_n(z)$ and γ_n is through the formula $\psi_n(1) = -\gamma_n$, where

$$\psi_n(x) = \frac{d}{dx} \ln(\Gamma_n(x)) = -\gamma_n - \sum_{k=0}^{\infty} \left(\frac{\ln(x+k)^n}{x+k} - \frac{\ln(k+1)^n}{k+1}\right)$$

is the generalized digamma function. A generalized Stirling formula includes

$$\Gamma_0(x) \sim \sqrt{2\pi}\, x^{x-\frac{1}{2}} e^{-x}, \quad \Gamma_1(x) \sim C x^{\frac{1}{2}(x-\frac{1}{2})\ln(x) - x} e^{x}$$

as special cases, where [48, 49]

$$\begin{aligned} \ln(C) &= \ln\left(\Gamma_1\left(\frac{1}{2}\right)\right) - \frac{1}{4}\ln(2)^2 - \frac{1}{2}\ln(2)\ln(2\pi) \\ &= -\frac{\pi^2}{48} - \frac{1}{4}\ln(2\pi)^2 + \frac{\gamma_0^2}{4} + \frac{\gamma_1}{2} = -1.0031782279\ldots. \end{aligned}$$

Many more formulas of this kind can be found.

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2.22 Liouville–Roth Constants

We may study constants by means of other constants. Given a real number ξ, let R denote the set of all positive real numbers r for which the inequality

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^r}$$

has at most finitely many solutions (p, q), where p and $q > 0$ are integers. Define the

Liouville–Roth constant (or **irrationality measure**)

$$r(\xi) = \inf_{r \in R} r,$$

that is, the critical rate threshold above which ξ is **not approximable** by rational numbers [1–3]. It is known that

$$\begin{aligned}
&\xi \text{ is rational} && \Rightarrow r(\xi) = 1,\\
&\xi \text{ is algebraic irrational} && \Rightarrow r(\xi) = 2 \quad \text{(Thue-Siegel-Roth theorem [4, 5])},\\
&\xi \text{ is transcendental} && \Rightarrow r(\xi) \geq 2.
\end{aligned}$$

If ξ is a Liouville number, for example,

$$\sum_{n=1}^{\infty} \frac{1}{2^{n!}} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^6} + \frac{1}{2^{24}} + \frac{1}{2^{120}} + \cdots = 0.7656250596\ldots,$$

then $r(\xi) = \infty$. Similarly, one can construct ξ so that $r(\xi)$ assumes any value, $2 < r(\xi) < \infty$ (from series of rationals with appropriately fast convergence). Among famous constants, it is known that [2]

$$r(e) = 2$$

(in fact, much more precise inequalities are possible, but e is somewhat atypical), and

$$\begin{aligned}
&2 \leq r(\pi) \leq 8.016045\ldots && \text{(Hata [6, 7])},\\
&2 \leq r(\ln(2)) \leq 3.89139978\ldots && \text{(Rukhadze [8, 9])},\\
&2 \leq r(\pi^2) \leq 5.441243\ldots && \text{(Hata [10], Rhin \& Viola [11])},\\
&2 \leq r(\zeta(3)) \leq 5.513891\ldots && \text{(Hata [12], Rhin \& Viola [13])},
\end{aligned}$$

where $\zeta(3)$ is Apéry's constant [1.6]. Upper bounds for r corresponding to Catalan's constant G [1.7] or Khintchine's constant K [1.8] are not known. Whether G and K are even irrational remains open.

A consequence of Hata's work concerning π is that the two functions [14, 15]

$$C(x) = \inf_{n>0 \text{ integer}} n^x |\sin(n)|, \quad D(x) = \sup_{n>0 \text{ integer}} n^{-x} |\tan(n)|$$

satisfy $C(7.02) > 0$, $D(7.02) = 0$. If a conjecture [16] that $r(\pi) = 2$ is true, then $C(1 + \varepsilon) > 0$, $D(1 + \varepsilon) = 0$ for all $\varepsilon > 0$. Numerical evidence suggests that $C(1) = 0$, $D(1) = \infty$.

One can also examine multidimensional analogs of these constants. For example, let $1, \xi_1, \xi_2, \ldots, \xi_n$ be linearly independent over the rationals, where $\xi_1, \xi_2, \ldots, \xi_n$ are real algebraic numbers. Let R denote the set of all positive real numbers r for which the simultaneous system of inequalities

$$0 < \left| \xi_i - \frac{p_i}{q} \right| < \frac{1}{q^r}, \quad i = 1, 2, \ldots, n,$$

has at most finitely many solutions $(p_1, p_2, \ldots, p_n, q)$, where each p_i and $q > 0$ are integers. Define $r(\xi_1, \xi_2, \ldots, \xi_n)$ exactly as before. Schmidt [5,17,18] extended the Thue–Siegel–Roth theorem to deduce that

$$r(\xi_1, \xi_2, \ldots, \xi_n) = \frac{n+1}{n}.$$

Clearly the joint irrationality measure $r(e, \pi)$ satisfies $r(e, \pi) \leq \max\{r(e), r(\pi)\}$, but no one has improved on this bound. Of course, we do not even know whether e and π are linearly independent over the rationals!

A related subject, concerning the simultaneous Diophantine approximation constants [2.23], is similar yet possesses a different focus than that here.

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2.23 Diophantine Approximation Constants

In our essay on Liouville–Roth constants [2.22], we discussed rational approximations of a single irrational number ξ. Here we study the simultaneous rational approximation of n real numbers $\xi_1, \xi_2, \ldots, \xi_n$, of which at least one is irrational, by fractions all with the same denominator. Dirichlet's box principle [1,2] implies that, if $c \geq 1$, then the system of inequalities

$$\left|\xi_i - \frac{p_i}{q}\right| < c^{\frac{1}{n}} q^{-\frac{n+1}{n}}, \quad i = 1, 2, \ldots, n,$$

has infinitely many solutions $(p_1, p_2, \ldots, p_n, q)$, where $p_1, p_2, \ldots, p_n$ and $q > 0$ are integers. The focus of this essay is not on the exponent $(n+1)/n$ of the right-hand side, as it was earlier, but rather on the linear coefficient c.

As is traditional, rearrange the inequalities to

$$q \cdot |q\xi_i - p_i|^n < c$$

and define c_n to be the infimum of all $0 < c \leq 1$ for which the solution set $(p_1, p_2, \ldots, p_n, q)$ remains infinite. Then define the **n-dimensional simultaneous Diophantine approximation constant** γ_n to be the supremum of c_n over all such $\xi_1, \xi_2, \ldots, \xi_n$. So γ_n is not measuring the goodness of approximation of a *single* set of n numbers, but instead it is defined across *all* possible sets and thus depends only on the dimension n.

Here is a summary of what is known about the approximation constants γ_n:

$$\begin{array}{ll}
\gamma_1 = \frac{1}{\sqrt{5}} = 0.4472135955\ldots & \text{(Hurwitz [1])}, \\
0.2857142857\ldots = \frac{2}{7} \leq \gamma_2 \leq \frac{64}{169} = 0.378\ldots & \text{(Cassels [2], Nowak [3])}, \\
0.120\ldots = \frac{2}{5\sqrt{11}} \leq \gamma_3 \leq \delta_2 = \frac{1}{2}\frac{1}{\pi-2} = 0.437\ldots & \text{(Cusick [4], Spohn [5])}, \\
0.044\ldots = \frac{16}{9\sqrt{1609}} \leq \gamma_4 \leq \delta_3 = \frac{27}{4}\frac{1}{8\sqrt{3}\pi-27} = 0.408\ldots & \text{(Krass [6], Spohn [5])}, \\
0.010\ldots = \frac{16}{207\sqrt{53}} \leq \gamma_5 \leq \delta_4 = 0.390\ldots & \text{[5–7]}, \\
0.004\ldots = \frac{16}{9\sqrt{184607}} \leq \gamma_6 \leq \delta_5 = 0.379\ldots & \text{[5–7]},
\end{array}$$

where the upper bounds [5] are computed via the definite integrals

$$\frac{1}{\delta_k} = k2^{k+1} \int_0^1 \frac{x^{k-1}}{(1+x^k)(1+x)^k} dx.$$

There is a wealth of computational [8] and theoretical evidence [9, 10] that $\gamma_2 = 2/7$ but this cannot yet be regarded as a theorem. Adams [9] proved that $2/7$ is the correct value if we impose the constraint that $\xi_1 = 1, \xi_2, \xi_3$ form a basis of a real cubic number field. Cusick [10, 11] proved additional results under the hypothesis that the regular continued fraction expansion of $2\cos(2\pi/7)$ has certain finite partial denominator patterns occurring infinitely often. See also [12, 13].

With regard to γ_3, Szekeres [14] indicated that its true value might be as high as 0.170, substantially greater than the lower bound given here.

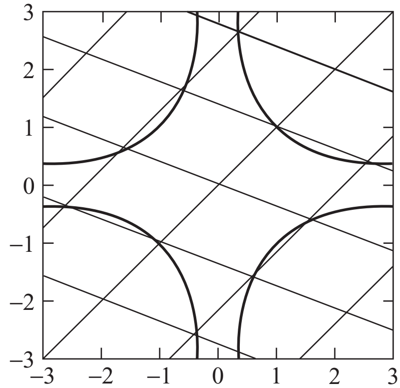

Figure 2.3. A star body S along with an S-admissible lattice L.

Nowak [15] obtained an improvement to Spohn's upper bounds, involving a function of δ_k, but numerical estimates are not possible at this time.

There is a remarkable connection between the values of γ_n and the geometry of numbers. We first illustrate this in the two-dimensional setting (see Figure 2.3). Consider the unbounded region S in the plane determined by $|xy| \le 1$ (which is an example of what is called a **star body**). Consider as well the lattice L with basis vectors $(1, 1)$ and $((1 + \sqrt{5})/2, (1 - \sqrt{5})/2)$. It can be proved that the only vertex of L that lies within the interior of S is the origin (0,0). Consequently L is said to be **S-admissible**.

The area of any single parallelogram cell of L is clearly $\sqrt{5}$. This is called the **determinant** of L, written $\det(L)$. It can be further proved that any other S-admissible lattice L must satisfy $\det(L) \ge \sqrt{5}$.

In the same way, consider the unbounded region S in $(n + 1)$-dimensional space determined by

$$|x_{n+1}| \cdot \max\{|x_1|^n, |x_2|^n, \ldots, |x_n|^n\} \le 1$$

and consider all $(n + 1)$-dimensional S-admissible lattices L. Davenport [16, 17] proved that the volume, $\det(L)$, of any single parallelepiped cell of L satisfies $\det(L) \ge 1/\gamma_n$ and, moreover, equality must occur for some choice of L. Therefore

$$\frac{1}{\gamma_n} = \min_{\substack{S\text{-admissible} \\ \text{lattices } L}} \det(L)$$

is also known as the **critical determinant** or **lattice constant** for the star body S. This geometric insight unfortunately offers only limited help in computing γ_n. Some sample computations are given in [18–24].

Here is a similar problem from the geometry of numbers (having nothing to do with γ_n as far as is known). Again, we illustrate this in the two-dimensional setting (see Figure 2.4). Let Z denote the standard integer lattice in the plane, that is, with basis vectors (1,0) and (0,1). Consider an arbitrary parallelogram P centered at the origin (0,0). P is called **Z-allowable** if the interior of P contains no other vertices of Z. Now,

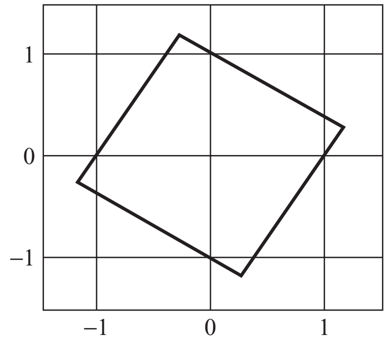

Figure 2.4. A Z-allowable parallelogram P.

given any basis v, w of the plane, there clearly exists a Z-allowable parallelogram P with sides perpendicular to v and w (just take P to have suitably small area). Define $\alpha(v, w)$ to be the supremum of the areas for all such P. Then define κ_2 to be the infimum of $\alpha(v, w)/4$ for all such bases v, w. Szekeres [25] proved that

$$\kappa_2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{5}}\right) = 0.7236067977\ldots.$$

The slopes of the "critical parallelogram," in this case a square, are $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$. It is interesting that the Golden mean [1.2] occurs here as well as with the computation of γ_2 earlier.

For higher dimensions, let Z denote the standard n-dimensional integer lattice and consider n-dimensional Z-allowable parallelepipeds P with faces normal to a given basis $v_1, v_2, \ldots, v_n$. As before, $2^n\kappa_n$ is the largest possible volume of P in the sense that P can have volume $2^n\kappa_n$ independent of the prescribed directions $v_1, v_2, \ldots, v_n$, but this fails for P of volume $2^n\kappa_n + \varepsilon$ for any $\varepsilon > 0$. It is known [26–28] that $\kappa_3 > 1/4$, $\kappa_4 > 1/16$, and there is theoretical evidence [29] that possibly

$$\kappa_3 = \frac{8}{7}\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)^2 = 0.5784167628\ldots.$$

Moreover, it has been proved that asymptotically [28, 30]

$$\frac{n}{(n!)^2}\left(\frac{1}{2}\right)^{\frac{n(n+1)}{2}} < \kappa_n < \left[\frac{1}{2}\left(1 + \frac{1}{\sqrt{5}}\right)\right]^{\frac{n-1}{2}}.$$

One might call $\kappa_2, \kappa_3, \kappa_4, \ldots$ the **Mordell constants** [31]; further discussion is found in [32–34].

Here is one more problem. Let K be a bounded convex body in n-space of volume $V(K)$ and symmetric with respect to the origin. Let $\Delta(K)$ denote the critical determinant of K and define

$$\rho_n = \inf_K \frac{V(K)}{\Delta(K)}.$$

For example, if $n = 2$ and K is a disk, then clearly $V(K)/\Delta(K) = 2\pi/\sqrt{3} = 3.627\ldots$. This is not optimal, for it is known [35–38] that

$$3.570624\ldots \leq \rho_2 \leq 4\frac{8 - 4\sqrt{2} - \ln(2)}{2\sqrt{2} - 1} = 3.6096567319\ldots$$

and further conjectured [39,40] that ρ_2 is equal to its upper bound (corresponding to a smoothed octagon K obtained by rounding off each corner with a hyperbolic arc). It is also known [35,41,42] that $\rho_3 \geq 4.216$, $\rho_4 \geq 4.721$, and $\rho_n > r = 4.921553\ldots$ for $n \geq 5$, where r is the unique solution > 1 of the equation $r\ln(r) = 2(r-1)$. Mahler [35], however, believed that $\rho_n \to \infty$ as $n \to \infty$, so there is considerable room for improvement. This theory is an outgrowth of the classical Minkowski–Hlawka theorem; by letting σ_n be the analog of ρ_n corresponding to bounded star bodies S, a parallel set of questions can be asked. For example [43], $\sigma_2 \leq 3.5128\ldots$ (corresponding to S bounded by eight hyperbolic arcs), but no one appears to have conjectured an exact value for σ_2.

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2.24 Self-Numbers Density Constant

Any nonnegative integer n has a unique binary representation:

$$n = \sum_{k=0}^{\infty} n_k 2^k, \quad n_k = 0 \text{ or } 1.$$

What happens if we slightly perturb this formula, for example, by replacing the exponential 2^k by $2^k + 1$? Things become noticeably different: The integers 1, 4, and 6 have *no* representations of the form

$$n_0 \cdot (2^0 + 1) + n_1 \cdot (2^1 + 1) + n_2 \cdot (2^2 + 1) = 2n_0 + 3n_1 + 5n_2, \quad n_k = 0 \text{ or } 1,$$

whereas 5 has *two* such representations, 5 and $2 + 3$.

Let us focus solely on the existence issue. Define S to be the set of all n for which a representation

$$n = \sum_{k=0}^{\infty} n_k \left(2^k + 1\right), \quad n_k = 0 \text{ or } 1$$

exists (including 0). Define T to be the complement of S relative to the nonnegative integers [1], thus $T = \{1, 4, 6, 13, 15, 18, 21, 23, 30, 32, 37, 39, \ldots\}$. These are known as **binary self numbers** (Kaprekar [2,3]) or **binary Columbian numbers** (Recamán [4]).

It can be proved that T is an infinite set. Let $\tau(N)$ denote the cardinality of binary self numbers not exceeding N. Zannier [5] proved that the limit

$$0 < \lambda = \lim_{N \to \infty} \frac{\tau(N)}{N} < 1$$

exists and moreover $\tau(N) = \lambda N + O(\ln(N)^2)$. The **self-numbers density constant** λ can be calculated by the formula

$$\lambda = \frac{1}{8} \left(\sum_{n \in S} \frac{1}{2^n} \right)^2 = 0.2526602590\ldots$$

and was recently proved by Troi & Zannier [6,7] to be a transcendental number.

We can extend this discussion to any base $b > 1$. Define S_b to be the set of all n for which a representation

$$n = \sum_{k=0}^{\infty} n_k \left(b^k + 1\right), \quad n_k = 0,\ 1, \ldots,\ b-2 \text{ or } b-1,$$

exists. Define T_b and $\tau_b(N)$ similarly. We have $\tau_b(N) = \lambda_b N + O(\ln(N)^2)$ as before [5] and numerical approximations $\lambda_4 = 0.209\ldots$ and $\lambda_{10} = 0.097\ldots$ but no fast infinite series for λ_b (analogous to the formula for λ_2) has yet been established for any $b > 2$. Likewise, no one has yet proved that λ_b, $b \geq 3$, is even irrational.

There is also the issue of uniqueness. Let us focus on the binary case only. Define U to be the set of all n for which the representation

$$n = \sum_{k=0}^{\infty} n_k(2^k + 1), \quad n_k = 0 \text{ or } 1,$$

exists and is unique. Define V to be the complement of U relative to S. The set V is trivially infinite because, for all $k > 2$,

$$1 \cdot (2^k + 1) + 1 \cdot (2^2 + 1) = 1 \cdot (2^k + 1) + 1 \cdot (2^0 + 1) + 1 \cdot (2^1 + 1)$$

and the set U is trivially infinite because, for each integer t in T,

$$\sum_{k=0}^{t+1} (2^k + 1) = (2^{t+2} + 1) + t$$

has no other admissible representations. What can be said about the densities of U and V? See also [8] for the density of self numbers within arithmetic progressions, and [9] for related discussion of digitaddition series.

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2.25 Cameron's Sum-Free Set Constants

A set S of positive integers is **sum-free** if the equation $x + y = z$ has no solutions $x, y, z \in S$. Equivalently, S is sum-free if and only if $(S + S) \cap S = \emptyset$, where $A + B$ denotes the set of all sums $a + b, a \in A, b \in B$. For example, the set of all odd positive integers is sum-free.

Consider now the collection of all sum-free sets. Cameron [1–3] defined a natural probability measure on this collection, which can informally be thought of as a recipe for constructing random sum-free sets S. The recipe is as follows:

- Set $S = \emptyset$ initially and look at each positive integer n one-by-one in order.
- If $n = a + b$ for some $a, b \in S$, then skip n and move ahead to $n + 1$.

- If $n = x + y$ has no solutions $x, y \in S$, then toss a fair coin; if heads, set $S = S \cup \{n\}$ and move ahead to $n + 1$; if tails, simply move ahead.

Observe, for example, that clearly

$$\mathrm{P}(S \text{ consists entirely of even integers}) = 0.$$

In contrast, Cameron [1] proved the remarkable fact that the constant

$$c = \mathrm{P}(S \text{ consists entirely of odd integers})$$

is *positive* and, in fact, $0.21759 \le c \le 0.21862$. Equivalently [2], if $N = \{0, 1, \ldots, n-1\}$ and

$$F(n) = 2^{-2n} \sum_{X \subseteq N} 2^{|(X+X)\cap N|},$$

then $F(n)$ is decreasing and $\lim_{n\to\infty} F(n) = c$. The summation is over all subsets X of N and $|E|$ denotes the cardinality of a set E. An alternative proof was given by Calkin [4].

Cameron [2] proved a more general result, which bounds (from below) the probability that S is contained entirely within certain sum-free unions of arithmetic progressions. Rather than state his general theorem, we simply provide a sample application:

$$\mathrm{P}(S \subseteq \{2,\ 7,\ 12,\ 17,\ 22,\ 27, \ldots\} \cup \{3,\ 8,\ 13,\ 18,\ 23,\ 28, \ldots\}) \ge \frac{c^2 d}{2} > 0.0066,$$

where $0.28295 \le d = \lim_{n\to\infty} G(n) \le 0.29484$ and the decreasing function $G(n)$ is defined by

$$G(n) = 2^{-3n} \sum_{X, Y \subseteq N} 2^{|(X+Y)\cap N|}.$$

This, however, is not close to his estimate of approximately 0.022 (based on computer simulation).

Calkin & Cameron [5] advanced our understanding of random sum-free sets even farther. Again, we do not present their theorem in general form, but merely give an example:

$$\mathrm{P}(S \text{ contains 2 and } S \text{ contains no other even integers}) > 0.$$

Computer simulations provide an estimate for this probability of approximately 0.00016.

Let us now turn away from probability and consider instead the number s_n of sum-free subsets of $\{1, 2, \ldots, n\}$. The first several terms [6] of s_n are 1, 2, 3, 6, 9, 16, 24, … . Cameron & Erdös [7,8] conjectured that $s_n 2^{-n/2}$ is bounded and, moreover, the following two limits exist and are approximately

$$\lim_{k\to\infty} s_{2k+1} 2^{-(k+\frac{1}{2})} = c_o = 6.8\ldots, \quad \lim_{k\to\infty} s_{2k} 2^{-k} = c_e = 6.0\ldots,$$

where

$$c_o = \sqrt{2} + \lim_{k\to\infty} H(2k+1), \quad c_e = 1 + \lim_{k\to\infty} H(2k),$$

$$H(n) = 2^{-n/2} \sum_{X\subseteq N'} 2^{-|(X+X)\cap N'|}, \quad N' = \{0, 1, \ldots, n\}.$$

Calkin [9], Alon [10], and Erdös & Granville independently demonstrated that

$$\lim_{n\to\infty} s_n 2^{-(\frac{1}{2}+\varepsilon)n} = 0$$

for every $\varepsilon > 0$. Additional evidence for boundedness appears in [11], and a generalization is found in [12–15].

We cannot resist presenting one more problem. A sum-free set S of positive integers is **complete** if, for all sufficiently large integers n, either $n \in S$ or there exist $s, t \in S$ such that $s + t = n$. Equivalently, S is complete if and only if it is constructed greedily from a finite set. A sum-free set S is **periodic** if there exists a positive integer m such that, for all sufficiently large integers n, $n \in S$ if and only if $n + m \in S$. Equivalently, S is periodic if and only if the elements of S, arranged in increasing order, give rise to an (eventually) periodic sequence of successive differences.

Is an arbitrary complete sum-free set necessarily periodic [16]? Cameron [3] gave the first potentially aperiodic example: the complete sum-free set starting with 3, 4, 13, 18, 24. Calkin & Finch [17] gave other potentially aperiodic examples, including $1, 3, 8, 20, 26, \ldots$ and $2, 15, 16, 23, 27, \ldots$. Calkin & Erdös [18] proved the existence of *incomplete* aperiodic sum-free sets – in fact, they exhibited uncountably many such sets, constructed in a natural way – but no one has yet established the existence of a single complete aperiodic sum-free set.

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2.26 Triple-Free Set Constants

A set S of positive integers is called **double-free** if, for any integer x, the set $\{x, 2x\} \not\subseteq S$. Equivalently, S is double-free if $x \in S$ implies $2x \notin S$. Consider the function

$$r(n) = \max\{|S| : S \subseteq \{1, 2, \ldots, n\} \text{ is double-free}\},$$

that is, the maximum cardinality of double-free sets with no element exceeding n. It is not difficult to prove that

$$\lim_{n\to\infty} \frac{r(n)}{n} = \frac{2}{3};$$

that is, the asymptotic maximal density of double-free sets is $2/3$. Wang [1] obtained both recursive and closed-form expressions for $r(n)$ and, moreover, demonstrated that $r(n) = 2n/3 + O(\ln(n))$ as $n \to \infty$.

Let us now discuss a much harder problem. Define a set S of positive integers to be

- **weakly triple-free** (or **triple-free**) if, for any integer x, the set $\{x, 2x, 3x\} \not\subseteq S$, and
- **strongly triple-free** if $x \in S$ implies $2x \notin S$ and $3x \notin S$.

Unlike the double-free case, the weak and strong senses of triple-free do not coincide. Consider the functions

$$p(n) = \max\{|S| : S \subseteq \{1, 2, \ldots, n\} \text{ is weakly triple-free}\},$$

$$q(n) = \max\{|S| : S \subseteq \{1, 2, \ldots, n\} \text{ is strongly triple-free}\}.$$

We wish to calculate the constants

$$\lambda = \lim_{n\to\infty} \frac{p(n)}{n}, \quad \mu = \lim_{n\to\infty} \frac{q(n)}{n}.$$

Define an infinite set

$$\begin{aligned} A &= \left\{2^i 3^j : i, j \geq 0\right\} = \{a_1 < a_2 < a_3 < \ldots\} \\ &= \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, \ldots\} \end{aligned}$$

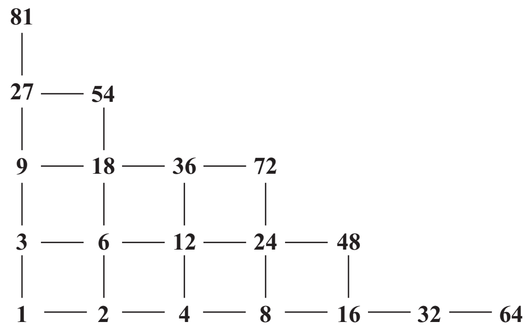

Figure 2.5. Grid graph associated with A_{19}, for which $g_{19} = 10$, $A_{19,0} = \{1, 4, 6, 9, 16, 24, 36, 54, 64, 81\}$, $f_{19} = 6 = h_{19}$, $B_{19,0} = \{1, 6, 8, 27, 36, 48, 64\}$, and $\tilde{B}_{19,0} = \{64\}$.

and A_n to be the first n terms of A; then λ and μ can be written as

$$\lambda = \frac{1}{3}\sum_{n=1}^{\infty}(n - f_n)\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right), \quad \mu = \frac{1}{3}\sum_{n=1}^{\infty} g_n\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right),$$

where the integer sequences

$$\{f_n\} = \{0, 0, 1, 1, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5, 5, 6, 6, 7, 7, 7, 8, 8, \ldots\},$$
$$\{g_n\} = \{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 11, \ldots\}$$

will be defined momentarily.

The constant μ has not attracted as much attention as λ. Eppstein [2] showed that g_n is the size of the largest set of nonadjacent vertices in the grid graph A_n (called an **independence number**). For each $k = 0, 1$, define $A_{n,k} \subseteq A_n$ to consist of all elements $2^i 3^j$ satisfying $i + j \equiv k \bmod 2$. Then $\{A_{n,0}, A_{n,1}\}$ is a partition of A_n and at least one of these is a maximal independent set, as found by Cassaigne [3]. (See Figure 2.5). From here, Zimmermann [3] computed the **triple-free set constant** to be $\mu = 0.6134752692\ldots$.

By way of contrast, the constant λ has intrigued people for over twenty-five years [4]. Graham, Spencer & Witsenhausen [5] were concerned with general conditions on sets, contained in $\{1, 2, \ldots, n\}$, that avoid the values of linear forms $\sum_{v=1}^{w} c_{uv}x_v$. Among many things, they asked whether λ is irrational. Starting from a table of f_n values in [5], Cassaigne [6] proved that $\lambda \geq 4/5$. Chung, Erdös & Graham [7] showed that f_n is the size of the smallest set of vertices in A_n that intersects every L-shaped vertex configuration of the form $\{2^i 3^j, 2^{i+1}3^j, 2^i 3^{j+1}\} \subseteq A_n$ (called an **L-hitting number**). For each $k = 0, 1, 2$, define $B_{n,k} \subseteq A_n$ to consist of all elements $2^i 3^j$ satisfying $i - j \equiv k \bmod 3$. Then $\{B_{n,0}, B_{n,1}, B_{n,2}\}$ is a partition of A_n. Define also $\tilde{B}_{n,k} \subseteq B_{n,k}$ to consist of all elements 2^i, $1 \leq i \equiv k \bmod 3$, for which $2^{i-1}3 \notin A_n$. It is known that

$$f_n \leq h_n = \min_{0 \leq k \leq 2} |B_{n,k}| - |\tilde{B}_{n,k}| \leq \left\lfloor \frac{n}{3} \right\rfloor,$$

and consequently $0.800319 < \lambda < 0.800962$. It is conjectured that $f_n = h_n$ for all n, which if true would imply that $\lambda = 0.8003194838\ldots = 1 - 0.1996805161\ldots$.

Given fixed $s > 1$, consider sets S of positive integers for which $\{x, 2x, 3x, \ldots, sx\} \not\subseteq S$ for all integers x. Denote the corresponding asymptotic maximal density by λ_s. What can be said about the asymptotics of λ_s as $s \to \infty$? Spencer & Erdös [8] proved that there exist constants c and C for which

$$1 - \frac{C}{s\ln(s)} < \lambda_s < 1 - \frac{c}{s\ln(s)}$$

for all suitably large s, although specific numerical values were not presented. Also, consider sets T of positive integers for which $\{x, 2x, 3x, 6x\} \not\subseteq T$ for all integers x. The corresponding asymptotic maximal density is exactly $11/12$ [7], which is surprising since the case $s = 3$ was so much more difficult.

More instances of the interplay between the numbers 2 and 3 occur in [2.30.1], which is concerned with powers of $3/2$ modulo 1.

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2.27 Erdös–Lebensold Constant

A strictly increasing sequence of positive integers $a_1, a_2, a_3, \ldots$ is **primitive** [1–3] if $a_i \nmid a_j$ for any $i \neq j$. That is, no term of the sequence divides any other. An example of a finite primitive sequence is the set of all integers m in the interval $\lceil \frac{n+1}{2} \rceil \leq m \leq n$, where n is a positive integer. An example of an infinite primitive sequence consists of all positive integers composed of exactly r prime factors, where r is fixed. We discuss the finite and infinite cases separately. See also [5.5] for a related note.

2.27.1 Finite Case

For each positive integer n, define

$$M(n) = \sup_{\substack{\text{primitive} \\ A \subseteq \{1,2,\ldots,n\}}} \sum_i 1$$

as the maximum possible number of terms, and

$$L(n) = \sup_{\substack{\text{primitive} \\ A \subseteq \{1,2,\ldots,n\}}} \sum_i \frac{1}{a_i}$$

as the maximum possible reciprocal sum. Clearly $M(n) = \left\lfloor \frac{n+1}{2} \right\rfloor$ and thus $\lim_{n\to\infty} M(n)/n = 1/2$. It is more difficult to establish [4, 5] that

$$\lim_{n\to\infty} \frac{\sqrt{\ln(\ln(n))}}{\ln(n)} L(n) = \frac{1}{\sqrt{2\pi}},$$

which is an unexpected appearance of Archimedes' constant [1.4].

2.27.2 Infinite Case

Any infinite primitive sequence satisfies

$$0 = \liminf_{n\to\infty} \frac{1}{n} \sum_{a_i \le n} 1 \le \limsup_{n\to\infty} \frac{1}{n} \sum_{a_i \le n} 1 < \frac{1}{2}.$$

Besicovitch [1, 6] proved that, for each $\varepsilon > 0$, there exists a primitive sequence such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{a_i \le n} 1 > \frac{1}{2} - \varepsilon.$$

In particular, a primitive sequence need not possess an asymptotic density! Maybe the limiting value 1/2 is not so surprising, given the earlier result about $M(n)$.

In contrast, Erdös, Sárkozy & Szemerédi [7] proved that

$$\lim_{n\to\infty} \frac{\sqrt{\ln(\ln(n))}}{\ln(n)} \sum_{a_i \le n} \frac{1}{a_i} = 0,$$

which is drastically different from the earlier result about $L(n)$. The finite and infinite cases behave independently in this respect.

Forging a new trail, Erdös [1, 8] proved that the series

$$\sum_i \frac{1}{a_i \ln(a_i)}$$

is convergent (except for the trivial primitive sequence $\{1\}$) and is, moreover, bounded by some absolute constant. He conjectured that

$$\sum_i \frac{1}{a_i \ln(a_i)} \le \sum_i \frac{1}{p_i \ln(p_i)} = 1.6366163233\ldots,$$

where the latter summation is over all primes. Several partial results are known. Zhang [9, 10] proved that the inequality is true for all primitive sequences whose terms contain at most four prime factors. Zhang [11] did likewise, hypothesizing a different, more technical set of conditions. Erdös & Zhang [12] proved that, for any primitive sequence,

$$\sum_i \frac{1}{a_i \ln(a_i)} \le 1.84$$

and Clark [13] strengthened this to

$$\sum_i \frac{1}{a_i \ln(a_i)} \le e^\gamma = 1.7810724179\ldots,$$

where γ is Euler's constant [1.5].

Incidently, the estimate 1.6366163233... given here for the prime series is due to Cohen [14].

2.27.3 Generalizations

Let k be a positive integer. A strictly increasing sequence of positive integers a_1, a_2, $a_3, \ldots$ is **k-primitive** if no term of the sequence divides k others. (This phraseology is new.) Let us consider only the finite case. Define $M(n, k)$ and $L(n, k)$ as before. An example of a 2-primitive sequence is the set of all integers m in the interval $\lceil \frac{n+1}{3} \rceil \le m \le n$; thus $\lim_{n\to\infty} M(n, 2)/n \ge 2/3$, but here improvement is possible. Lebensold [15] proved that

$$0.6725 \le \lim_{n\to\infty} \frac{M(n, 2)}{n} \le 0.6736$$

and observed that more accurate bounds could be achieved by additional computation in exactly the same manner. Erdös asked if the limit is irrational [10]. No one has examined $L(n, 2)$ or the case $k > 2$, as far as is known.

A strictly increasing sequence of positive integers $b_1, b_2, b_3, \ldots$ is **quasi-primitive** [16] if the equation $\gcd(b_i, b_j) = b_r$ is not solvable with $r < i < j$. An example of an infinite quasi-primitive sequence consists of all prime powers

$$q_1 = 2, q_2 = 3, q_3 = 2^2, q_4 = 5, q_5 = 7, q_6 = 2^3, q_7 = 3^2, q_8 = 11, \ \ldots.$$

Erdös & Zhang [16] conjectured that, for any quasi-primitive sequence,

$$\sum_i \frac{1}{b_i \ln(b_i)} \le \sum_i \frac{1}{q_i \ln(q_i)} = 2.006\ldots.$$

Clark [17] corrected a false claim in [16] and proved that

$$\sum_i \frac{1}{b_i \ln(b_i)} < 4.2022.$$

A more accurate estimate for the prime-power series is an unsolved problem.

The topics of k-primitive sequences and quasi-primitive sequences appear to be wide open areas for research, as are the allied topics of triple-free set constants [2.26] and Erdös' reciprocal sum constants [2.20].

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2.28 Erdös' Sum-Distinct Set Constant

A set $a_1 < a_2 < a_3 < \ldots < a_n$ of positive integers is called **sum-distinct** if the 2^n sums

$$\sum_{k=1}^{n} \varepsilon_k a_k \quad (\text{each } \varepsilon_k = 0 \text{ or } 1,\ 1 \le k \le n)$$

are all different. Equivalently, sum-distinctness holds if and only if any two subset sums are never equal [1–4]. The set of nonnegative powers of 2 is clearly sum-distinct and serves as a baseline for comparison. In 1931, Erdös examined the ratio

$$\alpha_n = \inf_A \frac{a_n}{2^n},$$

where the infimum is over all sum-distinct sets A of cardinality n, and conjectured that $\alpha = \inf_n \alpha_n$ is positive. No one knows whether this is true, but in 1955, Erdös and Moser [2, 5, 6] proved that, for all $n \geq 2$,

$$\alpha_n \geq \max\left(\frac{1}{n}, \frac{1}{4\sqrt{n}}\right),$$

and Elkies [7] proved that, for sufficiently large n,

$$\alpha_n \geq \frac{1}{\sqrt{\pi n}}.$$

Gleason & Elkies [8] subsequently removed the factor of π via a variance reduction technique. See also [9]. It is probably true that $\alpha > 1/8 = 0.125$. Significant progress in resolving Erdös' conjecture will almost certainly require a brand-new idea or as-yet-unseen insight.

Several interesting constructions provide upper bounds on α. In 1986, Atkinson, Negro & Santoro [10, 11] defined a sequence

$$u_0 = 0, \ \ u_1 = 1, \ \ u_{k+1} = 2u_k - u_{k-m}, \ \ m = \left\lfloor \tfrac{1}{2}k + 1 \right\rfloor$$

that gives rise to a sum-distinct set $a_k = u_n - u_{n-k}$, $1 \leq k \leq n$, for each n. Clearly $a_n = u_n$. Lunnon [11] calculated that

$$\lim_{n\to\infty} \frac{u_n}{2^n} = 0.3166841737\ldots = \frac{1}{2}(0.6333683473\ldots).$$

A smaller ratio is obtained via a sequence due to Conway & Guy [2, 11–13]:

$$v_0 = 0, \ \ v_1 = 1, \ \ v_{k+1} = 2v_k - v_{k-m}, \ \ m = \left\lfloor \tfrac{1}{2} + \sqrt{2k} \right\rfloor.$$

Only recently Bohman [14] proved that this sequence gives rise to a sum-distinct set $a_k = v_n - v_{n-k}$, $1 \leq k \leq n$, for each n. (Prior to 1996, we knew this claim to be true for only $n < 80$.) Lunnon [11] calculated that

$$\lim_{n\to\infty} \frac{v_n}{2^n} = 0.2351252848\ldots = \frac{1}{2}(0.4702505696\ldots).$$

Although the Atkinson–Negro–Santoro and Conway–Guy limiting ratios are interesting constants, they do not provide the best-known upper bounds on α. A frequently used trick for doing so is as follows: If $a_1 < a_2 < a_3 < \ldots < a_n$ is a sum-distinct set with n elements, then clearly $1 < 2a_1 < 2a_2 < 2a_3 < \ldots < 2a_n$ is a sum-distinct set with $n + 1$ elements. Enlarging as such can be continued indefinitely, of course. Thus if one has found a sum-distinct set with n elements and small ratio ρ, we immediately have an upper bound $\alpha \leq \rho$. For example, Lunnon [11] found a sum-distinct set with $n = 67$ and $\rho = 0.22096$ via computer search, which improves on the Conway–Guy bound. Generalizing the work of Conway, Guy, and Lunnon, Bohman [15] established the best-known upper bound $\alpha \leq 0.22002$. Additionally, Maltby [16] has shown, given a sum-distinct set, how to construct a larger sum-distinct set with a smaller ratio. Hence Erdös' constant α is not realized by any sum-distinct set; that is, the infimum is never achieved!

Bae [17] studied sum-distinct sets whose sums avoid r mod q, for given r and q. Also, consider the inequality

$$\sum_{k=1}^{n} \frac{1}{a_k} < 2 = \sum_{k=1}^{\infty} \frac{1}{2^{k-1}},$$

which is true for all sum-distinct sets A. It is curious that the upper bound 2 is sharp and elementary proofs are possible [9, 18, 19]. (Actually much more is known!) Elsewhere we discuss other such reciprocal sums [2.20], which are often exceedingly difficult to evaluate.

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2.29 Fast Matrix Multiplication Constants

Everyone knows that multiplying two arbitrary $n \times n$ matrices requires n^3 multiplications, at least if we do it using standard formulas.

In the mid 1960s, Pan and Winograd [1] discovered a way to reduce this to approximately $n^3/2$ multiplications for large n, and for a few years people believed that this might be the best possible reduction.

Define the **exponent of matrix multiplication** ω as the infimum of all real numbers τ such that multiplication of $n \times n$ matrices may be achieved with $O(n^\tau)$ multiplications. Clearly $\omega \leq 3$ and it can be proved that $\omega \geq 2$.

Strassen [2] discovered a surprising base algorithm to compute the product of 2×2 matrices with only seven multiplications. The technique can be recursively extended to large matrices via a tensor product construction. In this case, the construction is very simple: Large matrices are broken down recursively by partitioning the matrices into quarters, sixteenths, etc. This gives $\omega \leq \ln(7)/\ln(2) < 2.808$.

More sophisticated base algorithms and tensor product constructions permit further improvements. Many researchers have contributed to this problem, including Pan [3,4] who found $\omega < 2.781$ and Strassen [5] who found $\omega < 2.479$. See [6,7] for an overview and history.

Coppersmith & Winograd [8] presented a new method, based on a combinatorial theorem of Salem & Spencer [9], which gives dense sets of integers containing no three terms in arithmetic progression. They consequently obtained $\omega < 2.376$, which is the best-known upper bound today.

Is $\omega = 2$? Bürgisser [10] called this the central problem of algebraic complexity theory. Here is a closely related combinatorial problem [8,11].

Given an abelian additive group G of order n, find the least integer $f(n, G)$ with the following property. If a subset S of G has cardinality $\geq f(n, G)$, then there exist three subsets A, B, C of S, pairwise disjoint and not all empty, such that

$$\sum_{a \in A} a = \sum_{b \in B} b = \sum_{c \in C} c.$$

(Clearly $f(n, G)$ exists for $n \geq 5$, because if $S = G$, then consider $A = \{0\}$, $B = \{g, -g\}$, $C = \{h, -h\}$, where nonzero elements g and h satisfy $g \neq h$ and $g \neq -h$.) Now define another function

$$F(n) = \max_G f(n, G),$$

the maximum taken over all abelian groups G of order n, and examine the ratio

$$\rho = \lim_{n \to \infty} \frac{\ln(n)}{F(n)}.$$

Coppersmith & Winograd [8] demonstrated that if $\rho = 0$, then $\omega = 2$. A proof that $\rho = 0$, however, is still unknown. What (if any) numerical evidence exists in support of $\rho = 0$?

Coppersmith [12] further gave a constant $\alpha > 0.294$ and, for any $\varepsilon > 0$, an algorithm for multiplying an $n \times n$ matrix by an $n \times n^\alpha$ matrix with complexity $O(n^{2+\varepsilon})$. An improvement in the lower bound for α would provide more hope that $\omega = 2$. Research in this area continues [13, 14].

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2.30 Pisot–Vijayaraghavan–Salem Constants

Given any positive real number x, let $\{x\} = x \bmod 1$ denote the fractional part of x. For any positive integer n, clearly $\{n + x\} = x$ for all x, and the sequence $\{nx\}$ is periodic if x is rational. A consequence of Weyl's criterion [1–4] is that the sequence $\{nx\}$ is dense in the interval [0, 1] if x is irrational. Moreover, it is **uniformly distributed** in [0, 1], meaning that the probability of finding an arbitrary element in any subinterval is proportional to the subinterval length.

Having discussed addition and multiplication, let us turn to exponentiation. It can be proved [5, 6] that the sequence $\{x^n\}$ is uniformly distributed for *almost all* real numbers $x > 1$ (curiously, no specific such values x were known until recently [7, 8]). It is believed that the sequence for $x = 3/2$ is a typical example [2.30.1]. The measure-zero, uncountable set E of exceptions x to this behavior [9–12] includes the numbers $2, 3, 4, \ldots$ and $1 + \sqrt{2}$. What else can be said about E?

First, we review some terminology. A **monic polynomial** is a polynomial with a leading coefficient equal to 1. An **algebraic integer** α is a zero of a monic polynomial with integer coefficients. The **conjugates** of α are all zeros of the minimal polynomial of α. Define the set U to be all real algebraic integers $\alpha > 1$ whose conjugates $\gamma \neq \alpha$ each satisfy $|\gamma| \leq 1$. It is known that $U \subseteq E$ and that U is countably infinite. Let us study the exceptional behavior in more detail.

Define the set S of **Pisot–Vijayaraghavan (P-V) numbers** to be all real algebraic integers $\theta > 1$ whose conjugates $\gamma \neq \theta$ each satisfy $|\gamma| < 1$. Define the set T of **Salem numbers** to be all real algebraic integers $\tau > 1$ whose conjugates $\gamma \neq \tau$ each satisfy $|\gamma| \leq 1$ with at least one case of equality. Then clearly S and T determine a partition of U. Moreover, if θ is a P-V number, then

$$\lim_{n\to\infty} \{\theta^n\} = 0 \bmod 1,$$

whereas, if τ is a Salem number, then $\{\tau^n\}$ is dense but not uniformly distributed in the interval $[0, 1]$. There are many related results and we give an example [11]. Suppose we are given an algebraic real $\alpha > 1$ and a real $\lambda > 0$ for which $\{\lambda\alpha^n\}$ has at most finitely many limit points modulo one. Then α must be in S. Additionally, the limit points must each be rational. It is unknown whether anyone has exhibited explicitly a number that is in E but not in U (e.g., a transcendental exceptional x).

We turn attention to the set S, which is known to be countably infinite and closed, and which possesses an isolated minimum point $\theta_0 > 1$. Salem [13] and Siegel [14] proved that $\theta_0 = 1.3247179572\ldots$ is the real zero of the polynomial $x^3 - x - 1$, that is,

$$\theta_0 = \left(\frac{1}{2} + \frac{\sqrt{69}}{18}\right)^{\frac{1}{3}} + \frac{1}{3}\left(\frac{1}{2} + \frac{\sqrt{69}}{18}\right)^{-\frac{1}{3}} = \frac{2\sqrt{3}}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\right)\right).$$

This constant also appears in [1.2.2].

In fact, a complete listing of all P-V numbers up to $\varphi + \varepsilon$ is possible [15], where $\varphi = 1.6180339887\ldots$ is the Golden mean [1.2] and $0 < \varepsilon < 0.0004$. Also, let $S^{<1>}$ denote the set of all limit points of S, that is, the derived set of S. The minimum point of $S^{<1>}$ is φ and is isolated. More generally, let $S^{<k>}$ denote the derived set of $S^{<k-1>}$ for all $k \geq 2$. The minimum point of $S^{<2>}$ is 2, and the minimum point of $S^{<k>}$ is between $\sqrt{k}$ and $k + 1$, but no exact values of these points for $k \geq 3$ are known.

The set T is more difficult to study. We know that T is countably infinite and that U is a proper subset of the closure of T. The existence of a minimum Salem number remains an open problem, but it is conjectured to be $\tau_0 = 1.1762808182\ldots$, which is one of the zeros of **Lehmer's polynomial** [16]

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

It has been proved [17–20] that there are exactly forty-five Salem numbers less than 1.3 with degree at most 40. (There are only two known Salem numbers less than 1.3 with degree exceeding 40, but conceivably there may be more.) Is θ_0 the smallest limit point of T? The answer is not known to this question either.

The constants θ_0 and τ_0 appear in connection with a related conjecture, due to Lehmer, about Mahler's measure of a nonzero algebraic integer α. If α is of degree n with conjugates $\alpha_1 = \alpha, \alpha_2, \alpha_3, \ldots, \alpha_n$, define $M(\alpha)$ to be the absolute value of the product of all α_j satisfying $|\alpha_j| > 1$. Kronecker [21,22] proved that if $M(\alpha) = 1$, then α is a root of unity. Is it true that for every $\varepsilon > 0$, there exists α such that $1 < M(\alpha) < 1 + \varepsilon$?

If α is **non-reciprocal**, that is, if α and $1/\alpha$ are not conjugate, then Smyth [11,23] proved that the answer is no. More precisely, either $M(\alpha) \geq \theta_0 = 1.324\ldots$ or α is a root of unity.

For arbitrary α, Lehmer [16] conjectured that the answer remains no. More precisely, either $M(\alpha) \geq \tau_0 = 1.176\ldots$ or α is a root of unity. Despite extensive searches, no counterexamples to this inequality have been found. The best-known relevant estimate, if α is not a root of unity, is [24–30]

$$M(\alpha) > 1 + \left(\frac{9}{4} - \varepsilon\right)\left(\frac{\ln(\ln(n))}{\ln(n)}\right)^3$$

for sufficiently large n. For more about Mahler's measure, see [3.10]. We mention a related inequality [21,30–32] involving what is called the **house** of α:

$$\overline{|\alpha|} = \max_{1 \leq k \leq n} |\alpha_k| > 1 + \frac{1}{n}\left(\frac{64}{\pi^2} - \varepsilon\right)\left(\frac{\ln(\ln(n))}{\ln(n)}\right)^3$$

and a corresponding conjecture [33]: $\overline{|\alpha|} \geq 1 + \frac{3}{2}\ln(\theta_0)/n = 1 + (0.4217993614\ldots)/n$. See also [34,35].

2.30.1 Powers of 3/2 Modulo One

Pisot [9] and Vijayaraghavan [36] proved that $\{(3/2)^n\}$ has infinitely many accumulation points, that is, infinitely many convergent subsequences with distinct limits. The sequence is believed to be uniformly distributed, but no one has even proved that it is dense in [0, 1].

Here is a somewhat less ambitious problem: Prove that $\{(3/2)^n\}$ has infinitely many accumulation points in both [0, 1/2) and [1/2, 1]. In other words, prove that the sequence does not **prefer** one subinterval over the other. This problem remains unsolved, but Flatto, Lagarias & Pollington [37] recently made some progress. They proved that any subinterval of [0, 1] containing all but perhaps finitely many accumulation points of $\{(3/2)^n\}$ must have length at least 1/3. Therefore, the sequence cannot prefer $[0, 1/3 - \varepsilon)$ over $[1/3 - \varepsilon, 1]$ for any $\varepsilon > 0$. Likewise, it cannot prefer $[2/3 + \varepsilon, 1]$ over $[0, 2/3 + \varepsilon)$. To extend the proof to [0, 1/2) and [1/2, 1] would be a significant but formidable achievement.

Lagarias [38] mentioned the sequence $\{(3/2)^n\}$ and its loose connections with ergodic-theoretic aspects of the famous $3x + 1$ problem. The details are too elaborate to discuss here. What is fascinating is that the sequence is *also* fundamental to a seemingly distant area of number theory: Waring's problem on writing integers as sums of n^{th} powers.

Let $g(n)$ denote the smallest integer k for which every positive integer can be expressed as the sum of k n^{th} powers of nonnegative integers. Hilbert [39] proved that $g(n) < \infty$ for each n. For $2 \le n \le 6$, it is known that [40–44]

$$g(n) = 2^n + \left\lfloor \left(\frac{3}{2}\right)^n \right\rfloor - 2.$$

Dickson [45,46] and Pillai [47] independently proved that this formula is true for all $n > 6$, provided that the condition

$$\left\{ \left(\frac{3}{2}\right)^n \right\} \le 1 - \left(\frac{3}{4}\right)^n$$

is satisfied. Hence it is sufficient to study this inequality, the last remaining obstacle in the solution of Waring's problem.

Kubina & Wunderlich [48], extending the work of Stemmler [49], verified computationally that the inequality is met for all $2 \le n \le 471600000$. Mahler [50] moreover proved that it fails for at most finitely many n, using the Thue–Siegel–Roth theorem on rational approximations to algebraic numbers [2.22]. The proof is non-constructive and thus a computer calculation that rules out failure altogether is still not possible.

It appears that the inequality can be strengthened to

$$\left(\frac{3}{4}\right)^n < \left\{ \left(\frac{3}{2}\right)^n \right\} < 1 - \left(\frac{3}{4}\right)^n$$

for all $n > 7$ and generalized in certain ways [51,52]. Again, no proof is known apart from Mahler's argument. (The best effective results are due to Beukers [53], Dubickas [54], and Habsieger [55], with 3/4 replaced by 0.577.) The fact that so simple an inequality can defy all attempts at analysis is remarkable.

The calculation of $g(n)$ is sometimes called the "ideal" part of Waring's problem. Let $G(n)$ denote the smallest integer k for which all *sufficiently large* integers can be expressed as the sum of k n^{th} powers of nonnegative integers. Clearly $G(n) \le g(n)$, and Hurwitz [56] and Maillet [57] proved that $G(n) \ge n + 1$. In other words, there are arbitrarily large integers that are not the sum of n n^{th} powers. It is known [43,58–60] that $G(2) = 4$, $4 \le G(3) \le 7$, $G(4) = 16$, $6 \le G(5) \le 17$, and $9 \le G(6) \le 24$. See [61–63] for numerical evidence supporting a conjecture that $G(3) = 4$. See also [64,65] for the asymptotics of the number of representations of n as a sum of four cubes, which interestingly turns out to involve $\Gamma(4/3)$, where $\Gamma(x)$ is Euler's gamma function [1.5.4].

Here are several unrelated facts. Infinitely many integers of the form $\lfloor x^n \rfloor$ are composite [66,67] when $x = 3/2$. This is also true when $x = 4/3$. Are infinitely many such integers prime? What can be said for other values of x?

A conjecture is that, if t is a real number for which 2^t and 3^t are both integers, then t is rational. This would follow from the so-called four-exponentials conjecture [68,69]. A weaker result, the six-exponentials theorem, is known to be true.

Define an infinite sequence by $x_0 = 1$ and $x_n = \left\lceil \frac{3}{2} x_{n-1} \right\rceil$ for $n \geq 1$. Odlyzko & Wilf [70] proved that

$$x_n = \left\lfloor K \cdot \left(\frac{3}{2}\right)^n \right\rfloor$$

for all n, where the constant $K = 1.6222705028\ldots$ (in fact, they proved much more). Their work is connected to the solution of the ancient Josephus problem. The constant K is analogous to Mills' constant [2.13], in the sense that the formula is useless computationally (unless an exact value for K somehow became available), but its mere existence is remarkable.

A **3-smooth number** is a positive integer whose only prime divisors are 2 or 3. A positive integer n possesses a **3-smooth representation** if n can be written as a sum of 3-smooth numbers, where no summand divides another. Let $r(n)$ denote the number of 3-smooth representations of n. Some recent papers [71–73] answer the question of the maximal and average orders of $r(n)$. See also [5.4].

Let n be an integer larger than 8. Need the base-3 expansion of 2^n possess a digit equal to 2 somewhere? Erdös [74] conjectured that the answer is yes, and Vardi [75] verified this up to $n = 2 \cdot 3^{20}$. More instances of the interplay between the numbers 2 and 3 occur in [2.26].

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2.31 Freiman's Constant

2.31.1 Lagrange Spectrum

In our essay on Diophantine approximation constants [2.23], we discussed Hurwitz's [1, 2] theorem that, for any irrational number ξ, the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{\sqrt{5}}\frac{1}{q^2}$$

has infinitely many solutions (p, q), where p and q are integers. Can this result be improved? That is, can $\sqrt{5}$ be replaced by a larger quantity? The answer is no for certain special numbers ξ, but it is yes otherwise. We now elaborate.

For each number ξ, define $\lambda(\xi)$ to be the supremum of all quantities c for which the integer solution set (p, q) of

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{c}\frac{1}{q^2}$$

remains infinite. The set of values L taken by the function $\lambda(\xi)$ is called the **Lagrange spectrum** [3]. Clearly the smallest value in L is $\sqrt{5}$. It can be proved that the set $L \cap [2, 3]$ is countably infinite, with 3 as its only limit point, but $[\theta, \infty) \subseteq L$ for some point $\theta > 4$. Much more will be said about L shortly.

2.31.2 Markov Spectrum

A two-variable quadratic form with real coefficients $f(x, y) = \alpha x^2 + \beta xy + \gamma y^2$ is **indefinite** if f assumes both positive and negative values. If the **discriminant**

$d(f) = \beta^2 - 4\alpha\gamma$ is positive, then the plot of $z = f(x, y)$ in real xyz-space is a saddle surface, that is, with no maximum or minimum points.

For each such f, define

$$\mu(f) = \frac{\sqrt{d(f)}}{\inf\limits_{(m,n)\neq(0,0)} |f(m,n)|},$$

where the infimum ranges over all nonzero integer pairs. The set of values M taken by the function $\mu(f)$ is called the **Markov spectrum** [3]. It can be proved that $L \subseteq M$ and further that $M \cap [2, 3] = L \cap [2, 3]$ and $[\theta, \infty) \subseteq M$ for the same point $\theta > 4$ mentioned for L. However, $M \cap [3, \theta] \neq L \cap [3, \theta]$; that is, L is a *proper* subset of M, which gives rise to some interesting unresolved issues.

2.31.3 Markov–Hurwitz Equation

Let us return to Hurwitz's theorem. First, define two numbers ξ and η to be **equivalent** if there are integers a, b, c, d such that

$$\xi = \frac{a\eta + b}{c\eta + d}, \quad |ad - bc| = 1.$$

This relation permits the partitioning of numbers into equivalence classes. Two irrational numbers ξ and η are equivalent if and only if, after some point, their respective sequences of continued fraction partial denominators are identical.

Now, it can be proved that $\lambda(\xi) = \sqrt{5}$ for all ξ equivalent to the Golden mean φ [1.2], that is, possessing partial denominators that are eventually all 1s. Such numbers can be thought of as "simplest," but from the point of view of rational approximations, the simplest numbers are the "worst" [1,4]. If we leave these out, the next level of approximation difficulty is given by $\lambda(\xi) = \sqrt{8}$ for all ξ equivalent to Pythagoras' constant $\sqrt{2}$ [1.1], that is, possessing partial denominators that are eventually all 2s. If we leave these out as well, the next level is $\lambda(\xi) = \sqrt{221}/5$ and so on. See [3] for a table of smallest numbers in the Lagrange spectrum, as well as an algorithm for computing a corresponding representative quadratic form $f(x, y)$.

The values $\sqrt{5}$, $\sqrt{8}$, $\sqrt{221}/5$, $\sqrt{1517}/13$, $\sqrt{7565}/29, \ldots$ are all of the form $\sqrt{9w^2 - 4}/w$, where u, v, w are positive integers satisfying the Diophantine equation

$$u^2 + v^2 + w^2 = 3uvw, \quad 1 \leq u \leq v \leq w.$$

The first several admissible triples are

$$(u, v, w) = (1, 1, 1), (1, 1, 2), (1, 2, 5), (1, 5, 13), (2, 5, 29), (1, 13, 34), \ldots$$

and the infinite sequence of ws

$$1,\ 2,\ 5,\ 13,\ 29,\ 34,\ 89,\ 169,\ 194,\ 233,\ 433,\ 610,\ 985, \ldots$$

are called **Markov numbers** [5]. It is unknown [6–12] whether every w_k determines a *unique* admissible triple (u_k, v_k, w_k). Note that, clearly, the limit of $\lambda(w_k)$ as $k \to \infty$ is 3. This proves that $L \cap [2, 3]$ accumulates at 3, as was to be shown.

Here is a side topic. The number $N(n)$ of admissible triples (u, v, w) with $w \le n$ was proved by Zagier [7, 8] to be

$$N(n) = C \cdot \ln(n)^2 + O\left[\ln(n) \cdot \ln(\ln(n))^2\right],$$

where

$$\begin{aligned} C &= \frac{3}{\pi^2}\frac{1}{2}\left(\frac{1}{g(1)^2} + \frac{2g(1) - g(2)}{g(1)^2 g(2)}\right) + \frac{3}{\pi^2} \sum_{\substack{\text{admissible} \\ (u,v,w) \text{ with} \\ u<v<w}} \frac{g(u) + g(v) - g(w)}{g(u)g(v)g(w)} \\ &= 0.1807171047\ldots \end{aligned}$$

and

$$g(x) = \ln\left(\frac{3x + \sqrt{9x^2 - 4}}{2}\right) = \operatorname{arccosh}\left(\frac{3x}{2}\right), \quad x \ge \frac{2}{3}.$$

He conjectured that this asymptotic result can be strengthened to

$$N(n) = C \cdot \ln(3n)^2 + o\left(\ln(n)\right),$$

which, if the uniqueness conjecture is true, may be rewritten as

$$w_k = \left(\frac{1}{3} + o(1)\right) \exp\left(\sqrt{\frac{k}{C}}\right) = \left(\frac{1}{3} + o(1)\right) (10.5101504239\ldots)^{\sqrt{k}}.$$

Here is a generalization of the side topic. Let $m \ge 3$. Consider the **Markov–Hurwitz equation**

$$u_1^2 + u_2^2 + \cdots + u_m^2 = m u_1 u_2 \cdots u_m, \quad 1 \le u_1 \le u_2 \le \cdots \le u_m,$$

and define $N_m(n)$ to be the number of admissible m-tuples $(u_1, u_2, \ldots, u_m)$ of positive integers with $u_m \le n$. It is surprising that the growth rate of $N_m(n)$ is not $O(\ln(n)^{m-1})$, but rather $O(\ln(n)^{\alpha(m)+\varepsilon})$ for any $\varepsilon > 0$, where the exponents $\alpha(m)$ satisfy [13–15]

$$\alpha(3) = 2, \;\; 2.430 < \alpha(4) < 2.477, \;\; 2.730 < \alpha(5) < 2.798, \;\; 2.963 < \alpha(6) < 3.048$$

and $\lim_{m\to\infty} \alpha(m)/\ln(m) = 1/\ln(2)$. The analog of Zagier's constant C for $m \ge 4$ is not known.

2.31.4 Hall's Ray

Our knowledge of $L \cap [3, \infty)$ and $M \cap [3, \infty)$ is much less complete than the aforementioned information for $L \cap [2, 3]$. Each of L and M is a closed subset of the real line; hence the complement of each spectrum is a countable union of open intervals, that is, of **gaps**. A gap is **maximal** if its endpoints are in the spectrum under consideration. Here are several maximal gaps (with regard to both L and M):

$$\left(\sqrt{12}, \sqrt{13}\right) = (3.464101\ldots, 3.605551\ldots),$$

$$\left(\sqrt{13}, \frac{65 + 9\sqrt{3}}{22}\right) = (3.605551\ldots, 3.663111\ldots),$$

$$\left(\frac{\sqrt{480}}{7}, \sqrt{10}\right) = (3.129843\ldots, 3.162277\ldots).$$

The first two were discovered by Perron [16]; many others are listed in [3]. Evidently there is no "first" gap with left-hand endpoint ≥ 3.

Hall [17] proved that any real number in the interval $[\sqrt{2}-1, 4\sqrt{2}-4]$ can be written as a sum of two numbers whose continued fraction partial denominators never exceed 4. It follows that L and M contain all sufficiently large real numbers; this portion of these spectra is called **Hall's ray**. Freiman [18] succeeded in computing the precise point θ at which Hall's ray begins (which is the same for both L and M) and its exact expression is [3, 6]

$$\theta = 4 + \frac{253589820 + 283748\sqrt{462}}{491993569} = 4.5278295661\ldots.$$

In fact, the "last" gap with right-hand endpoint $< \infty$ is $(4.527829538\ldots, 4.527829566\ldots)$, true for both L and M.

By way of contrast, Bumby [3, 19] determined that $M \cap [3, 3.33437\ldots]$ has Lebesgue measure zero! Can the endpoint $3.33437\ldots$ be shifted any farther to the right and yet preserve the measure-zero property? Can an exact expression for this endpoint be found?

2.31.5 L and M Compared

This is perhaps the most mysterious area of this study, and we shall be very brief [3]. Freiman [20] constructed a quadratic irrational $\xi = 3.118120178\ldots$ that is in M but not in L. Freiman [21] later found another example: $\eta = 3.293044265\ldots$. Infinitely many more such examples are now known. Berstein [22, 23] determined the largest intervals containing Freiman's points ξ and η but not containing any elements of L. The interval for ξ has approximate length 1.7×10^{-10} whereas that for η has approximate length 2×10^{-7}. Freiman additionally showed that these intervals each contain countably infinite elements of M.

Cusick & Flahive [3] conjectured that L and M coincide above $\sqrt{12} = 3.464101\ldots$. The largest known number in M but not in L is $3.29304\ldots$. Much more on this fascinating subject is found in [24].

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2.32 De Bruijn–Newman Constant

We discuss a constant here that is unlike any other in this collection: It is positive if and only if the notorious Riemann hypothesis [1.6.2] is false. It is, moreover, defined in a manner that permits the computer calculation of precise numerical bounds [1].

Starting with the Riemann zeta function $\zeta(z)$, define [2]

$$\xi(z) = \tfrac{1}{2}z(z-1)\pi^{-\frac{1}{2}z}\Gamma(\tfrac{1}{2}z)\zeta(z), \quad \Xi(z) = \xi(iz + \tfrac{1}{2}), \quad z \text{ complex.}$$

It is trivial to prove that the Riemann hypothesis is true if and only if the zeros of $\Xi(z)$ are all real. This restatement of the conjecture will be useful to us in what follows.

Think of $\Xi(z/2)/8$ as a complex frequency function, that is, as the Fourier cosine transform of a time signal $\Phi(t)$. The signal can be calculated to be

$$\Phi(t) = \sum_{n=1}^{\infty}(2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t})\exp(-\pi n^2 e^{4t}), \quad t \text{ real}, t \geq 0.$$

Given a real parameter λ, consider the modified signal $\Phi(t)\exp(\lambda t^2)$ and then carry it back into the frequency domain, that is, returning to where we were initially. The resulting family of Fourier cosine transforms, $H_\lambda(z)$, contains $H_0(z) = \Xi(z/2)/8$ as a special case.

What is known about the zeros of $H_\lambda(z)$, for fixed λ? De Bruijn [3] proved, among other things, that H_λ has only real zeros for $\lambda \geq 1/2$. Newman [4] established further that there is a constant, Λ, such that H_λ has only real zeros if and only if $\lambda \geq \Lambda$. Of course, $\Lambda \leq 1/2$ follows immediately from de Bruijn's result. The Riemann hypothesis is equivalent to the conjecture that $\Lambda \leq 0$. Newman conjectured that $\Lambda \geq 0$, emphasizing nicely that the Riemann hypothesis, if it is true, is just barely so.

Lower bounds on Λ are clearly of enormous interest to everybody concerned. Elaborate computations in [1,5–8] gave $\Lambda > -0.0991$. Csordas, Smith & Varga [9,10] proved a theorem, involving certain "close" consecutive zeros of the Riemann xi function (known as Lehmer pairs), that dramatically sharpened estimates of the de Bruijn–Newman constant. The current best lower bound [11,12] is $\Lambda > -2.7 \times 10^{-9}$. No progress has been made, as far as is known, on improving the upper bound 1/2 on Λ.

As an aside, we mention one other criterion equivalent to the Riemann hypothesis. Define, for each positive integer n, the series

$$\lambda_n = \sum_{\rho}\left[1 - \left(1 - \frac{1}{\rho}\right)^n\right]$$
$$= \begin{cases} -\frac{1}{2}\ln(4\pi) + \frac{\gamma_0}{2} + 1 = 0.0230957089\ldots & \text{if } n = 1, \\ \frac{\pi^2}{8} - \ln(4\pi) + \gamma_0 - \gamma_0^2 - 2\gamma_1 + 1 = 0.0923457352\ldots & \text{if } n = 2, \\ 0.2076389205\ldots & \text{if } n = 3, \end{cases}$$

where each sum is over all nontrivial zeros ρ of $\zeta(z)$ and γ_k is the k^{th} Stieltjes constant [2.21]. Li [13] proved that $\lambda_n \geq 0$ for all n if and only if the Riemann hypothesis is true. See the related constants σ_n in [2.21] and insightful discussion in [14,15].

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2.33 Hall–Montgomery Constant

A complex-valued function f defined on the positive integers is **completely multiplicative** if $f(mn) = f(m)f(n)$ for all m and n. Clearly such a function is determined by its values on $1 \cup \{\text{primes}\}$. Simple examples include $f(n) = 0$, $f(n) = 1$, and $f(n) = n^r$ for some fixed $r > 0$. A more complicated example, for a fixed odd prime p, is the Legendre symbol

$$f_p(n) = \left(\frac{n}{p}\right) = \begin{cases} 0 & \text{if } p|n, \\ 1 & \text{if } p \nmid n \text{ and } n \text{ is a quadratic residue modulo } p, \\ -1 & \text{otherwise;} \end{cases}$$

for example, $(6/19) = 1$ since $5^2 \equiv 6 \bmod 19$, but $(39/47) = -1$ since the congruence $x^2 \equiv 39 \bmod 47$ has no solution.

To illustrate, define $g(N)$ to be the cardinality of the set $\{1 \le n \le N : f_p(n) = 1\}$. It is known [1] that, from the integers $\{1, 2, \dots, p-1\}$, $(p-1)/2$ are quadratic residues and $(p-1)/2$ are nonresidues. Hence $g(N)/N \to 1/2$ as $N \to \infty$ through multiples of p. It is natural to ask about other possible limiting values of $g(N)/N$ for different choices of N. We will return to this issue shortly.

Consider the class F of all completely multiplicative functions whose values are constrained to the closed real interval $[-1, 1]$. What numbers arise as mean values of functions in F? More precisely, what is the set Γ of limit points of

$$\mu_N(f) = \frac{1}{N}\sum_{n=1}^{N} f(n)$$

as f varies over F and as $N \to \infty$? The set Γ is called the **multiplicative spectrum** of $[-1, 1]$, and an understanding of its structure has been reached only recently.

Granville & Soundararajan [2, 3], building upon independent work by Hall & Montgomery [4], proved that Γ is a closed interval and, in fact,

$$\Gamma = [\delta_1, 1] = [-0.6569990137\ldots, 1],$$

where $\delta_1 = 2\delta_0 - 1$,

$$\delta_0 = 1 - \frac{\pi^2}{6} - \ln(1+\sqrt{e})\ln\left(\frac{e}{1+\sqrt{e}}\right) + 2\operatorname{Li}_2\left(\frac{1}{1+\sqrt{e}}\right) = 0.1715004931\ldots,$$

and $\operatorname{Li}_2(x)$ is the dilogarithm function [1.6.8]. By analytic continuation, the expression for δ_0 can be simplified to $1 + \pi^2/6 + 2\operatorname{Li}_2(-\sqrt{e})$. This remarkable formula is only the tip of a larger theory: Much can also be said about $\Gamma(S)$, where S is an arbitrary subset of the unit disk D in the complex plane (rather than just the interval $[-1, 1]$). An important role in the proofs is played by differential and integral equations with delay [5.4].

Returning to the special case of $f_p(n)$, by the aforementioned theorem,

$$g(N) - (N - g(N)) \geq (\delta_1 + o(1))N;$$

that is, $g(N) \geq (\delta_0 + o(1))N$. In other words, the proportion of integers not exceeding N that are quadratic residues mod p is at least δ_0, independent of the choice of p:

$$\delta_0 \leq \liminf_{N\to\infty} \frac{g(N)}{N} \leq \frac{1}{2} \leq \limsup_{N\to\infty} \frac{g(N)}{N} \leq 1.$$

This proves a 1994 conjecture of Heath-Brown [4]. Additionally, the constant δ_0 is the best possible and, in fact, the limit inferior is equal to δ_0 for infinitely many primes p.

Likewise, the limit superior is equal to 1 for infinitely many primes p. Here is a proof. For fixed N, select a prime $p \equiv 1 \bmod M$, where M is $8\times$ the product of all odd primes $\leq N$. This is possible by Dirichlet's theorem on primes in arithmetic progressions. Thus $(2/p) = 1$ and, if q is an odd prime $\leq N$, then $(q/p) = (p/q) = (1/q) = 1$ by the law of quadratic reciprocity. Any $n \leq N$ is the product of primes $\leq N$; hence $(n/p) = 1$. Therefore, all $n \leq N$ are quadratic residues mod p. Infinitely many choices of p are possible, of course, so the result follows.

Let us examine a generalization. A complex-valued function f defined on the positive integers is **multiplicative** if $f(mn) = f(m)f(n)$ whenever m and n are relatively prime. (If f is completely multiplicative, then clearly f is multiplicative.) Assume that $-1 \leq f(n) \leq 1$ for all n (as before); then its mean value exists and is equal to [5–9]

$$\lim_{N\to\infty} \mu_N(f) = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^k}\right),$$

where the product is over all primes p. For example, if $f(n) = \varphi(n)/n$, where φ is the Euler totient function [2.7], then $\lim_{N\to\infty} \mu_N(f) = 6/\pi^2$. Note that, in this example, $f(p^k) = f(p)$ for any $k \geq 1$. Complicated conditions for the existence of $\lim_{N\to\infty} \mu_N(f)$ arise if we weaken our assumption to only $f(n) \in D$ for all n.

Here is an (unrelated) asymptotic result corresponding to a rather artificial example [10]. Define a multiplicative function f by the recursive formula

$$f(n) = \begin{cases} 1 & \text{if } n = 1, \\ pf(k) & \text{if } n = p^k \text{ for any prime } p; \end{cases}$$

then

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{n=1}^{N} f(n) = \frac{1}{2} \prod_{p} \left(1 - \frac{1}{p^2} + (p-1) \sum_{n=2}^{\infty} \frac{f(n)}{p^{2n}}\right)$$
$$= \frac{1}{2}(0.8351076361\ldots).$$

By way of contrast, the **completely additive** function $\Omega(n)$ introduced in [2.2] satisfies $\Omega(p^k) = k\Omega(p)$ for any prime p and has quite dissimilar asymptotics.

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3

Constants Associated with Analytic Inequalities

3.1 Shapiro–Drinfeld Constant

Consider the cyclic sum

$$f_n(x_1, x_2, \ldots, x_n) = \frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \cdots + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2},$$

where each x_j is nonnegative and each denominator is positive. Shapiro [1] asked if $f_n(x_1, x_2, \ldots, x_n) \geq n/2$ for all n. Lighthill [2] gave a counterexample for $n = 20$. Other counterexamples were subsequently discovered for $n = 14$ [3,4] and for $n = 25$ [5,6]. See [7–9] for a history of progress in understanding cyclic sums. We will only summarize: Shapiro's inequality is true for even $n \leq 12$ and odd $n \leq 23$ (using a computer-based proof [10]) and is false otherwise. This result has been analytically proved in the even case [11] but not yet for odd $13 \leq n \leq 23$.

It is interesting to examine the tools mathematicians used to unravel Shapiro's inequality early on. We look at just one. Let

$$f(n) = \inf_{x \geq 0} f_n(x_1, x_2, \ldots, x_n).$$

Rankin [12] studied the expression

$$\lambda = \lim_{n \to \infty} \frac{f(n)}{n} = \inf_{n \geq 1} \frac{f(n)}{n}$$

and proved that $\lambda < 0.49999993 < 1/2$. From this he deduced immediately that Shapiro's inequality is false for all sufficiently large n. Others took interest in the constant λ and attempted to calculate it to increasing accuracy [7]. Note that such efforts had no bearing on the truth of Shapiro's inequality for finite n. As is often the case, a tool for one person's use becomes the object of study for another.

Drinfeld [13] discovered a *geometric* interpretation of λ that also provides means for computing λ to arbitrary precision. Consider the two curves

$$y = \frac{1}{\exp(x)}, \quad y = \frac{2}{\exp(x) + \exp(x/2)}$$

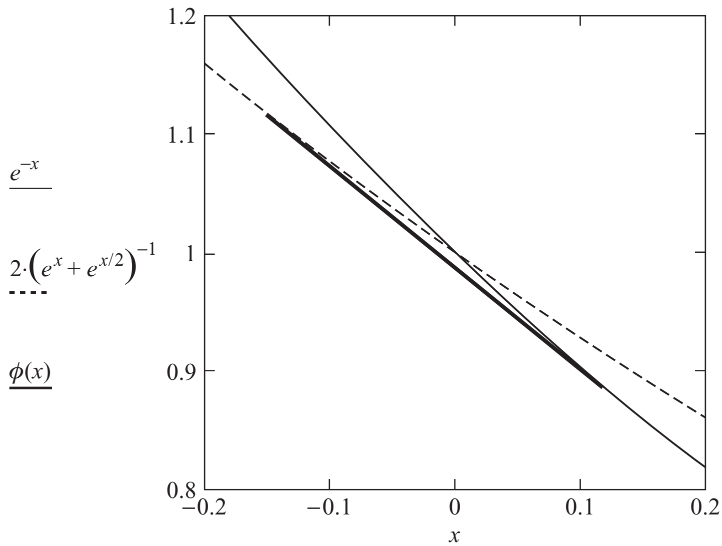

Figure 3.1. In a neighborhood of $x = 0$, the graph of $y = \varphi(x)$ is a joint tangent to the other two curves.

in the xy-plane. Let $\varphi(x)$ be the convex support of these two functions. That is, $\varphi(x)$ is the largest concave up function not exceeding the others (see Figure 3.1). Then

$$\lambda = \frac{\varphi(0)}{2} = 0.4945668172\ldots = \frac{1}{2}(0.9891336344\ldots).$$

Many modifications of Shapiro's sum have been studied [7]. We mention only two. Consider first the cyclic sum

$$g_n(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_3}{x_1 + x_2} + \frac{x_2 + x_4}{x_2 + x_3} + \cdots + \frac{x_{n-1} + x_1}{x_{n-1} + x_n} + \frac{x_n + x_2}{x_n + x_1}$$

under the same conditions for x_j. The inequality $g_n(x_1, x_2, \ldots, x_n) \geq n$ is, like Shapiro's inequality, false in general. Elbert [14] studied the expression

$$\mu = \lim_{n\to\infty} \frac{g(n)}{n}, \quad \text{where } g(n) = \inf_{x\geq 0} g_n(x_1, x_2, \ldots, x_n).$$

Using Drinfeld's method, he found that $\mu = \psi(0) = 0.9780124781\ldots$, where $y = \psi(x)$ is the convex support of the two functions

$$y = \frac{1 + \exp(x)}{2}, \quad y = \frac{1 + \exp(x)}{1 + \exp(x/2)}.$$

Recent computations of λ and μ include [15, 16]; generalizations are found in [17, 18]. Consider also the difference of cyclic sums $\Delta_n = f_n - h_n$, where f_n is as before and

$$h_n(x_1, x_2, \ldots, x_n) = \frac{x_1}{x_1 + x_2} + \frac{x_2}{x_2 + x_3} + \cdots + \frac{x_{n-1}}{x_{n-1} + x_n} + \frac{x_n}{x_n + x_1}.$$

Gauchman [19,20] obtained that

$$\inf_{n\geq 1}\inf_{x\geq 0}\frac{\Delta_n(x_1,x_2,\ldots,x_n)}{n}=-0.0219875218\ldots,$$

and the corresponding two curves are

$$y=\frac{1-\exp(x/2)}{\exp(x)+\exp(x/2)},\quad y=\frac{\exp(-x)-1}{2}.$$

We mention one other (non-cyclic) sum, due to Shallit [15,21]:

$$s_n(x_1,x_2,\ldots,x_n)=\sum_{i=1}^{n}x_i+\sum_{1\leq i\leq k\leq n}\prod_{j=i}^{k}\frac{1}{x_k},$$

which can be proved to satisfy

$$\lim_{n\to\infty}\inf_{x>0}s_n(x_1,x_2,\ldots,x_n)-3n=-1.3694514039\ldots$$

by numerical (non-geometric) means. Many variations of these sums f_n, Δ_n, and s_n suggest themselves.

3.1.1 Djokovic's Conjecture

Djokovic's conjecture, like Shapiro's, began as a *Monthly* problem and ultimately gave rise to an interesting constant. Assuming $x_1<x_2<\ldots<x_n$, define

$$P(x_1,x_2,\ldots,x_n)=\frac{1}{M}\int_{x_1}^{x_n}\left(\prod_{k=1}^{n}(t-x_k)\right)dt,\text{ where } M=\max_{x_1\leq t\leq x_n}\left|\prod_{k=1}^{n}(t-x_k)\right|.$$

Djokovic [22] conjectured that $(-1)^{n+1-k}(\partial P/\partial x_k)>0$ for each k. It is now known that this is not generally valid [23,24], even for $n=3$. Let $a_1=0.1824878875\ldots$ be the unique real zero of the cubic $12a^3-16a^2+8a-1$ and $a_2=1-a_1=0.8175121124\ldots$. Then Djokovic's inequality is true if $a_1(x_3-x_1)<x_2-x_1<a_2(x_3-x_1)$ and false otherwise. Similarly, for $n\geq 4$, the validity of the inequality depends on the distribution of the xs. If the xs are uniformly spaced, then for $n\leq 6$, the inequality is true, but for sufficiently large n, it is false.

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3.2 Carlson–Levin Constants

Let f be a nonnegative real-valued function on $[0, \infty)$. We wish to determine bounds for the integral of $f(x)$, given the existence of the integrals of $x^a f(x)^p$ and $x^b f(x)^q$. In the special case $a = 0$, $b = 2$, $p = q = 2$, Carlson [1–3] determined that

$$\int_0^\infty f(x)dx \leq \sqrt{\pi}\left(\int_0^\infty f(x)^2dx\right)^{1/4}\left(\int_0^\infty x^2 f(x)^2dx\right)^{1/4}$$

and that the constant $\sqrt{\pi}$ is the best possible. By "best possible" we mean that $\sqrt{\pi}$ is the smallest real coefficient for which the inequality is true. (If we attempt to sharpen the inequality by making the coefficient less than $\sqrt{\pi}$, then there is an admissible function f that will be a counterexample.)

For the general case, with $p > 1, q > 1, \lambda > 0$, and $\mu > 0$, Levin [2–4] discovered that

$$\int_0^\infty f(x)dx \le C\left(\int_0^\infty x^{p-1-\lambda} f(x)^p dx\right)^s \left(\int_0^\infty x^{q-1+\mu} f(x)^q dx\right)^t$$

and the best constant is

$$C = \frac{1}{(ps)^s}\frac{1}{(qt)^t}\left[\frac{\Gamma(\frac{s}{r})\Gamma(\frac{t}{r})}{(\lambda+\mu)\Gamma(\frac{s+t}{r})}\right]^r,$$

where

$$r = 1 - s - t,\ \ s = \frac{\mu}{p\mu + q\lambda},\ \ t = \frac{\lambda}{p\mu + q\lambda},$$

and $\Gamma(x)$ is Euler's gamma function [1.5.4]. It is interesting that such a closed-form expression for the best constant even *exists*: Many inequalities cannot be evaluated so completely. See extensions in [5–8].

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3.3 Landau–Kolmogorov Constants

There is a vast literature on inequalities involving the norms of a function f and its derivatives $f^{(k)}$. We state just enough here to define certain constants $C(n, k)$ in four separate cases. The constants correspond to the inequality (to be explained in each case)

$$||f^{(k)}|| \le C(n,k)||f||^{1-\frac{k}{n}}||f^{(n)}||^{\frac{k}{n}},\ \ 1 \le k < n,$$

which is henceforth called "inequality I."

3.3.1 $L_\infty(0, \infty)$ *Case*

Let $||f||$ here denote the supremum of $|f(x)|$, where the real-valued function f is defined on $(0, \infty)$. Landau [1] proved that if f is twice-differentiable and both f and

f'' are bounded, then

$$||f'|| \le 2||f||^{\frac{1}{2}}||f''||^{\frac{1}{2}}$$

and the constant 2 is the best possible. By this, we mean that replacing 2 by $2-\varepsilon$ for any positive number ε would necessarily lead to a counterexample f.

Schoenberg & Cavaretta [2, 3] extended this inequality to a setting where the n^{th} derivative of f exists and both f and $f^{(n)}$ are bounded. They determined best constants $C(n, k)$, $1 \le k < n$, for inequality I and characterized $C(n, k)$ in terms of norms of Euler splines. For example,

$$\begin{aligned} &C(3, 1) = \left(\tfrac{243}{8}\right)^{\frac{1}{3}} = 4.35622\ldots, && C(3, 2) = 24^{\frac{1}{3}} = 2.88449\ldots, && \\ &C(4, 1) = 4.288\ldots, && C(4, 2) = 5.750\ldots, && C(4, 3) = 3.708\ldots. \end{aligned}$$

An explicit formula for all n and k is not available [4, 5].

3.3.2 $L_\infty(-\infty, \infty)$ Case

Let $||f||$ here denote the supremum of $|f(x)|$, where the real-valued function f is defined on $(-\infty, \infty)$. Hadamard [6] proved that if f is twice-differentiable and both f and f'' are bounded, then

$$||f'|| \le \sqrt{2}||f||^{\frac{1}{2}}||f''||^{\frac{1}{2}}$$

and the constant $\sqrt{2}$ is the best possible.

Kolmogorov [7] determined best constants $C(n, k)$, $1 \le k < n$, for inequality I in terms of Favard constants [4.3]:

$$C(n, k) = a_{n-k}a_n^{-1+\frac{k}{n}}, \text{ where } a_n = \frac{4}{\pi}\sum_{j=0}^{\infty}\left[\frac{(-1)^j}{2j+1}\right]^{n+1}.$$

These formulas include special cases discovered by Shilov [8]:

$$\begin{aligned} &C(3, 1) = \left(\tfrac{9}{8}\right)^{\frac{1}{3}}, && C(3, 2) = 3^{\frac{1}{3}}, && \\ &C(4, 1) = \left(\tfrac{512}{375}\right)^{\frac{1}{4}}, && C(4, 2) = \left(\tfrac{6}{5}\right)^{\frac{1}{2}}, && C(4, 3) = \left(\tfrac{24}{5}\right)^{\frac{1}{4}}, \\ &C(5, 1) = \left(\tfrac{1953125}{1572864}\right)^{\frac{1}{5}}, && C(5, 2) = \left(\tfrac{125}{72}\right)^{\frac{1}{5}}. && \end{aligned}$$

Observe that this case, involving functions on the whole line, is easier than the previous case involving functions on the half line [4, 5].

3.3.3 $L_2(-\infty, \infty)$ Case

Given a real-valued function f defined on $(-\infty, \infty)$, define

$$||f|| = \left(\int_{-\infty}^{\infty} f(x)^2 dx\right)^{\frac{1}{2}}.$$

Hardy, Littlewood & Pólya [9] proved, assuming the n^{th} derivative of f exists and both f and $f^{(n)}$ are square-integrable, that $C(n,k)=1$ is the best possible for $1 \le k < n$.

3.3.4 $L_2(0,\infty)$ Case

As before, the half-line case is more difficult than the corresponding whole-line case. Given a real-valued function f defined on $(0,\infty)$, define

$$||f|| = \left(\int_0^\infty f(x)^2 dx \right)^{\frac{1}{2}}.$$

Hardy & Littlewood [9] proved, assuming f is twice-differentiable and both f and f'' are square-integrable, that

$$||f'|| \le \sqrt{2} ||f||^{\frac{1}{2}} ||f''||^{\frac{1}{2}}$$

and the constant $\sqrt{2}$ is the best possible.

Ljubic [10] and Kupcov [11] extended this inequality to I and gave a remarkable algorithm for finding best constants $C(n,k)$ in terms of zeros of certain explicit polynomials. For example [12, 13],

$$C(3,1) = C(3,2) = 3^{\frac{1}{2}} \left[2 \left(2^{\frac{1}{2}} - 1 \right) \right]^{-\frac{1}{3}} = 1.84420\ldots,$$

$$C(4,1) = C(4,3) = \left[\frac{1}{a} \left(3^{\frac{1}{4}} + 3^{-\frac{3}{4}} \right) \right]^{\frac{1}{2}} = 2.27432\ldots,$$

$$C(4,2) = \left(\frac{2}{b} \right)^{\frac{1}{2}} = 2.97963\ldots,$$

where a is the least positive root of $x^8 - 6x^4 - 8x^2 + 1 = 0$ and b is the least positive root of $x^4 - 2x^2 - 4x + 1 = 0$, and

$$C(5,1) = C(5,4) = 2.70247\ldots, \quad C(5,2) = C(5,3) = 4.37800\ldots.$$

In the special case $k = 1$, it can also be shown that

$$C(n,1) = \left[\frac{(n-1)^{\frac{1}{n}} + (n-1)^{-1+\frac{1}{n}}}{c} \right]^{\frac{1}{2}},$$

where c is the least positive root of

$$\int_0^c \int_0^\infty \frac{1}{(x^{2n} - yx^2 + 1)\sqrt{y}} dx\, dy = \frac{\pi^2}{2n}.$$

A similar formula for $k > 1$ is not known. A consequence of Ljubic and Kupcov's work is that all $C(n,k)$ for this case must be algebraic numbers. This assertion appears to be true for the $L_\infty(0,\infty)$ case as well.

Among the topics we have omitted are:

- best constants associated with the $L_p(0, \infty)$ and $L_p(-\infty, \infty)$ norms, where $p \neq 2$ and $p \neq \infty$, or the same over a finite interval [14, 15];
- best constants in the discrete case, specifically, those associated with one-way and two-way infinite real sequences with the l_p norm and where derivatives are replaced by differences [16, 17].

It turns out that $p = 1, 2, \infty$ are the only cases for which best constants have exact formulas. For all other values of p, numerical approximation is evidently required.

Here is an unsolved problem, which concerns a slight variant of $L_2(0, \infty)$. Assuming f to be twice-differentiable and both f and f'' to be square-integrable with respect to a weighting function $w(x) = x$, Everitt & Guinand [5, 18] proved that

$$\left(\int_0^\infty x f'(x)^2 dx\right)^2 \leq K \cdot \int_0^\infty x f(x)^2 dx \cdot \int_0^\infty x f''(x)^2 dx,$$

where the best possible constant satisfies $2.35070 < K < 2.35075$. An exact expression for K remains undiscovered.

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3.4 Hilbert's Constants

Let $p > 1$ and $q = p/(p-1)$. If $\{a_n\}$, $\{b_n\}$ are nonnegative sequences and $f(x)$, $g(x)$ are nonnegative integrable functions, then Hilbert's inequality [1–3] for series is

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\csc\left(\frac{\pi}{p}\right)\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}},$$

unless all a_n are zero or all b_n are zero, and Hilbert's inequality for integrals is

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dx\,dy < \pi\csc\left(\frac{\pi}{p}\right)\left(\int_0^{\infty}f(x)^p dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}g(y)^q dy\right)^{\frac{1}{q}},$$

unless f is identically zero or g is identically zero. The constant $\pi\csc(\pi/p)$ is the best possible in the sense that, if one replaces it by a smaller constant, then there exist counterexamples.

We are concerned with the following two-parameter extension of Hilbert's inequality. Let $p > 1$, $q > 1$ and

$$\tfrac{1}{p}+\tfrac{1}{q}\geq 1, \quad \text{so that } 0 < \lambda = 2-\tfrac{1}{p}-\tfrac{1}{q}\leq 1.$$

Levin [4], Steckin [5], and Bonsall [6] showed that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} \leq \left[\pi\csc\left(\frac{\pi(q-1)}{\lambda q}\right)\right]^{\lambda}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}},$$

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{(x+y)^{\lambda}}dx\,dy \leq \left[\pi\csc\left(\frac{\pi(q-1)}{\lambda q}\right)\right]^{\lambda}\left(\int_0^{\infty}f(x)^p dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}g(y)^q dy\right)^{\frac{1}{q}},$$

but it is not known whether the indicated constant is the best possible.

There appears to be some confusion on the last point. Boas [7] indicated in 1949 that Steckin had proved the constant is the best possible in the discrete case; in 1950 Boas corrected himself and wrote that the bound is *not* exact. Mitrinovic, Pecaric & Fink [1] wrote that Steckin had established the constant to be the best possible. However, both Levin & Steckin [8] and Walker [9] wrote that the problem is still open.

As far as is known, no one has calculated the best constant even for the case $\lambda = 1/2$ and $p = q = 4/3$. Is a computation possible analogous to that discussed with the Copson–de Bruijn constant [3.5]?

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3.5 Copson–de Bruijn Constant

The interplay between series and integrals is sometimes very natural, but sometimes not. Let $\{a_n\}$ be a nonnegative sequence and $f(x)$ a nonnegative integrable function. Define

$$A_n = \sum_{k=1}^{n} a_k, \qquad B_n = \sum_{k=n}^{\infty} a_k,$$

$$F(x) = \int_0^x f(t)dt, \quad G(x) = \int_x^\infty f(t)dt.$$

Assume throughout that all infinite series and improper integrals under consideration are convergent and finite. We will examine two examples, the first for which all is as expected and the second for which all is not. Given $p > 1$, Hardy's inequality [1] is of the form

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p,$$

which always holds unless all a_n are zero. The corresponding theorem for integrals is

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx,$$

which always holds unless f is identically zero. The constant $(p/(p-1))^p$ is the best possible in the sense that, if one replaces it by a smaller constant, then there exist $\{a_n\}$ and $f(x)$ that are counterexamples.

Given $0 < p < 1$, one of Copson's integral inequalities [2,3] is of the form

$$\int_0^\infty \left(\frac{G(x)}{x}\right)^p dx > \left(\frac{p}{1-p}\right)^p \int_0^\infty f(x)^p dx,$$

unless f is identically zero. The corresponding theorem for series, curiously, is

$$\left(1+\frac{1}{p-1}\right)\left(\frac{B_1}{1}\right)^p + \sum_{n=2}^\infty \left(\frac{B_n}{n}\right)^p > \left(\frac{p}{1-p}\right)^p \sum_{n=1}^\infty a_n^p,$$

unless all a_n are zero. The constant is the best possible, as found by Elliott. What is surprising is the correction term (or "gloss" as described in [2]) required to achieve the correspondence.

If one removes the correction term, the following inequality emerges [2,4]:

$$\sum_{n=1}^\infty \left(\frac{B_n}{n}\right)^p > p^p \sum_{n=1}^\infty a_n^p,$$

unless all a_n are zero. The constant p^p is, however, *not* the best possible. Hence by removing the "gloss" we have wrecked the precision of the inequality.

Levin & Steckin [5] proved, for $0 < p < 1/3$, that the best constant is $(p/(1-p))^p$, but they could not do likewise for $p > 1/3$.

Consider the special case when $p = 1/2$:

$$\sum_{n=1}^\infty \left(\frac{a_n + a_{n+1} + a_{n+2} + \cdots}{n}\right)^{\frac{1}{2}} \geq C \sum_{n=1}^\infty a_n^{\frac{1}{2}}$$

and rearrange the inequality by replacing a_n by a_n^2:

$$\sum_{n=1}^\infty a_n \leq c \sum_{n=1}^\infty \left(\frac{a_n^2 + a_{n+1}^2 + a_{n+2}^2 + \cdots}{n}\right)^{\frac{1}{2}}.$$

Steckin [6] proved that $c \leq 2/\sqrt{3}$ and Boas & de Bruijn [7] improved this to $1.08 < c < 17/15$. To estimate c more accurately, de Bruijn [8] defined a sequence of complex numbers via the recurrence

$$u_1 = x, \quad u_n = n^{-\frac{1}{2}} x + \left(u_{n-1}^2 - 1\right)^{\frac{1}{2}} \quad \text{for } n \geq 2.$$

It can be proved that $c = 1.1064957714\ldots$ is the smallest real number for which $x \geq c$ implies $u_n \geq 1$ (in particular, $\mathrm{Im}(u_n) = 0$) for all $n \geq 1$. Further, if $x \geq c$, then

$$\lim_{n\to\infty} n^{-\frac{1}{2}} u_n = \begin{cases} x + \left(x^2 - 1\right)^{\frac{1}{2}} & \text{if } x > c, \\ c - \left(c^2 - 1\right)^{\frac{1}{2}} & \text{if } x = c. \end{cases}$$

Whether de Bruijn's procedure can be applied for other values of $p > 1/3$ is open.

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3.6 Sobolev Isoperimetric Constants

The area A enclosed by a simple closed curve C in the plane with perimeter P satisfies $4\pi A \leq P^2$, and equality holds if and only if C is a circle. We first generalize this **isoperimetric property** from two to n dimensions and then relate it to a certain **Sobolev inequality**.

Let Ω be the closure of a bounded, open, connected set in Euclidean space $\mathbb{R}^n$ with piecewise continuously differentiable boundary and surface area S. Let f be a continuously differentiable function defined on $\mathbb{R}^n$ with compact support, meaning that $f = 0$ identically outside of a ball, and let ∇f denote the gradient of f. Also define $\omega_n = \pi^{n/2}\Gamma(n/2+1)^{-1}$, the volume enclosed by the unit sphere in $\mathbb{R}^n$. The following two statements are equivalent [1–4]:

- The volume V of Ω satisfies $n^n \omega_n V^{n-1} \leq S^n$ with equality if and only if Ω is a ball.
- The $L_{n/(n-1)}$ norm of f is related to the L_1 norm of its gradient via

$$\left(\int\limits_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq \frac{1}{n\omega_n^{1/n}} \int\limits_{\mathbb{R}^n} |\nabla f(x)| dx$$

and the constant $n^{-1}\omega_n^{-1/n}$ is sharp.

The former is geometric in nature, whereas the latter falls within functional analysis. As a consequence, there is an extended interpretation of the phrase "isoperimetric problem" to encompass Sobolev inequalities and hence eigenvalues of differential equations with boundary conditions. We cannot even hope to summarize such a massive field [5–7] but attempt only to introduce a few constants.

Several authors [8, 9] have commented that Sobolev inequalities act as uncertainty principles: The size of the gradient of a function f is bounded from below in terms of the size of f. Note that the constants ω_n are interesting in themselves; for example, $\lim_{n\to\infty} n^{1/2}\omega_n^{1/n} = \sqrt{2\pi e} = 4.1327313541\ldots$ by Stirling's formula. We turn to four sample exercises from physics.

3.6.1 String Inequality

If smooth functions f are constrained to satisfy $f(0) = f(1) = 0$, then

$$\int_0^1 f(x)^2 dx \le \frac{1}{\pi^2}\int_0^1 \left(\frac{df}{dx}\right)^2 dx$$

and the constant $1/\pi^2 = 0.1013211836\ldots$ is the best possible [10]. This corresponds, via the calculus of variations, to the fact that the smallest eigenvalue of the ordinary differential equation (ODE)

$$\frac{d^2g}{dx^2} + \lambda g(x) = 0, \quad g(0) = g(\pi) = 0,$$

is $\lambda = 1$. This ODE, in turn, arises from the study of a vibrating, homogeneous string that is pulled taut on the x-axis and is fastened at the endpoints [11, 12]. The value $\sqrt{\lambda} = 1$ has the physical interpretation as the principal frequency of the sound one hears when the string is plucked.

A generalization of this is due to Talenti [3]:

$$\left(\int_0^1 |f(x)|^q dx\right)^{\frac{1}{q}} \le \frac{q}{2}\left(1+\frac{r}{q}\right)^{\frac{1}{p}}\left(1+\frac{q}{r}\right)^{-\frac{1}{q}} \frac{\Gamma(\frac{1}{q}+\frac{1}{r})}{\Gamma(\frac{1}{q})\Gamma(\frac{1}{r})}\left(\int_0^1 \left|\frac{df}{dx}\right|^p dx\right)^{\frac{1}{p}},$$

where $f(0) = f(1) = 0$, $p > 1$, $q \ge 1$, and $r = p/(p-1)$. The indicated constant is sharp.

3.6.2 Rod Inequality

A second-order version of the "string inequality" follows. If suitably smooth f are constrained to satisfy

$$f(0) = \frac{df}{dx}(0) = f(1) = \frac{df}{dx}(1) = 0,$$

then

$$\int_0^1 f(x)^2 dx \le \mu \int_0^1 \left(\frac{d^2 f}{dx^2}\right)^2 dx,$$

where $\mu = 1/\theta^4 = 0.0019977469\ldots$ and $\theta = 4.7300407448\ldots$ is the smallest positive root of the equation

$$\cos(\theta)\cosh(\theta) = 1.$$

Moreover, the constant μ is the best possible [12–14]. This corresponds to the fact that the smallest eigenvalue of the ODE

$$\frac{d^4 g}{dx^4} - \lambda g(x) = 0, \quad g(0) = \frac{dg}{dx}(0) = g(\pi) = \frac{dg}{dx}(\pi) = 0,$$

is $\lambda = \theta^4/\pi^4 = 5.1387801326\ldots$. This ODE, in turn, arises from the study of a vibrating, homogeneous rod or bar that is clamped at the endpoints.

3.6.3 Membrane Inequality

A two-dimensional version of the "string inequality" follows. If smooth f are constrained to vanish on the boundary C of the unit disk D, then

$$\int\limits_D f^2 dx dy \le \mu \int\limits_D \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] dx\, dy,$$

where $\mu = 1/\theta^2 = 0.1729150690\ldots$ and $\theta = 2.4048255576\ldots$ is the smallest positive zero of the zeroth Bessel function

$$J_0(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2} \left(\frac{z}{2} \right)^{2j}.$$

Moreover, the constant μ is the best possible [11, 12, 15]. This corresponds to the fact that the smallest eigenvalue of the ODE

$$r^2 \frac{d^2 g}{dr^2} + r \frac{dg}{dr} + \lambda r^2 g(r) = 0, \quad g(0) = 1, \quad g(1) = 0,$$

is $\lambda = \theta^2 = 5.7831859629\ldots$. This ODE, in turn, arises from the study of a vibrating, homogeneous membrane that is uniformly stretched across D and fastened at the boundary C. The value $\sqrt{\lambda} = \theta$ is the principal frequency of the sound one hears when a kettledrum is struck.

Consider the Laplace partial differential equation (PDE)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \Lambda u = 0$$

for a vibrating membrane on an arbitrary region D of fixed area A with $u = 0$ on the boundary C. Rayleigh [16, 17] conjectured in 1877 that the first eigenvalue Λ is least when C is a circle. This conjecture was proved independently in 1923 by Faber [18] and Krahn [19]: $\Lambda \ge (\pi/A)\theta^2$ with equality if and only if C is a circle. Interestingly, the same is *not* true for the second eigenvalue: The critical boundary is not a circle, but a figure-eight [20–22].

3.6.4 Plate Inequality

A two-dimensional, second-order version of the "string inequality" follows. Assume that suitably smooth f and its outward normal derivative $\partial f/\partial n$ are both constrained to vanish on the boundary C of the unit disk D. Then

$$\int_D f^2 dxdy \le \mu \int_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)^2 dx\, dy$$

where $\mu = 1/\theta^4 = 0.0095819302\ldots$, $\theta = 3.1962206165\ldots$ is the smallest positive root of the equation

$$J_0(\theta)I_1(\theta) + I_0(\theta)J_1(\theta) = 0,$$

and $I_0(z)$ is the zeroth modified Bessel function

$$I_0(z) = \sum_{j=0}^{\infty} \frac{1}{(j!)^2}\left(\frac{z}{2}\right)^{2j}, \quad I_1(z) = \frac{dI_0}{dz}, \quad J_1(z) = -\frac{dJ_0}{dz}.$$

Moreover, the constant μ is the best possible [12, 14–16, 23]. This is associated with the study of a vibrating, homogeneous plate clamped at the boundary C.

As with the membrane case, we state a related isoperimetric inequality. Consider the PDE

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \Lambda u = 0$$

for a vibrating plate on an arbitrary region of fixed area A with $u = \partial u/\partial n = 0$ on the boundary. Rayleigh [16] conjectured that $\Lambda \ge (\pi^2/A^2)\theta^4$ and Szegö [24–26] proved this to be true under a special hypothesis. The general conjecture was proved only recently [27, 28].

3.6.5 Other Variations

Let $||f||$ denote the supremum of $|f(x, y)|$, where the function f is defined on all of $\mathbb{R}^2$ and is twice continuously differentiable. Then $||f||$ is related to the integral of the sum of squares of all partial derivatives of f via

$$||f|| \le \alpha_{2,2}\left[\int_{\mathbb{R}^2} \left(f^2 + f_x^2 + f_y^2 + f_{xx}^2 + f_{xy}^2 + f_{yy}^2\right) dxdy\right]^{\frac{1}{2}},$$

where the best constant $\alpha_{2,2} = 0.3187590609\ldots$ is given by [29]

$$\alpha_{2,2} = \left(\frac{1}{\pi^2}\int_0^{\infty}\int_0^{\infty} \frac{dxdy}{1 + x^2 + y^2 + x^4 + x^2y^2 + y^4}\right)^{\frac{1}{2}} = \left(\frac{1}{2\pi}\int_1^{\infty} \frac{dt}{\sqrt{t^2 + 2\sqrt{t^2+3}}}\right)^{\frac{1}{2}}.$$

Such formulation is naturally extended to m-times continuously differentiable functions f defined on all of $\mathbb{R}^n$, with corresponding constant $\alpha_{m,n}$. For example,

$$\alpha_{1,1} = \left(\frac{1}{\pi} \int\limits_0^\infty \frac{dx}{1+x^2} \right)^{\frac{1}{2}} = \frac{\sqrt{2}}{2}, \quad \alpha_{2,3} = 0.231522\ldots, \quad \alpha_{3,3} = 0.142892\ldots.$$

If instead f is defined only on the unit cube in $\mathbb{R}^n$, then among the associated constants $\tilde{\alpha}_{m,n}$, we have [30–32]

$$\tilde{\alpha}_{1,1} = \tanh(1)^{-\frac{1}{2}} = 1.1458775176\ldots, \quad \tilde{\alpha}_{2,2} = 1.24796\ldots.$$

In fact, for arbitrary $m \geq 1$,

$$\alpha_{m,1} = \left[\frac{1}{m+1} \frac{\cos(\frac{\pi}{2m+2})}{\sin(\frac{3\pi}{2m+2})} \right]^{\frac{1}{2}}, \quad \tilde{\alpha}_{m,1} = \left[\frac{2}{m+1} \sum_{k=1}^{m} \frac{\sin(\frac{\pi k}{m+1})^3}{\tanh(\sin(\frac{\pi k}{m+1}))} \right]^{\frac{1}{2}}.$$

These inequalities are useful in the study of the finite element method in numerical analysis.

A related idea is Friedrichs' inequality [33], which involves continuously differentiable functions f on the closed interval $[0, 1] \subseteq \mathbb{R}$:

$$\left[\int\limits_0^1 \left(f(x)^2 + f'(x)^2 \right) dx \right]^{\frac{1}{2}} \leq \beta \left[f(0)^2 + f(1)^2 + \int\limits_0^1 f'(x)^2 dx \right]^{\frac{1}{2}}.$$

The best constant $\beta = 1.0786902162\ldots$ satisfies $\beta = \sqrt{1+\theta^{-2}}$, where $\theta = 2.4725480752\ldots$ is the unique solution of the equation

$$\cos(\theta) - \theta(\theta^2+1)^{-1}\sin(\theta) = -1, \quad 0 < \theta < \pi.$$

Many more examples are possible [34–45].

Let us return to geometry for one more problem. Consider a simple closed curve C in $\mathbb{R}^3$ with perimeter P. Let V denote the volume of its convex hull, that is, the intersection of all convex sets in $\mathbb{R}^3$ containing C. Then $V \leq \gamma_3 P^3$ and the best constant is $\gamma_3 = 0.0031816877\ldots$ (obtained in [46,47] via numerical solution of a system of ODEs). No closed-form expression for γ_3 is known. If the setting is changed from $\mathbb{R}^3$ to $\mathbb{R}^n$, where the integer n is even, then curiously the best constant [48] is exactly given by $\gamma_n = [(\pi n)^{n/2} n!(n/2)!]^{-1}$. The case for odd $n \geq 5$ remains open.

A deeper connection between Sobolev inequalities and isoperimetric properties within Riemannian manifolds ($\mathbb{R}^n$ being the simplest example) is beyond the scope of this book.

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3.7 Korn Constants

Let $u(x)$ be a smooth vector field defined on the closure of a bounded, open, connected set Ω in n-dimensional space. Then $\nabla u(x)$ is the $n \times n$ matrix made up of partial derivatives of $u(x)$. By the norm $|M|$ of a matrix M, we mean the Euclidean norm of M, that is, the square root of the sum of squares of all entries. Let also M^{T} denote the transpose of M.

Consider the so-called **second case of Korn's inequality** [1–3]

$$\int_\Omega |\nabla u(x)|^2 dx \le K \int_\Omega \left| \frac{\nabla u(x) + \nabla u(x)^{\mathrm{T}}}{2} \right|^2 dx$$

with the side condition

$$\int_{\Omega} \left(\nabla u(x) - \nabla u(x)^{\mathrm{T}}\right) dx = 0.$$

The best constants $K(\Omega)$ for various domains Ω are important in linear elasticity theory and in incompressible fluid dynamics. If B_n is an n-dimensional ball [4, 5], then $K(B_2) = 4$ and $K(B_3) = 56/13$. The corresponding values for $n \geq 4$ are not known. Let P_m denote a two-dimensional m-sided regular polygonal region. For a square P_4, it can be proved that [2]

$$5 \leq K(P_4) \leq 4(2 + \sqrt{2}),$$

and Horgan & Payne [6] conjectured that $K(P_4) = 7$. For an equilateral triangle P_3, we have

$$6 \leq K(P_3) \leq 8(2 + \sqrt{3})$$

using Laplacian eigenvalue formulas in [7–9]. For arbitrary m, we have the upper bound [2]

$$K(P_m) \leq \frac{4}{1 - \sin(\pi/m)},$$

and a lower bound for $K(P_6)$ is possible using eigenvalue numerical estimates in [9]. Korn constants for ellipses and limacons are given in [2, 10]; for circular rings and spherical shells, see [11, 12].

Here is a related problem (for $n = 2$ only). Let $z = x + iy$, where i is the imaginary unit, and let $f(x, y)$ and $g(x, y)$ denote the real and imaginary parts of an analytic function $w(z)$. In other words, $f(x, y)$ and $g(x, y)$ are **harmonic conjugates**. Consider Friedrichs' inequality [6, 10, 13–15]

$$\int_{\Omega} f(x, y)^2 dx dy \leq \Gamma \int_{\Omega} g(x, y)^2 dx dy$$

with the side condition

$$\int_{\Omega} f(x, y) dx dy = 0.$$

The best constants Γ for various simply-connected domains Ω are related to the Korn constants K by $K = 2(1 + \Gamma)$, assuming Ω has a continuously differentiable boundary. In the event Ω is a square region, Horgan & Payne [6] conjectured that the optimizing functions are

$$f(x, y) = 2xy, \quad g(x, y) = y^2 - x^2$$

and hence $\Gamma = 5/2$. This would lead immediately to $K = 7$ if it were not for the smoothness requirement.

Horgan's survey [2] is a valuable starting point for research. Related topics appear in [16, 17].

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3.8 Whitney–Mikhlin Extension Constants

Let $B_{n,r}$ denote the n-dimensional closed ball of radius r centered at the origin. Assume throughout that $r > 1$ is fixed. A function F defined on all of n-dimensional space is called an r**-extension** of a given function f defined on $B_{n,1}$ if $F(x) = f(x)$ for all $|x| \le 1$ and $F(x) = 0$ for all $|x| \ge r$.

We are interested in procedures for building F, given f, and we want to do this in such a way as to "minimize waste." Here are two ways (among many) to interpret the phrase "minimize waste":

- To every continuous f, construct a continuous r-extension F such that

$$\max_{x \in B_{n,r}} |F(x)| \le c \cdot \max_{x \in B_{n,1}} |f(x)|,$$

  where c is a constant (independent of f) and is the smallest possible.

- To every continuously differentiable f, construct a continuously differentiable r-extension F such that

$$\left[\int_{B_{n,r}}\left(F(x)^2+\sum_{k=1}^{n}\left(\frac{\partial F}{\partial x_k}\right)^2\right)dx\right]^{\frac{1}{2}} \le \chi\cdot\left[\int_{B_{n,1}}\left(f(x)^2+\sum_{k=1}^{n}\left(\frac{\partial f}{\partial x_k}\right)^2\right)dx\right]^{\frac{1}{2}},$$

where (again) χ is a constant and is the smallest possible.

Another way of phrasing this is as follows: Given two Banach spaces of functions defined on $B_{n,1}$ and $B_{n,r}$, determine the r-extension operator from one to the other of minimal norm. In the first case, the Banach space norm is the L_∞ or supremum norm; in the second, it is the Sobolev W_2^1 integral norm, which penalizes misbehaved derivatives as well.

Whitney [1] proved that $c = 1$ in the first case by a partition-of-unity argument. The calculus of variations provides that [2, 3]

$$\chi = \sqrt{1+\coth(1)\coth(r-1)}$$

when $n = 1$ for the second case (note that this depends on r).

Mikhlin [4–6] determined best constants $\chi = \chi(n,r)$ when $n \ge 2$ for the second case. Earlier relevant work included Hestenes [7], Calderón [8], and Stein [9]. Define, for convenience, $\nu = (n-2)/2$ and modified Bessel functions

$$I_\nu(r) = \left(\frac{r}{2}\right)^{\nu}\sum_{j=0}^{\infty}\frac{1}{j!\,\Gamma(\nu+j+1)}\left(\frac{r}{2}\right)^{2j}, \quad K_\nu(r) = \frac{\pi}{2}\frac{I_{-\nu}(r)-I_\nu(r)}{\sin(\nu\pi)}.$$

See [4] for a table of numerical estimates of $\chi(n,r)$, based on algebraic formulas involving $I_\nu(r)$ and $K_\nu(r)$. Our interest is solely in the asymptotic values

$$\chi_n = \lim_{r\to\infty}\chi(n,r) = \sqrt{1+\frac{I_\nu(1)}{I_{\nu+1}(1)}\frac{K_{\nu+1}(1)}{K_\nu(1)}},$$

and clearly

$$\chi_1 = \sqrt{\tfrac{2e^2}{e^2-1}},\ \chi_3 = e,\ \chi_5 = \sqrt{\tfrac{e^2}{e^2-7}},\ \chi_7 = \sqrt{\tfrac{2}{7}}\sqrt{\tfrac{e^2}{37-5e^2}},\ \chi_9 = \sqrt{\tfrac{1}{37}}\sqrt{\tfrac{e^2}{18e^2-133}}$$

for odd dimensions n, an unexpected occurrence of the natural logarithmic base e. Similar formulation, in terms not of e but of $I_0(1)$, $I_1(1)$, $K_0(1)$, and $K_1(1)$, can be written for even dimensions n.

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3.9 Zolotarev–Schur Constant

Let n be a positive integer. Define S_n to be the set of n^{th} degree polynomials $p(x)$ with real coefficients satisfying $|p(x)| \le 1$ for all $-1 \le x \le 1$.

Markov [1,2] proved that, if $p \in S_n$, then $|p'(x)| \le n^2$ for all $-1 \le x \le 1$, where p' is the derivative of p. Equality occurs if and only if $x = \pm 1$ and $p(x) = \pm T_n(x)$, the n^{th} Chebyshev polynomial [4.9].

Let $-1 \le \xi \le 1$ be a real number and $n \ge 3$ be an integer. Define $S_{n,\xi}$ to be the subset of S_n characterized by the additional restriction $p''(\xi) = 0$. Note that $T_n \notin S_{n,\pm 1}$; hence maximizing the quantity $|p'(\pm 1)|$ over the set $S_{n,\pm 1}$ leads to quite different solutions than before.

Schur [3,4] proved that, if $p \in S_{n,\xi}$, then $|p'(\xi)| < \frac{1}{2}n^2$. Further, letting

$$s_n = \sup_{-1 \le \xi \le 1} \sup_{p \in S_{n,\xi}} \frac{|p'(\xi)|}{n^2} \quad \text{and} \quad \sigma = \limsup_{n \to \infty} s_n$$

he obtained the bounds $0.217 \le \sigma \le 0.465$.

It turns out that identifying the constant σ is an outcome of work performed by Zolotarev [5–12]. Just as $T_n(x)$ arise as extremal polynomials in Markov's theorem, a new set of polynomials $Z_n(x)$ are required to fully understand Schur's theorem. Zolotarev determined in 1877 a number of exact solutions to various polynomial approximation problems using elliptic functions, in research that was far ahead of its time.

Erdös & Szegö [4] established the connection between Schur's theorem and Zolotarev's polynomials. They proved that

$$\sigma = \frac{1}{c^2}\left(1 - \frac{E(c)}{K(c)}\right)^2 = 0.3110788667\ldots,$$

where $K(x)$ and $E(x)$ are complete elliptic integrals of the first and second kind [1.4.6], and c is the unique solution of the equation

$$[K(c) - E(c)]^3 + (1 - c^2)K(c) - (1 + c^2)E(c) = 0, \quad 0 < c < 1.$$

The extremum $s_n n^2$ is attained for $n > 3$ at $\xi = 1$ and $p(x) = \pm Z_n(x)$, or at $\xi = -1$ and $p(x) = \pm Z_n(-x)$. To discuss Zolotarev's polynomials and the associated differential equation would take us too far afield, so we stop here.

3.9.1 Sewell's Problem on an Ellipse

Here is an extension of Markov's problem. Let $p(z)$ be a complex polynomial of degree n in $z = x + iy$ and assume that $|p(z)| \le 1$ on the elliptical region E given by $x^2 + (y/g)^2 \le 1$, where $0 < g \le 1$. What is the smallest constant $K(g)$, independent of n, for which $|p'(z)| \le n \cdot K(g)$ over all of E?

It is known [13–16] that $K(1) = 1$ and $K(g) \le 1/g$. From the quadratic example $p(z) = (8z^2 - 3)/5$, van Delden [17] deduced that $K(1/2) \ge 8/5$. He further utilized the generalized Chebyshev polynomial sequence [4.9]

$$T_n(z, g) = \cos(n \arccos(\tilde{z})) = \frac{(\tilde{z} + \sqrt{\tilde{z}^2 - 1})^n + (\tilde{z} - \sqrt{\tilde{z}^2 - 1})^n}{2}, \quad \tilde{z} = \frac{z}{\sqrt{1 - g^2}},$$

to suggest that $K(g)$ is equal to its upper bound $1/g$.

Analogous constants can be defined over other boundary curves as well [18–20]. See also [21–25].

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3.10 Kneser–Mahler Polynomial Constants

Given a polynomial, what can be said about the size of its factors? Let $||p||$ denote the supremum norm of an n^{th} degree polynomial $p(x)$ with complex coefficients, defined on the closed real interval $[-1, 1]$. Suppose $p(x) = q(x)r(x)$, where $q(x)$ is of degree k and $r(x)$ is of degree $n - k$. Then Kneser [1], building upon the work of Aumann [2], proved that [3–5]

$$||q|| \cdot ||r|| \le \frac{1}{2} C_{n,k} C_{n,n-k} \cdot ||p||,$$

where

$$C_{n,k} = 2^k \prod_{j=1}^{k} \left[1 + \cos\left(\frac{(2j-1)\pi}{2n} \right) \right].$$

Furthermore, for any n and $k \le n$, the constant is the best possible. Observe that here, the right-hand "knows" the degree k of $q(x)$.

Suppose information on the degree k of $q(x)$ is not available. Borwein [4, 5] observed as a corollary of Kneser's result that $k = \lfloor n/2 \rfloor$ maximizes $C_{n,k}$ and thus

$$||q|| \cdot ||r|| \le \delta^{2n} ||p||$$

asymptotically as $n \to \infty$, where

$$\delta = \exp\left(\frac{2G}{\pi}\right) = 1.7916228120\ldots$$

is the dimer constant [5.23] and G is Catalan's constant [1.7]. Moreover, the inequality is sharp, meaning

$$\limsup_{n\to\infty}\left(\frac{||q||\cdot||r||}{||p||}\right)^{\frac{1}{n}} = \delta^2 = 3.2099123007\ldots,$$

where the supremum is over all polynomials p of degree n and factors q and r.

The remarkable occurrence of δ in this expression was anticipated several years earlier by Boyd [6], working over a different domain. Henceforth, define $||p||$ to be the supremum norm of $p(z)$ defined on the unit disk D in the complex plane. Boyd proved, if $p(z) = q(z)r(z)$, then asymptotically

$$||q||\cdot||r|| \leq \delta^n ||p||$$

and this is sharp. It is interesting that δ^2 occurs for $[-1, 1]$ but δ occurs for D.

Suppose we remove $||r||$ from this inequality. To avoid frivolous multiplication of q by a large constant, we assume that p and q and hence r are monic. Boyd [6] proved here that asymptotically

$$||q|| \leq \beta^n ||p||$$

and this is sharp, where

$$\beta = \exp\left(\frac{1}{\pi} I(\frac{2}{3}\pi)\right) = 1.3813564445\ldots$$

and

$$I(\theta) = \int_0^\theta \ln\left(2\cos(\frac{x}{2})\right) dx.$$

The integral is simply $\mathrm{Cl}(\pi - \theta)$, where $\mathrm{Cl}(\theta)$ is Clausen's integral [7,8]. We note a similar representation [6,9]

$$\delta = \exp\left(\frac{2}{\pi} I(\frac{1}{2}\pi)\right)$$

and also two series [10,11]

$$\ln(\delta) = \frac{2}{\pi}\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + - \cdots\right) = 0.5831218080\ldots,$$

$$\ln(\beta) = \frac{3\sqrt{3}}{4\pi}\left(1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} + - \cdots\right) = 0.3230659472\ldots.$$

The constant β has occurred in several places in the literature, the first in Mahler [12] with regard to an apparently unrelated polynomial inequality. In [13, 14], it appears in the asymptotics of what are called binomial circulant determinants. In [15], $\ln(\beta)$ is the entropy of a simple two-dimensional shift and in [16], $\pi\ln(\beta) = 1.0149416064\ldots$ is the largest possible volume of a hyperbolic tetrahedron. See also [5.23] and [8.9]. An amusing recent account of $\pi\ln(\beta)$ is found in [17], where it is called **Gieseking's constant**.

Likewise, δ has occurred throughout the literature. We already mentioned the connection to the dimer packing of a two-dimensional integer lattice. In [18, 19], $\ln(\delta)$ appears with regard to Schmidt's Gaussian integer continued fractions. Other ways δ plays a role in mathematical physics include those described in [20, 21].

Boyd [9] extended this discussion from two factors to m factors. If $p(z) = p_1(z)p_2(z)\cdots p_m(z)$, with m fixed, then asymptotically

$$||p_1||\cdot||p_2||\cdots||p_m|| \le c_m^n \cdot ||p||$$

and this is sharp, where

$$c_m = \exp\left(\frac{m}{\pi}I(\frac{1}{m}\pi)\right).$$

Observe that $c_2 = \delta$ and, since $I(\pi/3) = (2/3)I(2\pi/3)$, we have $c_3 = \beta^2 = 1.9081456268\ldots$ [8]. We also have $c_4 = 1.9484547890\ldots$, $c_5 = 1.9670449011\ldots$, and $c_6 = 1.9771268308\ldots$.

Boyd [9] considered the case when $p(z)$ and all $p_i(z)$ have real coefficients, but are defined on D. Here the constant c_m is simply replaced by δ and this is sharp. That is, in the real case, the best constant does not depend on m. Borwein [4, 5] considered the case of complex $p(x)$ and $p_i(x)$ defined on the interval $[-1, 1]$. Here the constant c_m is simply replaced by δ^2 and again this is sharp. Pritsker [22, 23] obtained a general formula for the analog, $B(a)$, of β for Boyd's inequality [6] on the interval $[-a, a]$. For example, $B(2) = \beta^2 = 1.90815\ldots$ and $B(1) = \sqrt{2}\delta = 2.53373\ldots$. See also [24, 25].

In [2.30], we discuss **Mahler's measure** $M(\alpha)$ for algebraic integers α. This is, in essence, equivalent to Mahler's measure $M(f)$ for univariate polynomials [26]

$$f(z) = \alpha_0\prod_{j=1}^{n}(z-\alpha_j),$$

which is given by

$$M(f) = \exp\left(\int_0^1 \ln(|f(e^{2\pi i\theta})|)d\theta\right) = |\alpha_0|\prod_{j=1}^{n}\max(|\alpha_j|, 1)$$

as a consequence of Jensen's formula [27].

An important generalization to multivariate functions $f(z_1, z_2, \ldots, z_m)$ is given by

$$M(f) = \exp\left(\int_0^1\int_0^1\cdots\int_0^1 \ln(|f(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, \ldots, e^{2\pi i\theta_m})|)d\theta_1 d\theta_2\cdots d\theta_m\right).$$

Some examples are

$$M(1+x)=1, \quad M(1+x+y)=\beta=M(\max(1,|x+1|)),$$
$$M(1+x+y+z)=\exp\left(\frac{7\zeta(3)}{2\pi^2}\right),$$
$$M(1+x+y-xy)=\delta=M(\max(|x-1|,|x+1|)),$$

where $\zeta(3)$ is Apéry's constant [1.6]. Two asymptotic results are [10]

$$\lim_{m\to\infty}\frac{M(z_1+z_2+\cdots+z_m)}{\sqrt{m}}=\exp\left(-\frac{1}{2}\gamma\right)=0.7493060013\ldots,$$

involving the Euler–Mascheroni constant γ [1.5], and

$$\lim_{m\to\infty}M\left(z_1+(1+z_2)(1+z_3)\cdots(1+z_m)\right)^{\frac{1}{\sqrt{m}}}=\exp\left(\sqrt{\frac{\pi}{24}}\right).$$

Finally, we discuss **Bombieri's supremum norm**: If $p(z)=\sum_{j=0}^{n}a_jz^j$, then

$$[p]=\max_{0\le j\le n}|a_j|\frac{n!}{j!(n-j)!}.$$

If $p(z)$ and $q(z)$ are complex monic polynomials on D, $\deg(p)=n$, and q is a factor of p, we are interested in the size of $||q||$ relative to $[p]$. It is known that asymptotically [28–31]

$$||q||\le K^n\cdot[p],$$

where

$$K=M(1+|x+1|)=M\left((1+x+x^2+y)^2\right)=2.1760161352\ldots,$$

but a proof that K is the best possible remains undiscovered.

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3.11 Grothendieck's Constants

For any integer $n \geq 2$, there is a constant $k(n)$ with the following property [1,2]: Let A be any $m \times m$ matrix for which

$$\left|\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}s_i t_j\right| \leq 1$$

is satisfied for all scalars $s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_m$ with $|s_i| \le 1$, $|t_j| \le 1$. Then

$$\left| \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \langle x_i, y_j \rangle \right| \le k(n)$$

for all vectors $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$ in an n-dimensional Hilbert space with $||x_i|| \le 1$, $||y_j|| \le 1$. As usual, $\langle x, y \rangle$ is the inner product of x and y and $||x|| = \sqrt{\langle x, x \rangle}$. The constant $k(n)$ is taken to be the least possible.

This definition actually covers two possible cases:

- Scalars and matrices are real, and vectors are in a real Hilbert space.
- Scalars and matrices are complex, and vectors are in a complex Hilbert space.

We denote the two corresponding constants by $k_R(n)$ and $k_C(n)$. It is known [3–6] that

$$k_R(2) = \sqrt{2}, \quad k_R(3) < 1.517, \quad k_R(4) \le \pi/2$$

but

$$1.1526 \le k_C(2) \le 1.2157, \quad 1.2108 \le k_C(3) \le 1.2744, \quad 1.2413 \le k_C(4) \le 1.3048.$$

Each sequence clearly increases with n. For both real and complex cases, define $\kappa = \lim_{n\to\infty} k(n)$. It is not hard to show that [2], in the limit,

$$\frac{1}{2}\kappa_R \le \kappa_C \le 2\kappa_R.$$

The best-known numerical bounds are [3, 4, 7–9]

$$1.67696 \le \kappa_R \le \frac{\pi}{2\ln(1+\sqrt{2})} = 1.7822139781\ldots,$$
$$1.33807 \le \kappa_C \le \frac{8}{\pi \cdot (x_0 + 1)} = 1.40491\ldots,$$

where x_0 is the solution of a certain equation involving complete elliptic integrals $K(x)$ and $E(x)$ of the first and second kind [1.4.6]:

$$\psi(x) = \frac{\pi}{8}(x+1), \quad -1 < x < 1,$$

where

$$\psi(x) = x \int_0^{\frac{\pi}{2}} \frac{\cos(\theta)^2}{\sqrt{1 - x^2 \sin(\theta)^2}} d\theta = \frac{1}{x}\left[E(x) - (1 - x^2)K(x)\right].$$

The upper estimate for κ_R was conjectured by Krivine [3, 4, 10] to be the exact value. In contrast, Haagerup [7] doubted whether 1.40491 is the exact value for κ_C and thought that

$$\frac{1}{|\psi(i)|} = \left(\int_0^{\frac{\pi}{2}} \frac{\cos(\theta)^2}{\sqrt{1+\sin(\theta)^2}} d\theta \right)^{-1} = 1.4045759346\ldots$$

is a more plausible candidate. His reasoning was by analogy: The function $\psi(x)$ for the complex case is like the function $\varphi(x)$ employed by Krivine for the real case,

$$\varphi(x) = \frac{2}{\pi} \arcsin(x),$$

and one sees that

$$\frac{1}{|\varphi(i)|} = \frac{\pi}{2\,\mathrm{arcsinh}(1)} = \frac{\pi}{2\ln(1+\sqrt{2})}.$$

A different approach for bounding κ_R is given in [11].

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3.12 Du Bois Reymond's Constants

Abel's theorem from advanced calculus implies that if the series of real numbers $\sum_{n=0}^{\infty} a_n$ converges, then the corresponding power series satisfies

$$\lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n r^n = \sum_{n=0}^{\infty} a_n.$$

This is a consequence of uniform convergence on the interval [0, 1]. We start with a question: What happens if $\sum_{n=0}^{\infty} a_n$ *diverges*?

Define the sequence of partial sums $s_n = \sum_{k=0}^{n} a_k$ and assume

$$s = \liminf_{n\to\infty} s_n, \quad S = \limsup_{n\to\infty} s_n$$

are both finite. That is, the series is bounded and oscillates between two finite limits. It is natural to believe here that

$$s \le \lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n r^n \le S$$

and this is indeed true [1].

In fact, much more is true. Let $\varphi(x)$ be a continuously differentiable function for $x > 0$ that satisfies the conditions

$$\lim_{x\to 0^+} \varphi(x) = 1, \quad \lim_{x\to\infty} \varphi(x) = 0, \quad I = \int_0^\infty \left| \frac{d}{dx}\varphi(x) \right| dx < \infty$$

and

$$f(x) = \sum_{n=0}^{\infty} a_n \varphi(nx) \ \text{ is convergent for all } x > 0.$$

Then it can be proved that [1,2]

$$\frac{1}{2}(S+s) - \frac{1}{2}(S-s)\cdot I \le \lim_{x\to 0^+} f(x) \le \frac{1}{2}(S+s) + \frac{1}{2}(S-s)\cdot I.$$

Moreover, this truly extends what was discussed before: Set $r = \varphi(x) = \exp(-x)$ to see why.

Another important case arises if we instead set $\varphi(x) = (\sin(x)/x)^m$ for an integer $m \ge 2$. Define the m^{th} **Du Bois Reymond constant** by

$$c_m = I - 1 = \int_0^\infty \left| \frac{d}{dx}\left(\frac{\sin(x)}{x}\right)^m \right| dx - 1.$$

Watson [2–6] proved that

$$c_2 = \tfrac{1}{2}(e^2 - 7) = 0.1945280495\ldots, \quad c_4 = \tfrac{1}{8}(e^4 - 4e^2 - 25) = 0.0052407047\ldots,$$

$$c_6 = \tfrac{1}{32}(e^6 - 6e^4 + 3e^2 - 98) = 0.0002206747\ldots$$

and that c_{2k} is expressible as a polynomial of degree k in e^2 with rational coefficients. No such expression is known for c_{2k+1}, but there is an interesting series available for all c_m. Let $\xi_1, \xi_2, \xi_3, \ldots$ denote all positive solutions of the equation $\tan(x) = x$. Then

$$c_m = 2\sum_{j=1}^{\infty} \frac{1}{(1+\xi_j^2)^{m/2}}$$

and, in particular, $c_3 = 0.0282517642\ldots$. It is possible to numerically evaluate c_5, $c_7, \ldots$ as well. Watson also determined that

$$c_3 = -\frac{2}{\pi}\int_1^\infty \frac{x}{\sqrt{x^2-1}}\frac{d}{dx}\left(\frac{\tanh(x)^2}{x-\tanh(x)}\right)dx,$$

but there appears to be no further simplification of this integral.

The sequence $\xi_1, \xi_2, \xi_3, \ldots$ arose in a recent *Monthly* problem:

$$\sum_{n=1}^\infty \frac{1}{\xi_n^2} = \frac{1}{10}$$

and attracted much attention [7]. This formula parallels that just discussed and Watson's other results, namely,

$$b_m = 2\sum_{j=1}^\infty \frac{(-1)^{j+1}}{(1+\xi_j^2)^{m/2}}, \quad b_3 = -\tfrac{1}{4}(e^3 - 3e - 12) = 0.0173271405\ldots,$$

and b_{2k+1} is expressible as a polynomial of degree $2k+1$ in e with rational coefficients. Note that similar expressions in e appear in [3.8].

Here are other constants involving equations with the tangent function. The maximum value $M(n)$ of the function

$$\left(\sum_{k=1}^n \frac{x_k}{k}\right)^2 + \sum_{k=1}^n \left(\frac{x_k}{k}\right)^2,$$

subject to the constraint $\sum_{k=1}^n x_k^2 \le 1$, satisfies the following asymptotic result [8]:

$$\lim_{n\to\infty} M(n) = \left(\frac{\pi}{\xi}\right)^2 = 2.3979455861\ldots,$$

where $\xi = 2.0287578381\ldots$ is the smallest positive solution of the equation $x + \tan(x) = 0$. Another example [9], described in [3.14], involves the equation $\pi + x = \tan(x)$.

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3.13 Steinitz Constants

3.13.1 Motivation

If $\sum x_i$ is an absolutely convergent series of real numbers, then any rearrangement of the terms x_i of the series will have no impact on the sum.

By contrast, if $\sum x_i$ is a conditionally convergent series of real numbers, then the terms x_i may be rearranged to produce a series that has any desired sum (even ∞ or $-\infty$). This is a well-known theorem due to Riemann.

Suppose instead that the terms x_i are elements of a finite-dimensional normed real space; that is, the x_i are real vectors but possibly with a different notion of length (choice of metric). Assume nothing about the nature of $\sum x_i$. Let C denote the set of all sums of convergent rearrangements of the terms x_i. Steinitz [1–3] proved that C is either empty or of the form $y + L$, for some vector y and some linear subspace L. (Note that $L = \{0\}$, the zero subspace, is one possibility.)

To prove this theorem, Steinitz needed bounds on certain constants $K(0, 0)$, defined in the next section. For details on the precise connection, see [4–6].

3.13.2 Definitions

Let a and b be nonnegative real numbers. In an m-dimensional normed real space, define a set $S = \{u, v_1, v_2, \ldots, v_{n-1}, v_n, w\}$ of $n + 2$ vectors satisfying $|u| \le a$, $|v_j| \le 1$ for each $1 \le j \le n$, $|w| \le b$, and $u + \sum_{j=1}^{n} v_j + w = 0$ (see Figure 3.2).

Let π denote a permutation of the indices $\{1, 2, \ldots, n\}$ and define a function

$$F(\pi, n, S) = \max_{1 \le k \le n} \left| u + \sum_{j=1}^{k} v_{\pi(j)} \right|.$$

In words, F is the radius of the smallest sphere, with center at 0, circumscribing the

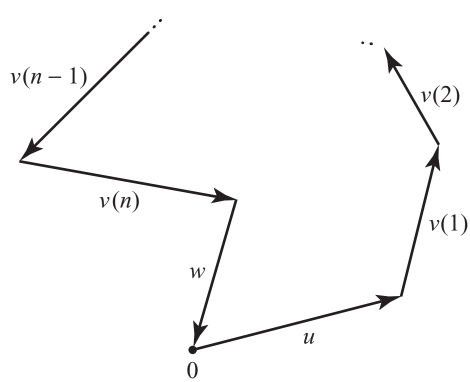

Figure 3.2. A set S of vectors satisfying $u + \sum_{j=1}^{n} v_j + w = 0$.

polygon with sides $u, v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)}$. Of the vector orderings determined by all possible π, there is (at least) one that minimizes the spherical radius. Define

$$K_m(a, b) = \max_{n,S} \min_{\pi} F(\pi, n, S);$$

that is, $K_m(a, b)$ is the least number for which $|u + \sum_{j=1}^{k} v_{\pi(j)}| \leq K_m(a, b)$ for some permutation π, for all integers n and sets S.

3.13.3 Results

In the general setting just described (with no restrictions on the norm), the best-known upper bound on the m**-dimensional Steinitz constant** is

$$K_m(0, 0) \leq m - 1 + \frac{1}{m}$$

due to Banaszczyk [7], improving on the work in [8]. Further, Grinberg & Sevastyanov [8] observed that, for $m = 2$, the upper bound $3/2$ is the best possible. In other words, there exists a norm for which equality holds. Whether this observation holds for larger m is unknown.

Henceforth let us assume the norm is Euclidean. Banaszczyk [9] proved that

$$K_2(a, b) = \sqrt{1 + \max(a^2, b^2, 1/4)},$$

which extends the results $K_2(1, 0) = K_2(1, 1) = \sqrt{2}$, $K_2(0, 0) = \sqrt{5}/2 = 1.1180339887\ldots$ known to earlier authors. Damsteeg & Halperin [4] demonstrated that

$$K_m(0, 0) \geq \frac{1}{2}\sqrt{m + 3}$$

and, for $m \geq 2$,

$$K_m(1, 1) \geq K_m(1, 0) \geq \frac{1}{2}\sqrt{m + 6}.$$

Behrend [10] proved that

$$K_m(1, 0) \leq K_m(1, 1) < m, \quad K_3(1, 0) \leq K_3(1, 1) < \sqrt{5 + 2\sqrt{3}} = 2.9093129112\ldots,$$

but an exact value for any $m > 2$ remains unknown. (Note: There seems to be some confusion in [11] between $K(0, 0)$ and $K(1, 0)$, but not in the earlier reference [12].) Behrend believed it to be likely that the true order of these constants is $\sqrt{m}$. See also [13–18] for related ideas.

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3.14 Young–Fejér–Jackson Constants

3.14.1 Nonnegativity of Cosine Sums

In the following, n is a positive integer, $0 \le \theta \le \pi$, and a is a parameter to be studied. Young [1] proved that the cosine sum

$$C(\theta, a, n) = \frac{1}{1+a} + \sum_{k=1}^{n} \frac{\cos(k\theta)}{k+a} \ge 0$$

for $-1 < a \le 0$. Rogosinski & Szegö [2] extended this result to $-1 < a \le 1$ and proved that there is a best upper limit A, $1 \le A \le 2(1+\sqrt{2})$, in the sense that

- $C(\theta, a, n) \ge 0$ for $-1 < a \le A$, for all n and all θ,
- $C(\theta, a, n) < 0$ for $a > A$, for some n and some θ.

Gasper [3,4] proved that $A = 4.5678018826\ldots$ and has minimal polynomial

$$9x^7 + 55x^6 - 14x^5 - 948x^4 - 3247x^3 - 5013x^2 - 3780x - 1134.$$

In fact, if $a > A$, then $C(\theta, a, 3) < 0$ for some θ. This completes the story for cosine sums.

3.14.2 Positivity of Sine Sums

Here, n is a positive integer, $0 < \theta < \pi$, and b is the parameter of interest. Fejér [5], Gronwall [6, 7], and Jackson [8] obtained that the corresponding sine series

$$S(\theta, b, n) = \sum_{k=1}^{n} \frac{\sin(k\theta)}{k+b} > 0$$

for $b = 0$. See [9] for a quick proof; see also [10–13]. Brown & Wang [14] extended this result to $-1 < b \leq B$ for odd integers n, where B is the best upper limit. For even integers n, the story is more complicated and we shall explain later.

Two intermediate constants need to be defined:

- $\lambda = 0.4302966531\ldots$, a solution of the equation $(1+\lambda)\pi = \tan(\lambda\pi)$,
- $\mu = 0.8128252421\ldots$, a solution of the equation $(1+\lambda)\sin(\mu\pi) = \mu\sin(\lambda\pi)$.

With these, define $B = 2.1102339661\ldots$ to be a solution of the equation [14, 15]

$$(1+\lambda)\cdot\pi\cdot\left((B-1)\psi(1+\tfrac{B-1}{2}) - 2B\psi(1+\tfrac{B}{2}) + (B+1)\psi(1+\tfrac{B+1}{2})\right) = 2\sin(\lambda\pi),$$

where $\psi(x)$ is the digamma function [1.5.4]. Is B algebraic? The answer is unknown.

We now discuss the case of even n. Define $c_n(x) = 1 - 2x/(4n+1)$. If $-1 < b \leq B$ and n is even, then $S(\theta, b, n) > 0$ for $0 < \theta \leq \pi c_n(\mu)$. Further, the constant μ is the best possible, meaning that $0 < \nu < \mu$ implies $S(\pi c_n(\nu), b, n) < 0$ for some $b < B$ and infinitely many n.

Wilson [16] indicated that $S < 0$ can be expected on the basis of Belov's work [17].

3.14.3 Uniform Boundedness

Fix a parameter value $0 < r < 1$. Consider the sequence of functions

$$F_n(\theta, r) = \sum_{k=1}^{n} k^{-r}\cos(k\theta), \quad n = 1, 2, 3, \ldots.$$

This sequence is said to be **uniformly bounded below** if there exists a constant $m > -\infty$ such that $m < F_n(\theta, r)$ for all θ and all n. Note that m depends on the choice of r.

Zygmund [11] proved that there is a best lower limit $0 < R < 1$ for r, in the sense that

- $F_n(\theta, r)$ is uniformly bounded below for $r \geq R$ and
- $F_n(\theta, r)$ is not uniformly bounded below for $r < R$.

The constant $R = 0.3084437795\ldots$ is the unique solution of the equation [15, 18–22]

$$\int_0^{\frac{3\pi}{2}} x^{-R}\cos(x)dx = 0$$

and this plays a role in Belov's papers [17, 23] as well. Interestingly, the sequence of functions

$$G_n(\theta, r) = \sum_{k=1}^{n} k^{-r}\sin(k\theta), \quad n = 1, 2, 3, \ldots,$$

is uniformly bounded below for all $r > 0$; hence there is no analog of R for the sequence $G_n(\theta, r)$.

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3.15 Van der Corput's Constant

Let f be a real twice-continuously differentiable function on the interval $[a, b]$ with the property that $|f''(x)| \geq r$ for all x. There exists a smallest constant m, independent of a and b as well as f, such that

$$\left| \int_a^b \exp(i \cdot f(x))dx \right| \leq \frac{m}{\sqrt{r}},$$

where i is the imaginary unit [1–3]. This inequality was first proved by van der Corput [1] and has several applications in analytic number theory. Kershner [4, 5], following a suggestion of Wintner, proved that the maximizing function f is the parabola $f(x) = rx^2/2 + c$, with domain endpoints given by

$$-a = b = \sqrt{\frac{\pi - 2c}{r}}$$

and coefficient $c = -0.7266432468\ldots$ given as the only solution of the equation

$$\int_0^{\sqrt{\frac{\pi}{2}-c}} \sin(x^2 + c)dx = 0, \quad -\frac{\pi}{2} \leq c \leq \frac{\pi}{2}.$$

From this, it follows that van der Corput's constant m is

$$m = 2\sqrt{2} \int_0^{\sqrt{\frac{\pi}{2}-c}} \cos(x^2 + c)dx = 3.3643175781\ldots.$$

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3.16 Turán's Power Sum Constants

For fixed complex numbers $z_1, z_2, \ldots, z_n$, define [1]

$$S(z) = \max_{1 \le k \le n} \left| \sum_{j=1}^{n} z_j^k \right|$$

to be the maximum modulus of power sums of degree $\le n$. Define also the $(n-1)$-dimensional complex region

$$K_n = \left\{ z \in \mathbb{C}^n : z_1 = 1 \text{ and } |z_j| \le 1 \text{ for } 2 \le j \le n \right\}.$$

Consider the problem of minimizing $S(z)$ subject to $z \in K_n$. The optimal value σ_n of $S(z)$ is [2–4]

$$\frac{\sqrt{5}-1}{\sqrt{2}} = 0.8740320488\ldots \text{ if } n = 2, \text{ and } x = 0.8247830309\ldots \text{ if } n = 3,$$

where x has minimal polynomial [5]

$$\begin{aligned} & x^{30} - 81x^{28} + 2613x^{26} - 43629x^{24} + 417429x^{22} - 2450985x^{20} + 9516137x^{18} \\ & \quad - 26203659x^{16} + 53016480x^{14} - 83714418x^{12} + 112601340x^{10} - 140002992x^{8} \\ & \quad + 156204288x^{6} - 124361568x^{4} + 55427328x^{2} - 10077696. \end{aligned}$$

Exact values of σ_n for $n \ge 4$ are not known, but we have bounds $0.3579 < \sigma_n < 1 - (250n)^{-1}$ for all sufficiently large n [1,6,7]. It is conjectured that $\lim_{n\to\infty} \sigma_n$ exists, but no one has numerically explored this issue, as far as is known.

Define instead [1,8]

$$T(z) = \max_{2 \le k \le n+1} \left| \sum_{j=1}^{n} z_j^k \right|$$

and consider the problem of minimizing $T(z)$ subject to $z \in K_n$. The minimum value τ_n of $T(z)$ surprisingly satisfies $\tau_n < 1.321^{-n}$ for all sufficiently large n. This is very different behavior from that of σ_n. If we replace the exponent range $2 \le k \le n+1$ by $3 \le k \le n+2$, then the constant 1.321 can be replaced by 1.473.

Turán's book [1] is a gold mine of related theory and applications.

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4

Constants Associated with the Approximation of Functions

4.1 Gibbs–Wilbraham Constant

Let f be a piecewise smooth function defined on the half-open interval $[-\pi, \pi)$, extended to the real line via periodicity, and possessing at most finitely many discontinuities (all finite jumps). Let

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt$$

denote the Fourier coefficients of f and let

$$S_n(f, x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos(kx) + b_k\sin(kx))$$

be the n^{th} partial sum of the Fourier series of f. Let $x = c$ denote one of the discontinuities. Define

$$\delta = \left(\lim_{x\to c^-} f(x)\right) - \left(\lim_{x\to c^+} f(x)\right), \quad \mu = \frac{1}{2}\left[\left(\lim_{x\to c^-} f(x)\right) + \left(\lim_{x\to c^+} f(x)\right)\right]$$

and assume without loss of generality that $\delta > 0$. Let $x_n < c$ denote the first local maximum of $S_n(f, x)$ to the left of c, and let $\xi_n > c$ denote the first local minimum of $S_n(f, x)$ to the right of c. Then

$$\lim_{n\to\infty} S_n(f, x_n) = \mu + \frac{\delta}{\pi}G, \quad \lim_{n\to\infty} S_n(f, \xi_n) = \mu - \frac{\delta}{\pi}G,$$

where

$$G = \int_0^{\pi} \frac{\sin(\theta)}{\theta}\,d\theta = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n+1)(2n+1)!} = 1.8519370519\ldots$$

$$= \frac{\pi}{2}(1.1789797444\ldots)$$

is the **Gibbs–Wilbraham constant** [1–5].

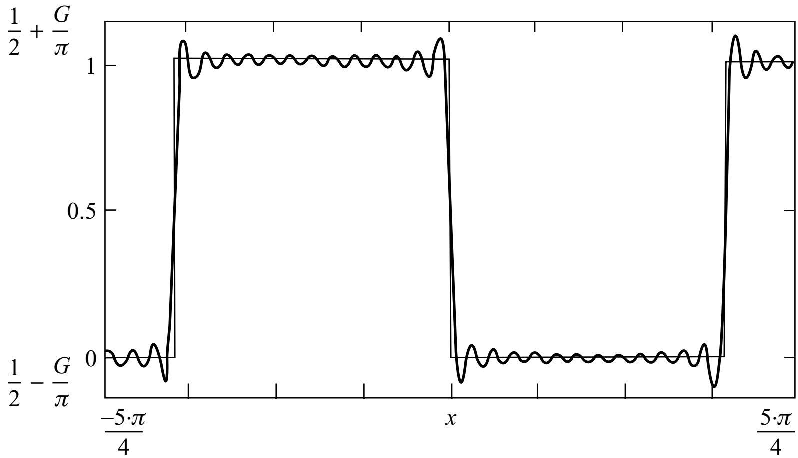

Figure 4.1. The Fourier series approximation of a square wave exhibits both overshooting and undershooting.

Consider the graph in Figure 4.1, with $f(x) = 1$ for $-\pi \le x < 0$ and $f(x) = 0$ for $0 \le x < \pi$. The limiting crest of the highest oscillation converges not to 1 but to $1/2 + G/\pi = 1.0894898722\ldots$. Similarly, the deepest trough converges not to 0 but to $1/2 - G/\pi = -0.0894898722\ldots$. In words, the Gibbs–Wilbraham constant quantifies the degree to which the Fourier series of a function overshoots or undershoots the function value at a jump discontinuity.

These phenomena were first observed by Wilbraham [6] and Gibbs [7]. Bôcher [8] generalized such observations to arbitrary functions f.

More generally, if $x_{n,2r-1} < c$ denotes the r^{th} local maximum of $S_n(f, x)$ to the left of c, if $x_{n,2r} < c$ denotes the r^{th} local minimum to the left of c, and if likewise for $\xi_{n,2r}$ and $\xi_{n,2r-1}$, then

$$\lim_{n\to\infty} S_n(f, x_{n,s}) = \mu + \frac{\delta}{\pi}\int_0^{s\pi} \frac{\sin(\theta)}{\theta}\, d\theta, \quad \lim_{n\to\infty} S_n(f, \xi_{n,s}) = \mu - \frac{\delta}{\pi}\int_0^{s\pi} \frac{\sin(\theta)}{\theta}\, d\theta.$$

The sine integral decreases to $\pi/2$ for increasing integer values of $s = 2r - 1$, but it increases to $\pi/2$ for $s = 2r$. For large enough r, the limiting values become $\mu \pm \delta/2$, which is consistent with intuition.

Fourier series are best L_2 (least-squares) trigonometric polynomial fits; Gibbs–Wilbraham phenomena appear in connection with splines [5, 9–11], wavelets [5, 12], and generalized Padé approximants [13] as well. Hence there are many Gibbs–Wilbraham constants! Moskona, Petrushev & Saff [5, 14] studied best L_1 trigonometric polynomial fits and determined the analog of $2G/\pi - 1 = 0.1789797444\ldots$ in this setting; its value is $\max_{x\ge 1} g(x) = 0.0657838882\ldots$, where

$$g(x) = -\frac{\sin(\pi x)}{\pi}\int_0^1 t^{x-1}\frac{1-t}{1+t}\, dt = -\frac{\sin(\pi x)}{\pi x}\sum_{k=1}^{\infty}\frac{k!\cdot 2^{-k}}{(x+1)(x+2)\cdots(x+k)}$$

for $x > 0$. The case of L_p approximation, where $1 < p \neq 2$, was investigated only recently [15].

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4.2 Lebesgue Constants

4.2.1 Trigonometric Fourier Series

If a function f is integrable over the interval $[-\pi, \pi]$, let

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt$$

denote the Fourier coefficients of f and let

$$S_n(f, x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos(kx) + b_k\sin(kx))$$

be the n^{th} partial sum of the Fourier series of f. Assuming further that $|f(x)| \le 1$ for all x, it follows that

$$|S_n(f,x)| \le \frac{1}{\pi}\int_0^\pi \frac{\left|\sin(\frac{2n+1}{2}\theta)\right|}{\sin(\frac{\theta}{2})}\,d\theta = L_n$$

for all x, where L_n is the n^{th} **Lebesgue constant** [1,2]. The values of the first several Lebesgue constants are

$$L_0 = 1,\ L_1 = \tfrac{1}{3} + \tfrac{2\sqrt{3}}{\pi} = 1.4359911241\ldots,\ L_2 = 1.6421884352\ldots,$$
$$L_3 = 1.7783228615\ldots.$$

Several alternative formulas are due to Fejér [3,4] and Szegö [5]:

$$L_n = \frac{1}{2n+1} + \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k}\tan\left(\frac{\pi k}{2n+1}\right) = \frac{16}{\pi^2}\sum_{k=1}^{\infty}\sum_{j=1}^{(2n+1)k}\frac{1}{4k^2-1}\frac{1}{2j-1}$$

The latter expression demonstrates that $\{L_n\}$ is monotonically increasing.

The Lebesgue constants are the best possible, in the sense that $L_n = \sup_f |S_n(f,0)|$ and the supremum is taken over all continuous f satisfying $|f(x)| \le 1$ for all x. It can be easily shown [6,7] that

$$\frac{4}{\pi^2}\ln(n) < L_n < 3 + \frac{4}{\pi^2}\ln(n).$$

This implies that $L_n \to \infty$ and, consequently, the Fourier series for f can be unbounded even if f is continuous [8–10]. It also implies that if the **modulus of continuity** of f,

$$\omega(f,\delta) = \sup_{|x-y|<\delta}|f(x) - f(y)|,$$

satisfies $\lim_{\delta\to 0}\omega(f,\delta)\ln(\delta) = 0$, then the Fourier series for f converges uniformly to f. This is known as the Dini–Lipschitz theorem [2,7]. In words, while mere continuity is not enough, continuity *plus* additional conditions (e.g., differentiability) ensure uniform convergence.

Much greater precision in estimating the Lebesgue constants is possible. Watson [11] proved that

$$\lim_{n\to\infty}\left(L_n - \frac{4}{\pi^2}\ln(2n+1)\right) = c,$$

where

$$\begin{aligned} c &= \frac{8}{\pi^2}\left(\sum_{k=1}^{\infty}\frac{\ln(k)}{4k^2-1}\right) - \frac{4}{\pi^2}\psi(\frac{1}{2}) \\ &= \frac{8}{\pi^2}\left(\sum_{j=0}^{\infty}\frac{\lambda(2j+2)-1}{2j+1}\right) + \frac{4}{\pi^2}(2\ln(2)+\gamma) \\ &= 0.9894312738\ldots = \frac{4}{\pi^2}(2.4413238136\ldots), \end{aligned}$$

γ is the Euler–Mascheroni constant [1.5], $\psi(x)$ is the digamma function [1.5.4], and $\lambda(x)$ appears in [1.7]. Higher-order coefficients in the asymptotic expansion of L_n can be written as finite combinations of Bernoulli numbers [1.6.1]. Galkin [12] further proved that

$$L_n - \frac{4}{\pi^2}\ln(2n+1) \text{ decreases to } c, \text{ whereas } L_n - \frac{4}{\pi^2}\ln(2n+2) \text{ increases to } c$$

as $n \to \infty$. More asymptotics appear in [13, 14]. We mention two integral formulas discovered by Hardy [15]:

$$\begin{aligned} L_n &= 4\int_0^\infty \frac{\tanh((2n+1)x)}{\tanh(x)} \frac{1}{\pi^2+4x^2}dx \\ &= \frac{4}{\pi^2}\int_0^\infty \frac{\sinh((2n+1)x)}{\sinh(x)} \ln\left(\coth(\tfrac{2n+1}{2}x)\right)dx. \end{aligned}$$

See a related discussion in our essay on Favard constants [4.3].

There are many possible extensions of L_n; it is interesting to ascertain which properties for Fourier series carry over to the case in question. For example, the monotonicity of Lebesgue constants for Legendre series has been proved [16], confirming a conjecture of Szegö.

Here is a related idea. If f is complex analytic inside the unit disk, continuous on the boundary, and $|f(z)| < 1$ for all $|z| < 1$, then [17]

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \text{ implies that } \left|\sum_{k=0}^{n} a_k\right| \le G_n,$$

where

$$G_n = \sum_{m=0}^{n} \frac{1}{2^{4m}}\binom{2m}{m}^2 = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 + \cdots + \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2$$

is the n^{th} **Landau constant** (note the similarity with [1.5.4]). The constant G_n is the best possible for each n. It is known that [11]

$$\lim_{n\to\infty}\left(G_n - \frac{1}{\pi}\ln(n+1)\right) = \frac{1}{\pi}(4\ln(2)+\gamma) = 1.0662758532\ldots,$$

$$G_{2n} \le L_n < \frac{4}{\pi}G_{2n},$$

and both sequences $\{G_n\}$ and $\{L_n/G_{2n}\}$ are monotonically increasing. More refinements are found in [18–21].

4.2.2 Lagrange Interpolation

Here is a different sense in which the same phrase "Lebesgue constants" is used. Given real-valued data $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ with $-1 \le x_1 < x_2 < \ldots <$

$x_n \le 1$, there is a unique polynomial $p_{X,Y}(x)$ of degree at most $n-1$ such that

$$p_{X,Y}(x_i) = y_i, \quad i = 1, 2, \ldots, n,$$

called the **Lagrange interpolating polynomial**, given X and Y. The formula for $p_{X,Y}(x)$ is

$$p_{X,Y}(x) = \sum_{k=1}^{n} \left(y_k \cdot \prod_{j \ne k} \frac{x - x_j}{x_k - x_j} \right).$$

We wish to understand the approximating power of interpolating polynomials as the spatial arrangement of $\{x_i\}$ varies or as n increases [6,22]. The expression

$$\Lambda_n(X) = \max_{-1 \le x \le 1} \sum_{k=1}^{n} \left| \prod_{j \ne k} \frac{x - x_j}{x_k - x_j} \right|$$

is useful for this purpose and is called the n^{th} **Lebesgue constant** corresponding to X. Note that Λ_n does not depend on Y. It can be easily shown that

$$\Lambda_n > \frac{4}{\pi^2} \ln(n) - 1$$

for all n and hence $\lim_{n\to\infty} \Lambda_n = \infty$, regardless of the choice of X. This means that, given any X, there exists a continuous function f such that $p_{X,f(X)}(x)$ does *not* converge uniformly to f as n increases. In words, there is no "universal" set X guaranteeing uniform convergence for all continuous functions f.

Erdös [23] further tightened the lower bound on the Lebesgue constants. He proved that there must exist a constant C such that

$$\Lambda_n > \frac{2}{\pi} \ln(n) - C$$

for all n, for arbitrary X. We will exhibit the smallest possible value of C shortly. Erdös' result cannot be improved because, if T consists of the n zeros

$$x_j = -\cos\left(\frac{(2j-1)\pi}{2n} \right) \quad j = 1, 2, \ldots, n,$$

of the n^{th} Chebyshev polynomial [4.9], then

$$\Lambda_n(T) = \frac{1}{n} \sum_{j=1}^{n} \cot\left(\frac{(2j-1)\pi}{4n} \right) \le \frac{2}{\pi} \ln(n) + 1.$$

In fact, $\{\Lambda_n(T) - \frac{2}{\pi}\ln(n)\}$ is monotonically decreasing with [24–26]

$$\lim_{n\to\infty} \left(\Lambda_n(T) - \frac{2}{\pi} \ln(n) \right) = \frac{2}{\pi}(3\ln(2) - \ln(\pi) + \gamma) = 0.9625228267\ldots.$$

A complete asymptotic expansion (again involving Bernoulli numbers) was obtained in [27–30].

What is the optimal set X^* for which Λ_n is smallest [22]? Certainly the Chebyshev zeros are a good candidate for X^* but it can be shown that other choices of X will do even better. Kilgore [31] and de Boor & Pinkus [32] proved Bernstein's equioscillatory conjecture [33] regarding such X^*. A more precise, analytical description of X^* is not known.

A less hopeless problem is to estimate $\Lambda_n^* = \Lambda_n(X^*)$. Vértesi [34–36], building upon the work of Erdös [23], proved that

$$\lim_{n\to\infty}\left(\Lambda_n^* - \frac{2}{\pi}\ln(n)\right) = \frac{2}{\pi}(2\ln(2) - \ln(\pi) + \gamma) = 0.5212516264\ldots.$$

This resolves the identity of C, but higher-order asymptotics and monotonicity issues remain open.

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4.3 Achieser–Krein–Favard Constants

In this essay, we presuppose knowledge of the Lebesgue constants L_n [4.2]. Assume a function f to be integrable over the interval $[-\pi, \pi]$ and $S_n(f, x)$ to be the n^{th} partial sum of the Fourier series of f. If $|f(x)| \le 1$ for all x, then we know that

$$|S_n(f, x)| \le L_n = \frac{4}{\pi^2}\ln(n) + O(1)$$

and, moreover, L_n is best possible (it is a maximum). If we restrict attention to continuous functions f, that is, a subclass of the integrable functions, then L_n is still best possible (although it is only a supremum).

This may be considered as an extreme case ($r = 0$) of the following result due to Kolmogorov [1–3]. Fix an integer $r \geq 1$. If a function f is r-times differentiable and satisfies $|f^{(r)}(x)| \leq 1$ for all x, then

$$|f(x) - S_n(f, x)| \leq L_{n,r} = \frac{4}{\pi^2}\frac{\ln(n)}{n^r} + O\left(\frac{1}{n^r}\right),$$

where

$$L_{n,r} = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\left|\sum_{k=n+1}^{\infty}\frac{\sin(k\theta)}{k^r}\right| d\theta & \text{if } r \geq 1 \text{ is odd,} \\ \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\left|\sum_{k=n+1}^{\infty}\frac{\cos(k\theta)}{k^r}\right| d\theta & \text{if } r \geq 2 \text{ is even} \end{cases}$$

is best possible.

All this is a somewhat roundabout way for introducing the **Achieser–Krein–Favard constants**, which are often simply called **Favard constants**. In the preceding, we focused solely on the quality of the Fourier estimate $S_n(f, x)$ of f. Suppose we replace $S_n(f, x)$ by an arbitrary trigonometric polynomial

$$P_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos(kx) + b_k\sin(kx)),$$

where no conditions are placed on the coefficients (apart from being real). If, as before, the r^{th} derivative of f is bounded between -1 and 1, then there *exists* a polynomial $P_n(x)$ for which

$$|f(x) - P_n(x)| \leq \frac{K_r}{(n+1)^r}$$

for all x, where the r^{th} Favard constant [4–6]

$$K_r = \frac{4}{\pi}\sum_{j=0}^{\infty}\left[\frac{(-1)^j}{2j+1}\right]^{r+1}$$

is the smallest numerator possible. In other words, whereas Lebesgue constants are connected to approximations that are best in a least-squares sense (Fourier series), Favard constants are connected to approximations that are best in a pointwise sense.

Observe that

$$K_r = \begin{cases} \dfrac{4}{\pi}\lambda(r+1) & \text{if } r \text{ is odd,} \\ \dfrac{4}{\pi}\beta(r+1) & \text{if } r \text{ is even,} \end{cases}$$

where both the lambda and beta functions are discussed in [1.7]. Each Favard constant is hence a rational multiple of π^r, for example,

$$K_0 = 1, \quad K_1 = \frac{\pi}{2}, \quad K_2 = \frac{\pi^2}{8}, \quad K_3 = \frac{\pi^3}{24},$$

and $1 = K_0 < K_2 < \ldots < 4/\pi < \ldots < K_3 < K_1 = \pi/2$.

This is the first of many sharp results for various classes of functions and methods of approximation that involve the constants K_r. The theorems are rather technical and so will not be discussed here. We mention, however, the Bohr–Favard inequality [7–9] and the Landau–Kolmogorov constants [3.3]. See also [10, 11].

Here is an unsolved problem. For an arbitrary trigonometric polynomial $P_n(\theta)$, it is known that [12, 13]

$$\max_{-\pi \le \theta \le \pi} |P_n(\theta)| \le C \frac{n}{2\pi} \int_{-\pi}^{\pi} |P_n(\theta)|\, d\theta,$$

and the best possible constant asymptotically satisfies $0.539 \le C \le 0.58$ as $n \to \infty$. An exact expression for C is not known.

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4.4 Bernstein's Constant

For any real function $f(x)$ with domain $[-1, 1]$, let $E_n(f)$ denote the error of best uniform approximation to f by real polynomials of degree at most n. That is,

$$E_n(f) = \inf_{p \in P_n} \sup_{-1 \le x \le 1} |f(x) - p(x)|,$$

where $P_n = \left\{\sum_{k=0}^{n} a_k x^k : a_k \text{ real}\right\}$. Consider the special case $\alpha(x) = |x|$, for which Jackson's theorem [1,2] implies $E_n(\alpha) \le 6/n$. Since $|x|$ is an even continuous function on $[-1, 1]$, then so is its (unique) best uniform approximation from P_n on $[-1, 1]$. It follows that $E_{2n}(\alpha) = E_{2n+1}(\alpha)$, so we consider only the even-subscript case henceforth. Bernstein [3] strengthened the Jackson inequality

$$2nE_{2n}(\alpha) \le 6$$

to

$$2nE_{2n}(\alpha) \le \frac{4n}{\pi(2n+1)} < \frac{2}{\pi} = 0.636\ldots$$

using Chebyshev polynomials [4.9]. He proved the existence of the following limit and obtained the indicated bounds:

$$0.278\ldots < \beta = \lim_{n\to\infty} 2nE_{2n}(\alpha) < 0.286\ldots.$$

Bernstein conjectured that $\beta = 1/(2\sqrt{\pi}) = 0.2821\ldots$. This conjecture remained unresolved for seventy years, owing to the difficulty in computing $E_{2n}(\alpha)$ for large n and to the slow convergence of $2nE_{2n}(\alpha)$ to β.

Varga & Carpenter [4,5] computed $\beta = 0.2801694990\ldots$ to fifty decimal places, disproving Bernstein's conjecture. They required calculations of $2nE_{2n}(\alpha)$ up to $n = 52$ with accuracies of nearly 95 places and a number of other techniques. At the end of [4], they indicated that it is not implausible to believe that β might admit a closed-form expression in terms of the classical hypergeometric function or other known constants.

Since we have just discussed the problem of the best uniform *polynomial* approximation to $|x|$, it is natural to consider the problem of the best uniform *rational* approximation as well. Define, for arbitrary f on $[-1, 1]$,

$$E_{m,n}(f) = \inf_{r\in R_{m,n}} \sup_{-1\le x\le 1} |f(x) - r(x)|,$$

where $R_{m,n} = \{p(x)/q(x) : p \in P_m,\ q \in P_n,\ q \ne 0\}$. Newman [6] proved that

$$\tfrac{1}{2}e^{-9\sqrt{n}} \le E_{n,n}(\alpha) \le 3e^{-\sqrt{n}},\quad n \ge 4,$$

equivalently, that $E_{n,n} \to 0$ incomparably faster than E_n. Newman's work created a sensation among researchers [5,7]. Bulanov [8], extending results of Gonchar [9], proved that the lower bound could be improved to

$$e^{-\pi\sqrt{n+1}} \le E_{n,n}(\alpha)$$

and Vjacheslavov [10] proved the existence of positive constants m and M such that

$$m \le e^{\pi\sqrt{n}}E_{n,n}(\alpha) \le M.$$

(Petrushev & Popov [7] remarked on the interesting juxtaposition of the constants e and π here in a seemingly unrelated setting.) As before, $E_{2n,2n}(\alpha) = E_{2n+1,2n+1}(\alpha)$, so we focus on the even-subscript case. Varga, Ruttan & Carpenter [11] conjectured, on the basis of careful computations, that

$$\lim_{n\to\infty} e^{\pi\sqrt{2n}} E_{2n,2n}(\alpha) = 8,$$

which Stahl [12, 13] recently proved. The contrast between the polynomial and rational cases is fascinating!

Gonchar [9] pointed out the relevance of Zolotarev's work [3.9] to this line of research.

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4.5 The "One-Ninth" Constant

We are concerned here with the rational approximation of $\exp(-x)$ on the half-line $[0, \infty)$. Let $\lambda_{m,n}$ denote the error of best uniform approximation:

$$\lambda_{m,n} = \inf_{r\in R_{m,n}} \sup_{x\geq 0} |e^{-x} - r(x)|,$$

where $R_{m,n}$ is the set of real rational functions $p(x)/q(x)$ with $\deg(p) \leq m, \deg(q) \leq n$, and $q \neq 0$, as defined in [4.4].

There are two cases of special interest, when $m = 0$ and $m = n$, since clearly

$$0 < \lambda_{n,n} \leq \lambda_{n-1,n} \leq \lambda_{n-2,n} \leq \ldots \leq \lambda_{2,n} \leq \lambda_{1,n} \leq \lambda_{0,n}.$$

Many researchers [1–4] have studied these constants $\lambda_{m,n}$, referred to as **Chebyshev constants** in [4]. We mention the work of only a few. Schönhage [5] proved that

$$\lim_{n\to\infty} \lambda_{0,n}^{\frac{1}{n}} = \frac{1}{3},$$

which led several people to conjecture that

$$\lim_{n\to\infty} \lambda_{n,n}^{\frac{1}{n}} = \frac{1}{9}.$$

Numerical evidence uncovered by Schönhage [6] and Trefethen & Gutknecht [7] suggested that the conjecture is false. Carpenter, Ruttan & Varga [8] calculated the Chebyshev constants to an accuracy of 200 digits up to $n = 30$ and carefully obtained

$$\lim_{n\to\infty} \lambda_{n,n}^{\frac{1}{n}} = \frac{1}{9.2890254919\ldots} = 0.1076539192\ldots,$$

although a proof that the limit even *existed* was still to be found.

Building upon the work of Opitz & Scherer [9] and Magnus [10–12], Gonchar & Rakhmanov [4, 13] proved that the limit exists and that it equals

$$\Lambda = \exp\left(\frac{-\pi\, K(\sqrt{1-c^2})}{K(c)}\right),$$

where $K(x)$ is the complete elliptic integral of the first kind [1.4.6] and the constant c is defined as follows. Let $E(x)$ be the complete elliptic integral of the second kind [1.4.6]; then $0 < c < 1$ is the unique solution of the equation $K(c) = 2E(c)$.

Gonchar and Rakhmanov's exact disproof of the "one-ninth" conjecture utilized ideas from complex potential theory, which seems far removed from the rational approximation of $\exp(-x)$! They also obtained a number-theoretic characterization of the "one-ninth" constant Λ. If

$$f(z) = \sum_{j=1}^{\infty} a_j z^j, \text{ where } a_j = \left|\sum_{d|j} (-1)^d d\right|,$$

then f is complex-analytic in the open unit disk. The unique positive root of the equation $f(z) = 1/8$ is the constant Λ. Another way of writing a_j is as follows [14]: If

$$j = 2^m p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$$

is the prime factorization of the integer j, where $p_1 < p_2 < \ldots < p_k$ are odd primes, $m \geq 0$, and $m_i \geq 1$, then

$$a_j = |2^{m+1} - 3| \frac{p_1^{m_1+1} - 1}{p_1 - 1} \frac{p_2^{m_2+1} - 1}{p_2 - 1} \cdots \frac{p_k^{m_k+1} - 1}{p_k - 1}.$$

Carpenter [4] computed Λ to 101 digits using this equation.

Here is another expression due to Magnus [10]. The one-ninth constant Λ is the unique solution of the equation

$$\sum_{k=0}^{\infty}(2k+1)^2(-x)^{\frac{k(k+1)}{2}} = 0, \quad 0 < x < 1,$$

which turns out to have been studied one hundred years earlier by Halphen [15]. Halphen was interested in theta functions and computed Λ to six digits, clearly unaware that this constant would become prominent a century later! Varga [4] suggested that Λ be renamed the Halphen constant. So many researchers have contributed to the solution of this approximation problem, however, that retaining the amusingly inaccurate "one-ninth" designation might be simplest.

The constant $c = 0.9089085575\ldots$ defining Λ arises in a completely unrelated field: the study of *Euler elasticae* [16–18]. A quotient of elliptic functions, similar to that discussed here, occurs in [7.8].

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4.6 Fransén–Robinson Constant

For increasing x, the **reciprocal gamma function** $1/\Gamma(x)$ decreases more rapidly than $\exp(-cx)$ for any constant c, and thus may be useful as a one-sided density function for certain probability models. As a consequence, the value

$$I = \int_0^\infty \frac{1}{\Gamma(x)} dx = 2.8077702420\ldots$$

is needed for the sake of normalization.

One way to compute this integral is via the limit of Riemann sums I_n as $n \to \infty$, where [1]

$$I_n = \frac{1}{n} \sum_{k=1}^{\infty} \frac{1}{\Gamma(\frac{k}{n})} = \begin{cases} e = 2.7182818284\ldots & \text{if } n = 1, \\ \frac{1}{2}\left(\frac{1}{\sqrt{\pi}} + e\,\mathrm{erfc}(-1)\right) = 2.7865848321\ldots & \text{if } n = 2, \end{cases}$$

and

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt = 1 - \mathrm{erfc}(x)$$

is the **error function**. This is, however, too slow a procedure for computing I to high precision.

Fransén [2] computed I to 65 decimal digits, using Euler–Maclaurin summation and the formula

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right)^{-1} e^{\frac{x}{n}} = \frac{1}{x} \exp\left[\sum_{k=1}^{\infty} \frac{(-1)^k s_k}{k} x^k\right],$$

where $s_1 = \gamma$ and $s_k = \zeta(k)$, $k \geq 2$. Background on the Euler–Mascheroni constant γ appears in [1.5] and that on the Riemann zeta function $\zeta(z)$ in [1.6].

Robinson [2] independently obtained an estimate of I to 36 digits using an 11-point Newton–Coates approach. Fransén & Wrigge [3,4], via Taylor series and other analytical tools, achieved 80 digits, and Johnson [5] subsequently achieved 300 decimal places.

Sebah [6] utilized the Clenshaw–Curtis method (based on Chebyshev polynomials) to compute the Fransén–Robinson constant to over 600 digits. He also noticed the elementary fact that

$$I = \int_1^2 \frac{f(x)}{\Gamma(x)} dx,$$

where $f(x)$ is defined by the fast converging series

$$f(x) = x + \sum_{k=0}^{\infty} \left(\prod_{j=0}^{k} \frac{1}{x+j} \right) = x + e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(x+k)}$$

and $f(1) = f(2) = e$, $f(3/2) = (1 + e\sqrt{\pi}\,\mathrm{erf}(1))/2$. Using this, I is now known to 1025 digits.

Ramanujan [7, 8] observed that

$$\int_0^{\infty} \frac{w^x}{\Gamma(1+x)} dx = e^w - \int_{-\infty}^{\infty} \frac{\exp(-we^y)}{y^2+\pi^2} dy,$$

which has value 2.2665345077 ... when $w = 1$. Differentiating with respect to w gives the analogous expression that generalizes I:

$$\frac{1}{w} \int_0^{\infty} \frac{w^x}{\Gamma(x)} dx = e^w + \int_{-\infty}^{\infty} \frac{\exp(-we^y + y)}{y^2+\pi^2} dy.$$

Such formulas play a role in the computation of moments for the reciprocal gamma distribution [5, 9].

The function x^x grows even more quickly than $\Gamma(x)$ and we compute [10]

$$\int_0^{\infty} \frac{1}{x^x} dx = 1.9954559575\ldots, \quad \int_1^{\infty} \frac{1}{x^x} dx = 0.7041699604\ldots.$$

More about iterated exponentials is found in [6.11]. Reciprocal distributions could be based on the multiple Barnes functions [2.15] or generalized gamma functions [2.21] as well.

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4.7 Berry–Esseen Constant

Let $X_1, X_2, \ldots, X_n$ be independent random variables with moments

$$E(X_k) = 0, \quad E(X_k^2) = \sigma_k^2 > 0, \quad E(|X_k|^3) = \beta_k < \infty$$

for each $1 \le k \le n$. Let Φ_n be the probability distribution function of the random variable

$$X = \frac{1}{\sigma}\sum_{k=1}^{n} X_k, \quad \text{where } \sigma^2 = \sum_{k=1}^{n} \sigma_k^2.$$

Define the Lyapunov ratio

$$\lambda = \frac{\beta}{\sigma^3}, \quad \text{where } \beta = \sum_{k=1}^{n} \beta_k.$$

Let Φ denote the standard normal distribution function. Berry [1] and Esseen [2,3] proved that there exists a constant C such that

$$\sup_n \sup_{F_k} \sup_x |\Phi_n(x) - \Phi(x)| \le C\,\lambda,$$

where, for all k, F_k denotes the distribution function of X_k. The smallest such constant C has bounds [4–12]

$$0.4097321837\ldots = \frac{3+\sqrt{10}}{6\sqrt{2\pi}} \le C < 0.7915$$

under the conditions given here. If $X_1, X_2, \ldots, X_n$ are identically distributed, then the upper bound for C can be improved to 0.7655. Furthermore, there is asymptotic evidence that C is equal to the indicated lower bound.

Related studies include [13–22]. In words, the Berry–Esseen inequality quantifies the rate of convergence in the Central Limit Theorem, that is, how close the normal distribution is to the distribution of a sum of independent random variables [23–26]. Hall & Barbour [27], by way of contrast, presented an inequality that describes how far apart the two distributions must be. Another constant arises here too, but little seems to be known about it.

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4.8 Laplace Limit Constant

Given real numbers M and ε, $|\varepsilon| \leq 1$, the accurate solution of **Kepler's equation**

$$M = E - \varepsilon \sin(E)$$

is critical in celestial mechanics [1–4]. It relates the **mean anomaly** M of a planet, in elliptical orbit around the sun, to the planet's **eccentric anomaly** E and to the eccentricity ε of the ellipse. It is a transcendental equation, that is, without an algebraic solution in terms of M and ε. Computing E is a commonly-used intermediate step to the calculation of planetary position as a function of time. Therefore it is not hard to see why hundreds of mathematicians from Newton to present have devoted thought to this problem.

We will not give the orbital mechanics underlying Kepler's equation but instead give a simple geometric motivational example. Pick an arbitrary point F inside the unit circle. Let P be the point on the circle closest to F and pick another point Q elsewhere on the circle. Define E and ε as pictured in Figure 4.2. Let M be twice the area of the shaded sector PFQ. Then

$$\frac{M}{2} = (\text{area of sector } POQ) - (\text{area of triangle } FOQ) = \frac{1}{2}E - \frac{1}{2}\varepsilon \sin(E).$$

So the solution of Kepler's equation allows us to compute the angle E, given the area M and the length ε.

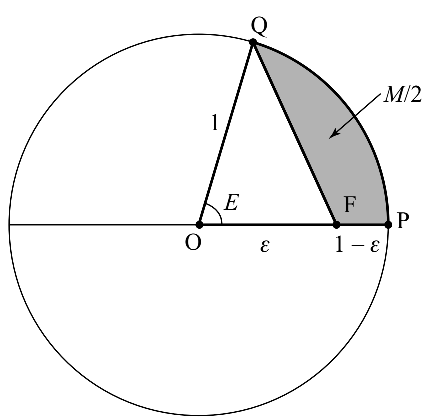

Figure 4.2. Geometric motivational example for Kepler's equation.

Kepler's equation has a unique solution, here given as a power series in ε (via the inversion method of Lagrange):

$$E = M + \sum_{n=1}^{\infty} a_n \varepsilon^n,$$

where [1, 5–7]

$$a_n = \frac{1}{n!2^{n-1}} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} (n-2k)^{n-1} \sin((n-2k)M).$$

Power series solutions as such were the preferred way to do calculations in the pre-computer nineteenth century. So it perhaps came as a shock that this series diverges for $|\varepsilon| > 0.662$ as evidently first discovered by Laplace. Arnold [8] wrote, "This plays an important part in the history of mathematics ... The investigation of the origin of this mysterious constant led Cauchy to the creation of complex analysis."

In fact, the power series for E converges like a geometric series with ratio

$$f(\varepsilon) = \frac{\varepsilon}{1+\sqrt{1+\varepsilon^2}} \exp(\sqrt{1+\varepsilon^2}).$$

The value $\lambda = 0.6627434193\ldots$ for which $f(\lambda) = 1$ is called the **Laplace limit**. A closed-form expression for λ in terms of elementary functions is not known. An infinite series or definite integral expression for λ is likewise not known.

The story does not end here. A Bessel function series for E is as follows [5, 6, 9]:

$$E = M + \sum_{n=1}^{\infty} \frac{2}{n} J_n(n\varepsilon) \sin(nM),$$

where

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(p+k)!} \left(\frac{x}{2}\right)^{p+2k}.$$

This series is better than the power series since it converges like a geometric series with ratio

$$g(\varepsilon) = \frac{\varepsilon}{1+\sqrt{1-\varepsilon^2}} \exp(\sqrt{1-\varepsilon^2}),$$

which satisfies $|g(\varepsilon)| \le 1$ for all $|\varepsilon| \le 1$.

Iterative methods, however, outperform both of these series expansion methods. Note that the function

$$T(E) = M + \varepsilon \sin(E) \quad \text{(for fixed } M \text{ and } \varepsilon)$$

is a contraction mapping; thus the method of successive approximations

$$E_0 = 0, \quad E_{i+1} = T(E_i) = M + \varepsilon \sin(E_i)$$

works well. Newton's method

$$E_0 = 0, \quad E_{i+1} = E_i + \frac{M + \varepsilon \sin(E_i) - E_i}{1 - \varepsilon \cos(E_i)}$$

converges even more quickly. Variations of these abound. Putting practicality aside, there are some interesting definite integral expressions [10–13] that solve Kepler's equation. These cannot be regarded as competitive in the race for quick accuracy, as far as is known.

An alternative representation of λ is as follows [7, 14, 15]: Let $\mu = 1.1996786402\ldots$ be the unique positive solution of $\coth(\mu) = \mu$, then $\lambda = \sqrt{\mu^2 - 1}$.

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4.9 Integer Chebyshev Constant

Consider the class P_n of all real, monic polynomials of degree n. Which nonzero member of this class deviates least from zero in the interval [0, 1]? That is, what is the solution of the following optimization problem:

$$\min_{\substack{p\in P_n \\ p\neq 0}} \max_{0\leq x\leq 1} |p(x)| = f(n)?$$

The unique answer is $p_n(x) = 2^{1-2n} T_n(2x-1)$, where [1,2]

$$T_n(x) = \cos(n\arccos(x)) = \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2}$$

and

$$\lim_{n\to\infty} f(n)^{\frac{1}{n}} = \frac{1}{4}.$$

Table 4.1. *Real Chebyshev Polynomials*

n	$p_n(x)$	$f(n)$	$f(n)^{1/n}$
1	$x-\frac{1}{2}$	$\frac{2}{4^1}=\frac{1}{2}$	0.500
2	$x^2-x+\frac{1}{8}$	$\frac{2}{4^2}=\frac{1}{8}$	0.353
3	$x^3-\frac{3}{2}x^2+\frac{9}{16}x-\frac{1}{32}$	$\frac{2}{4^3}=\frac{1}{32}$	0.314
4	$x^4-2x^3+\frac{5}{4}x^2-\frac{1}{4}x+\frac{1}{128}$	$\frac{2}{4^4}=\frac{1}{128}$	0.297
5	$x^5-\frac{5}{2}x^4+\frac{35}{16}x^3-\frac{25}{32}x^2+\frac{25}{256}x-\frac{1}{512}$	$\frac{2}{4^5}=\frac{1}{512}$	0.287

The first several polynomials $p_n(x)$, which we call **real Chebyshev polynomials** (defying tradition), are listed in Table 4.1. (The phrase "Chebyshev polynomial" is more customarily used to denote the polynomial $T_n(x)$.) In the definition of $f(n)$, note that we could just as well replace the word "monic" by the phrase "leading coefficient at least 1."

Consider instead the class Q_n of all integer polynomials of degree n, with positive leading coefficient. Again, which nonzero member of this class deviates least from zero in the interval [0, 1]? That is, what is the solution of

$$\min_{\substack{q\in Q_n\\ q\neq 0}} \max_{0\leq x\leq 1} |q(x)| = g(n)\,?$$

Clearly this is a more restrictive version of the earlier problem. Here we do not have a complete solution nor do we have uniqueness. The first several polynomials $q_n(x)$, which we call **integer Chebyshev polynomials**, are listed in Table 4.2 [3,4]. Define the **integer Chebyshev constant** (or **integer transfinite diameter** or **integer logarithmic capacity** [4.9.1]) to be

$$\chi = \lim_{n\to\infty} g(n)^{\frac{1}{n}}.$$

What can be said about χ? On the one hand, we have a lower bound [3–5]

$$\exp(-0.8657725922\ldots) = \frac{1}{2.3768417063\ldots} = 0.4207263771\ldots = \alpha \leq \chi,$$

Table 4.2. *Integer Chebyshev Polynomials*

n	$q_n(x)$	$g(n)$	$g(n)^{1/n}$
1	x or $x-1$ or $2x-1$	1	1.000
2	$x(x-1)$	$\frac{1}{4}$	0.500
3	$x(x-1)(2x-1)$	$\frac{\sqrt{3}}{18}$	0.458
4	$x^2(x-1)^2$ or $x(x-1)(2x-1)^2$ or $x(x-1)(5x^2-5x+1)$	$\frac{1}{16}$	0.500
5	$x^2(x-1)^2(2x-1)$	$\frac{\sqrt{5}}{125}$	0.447
6	$x^2(x-1)^2(2x-1)^2$	$\frac{1}{108}$	0.458
7	$x^3(x-1)^3(2x-1)$	$\frac{27\sqrt{7}}{19208}$	0.449

where

$$\alpha_0 = 2, \ \ \alpha_k = \alpha_{k-1} + \frac{1}{\alpha_{k-1}}, \ k \geq 1, \ \ \alpha = \frac{1}{2}\prod_{j=0}^{\infty}\left(1 + \frac{1}{\alpha_j^2}\right)^{-\frac{1}{2^{j+1}}}.$$

This recursion was obtained with the help of what are known as Gorshkov–Wirsing polynomials [3, 6]. It was conjectured [5] that $\chi = \alpha$ until Borwein & Erdélyi [4] proved to everyone's surprise that $\chi > \alpha$. On the other hand, we have an upper bound

$$\chi \leq \beta = 0.42347945 = \frac{1}{2.36138964} = \exp(-0.85925028)$$

due to Habsieger & Salvy [7], who succeeded in computing an integer Chebyshev polynomial for each degree up to 75. Better algorithms will be needed to find such polynomials to significantly higher degree and to determine β in this manner. By a different approach, however, Pritsker [8] recently obtained improved bounds $0.4213 < \chi < 0.4232$.

Thus far we have focused all attention on the interval [0, 1], that is, on the constant $\chi = \chi(0, 1)$. What can be said about other intervals $[a, b]$? It is known [4, 9] that

$$\chi(-1, 1)^4 = \chi(0, 1)^2 = \chi(0, \tfrac{1}{4});$$

hence the preceding bounds can be applied. The exact value of $\chi(a, b)$ for any $0 < b - a < 4$ remains an open question [4]. However, $\chi(a, b) = 1$ if $b - a \geq 4$ and $\chi(0, c) = \chi(0, 1)$ for all $1 - 0.17^2 \leq c < 1 + \varepsilon$ for some $\varepsilon > 0$, that is, $\chi(0, c)$ is locally constant at $c = 1$. Also [10], we have

$$\chi(0, 1) = \chi(1, 2) > 0.42,$$

but, from elementary considerations,

$$\chi(0, 2) \leq \tfrac{1}{\sqrt{2}} < 0.71 < 0.84 = 2(0.42);$$

that is, $\chi(0, 2)$ is not the same as either $2\chi(0, 1)$ or $\chi(0, 1) + \chi(1, 2)$. The relation $\chi(0, 1) = \chi(d, d + 1)$ also fails for non-integer d. So scaling, additivity, and translation-invariance do not hold for the integer Chebyshev case (unlike the real case).

There is an interesting connection between calculating $\chi(0, 1)$ and prime number theory [3, 5] due to Gel'fond and Schnirelmann. If it were true that $\chi = 1/e = 0.36\ldots$, then one would have a new proof of the famous Prime Number Theorem. Unfortunately, this is false (as our bounds clearly indicate).

Finally, on the interval [0, 1], Aparicio Bernardo [11] observed that integer Chebyshev polynomials $q_n(x)$ always have factors

$$x(x - 1), \ 2x - 1, \ \text{and} \ 5x^2 - 5x + 1$$

that tend to repeat and increase in power as n grows. The relative rates at which this occurs, that is, the asymptotic structure of the polynomial $q_n(x)$, gives rise to more interesting constants [4, 6, 8].

4.9.1 Transfinite Diameter

We utilized some language earlier from potential theory that deserves elaboration. Let E be a compact set in the complex plane. The **(real) transfinite diameter** or **(real) logarithmic capacity** is defined to be

$$\gamma(E) = \lim_{n\to\infty} \max_{z_1,z_2,\ldots,z_n\in E} \left(\prod_{j<k} |z_j - z_k|\right)^{\frac{2}{n(n-1)}},$$

that is, the maximal geometric mean of pairwise distances for n points in E, in the limit as $n \to \infty$. For example,

$$\gamma([0, 1]) = \frac{1}{4} = \lim_{n\to\infty} f(n)^{\frac{1}{n}},$$

and this equality is not an accidental coincidence. For arbitrary E, the phrases transfinite diameter, logarithmic capacity, and (real) Chebyshev constant are interchangeable [1, 12]. See [13–15] for sample computations. Relevant discussions of what are known as Robin constants appear in [16–18].

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5

Constants Associated with Enumerating Discrete Structures

5.1 Abelian Group Enumeration Constants

Every finite abelian group is a direct sum of cyclic subgroups. A corollary of this fundamental theorem is the following. Given a positive integer n, the number $a(n)$ of non-isomorphic abelian groups of order n is given by [1,2]

$$a(n) = P(\alpha_1)P(\alpha_2)P(\alpha_3)\cdots P(\alpha_r),$$

where $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$ is the prime factorization of n, $p_1, p_2, p_3, \ldots, p_r$ are distinct primes, each α_k is positive, and $P(\alpha_k)$ denotes the number of unrestricted partitions of α_k. For example, $a(p^4) = 5$ for any prime p since there are five partitions of 4:

$$4 = 1 + 3 = 2 + 2 = 1 + 1 + 2 = 1 + 1 + 1 + 1.$$

As another example, $a(p^4 q^4) = 25$ for any distinct primes p and q, but $a(p^8) = 22$.

It is clear that

$$\liminf_{n\to\infty} a(n) = 1,$$

but it is more difficult to see that [3–6]

$$\limsup_{n\to\infty} \ln(a(n))\frac{\ln(\ln(n))}{\ln(n)} = \frac{\ln(5)}{4}.$$

A number of authors have examined the average behavior of $a(n)$ over all positive integers. The most precise known results are [7–10]

$$\sum_{n=1}^{N} a(n) = A_1 N + A_2 N^{\frac{1}{2}} + A_3 N^{\frac{1}{3}} + O\left(N^{\frac{50}{199}+\varepsilon}\right),$$

where $\varepsilon > 0$ is arbitrarily small,

$$A_k = \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \zeta\left(\frac{j}{k}\right) = \begin{cases} 2.2948565916\ldots & \text{if } k = 1, \\ -14.6475663016\ldots & \text{if } k = 2, \\ 118.6924619727\ldots & \text{if } k = 3, \end{cases}$$

and $\zeta(x)$ is Riemann's zeta function [1.6]. We cannot help but speculate about the following estimate:

$$\sum_{n=1}^{N} a(n) \sim \sum_{k=1}^{\infty} A_k N^{\frac{1}{k}} + \Delta(N),$$

but an understanding of the error $\Delta(N)$ has apparently not yet been achieved [11, 12]. Similar enumeration results for finite semisimple associative rings appear in [5.1.1].

If, instead, focus is shifted to the sum of the reciprocals of $a(n)$, then [13, 14]

$$\sum_{n=1}^{N} \frac{1}{a(n)} = A_0 N + O\left(N^{\frac{1}{2}} \ln(N)^{-\frac{1}{2}}\right),$$

where A_0 is an infinite product over all primes p:

$$A_0 = \prod_{p} \left[1 - \sum_{k=2}^{\infty} \left(\frac{1}{P(k-1)} - \frac{1}{P(k)}\right) \frac{1}{p^k}\right] = 0.7520107423\ldots.$$

In summary, the average number of non-isomorphic abelian groups of any given order is $A_1 = 2.2948$ if "average" is understood in the sense of arithmetic mean, and $A_0^{-1} = 1.3297$ if "average" is understood in the sense of harmonic mean. We cannot even hope to obtain analogous statistics for the general (not necessarily abelian) case at present. Some interesting bounds are known [15–19] and are based on the classification theorem of finite simple groups.

The constant A_1 also appears in [20] in connection with the arithmetical properties of class numbers of quadratic fields.

Erdös & Szekeres [21, 22] examined $a(n)$ and the following generalization: $a(n, i)$ is the number of representations of n as a product (of an arbitrary number of terms, with order ignored) of factors of the form p^j, where $j \geq i$. They proved that

$$\sum_{n=1}^{N} a(n, i) = C_i N^{\frac{1}{i}} + O\left(N^{\frac{1}{i+1}}\right), \quad \text{where } C_i = \prod_{k=1}^{\infty} \zeta\left(1 + \frac{k}{i}\right),$$

and surely someone has tightened this estimate by now. See also the discussion of square-full and cube-full integers in [2.6.1].

5.1.1 Semisimple Associative Rings

A finite associative ring R with identity element $1 \neq 0$ is said to be **simple** if R has no proper (two-sided) ideals and is **semisimple** if R is a direct sum of simple ideals.

Simple rings generalize fields. Semisimple rings, in turn, generalize simple rings. Every (finite) semisimple ring is, in fact, a direct sum of full matrix rings over finite fields. Consequently, given a positive integer n, the number $s(n)$ of non-isomorphic semisimple rings of order n is given by

$$s(n) = Q(\alpha_1)Q(\alpha_2)Q(\alpha_3)\cdots Q(\alpha_r),$$

where $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$ is the prime factorization of n, $p_1, p_2, p_3, \ldots, p_r$ are distinct primes, each α_k is positive, and $Q(\alpha_k)$ denotes the number of (unordered) sets of integer pairs (r_j, m_j) for which

$$\alpha_k = \sum_j r_j m_j^2 \text{ and } r_j m_j^2 > 0 \text{ for all } j.$$

As an example, $s(p^5) = 8$ for any prime p since

$$\begin{aligned} 5 &= 1\cdot 1^2 + 1\cdot 2^2 = 5\cdot 1^2 = 2\cdot 1^2 + 3\cdot 1^2 = 1\cdot 1^2 + 4\cdot 1^2 \\ &= 1\cdot 1^2 + 1\cdot 1^2 + 3\cdot 1^2 = 1\cdot 1^2 + 2\cdot 1^2 + 2\cdot 1^2 \\ &= 1\cdot 1^2 + 1\cdot 1^2 + 1\cdot 1^2 + 2\cdot 1^2 = 1\cdot 1^2 + 1\cdot 1^2 + 1\cdot 1^2 + 1\cdot 1^2 + 1\cdot 1^2. \end{aligned}$$

Asymptotically, there are extreme results [23, 24]:

$$\liminf_{n\to\infty} s(n) = 1,$$

$$\limsup_{n\to\infty} \ln(s(n))\frac{\ln(\ln(n))}{\ln(n)} = \frac{\ln(6)}{4}$$

and average results [25–30]:

$$\sum_{n=1}^{N} s(n) = A_1B_1N + A_2B_2N^{\frac{1}{2}} + A_3B_3N^{\frac{1}{3}} + O\left(N^{\frac{50}{199}+\varepsilon}\right),$$

where $\varepsilon > 0$ is arbitrarily small, A_k is as defined in the preceding, and

$$B_k = \prod_{r=1}^{\infty}\prod_{m=2}^{\infty} \zeta\left(\frac{rm^2}{k}\right).$$

In particular, there are, on average,

$$A_1B_1 = \prod_{rm^2>1} \zeta\left(rm^2\right) = 2.4996161129\ldots$$

non-isomorphic semisimple rings of any given order ("average" in the sense of arithmetic mean).

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5.2 Pythagorean Triple Constants

The positive integers a, b, c are said to form a **primitive Pythagorean triple** if $a \le b$, $\gcd(a, b, c) = 1$, and $a^2 + b^2 = c^2$. Clearly any such triple can be interpreted geometrically as the side lengths of a right triangle with commensurable sides. Define $P_h(n)$, $P_p(n)$, and $P_a(n)$ respectively as the number of primitive Pythagorean triples whose hypotenuses, perimeters, and areas do not exceed n. D. N. Lehmer [1] showed that

$$\lim_{n\to\infty} \frac{P_h(n)}{n} = \frac{1}{2\pi}, \quad \lim_{n\to\infty} \frac{P_p(n)}{n} = \frac{\ln(2)}{\pi^2}$$

and Lambek & Moser [2] showed that

$$\lim_{n\to\infty} \frac{P_a(n)}{\sqrt{n}} = C = \frac{1}{\sqrt{2\pi^5}}\Gamma\left(\frac{1}{4}\right)^2 = 0.5313399499\ldots,$$

where $\Gamma(x)$ is the Euler gamma function [1.5.4].

What can be said about the error terms? D. H. Lehmer [3] demonstrated that

$$P_p(n) = \frac{\ln(2)}{\pi^2}n + O\left(n^{\frac{1}{2}}\ln(n)\right),$$

and Lambek & Moser [2] and Wild [4] further demonstrated that

$$P_h(n) = \frac{1}{2\pi}n + O\left(n^{\frac{1}{2}}\ln(n)\right), \quad P_a(n) = Cn^{\frac{1}{2}} - Dn^{\frac{1}{3}} + O\left(n^{\frac{1}{4}}\ln(n)\right),$$

where

$$D = -\frac{1+2^{-\frac{1}{3}}}{1+4^{-\frac{1}{3}}}\frac{\zeta(\frac{1}{3})}{\zeta(\frac{4}{3})} = 0.2974615529\ldots$$

and $\zeta(x)$ is the Riemann zeta function [1.6]. Sharper estimates for $P_a(n)$ were obtained in [5–8].

It is obvious that the hypotenuse c and the perimeter $a + b + c$ of a primitive Pythagorean triple a, b, c must both be integers. If ab was odd, then both a and b would be odd and hence $c^2 \equiv 2 \bmod 4$, which is impossible. Thus the area $ab/2$ must also be an integer. If $P_a'(n)$ is the number of primitive Pythagorean triples whose areas $\le n$ are integers, then $P_a'(n) = P_a(n)$. Such an identity does not hold for non-right triangles, of course.

A somewhat related matter is the ancient **congruent number problem** [9], the solution of which Tunnell [10] has reduced to a weak form of the Birch–Swinnerton–Dyer conjecture from elliptic curve theory. In the congruent number problem, the right triangles are permitted to have rational sides (rather than just integer sides). For a prescribed integer n, does there exist a rational right triangle with area n?

There is also the problem of enumerating **primitive Heronian triples**, equivalently, coprime integers $a \le b \le c$ that are side lengths of an *arbitrary* triangle with commensurable sides. What can be said asymptotically about the numbers $H_h(n)$, $H_p(n)$, $H_a(n)$, and $H_a'(n)$ (analogously defined)? A starting point for answering this question might be [11, 12].

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5.3 Rényi's Parking Constant

Consider the one-dimensional interval $[0, x]$ with $x > 1$. Imagine it to be a street for which parking is permitted on one side. Cars of unit length are one-by-one parked *completely at random* on the street and obviously no overlap is allowed with cars already in place. What is the mean number, $M(x)$, of cars that can fit?

Rényi [1–3] determined that $M(x)$ satisfies the following integrofunctional equation:

$$M(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 + \dfrac{2}{x-1} \displaystyle\int_0^{x-1} M(t)\,dt & \text{if } x \ge 1. \end{cases}$$

By a Laplace transform technique, Rényi proved that the limiting mean density, m, of cars in the interval $[0, x]$ is

$$m = \lim_{x\to\infty} \frac{M(x)}{x} = \int_0^\infty \beta(x)\,dx = 0.7475979202\ldots,$$

where

$$\beta(x) = \exp\left(-2\int_0^x \frac{1-e^{-t}}{t}\,dt\right) = e^{-2(\ln(x)-\mathrm{Ei}(-x)+\gamma)}, \quad \alpha(x) = m - \int_0^x \beta(t)\,dt,$$

γ is the Euler–Mascheroni constant [1.5], and Ei is the exponential integral [6.2.1]. Several alternative proofs appear in [4,5].

What can be said about the variance, $V(x)$, of the number of cars that can fit on the street? Mackenzie [6], Dvoretzky & Robbins [7], and Mannion [8,9] independently addressed this question and deduced that

$$v = \lim_{x\to\infty} \frac{V(x)}{x} = 4\int_0^\infty \left[e^{-x}(1-e^{-x})\frac{\alpha(x)}{x} - e^{-2x}(x+e^{-x}-1)\frac{\alpha(x)^2}{\beta(x)x^2} \right] dx - m$$
$$= 0.0381563991\ldots.$$

A central limit theorem holds [7], that is, the total number of cars is approximately normally distributed with mean mx and variance vx for large enough x.

It is natural to consider the parking problem in a higher dimensional setting. Consider the two-dimensional rectangle of length $x > 1$ and width $y > 1$ and imagine cars to be unit squares with sides parallel to the sides of the parking rectangle. What is the mean number, $M(x, y)$, of cars that can fit? Palasti [10–12] conjectured that

$$\lim_{x\to\infty}\lim_{y\to\infty} \frac{M(x,y)}{xy} = m^2 = (0.7475979202\ldots)^2 = 0.558902\ldots.$$

Despite some determined yet controversial attempts at analysis [13,14], the conjecture remains unproven. The mere existence of the limiting parking density was shown only recently [15]. Intensive computer simulation [16–18] suggests, however, that the conjecture is false and the true limiting value is $0.562009\ldots$.

Here is a variation in the one-dimensional setting. In Rényi's problem, a car that lands in a parking position overlapping with an earlier car is discarded. Solomon [14,19–21] studied a revised rule in which the car "rolls off" the earlier car immediately to the left or to the right, whichever is closer. It is then parked if there exists space for it; otherwise it is discarded. The mean car density is larger:

$$m = \int_0^\infty (2x+1)\exp\left[-2(x+e^{-x}-1)\right]\beta(x)\,dx = 0.8086525183\ldots$$

since cars are permitted greater flexibility to park bumper to bumper. If Rényi's problem is thought of as a model for sphere packing in a three-dimensional volume, then Solomon's variation corresponds to packing with "shaking" allowed for the spheres to settle, hence creating more space for additional spheres.

Another variation involves random car lengths [22,23]. If the left and right endpoints of the k^{th} arriving car are taken as the smaller and larger of two independent uniform draws from $[0, x]$, then the asymptotic expected number of cars successfully parked is $C \cdot k^{(\sqrt{17}-3)/4}$, where [24,25]

$$C = \left(1 - \frac{1}{2^{(\sqrt{17}-1)/4}}\right)\sqrt{\pi}\,\frac{\Gamma\left(\frac{\sqrt{17}}{2}\right)}{\Gamma\left(\frac{\sqrt{17}+1}{4}\right)\Gamma\left(\frac{\sqrt{17}+3}{4}\right)^2} = 0.9848712825\ldots$$

and Γ is the gamma function [1.5.4]. Note that x is only a scale factor in this variation and does not figure in the result.

Applications of the parking problem (or, more generally, the sequential packing or space-filling problem) include such widely separated disciplines as:

- Physics: models of liquid structure [26–29];
- Chemistry: adsorption of a fluid film on a crystal surface [5.3.1];
- Monte Carlo methods: evaluation of definite integrals [30];
- Linguistics: frequency of one-syllable, length-n English words [31];
- Sociology: models of elections in Japan and lengths of gaps generated in parking problems [32–35];
- Materials science: intercrack distance after multiple fracture of reinforced concrete [36];
- Computer science: optimal data placement on a CD [37] and linear probing hashing [38].

See also [39–41]. Note the similarities in formulation between the Golomb–Dickman constant [5.4] and the Rényi constant.

5.3.1 Random Sequential Adsorption

Consider the case in which the interval $[0, x]$ is replaced by the discrete finite linear lattice $1, 2, 3, \ldots, n$. Each car is a line segment of unit length and covers two lattice points when it parks. No car is permitted to touch points that have already been covered. The process stops when no adjacent pairs of lattice points are left uncovered. It can be proved that, as $n \to \infty$ [19,42–45],

$$m = \frac{1 - e^{-2}}{2} = 0.4323323583\ldots, \quad v = e^{-4} = 0.0183156388\ldots,$$

both of which are smaller than their continuous-case counterparts. The two-dimensional discrete analog involves unit square cars covering four lattice points, and is analytically intractable just like the continuous case. Palasti's conjecture appears to be false here too: The limiting mean density in the plane is not $m^2 = 0.186911\ldots$ but rather $0.186985\ldots$ [46–48].

For simplicity's sake, we refer to the infinite linear lattice $1, 2, 3, \ldots$ as the $1 \times \infty$ strip. The $2 \times \infty$ strip is the infinite ladder lattice with two parallel lines and crossbeams, the $3 \times \infty$ strip is likewise with three parallel lines, and naturally the $\infty \times \infty$ strip is the infinite square lattice. Thus we have closed-form expressions for m and v on $1 \times \infty$, but only numerical corrections to Palasti's estimate on $\infty \times \infty$.

If a car is a unit line segment (**dimer**) on the $2 \times \infty$ strip, then the mean car density is $\frac{1}{2}(0.91556671\ldots)$. If instead the car is on the $\infty \times \infty$ strip, then the corresponding mean density is $\frac{1}{2}(0.90682\ldots)$ [49–55]. Can exact formulas be found for these two quantities?

If the car is a line segment of length two (linear **trimer**) on the $1 \times \infty$ strip, then the mean density of vacancies is $\mu(3) = 0.1763470368\ldots$, where [6, 56–58]

$$\mu(r) = 1 - r \int_0^1 \exp\left(-2 \sum_{k=1}^{r-1} \frac{1 - x^k}{k}\right) dx.$$

More generally, $\mu(r)$ is the mean density of vacancies for linear r-mers on the $1 \times \infty$ strip, for any integer $r \geq 2$. A corresponding formula for the variance is not known.

Now suppose that the car is a single particle and that no other cars are allowed to park in any adjacent lattice points (**monomer** with **nearest neighbor exclusion**). The mean car density for the $1 \times \infty$ strip is $m_1 = \frac{1}{2}(1 - e^{-2})$ as before, of course. The mean densities for the $2 \times \infty$ and $3 \times \infty$ strips are [59–61]

$$m_2 = \frac{2 - e^{-1}}{4} = 0.4080301397\ldots, \quad m_3 = \frac{1}{3} = 0.3333333333\ldots,$$

and the corresponding density for the $\infty \times \infty$ strip is $m_\infty = 0.364132\ldots$ [47, 48, 50, 53, 55, 62]. Again, can exact formulas for m_4 or m_∞ be found?

The continuous case can be captured from the discrete case by appropriate limiting arguments [6, 58, 63]. Exhaustive surveys of random sequential adsorption models are provided in [64–66].

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5.4 Golomb–Dickman Constant

Every permutation on n symbols can be uniquely expressed as a product of disjoint cycles. For example, the permutation π on $\{0, 1, 2, \ldots, 9\}$ defined by $\pi(x) = 3x$ mod 10 has cycle structure

$$\pi = (0)\ (1\ 3\ 9\ 7)\ (2\ 6\ 8\ 4)\ (5).$$

In this case, the permutation π has $\alpha_1(\pi) = 2$ cycles of length 1, $\alpha_2(\pi) = 0$ cycles of length 2, $\alpha_3(\pi) = 0$ cycles of length 3, and $\alpha_4(\pi) = 2$ cycles of length 4. The total number $\sum_{j=1}^{\infty} \alpha_j$ of cycles in π is equal to 4 in the example.

Assume that n is fixed and that the $n!$ permutations on $\{0, 1, 2, \ldots, n-1\}$ are assigned equal probability. Picking π at random, we have the classical results [1–4]:

$$\mathrm{E}\left(\sum_{j=1}^{\infty} \alpha_j\right) = \sum_{i=1}^{n} \frac{1}{i} = \ln(n) + \gamma + O\left(\frac{1}{n}\right),$$

$$\mathrm{Var}\left(\sum_{j=1}^{\infty} \alpha_j\right) = \sum_{i=1}^{n} \frac{i-1}{i^2} = \ln(n) + \gamma - \frac{\pi^2}{6} + O\left(\frac{1}{n}\right),$$

$$\lim_{n\to\infty} \mathrm{P}(\alpha_j = k) = \frac{1}{k!} \exp\left(-\frac{1}{j}\right)\left(\frac{1}{j}\right)^k \quad \text{(asymptotic Poisson distribution)},$$

$$\lim_{n\to\infty} \mathrm{P}\left(\frac{\sum\limits_{j=1}^{\infty} \alpha_j - \ln(n)}{\sqrt{\ln(n)}} \le x\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt \quad \text{(asymptotic normal distribution)},$$

where γ is the Euler–Mascheroni constant [1.5].

What can be said about the limiting distribution of the **longest cycle** and the **shortest cycle**,

$$M(\pi) = \max\{j \ge 1 : \alpha_j > 0\}, \quad m(\pi) = \min\{j \ge 1 : \alpha_j > 0\},$$

given a random permutation π? Goncharov [1,2] and Golomb [5–7] both considered the average value of $M(\pi)$. Golomb examined the constant [8–10]

$$\lambda = \lim_{n\to\infty} \frac{\mathrm{E}(M(\pi))}{n} = 1 - \int_{1}^{\infty} \frac{\rho(x)}{x^2} dx = 0.6243299885\ldots,$$

where $\rho(x)$ is the unique continuous solution of the following delay-differential equation:

$$\rho(x) = 1 \text{ for } 0 \le x \le 1, \; x\rho'(x) + \rho(x-1) = 0 \quad \text{for } x > 1.$$

(Actually, he worked with the function $\rho(x-1)$.) Shepp & Lloyd [11] and others [6] discovered additional expressions:

$$\lambda = \int_0^\infty e^{-x+\mathrm{Ei}(-x)}\, dx = \int_0^1 e^{\mathrm{Li}(x)}\, dx = G(1,1),$$

where

$$G(a,r) = \frac{1}{a}\int_0^\infty \left(1 - \exp(a\,\mathrm{Ei}(-x)) \sum_{k=0}^{r-1} \frac{(-a)^k}{k!}\,\mathrm{Ei}(-x)^k\right) dx,$$

Ei is the exponential integral [6.2.1], and Li is the logarithmic integral [6.2.2]. Gourdon [12] determined the complete asymptotic expansion for $\mathrm{E}(M(\pi))$:

$$\begin{aligned}\mathrm{E}(M(\pi)) = \lambda n + \frac{\lambda}{2} - \frac{e^\gamma}{24}\frac{1}{n} + \left[\frac{e^\gamma}{48} - \frac{(-1)^n}{8}\right]\frac{1}{n^2} \\ + \left[\frac{17e^\gamma}{3840} + \frac{(-1)^n}{8} + \frac{e^{\frac{2(2n+1)\pi}{3}i}}{6} + \frac{e^{\frac{2(n+2)\pi}{3}i}}{6}\right]\frac{1}{n^3} + O\left(\frac{1}{n^4}\right).\end{aligned}$$

Note the periodic fluctuations involving roots of unity.

A similar integral formula for $\lim_{n\to\infty} \mathrm{Var}(M(\pi))/n^2 = 0.0369078300\ldots = H(1,1)$ holds, where [12]

$$H(a,r) = \frac{2}{a(a+1)}\int_0^\infty \left(1 - \exp(a\,\mathrm{Ei}(-x)) \sum_{k=0}^{r-1} \frac{(-a)^k}{k!}\,\mathrm{Ei}(-x)^k\right) x\, dx - G(a,r)^2.$$

We will need values of $G(a,r)$ and $H(a,r)$, $a \neq 1 \neq r$, later in this essay. An analog of λ appears in [13, 14] in connection with polynomial factorization.

The arguments leading to asymptotic average values of $m(\pi)$ are more complicated. Shepp & Lloyd [11] proved that

$$\lim_{n\to\infty} \frac{\mathrm{E}(m(\pi))}{\ln(n)} = e^{-\gamma} = 0.5614594835\ldots$$

as well as formulas for higher moments. A complete asymptotic expansion for $\mathrm{E}(m(\pi))$, however, remains open.

The mean and variance of the r^{th} longest cycle (normalized by n and n^2, as $n \to \infty$) are given by $G(1,r)$ and $H(1,r)$. For example, $G(1,2) = 0.2095808742\ldots$, $H(1,2) = 0.0125537906\ldots$ and $G(1,3) = 0.0883160988\ldots$, $H(1,3) = 0.0044939231\ldots$ [11, 12].

There is a fascinating connection between λ and prime factorization algorithms [15, 16]. Let $f(n)$ denote the largest prime factor of n. By choosing a random integer

n between 1 and N, Dickman [17–20] determined that

$$\lim_{N\to\infty} \mathrm{P}(f(n) \le n^x) = \rho(\tfrac{1}{x})$$

for $0 < x \le 1$. With this in mind, what is the average value of x such that $f(n) = n^x$? Dickman obtained numerically that

$$\mu = \lim_{N\to\infty} \mathrm{E}(x) = \lim_{N\to\infty} \mathrm{E}\left(\frac{\ln(f(n))}{\ln(n)}\right) = \int_0^1 x\, d\rho\left(\frac{1}{x}\right) = 1 - \int_1^\infty \frac{\rho(y)}{y^2} dy = \lambda,$$

which is indeed surprising! Dickman's constant μ and Golomb's constant λ are identical! Knuth & Trabb Pardo [15] described this result as follows: λn is the *asymptotic average number of digits* in the largest prime factor of an n-digit number. More generally, if we are factoring a random n-digit number, the distribution of digits in its prime factors is approximately the same as the distribution of the cycle lengths in a random permutation on n elements. This remarkable and unexpected fact is explored in greater depth in [21, 22].

Other asymptotic formulas involving the largest prime factor function $f(n)$ include [15, 23, 24]

$$\mathrm{E}(f(n)^k) \sim \frac{\zeta(k+1)}{k+1} \frac{N^k}{\ln(N)}, \quad \mathrm{E}(\ln(f(n))) \sim \lambda \ln(N) - \lambda(1-\gamma),$$

where $\zeta(x)$ is the zeta function [1.6]. See also [25–29]. Note the curious coincidence [15] involving integral and sum:

$$\int_0^\infty \rho(x)\, dx = e^\gamma = \sum_{n=1}^\infty n\rho(n).$$

Dickman's function is important in the study of y-smooth numbers [24, 30–32], that is, integers whose prime divisors never exceed y. It appears in probability theory as the density function (normalized by e^γ) of [33, 34]

$$X_1 + X_1 X_2 + X_1 X_2 X_3 + \cdots, \quad X_j \text{ independent uniform random variables on } [0, 1].$$

See [35–40] for other applications of $\rho(x)$. A closely-allied function, due to Buchstab, satisfies [24, 34, 41–45]

$$\omega(x) = \frac{1}{x} \text{ for } 1 \le x \le 2, \; x\omega'(x) + \omega(x) - \omega(x-1) = 0 \text{ for } x > 2,$$

which arises when estimating the frequency of integers n whose *smallest* prime factor $\ge n^x$. Both functions are positive everywhere, and special values include [46]

$$\rho(\tfrac{3+\sqrt{5}}{2}) = 1 - \ln(\tfrac{3+\sqrt{5}}{2}) + \ln(\tfrac{1+\sqrt{5}}{2})^2 - \tfrac{\pi^2}{60}, \qquad \lim_{x\to\infty} \rho(x) = 0,$$
$$\tfrac{5+\sqrt{5}}{2}\omega(\tfrac{5+\sqrt{5}}{2}) = 1 + \ln(\tfrac{3+\sqrt{5}}{2}) + \ln(\tfrac{1+\sqrt{5}}{2})^2 - \tfrac{\pi^2}{60}, \quad \lim_{x\to\infty} \omega(x) = e^{-\gamma}.$$

Whereas $\rho(x)$ is nonincreasing, the difference $\omega(x) - e^{-\gamma}$ changes sign (at most twice) in every interval of length 1. Its oscillatory behavior plays a role in understanding irregularities in the distribution of primes.

Note the similarity in formulation between the Golomb–Dickman constant and Rényi's parking constant [5.3].

5.4.1 Symmetric Group

Here are several related questions. Given π, a permutation on n symbols, define its **order** $\theta(\pi)$ to be the least positive integer m such that $\pi^m =$ identity. Clearly $1 \le \theta(\pi) \le n!$. What is its mean value, $\mathrm{E}(\theta(\pi))$? Goh & Schmutz [47], building upon the work of Erdös & Turán [48], proved that

$$\ln(\mathrm{E}(\theta(\pi))) = B\sqrt{\frac{n}{\ln(n)}} + o(1),$$

where $B = 2\sqrt{2b} = 2.9904703993\ldots$ and

$$b = \int_0^\infty \ln\left(1 - \ln(1 - e^{-x})\right) dx = 1.1178641511\ldots.$$

Stong [49] improved the $o(1)$ estimate and gave alternative representations for b:

$$b = \int_0^\infty \frac{xe^{-x}}{(1 - e^{-x})(1 - \ln(1 - e^{-x}))}\, dx = \int_0^\infty \frac{\ln(x+1)}{e^x - 1}\, dx = -\sum_{k=1}^\infty \frac{e^k}{k}\,\mathrm{Ei}(-k).$$

A typical permutation π can be shown to satisfy $\ln(\theta(\pi)) \sim \frac{1}{2}\ln(n)^2$; hence a few exceptional permutations contribute significantly to the mean. What can be said about the variance of $\theta(\pi)$?

Also, define $g(n)$ to be the maximum order $\theta(\pi)$ of all n-permutations π. Landau [50, 51] proved that $\ln(g(n)) \sim \sqrt{n\ln(n)}$, and greatly refined estimates of $g(n)$ appeared in [52].

A natural equivalence relation can be defined on the symmetric group S_n via conjugacy. In the limit as $n \to \infty$, for almost all conjugacy classes C, the elements of C have order equal to $\exp(\sqrt{n}(A + o(1)))$, where [48, 53, 54]

$$A = \frac{2\sqrt{6}}{\pi}\sum_{j\neq 0}\frac{(-1)^{j+1}}{3j^2 + j} = 4\sqrt{2} - \frac{6\sqrt{6}}{\pi}.$$

Note that the summation involves reciprocals of nonzero pentagonal numbers.

Let s_n denote the probability that two elements of the symmetric group, S_n, chosen at random (with replacement) actually generate S_n. The first several values are $s_1 = 1$, $s_2 = 3/4$, $s_3 = 1/2$, $s_4 = 3/8, \ldots$ [55]. What can be said about the asymptotics of s_n? Dixon [56] proved an 1892 conjecture by Netto [57] that $s_n \to 3/4$ as $n \to \infty$. Babai [58] gave a more refined estimate.

5.4.2 Random Mapping Statistics

We now generalize the discussion from permutations (bijective functions) on n symbols to arbitrary mappings on n symbols. For example, the function φ on $\{0, 1, 2, \ldots, 9\}$

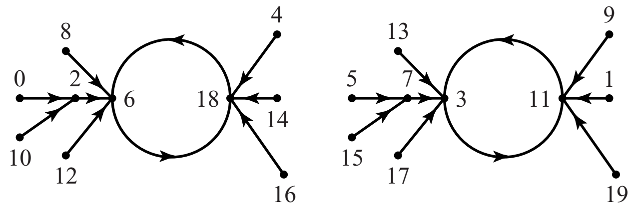

Figure 5.1. The functional graph for $\psi(x) = x^2 + 2 \bmod 20$ has two components, each containing a cycle of length 2.

defined by $\varphi(x) = 2x$ mod 10 has cycles (0) and (2 4 8 6). The remaining symbols 1, 3, 5, 7, and 9 are transient in the sense that if one starts with 3, one is absorbed into the cycle (2 4 8 6) and never returns to 3. We can nevertheless define cycle lengths α_j as before; in this simple example, $\alpha_1(\varphi) = 1$, $\alpha_2(\varphi) = \alpha_3(\varphi) = 0$, and $\alpha_4(\varphi) = 1$.

The lengths of the longest and shortest cycles, $M(\varphi)$ and $m(\varphi)$, are clearly of interest in pseudo-random number generation. Purdom & Williams [59–61] found that

$$\lim_{n\to\infty} \frac{\mathrm{E}(M(\varphi))}{\sqrt{n}} = \lambda\sqrt{\frac{\pi}{2}} = 0.7824816009\ldots, \quad \lim_{n\to\infty} \frac{\mathrm{E}(m(\varphi))}{\ln(n)} = \frac{1}{2}e^{-\gamma}.$$

Observe that $\mathrm{E}(M(\varphi))$ grows on the order of only $\sqrt{n}$ rather than n as earlier.

As another example, consider the function ψ on $\{0, 1, 2, \ldots, 19\}$ defined by $\psi(x) = x^2 + 2$ mod 20. From Figure 5.1, clearly $\alpha_2(\psi) = 2$. Here are other interesting quantities [62]. Note that the transient symbols 0, 5, 10, and 15 each require 2 steps to reach a cycle, and this is the maximum such distance. Thus define the **longest tail** $L(\psi) = 2$. Note also that 4 is the number of vertices in the nonrepeating trajectory for each of 0, 5, 10, and 15, and this is the maximum such length. Thus define the **longest rho-path** $R(\psi) = 4$. Clearly, for the earlier example, $L(\varphi) = 1$ and $R(\varphi) = 5$. It can be proved that, for arbitrary n-mappings φ [61],

$$\lim_{n\to\infty} \frac{\mathrm{E}(L(\varphi))}{\sqrt{n}} = \sqrt{2\pi}\ln(2) = 1.7374623212\ldots,$$

$$\lim_{n\to\infty} \frac{\mathrm{E}(R(\varphi))}{\sqrt{n}} = \sqrt{\frac{\pi}{2}} \int_0^\infty \left(1 - e^{\mathrm{Ei}(-x) - I(x)}\right) dx = 2.4149010237\ldots,$$

where

$$I(x) = \int_0^x \frac{e^{-y}}{y}\left(1 - \exp\left(\frac{-2y}{e^{x-y} - 1}\right)\right) dy.$$

Another quantity associated with a mapping φ is the **largest tree** $P(\varphi)$. Each vertex in each cycle of φ is the root of a unique maximal tree [5.6]. Select the tree with the greatest number of vertices, and call this number $P(\varphi)$. For the two examples, clearly

$P(\varphi) = 2$ and $P(\psi) = 6$. It is known that, for arbitrary n-mappings φ [12, 61],

$$\nu = \lim_{n\to\infty} \frac{\mathrm{E}(P(\varphi))}{n} = 2\int_0^\infty \left[1 - (1 - F(x))^{-1}\right] dx = 0.4834983471\ldots,$$

$$\lim_{n\to\infty} \frac{\mathrm{Var}(P(\varphi))}{n^2} = \frac{8}{3}\int_0^\infty \left[1 - (1 - F(x))^{-1}\right] x\, dx - \nu^2 = 0.0494698522\ldots,$$

where

$$F(x) = \frac{-1}{2\sqrt{\pi}} \int_x^\infty e^{-t} t^{-\frac{3}{2}}\, dt = 1 - \frac{1}{\sqrt{\pi x}} \exp(-x) - \mathrm{erf}(\sqrt{x})$$

and erf is the error function [4.6]. Gourdon [63] mentioned a coin-tossing game, the analysis of which yields the preceding two constants.

Finally, let us examine the connected component structure of a mapping. We have come full circle, in a sense, because components relate to mappings as cycles relate to permutations. For the two examples, the counting function is $\beta_2(\varphi) = 1$, $\beta_8(\varphi) = 1$ while $\beta_{10}(\psi) = 2$. In the interest of analogy, here are more details. The total number $\sum_{j=1}^\infty \beta_j$ of components is equal to 2 in both cases. Picking φ at random, we have [64–67]

$$\mathrm{E}\left(\sum_{j=1}^\infty \beta_j\right) = \sum_{i=1}^n c_{n,0,i} = \frac{1}{2}\ln(n) + \frac{1}{2}(\ln(2) + \gamma) + o(1),$$

$$\begin{aligned}\mathrm{Var}\left(\sum_{j=1}^\infty \beta_j\right) &= \sum_{i=1}^n c_{n,0,i} - \left(\sum_{i=1}^n c_{n,0,i}\right)^2 + \sum_{i=1}^n c_{n,0,i} \sum_{j=1}^{n-i} c_{n,i,j}\\ &= \frac{1}{2}\ln(n) + o(\ln(n)),\end{aligned}$$

$$\lim_{n\to\infty} \mathrm{P}(\beta_j = k) = \frac{1}{k!}\exp(-d_j)\, d_j^k, \quad \text{(asymptotic Poisson distribution)},$$

where

$$c_{n,p,q} = \binom{n-p}{q}\frac{(q-1)!}{n^q}, \quad d_j = \frac{e^{-j}}{j}\sum_{i=0}^{j-1}\frac{j^i}{i!},$$

and a corresponding Gaussian limit also holds. Define the **largest component** $Q(\varphi) = \max\{j \geq 1 : \beta_j > 0\}$; then [12, 61, 68]

$$\lim_{n\to\infty} \frac{\mathrm{E}(Q(\varphi))}{n} = G(\tfrac{1}{2}, 1) = 0.7578230112\ldots,$$

$$\lim_{n\to\infty} \frac{\mathrm{Var}(Q(\varphi))}{n^2} = H(\tfrac{1}{2}, 1) = 0.0370072165\ldots.$$

Such results answer questions raised in [69–71]. It seems fitting to call 0.75782... the **Flajolet–Odlyzko constant**, owing to its importance. The mean and variance of the r^{th} largest component (again normalized by n and n^2, as $n \to \infty$) are given by $G(\frac{1}{2}, r)$ and $H(\frac{1}{2}, r)$. For example, $G(\frac{1}{2}, 2) = 0.1709096198\ldots$ and $H(\frac{1}{2}, 2) = 0.0186202233\ldots$. A discussion of smallest components appears in [72].

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5.5 Kalmár's Composition Constant

An **additive composition** of an integer n is a sequence $x_1, x_2, \ldots, x_k$ of integers (for some $k \geq 1$) such that

$$n = x_1 + x_2 + \cdots + x_k, \quad x_j \geq 1 \text{ for all } 1 \leq j \leq k.$$

A **multiplicative composition** of n is the same except

$$n = x_1 x_2 \cdots x_k, \;\; x_j \geq 2 \text{ for all } 1 \leq j \leq k.$$

The number $a(n)$ of additive compositions of n is trivially 2^{n-1}. The number $m(n)$ of multiplicative compositions does not possess a closed-form expression, but asymptotically satisfies

$$\sum_{n=1}^{N} m(n) \sim \frac{-1}{\rho \zeta'(\rho)} N^{\rho} = (0.3181736521\ldots) \cdot N^{\rho},$$

where $\rho = 1.7286472389\ldots$ is the unique solution of $\zeta(x) = 2$ with $x > 1$ and $\zeta(x)$ is Riemann's zeta function [1.6]. This result was first deduced by Kalmár [1,2] and refined in [3–8].

An **additive partition** of an integer n is a sequence $x_1, x_2, \ldots, x_k$ of integers (for some $k \geq 1$) such that

$$n = x_1 + x_2 + \cdots + x_k, \;\; 1 \leq x_1 \leq x_2 \leq \cdots \leq x_k.$$

Partitions naturally represent equivalence classes of compositions under sorting. The number $A(n)$ of additive partitions of n is mentioned in [1.4.2], while the number $M(n)$ of **multiplicative partitions** asymptotically satisfies [9, 10]

$$\sum_{n=1}^{N} M(n) \sim \frac{1}{2\sqrt{\pi}} N \exp\left(2\sqrt{\ln(N)}\right) \ln(N)^{-\frac{3}{4}}.$$

Thus far we have dealt with *unrestricted* compositions and partitions. Of many possible variations, let us focus on the case in which each x_j is restricted to be a prime number. For example, the number $M_{\mathrm{p}}(n)$ of **prime multiplicative partitions** is trivially 1 for $n \geq 2$. The number $a_{\mathrm{p}}(n)$ of **prime additive compositions** is [11]

$$a_{\mathrm{p}}(n) \sim \frac{1}{\xi f'(\xi)} \left(\frac{1}{\xi}\right)^{n} = (0.3036552633\ldots) \cdot (1.4762287836\ldots)^{n},$$

where $\xi = 0.6774017761\ldots$ is the unique solution of the equation

$$f(x) = \sum_{p} x^{p} = 1, \;\; x > 0,$$

and the sum is over all primes p. The number $m_{\mathrm{p}}(n)$ of **prime multiplicative compositions** satisfies [12]

$$\sum_{n=1}^{N} m_{\mathrm{p}}(n) \sim \frac{-1}{\eta g'(\eta)} N^{-\eta} = (0.4127732370\ldots) \cdot N^{-\eta},$$

where $\eta = -1.3994333287\ldots$ is the unique solution of the equation

$$g(y) = \sum_{p} p^{y} = 1, \;\; y < 0.$$

Not much is known about the number $A_{\mathrm{p}}(n)$ of **prime additive partitions** [13–16] except that $A_{\mathrm{p}}(n+1) > A_{\mathrm{p}}(n)$ for $n \geq 8$.

Here is a related, somewhat artificial topic. Let p_n be the n^{th} prime, with $p_1 = 2$, and define formal series

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad Q(z) = \frac{1}{P(z)} = \sum_{n=0}^{\infty} q_n z^n.$$

Some people may be surprised to learn that the coefficients q_n obey the following asymptotics [17]:

$$q_n \sim \frac{1}{\theta P'(\theta)} \left(\frac{1}{\theta}\right)^n = (-0.6223065745\ldots) \cdot (-1.4560749485\ldots)^n.$$

where $\theta = -0.6867778344\ldots$ is the unique zero of $P(z)$ inside the disk $|z| < 3/4$. By way of contrast, $p_n \sim n \ln(n)$ by the Prime Number Theorem. In a similar spirit, consider the coefficients c_k of the $(n-1)^{\text{st}}$ degree polynomial fit

$$c_0 + c_1(x-1) + c_2(x-1)(x-2) + \cdots + c_{n-1}(x-1)(x-2)(x-3)\cdots(x-n+1)$$

to the dataset [18]

$$(1,2), (2,3), (3,5), (4,7), (5,11), (6,13), \ldots, (n, p_n).$$

In the limit as $n \to \infty$, the sum $\sum_{k=0}^{n-1} c_k$ converges to $3.4070691656\ldots$.

Let us return to the counting of compositions and partitions, and merely mention variations in which each x_j is restricted to be square-free [12] or where the xs must be distinct [8]. Also, compositions/partitions $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_l$ of n are said to be **independent** if proper subsequence sums/products of xs and ys never coincide. How many such pairs are there (as a function of n)? See [19] for an asymptotic answer.

Cameron & Erdös [20] pointed out that the number of sequences $1 \leq z_1 < z_2 < \cdots < z_k = n$ for which $z_i | z_j$ whenever $i < j$ is $2m(n)$. The factor 2 arises because we can choose whether or not to include 1 in the sequence. What can be said about the number $c(n)$ of sequences $1 \leq w_1 < w_2 < \cdots < w_k \leq n$ for which $w_i \nmid w_j$ whenever $i \neq j$? It is conjectured that $\lim_{n\to\infty} c(n)^{1/n}$ exists, and it is known that $1.55967^n \leq c(n) \leq 1.59^n$ for sufficiently large n. For more about such sequences, known as **primitive sequences**, see [2.27].

Finally, define $h(n)$ to be the number of ways to express 1 as a sum of $n+1$ elements of the set $\{2^{-i} : i \geq 0\}$, where repetitions are allowed and order is immaterial. Flajolet & Prodinger [21] demonstrated that

$$h(n) \sim (0.2545055235\ldots)\kappa^n,$$

where $\kappa = 1.7941471875\ldots$ is the reciprocal of the smallest positive root x of the equation

$$\sum_{j=1}^{\infty} (-1)^{j+1} \frac{x^{2^{j+1}-2-j}}{(1-x)(1-x^3)(1-x^7)\cdots(1-x^{2^j-1})} - 1 = 0.$$

This is connected to enumerating level number sequences associated with binary trees [5.6].

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5.6 Otter's Tree Enumeration Constants

A **graph** of order n consists of a set of n **vertices** (points) together with a set of **edges** (unordered pairs of distinct points). Note that loops and multiple parallel edges are automatically disallowed. Two vertices joined by an edge are called **adjacent**.

A **forest** is a graph that is **acyclic**, meaning that there is no sequence of adjacent vertices $v_0, v_1, \ldots, v_m$ such that $v_i \neq v_j$ for all $i < j < m$ and $v_0 = v_m$.

A **tree** (or **free tree**) is a forest that is **connected**, meaning that for any two distinct vertices u and w, there is a sequence of adjacent vertices $v_0, v_1, \ldots, v_m$ such that $v_0 = u$ and $v_m = w$.

Two trees σ and τ are **isomorphic** if there is a one-to-one map from the vertices of σ to the vertices of τ that preserves adjacency (see Figure 5.2). Diagrams for all non-isomorphic trees of order < 11 appear in [1]. Applications are given in [2].

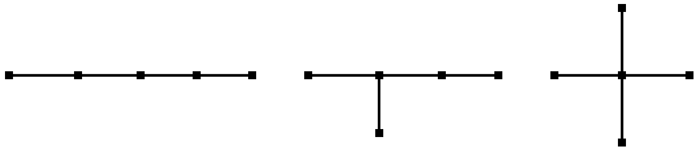

Figure 5.2. There exist three non-isomorphic trees of order 5.

What can be said about the asymptotics of t_n, the number of non-isomorphic trees of order n? Building upon the work of Cayley and Pólya, Otter [3–6] determined that

$$\lim_{n\to\infty} \frac{t_n n^{\frac{5}{2}}}{\alpha^n} = \beta,$$

where $\alpha = 2.9557652856\ldots = (0.3383218568\ldots)^{-1}$ is the unique positive solution of the equation $T(x^{-1}) = 1$ involving a certain function T to be defined shortly, and

$$\beta = \frac{1}{\sqrt{2\pi}}\left(1 + \sum_{k=2}^{\infty} \frac{1}{\alpha^k} T'\left(\frac{1}{\alpha^k}\right)\right)^{\frac{3}{2}} = 0.5349496061\ldots$$

where T' denotes the derivative of T. Although α and β can be calculated efficiently to great accuracy, it is not known whether they are algebraic or transcendental [6, 7].

A **rooted tree** is a tree in which precisely one vertex, called the **root**, is distinguished from the others (see Figure 5.3). We agree to draw the root as a tree's topmost vertex and that an isomorphism of rooted trees maps a root to a root. What can be said about the asymptotics of T_n, the number of non-isomorphic rooted trees of order n? Otter's corresponding result is

$$\lim_{n\to\infty} \frac{T_n n^{\frac{3}{2}}}{\alpha^n} = \left(\frac{\beta}{2\pi}\right)^{\frac{1}{3}} = 0.4399240125\ldots = \left(\frac{1}{4\pi\alpha}\right)^{\frac{1}{2}}(2.6811281472\ldots).$$

In fact, the generating functions

$$\begin{aligned} t(x) &= \sum_{n=1}^{\infty} t_n x^n \\ &= x + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + 11x^7 + 23x^8 + 47x^9 + 106x^{10} + \cdots, \end{aligned}$$

$$\begin{aligned} T(x) &= \sum_{n=1}^{\infty} T_n x^n \\ &= x + x^2 + 2x^3 + 4x^4 + 9x^5 + 20x^6 + 48x^7 + 115x^8 + 286x^9 + \cdots \end{aligned}$$

are related by the formula $t(x) = T(x) - \frac{1}{2}(T(x)^2 - T(x^2))$, the constant α^{-1} is the radius of convergence for both, and the coefficients T_n can be computed using

$$T(x) = x\exp\left(\sum_{k=1}^{\infty} \frac{T(x^k)}{k}\right), \quad T_{n+1} = \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{d|k} dT_d\right) T_{n-k+1}.$$

There are many varieties of trees and the elaborate details of enumerating them are best left to [4, 5]. Here is the first of many examples. A **weakly binary tree** is a rooted

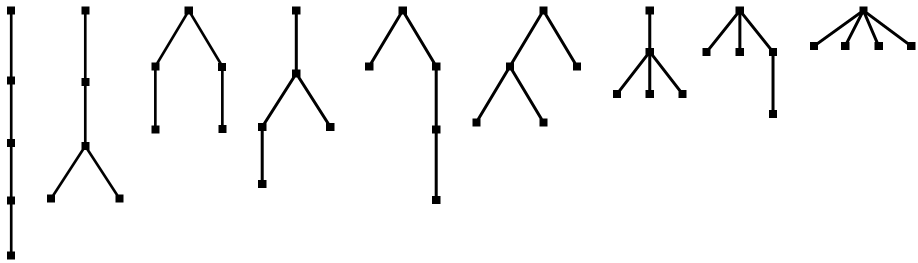

Figure 5.3. There exist nine non-isomorphic rooted trees of order 5.

tree for which the root is adjacent to at most two vertices and all non-root vertices are adjacent to at most three vertices. For instance, there exist six non-isomorphic weakly binary trees of order 5. The asymptotics of B_n, the number of non-isomorphic weakly binary trees of order n, were obtained by Otter [3, 8–10]:

$$\lim_{n\to\infty} \frac{B_n n^{\frac{3}{2}}}{\xi^n} = \eta,$$

where $\xi^{-1} = 0.4026975036\ldots = (2.4832535361\ldots)^{-1}$ is the radius of convergence for

$$\begin{aligned} B(x) &= \sum_{n=0}^{\infty} B_n x^n \\ &= 1 + x + x^2 + 2x^3 + 3x^4 + 6x^5 + 11x^6 + 23x^7 + 46x^8 + 98x^9 + \cdots \end{aligned}$$

and

$$\begin{aligned} \eta &= \sqrt{\frac{\xi}{2\pi}} \left(1 + \frac{1}{\xi} B(\frac{1}{\xi^2}) + \frac{1}{\xi^3} B'(\frac{1}{\xi^2})\right)^{\frac{1}{2}} \\ &= 0.7916031835\ldots = (0.3187766258\ldots)\xi. \end{aligned}$$

The series coefficients arise from

$$B(x) = 1 + \frac{1}{2}x\left(B(x)^2 + B(x^2)\right),$$

$$B_k = \begin{cases} \dfrac{B_i(B_i+1)}{2} + \displaystyle\sum_{j=0}^{i-1} B_{k-j-1}B_j & \text{if } k = 2i+1, \\ \displaystyle\sum_{j=0}^{i-1} B_{k-j-1}B_j & \text{if } k = 2i. \end{cases}$$

Otter showed, in this special case, that $\xi = \lim_{n\to\infty} c_n^{2^{-n}}$, where the sequence $\{c_n\}$ obeys the quadratic recurrence

$$c_0 = 2, \qquad c_n = c_{n-1}^2 + 2 \quad \text{for } n \ge 1,$$

and consequently

$$\eta = \frac{1}{2}\sqrt{\frac{\xi}{\pi}}\sqrt{3 + \frac{1}{c_1} + \frac{1}{c_1c_2} + \frac{1}{c_1c_2c_3} + \frac{1}{c_1c_2c_3c_4} + \cdots}.$$

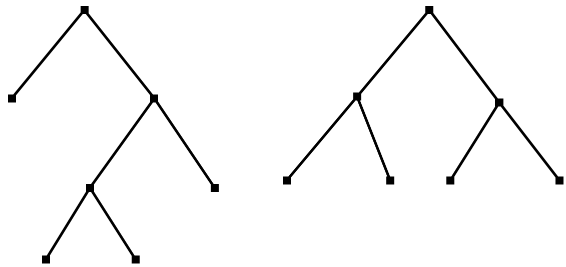

Figure 5.4. There exist two non-isomorphic strongly binary trees of order 7.

Here is a slight specialization of the preceding. Define a **strongly binary tree** to be a rooted tree for which the root is adjacent to either zero or two vertices, and all non-root vertices are adjacent to either one or three vertices (see Figure 5.4). These trees, also called **binary trees**, are discussed further in [5.6.9] and [5.13]. The number of non-isomorphic strongly binary trees of order $2n+1$ turns out to be exactly B_n. The one-to-one correspondence is obtained, in the forward direction, by deleting all the **leaves** (terminal nodes) of a strongly binary tree. To go in reverse, starting with a weakly binary tree, add two leaves to any vertex of degree 1 (or to the root if it has degree 0), and add one leaf to any vertex of degree 2 (or to the root if it has degree 1). Hence the same asymptotics apply in both weak and strong cases.

Also, in a commutative non-associative algebra, the expression x^4 is ambiguous and could be interpreted as xx^3 or x^2x^2. The expression x^5 likewise could mean xxx^3, xx^2x^2, or x^2x^3. Clearly B_{n-1} is the number of possible interpretations of x^n; thus $\{B_n\}$ is sometimes called the Wedderburn-Etherington sequence [11–15].

5.6.1 Chemical Isomers

A **weakly ternary tree** is a rooted tree for which the root is adjacent to at most three vertices and all non-root vertices are adjacent to at most four vertices. For instance, there exist eight non-isomorphic weakly ternary trees of order 5. The asymptotics of R_n, the number of non-isomorphic weakly ternary trees of order n, were again obtained by Otter [3, 15–17]:

$$\lim_{n\to\infty} \frac{R_n n^{\frac{3}{2}}}{\xi_R^n} = \eta_R,$$

where $\xi_R^{-1} = 0.3551817423\ldots = (2.8154600332\ldots)^{-1}$ is the radius of convergence for

$$\begin{aligned} R(x) &= \sum_{n=0}^{\infty} R_n x^n \\ &= 1 + x + x^2 + 2x^3 + 4x^4 + 8x^5 + 17x^6 + 39x^7 + 89x^8 + 211x^9 + \cdots, \\ \eta_R &= \sqrt{\frac{\xi_R}{2\pi}} \left(-1 + \rho + \frac{1}{\xi_R^3} R'\left(\frac{1}{\xi_R^2}\right) \rho + \frac{1}{\xi_R^4} R'\left(\frac{1}{\xi_R^3}\right) \right)^{\frac{1}{2}} \rho^{-\frac{1}{2}} \\ &= 0.5178759064\ldots, \end{aligned}$$

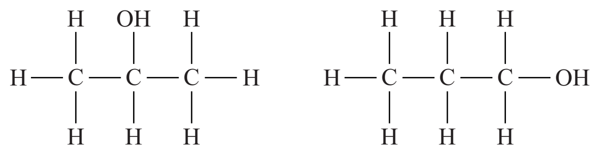

Figure 5.5. The formula C_3H_7OH (propanol) has two isomers.

and $\rho = R(\xi_R^{-1})$. The series coefficients arise from

$$R(x) = 1 + \frac{1}{6}x\left(R(x)^3 + 3R(x)R(x^2) + 2R(x^3)\right).$$

An application of this material involves organic chemistry [18–21]: R_n is the number of **constitutional isomers** of the molecular formula $C_nH_{2n+1}OH$ (alcohols – see Figure 5.5). Constitutional isomeric pairs differ in their atomic connectivity, but the relative positioning of the OH group is immaterial.

Further, if we define [18, 19, 22, 23]

$$\begin{aligned} r(x) = {} & \frac{1}{24}x\left(R(x)^4 + 6R(x)^2R(x^2) + 8R(x)R(x^3) + 3R(x^2)^2 + 6R(x^4)\right) \\ & - \frac{1}{2}\left(R(x)^2 - R(x^2)\right) + R(x) \end{aligned}$$

then

$$\begin{aligned} r(x) &= \sum_{n=0}^{\infty} r_n x^n \\ &= 1 + x + x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 + 9x^7 + 18x^8 + 35x^9 + 75x^{10} + \cdots \end{aligned}$$

and r_n is the number of constitutional isomers of the molecular formula C_nH_{2n+2} (alkanes – see Figure 5.6). The series $r(x)$ is related to $R(x)$ as $t(x)$ is related to $T(x)$ (in the sense that r, t are free and R, T are rooted); its radius of convergence is likewise ξ_R^{-1} and

$$\lim_{n\to\infty} \frac{r_n n^{\frac{5}{2}}}{\xi_R^n} = 2\pi \frac{\eta_R^3}{\xi_R}\rho = 0.6563186958\ldots.$$

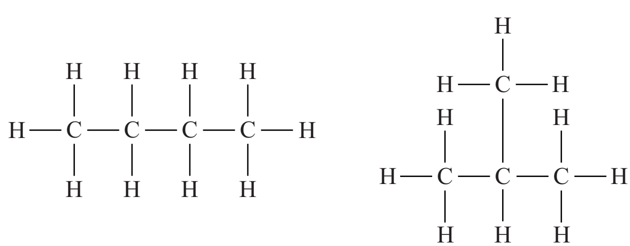

Figure 5.6. The formula C_4H_{10} (butane) has two isomers.

A carbon atom is **chiral** or **asymmetric** if it is attached to four distinct substituents (atoms or groups). If Q_n is the number of constitutional isomers of $C_nH_{2n+1}OH$ without chiral C atoms, then [18, 24]

$$\lim_{n\to\infty} \frac{Q_n}{\xi_Q^n} = \eta_Q,$$

where $\xi_Q^{-1} = 0.5947539639\ldots = (1.6813675244\ldots)^{-1}$ is the radius of convergence for

$$\begin{aligned} Q(x) &= \sum_{n=0}^{\infty} Q_n x^n \\ &= 1 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 14x^7 + 23x^8 + 39x^9 + \cdots. \end{aligned}$$

The coefficients arise from $Q(x) = 1 + x\,Q(x)Q(x^2)$, so that

$$Q(x) = \frac{1|}{|1} - \frac{x|}{|1} - \frac{x^2|}{|1} - \frac{x^4|}{|1} - \frac{x^8|}{|1} - \frac{x^{16}|}{|1} - \cdots,$$

which is an interesting continued fraction. From this, it easily follows that $Q(x) = \psi(x^2)/\psi(x)$ uniquely (assuming ψ is analytic and $\psi(0) = 1$) and hence

$$\eta_Q = -\xi_Q\, \psi\left(\frac{1}{\xi_Q^2}\right)\left(\psi'\left(\frac{1}{\xi_Q}\right)\right)^{-1} = 0.3607140971\ldots.$$

Let S_n denote the number of **stereoisomers** of $C_nH_{2n+1}OH$. The relative positioning of the hydroxyl group now matters as well [18, 19, 25]; for instance, the illustrated stereoisomeric pair (represented by two tetrahedra – see Figure 5.7) are non-superimposable. The generating function for S_n is

$$\begin{aligned} S(x) &= \sum_{n=0}^{\infty} S_n x^n \\ &= 1 + x + x^2 + 2x^3 + 5x^4 + 11x^5 + 28x^6 + 74x^7 + 199x^8 + 551x^9 + \cdots, \end{aligned}$$

$$S(x) = 1 + \frac{1}{3}x\left(S(x)^3 + 2S(x^3)\right), \quad S_n \sim \eta_S\, n^{-\frac{3}{2}}\, \xi_S^n,$$

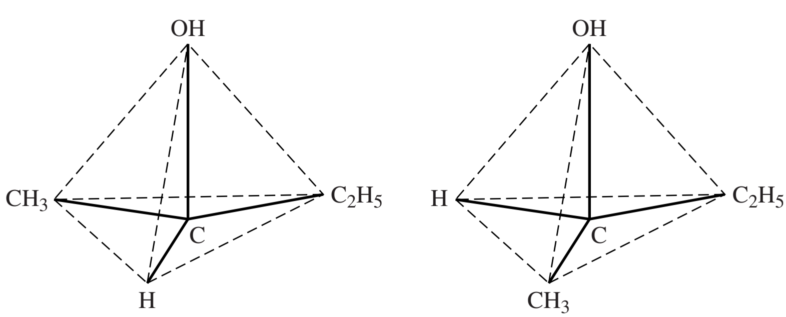

Figure 5.7. The simplest alcohol for which there are (nontrivial) stereoisomers is C_4H_9OH.

with radius of convergence $\xi_S^{-1} = 0.3042184090\ldots = (3.2871120555\ldots)^{-1}$. We omit the value of η_S for brevity's sake.

5.6.2 More Tree Varieties

An **identity tree** is a tree for which the only automorphism is the identity map. There clearly exist unique identity trees of orders 7 and 8 but no nontrivial cases of order ≤ 6. The generating function for identity trees is [4, 26]

$$\begin{aligned} u(x) &= \sum_{n=1}^{\infty} u_n x^n \\ &= x + x^7 + x^8 + 3x^9 + 6x^{10} + 15x^{11} + 29x^{12} + 67x^{13} + 139x^{14} + \cdots. \end{aligned}$$

A **rooted identity tree** is a rooted tree for which the identity map is the only automorphism that fixes the root. With this additional condition, rooted identity trees exist of all orders, and the associated generating function is

$$U(x) = \sum_{n=1}^{\infty} U_n x^n = x + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + 12x^7 + 25x^8 + 52x^9 + \cdots.$$

See the pictures of rooted identity trees in [6.11]. Such trees are also said to be **asymmetric**, in the sense that every vertex and edge is unique, that is, isomorphic siblings are forbidden. It can be proved that [5, 27]

$$\lim_{n\to\infty} \frac{U_n n^{\frac{3}{2}}}{\xi_U^n} = \eta_U = \frac{1}{\sqrt{2\pi}} \left(1 - \sum_{k=2}^{\infty} \frac{(-1)^k}{\xi_U^k} U'\left(\frac{1}{\xi_U^k}\right)\right)^{\frac{1}{2}} = 0.3625364234\ldots,$$

$$\lim_{n\to\infty} \frac{u_n n^{\frac{5}{2}}}{\xi_U^n} = 2\pi\eta_U^3 = 0.2993882877\ldots,$$

where $\xi_U^{-1} = 0.3972130965 = (2.5175403550\ldots)^{-1}$ is the radius of convergence for both $U(x)$ and $u(x)$, and further

$$U(x) = x \exp\left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{U(x^k)}{k}\right), \quad u(x) = U(x) - \tfrac{1}{2}(U(x)^2 + U(x^2)).$$

A tree is **homeomorphically irreducible** (or **series-reduced**) if no vertex is adjacent to exactly two other vertices. Clearly no such tree of order 3 exists, and the generating function is [4, 26, 28]

$$\begin{aligned} h(x) &= \sum_{n=1}^{\infty} h_n x^n \\ &= x + x^2 + x^4 + x^5 + 2x^6 + 2x^7 + 4x^8 + 5x^9 + 10x^{10} + 14x^{11} + \cdots. \end{aligned}$$

A **planted homeomorphically irreducible tree** is a rooted tree that is homeomorphically irreducible and whose root is adjacent to exactly one other vertex. The associated generating function is

$$H(x) = \sum_{n=1}^{\infty} H_n x^n$$
$$= x^2 + x^4 + x^5 + 2x^6 + 3x^7 + 6x^8 + 10x^9 + 19x^{10} + 35x^{11} + \cdots = x\tilde{H}(x).$$

It can be proved that [5,29]

$$\lim_{n\to\infty} \frac{H_n n^{\frac{3}{2}}}{\xi_H^n} = \eta_H = \frac{1}{\xi_H\sqrt{2\pi}} \left(\frac{\xi_H}{\xi_H + 1} + \sum_{k=2}^{\infty} \frac{1}{\xi_H^k} \tilde{H}'\left(\frac{1}{\xi_H^k}\right)\right)^{\frac{1}{2}} = 0.1924225474\ldots,$$

$$\lim_{n\to\infty} \frac{h_n n^{\frac{5}{2}}}{\xi_H^n} = 2\pi\xi_H^2(\xi_H + 1)\eta_H^3 = 0.6844472720\ldots,$$

where $\xi_H^{-1} = 0.4567332095\ldots = (2.1894619856\ldots)^{-1}$ is the radius of convergence for both $H(x)$ and $h(x)$, and further

$$\tilde{H}(x) = \frac{x}{x+1} \exp\left(\sum_{k=1}^{\infty} \frac{\tilde{H}(x^k)}{k} \right),$$
$$h(x) = (x+1)\tilde{H}(x) - \frac{x+1}{2}\tilde{H}(x)^2 - \frac{x-1}{2}\tilde{H}(x^2).$$

If we take into account the ordering (from left to right) of the subtrees of any vertex, then **ordered trees** arise and different enumeration problems occur. For example, define two ordered rooted trees σ and τ to be **cyclically isomorphic** if σ and τ are isomorphic as rooted trees, and if τ can be obtained from σ by circularly rearranging all the subtrees of any vertex, or likewise for each of several vertices. The equivalence classes under this relation are called **mobiles**. There exist fifty-one mobiles of order 7 but only forty-eight rooted trees of order 7 (see Figure 5.8).

The generating function for mobiles is [22,26,30]

$$M(x) = \sum_{n=1}^{\infty} M_n x^n$$
$$= x + x^2 + 2x^3 + 4x^4 + 9x^5 + 20x^6 + 51x^7 + 128x^8 + 345x^9 + \cdots,$$

$$M(x) = x\left(1 - \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \ln(1 - M(x^k))\right), \quad M_n \sim \eta_M\, n^{-\frac{3}{2}}\, \xi_M^n,$$

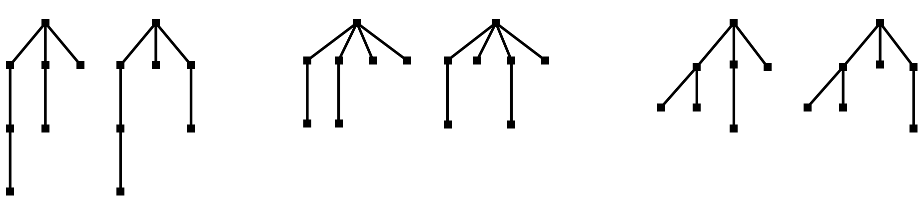

Figure 5.8. There exist three pairs of distinct mobiles (of order 7) that are identical as rooted trees.

where φ is the Euler totient function [2.7] and $\xi_M^{-1} = 0.3061875165\ldots = (3.2659724710\ldots)^{-1}$.

If we **label** the vertices of a graph distinctly with the integers $1, 2, \ldots, n$, the corresponding enumeration problems often simplify; for example, there are exactly n^{n-2} labeled free trees and n^{n-1} labeled rooted trees. For labeled mobiles, the problem becomes quite interesting, with exponential generating function [31]

$$\begin{aligned}\hat{M}(x) &= \sum_{n=1}^{\infty} \frac{\hat{M}_n}{n!} x^n \\ &= x + \frac{2}{2!}x^2 + \frac{9}{3!}x^3 + \frac{68}{4!}x^4 + \frac{730}{5!}x^5 + \frac{10164}{6!}x^6 + \frac{173838}{7!}x^7 + \cdots,\end{aligned}$$

$$\hat{M}(x) = x\,(1 - \ln(1 - \hat{M}(x))), \quad \hat{M}_n \sim \hat{\eta}\ \hat{\xi}^n n^{n-1},$$

where $\hat{\xi} = e^{-1}(1-\mu)^{-1} = 1.1574198038\ldots$, $\hat{\eta} = \sqrt{\mu(1-\mu)} = 0.4656386467\ldots$, and $\mu = 0.6821555671\ldots$ is the unique solution of the equation $\mu(1-\mu)^{-1} = 1 - \ln(1-\mu)$.

An **increasing tree** is a labeled rooted tree for which the labels along any branch starting at the root are increasing. The root must be labeled 1. Again, for increasing mobiles, enumeration provides interesting constants [32]:

$$\begin{aligned}\tilde{M}(x) &= \sum_{n=1}^{\infty} \frac{\tilde{M}_n}{n!} x^n \\ &= x + \frac{1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{7}{4!}x^4 + \frac{36}{5!}x^5 + \frac{245}{6!}x^6 + \frac{2076}{7!}x^7 + \cdots,\end{aligned}$$

$$\tilde{M}'(x) = 1 - \ln(1 - \tilde{M}(x)), \quad \tilde{M}_n \sim \tilde{\xi}^{n-1} n! \left(\frac{1}{n^2} - \frac{1}{n^2 \ln(n)} + O\left(\frac{1}{n^2 \ln(n)^2} \right) \right)$$

where $\tilde{\xi}^{-1} = -e\,\mathrm{Ei}(-1) = 0.5963473623\ldots = e^{-1}(0.6168878482\ldots)^{-1}$ is the Euler–Gompertz constant [6.2]. See a strengthening of these asymptotics in [31, 33].

5.6.3 Attributes

Thus far, we have discussed only enumeration issues. Otter's original constants α and β, however, appear in several asymptotic formulas governing other attributes of trees. By the **degree** (or **valency**) of a vertex, we mean the number of vertices that are adjacent to it. Given a random rooted tree with n vertices, the expected degree of the root is [34]

$$\theta = 1 + \sum_{i=1}^{\infty} T\left(\frac{1}{\alpha^i}\right) = 2 + \sum_{j=1}^{\infty} T_j \frac{1}{\alpha^j(\alpha^j - 1)} = 2.1918374031\ldots$$

as $n \to \infty$, and the variance of the degree of the root is

$$\sum_{i=1}^{\infty} i\,T\left(\frac{1}{\alpha^i}\right) = 1 + \sum_{j=1}^{\infty} T_j \frac{2\alpha^j - 1}{\alpha^j(\alpha^j - 1)^2} = 1.4741726868\ldots.$$

By the **distance** between two vertices, we mean the number of edges in the shortest path connecting them. The average distance between a vertex and the root is

$$\frac{1}{2}\left(\frac{2\pi}{\beta}\right)^{\frac{1}{3}} n^{\frac{1}{2}} = (1.1365599187\ldots)n^{\frac{1}{2}}$$

as $n \to \infty$, and the variance of the distance is

$$\frac{4-\pi}{4\pi}\left(\frac{2\pi}{\beta}\right)^{\frac{2}{3}} n = (0.3529622229\ldots)n.$$

Let v be an arbitrary vertex in a random free tree with n vertices and let p_m denote the probability, in the limit as $n \to \infty$, that v is of degree m. Then [35]

$$p_1 = \frac{\alpha^{-1} + \sum\limits_{k=1}^{\infty} D_k \dfrac{\alpha^{-2k}}{1-\alpha^{-k}}}{1 + \sum\limits_{k=1}^{\infty} k T_k \dfrac{\alpha^{-2k}}{1-\alpha^{-k}}} = 0.4381562356\ldots,$$

where $D_1 = 1$ and $D_{k+1} = \sum_{j=1}^{n}\left(\sum_{d|j} D_d\right) T_{k-j+1}$. Clearly $p_m \to 0$ as $m \to \infty$. More precisely, if

$$\omega = \prod_{i=1}^{\infty}\left(1 - \frac{1}{\alpha^i}\right)^{-T_{i+1}} = \exp\left(\sum_{j=1}^{\infty} \frac{1}{j}\left[\alpha^j T(\frac{1}{\alpha^j}) - 1\right]\right) = 7.7581602911\ldots$$

then $\lim_{m\to\infty} \alpha^m p_m$ is given by [36, 37]

$$(2\pi\beta^2)^{-\frac{1}{3}}\omega = (1.2160045618\ldots)^{-1}\omega = 6.3800420942\ldots.$$

We will need both θ and ω later. See also [38, 39].

Let G be a graph and let $A(G)$ be the automorphism group of G. A vertex v of G is a **fixed point** if $\varphi(v) = v$ for every $\varphi \in A(G)$. Let q denote the probability, in the limit as $n \to \infty$, that an arbitrary vertex in a random tree of order n is a fixed point. Harary & Palmer [7, 40] proved that

$$q = (2\pi\beta^2)^{-\frac{1}{3}}\left(1 - E\left(\frac{1}{\alpha^2}\right)\right) = 0.6995388700\ldots,$$

where $E(x) = T(x)(1 + F(x) - F(x^2))$. Interestingly, the same value q applies for rooted trees as well.

For reasons of space, we omit discussion of constants associated with covering and packing [41–43], as well as counting maximally independent sets of vertices [44–47], games [48], and equicolorable trees [49].

5.6.4 Forests

Let f_n denote the number of non-isomorphic forests of order n; then the generating function [26]

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} f_n x^n \\ &= x + 2x^2 + 3x^3 + 6x^4 + 10x^5 + 20x^6 + 37x^7 + 76x^8 + 153x^9 + 329x^{10} + \cdots \end{aligned}$$

satisfies

$$1 + f(x) = \exp\left(\sum_{k=1}^{\infty} \frac{t(x^k)}{k}\right), \quad f_n = \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{d|k} d t_d\right) f_{n-k}$$

and $f_0 = 1$ for the sake only of the latter formula. Palmer & Schwenk [50] showed that

$$f_n \sim c\, t_n = \left(1 + f\left(\frac{1}{\alpha}\right)\right) t_n = (1.9126258077\ldots) t_n.$$

If a forest is chosen at random, then as $n \to \infty$, the expected number of trees in the forest is

$$1 + \sum_{i=1}^{\infty} t\left(\frac{1}{\alpha^i}\right) = \frac{3}{2} + \frac{1}{2}T\left(\frac{1}{\alpha^2}\right) + \sum_{j=1}^{\infty} t_j \frac{1}{\alpha^j(\alpha^j - 1)} = 1.7555101394\ldots.$$

The corresponding number for rooted trees is $\theta = 2.1918374031\ldots$, a constant that unsurprisingly we encountered earlier [5.6.3]. The probability of exactly k rooted trees in a random forest is asymptotically $\omega\alpha^{-k} = (7.7581602911\ldots)\alpha^{-k}$. For free trees, the analogous probability likewise drops off geometrically as α^{-k} with coefficient

$$\frac{\alpha}{c}\prod_{i=1}^{\infty}\left(1 - \frac{1}{\alpha^i}\right)^{-t_{i+1}} = \frac{\alpha}{c}\exp\left(\sum_{j=1}^{\infty}\frac{1}{j}\left[\alpha^j t\left(\frac{1}{\alpha^j}\right) - 1\right]\right) = 3.2907434386\ldots.$$

Also, the asymptotic probability that two rooted forests of order n have no tree in common is [51]

$$\prod_{i=1}^{\infty}\left(1 - \frac{1}{\alpha^{2i}}\right)^{T_i} = \exp\left(-\sum_{j=1}^{\infty}\frac{1}{j}T\left(\frac{1}{\alpha^{2j}}\right)\right) = 0.8705112052\ldots.$$

5.6.5 Cacti and 2-Trees

We now examine graphs that are not trees but are nevertheless tree-like. A **cactus** is a connected graph in which no edge lies on more than one (minimal) cycle [52–54]. See Figure 5.9. If we further assume that every edge lies on exactly one cycle and that all cycles are polygons with m sides for a fixed integer m, the cactus is called an m**-cactus**. By convention, a 2-cactus is simply a tree. Discussions of 3-cacti appear in [4], 4-cacti in [55], and m-cacti with vertex coloring in [56]; we will not talk about such special

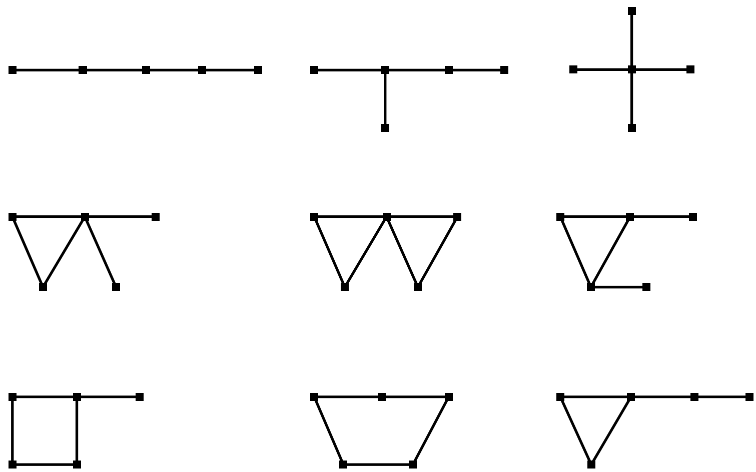

Figure 5.9. There exist nine non-isomorphic cacti of order 5.

cases. The generating functions for cacti and rooted cacti are [57]

$$\begin{aligned} c(x) &= \sum_{n=1}^{\infty} c_n x^n \\ &= x + x^2 + 2x^3 + 4x^4 + 9x^5 + 23x^6 + 63x^7 + 188x^8 + 596x^9 + 1979x^{10} + \cdots, \\ C(x) &= \sum_{n=1}^{\infty} C_n x^n \\ &= x + x^2 + 3x^3 + 8x^4 + 26x^5 + 84x^6 + 297x^7 + 1066x^8 + 3976x^9 + \cdots, \end{aligned}$$

and these satisfy [58–60]

$$C(x) = x \exp\left[-\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{C(x^k)^2 - 2 + C(x^{2k})}{2(C(x^k)-1)(C(x^{2k})-1)} + 1\right)\right],$$

$$c(x) = C(x) - \frac{1}{2}\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \ln(1 - C(x^k)) + \frac{(C(x)+1)(C(x)^2 - 2C(x) + C(x^2))}{4(C(x)-1)(C(x^2)-1)},$$

with radius of convergence 0.2221510651 .... For the labeled case, we have

$$\begin{aligned} \hat{c}(x) &= \sum_{n=1}^{\infty} \frac{\hat{c}_n}{n!} x^n \\ &= x + \frac{1}{2!}x^2 + \frac{4}{3!}x^3 + \frac{31}{4!}x^4 + \frac{362}{5!}x^5 + \frac{5676}{6!}x^6 + \frac{111982}{7!}x^7 + \cdots, \\ \hat{C}(x) &= \sum_{n=1}^{\infty} \frac{\hat{C}_n}{n!} x^n \\ &= x + \frac{2}{2!}x^2 + \frac{12}{3!}x^3 + \frac{124}{4!}x^4 + \frac{1810}{5!}x^5 + \frac{34056}{6!}x^6 + \frac{783874}{7!}x^7 + \cdots, \end{aligned}$$

and these satisfy

$$\hat{C}(x) = x \exp\left(\frac{\hat{C}(x)}{2}\frac{2-\hat{C}(x)}{1-\hat{C}(x)}\right), \quad x\hat{c}'(x) = \hat{C}(x),$$

with radius of convergence $0.2387401436\ldots$.

A **2-tree** is defined recursively as follows [4]. A 2-tree of rank 1 is a triangle (a graph with three vertices and three edges), and a 2-tree of rank $n \geq 2$ is built from a 2-tree of rank $n-1$ by creating a new vertex of degree 2 adjacent to each of two existing adjacent vertices. Hence a 2-tree of rank n has $n+2$ vertices and $2n+1$ edges. The generating function for 2-trees is [61]

$$\begin{aligned} w(x) &= \sum_{n=0}^{\infty} w_n x^n \\ &= 1 + x + x^2 + 2x^3 + 5x^4 + 12x^5 + 39x^6 + 136x^7 + 529x^8 + 2171x^9 + \cdots \\ w(x) &= \frac{1}{2}\left[W(x) + \exp\left(\sum_{k=1}^{\infty}\frac{1}{2k}(2x^k W(x^{2k}) + x^{2k}W(x^{2k})^2 - x^{2k}W(x^{4k}))\right)\right] \\ &\quad + \frac{1}{3}x\left(W(x^3) - W(x)^3\right), \end{aligned}$$

where $W(x)$ is the generating function for 2-trees with a distinguished and oriented edge:

$$\begin{aligned} W(x) &= \sum_{n=0}^{\infty} W_n x^n \\ &= 1 + x + 3x^2 + 10x^3 + 39x^4 + 160x^5 + 702x^6 + 3177x^7 + 14830x^8 + \cdots \end{aligned}$$

$$W(x) = \exp\left(\sum_{k=1}^{\infty}\frac{x^k W(x^k)^2}{k}\right), \quad w_n \sim \eta_w\, n^{-\frac{5}{2}}\, \xi_w^n.$$

Further, $w(x)$ has radius of convergence $\xi_w^{-1} = 0.1770995223\ldots = (5.6465426162\ldots)^{-1}$ and

$$\eta_w = \frac{1}{16\xi\sqrt{\pi}}\left(\xi + 2\tilde{W}'\left(\frac{1}{\xi}\right)\tilde{W}\left(\frac{1}{\xi}\right)^{-1}\right)^{\frac{3}{2}} = 0.0948154165\ldots,$$

$$\tilde{W}(x) = e^{-xW(x)^2}W(x).$$

5.6.6 Mapping Patterns

We studied labeled functional graphs on n vertices in [5.4]. Let us remove the labels and consider only graph isomorphism classes, called **mapping patterns**. Observe that the original Otter constants α and β play a crucial role here. The generating function of mapping patterns is [57, 62]

$$\begin{aligned} P(x) &= \sum_{n=1}^{\infty} P_n x^n \\ &= x + 3x^2 + 7x^3 + 19x^4 + 47x^5 + 130x^6 + 343x^7 + 951x^8 + 2615x^9 + \cdots, \end{aligned}$$

$$1 + P(x) = \prod_{k=1}^{\infty} \left(1 - T(x^k)\right)^{-1}, \quad P_n \sim \eta_P \, n^{-\frac{1}{2}} \alpha^n,$$

where

$$\begin{aligned} \eta_P &= \frac{1}{2\pi} \left(\frac{2\pi}{\beta}\right)^{\frac{1}{3}} \prod_{i=2}^{\infty} \left(1 - T(\frac{1}{\alpha^i})\right)^{-1} = 0.4428767697\ldots \\ &= (1.2241663491\ldots)(4\pi^2\beta)^{-\frac{1}{3}}. \end{aligned}$$

From this, it follows that the expected length of an arbitrary cycle in a random mapping pattern is

$$\frac{1}{2}\left(\frac{2\pi}{\beta}\right)^{\frac{1}{3}} n^{\frac{1}{2}} = (1.1365599187\ldots)n^{\frac{1}{2}}, \quad n \to \infty$$

(an expression that we saw in [5.6.3], by coincidence) and the asymptotic probability that the mapping pattern is connected is

$$\frac{1}{2\eta_P} n^{-\frac{1}{2}} = (1.1289822228\ldots)n^{-\frac{1}{2}}.$$

If we further restrict attention to connected mapping patterns, the associated generating function is

$$\begin{aligned} K(x) &= \sum_{n=1}^{\infty} K_n x^n \\ &= x + 2x^2 + 4x^3 + 9x^4 + 20x^5 + 51x^6 + 125x^7 + 329x^8 + 862x^9 + \cdots \end{aligned}$$

$$K(x) = -\sum_{j=1}^{\infty} \frac{\varphi(j)}{j} \ln(1 - T(x^j)), \quad K_n \sim \frac{1}{2} n^{-1} \alpha^n.$$

It follows that the expected length of the (unique) cycle in a random connected mapping pattern is

$$\frac{1}{\pi}\left(\frac{2\pi}{\beta}\right)^{\frac{1}{3}} n^{\frac{1}{2}} = (0.7235565167\ldots)n^{\frac{1}{2}}, \quad n \to \infty,$$

which is less than before. A comparison between such statistics for both unlabeled and labeled cases (the numerical results are indeed slightly different) appears in [62]. See [63, 64] for more recent work in this area.

5.6.7 More Graph Varieties

A graph G is an **interval graph** if it can be represented as follows: Each vertex of G corresponds to a subinterval of the real line in such a way that two vertices are adjacent if and only if their corresponding intervals have nonempty intersection. It is a **unit interval graph** if the intervals can all be chosen to be of length 1. The generating function of unit interval graphs, for example, is [65, 66]

$$\begin{aligned} I(x) &= \sum_{n=1}^{\infty} I_n x^n \\ &= x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 55x^6 + 151x^7 + 447x^8 + 1389x^9 + \cdots, \end{aligned}$$

$$1 + I(x) = \exp\left(\sum_{k=1}^{\infty} \frac{\psi(x^k)}{k}\right), \quad \psi(x) = \frac{1 + 2x - \sqrt{1-4x}\sqrt{1-4x^2}}{4\sqrt{1-4x^2}},$$

with asymptotics

$$I_n \sim \frac{1}{8\kappa\sqrt{\pi}} n^{-\frac{3}{2}} 4^n, \quad \kappa = \exp\left(-\frac{\sqrt{3}}{4}\right) \exp\left(-\sum_{j=2}^{\infty} \frac{\psi(4^{-j})}{j}\right) = 0.6231198963\ldots.$$

Interval graphs have found applications in genetics and other fields [67, 68].

A graph is **2-regular** if every vertex has degree two. The number J_n of 2-regular graphs on n vertices is equal to the number of partitions of n into parts ≥ 3, whereas the exponential generating function of 2-regular labeled graphs is [69]

$$\begin{aligned} \hat{J}(x) &= \sum_{n=0}^{\infty} \frac{\hat{J}_n}{n!} x^n = 1 + \frac{1}{3!}x^3 + \frac{3}{4!}x^4 + \frac{12}{5!}x^5 + \frac{70}{6!}x^6 + \cdots \\ &= \frac{1}{\sqrt{1-x}} \exp\left(-\frac{1}{2}x - \frac{1}{4}x^2\right); \end{aligned}$$

therefore

$$J_n \sim \frac{\pi^2}{12\sqrt{3}n^2} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \quad \hat{J}_n \sim \sqrt{2}e^{-\frac{3}{4}} \left(\frac{n}{e}\right)^n.$$

The latter has an interesting geometric interpretation [14, 70]. Given n planar lines in general position with $\binom{n}{2}$ intersecting points, a **cloud** of size n is a (maximal) set of n intersecting points, no three of which are collinear. The number of clouds of size n is clearly $\hat{J}_n$.

A **directed graph** or **digraph** is a graph for which the edges are ordered pairs of distinct vertices (rather than unordered pairs). Note that loops are automatically disallowed. An **acyclic digraph** further contains no directed cycles; in particular, it has no multiple parallel edges. The (transformed) exponential generating function of labeled acyclic digraphs is [65, 71–74]

$$A(x) = \sum_{n=0}^{\infty} \frac{A_n}{n!2^{\binom{n}{2}}} x^n = 1 + x + \frac{3}{2! \cdot 2} x^2 + \frac{25}{3! \cdot 2^3} x^3 + \frac{543}{4! \cdot 2^6} x^4 + \frac{29281}{5! \cdot 2^{10}} x^5 + \cdots,$$

$$A'(x) = A(x)^2 A(\tfrac{1}{2}x)^{-1}, \quad A_n \sim \frac{n!2^{\binom{n}{2}}}{\eta_A \ \xi_A^n},$$

where $\xi_A = 1.4880785456\ldots$ is the smallest positive zero of the function

$$\lambda(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^{\binom{n}{2}}} x^n = A(x)^{-1}, \quad \lambda'(x) = -\lambda(\tfrac{1}{2}x),$$

and $\eta_A = \xi_A \lambda(\xi_A/2) = 0.5743623733\ldots = (1.7410611252\ldots)^{-1}$. It is curious that the function $\lambda(-x)$ was earlier studied by Mahler [75] with regard to enumerating partitions of integers into powers of 2. See [76, 77] for discussion of the unlabeled acyclic digraph analog.

5.6.8 Data Structures

To a combinatorialist, the phrase "(strongly) binary tree with $2n+1$ vertices" means an isomorphism class of trees. To a computer scientist, however, the same phrase virtually always includes the word "ordered," whether stated explicitly or not. Hence the phrase "random binary tree" is sometimes ambiguous in the literature: The sample space has B_n elements for the former person but $\binom{2n}{n}/(n+1)$ elements for the latter! We cannot hope here to survey the role of trees in computer algorithms, only to provide a few constants.

A **leftist tree** of size n is an ordered binary tree with n leaves such that, in any subtree σ, the leaf closest to the root of σ is in the right subtree of σ. The generating function of leftist trees is [6, 65, 78, 79]

$$\begin{aligned} L(x) &= \sum_{n=0}^{\infty} L_n x^n \\ &= x + x^2 + x^3 + 2x^4 + 4x^5 + 8x^6 + 17x^7 + 38x^8 + 87x^9 + 203x^{10} + \cdots \end{aligned}$$

$$L(x) = x + \frac{1}{2} L(x)^2 + \frac{1}{2} \sum_{m=1}^{\infty} l_m(x)^2 = \sum_{m=1}^{\infty} l_m(x),$$

where the auxiliary generating functions $l_m(x)$ satisfy

$$l_1(x) = x, \quad l_2(x) = xL(x), \quad l_{m+1}(x) = l_m(x) \left(L(x) - \sum_{k=1}^{m-1} l_k(x) \right), \quad m \geq 2.$$

It can be proved (with difficulty) that

$$L_n \sim (0.2503634293\ldots) \cdot (2.7494879027\ldots)^n n^{-\frac{3}{2}}.$$

Leftist trees are useful in certain sorting and merging algorithms.

A **2,3-tree** of size n is a rooted ordered tree with n leaves satisfying the following:

- Each non-leaf vertex has either 2 or 3 successors.
- All of the root-to-leaf paths have the same length.

The generating function of 2,3-trees (no relation to 2-trees!) is [65, 80, 81]

$$\begin{aligned} Z(x) &= \sum_{n=0}^{\infty} Z_n x^n \\ &= x + x^2 + x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 4x^8 + 5x^9 + 8x^{10} + 14x^{11} + \cdots \end{aligned}$$

$$Z(x) = x + Z(x^2 + x^3), \quad Z_n = \sum_{k=\lceil n/3 \rceil}^{\lfloor n/2 \rfloor} \binom{k}{3k-n} Z_k \sim \varphi^n n^{-1} f(\ln(n)),$$

where φ is the Golden mean [1.2] and $f(x)$ is a nonconstant, positive, continuous function that is periodic with period $\ln(4-\varphi) = 0.867\ldots$, has mean $(\varphi \ln(4-\varphi))^{-1} = 0.712\ldots$, and oscillates between $0.682\ldots$ and $0.806\ldots$. These are also a particular type of **B-trees**. A similar analysis [82] uncovers the asymptotics of what are known as AVL-trees (or height-balanced trees). Such trees support efficient database searches, deletions, and insertions; other varieties are too numerous to mention.

If τ is an ordered binary tree, then its **height** and **register functions** are recursively defined by [83]

$$\operatorname{ht}(\tau) = \begin{cases} 0 & \text{if } \tau \text{ is a point,} \\ 1 + \max(\operatorname{ht}(\tau_L), \operatorname{ht}(\tau_R)) & \text{otherwise,} \end{cases}$$

$$\operatorname{rg}(\tau) = \begin{cases} 0 & \text{if } \tau \text{ is a point,} \\ 1 + \operatorname{rg}(\tau_L) & \text{if } \operatorname{rg}(\tau_L) = \operatorname{rg}(\tau_R), \\ \max(\operatorname{rg}(\tau_L), \operatorname{rg}(\tau_R)) & \text{otherwise,} \end{cases}$$

where τ_L and τ_R are the left and right subtrees of the root. That is, $\operatorname{ht}(\tau)$ is the number of edges along the longest branch from the root, whereas $\operatorname{rg}(\tau)$ is the minimum number of registers needed to evaluate the tree (thought of as an arithmetic expression). If we randomly select a binary tree τ with $2n+1$ vertices, then the asymptotics of $\mathrm{E}(\operatorname{ht}(\tau))$ involve $2\sqrt{\pi n}$ as mentioned in [1.4], and those of $\mathrm{E}(\operatorname{rg}(\tau))$ involve $\ln(n)/\ln(4)$ plus a zero mean oscillating function [2.16]. Also, define $\operatorname{ym}(\tau)$ to be the number of maximal subtrees of τ having register function exactly 1 less than $\operatorname{rg}(\tau)$. Prodinger [84], building upon the work of Yekutiele & Mandelbrot [85], proved that $\mathrm{E}(\operatorname{ym}(\tau))$ is asymptotically

$$\frac{2G}{\pi \ln(2)} + \frac{5}{2} = 3.3412669407\ldots$$

plus a zero mean oscillating function, where G is Catalan's constant [1.7]. This is also known as the **bifurcation ratio** at the root, which quantifies the hierarchical complexity of more general branching structures.

5.6.9 *Galton–Watson Branching Process*

Thus far, by "random binary trees," it is meant that we select binary trees with n vertices from a population endowed with the uniform probability distribution. The integer n is fixed.

It is also possible, however, to *grow* binary trees (rather than to merely select them). Fix a probability $0 < p < 1$ and define recursively a (strongly) binary tree τ in terms of left and right subtrees of the root as follows: Take $\tau_L = \emptyset$ with probability $1 - p$, and independently take $\tau_R = \emptyset$ with probability $1 - p$. It can be shown [86–88] that this process terminates, that is, τ is a finite tree, with **extinction probability** 1 if $p \leq 1/2$ and $1/p - 1$ if $p > 1/2$. Of course, the number of vertices N is here a random variable, called the **total progeny**.

Much can be said about the Bienaymé–Galton–Watson process (which is actually more general than described here). We focus on just one detail. Let N_k denote the number of vertices at distance k from the root, that is, the size of the k^{th} generation. Consider the subcritical case $p < 1/2$. Let a_k denote the probability that $N_k = 0$; then the sequence $a_0, a_1, a_2, \ldots$ obeys the quadratic recurrence [6.10]

$$a_0 = 0, \qquad a_k = (1-p) + p a_{k-1}^2 \quad \text{for } k \geq 1, \qquad \lim_{k\to\infty} a_k = 1.$$

What can be said about the convergence rate of $\{a_k\}$? It can be proved that

$$C(p) = \lim_{k\to\infty} \frac{1-a_k}{(2p)^k} = \prod_{l=0}^{\infty} \frac{1+a_l}{2},$$

which has no closed-form expression in terms of p, as far as is known. This is over and beyond the fact, of greatest interest to us here, that $\mathrm{P}(N_k > 0) \sim C(p)(2p)^k$ for $0 < p < 1/2$. Other interesting parameters are the **moment of extinction** $\min\{k : N_k = 0\}$ or tree height, and the **maximal generation size** $\max\{N_k : k \geq 0\}$ or tree width.

5.6.10 *Erdös–Rényi Evolutionary Process*

Starting with n initially disconnected vertices, define a random graph by successively adding edges between pairs of distinct points, chosen uniformly from $\binom{n}{2}$ candidates without replacement. Continue with this process until no candidate edges are left [89–92].

At some stage of the evolution, a **complex component** emerges, that is, the first component possessing more than one cycle. It is remarkable that this complex component will usually remain unique throughout the entire process, and the probability that this is true is $5\pi/18 = 0.8726\ldots$ as $n \to \infty$. In other words, the first component that acquires more edges than vertices is quite likely to become the **giant component** of the random graph. The probability that exactly two complex components emerge is $50\pi/1296 = 0.1212\ldots$, but the probability ($> 0.9938\ldots$) that the evolving graph never has more than two complex components at any time is not precisely known [93].

There are many related results, but we mention only one. Start with an $m \times n$ rectangular grid of rooms, each with four walls. Successively remove interior walls in a random manner such that, at some step in the procedure, the associated graph (with all

mn rooms as vertices and all neighboring pairs of rooms with open passage as edges) becomes a tree. Stop when this condition is met; the result is a **random maze** [94]. The difficulty lies in detecting whether the addition of a new edge creates an unwanted cycle. An efficient way of doing this (maintaining equivalence classes that change over time) is found in QF and QFW, two of a class of **union-find algorithms** in computer science. Exact performance analyses of QF and QFW appear in [95–97], using random graph theory and a variant of the Erdös-Rényi process.

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5.7 Lengyel's Constant

5.7.1 Stirling Partition Numbers

Let S be a set with n elements. The set of all subsets of S has 2^n elements. By a **partition** of S we mean a disjoint set of nonempty subsets (called **blocks**) whose union is S. The set of partitions of S that possess exactly k blocks has $S_{n,k}$ elements, where $S_{n,k}$ is a

Stirling number of the second kind. The set of *all* partitions of S has B_n elements, where B_n is a **Bell number**:

$$B_n = \sum_{k=1}^{n} S_{n,k} = \frac{1}{e}\sum_{j=0}^{\infty}\frac{j^n}{j!} = \frac{d^n}{dx^n}\exp\left(e^x - 1\right)\bigg|_{x=0}.$$

For example, $S_{4,1} = 1$, $S_{4,2} = 7$, $S_{4,3} = 6$, $S_{4,4} = 1$, and $B_4 = 15$. More generally, $S_{n,1} = 1$, $S_{n,2} = 2^{n-1} - 1$, and $S_{n,3} = \frac{1}{2}(3^{n-1}+1) - 2^{n-1}$. The following recurrences are helpful [1–4]:

$$S_{n,0} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \geq 1, \end{cases} \quad S_{n,k} = kS_{n-1,k} + S_{n-1,k-1} \ \text{ if } n \geq k \geq 1,$$

$$B_0 = 1, \quad B_n = \sum_{k=0}^{n-1}\binom{n-1}{k}B_k,$$

and corresponding asymptotics are discussed in [5–9].

5.7.2 Chains in the Subset Lattice of S

If U and V are subsets of S, write $U \subset V$ if U is a proper subset of V. This endows the set of all subsets of S with a **partial ordering**; in fact, it is a **lattice** with maximum element S and minimum element $\emptyset$. The number of **chains** $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_{k-1} \subset U_k = S$ of length k is $k!S_{n,k}$. Hence the number of all chains from $\emptyset$ to S is [1, 6, 10]

$$\sum_{k=0}^{n} k!S_{n,k} = \sum_{j=0}^{\infty}\frac{j^n}{2^{j+1}} = \frac{1}{2}\operatorname{Li}_{-n}\left(\frac{1}{2}\right) = \frac{d^n}{dx^n}\frac{1}{2-e^x}\bigg|_{x=0} \sim \frac{n!}{2}\left(\frac{1}{\ln(2)}\right)^{n+1},$$

where $\operatorname{Li}_m(x)$ is the polylogarithm function. Wilf [10] marveled at how accurate this asymptotic approximation is.

If we further insist that the chains are **maximal**, equivalently, that additional proper insertions are impossible, then the number of such chains is $n!$ A general technique due to Doubilet, Rota & Stanley [11], involving what are called *incidence algebras*, can be used to obtain the two aforementioned results, as well as to enumerate chains within more complicated posets [12].

As an aside, we give a deeper application of incidence algebras: to enumerating chains of linear subspaces within finite vector spaces [6]. Define the q**-binomial coefficient** and q**-factorial** by

$$\binom{n}{k}_q = \frac{\prod_{j=1}^{n}(q^j - 1)}{\prod_{j=1}^{k}(q^j - 1)\cdot\prod_{j=1}^{n-k}(q^j - 1)},$$

$$[n!]_q = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}),$$

where $q > 1$. Note the special case in the limit as $q \to 1^+$. Consider the n-dimensional vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$, where q is a prime power [12–16]. The number of k-dimensional linear subspaces of $\mathbb{F}_q^n$ is $\binom{n}{k}_q$ and the total number of linear subspaces of $\mathbb{F}_q^n$ is asymptotically $c_e q^{n^2/4}$ if n is even and $c_o q^{n^2/4}$ if n is odd, where [17, 18]

$$c_e = \frac{\sum\limits_{k=-\infty}^{\infty} q^{-k^2}}{\prod\limits_{j=1}^{\infty}(1-q^{-j})}, \quad c_o = \frac{\sum\limits_{k=-\infty}^{\infty} q^{-\left(k+\frac{1}{2}\right)^2}}{\prod\limits_{j=1}^{\infty}(1-q^{-j})}.$$

We give a recurrence for the number χ_n of chains of proper subspaces (again, ordered by inclusion):

$$\chi_1 = 1, \quad \chi_n = 1 + \sum_{k=1}^{n-1} \binom{n}{k}_q \chi_k \quad \text{for } n \ge 2.$$

For the asymptotics, it follows that [6, 17]

$$\chi_n \sim \frac{1}{\zeta_q'(r) r} \left(\frac{1}{r}\right)^n \prod_{j=1}^{n}(q^j - 1) = \frac{A}{r^n}(q-1)(q^2-1)(q^3-1)\cdots(q^n-1),$$

where $\zeta_q(x)$ is the zeta function for the poset of subspaces:

$$\zeta_q(x) = \sum_{k=1}^{\infty} \frac{x^k}{(q-1)(q^2-1)(q^3-1)\cdots(q^k-1)}$$

and $r > 0$ is the unique solution of the equation $\zeta_q(r) = 1$. In particular, when $q = 2$, we have $c_e = 7.3719688014\ldots$, $c_o = 7.3719494907\ldots$, and

$$\chi_n \sim \frac{A}{r^n} \cdot Q \cdot 2^{\frac{n(n+1)}{2}},$$

where $r = 0.7759021363\ldots$, $A = 0.8008134543\ldots$, and

$$Q = \prod_{k=1}^{\infty}\left(1 - \frac{1}{2^k}\right) = 0.2887880950\ldots$$

is one of the digital search tree constants [5.14]. If we further insist that the chains are maximal, then the number of such chains is $[n!]_q$.

5.7.3 Chains in the Partition Lattice of S

We have discussed chains in the poset of subsets of the set S. There is, however, another poset associated naturally with S that is less familiar and more difficult to study: the **poset of partitions** of S. Here is the partial ordering: Assuming P and Q are two partitions of S, then $P < Q$ if $P \neq Q$ and if $p \in P$ implies that p is a subset of q for some $q \in Q$. In other words, P is a *refinement* of Q in the sense that each of its blocks fits within a block of Q. For arbitrary n, the poset is, in fact, a lattice with minimum element $m = \{\{1\}, \{2\}, \ldots, \{n\}\}$ and maximum element $M = \{\{1, 2, \ldots, n\}\}$.

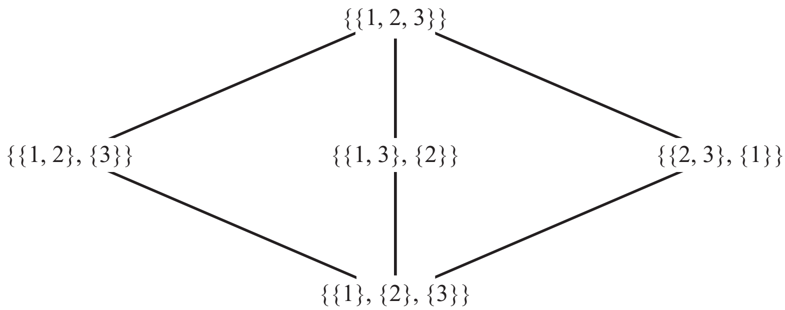

Figure 5.10. The number of chains $m < P_1 < M$ in the partition lattice of the set $\{1, 2, 3\}$ is three.

What is the number of chains $m = P_0 < P_1 < P_2 < \cdots < P_{k-1} < P_k = M$ of length k in the partition lattice of S? In the case $n = 3$, there is only one chain for $k = 1$, specifically, $m < M$. For $k = 2$, there are three such chains as pictured in Figure 5.10.

Let Z_n denote the number of all chains from m to M of any length; clearly $Z_1 = Z_2 = 1$ and, by the foregoing, $Z_3 = 4$. We have the recurrence

$$Z_n = \sum_{k=1}^{n-1} S_{n,k} Z_k$$

and exponential generating function

$$Z(x) = \sum_{n=1}^{\infty} \frac{Z_n}{n!} x^n, \quad 2Z(x) = x + Z(e^x - 1),$$

but techniques of Doubilet, Rota & Stanley and Bender do not apply here to give asymptotic estimates of Z_n. The partition lattice is the first natural lattice without the structure of a *binomial lattice*, which implies that well-known generating function techniques are no longer helpful.

Lengyel [19] formulated a different approach to prove that the quotient

$$r_n = \frac{Z_n}{(n!)^2 (2\ln(2))^{-n} n^{-1-\ln(2)/3}}$$

must be bounded between two positive constants as $n \to \infty$. He presented numerical evidence suggesting that r_n tends to a unique value. Babai & Lengyel [20] then proved a fairly general convergence criterion that enabled them to conclude that $\Lambda = \lim_{n\to\infty} r_n$ exists and $\Lambda = 1.09\ldots$. The analysis in [19] involves intricate estimates of the Stirling numbers; in [20], the focus is on nearly convex linear recurrences with finite retardation and active predecessors.

In an ambitious undertaking, Flajolet & Salvy [21] computed $\Lambda = 1.0986858055\ldots$. Their approach is based on (complex fractional) analytic iterates of $\exp(x) - 1$ and much more, but unfortunately their paper is presently incomplete. See [5.8] for related discussion of the Takeuchi-Prellberg constant.

By way of contrast, the number of *maximal* chains is given exactly by $n!(n-1)!/2^{n-1}$ and Lengyel [19] observed that Z_n exceeds this by an exponentially large factor.

5.7.4 Random Chains

Van Cutsem & Ycart [22] examined random chains in both the subset and partition lattices. It is remarkable that a common framework exists for studying these and that, in a certain sense, the limiting distributions of both types of chains are *identical*. We mention only one consequence: If $\kappa_n = k/n$ is the normalized length of the random chain, then

$$\lim_{n\to\infty} \mathrm{E}(\kappa_n) = \frac{1}{2\ln(2)} = 0.7213475204\ldots$$

and a corresponding Central Limit Theorem also holds.

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5.8 Takeuchi–Prellberg Constant

In 1978, Takeuchi defined a triply recursive function [1,2]

$$t(x,y,z)=\begin{cases} y & \text{if } x\le y,\\ t(t(x-1,y,z),t(y-1,z,x),t(z-1,x,y)) & \text{otherwise}\end{cases}$$

that is useful for benchmark testing of programming languages. The value of $t(x,y,z)$ is of no practical significance; in fact, McCarthy [1,2] observed that the function can be described more simply as

$$t(x,y,z)=\begin{cases} y & \text{if } x\le y,\\ \begin{cases} z & \text{if } y\le z,\\ x & \text{otherwise,}\end{cases} & \text{otherwise.}\end{cases}$$

The interesting quantity is not $t(x,y,z)$, but rather $T(x,y,z)$, defined to be the number of times the *otherwise* clause is invoked in the recursion. We assume that the program is memoryless in the sense that previously computed results are not available at any time in the future. Knuth [1,3] studied the **Takeuchi numbers** $T_n=T(n,0,n+1)$:

$$T_0=0,\ T_1=1,\ T_2=4,\ T_3=14,\ T_4=53,\ T_5=223,\ldots$$

and deduced that

$$e^{n\ln(n)-n\ln(\ln(n))-n}<T_n<e^{n\ln(n)-n+\ln(n)}$$

for all sufficiently large n. He asked for more precise asymptotic information about the growth of T_n.

Starting with Knuth's recursive formula for the Takeuchi numbers

$$T_{n+1}=\sum_{k=0}^{n}\left[\binom{n+k}{n}-\binom{n+k}{n+1}\right]T_{n-k}+\sum_{k=1}^{n-1}\binom{2k}{k}\frac{1}{k+1}$$

and the somewhat related Bell numbers [5.7]

$$B_{n+1}=\sum_{k=0}^{n}\binom{n}{k}B_{n-k},\ \ B_0=1,\ B_1=1,\ B_2=2,\ B_3=5,\ B_4=15,\ B_5=52,\ldots,$$

Prellberg [4] observed that the following limit exists:

$$c=\lim_{n\to\infty}\frac{T_n}{B_n\exp\left(\frac{1}{2}W_n^2\right)}=2.2394331040\ldots,$$

where $W_n\exp(W_n)=n$ are special values of the Lambert W function [6.11].

Since both the Bell numbers and the W function are well understood, this provides an answer to Knuth's question. The underlying theory is still under development, but

Prellberg's numerical evidence is persuasive. Recent theoretical work [5] relates the constant c to an associated functional equation,

$$T(z)=\sum_{n=0}^{\infty}T_nz^n,\quad T(z)=\frac{T(z-z^2)}{z}-\frac{1}{(1-z)(1-z+z^2)},$$

in a manner parallel to how Lengyel's constant [5.7] is obtained.

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5.9 Pólya's Random Walk Constants

Let L denote the d-dimensional cubic lattice whose vertices are precisely all integer points in d-dimensional space. A **walk** ω on L, beginning at the origin, is an infinite sequence of vertices $\omega_0, \omega_1, \omega_2, \omega_3, \ldots$ with $\omega_0 = 0$ and $|\omega_{j+1} - \omega_j| = 1$ for all j. Assume that the walk is random and symmetric in the sense that, at each time step, all $2d$ directions of possible travel have equal probability. What is the likelihood that $\omega_n = 0$ for some $n > 0$? That is, what is the **return probability** p_d?

Pólya [1–4] proved the remarkable fact that $p_1 = p_2 = 1$ but $p_d < 1$ for $d > 2$. McCrea & Whipple [5], Watson [6], Domb [7] and Glasser & Zucker [8] each contributed facets of the following evaluations of $p_3 = 1 - 1/m_3 = 0.3405373295\ldots$, where the expected number m_3 of returns to the origin, plus one, is

$$\begin{aligned}
m_3 &= \frac{3}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{3-\cos(\theta)-\cos(\varphi)-\cos(\psi)}\,d\theta\,d\varphi\,d\psi \\
&= \frac{12}{\pi^2}\left(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6}\right)K\left[(2-\sqrt{3})(\sqrt{3}-\sqrt{2})\right]^2 \\
&= 3\left(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6}\right)\left[1+2\sum_{k=1}^{\infty}\exp(-\sqrt{6}\pi k^2)\right]^4 \\
&= \frac{\sqrt{6}}{32\pi^3}\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{11}{24}\right) = 1.5163860591\ldots.
\end{aligned}$$

Hence the **escape probability** for a random walk on the three-dimensional cubic lattice is $1 - p_3 = 0.6594626704\ldots$. In these expressions, K denotes the complete elliptic integral of the first kind [1.4.6] and Γ denotes the gamma function [1.5.4]. Return and escape probabilities can also be computed for the body-centered or face-centered cubic

Table 5.1. *Expected Number of Returns and Return Probabilities*

d	m_d	p_d
4	1.2394671218...	0.1932016732...
5	1.1563081248...	0.1351786098...
6	1.1169633732...	0.1047154956...
7	1.0939063155...	0.0858449341...
8	1.0786470120...	0.0729126499...

lattices (as opposed to the simple cubic lattice), but we will not discuss these or other generalizations [9].

What can be said about p_d for $d > 3$? Closed-form expressions do not appear to exist here. Montroll [10–12] determined that $p_d = 1 - 1/m_d$, where

$$m_d = \frac{d}{(2\pi)^d} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left(d - \sum_{k=1}^{d} \cos(\theta_k)\right)^{-1} d\theta_1\, d\theta_2 \cdots d\theta_d = \int_0^\infty e^{-t} \left(I_0\left(\frac{t}{d}\right)\right)^d dt$$

and $I_0(x)$ denotes the zeroth modified Bessel function [3.6]. The corresponding numerical approximations, as functions of d, are listed in Table 5.1 [10, 13–17].

What is the length of travel required for a return? Let $U_{d,l,n}$ be the number of d-dimensional n-step walks that start from the origin and end at a lattice point l. Let $V_{d,l,n}$ be the number of d-dimensional n-step walks that start from the origin and reach the lattice point $l \neq 0$ for the *first time* at the end (second time if $l = 0$). Then the generating functions

$$U_{d,l}(x) = \sum_{n=0}^{\infty} \frac{U_{d,l,n}}{(2d)^n} x^n, \quad V_{d,l}(x) = \sum_{n=0}^{\infty} \frac{V_{d,l,n}}{(2d)^n} x^n$$

satisfy $V_{d,l}(x) = U_{d,l}(x)/U_{d,0}(x)$ if $l \neq 0$, $V_{d,l}(x) = 1 - 1/U_{d,0}(x)$ if $l = 0$, and $U_{d,0}(1) = m_d$, $V_{d,0}(1) = p_d$. For example,

$$U_{1,l}(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \binom{n}{\frac{l+n}{2}} x^n, \quad U_{2,l}(x) = \sum_{n=0}^{\infty} \frac{1}{4^n} \binom{n}{\frac{l_1+l_2+n}{2}} \binom{n}{\frac{l_1-l_2+n}{2}} x^n,$$

where we agree to set the binomial coefficients equal to 0 if $l + n$ is odd for $d = 1$ or $l_1 + l_2 + n$ is odd for $d = 2$. If $d = 3$, then $a_n = U_{3,0,2n}$ satisfies [18]

$$a_n = \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} = \sum_{k=0}^{n} \frac{(2n)!(2k)!}{(n-k)!^2 k!^4}, \quad \sum_{n=0}^{\infty} \frac{a_n}{(2n)!} y^{2n} = I_0(2y)^3,$$

$$(n+2)^3 a_{n+2} - 2(2n+3)(10n^2 + 30n + 23) a_{n+1} + 36(n+1)(2n+1)(2n+3) a_n = 0,$$

and if $d = 4$, then $b_n = U_{4,0,2n}$ satisfies [19]

$$(n+2)^4 b_{n+2} - 4(2n+3)^2 (5n^2 + 15n + 12) b_{n+1} + 256(n+1)^2 (2n+1)(2n+3) b_n = 0.$$

For any d, the mean **first-passage time** to arrive at any lattice point l is infinite (in spite of the fact that the associated probability $V_{d,l}(1) = 1$ for $d = 1$ or 2). There are several alternative ways of quantifying the length of required travel. Using our formulas for $V_{d,l}(x)$, the median first-passage times are 2-4, 1-3, 6-8, and 17-19 steps for $l = 0$, 1, 2, and 3 when $d = 1$, and 2-4, 25-27, and 520-522 steps for $l = (0,0)$, $(1,0)$, and $(1,1)$ when $d = 2$. Hughes [3,20] examined the conditional mean time to return to the origin (conditional upon return eventually occurring). Also, for $d = 1$, the mean time for the earliest of three independent random walkers to return to the origin is finite and has value [6,21–23]

$$2\sum_{n=0}^{\infty}\frac{1}{2^{6n}}\binom{2n}{n}^3 = \frac{2}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\frac{1}{1-\cos(\theta)\cos(\varphi)\cos(\psi)}\,d\theta\,d\varphi\,d\psi$$

$$= \frac{8}{\pi^2}K\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2\pi^3}\Gamma\left(\frac{1}{4}\right)^4 = 2(1.3932039296\ldots),$$

whereas for $d = 2$, the mean time for the earliest of an *arbitrary* number of independent random walkers is infinite. More on multiple random walkers, of both the friendly and vicious kinds, is found in [24].

It is known that

$$U_{d,l}(x) = \frac{d}{(2\pi)^d}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left(d - x\sum_{k=1}^{d}\cos(\theta_k)\right)^{-1}$$
$$\times \exp\left(i\sum_{k=1}^{d}\theta_k l_k\right)d\theta_1\,d\theta_2\cdots d\theta_d,$$

which can be numerically evaluated for small d. Here are some sample probabilities [11,16] that a three-dimensional walk reaches a point l:

$$V_{3,l}(1) = \frac{U_{3,l}(1)}{m_3} = \begin{cases} 0.3405373295\ldots & \text{if } l = (1,0,0), \\ 0.2183801414\ldots & \text{if } l = (1,1,0), \\ 0.1724297877\ldots & \text{if } l = (1,1,1). \end{cases}$$

An asymptotic expansion for these probabilities is [11,12]

$$V_{3,l}(1) = \frac{3}{2\pi m_3|l|}\left[1 + \frac{1}{8|l|^2}\left(-3 + \frac{5(l_1^4+l_2^4+l_3^4)}{|l|^2} + \cdots\right)\right] \sim \frac{0.3148702313\ldots}{|l|}$$

and is valid as $|l|^2 = l_1^2 + l_2^2 + l_3^2 \to \infty$.

Let $W_{d,n}$ be the average number of distinct vertices visited during a d-dimensional n-step walk. It can be shown that [25–28]

$$W_d(x) = \sum_{n=0}^{\infty}W_{d,n}x^n = \frac{1}{(1-x)^2U_{d,0}(x)}, \quad W_{d,n} \sim \begin{cases} \sqrt{\dfrac{8n}{\pi}} & \text{if } d = 1, \\ \dfrac{\pi n}{\ln(n)} & \text{if } d = 2, \\ (1-p_3)n & \text{if } d = 3 \end{cases}$$

as $n \to \infty$. Higher-order asymptotics for $W_{3,n}$ are possible using the expansion [11, 12, 29–31]

$$U_{3,0}(x) = m_3 - \frac{3\sqrt{3}}{2\pi}(1-x^2)^{\frac{1}{2}} + c(1-x^2) - \frac{3\sqrt{3}}{4\pi}(1-x^2)^{\frac{3}{2}} + \cdots,$$

where $x \to 1^-$ and

$$c = \frac{9}{32}\left(m_3 + \frac{6}{\pi^2 m_3}\right) = 0.5392381750\ldots.$$

Other parameters, for example, the average growth of distance from the origin [32],

$$\lim_{n\to\infty} \frac{1}{\ln(n)} \sum_{j=1}^{n} \frac{j^{-1/2}}{1+|\omega_j|} = \lambda_1 \quad \text{with probability 1, if } d = 1,$$

$$\lim_{n\to\infty} \frac{1}{\ln(n)^2} \sum_{j=1}^{n} \frac{1}{1+|\omega_j|^2} = \lambda_2 \quad \text{with probability 1, if } d = 2,$$

$$\lim_{n\to\infty} \frac{1}{\ln(n)} \sum_{j=1}^{n} \frac{1}{1+|\omega_j|^2} = \lambda_d \quad \text{with probability 1, if } d \geq 3,$$

are more difficult to analyze. The constants λ_d are known only to be finite and positive.

For a one-dimensional n-step walk ω, define M_n^+ to be the maximum value of ω_j and M_n^- to be the maximum value of $-\omega_j$. Then M_n^+ and M_n^- each follow the half-normal distribution [6.2] in the limit as $n \to \infty$, and [33, 34]

$$\lim_{n\to\infty} \mathrm{E}\left(n^{-\frac{1}{2}} M_n^+\right) = \sqrt{\frac{2}{\pi}} = \lim_{n\to\infty} \mathrm{E}\left(n^{-\frac{1}{2}} M_n^-\right).$$

Further, if T_n^+ is the smallest value of j for which $\omega_j = M_n^+$ and T_n^- is the smallest value of k for which $-\omega_k = M_n^-$, then the **arcsine law** applies:

$$\lim_{n\to\infty} \mathrm{P}\left(n^{-1} T_n^+ < x\right) = \frac{2}{\pi} \arcsin\sqrt{x} = \lim_{n\to\infty} \mathrm{P}\left(n^{-1} T_n^- < x\right),$$

which implies that a one-dimensional random walk tends to be either highly negative or highly positive (not both). Such detailed information about d-dimensional walks is not yet available. Define also $\tau_{d,r}$ to be the smallest value of j for which $|\omega_j| \geq r$, for any positive integer r. Then [35]

$$\tau_{1,r} = r^2, \quad \tau_{2,2} = \tfrac{9}{2}, \quad \tau_{2,3} = \tfrac{135}{13}, \quad \tau_{2,4} = \tfrac{11791}{668},$$

but a pattern is not evident. What precisely can be said about $\tau_{d,r}$ as $r \to \infty$?

As a computational aside, we mention a result of Odlyzko's [36–38]: Any algorithm that determines M_n^+ (or M_n^-) exactly must examine at least $(A + o(1))\sqrt{n}$ of the ω_j values on average, where $A = \sqrt{8/\pi}\ln(2) = 1.1061028674\ldots$.

On the one hand, the waiting time N_n for a one-dimensional random walk to hit a new vertex, not visited in the first n steps, satisfies [39]

$$\limsup_{n\to\infty} \frac{N_n}{n\ln(\ln(n))^2} = \frac{1}{\pi^2} \text{ with probability 1.}$$

On the other hand, if F_n denotes the set of vertices that are maximally visited by the random walk up to step n, called **favorite sites**, then $|F_n| \geq 4$ only finitely often, with probability 1 [40].

For two-dimensional random walks, we may define F_n analogously. The number of visits to a selected point in F_n within the first n steps is $\sim \ln(n)^2/\pi$ with probability 1, as $n \to \infty$. This can be rephrased as the asymptotic number of times a drunkard drops by his favorite watering hole [41,42]. Dually, the length of time C_r required to totally cover all vertices of the $r \times r$ torus (square with opposite sides identified) satisfies [43]

$$\lim_{r\to\infty} \mathrm{P}\left(\left|\frac{C_r}{r^2\ln(r)^2} - \frac{4}{\pi}\right| < \varepsilon\right) = 1$$

for every $\varepsilon > 0$ (convergence in probability). This solves what is known as the "white screen problem" [44].

If a three-dimensional random walk ω is restricted to the region $x \geq y \geq z$, then the analogous series coefficients are

$$\bar{a}_n = \sum_{k=0}^{n} \frac{(2n)!(2k)!}{(n-k)!(n+1-k)!k!^2(k+1)!^2},$$

and from this we have [45]

$$\bar{m}_3 = \sum_{n=0}^{\infty} \frac{\bar{a}_n}{6^{2n}} = 1.0693411205\ldots, \quad \bar{p}_3 = 1 - \frac{1}{\bar{m}_3} = 0.0648447153\ldots$$

characterizing the return. What can be said concerning other regions, for example, a half-space, quarter-space, or octant?

Here is one variation. Let $X_1, X_2, X_3, \ldots$ be independent normally distributed random variables with mean μ and variance 1. Consider the partial sums $S_j = \sum_{k=1}^{j} X_k$, which constitute a random walk on the real line (rather than the one-dimensional lattice) with Gaussian increments (rather than Bernoulli increments). There is an enormous literature on $\{S_j\}$, but we shall mention only one result. Let H be the first positive value of S_j, called the **first ladder height** of the process; then the moments of H when $\mu = 0$ are [46]

$$\mathrm{E}_0(H) = \frac{1}{\sqrt{2}}, \quad \mathrm{E}_0(H^2) = -\frac{\zeta(\frac{1}{2})}{\sqrt{\pi}} = \sqrt{2}\rho = \sqrt{2}(0.5825971579\ldots)$$

and, for arbitrary μ in a neighborhood of 0,

$$\mathrm{E}_\mu(H) = \frac{1}{\sqrt{2}} \exp\left[-\frac{\mu}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{\zeta(\frac{1}{2}-k)}{k!(2k+1)} \left(-\frac{\mu^2}{2}\right)^k\right],$$

where $\zeta(x)$ is the Riemann zeta function [1.6]. Other occurrences of the interesting constant ρ in the statistical literature are in [47–50].

Here is another variation. Let $Y_1, Y_2, Y_3, \ldots$ be independent Uniform $[-1, 1]$ random variables, $S_0 = 0$, and $S_j = \sum_{k=1}^{j} Y_k$. Then the expected maximum value of $\{S_0, S_1, \ldots, S_n\}$ is [51]

$$\mathrm{E}\left(\max_{0\le j\le n} S_j\right) = \sqrt{\frac{2}{3\pi}}\, n^{\frac{1}{2}} + \sigma + \frac{1}{5}\sqrt{\frac{2}{3\pi}}\, n^{-\frac{1}{2}} + O\left(n^{-\frac{3}{2}}\right)$$

as $n \to \infty$, where $\sigma = -0.2979521902\ldots$ is given by

$$\sigma = \frac{\zeta(\frac{1}{2})}{\sqrt{6\pi}} + \frac{\zeta(\frac{3}{2})}{20\sqrt{6\pi}} + \sum_{k=1}^{\infty}\left(\frac{t_k}{k} - \frac{k^{-\frac{1}{2}}}{\sqrt{6\pi}} - \frac{k^{-\frac{3}{2}}}{20\sqrt{6\pi}}\right)$$

and

$$t_k = \frac{2(-1)^k}{(k+1)!} \sum_{k/2\le j\le k} (-1)^j \binom{k}{j} \left(j - \frac{k}{2}\right)^{k+1}.$$

A deeper connection between $\zeta(x)$ and random walks is discussed in [52].

5.9.1 Intersections and Trappings

A walk ω on the lattice L is **self-intersecting** if $\omega_i = \omega_j$ for some $i < j$, and the **self-intersection time** is the smallest value of j for which this happens. Computing self-intersection times is more difficult than first-passage times since the entire history of the walk requires memorization. If $d = 1$, then clearly the mean self-intersection time is 3. If $d = 2$, the mean self-intersection time is [53]

$$\begin{aligned}\frac{2\cdot 4}{4^2} + \frac{3\cdot 12}{4^3} + \frac{4\cdot 44}{4^4} + \frac{5\cdot 116}{4^5} + \cdots &= \sum_{n=2}^{\infty} \frac{n(4c_{n-1} - c_n)}{4^n} \\ &= \frac{c_1}{2} + \sum_{n=2}^{\infty} \frac{c_n}{4^n} = 4.5860790989\ldots,\end{aligned}$$

where the sequence $\{c_n\}$ is defined in [5.10]. When n is large, no exact formula for evaluating c_n is known, unlike the sequences $\{a_n\}$, $\{\bar{a}_n\}$, and $\{b_n\}$ discussed earlier. We are, in this example, providing foreshadowing of difficulties to come later. See the generalization in [54, 55].

A walk ω is **self-trapping** if, for some k, $\omega_i \ne \omega_j$ for all $i < j \le k$ and ω_k is completely surrounded by previously visited vertices. If $d = 2$, there are eight self-trapping walks when $k = 7$ and sixteen such walks when $k = 8$. A Monte Carlo simulation in [56, 57] gave a mean self-trapping time of approximately $70.7\ldots$.

Two walks ω and ω' **intersect** if $\omega_i = \omega'_j$ for some nonzero i and j. The probability q_n that two n-step independent random walks never intersect satisfies [58–61]

$$\ln(q_n) \sim \begin{cases} -\frac{5}{8}\ln(n) & \text{if } d = 2, \\ -\xi \ln(n) & \text{if } d = 3, \\ -\frac{1}{2}\ln(\ln(n)) & \text{if } d = 4 \end{cases}$$

as $n \to \infty$, where the exponent ξ is approximately $0.29\ldots$ (again obtained by simulation). For each $d \geq 5$, it can be shown [62] that $\lim_{n\to\infty} q_n$ lies strictly between 0 and 1. Further simulation [63] yields $q_5 = 0.708\ldots$ and $q_6 = 0.822\ldots$, and we shall refer to these in [5.10].

5.9.2 Holonomicity

A **holonomic function** (in the sense of Zeilberger [45, 64, 65]) is a solution $f(z)$ of a linear homogeneous differential equation

$$f^{(n)}(z) + r_1(z) f^{(n-1)}(z) + \cdots + r_{n-1}(z) f'(z) + r_n(z) f(z) = 0,$$

where each $r_k(z)$ is a rational function with rational coefficients. **Regular holonomic constants** are values of f at algebraic points z_0 where each r_k is analytic; f can be proved to be analytic at z_0 as well. **Singular holonomic constants** are values of f at algebraic points z_0 where each r_k has, at worst, a pole of order k at z_0 (called Fuchsian or "regular" singularities [66–68]). The former include π, $\ln(2)$, and the tetralogarithm $\mathrm{Li}_4(1/2)$; the latter include Apéry's constant $\zeta(3)$, Catalan's constant G, and Pólya's constants p_d, $d > 2$. Holonomic constants of either type fall into the class of polynomial-time computable constants [69]. We merely mention a somewhat related theory of EL numbers due to Chow [70].

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5.10 Self-Avoiding Walk Constants

Let L denote the d-dimensional cubic lattice whose vertices are precisely all integer points in d-dimensional space. An n-step **self-avoiding walk** ω on L, beginning at the origin, is a sequence of vertices $\omega_0, \omega_1, \omega_2, \ldots, \omega_n$ with $\omega_0 = 0$, $|\omega_{j+1} - \omega_j| = 1$ for all j and $\omega_i \neq \omega_j$ for all $i \neq j$. The number of such walks is denoted by c_n. For example, $c_0 = 1, c_1 = 2d, c_2 = 2d(2d-1), c_3 = 2d(2d-1)^2$, and $c_4 = 2d(2d-1)^3 - 2d(2d-2)$. Self-avoiding walks are vastly more difficult to study than ordinary walks [1–6], and historically arose as a model for linear polymers in chemistry [7, 8]. No exact combinatorial enumerations are possible for large n. The methods for analysis hence include finite series expansions and Monte Carlo simulations.

For simplicity's sake, we have suppressed the dependence of c_n on d; we will do this for associated constants too whenever possible.

What can be said about the asymptotics of c_n? Since $c_{n+m} \leq c_n c_m$, on the basis of Fekete's submultiplicativity theorem [9–12], it is known that the **connective constant**

$$\mu_d = \lim_{n\to\infty} c_n^{\frac{1}{n}} = \inf_n c_n^{\frac{1}{n}}$$

exists and is nonzero. Early attempts to estimate $\mu = \mu_d$ included [13–15]; see [2] for a detailed survey. The current best rigorous lower and upper bounds for μ, plus the best-known estimate, are given in Table 5.2 [16–24]. The extent of our ignorance is fairly surprising: Although we know that $\mu^2 = \lim_{n\to\infty} c_{n+2}/c_n$ and $c_{n+1} \geq c_n$ for all n and all d, proving that $\mu = \lim_{n\to\infty} c_{n+1}/c_n$ for $2 \leq d \leq 4$ remains an open problem [25, 26].

Table 5.2. *Estimates for Connective Constant* μ

d	Lower Bound	Best Estimate for μ	Upper Bound
2	2.6200	2.6381585303	2.6792
3	4.5721	4.68404	4.7114
4	6.7429	6.77404	6.8040
5	8.8285	8.83854	8.8602
6	10.8740	10.87809	10.8886

It is believed that there exists a positive constant $\gamma = \gamma_d$ such that the following limit exists and is nonzero:

$$A = \begin{cases} \lim\limits_{n\to\infty} \dfrac{c_n}{\mu^n n^{\gamma-1}} & \text{if } d \neq 4, \\ \lim\limits_{n\to\infty} \dfrac{c_n}{\mu^n n^{\gamma-1}\ln(n)^{1/4}} & \text{if } d = 4. \end{cases}$$

The **critical exponent** γ is conjectured to be [27–29]

$$\gamma_2 = \tfrac{43}{32} = 1.34375, \quad \gamma_3 = 1.1575\ldots, \quad \gamma_4 = 1$$

and has been proved [1,30] to equal 1 for $d > 4$. For small d, we have bounds [1,25,31]

$$c_n \le \begin{cases} \mu^n \exp\left(C n^{1/2}\right) & \text{if } d = 2, \\ \mu^n \exp\left(C n^{2/(d+2)}\ln(n)\right) & \text{if } 3 \le d \le 4, \end{cases}$$

which do not come close to proving the existence of A. It is known [32] that, for $d = 5$, $1 \le A \le 1.493$ and, for sufficiently large d, $A = 1 + (2d)^{-1} + d^{-2} + O(d^{-3})$.

Another interesting object of study is the **mean square end-to-end distance**

$$r_n = \mathrm{E}\left(|\omega_n|^2\right) = \frac{1}{c_n}\sum_{\omega} |\omega_n|^2,$$

where the summation is over all n-step self-avoiding walks ω on L. Like c_n, it is believed that there is a positive constant $\nu = \nu_d$ such that the following limit exists and is nonzero:

$$B = \begin{cases} \lim\limits_{n\to\infty} \dfrac{r_n}{n^{2\nu}} & \text{if } d \neq 4, \\ \lim\limits_{n\to\infty} \dfrac{r_n}{n^{2\nu}\ln(n)^{1/4}} & \text{if } d = 4. \end{cases}$$

As before, it is conjectured that [27,33,34]

$$\nu_2 = \tfrac{3}{4} = 0.75, \quad \nu_3 = 0.5877\ldots, \quad \nu_4 = \tfrac{1}{2} = 0.5$$

and has been proved [1,30] that $\nu = 1/2$ for $d > 4$. This latter value is the same for Pólya walks, that is, the self-avoidance constraint has little effect in high dimensions. It is known [32] that, for $d = 5$, $1.098 \le B \le 1.803$ and, for sufficiently large d, $B = 1 + d^{-1} + 2d^{-2} + O(d^{-3})$. Hence a self-avoiding walk moves away from the origin faster than a Pólya walk, but only at the level of the amplitude and not at the level of the exponent.

If we accept the conjectured asymptotics $c_n \sim A\mu^n n^{\gamma-1}$ and $r_n \sim Bn^{2\nu}$ as truth (for $d \neq 4$), then the calculations shown in Table 5.3 become possible [23,24,33,35–37].

Table 5.3. *Estimates for Amplitudes A and B*

d	Estimate for A	Estimate for B	d	Estimate for A	Estimate for B
2	1.177043	0.77100	5	1.275	1.4767
3	1.205	1.21667	6	1.159	1.2940

(The logarithmic correction for $d = 4$ renders any reliable estimation of A or B very difficult.) Here is an application. Two walks ω and ω' **intersect** if $\omega_i = \omega'_j$ for some nonzero i and j. The probability that two n-step independent random self-avoiding walks never intersect is [1,38]

$$\frac{c_{2n}}{c_n^2} \sim \begin{cases} A^{-1}2^{\gamma-1}n^{1-\gamma} \to 0 & \text{if } 2 \le d \le 3, \\ A^{-1}\ln(n)^{-1/4} \to 0 & \text{if } d = 4, \\ A^{-1} > 0 & \text{if } d \ge 5 \end{cases}$$

as $n \to \infty$. This conjectured behavior is consistent with intuition: c_{2n}/c_n^2 is (slightly) larger than the corresponding probability q_n for ordinary walks [5.9.1] since self-avoiding walks tend to be more thinly dispersed in space.

Other interesting measures of the size of a walk include the **mean square radius of gyration**,

$$s_n = \mathrm{E}\left(\frac{1}{n+1}\sum_{i=0}^{n}\left|\omega_i - \frac{1}{n+1}\sum_{j=0}^{n}\omega_j\right|^2\right) = \mathrm{E}\left(\frac{1}{2(n+1)^2}\sum_{i=0}^{n}\sum_{j=0}^{n}\left|\omega_i - \omega_j\right|^2\right),$$

and the **mean square distance of a monomer from the endpoints**,

$$t_n = \mathrm{E}\left(\frac{1}{n+1}\sum_{i=0}^{n}\frac{|\omega_i|^2 + |\omega_n - \omega_i|^2}{2}\right).$$

The radius of gyration, for example, can be experimentally measured for polymers in a dilute solution via light scattering, but the end-to-end distance is preferred for theoretical simplicity [33,39–41]. It is conjectured that $s_n \sim En^{2\nu}$ and $t_n \sim Fn^{2\nu}$, where ν is the same exponent as for r_n, and $E/B = 0.14026\ldots$, $F/B = 0.43961\ldots$ for $d = 2$ and $E/B = 0.1599\ldots$ for $d = 3$.

One can generalize this discussion to arbitrary lattices L in d-dimensional space. For example, in the case $d = 2$, there is a rigorous upper bound $\mu < 4.278$ and an estimate $\mu = 4.1507951\ldots$ for the equilateral triangular lattice [17,35,42–45], and it is conjectured that $\mu = \sqrt{2+\sqrt{2}} = 1.8477590650\ldots$ for the hexagonal (honeycomb) lattice [46–48]. The critical exponents γ, ν and amplitude ratios E/B, F/B, however, are thought to be *universal* in the sense that they are lattice-independent (although dimension-dependent). An important challenge, therefore, is to better understand the nature of such exponents and ratios, and certainly to prove their existence in low dimensions.

5.10.1 Polygons and Trails

The connective constant μ values given previously apply not only to the asymptotic growth of the number of self-avoiding walks, but also to the asymptotic growth of numbers of **self-avoiding polygons** and of self-avoiding walks with prescribed endpoints [2,49]. See [5.19] for discussion of lattice animals or polyominoes, which are related to self-avoiding polygons.

No site or bond may be visited more than once in a self-avoiding walk. By way of contrast, a **self-avoiding trail** may revisit sites, but not bonds. Thus walks are a proper subset of trails [50–55]. The number h_n of trails is conjectured to satisfy $h_n \sim G\lambda^n n^{\gamma-1}$, where γ is the same exponent as for c_n. The connective constant λ provably exists as before and, in fact, satisfies $\lambda \geq \mu$. For the square lattice, there are rigorous bounds $2.634 < \lambda < 2.851$ and an estimate $\lambda = 2.72062\ldots$; the amplitude is approximately $G = 1.272\ldots$. For the cubic lattice, there is an upper bound $\lambda < 4.929$ and an estimate $\lambda = 4.8426\ldots$. Many related questions can be asked.

5.10.2 Rook Paths on a Chessboard

How many self-avoiding walks can a rook take from a fixed corner of an $m \times n$ chessboard to the opposite corner without ever leaving the chessboard? Denote the number of such **paths** by $p_{m-1,n-1}$; clearly $p_{k,1} = 2^k$, $p_{2,2} = 12$, and [56–58]

$$p_{k,2} \sim \frac{4+\sqrt{13}}{2\sqrt{13}}\left(\sqrt{\frac{3+\sqrt{13}}{2}}\right)^{2k} = 1.0547001962\ldots \cdot (1.8173540210\ldots)^{2k}$$

as $k \to \infty$. More broadly, the generating function for the sequence $\{p_{k,l}\}_{k=1}^{\infty}$ is rational for any integer $l \geq 1$ and thus relevant asymptotic coefficients are all algebraic numbers. What can be said about the asymptotics of $p_{k,k}$ as $k \to \infty$? Whittington & Guttmann [59] proved that

$$p_{k,k} \sim (1.756\ldots)^{k^2}$$

and conjectured the following [60, 61]. If $\pi_{j,k}$ is the number of j-step paths with generating function

$$P_k(x) = \sum_{j=1}^{\infty} \pi_{j,k} x^j, \quad P_k(1) = p_{k,k}$$

then there is a *phase transition* in the sense that

$$\begin{aligned}
&0 < \lim_{k\to\infty} P_k(x)^{\frac{1}{k}} < 1 \quad \text{exists for } 0 < x < \mu^{-1} = 0.3790522777\ldots, \\
&\lim_{k\to\infty} P_k(\mu^{-1})^{\frac{1}{k}} = 1, \\
&1 < \lim_{k\to\infty} P_k(x)^{\frac{1}{k^2}} < \infty \text{ exists for } x > \mu^{-1}.
\end{aligned}$$

A proof was given by Madras [62]. This is an interesting occurrence of the connective constant $\mu = \mu_2$; an analogous theorem involving a d-dimensional chessboard also holds and naturally makes use of μ_d.

5.10.3 Meanders and Stamp Foldings

A **meander** of order n is a planar self-avoiding loop (road) crossing an infinite line (river) $2n$ times ($2n$ bridges). Define two meanders as equivalent if one may be deformed continuously into the other, keeping the bridges fixed. The number of inequivalent meanders M_n of order n satisfy $M_1 = 1$, $M_2 = 2$, $M_3 = 8$, $M_4 = 42$, $M_5 = 262, \ldots$.

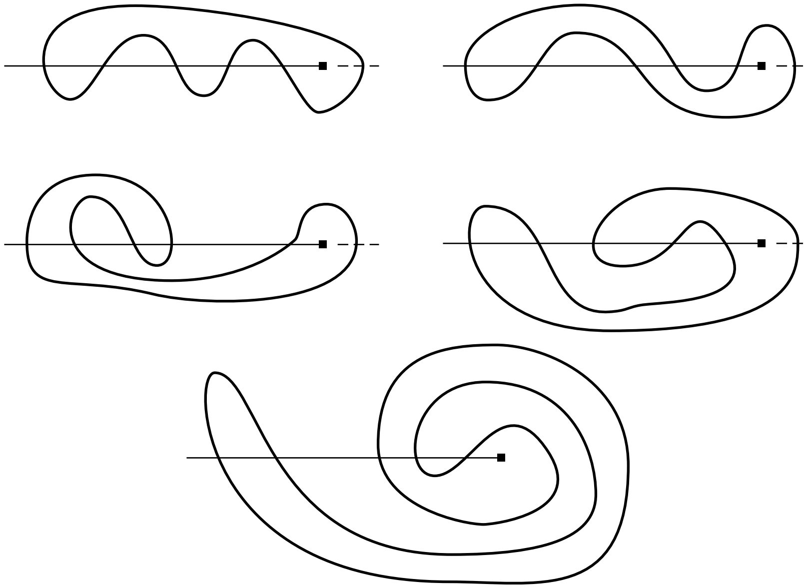

Figure 5.11. There are eight meanders of order 3 and ten semi-meanders of order 5; reflections across the river are omitted.

A **semi-meander** of order n is a planar self-avoiding loop (road) crossing a semi-infinite line (river with a source) n times (n bridges). Equivalence of semi-meanders is defined similarly. The number of inequivalent semi-meanders $\tilde{M}_n$ of order n satisfy $\tilde{M}_1 = 1$, $\tilde{M}_2 = 1$, $\tilde{M}_3 = 2$, $\tilde{M}_4 = 4$, $\tilde{M}_5 = 10, \ldots$.

Counting meanders and semi-meanders has attracted much attention [63–73]. See Figure 5.11. As before, we expect asymptotic behavior

$$M_n \sim C\frac{R^{2n}}{n^{\alpha}}, \quad \tilde{M}_n \sim \tilde{C}\frac{R^n}{n^{\tilde{\alpha}}},$$

where $R = 3.501838\ldots$, that is, $R^2 = 12.262874\ldots$. No exact formula for the connective constant R is known. In contrast, there is a conjecture [74–76] that the critical exponents are given by

$$\alpha = \sqrt{29}\frac{\sqrt{29}+\sqrt{5}}{12} = 3.4201328816\ldots,$$

$$\tilde{\alpha} = 1 + \sqrt{11}\frac{\sqrt{29}+\sqrt{5}}{24} = 2.0531987328\ldots,$$

but doubt has been raised [77–79] about the semi-meander critical exponent value. The sequences $\tilde{M}_n$ and M_n are also related to enumerating the ways of folding a linear or circular row of stamps onto one stamp [80–87].

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5.11 Feller's Coin Tossing Constants

Let w_n denote the probability that, in n independent tosses of an ideal coin, no run of three consecutive heads appears. Clearly $w_0 = w_1 = w_2 = 1$, $w_n = \frac{1}{2}w_{n-1} + \frac{1}{4}w_{n-2} + \frac{1}{8}w_{n-3}$ for $n \geq 3$, and $\lim_{n\to\infty} w_n = 0$. Feller [1] proved the following more precise asymptotic result:

$$\lim_{n\to\infty} w_n \alpha^{n+1} = \beta,$$

where

$$\alpha = \frac{\left(136 + 24\sqrt{33}\right)^{\frac{1}{3}} - 8\left(136 + 24\sqrt{33}\right)^{-\frac{1}{3}} - 2}{3} = 1.0873780254\ldots$$

and

$$\beta = \frac{2-\alpha}{4-3\alpha} = 1.2368398446\ldots.$$

We first examine generalizations of these formulas. If runs of k consecutive heads, $k > 1$, are disallowed, then the analogous constants are [1,2]

$$\alpha \text{ is the smallest positive root of } 1 - x + \left(\frac{x}{2}\right)^{k+1} = 0$$

and

$$\beta = \frac{2-\alpha}{k+1-k\alpha}.$$

Equivalently, the generating function that enumerates coin toss sequences with no runs of k consecutive heads is [3]

$$S_k(z) = \frac{1-z^k}{1-2z+z^{k+1}}, \quad \frac{1}{n!}\frac{d^n}{dz^n}S_k(z)\bigg|_{z=0} \sim \frac{\beta}{\alpha}\left(\frac{2}{\alpha}\right)^n.$$

See [4–8] for more material of a combinatorial nature.

If the coin is non-ideal, that is, if $\mathrm{P}(H) = p$, $\mathrm{P}(T) = q$, $p + q = 1$, but p and q are not equal, then the asymptotic behavior of w_n is governed by

$$\alpha \text{ is the smallest positive root of } 1 - x + qp^k x^{k+1} = 0$$

and

$$\beta = \frac{1-p\alpha}{(k+1-k\alpha)q}.$$

A further generalization involves time-homogeneous two-state Markov chains. It makes little sense here to talk of coin tosses, so we turn attention to a different application. Imagine that a ground-based sensor determines once per hour whether a fixed line-of-sight through the atmosphere is cloud-obscured (0) or clear (1). Since meteorological events often display persistence through time, the sensor observations are not independent. A simple model for the time series $X_1, X_2, X_3, \ldots$ of observations might be a Markov chain with transition probability matrix

$$\begin{pmatrix} \mathrm{P}(X_{j+1}=0|X_j=0) & \mathrm{P}(X_{j+1}=1|X_j=0) \\ \mathrm{P}(X_{j+1}=0|X_j=1) & \mathrm{P}(X_{j+1}=1|X_j=1) \end{pmatrix} = \begin{pmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{pmatrix},$$

where conditional probability parameters satisfy $\pi_{00}+\pi_{01}=1=\pi_{10}+\pi_{11}$. The special case when $\pi_{00}=\pi_{10}$ and $\pi_{01}=\pi_{11}$ is equivalent to the Bernoulli trials scenario discussed in connection with coin tossing. Let $w_{n,k}$ denote the probability that no cloudy intervals of length $k>1$ occur, and assume that initially $\mathrm{P}(X_0=1)=\theta_1$. The asymptotic behavior is similar to before, where α is the smallest positive root of [9, 10]

$$1-(\pi_{11}+\pi_{00})x+(\pi_{11}-\pi_{01})x^2+\pi_{10}\pi_{01}\pi_{11}^{k-1}x^{k+1}=0$$

and

$$\beta=\frac{[-1+(2\pi_{11}-\pi_{01})\alpha-(\pi_{11}-\pi_{01})\pi_{11}\alpha^2][\theta_1+(\pi_{01}-\theta_1)\alpha]}{\pi_{10}\pi_{01}[-1-k+(\pi_{11}+\pi_{00})k\alpha+(\pi_{11}-\pi_{01})(1-k)\alpha^2]}.$$

See [11] for a general technique for analysis of pattern statistics, with applications in molecular biology.

Of many possible variations on this problem, we discuss one. How many patterns of n children in a row are there if every girl is next to at least one other girl? If we denote the answer by Y_n, then $Y_1=1$, $Y_2=2$, $Y_3=4$, and $Y_n=2Y_{n-1}-Y_{n-2}+Y_{n-3}$ for $n\geq 4$; hence

$$\lim_{n\to\infty}\frac{Y_{n+1}}{Y_n}=\frac{\left(100+12\sqrt{69}\right)^{\frac{1}{3}}+4\left(100+12\sqrt{69}\right)^{-\frac{1}{3}}+4}{6}=1.7548776662\ldots.$$

A generalization of this, in which the girls must appear in groups of at least k, is given in [12, 13]. Similar cubic irrational numbers occur in [1.2.2].

Let us return to coin tossing. What is the expected length of the longest run of consecutive heads in a sequence of n ideal coin tosses? The answer is surprisingly complicated [14–21]:

$$\sum_{k=1}^{n}(1-w_{n,k})=\frac{\ln(n)}{\ln(2)}-\left(\frac{3}{2}-\frac{\gamma}{\ln(2)}\right)+\delta(n)+o(1)$$

as $n\to\infty$, where γ is the Euler-Mascheroni constant and

$$\delta(n)=\frac{1}{\ln(2)}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\Gamma\left(\frac{2\pi ik}{\ln(2)}\right)\exp\left(-2\pi ik\frac{\ln(n)}{\ln(2)}\right).$$

That is, the expected length is $\ln(n)/\ln(2)-0.6672538227\ldots$ plus an oscillatory, small-amplitude correction term. The function $\delta(n)$ is periodic ($\delta(n)=\delta(2n)$), has zero mean, and is "negligible" ($|\delta(n)|<1.574\times10^{-6}$ for all n). The corresponding variance is $C + c + \varepsilon(n) + o(1)$, where $\varepsilon(n)$ is another small-amplitude function and

$$C = \frac{1}{12} + \frac{\pi^2}{6\ln(2)^2} = 3.5070480758\ldots,$$

$$c = \frac{2}{\ln(2)} \sum_{k=0}^{\infty} \ln\left[1 - \exp\left(-\frac{2\pi^2}{\ln(2)}(2k+1)\right)\right] = (-1.237412\ldots) \times 10^{-12}.$$

Functions similar to $\delta(n)$ and $\varepsilon(n)$ appear in [2.3], [2.16], [5.6], and [5.14].

Also, if we toss n ideal coins, then toss those which show tails after the first toss, then toss those which show tails after the second toss, etc., what is the probability that the final toss involves exactly one coin? Again, the answer is complicated [22–25]:

$$\frac{n}{2}\sum_{j=0}^{\infty} 2^{-j}(1-2^{-j})^{n-1} \sim \frac{1}{2\ln(2)} + \rho(n) + o(1)$$

as $n \to \infty$, where

$$\rho(n) = \frac{1}{2\ln(2)} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \Gamma\left(1 - \frac{2\pi i k}{\ln(2)}\right) \exp\left(2\pi i k \frac{\ln(n)}{\ln(2)}\right).$$

That is, the probability of a unique survivor (no ties) at the end is $1/(2\ln(2)) = 0.7213475204\ldots$ plus an oscillatory function satisfying $|\rho(n)| < 7.131 \times 10^{-6}$ for all n. The expected length of the longest of the n coin toss sequences is $\sum_{j=0}^{\infty}[1-(1-2^{-j})^n]$ and can be analyzed similarly [26]. Related discussion is found in [27–31].

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5.12 Hard Square Entropy Constant

Consider the set of all $n \times n$ binary matrices. What is the number $F(n)$ of such matrices with no pairs of adjacent 1s? Two 1s are said to be adjacent if they lie in positions (i, j) and $(i + 1, j)$, or if they lie in positions (i, j) and $(i, j + 1)$, for some i, j. Equivalently, $F(n)$ is the number of configurations of non-attacking Princes on an $n \times n$ chessboard, where a "Prince" attacks the four adjacent, non-diagonal places. Let $N = n^2$; then [1–3]

$$\kappa = \lim_{n\to\infty} F(n)^{\frac{1}{N}} = 1.5030480824\ldots = \exp(0.4074951009\ldots)$$

is the **hard square entropy constant**. Earlier estimates were obtained by both physicists [4–9] and mathematicians [10–13]. Some related combinatorial enumeration problems appear in [14–16].

Instead of an $n \times n$ binary matrix, consider an $n \times n$ binary array that looks like

$$\begin{pmatrix} a_{11} & & a_{23} & \\ & a_{22} & & a_{34} \\ a_{21} & & a_{33} & \\ & a_{32} & & a_{44} \\ a_{31} & & a_{43} & \\ & a_{42} & & a_{54} \\ a_{41} & & a_{53} & \\ & a_{52} & & a_{64} \end{pmatrix}$$

(here $n = 4$). What is the number $G(n)$ of such arrays with no pairs of adjacent 1s? Two 1s here are said to be adjacent if they lie in positions (i, j) and $(i+1, j)$, or in (i, j) and $(i, j+1)$, or in (i, j) and $(i+1, j+1)$, for some i, j. Equivalently, $G(n)$ is the number of configurations of non-attacking Kings on an $n \times n$ chessboard with regular hexagonal cells. It is surprising that the **hard hexagon entropy constant**

$$\kappa = \lim_{n\to\infty} G(n)^{\frac{1}{N}} = 1.3954859724\ldots = \exp(0.3332427219\ldots)$$

is *algebraic* (in fact, is solvable in radicals [17–22]) with minimal integer polynomial [23]

$$\begin{aligned} &25937424601x^{24} + 2013290651222784x^{22} + 2505062311720673792x^{20} \\ &\quad + 797726698866658379776x^{18} + 74494883101310831001 60x^{16} \\ &\quad + 29580150383769582305 28x^{14} - 72405670285649161617408x^{12} \\ &\quad + 107155448150443388043264x^{10} - 71220809441400405884928x^{8} \\ &\quad - 73347491183630103871488x^{6} + 97143135277377575190528x^{4} \\ &\quad - 32751691810479015985152. \end{aligned}$$

This is a consequence of Baxter's exact solution of the hard hexagon model [24–27] via theta elliptic functions and the Rogers–Ramanujan identities from number theory [28–31]! The expression for κ, in fact, comes out of a more general expression for

$$\kappa(z) = \lim_{n\to\infty} Z_n(z)^{\frac{1}{N}},$$

where $Z_n(z)$ is known as the **partition function** for the model and $G(n) = Z_n(1)$, $\kappa = \kappa(1)$. More on the physics of phase transitions in lattice gas models is found in [5.12.1].

McKay and Calkin independently calculated that, if we replace Princes by Kings on the chessboard with square cells, then the corresponding constant κ is $1.3426439511\ldots$; see also [32–34]. Note that the distinction between Princes and Kings on a chessboard with regular hexagonal cells is immaterial. (Clarification: If a Prince occupies cell c, then any cell sharing an edge with c is vulnerable to attack. If a King occupies cell c, by contrast, then any cell sharing *either* an edge or corner with c is vulnerable.)

If we examine instead a chessboard with equilateral triangular cells, then $\kappa = 1.5464407087\ldots$ for Princes [3]. This may be called the **hard triangle entropy constant**. The value of κ when replacing Princes by Kings here is not known.

What are the constants κ for non-attacking Knights or Queens on chessboards with square cells? The analysis for Knights should be similar to that for Princes and Kings, but for Queens everything is different since interactions are no longer local [35].

The hard square entropy constant also appears in the form $\ln(\kappa)/\ln(2) = 0.5878911617\ldots$ in several coding-theoretic papers [36–41], with applications including holographic data storage and retrieval.

5.12.1 Phase Transitions in Lattice Gas Models

Statistical mechanics is concerned with the average properties of a large system of particles. We consider here, for example, the phase transition from a disordered fluid state to an ordered solid state, as temperature falls or density increases.

A simple model for this phenomenon is a **lattice gas**, in which particles are placed on the sites of a regular lattice and only adjacent particles interact. This may appear to be hopelessly idealized, as rigid molecules could not possibly satisfy such strict symmetry requirements. The model is nevertheless useful in understanding the link between microscopic and macroscopic descriptions of matter.

Two types of lattice gas models that have been studied extensively are the **hard square** model and the **hard hexagon** model. Once a particle is placed on a lattice site, no other particle is allowed to occupy the same site or any next to it, as pictured in Figure 5.12. Equivalently, the indicated squares and hexagons cannot overlap, hence giving rise to the adjective "hard."

Given a (square or triangular) lattice of N sites, assign a variable $\sigma_i = 1$ if site i is occupied and $\sigma_i = 0$ if it is vacant, for each $1 \le i \le N$. We study the **partition function**

$$Z_n(z) = \sum_{\sigma} \left(z^{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_N} \cdot \prod_{(i,j)} (1 - \sigma_i \sigma_j) \right),$$

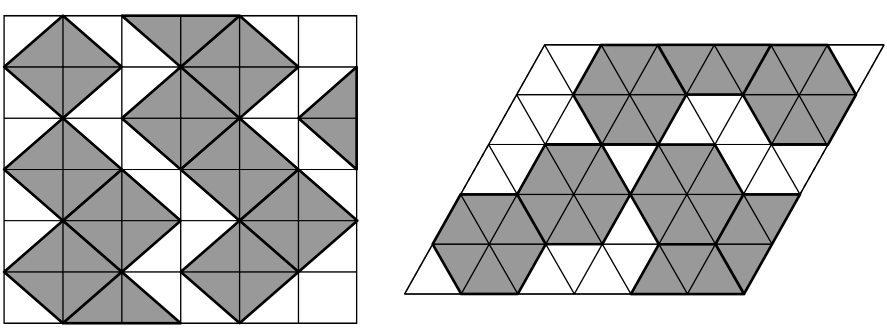

Figure 5.12. Hard squares and hard hexagons sit, respectively, on the square lattice and triangular lattice.

where the sum is over all 2^N possible values of the vector $\sigma = (\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_N)$ and the product is over all edges of the lattice (sites i and j are distinct and adjacent). Observe that the product enforces the nearest neighbor exclusion: If a configuration has two particles next to each other, then zero contribution is made to the partition function.

It is customary to deal with boundary effects by wrapping the lattice around to form a torus. More precisely, for the square lattice, $2n$ new edges are created to connect the n rightmost and n topmost points to corresponding n leftmost and n bottommost points. Hence there are a total of $2N$ edges in the square lattice, each site "looking like" every other. For the triangular lattice, $4n - 1$ new edges are created, implying a total of $3N$ edges. In both cases, the number of boundary sites, relative to N, is vanishingly small as $n \to \infty$, so this convention does not lead to any error.

Clearly the following combinatorial expressions are true [4, 42, 43]: For the square lattice,

$$Z_n = \sum_{k=0}^{\lfloor N/2 \rfloor} f_{k,n} z^k, \quad f_{0,n} = 1, \quad f_{1,n} = N, \quad f_{2,n} = \begin{cases} 2 & \text{if } n = 2, \\ \frac{1}{2} N(N-5) & \text{if } n \geq 3, \end{cases}$$

$$f_{3,n} = \begin{cases} 6 & \text{if } n = 3, \\ \frac{1}{6}\left(N(N-10)(N-13) + 4N(N-9) + 4N(N-8)\right) & \text{if } n \geq 4, \end{cases}$$

where $f_{k,n}$ denotes the number of allowable tilings of the N-site lattice with k squares, and for the triangular lattice,

$$Z_n = \sum_{k=0}^{\lfloor N/3 \rfloor} g_{k,n} z^k, \quad g_{0,n} = 1, \quad g_{1,n} = N, \; g_{2,n} = \tfrac{1}{2} N(N-7),$$

$$g_{3,n} = \begin{cases} 0 & \text{if } n = 3, \\ \frac{1}{6}\left(N(N-14)(N-19) + 6N(N-13) + 6N(N-12)\right) & \text{if } n \geq 4, \end{cases}$$

where $g_{k,n}$ denotes the corresponding number of hexagonal tilings.

Returning to physics, we remark that the partition function is important since it acts as the "denominator" in probability calculations. For example, consider the two sublattices A and B of the square lattice with sites as shown in Figure 5.13. The probability that an arbitrary site α in the sublattice A is occupied is

$$\rho_A(z) = \lim_{n \to \infty} \frac{1}{Z_n} \sum_{\sigma} \left(\sigma_\alpha \cdot z^{\sigma_1 + \sigma_2 + \sigma_3 + \cdots + \sigma_N} \cdot \prod_{(i,j)} (1 - \sigma_i \sigma_j) \right),$$

which is also called the local density at α. We can define analogous probabilities for the three sublattices A, B, and C of the triangular lattice.

We are interested in the behavior of these models as a function of the positive variable z, known as the **activity**. Figure 5.14, for example, exhibits a graph of the mean density for the hard hexagon case:

$$\rho(z) = z \frac{d}{dz}(\ln(\kappa(z))) = \frac{\rho_A(z) + \rho_B(z) + \rho_C(z)}{3}$$

using the exact formulation given in [18].

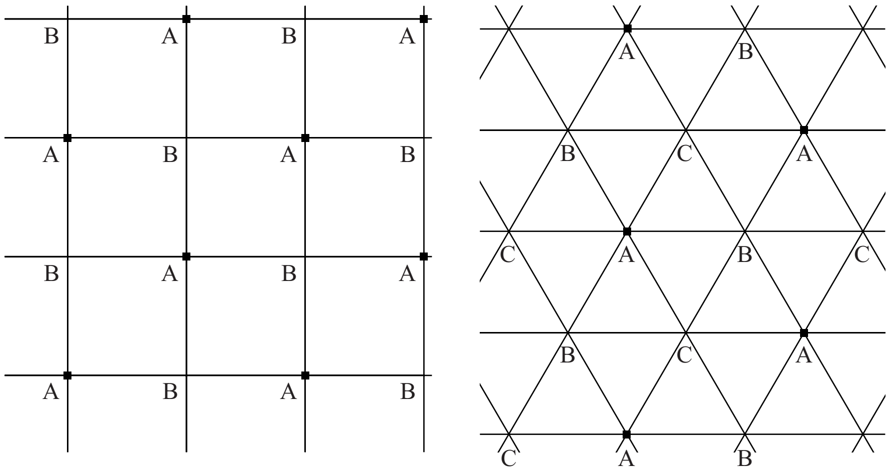

Figure 5.13. Two sublattices of the square lattice and three sublattices of the triangular lattice.

The existence of a phase transition is visually obvious. Let us look at the extreme cases: closely-packed configurations (large z) and sparsely-distributed configurations (small z). For infinite z, one of the possible sublattices is completely occupied, assumed to be the A sublattice, and the others are completely vacant; that is,

$$\rho_A = 1, \ \rho_B = 0 \ \ \text{(for the square model)}$$

and

$$\rho_A = 1, \ \rho_B = \rho_C = 0 \ \ \text{(for the hexagon model).}$$

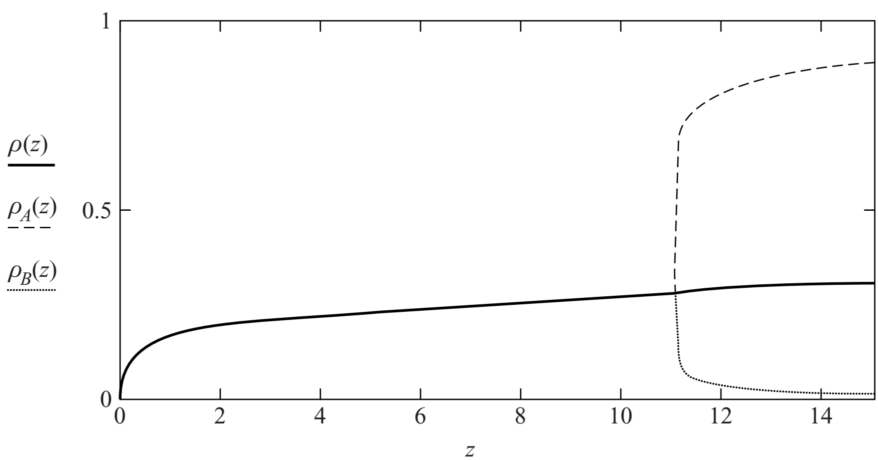

Figure 5.14. Graph of the mean density and sublattice densities, as functions of z, for the hexagon model.

For z close to zero, there is no preferential ordering on the sublattices; that is,

$$\rho_A = \rho_B \text{ (for the square model) and } \rho_A = \rho_B = \rho_C \text{ (for the hexagon model).}$$

Low activity corresponds to homogeneity and high activity corresponds to heterogeneity; thus there is a critical value, z_c, at which a phase transition occurs. Define the **order parameter**

$$R = \rho_A - \rho_B \text{ (for squares) and } R = \rho_A - \rho_B = \rho_A - \rho_C \text{ (for hexagons);}$$

then $R = 0$ for $z < z_c$ and $R > 0$ for $z > z_c$.

Elaborate numerical computations [7,44,45] have shown that, in the limit as $n \to \infty$,

$$z_c = 3.7962\ldots \text{ (for squares) and } z_c = 11.09\ldots \text{ (for hexagons),}$$

assuming site α to be infinitely deep within the lattice. The computations involved highly-accurate series expansions for R and what are known as corner transfer matrices, which we cannot discuss here for reasons of space.

In a beautiful development, Baxter [24,25] provided an exact solution of the hexagon model. The full breadth of this accomplishment cannot be conveyed here, but one of many corollaries is the exact formula

$$z_c = \frac{11 + 5\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^5 = 11.0901699437\ldots$$

for the hexagon model. No similar theoretical breakthrough has occurred for the square model and thus the identity of $3.7962\ldots$ remains masked from sight. The critical value $z_c = 7.92\ldots$ for the triangle model (on the hexagonal or honeycomb lattice) likewise is not exactly known [46].

For hard hexagons, the behavior of $\rho(z)$ and $R(z)$ at criticality is important [24,26,27]:

$$\rho \sim \rho_c - 5^{-3/2}\left(1 - \frac{z}{z_c}\right)^{2/3} \text{ as } z \to z_c^-, \quad \rho_c = \frac{5-\sqrt{5}}{10} = 0.2763932022\ldots,$$

$$R \sim \frac{3}{\sqrt{5}}\left[\frac{1}{5\sqrt{5}}\left(\frac{z}{z_c} - 1\right)\right]^{1/9} \text{ as } z \to z_c^+,$$

and it is conjectured that the exponents $1/3$ and $1/9$ are universal. For hard squares and hard triangles, we have only numerical estimates $\rho_c = 0.368\ldots$ and $0.422\ldots$, respectively. Far away from criticality, computations at $z = 1$ are less difficult [3,47]:

$$\rho(1) = \begin{cases} 0.1624329213\ldots & \text{for hard hexagons,} \\ 0.2265708154\ldots & \text{for hard squares,} \\ 0.2424079763\ldots & \text{for hard triangles,} \end{cases}$$

and the first of these is algebraic of degree 12 [18,22]. A generalization of $\rho(1)$ is the probability that an arbitrary point α and a specified configuration of neighboring points α' are all occupied; sample computations can be found in [3].

Needless to say, three-dimensional analogs of the models discussed here defy any attempt at exact solution [44].

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5.13 Binary Search Tree Constants

We first define a certain function f. The formulation may seem a little abstruse, but f has a natural interpretation as a path length along a type of weakly binary tree (an application of which we will discuss subsequently) [5.6].

Given a vector $V = (v_1, v_2, \ldots, v_k)$ of k distinct integers, define two subvectors V_L and V_R by

$$V_L = (v_j : v_j < v_1,\ 2 \le j \le k),\quad V_R = (v_j : v_j > v_1,\ 2 \le j \le k).$$

The subscripts L and R mean "left" and "right"; we emphasize that the sublists V_L and V_R preserve the ordering of the elements as listed in V.

Now, over all integers x, define the recursive function

$$f(x,V)=\begin{cases} \qquad 0 & \text{if } V=\emptyset \qquad (\emptyset \text{ is the empty vector}), \\ \begin{cases} 1 & \text{if } x=v_1, \\ 1+f(x,V_L) & \text{if } x<v_1, \\ 1+f(x,V_R) & \text{if } x>v_1. \end{cases} & \text{otherwise} \quad (v_1 \text{ is the first vector component}), \end{cases}$$

Clearly $0 \le f(x,V) \le k$ always and the ordering of $v_1, v_2, \ldots, v_k$ is crucial in determining the value of $f(x,V)$. For example, $f(7,(3,9,5,1,7)) = 4$ and $f(4,(3,9,5,1,7)) = 3$.

Let V be a random permutation of $(1,3,5,\ldots,2n-1)$. We are interested in the probability distribution of $f(x,V)$ in two regimes:

- random odd x satisfying $1 \le x \le 2n-1$ (successful search),
- random even x satisfying $0 \le x \le 2n$ (unsuccessful search).

Note that both V and x are random; it is assumed that they are drawn independently with uniform sampling. The expected value of $f(x,V)$ is, in the language of computer science [1–3],

- the average number of comparisons required to *find* an existing random record x in a data structure with n records,
- the average number of comparisons required to *insert* a new random record x into a data structure with n records,

where it is presumed the data structure follows that of a **binary search tree**. Figure 5.15 shows how such a tree is built starting with V as prescribed. Define also $g(l,V) = |\{x : f(x,V)=l,\ 1 \le x \le 2n-1,\ x \text{ odd}\}|$, the number of vertices occupying the l^{th} level of the tree ($l=1$ is the root level). For example, $g(2,(3,9,5,1,7)) = 2$ and $g(3,(3,9,5,1,7)) = 1$.

$$V=\begin{pmatrix}3\\9\\5\\1\\7\end{pmatrix}$$

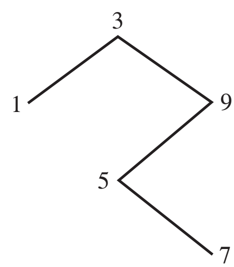

Figure 5.15. Binary search tree constructed using V.

In addition to the two average-case parameters, we want the probability distribution of

$$h(V) = \max\{f(x, V) : 1 \le x \le 2n - 1,\ x \text{ odd}\} - 1,$$

the **height** of the tree (which captures the worst-case scenario for finding the record x, given V), and

$$s(V) = \max\{l : g(l, V) = 2^{l-1}\} - 1,$$

the **saturation level** of the tree (which provides the number of full levels of vertices in the tree, minus one). Thus $h(V)$ is the longest path length from the root of the tree to a leaf whereas $s(V)$ is the shortest such path. For example, $h(3, 9, 5, 1, 7) = 3$ and $s(3, 9, 5, 1, 7) = 1$.

Define, as is customary, the harmonic numbers

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \ln(n) + \gamma + \frac{1}{n} + O\left(\frac{1}{n^2}\right), \quad H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{1}{n} + O\left(\frac{1}{n^2}\right),$$

where γ is the Euler–Mascheroni constant [1.5]. Then the expected number of comparisons in a successful search (random, odd $1 \le x \le 2n - 1$) of a random tree is [2–4]

$$\mathrm{E}(f(x, V)) = 2\left(1 + \frac{1}{n}\right) H_n - 3 = 2\ln(n) + 2\gamma - 3 + O\left(\frac{\ln(n)}{n}\right),$$

and in an unsuccessful search (random, even $0 \le x \le 2n$) the expected number is

$$\mathrm{E}(f(x, V)) = 2(H_{n+1} - 1) = 2\ln(n) + 2\gamma - 2 + O\left(\frac{1}{n}\right).$$

The corresponding variances are, for odd x,

$$\begin{aligned}\mathrm{Var}(f(x, V)) &= \left(2 + \frac{10}{n}\right) H_n - 4\left(1 + \frac{1}{n}\right)\left(H_n^{(2)} + \frac{H_n^2}{4}\right) + 4 \\ &\sim 2\left(\ln(n) + \gamma - \frac{\pi^3}{3} + 2\right)\end{aligned}$$

and, for even x,

$$\mathrm{Var}(f(x, V)) = 2(H_{n+1} - 2H_{n+1}^{(2)} + 1) \sim 2\left(\ln(n) + \gamma - \frac{\pi^3}{3} + 1\right).$$

A complete analysis of $h(V)$ and $s(V)$ remained unresolved until 1985 when Devroye [3, 5–7], building upon work of Robson [8] and Pittel [9], proved that

$$\frac{h(V)}{\ln(n)} \to c, \quad \frac{s(V)}{\ln(n)} \to d,$$

almost surely as $n \to \infty$, where $c = 4.3110704070\ldots$ and $d = 0.3733646177\ldots$ are the only two real solutions of the equation

$$\frac{2}{x}\exp\left(1-\frac{1}{x}\right) = 0.$$

Observe that the rate of convergence for $h(V)/\ln(n)$ and $s(V)/\ln(n)$ is slow; hence a numerical verification requires efficient simulation [10]. Considerable effort has been devoted to making these asymptotics more precise [11–14]. Reed [15, 16] and Drmota [17–19] recently proved that

$$\begin{aligned} \mathrm{E}(h(V)) &= c\ln(n) - \frac{3c}{2(c-1)}\ln(\ln(n)) + O(1),\\ \mathrm{E}(s(V)) &= d\ln(n) + O(\sqrt{\ln(n)}\ln(\ln(n))) \end{aligned}$$

and $\mathrm{Var}(h(V)) = O(1)$ as $n \to \infty$. No numerical estimates of the latter are yet available. See also [20].

It is curious that for digital search trees [5.14], which are somewhat more complicated than binary search trees, the analogous limits

$$\frac{h(V)}{\ln(n)} \to \frac{1}{\ln(2)}, \quad \frac{s(V)}{\ln(n)} \to \frac{1}{\ln(2)}$$

do not involve new constants. The fact that limiting values for $h(V)/\ln(n)$ and $s(V)/\ln(n)$ are equal means that the trees are almost perfect (with only a small "fringe" around $\log_2(n)$). This is a hint that search/insertion algorithms on digital search trees are, on average, more efficient than on binary search trees.

Here is one related subject [21–23]. Break a stick of length r into two parts at random. Independently, break each of the two substicks into two parts at random as well. Continue inductively, so that at the end of the n^{th} step, we have 2^n pieces. Let $P_n(r)$ denote the probability that all of the pieces have length < 1. For fixed r, clearly $P_n(r) \to 1$ as $n \to \infty$. More interestingly,

$$\lim_{n\to\infty} P_n(r^n) = \begin{cases} 0 & \text{if } r > e^{1/c}, \\ 1 & \text{if } 0 < r < e^{1/c}, \end{cases}$$

where $e^{1/c} = 1.2610704868\ldots$ and c is as defined earlier. The techniques for proving this are similar to those utilized in [5.3].

We merely mention a generalization of binary search trees called **quadtrees** [24–30], which also possess intriguing asymptotic constants. Quadtrees are useful for storing and retrieving multidimensional real data, for example, in cartography, computer graphics, and image processing [31–33].

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5.14 Digital Search Tree Constants

Prior acquaintance with binary search trees [5.13] is recommended before reading this essay. Given a binary $k \times n$ matrix $M = (m_{i,j}) = (m_1, m_2, \ldots, m_k)$ of k distinct rows, define two submatrices $M_{L,p}$ and $M_{R,p}$ by

$$M_{L,p} = (m_i : m_{i,p} = 0,\ 2 \le i \le k), \quad M_{R,p} = (m_i : m_{i,p} = 1,\ 2 \le i \le k)$$

for any integer $1 \le p \le n$. That is, the p^{th} column of $M_{L,p}$ is all zeros and the p^{th} column of $M_{R,p}$ is all ones. The subscripts L and R mean “left” and “right”; we emphasize that the sublists $M_{L,p}$ and $M_{R,p}$ preserve the ordering of the rows as listed in M.

Now, over all binary n-vectors x, define the recursive function

$$f(x, M, p) = \begin{cases} 0 & \text{if } M = \emptyset, \\ \begin{cases} 1 & \text{if } x = m_1, \\ 1 + f(x, M_{L,p}, p+1) & \text{if } x \ne m_1 \text{ and } x_p = 0, \\ 1 + f(x, M_{R,p}, p+1) & \text{if } x \ne m_1 \text{ and } x_p = 1. \end{cases} & \text{otherwise,} \end{cases}$$

Clearly $0 \le f(x, M, p) \le k$ always and the ordering of $m_1, m_2, \ldots, m_k$, as well as the value of p, is crucial in determining the value of $f(x, M, p)$.

Let $M = (m_1, m_2, \ldots, m_k)$ be a random binary $n \times n$ matrix with n distinct rows, and let x denote a binary n-vector. We are interested in the probability distribution of $f(x, M, 1)$ in two regimes:

- random x satisfying $x = m_i$ for some i, $1 \le i \le n$ (successful search),
- random x satisfying $x \ne m_i$ for all i, $1 \le i \le n$ (unsuccessful search).

There is double randomness here as with binary search trees [5.13], but note that x depends on M more intricately than before. The expected value of $f(x, M, 1)$ is, in the language of computer science, [1–6]

- the average number of comparisons required to *find* an existing random record x in a data structure with n records,
- the average number of comparisons required to *insert* a new random record x into a data structure with n records,

where it is presumed the data structure follows that of a **digital search tree**. Figure 5.16 shows how such a tree is built starting with M as prescribed.

$$M = \begin{pmatrix} 0\,1\,0\,0\,0\,0 \\ 1\,0\,1\,0\,1\,1 \\ 0\,1\,0\,1\,1\,0 \\ 0\,1\,1\,0\,1\,1 \\ 1\,1\,0\,1\,0\,0 \\ 1\,0\,0\,0\,1\,1 \end{pmatrix}$$

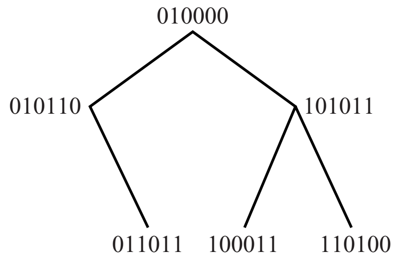

Figure 5.16. Digital search tree constructed using M.

Another parameter of some interest is the number A_n of non-root vertices of degree 1, that is, nodes without children. For binary search trees [3, 7], it is known that $\mathrm{E}(A_n) = (n+1)/3$. For digital search trees, the corresponding result is more complicated, as we shall soon see. Because digital search trees are usually better "balanced" than binary search trees, one anticipates a linear coefficient closer to $1/2$ than $1/3$.

Let γ denote the Euler–Mascheroni constant [1.5] and define a new constant

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{2^k - 1} = 1.6066951524\ldots.$$

Then the expected number of comparisons in a successful search (random, $x = m_i$ for some i) of a random tree is [3–6, 8, 9]

$$\begin{aligned} \mathrm{E}(f(x, M, 1)) &= \frac{1}{\ln(2)}\ln(n) + \frac{3}{2} + \frac{\gamma - 1}{\ln(2)} - \alpha + \delta(n) + O\left(\frac{\ln(n)}{n}\right) \\ &\sim \log_2(n) - 0.716644\ldots + \delta(n), \end{aligned}$$

and in an unsuccessful search (random $x \neq m_i$ for all i) the expected number is

$$\begin{aligned} \mathrm{E}(f(x, M, 1)) &= \frac{1}{\ln(2)}\ln(n) + \frac{1}{2} + \frac{\gamma}{\ln(2)} - \alpha + \delta(n) + O\left(\frac{\ln(n)}{n}\right) \\ &\sim \log_2(n) - 0.273948\ldots + \delta(n), \end{aligned}$$

where

$$\delta(n) = \frac{1}{\ln(2)} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \Gamma\left(-1 - \frac{2\pi i k}{\ln(2)}\right) \exp\left(2\pi i k \frac{\ln(n)}{\ln(2)}\right).$$

The function $\delta(n)$ is oscillatory ($\delta(n) = \delta(2n)$), has zero mean, and is "negligible" ($|\delta(n)| < 1.726 \times 10^{-7}$ for all n). Similar functions $\varepsilon(n)$, $\rho(n)$, $\sigma(n)$ and $\tau(n)$ will be needed later. These arise in the analysis of many algorithms [3, 4, 6], as well as in problems discussed in [2.3], [2.16], [5.6], and [5.11]. Although such functions can be safely ignored for practical purposes, they need to be included in certain treatments for the sake of theoretical rigor.

The corresponding variances are, for searching,

$$\mathrm{Var}(f(x, M, 1)) \sim \frac{1}{12} + \frac{\pi^2 + 6}{6\ln(2)^2} - \alpha - \beta + \varepsilon(n) \sim 2.844383\ldots + \varepsilon(n)$$

and, for inserting,

$$\mathrm{Var}(f(x, M, 1)) \sim \frac{1}{12} + \frac{\pi^2}{6\ln(2)^2} - \alpha - \beta + \varepsilon(n) \sim 0.763014\ldots + \varepsilon(n),$$

where the new constant β is given by

$$\beta = \sum_{k=1}^{\infty} \frac{1}{(2^k - 1)^2} = 1.1373387363\ldots.$$

Flajolet & Sedgewick [3, 8, 10] answered an open question of Knuth's regarding the parameter A_n:

$$\mathrm{E}(A_n) = \left[\theta + 1 - \frac{1}{Q}\left(\frac{1}{\ln(2)} + \alpha^2 - \alpha\right) + \rho(n)\right] n + O(n^{1/2}),$$

where the new constants Q and θ are given by

$$Q = \prod_{k=1}^{\infty}\left(1 - \frac{1}{2^k}\right) = 0.2887880950\ldots = (3.4627466194\ldots)^{-1},$$

$$\theta = \sum_{k=1}^{\infty} \frac{k 2^{k(k-1)/2}}{1 \cdot 3 \cdot 7 \cdots (2^k - 1)} \sum_{j=1}^{k} \frac{1}{2^j - 1} = 7.7431319855\ldots.$$

The linear coefficient of $\mathrm{E}(A_n)$ fluctuates around

$$c = \theta + 1 - \frac{1}{Q}\left(\frac{1}{\ln(2)} + \alpha^2 - \alpha\right) = 0.3720486812\ldots,$$

which is not as close to $1/2$ as one might have anticipated! Here also [11] is an integral representation for c:

$$c = \frac{1}{\ln(2)} \int_0^{\infty} \frac{x}{1+x}\left(1 + \frac{x}{1}\right)^{-1}\left(1 + \frac{x}{2}\right)^{-1}\left(1 + \frac{x}{4}\right)^{-1}\left(1 + \frac{x}{8}\right)^{-1} \cdots dx.$$

There are three main types of m-ary search trees: digital search trees, radix search tries (tries), and Patricia tries. We have assumed that $m = 2$ throughout. What, for example, is the variance for searching corresponding to Patricia tries? If we omit the fluctuation term, the remaining coefficient

$$\nu = \frac{1}{12} + \frac{\pi^2}{6\ln(2)^2} + \frac{2}{\ln(2)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k(2^k - 1)}$$

is interesting because, at first glance, it seems to be exactly 1! In fact, $\nu > 1 + 10^{-12}$ and this can be more carefully explained via the Dedekind eta function [12, 13].

5.14.1 Other Connections

In number theory, the divisor function $d(n)$ is the number of integers d, $1 \le d \le n$, that divide n. A special value of its generating function [4, 14, 15]

$$\sum_{n=1}^{\infty} d(n)q^n = \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} = \sum_{k=1}^{\infty} \frac{q^{k^2}(1+q^k)}{1-q^k}$$

is α when $q = 1/2$. Erdös [16, 17] proved that α is irrational; forty years passed while people wondered about constants such as

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 3} \text{ and } \sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

(the former appears in [18] whereas the latter is connected to tries [6] and mergesort asymptotics [19, 20]). Borwein [21, 22] proved that, if $|a| \ge 2$ is an integer, $b \ne 0$ is a rational number, and $b \ne -a^n$ for all n, then the series

$$\sum_{n=1}^{\infty} \frac{1}{a^n + b} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^n}{a^n + b}$$

are both irrational. Under the same conditions, the product

$$\prod_{n=1}^{\infty} \left(1 + \frac{b}{a^n}\right)$$

is irrational [23, 24], and hence so is Q. See [25] for recent computer-aided irrationality proofs.

On the one hand, from the combinatorics of integer partitions, we have Euler's pentagonal number theorem [14, 26–28]

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(3n+1)n} = 1 + \sum_{n=1}^{\infty}(-1)^n \left(q^{\frac{1}{2}(3n-1)n} + q^{\frac{1}{2}(3n+1)n}\right)$$

and

$$\begin{aligned}\prod_{n=1}^{\infty}(1-q^n)^{-1} &= 1 + \sum_{n=1}^{\infty} \frac{q^n}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^n)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2(1-q^3)^2\cdots(1-q^n)^2} = 1 + \sum_{n=1}^{\infty} p(n)q^n,\end{aligned}$$

where $p(n)$ denotes the number of unrestricted partitions of n. If $q = 1/2$, these specialize to Q and $1/Q$. On the other hand, in the theory of finite vector spaces, Q appears in the asymptotic formula [5.7] for the number of linear subspaces of $\mathbb{F}_{q,n}$ when $q = 2$.

A substantial theory has emerged involving q-analogs of various classical mathematical objects. For example, the constant α is regarded as a 1/2-analog of the Euler–Mascheroni constant [11]. Other constants (e.g., Apéry's constant $\zeta(3)$ or Catalan's constant G) can be similarly generalized.

Out of many more possible formulas, we mention three [4, 14, 26, 29]:

$$\begin{aligned} Q &= \frac{1}{3} - \frac{1}{3\cdot 7} + \frac{1}{3\cdot 7\cdot 15} - \frac{1}{3\cdot 7\cdot 15\cdot 31} + - \cdots \\ &= \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n(2^n-1)}\right) \\ &= \sqrt{\frac{2\pi}{\ln(2)}}\exp\left(\frac{\ln(2)}{24} - \frac{\pi^2}{6\ln(2)}\right)\prod_{n=1}^{\infty}\left[1-\exp\left(\frac{-4\pi^2 n}{\ln(2)}\right)\right]. \end{aligned}$$

The second makes one wonder if a simple relationship between Q and α exists. It can be shown that Q is the asymptotic probability that the determinant of a random $n \times n$ binary matrix is odd. A constant P similar to Q appears in [2.8]; exponents in P are constrained to be odd integers.

The reciprocal sum of repunits [30]

$$9\sum_{n=1}^{\infty}\frac{1}{10^n-1} = \frac{1}{1} + \frac{1}{11} + \frac{1}{111} + \frac{1}{1111} + \cdots = 1.1009181908\ldots$$

is irrational by Borwein's theorem. The reciprocal series of Fibonacci numbers can be expressed as [31–33]

$$\sum_{k=1}^{\infty}\frac{1}{f_k} = \sqrt{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{\varphi^{2n+1}-(-1)^n} = 3.3598856662\ldots,$$

where φ is the Golden mean, and this sum is known to be irrational [34–37]. Note that the subseries of terms with even subscripts can similarly be evaluated [26, 31]:

$$\sum_{k=1}^{\infty}\frac{1}{f_{2k}} = \sqrt{5}\left(\sum_{n=1}^{\infty}\frac{1}{\lambda^n-1} - \sum_{n=1}^{\infty}\frac{1}{\mu^n-1}\right) = 1.5353705088\ldots,$$

where $2\lambda = \sqrt{3}+5$ and $2\mu = 7+3\sqrt{5}$. A completely different connection to the Fibonacci numbers (this time resembling the constant Q) is found in [1.2].

A certain normalizing constant [38–40]

$$K = \sqrt{\prod_{n=0}^{\infty}\left(1+\frac{1}{2^{2n}}\right)} = 1.6467602581\ldots$$

occurs in efficient *binary cordic* implementations of two-dimensional vector rotation. Products such as Q and K, however, have no known closed-form expression except when $q = \exp(-\pi\xi)$, where $\xi > 0$ is an algebraic number [26, 41].

Observe that $2^{n+1}-1$ is the smallest positive integer not representable as a sum of n integers of the form 2^i, $i \geq 0$. Define h_n to be the smallest positive integer not representable as a sum of n integers of the form $2^i 3^j$, $i \geq 0$, $j \geq 0$, that is, $h_0 = 1$, $h_1 = 5$, $h_2 = 23$, $h_3 = 431, \ldots$ [42, 43]. What is the precise growth rate of h_n as $n \to \infty$? What is the numerical value of the reciprocal sum of h_n (what might be called the *2-3 analog* of the constant α)? This is vaguely related to our discussion in [2.26] and [2.30.1].

5.14.2 Approximate Counting

Returning to computer science, we discuss **approximate counting**, an algorithm due to Morris [44]. Approximate counting involves keeping track of a large number, N, of events in only $\log_2(\log_2(N))$ bit storage, where accuracy is not paramount. Consider the integer time series $X_0, X_1, \ldots, X_N$ defined recursively by

$$X_n = \begin{cases} 1 & \text{if } n = 0, \\ \begin{cases} 1 + X_{n-1} & \text{with probability } 2^{-X_{n-1}}, \\ X_{n-1} & \text{with probability } 1 - 2^{-X_{n-1}}. \end{cases} & \text{otherwise,} \end{cases}$$

It is not hard to prove that

$$\mathrm{E}(2^{X_N} - 2) = N \text{ and } \mathrm{Var}(2^{X_N}) = \tfrac{1}{2}N(N+1);$$

hence probabilistic updates via this scheme give an unbiased estimator of N. Flajolet [45–50] studied the distribution of X_N in much greater detail:

$$\begin{aligned} \mathrm{E}(X_N) &= \frac{1}{\ln(2)}\ln(N) + \frac{1}{2} + \frac{\gamma}{\ln(2)} - \alpha + \sigma(n) + O\left(\frac{\ln(N)}{N}\right) \\ &\sim \log_2(N) - 0.273948\ldots + \sigma(N), \end{aligned}$$

$$\mathrm{Var}(X_N) \sim \frac{1}{12} + \frac{\pi^2}{6\ln(2)^2} - \alpha - \beta - \chi + \tau(n) \sim 0.763014\ldots + \tau(N),$$

where α and β are as before, the new constant χ is given by

$$\chi = \frac{1}{\ln(2)} \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{csch}\left(\frac{2\pi^2 n}{\ln(2)}\right) = (1.237412\ldots) \times 10^{-12},$$

and $\sigma(n)$ and $\tau(n)$ are oscillatory "negligible" functions. In particular, since $\chi > 0$, the constant coefficient for $\mathrm{Var}(X_N)$ is (slightly) smaller than that for $\mathrm{Var}(f(x, M, 1))$ given earlier. Similar ideas in probabilistic counting algorithms are found in [6.8].

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5.15 Optimal Stopping Constants

Consider the well-known **secretary problem**. An unordered sequence of **applicants** (distinct real numbers) $s_1, s_2, \ldots, s_n$ are interviewed by you one at a time. You have no prior information about the ss. You know the value of n, and as s_k is being interviewed, you must either accept s_k and end the process, or reject s_k and interview s_{k+1}. The decision to accept or reject s_k must be based solely on whether $s_k > s_j$ for all $1 \le j < k$ (that is, on whether s_k is a **candidate**). An applicant once rejected cannot later be recalled.

If your objective is to select the most highly qualified applicant (the largest s_k), then the optimal strategy is to reject the first $m - 1$ applicants and accept the next candidate, where [1–4]

$$m = \min\left\{k \ge 1 : \sum_{j=k+1}^{n} \frac{1}{j-1} \le 1\right\} \sim \frac{n}{e}$$

as $n \to \infty$. The asymptotic probability of obtaining the best applicant via this strategy is hence $1/e = 0.3678794411\ldots$, where e is the natural logarithmic base [1.3]. See a generalization of this in [5–7].

If your objective is instead to minimize the expected rank R_n of the chosen applicant (the largest s_k has rank 1, the second-largest has rank 2, etc.), then different formulation applies. Lindley [8] and Chow et al. [9] derived the optimal strategy in this case and proved that [10]

$$\lim_{n\to\infty} R_n = \prod_{k=1}^{\infty}\left(1+\frac{2}{k}\right)^{\frac{1}{k+1}} = 3.8695192413\ldots = C.$$

A variation might include you knowing in advance that $s_1, s_2, \ldots, s_n$ are independent, uniformly distributed variables on the interval [0, 1]. This is known as a **full-information problem** (as opposed to the no-information problems just discussed). How does knowledge of the distribution improve your chances of success? For the "nothing but the best" objective, Gilbert & Mosteller [11] calculated the asymptotic probability of success to be [12, 13]

$$e^{-a} - (e^{a} - a - 1)\,\mathrm{Ei}(-a) = 0.5801642239\ldots,$$

where $a = 0.8043522628\ldots$ is the unique real solution of the equation $\mathrm{Ei}(a) - \gamma - \ln(a) = 1$, Ei is the exponential integral [6.2], and γ is the Euler-Mascheroni constant [1.5].

The full-information analog for $\lim_{n\to\infty} R_n$ appears to be an open problem [14–16]. Yet another objective, however, might be to maximize the hiree's expected quality Q_n itself (the k^{th} applicant has quality s_k). Clearly

$$Q_0 = 0, \quad Q_n = \tfrac{1}{2}(1 + Q_{n-1}^2) \text{ if } n \ge 1,$$

and $Q_n \to 1$ as $n \to \infty$. Moser [11, 17–19] deduced that

$$Q_n \sim 1 - \frac{2}{n + \ln(n) + b},$$

where the constant b is estimated [10] to be $1.76799378\ldots$.

Here is a closely related problem. Assume $s_1, s_2, \ldots, s_n$ are independent, uniformly distributed variables on the interval $[0, N]$. Your objective is to minimize the number T_N of interviews necessary to select an applicant of expected quality $\ge N - 1$. Gum [20] sketched a proof that $T_N = 2N - O(\ln(N))$ as $N \to \infty$. Alternatively, assume everything as before except that $s_1', s_2', \ldots, s_n'$ are drawn with replacement from the set $\{1, 2, \ldots, N\}$. It can be proved here that $T_N' = c\,N + O(\sqrt{N})$, where [10]

$$c = 2\sum_{k=3}^{\infty}\frac{\ln(k)}{k^2-1} - \frac{\ln(2)}{3} = 1.3531302722\ldots = \ln(C).$$

The secretary problem and its offshoots fall within the theory of **optimal stopping** [19]. Here is a sample exercise: We observe a fair coin being tossed repeatedly and can stop observing at any time. When we stop, the payoff is the average number of heads observed. What is the best strategy to maximize the expected payoff? Chow & Robbins [21,22] described a strategy that achieves an expected payoff $> 0.79 = (0.59 + 1)/2$.

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5.16 Extreme Value Constants

Let $X_1, X_2, \ldots, X_n$ denote a random sample from a population with continuous probability density function $f(x)$. Many interesting results exist concerning the distribution of the **order statistics**

$$X^{\langle 1\rangle} < X^{\langle 2\rangle} < \cdots < X^{\langle n\rangle},$$

where $X^{\langle 1\rangle} = \min\{X_1, X_2, \ldots, X_n\} = m_n$ and $X^{\langle n\rangle} = \max\{X_1, X_2, \ldots, X_n\} = M_n$. We will focus only on the extreme values M_n for brevity's sake.

If $X_1, X_2, \ldots, X_n$ are taken from a Uniform [0, 1] distribution (i.e., $f(x)$ is 1 for $0 \le x \le 1$ and is 0 otherwise), then the probability distribution of M_n is prescribed by

$$\mathrm{P}(M_n < x) = \begin{cases} 0 & \text{if } x < 0, \\ x^n & \text{if } 0 \le x \le 1, \\ 1 & \text{if } x > 1 \end{cases}$$

and its moments are given by

$$\mu_n = \mathrm{E}(M_n) = \frac{n}{n+1}, \quad \sigma_n^2 = \mathrm{Var}(M_n) = \frac{n}{(n+1)^2(n+2)}.$$

These are all exact results [1–3]. Note that clearly

$$\lim_{n\to\infty} \mathrm{P}\left(n\left(M_n - 1\right) < y\right) = \lim_{n\to\infty} \mathrm{P}\left(M_n < 1 + \frac{1}{n}y\right) = \begin{cases} e^y & \text{if } y < 0, \\ 1 & \text{if } y \ge 0. \end{cases}$$

This asymptotic result is a special case of a far more general theorem due to Fisher & Tippett [4] and Gnedenko [5]. Under broad circumstances, the asymptotic distribution of M_n (suitably normalized) must belong to one of just three possible families. We see another, less trivial, example in the following.

If $X_1, X_2, \ldots, X_n$ are from a Normal (0, 1) distribution, that is,

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad F(x) = \int_{-\infty}^{x} f(\xi)\, d\xi = \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{2},$$

then the probability distribution of M_n is prescribed by

$$\mathrm{P}(M_n < x) = F(x)^n = n \int_{-\infty}^{x} F(\xi)^{n-1} f(\xi)\, d\xi$$

and its moments are given by

$$\mu_n = n \int_{-\infty}^{\infty} x F(x)^{n-1} f(x)\, dx, \quad \sigma_n^2 = n \int_{-\infty}^{\infty} x^2 F(x)^{n-1} f(x)\, dx - \mu_n^2.$$

For small n, exact expressions are possible [2,3,6–11]:

$$\mu_2 = \frac{1}{\sqrt{\pi}} = 0.564\ldots, \qquad \sigma_2^2 = 1 - \mu_2^2 = 0.681\ldots,$$
$$\mu_3 = \frac{3}{2\sqrt{\pi}} = 0.846\ldots, \qquad \sigma_3^2 = 1 + \frac{\sqrt{3}}{2\pi} - \mu_3^2 = 0.559\ldots,$$
$$\mu_4 = \frac{3}{\sqrt{\pi}}(1 - 2S_2) = 1.029\ldots, \qquad \sigma_4^2 = 1 + \frac{\sqrt{3}}{\pi} - \mu_4^2 = 0.491\ldots,$$
$$\mu_5 = \frac{5}{\sqrt{\pi}}(1 - 3S_2) = 1.162\ldots, \qquad \sigma_5^2 = 1 + \frac{5\sqrt{3}}{2\pi}(1 - 2S_3) - \mu_5^2 = 0.447\ldots,$$
$$\mu_6 = \frac{15}{2\sqrt{\pi}}(1 - 4S_2 + 2T_2) = 1.267\ldots, \qquad \sigma_6^2 = 1 + \frac{5\sqrt{3}}{\pi}(1 - 3S_3) - \mu_6^2 = 0.415\ldots,$$
$$\mu_7 = \frac{21}{2\sqrt{\pi}}(1 - 5S_2 + 5T_2) = 1.352\ldots, \qquad \sigma_7^2 = 1 + \frac{35\sqrt{3}}{4\pi}(1 - 4S_3 + 2T_3) - \mu_7^2 = 0.391\ldots,$$

where

$$S_k = \frac{\sqrt{k}}{\pi}\int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{k + \sec(x)^2}} = \frac{1}{\pi}\arcsin\sqrt{\frac{k}{2(1+k)}},$$

$$T_k = \frac{\sqrt{k}}{\pi^2}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{dx\,dy}{\sqrt{k + \sec(x)^2 + \sec(y)^2}} = \frac{1}{\pi^2}\int_0^{\pi S(k)} \arcsin\sqrt{\frac{1}{2}\frac{k(k+1)}{k(k+2) - \tan(z)^2}}\,dz.$$

Similar expressions for $\mu_8 = 1.423\ldots$ and $\sigma_8^2 = 0.372\ldots$ remain to be found. Ruben [12] demonstrated a connection between moments of order statistics and volumes of certain hyperspherical simplices (generalized spherical triangles). Calkin [13] discovered a binomial identity that, in a limiting case, yields the exact expression for μ_3.

We turn now to the asymptotic distribution of M_n. Let

$$a_n = \sqrt{2\ln(n)} - \frac{1}{2}\frac{\ln(\ln(n)) + \ln(4\pi)}{\sqrt{2\ln(n)}}.$$

It can be proved [14–18] that

$$\lim_{n\to\infty} \mathrm{P}\left(\sqrt{2\ln(n)}(M_n - a_n) < y\right) = \exp(-e^{-y}),$$

and the resulting doubly exponential density function $g(y) = \exp(-y - e^{-y})$ is skewed to the right (called the Gumbel density or Fisher–Tippett Type I extreme values density). A random variable Y, distributed according to Gumbel's expression, satisfies [4]

$$\mathrm{E}(Y) = \gamma = 0.577215\ldots, \qquad \mathrm{Skew}(Y) = \frac{\mathrm{E}\left[(Y - \mathrm{E}(Y))^3\right]}{\mathrm{Var}(Y)^{3/2}} = \frac{12\sqrt{6}}{\pi^3}\zeta(3) = 1.139547\ldots,$$

$$\mathrm{Var}(Y) = \frac{\pi^2}{6} = 1.644934\ldots, \quad \mathrm{Kurt}(Y) = \frac{\mathrm{E}\left[(Y - \mathrm{E}(Y))^4\right]}{\mathrm{Var}(Y)^2} - 3 = \frac{12}{5} = 2.4,$$

where γ is the Euler–Mascheroni constant [1.5] and $\zeta(3)$ is Apéry's constant [1.6]. (Some authors report the *square* of skewness; this explains the estimate 1.2986 in [2] and 1.3 in [19].) The constant $\zeta(3)$ also appears in [20]. Doubly exponential functions like $g(y)$ occur elsewhere (see [2.13], [5.7], and [6.10]).

The well-known Central Limit Theorem implies an asymptotic normal distribution for the *sum* of many independent, identically distributed random variables, whatever their common original distribution. A similar situation holds in extreme value theory. The asymptotic distribution of M_n (normalized) must belong to one of the following families [2, 14–17]:

$$\begin{aligned} G_{1,\alpha}(y) &= \begin{cases} 0 & \text{if } y \le 0, \\ \exp(-y^{-\alpha}) & \text{if } y > 0, \end{cases} && \text{“Fréchet” or Type II,} \\ G_{2,\alpha}(y) &= \begin{cases} \exp(-(-y)^{\alpha}) & \text{if } y \le 0, \\ 1 & \text{if } y > 0, \end{cases} && \text{“Weibull” or Type III,} \\ G_3(y) &= \exp(-e^{-y}), && \text{“Gumbel” or Type I,} \end{aligned}$$

where $\alpha > 0$ is an arbitrary shape parameter. Note that $G_{2,1}(y)$ arose in our discussion of uniformly distributed X and $G_3(y)$ with regard to normally distributed X. It turns out to be unnecessary to know much about the distribution F of X to ascertain to which “domain of attraction” it belongs; the behavior of the tail of F is the crucial element. These three families can be further combined into a single one:

$$H_\beta(y) = \exp\left(-(1+\beta y)^{-1/\beta}\right) \text{ if } 1 + \beta y > 0, \quad H_0(y) = \lim_{\beta \to 0} H_\beta(y),$$

which reduces to the three cases accordingly as $\beta > 0$, $\beta < 0$, or $\beta = 0$.

There is a fascinating connection between the preceding and random matrix theory (RMT). Consider first an $n \times n$ diagonal matrix with random diagonal elements X_1, X_2, $\ldots$, X_n; of course, its largest eigenvalue is equal to M_n. Consider now a random $n \times n$ complex Hermitian matrix. This means $X_{ij} = \bar{X}_{ij}$, so diagonal elements are real and off-diagonal elements satisfy a symmetry condition; further, all eigenvalues are real. A “natural” way of generating such matrices follows what is called the Gaussian Unitary Ensemble (GUE) probability distribution [21]. Exact moment formulas for the largest eigenvalue exist here for small n just as for the diagonal normally-distributed case discussed earlier [22]. The eigenvalues are independent in the diagonal case, but they are strongly dependent in the full Hermitian case. RMT is important in several ways: First, the spacing distribution between nontrivial zeros of the Riemann zeta function appears to be close to the eigenvalue distribution coming from GUE [2.15.3]. Second, RMT is pivotal in solving the longest increasing subsequence problem discussed in [5.20], and its tools are useful in understanding the two-dimensional Ising model [5.22]. Finally, RMT is associated with the physics of atomic energy levels, but elaboration on this is not possible here.

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5.17 Pattern-Free Word Constants

Let a, b, c, ... denote the letters of a finite alphabet. A **word** is a finite sequence of letters; two examples are $abcacbacbc$ and $abcacbabcb$. A **square** is a word of the form xx, with x a nonempty word. A word is **square-free** if it contains no squares as factors. The first example contains the square $acbacb$ whereas the second is square-free. We ask the following question: How many square-free words of length n are there?

Over a two-letter alphabet, the only square-free words are a, b, ab, ba, aba, and bab; thus **binary** square-free words are not interesting. There do, however, exist arbitrarily long **ternary** square-free words, that is, over a three-letter alphabet. This fact was first proved by Thue [1, 2] using what is now called the Prouhet–Thue–Morse sequence [6.8]. Precise asymptotic enumeration of such words is complicated [3–7]. Brandenburg [8] proved that the number $s(n)$ of ternary square-free words of length $n > 24$ satisfies

$$6 \cdot 1.032^n < 6 \cdot 2^{\frac{n}{22}} \le s(n) \le 6 \cdot 1172^{\frac{n-2}{22}} < 3.157 \cdot 1.379^n,$$

and Brinkhuis [9] showed that $s(n) \le A \cdot 1.316^n$ for some constant $A > 0$. Noonan & Zeilberger [10] improved the upper bound to $A' \cdot 1.302128^n$ for some constant $A' > 0$, and obtained a non-rigorous estimate of the limit

$$S = \lim_{n\to\infty} s(n)^{\frac{1}{n}} = 1.302\ldots.$$

An independent computation [11] gave $S = \exp(0.263719\ldots) = 1.301762\ldots$, as well as estimates of S for k-letter alphabets, $k > 3$. Ekhad & Zeilberger [12] recently demonstrated that $1.041^n < 2^{n/17} \le s(n)$, the first improvement in the lower bound in fifteen years. Note that S is a connective constant in the same manner as certain constants μ associated with self-avoiding walks [5.10]. In fact, Noonan & Zeilberger's computation of S is based on the same Goulden-Jackson technology used in bounding μ.

A **cube-free word** is a word that contains no factors of the form xxx, where x is a nonempty word. The Prouhet–Thue–Morse sequence gives examples of arbitrarily long binary cube-free words. Brandenburg [8] proved that the number $c(n)$ of binary cube-free words of length $n > 18$ satisfies

$$2 \cdot 1.080^n < 2 \cdot 2^{\frac{n}{9}} \le c(n) \le 2 \cdot 1251^{\frac{n-1}{17}} < 1.315 \cdot 1.522^n,$$

and Edlin [13] improved the upper bound to $B \cdot 1.45757921^n$ for some constant $B > 0$. Edlin also obtained a non-rigorous estimate of the limit:

$$C = \lim_{n\to\infty} c(n)^{\frac{1}{n}} = 1.457\ldots.$$

A word is **overlap-free** if it contains no factor of the form $xyxyx$, with x nonempty. The Prouhet–Thue–Morse sequence, again, gives examples of arbitrarily long binary overlap-free words. Observe that a square-free word must be overlap-free, and that an overlap-free word must be cube-free. In fact, overlapping is the lowest pattern avoidable in arbitrarily long binary words. The number $t(n)$ of binary overlap-free words of length n satisfies [14, 15]

$$p \cdot n^{1.155} \le t(n) \le q \cdot n^{1.587}$$

for certain constants p and q. Therefore, $t(n)$ experiences only polynomial growth, unlike $s(n)$ and $c(n)$. Cassaigne [16] proved the interesting fact that $\lim_{n\to\infty} \ln(t(n))/\ln(n)$ does not exist, but

$$1.155 < T_L = \liminf_{n\to\infty} \frac{\ln(t(n))}{\ln(n)} < 1.276 < 1.332 < T_U = \limsup_{n\to\infty} \frac{\ln(t(n))}{\ln(n)} < 1.587$$

(actually, he proved much more). We observed similar asymptotic misbehavior in [2.16].

An **abelian square** is a word xx', with x a nonempty word and x' a permutation of x. A word is **abelian square-free** if it contains no abelian squares as factors. The word *abcacbabcb* contains the abelian square *abcacb*. In fact, any ternary word of length at least 8 must contain an abelian square. Pleasants [17] proved that arbitrarily long abelian square-free words, based on five letters, exist. The four-letter case remained an open question until recently. Keränen [18] proved that arbitrarily long quaternary abelian square-free words also exist. Carpi [19] went farther to show that their number $h(n)$ must satisfy

$$\liminf_{n\to\infty} h(n)^{\frac{1}{n}} > 1.000021,$$

and he wrote, "...the closeness of this value to 1 leads us to think that, probably, it is far from optimal."

A ternary word w is a **partially abelian square** if $w = xx'$, with x a nonempty word and x' a permutation of x that leaves the letter b fixed, and that allows only adjacent letters a and c to commute. For example, the word *bacbca* is a partially abelian square. A word is **partially abelian square-free** if it contains no partially abelian squares as factors. Cori & Formisano [20] used Kobayashi's inequalities for $t(n)$ to derive bounds for the number of partially abelian square-free words.

Kolpakov & Kucherov [21,22] asked: What is the minimal proportion of one letter in infinite square-free ternary words? Follow-on work by Tarannikov suggests [23] that the answer is $0.2746\ldots$.

A word over a k-letter alphabet is **primitive** if it is not a power of any subword [24]. The number of primitive words of length n is $\sum_{d|n} \mu(d)k^{n/d}$, where $\mu(d)$ is the Möbius mu function [2.2]. Hence, on the one hand, the proportion of words that are primitive is easily shown to approach 1 as $n \to \infty$. On the other hand, the problem of all counting words not *containing* a power is probably about as difficult as enumerating square-free words, cube-free words, etc.

A binary word $w_1w_2w_3\ldots w_n$ of length n is said to be **unforgeable** if it never matches a left or right shift of itself, that is, it is never the same as any of $u_1u_2\ldots u_m w_1 w_2 \ldots w_{n-m}$ or $w_{m+1}w_{m+2}\ldots w_n v_1 v_2 \ldots v_m$ for any possible choice of u_is or v_js and any $1 \le m \le n-1$. For example, we cannot have $w_1 = w_n$ because trouble would arise when $m = n-1$. Let $f(n)$ denote the number of unforgeable words of length n. The example shows immediately that

$$0 \le \rho = \lim_{n\to\infty} \frac{f(n)}{2^n} \le \frac{1}{2}.$$

Further, via generating functions [7,25–27],

$$\begin{aligned}\rho &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{2^{(2^{n+1}-1)}-1} \prod_{m=2}^{n} \frac{2^{(2^m-1)}}{2^{(2^m-1)}-1} = 0.2677868402\ldots \\ &= 1 - 0.7322131597\ldots,\end{aligned}$$

and this series is extremely rapidly convergent.

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5.18 Percolation Cluster Density Constants

Percolation theory is concerned with fluid flow in random media, for example, molecules penetrating a porus solid or wildfires consuming a forest. Broadbent & Hammersley [1–3] wondered about the probable number and structure of open channels in media for fluid passage. Answering their question has created an entirely new field of research [4–10]. Since the field is vast, we will attempt only to present a few constants.

Let $M = (m_{ij})$ be a random $n \times n$ binary matrix satisfying the following:

- $m_{ij} = 1$ with probability p, 0 with probability $1 - p$ for each i, j,
- m_{ij} and m_{kl} are independent for all $(i, j) \neq (k, l)$.

An s-**cluster** is an isolated grouping of s adjacent 1s in M, where adjacency means horizontal or vertical neighbors (not diagonal). For example, the 4×4 matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

has one 1-cluster, two 2-clusters, and one 4-cluster. The total number of clusters K_4 is 4 in this case. For arbitrary n, the total cluster count K_n is a random variable. The limit $\kappa_S(p)$ of the normalized expected value $\mathrm{E}(K_n)/n^2$ exists as $n \to \infty$, and $\kappa_S(p)$ is called the **mean cluster density** for the **site percolation model**. It is known that $\kappa_S(p)$ is twice continuously differentiable on $[0, 1]$; further, $\kappa_S(p)$ is analytic on $[0, 1]$ except possibly at one point $p = p_c$. Monte Carlo simulation and numerical Padé approximants can be used to compute $\kappa_S(p)$. For example [11], it is known that $\kappa_S(1/2) = 0.065770\ldots$.

Instead of an $n \times n$ binary matrix M, consider a binary array A of $2n(n-1)$ entries that looks like

$$A = \begin{pmatrix} & a_{12} & & a_{14} & & a_{16} & \\ a_{11} & & a_{13} & & a_{15} & & a_{17} \\ & a_{22} & & a_{24} & & a_{26} & \\ a_{21} & & a_{23} & & a_{25} & & a_{27} \\ & a_{32} & & a_{34} & & a_{36} & \\ a_{31} & & a_{33} & & a_{35} & & a_{37} \\ & a_{42} & & a_{44} & & a_{46} & \end{pmatrix}$$

(here $n = 4$). We associate a_{ij} not with a site of the $n \times n$ square lattice (as we do for m_{ij}) but with a bond. An s-cluster here is an isolated, connected subgraph of the graph of all bonds associated with 1s. For example, the array

$$A = \left(\begin{array}{ccccccc} & 1 & & 0 & & 0 & \\ 1 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ 0 & & 1 & & 1 & & 0 \\ & 0 & & 1 & & 0 & \\ 1 & & 0 & & 0 & & 0 \\ & 0 & & 0 & & 0 & \end{array} \right)$$

has one 1-cluster, one 2-cluster, and one 4-cluster. For **bond percolation models** such as this, we include 0-clusters in the total count as well, that is, isolated sites with no attached 1s bonds. In this case there are seven 0-clusters; hence the total number of clusters K_4 is 10. The mean cluster density $\kappa_B(p) = \lim_{n\to\infty} \mathrm{E}(K_n)/n^2$ exists and similar smoothness properties hold. Remarkably, however, an exact integral expression can be found at $p = 1/2$ for the mean cluster density [13, 14]:

$$\kappa_B\left(\frac{1}{2}\right) = -\frac{1}{8}\cot(y)\cdot\frac{d}{dy}\left\{\frac{1}{y}\int_{-\infty}^{\infty}\operatorname{sech}\left(\frac{\pi x}{2y}\right)\ln\left(\frac{\cosh(x)-\cos(2y)}{\cosh(x)-1}\right)dx\right\}\Bigg|_{y=\frac{\pi}{3}},$$

which Adamchik [11, 12] recently simplified to

$$\kappa_B\left(\frac{1}{2}\right) = \frac{3\sqrt{3}-5}{2} = 0.0980762113\ldots.$$

This constant is sometimes reported as $0.0355762113\ldots$, which is $\kappa_B(1/2) - 1/16$, if 0-clusters are not included in the total count. It may alternatively be reported as $0.0177881056\ldots$, which occurs if one normalizes not by the number of sites, n^2, but by the number of bonds, $2n(n-1)$. Caution is needed when reviewing the literature. Other occurrences of this integral are in [15–18].

An expression for the limiting variance of bond cluster density is not known, but a Monte Carlo estimate $0.164\ldots$ and relevant discussion appear in [11]. The bond percolation model on the *triangular* lattice gives a limiting mean cluster density $0.111\ldots$ at a specific value $p = 0.347\ldots$ (see the next section for greater precision). The associated variance $0.183\ldots$, again, is not known.

5.18.1 Critical Probability

Let us turn attention away from mean cluster density $\kappa(p)$ and instead toward **mean cluster size** $\sigma(p)$. In the examples given earlier, $S_4 = (1+2+2+4)/4 = 9/4$ for the site case, $S_4 = (1+2+4)/3 = 7/3$ for the bond case, and $\sigma(p)$ is the limiting value of $\mathrm{E}(S_n)$ as $n \to \infty$. The **critical probability** or **percolation threshold** p_c is defined to be [5, 6, 10]

$$p_c = \inf_{\substack{0<p<1 \\ \sigma(p)=\infty}} p,$$

that is, the concentration p at which an ∞-cluster appears in the infinite lattice. There are other possible definitions that turn out to be equivalent under most conditions. For example, if $\theta(p)$ denotes the **percolation probability**, that is, the probability that an ∞-cluster contains a prescribed site or bond, then p_c is the unique point for which $p < p_c$ implies $\theta(p) = 0$, and $p > p_c$ implies $\theta(p) > 0$. The critical probability indicates a phase transition in the system, analogous to that observed in [5.12] and [5.22].

For site percolation on the square lattice, there are rigorous bounds [19–24]

$$0.556 < p_c < 0.679492$$

and an estimate [25,26] $p_c = 0.5927460\ldots.$ based on extensive simulation. Ziff [11] additionally calculated that $\kappa_S(p_c) = 0.0275981\ldots$ via simulation. Parameter bounds for the cubic lattice and higher dimensions appear in [27–30].

In contrast, for bond percolation on the square and triangular lattices, there are exact results due to Sykes & Essam [31,32]. Kesten [33] proved that $p_c = 1/2$ on the square lattice, corresponding to the expression $\kappa_B(1/2)$ in the previous section. On the triangular lattice, Wierman [34] proved that

$$p_c = 2\sin\left(\frac{\pi}{18}\right) = 0.3472963553\ldots,$$

and this corresponds to another exact expression [11,35–37],

$$\begin{aligned}\kappa_B(p_c) &= -\frac{3}{8}\csc(2y)\cdot\frac{d}{dy}\left\{\int_{-\infty}^{\infty}\frac{\sinh((\pi-y)x)\sinh(\frac{2}{3}yx)}{x\sinh(\pi x)\cosh(yx)}dx\right\}\Bigg|_{y=\frac{\pi}{3}} + \frac{3}{2} - \frac{2}{1+p_c}\\ &= \frac{35}{4} - \frac{3}{p_c} = \frac{23}{4} - \frac{3}{2}\cdot\left\{\sqrt[3]{4\left(1+i\sqrt{3}\right)} + \sqrt[3]{4\left(1-i\sqrt{3}\right)}\right\}\\ &= 0.1118442752\ldots.\end{aligned}$$

Similar results apply for the hexagonal (honeycomb) lattice by duality.

It is also known that $p_c = 1/2$ for site percolation on the triangular lattice [10] and, in this case, $\kappa_S(1/2) = 0.0176255\ldots$ via simulation [11,38]. For site percolation on the hexagonal lattice, we have bounds [39]

$$0.6527 < 1 - 2\sin\left(\frac{\pi}{18}\right) \le p_c \le 0.8079$$

and an estimate $p_c = 0.6962\ldots$ [40,41].

5.18.2 Series Expansions

Here are details on how the functions $\kappa_S(p)$ and $\kappa_B(p)$ may be computed [6,42,43]. We will work on the square lattice, focusing mostly on site percolation. Let g_{st} denote the number of lattice animals [5.19] with area s and perimeter t, and let $q = 1 - p$. The probability that a fixed site is a 1-cluster is clearly pq^4. Because a 2-cluster can be oriented either horizontally or vertically, the average 2-cluster count per site is $2p^2q^6$. A 3-cluster can be linear (two orientations) or L-shaped (four orientations); hence the

Table 5.4. *Mean s-Cluster Densities*

s	Mean s-Cluster Density for Site Model	Mean s-Cluster Density for Bond Model
0	0	q^4
1	pq^4	$2pq^6$
2	$2p^2q^6$	$6p^2q^8$
3	$p^3(2q^8+4q^7)$	$p^3(18q^{10}+4q^9)$
4	$p^4(2q^{10}+8q^9+9q^8)$	$p^4(55q^{12}+32q^{11}+q^8)$

average 3-cluster count per site is $p^3(2q^8+4q^7)$. More generally, the mean s-cluster density is $\sum_t g_{st}p^sq^t$. Summing the left column entries in Table 5.4 [44,45] gives $\kappa_S(p)$ as the number of entries $\to\infty$:

$$\begin{aligned}\kappa_S(p) &= p-2p^2+p^4+p^8-p^9+2p^{10}-4p^{11}+11p^{12}+-\cdots\\ &\sim \kappa_S(p_c)+a_S(p-p_c)+b_S(p-p_c)^2+c_S\,|p-p_c|^{2-\alpha}\,.\end{aligned}$$

Likewise, summing the right column entries in the table gives $\kappa_B(p)$:

$$\begin{aligned}\kappa_B(p) &= q^4+2p-6p^2+4p^3+2p^6-2p^7+7p^8-12p^9+28p^{10}+-\cdots\\ &\sim \kappa_B(\tfrac{1}{2})+a_B(p-\tfrac{1}{2})+b_B(p-\tfrac{1}{2})^2+c_B\left|p-\tfrac{1}{2}\right|^{2-\alpha},\end{aligned}$$

where $a_B=-0.50\ldots$, $b_B=2.8\ldots$, and $c_B=-8.48\ldots$ [46]. The exponent α is conjectured to be $-2/3$, that is, $2-\alpha=8/3$.

If instead of $\sum_{s,t}g_{st}p^sq^t$, we examine $\sum_{s,t}s^2g_{st}p^{s-1}q^t$, then for the site model,

$$\begin{aligned}\sigma_S(p) &= 1+4p+12p^2+24p^3+52p^4+108p^5+224p^6+412p^7+-\cdots\\ &\sim C\,|p-p_c|^{-\gamma}\end{aligned}$$

is the mean cluster size series (for low concentration $p<p_c$). The exponent γ is conjectured to be $43/18$.

The expression $1-\sum_{s,t}sg_{st}p^{s-1}q^t$, when expanded in terms of q, gives

$$\begin{aligned}\theta_S(p) &= 1-q^4-4q^6-8q^7-23q^8-28q^9-186q^{10}+48q^{11}-+\cdots\\ &\sim D\,|p-p_c|^{\beta}\,,\end{aligned}$$

which is the site percolation probability series (for high concentration $p>p_c$). The exponent β is conjectured to be $5/36$.

Smirnov & Werner [47] recently proved that α, β, and γ indeed exist and are equal to their conjectured values, for site percolation on the triangular lattice. A proof of universality would encompass both site and bond cases on the square lattice, but this has not yet been achieved.

5.18.3 Variations

Let the sites of an infinite lattice be independently labeled A with probability p and B with probability $1-p$. Ordinary site percolation theory involves clusters of As. Let us instead connect adjacent sites that possess *opposite* labels and leave adjacent sites with the same labels disconnected. This is known as AB **percolation** or **antipercolation**. We wish to know what can be said of the probability $\theta(p)$ that an infinite AB cluster contains a prescribed site. It turns out that $\theta(p) = 0$ for all p for the infinite square lattice [48], but $\theta(p) > 0$ for all p lying in some nonempty subinterval containing 1/2, for the infinite triangular lattice [49]. The exact extent of this interval is not known: Mai & Halley [50] gave $[0.2145, 0.7855]$ via Monte Carlo simulation whereas Wierman [51] gave $[0.4031, 0.5969]$. The function $\theta(p)$, for the triangular lattice, is nondecreasing on $[0, 1/2]$ and therefore was deemed unimodal on [0, 1] by Appel [52].

Ordinary bond percolation theory is concerned with models in which any selected bond is either open (1) or closed (0). First-passage percolation [53] assigns not a binary random variable to each bond, but rather a nonnegative *real* random variable, thought of as length. Consider the square lattice in which each bond is independently assigned a length from the Uniform [0, 1] probability distribution. Let T_n denote the shortest length of all lattice path lengths starting at the origin (0, 0) and ending at $(n, 0)$; then it can be proved that the limit

$$\tau = \lim_{n\to\infty} \frac{\mathrm{E}(T_n)}{n} = \inf_n \frac{\mathrm{E}(T_n)}{n}$$

exists. Building upon earlier work [54–58], Alm & Parviainen [59] obtained rigorous bounds $0.243666 \leq \tau \leq 0.403141$ and an estimate $\tau = 0.312\ldots$ via simulation. If, instead, lengths are taken from the exponential distribution with unit mean, then we have bounds $0.300282 \leq \tau \leq 0.503425$ and an estimate $\tau = 0.402$. Godsil, Grötschel & Welsh [9] suggested the exact evaluation of τ to be a "hopelessly intractable problem."

We mention finally a constant $\lambda_c = 0.359072\ldots$ that arises in **continuum percolation** [5, 60]. Consider a homogeneous Poisson process of intensity λ on the plane, that is, points are uniformly distributed in the plane such that

- the probability of having exactly n points in a subset S of measure μ is $e^{-\lambda\mu}(\lambda\mu)^n/n!$ and
- the counts n_i of points in any collection of disjoint measurable subsets S_i are independent random variables.

Around each point, draw a disk of unit radius. The disks are allowed to overlap; that is, they are fully penetrable. There exists a unique critical intensity λ_c such that an unbounded connected cluster of disks develops with probability 1 if $\lambda > \lambda_c$ and with probability 0 if $\lambda < \lambda_c$. Hall [61] proved the best-known rigorous bounds $0.174 < \lambda_c < 0.843$, and the numerical estimate $0.359072\ldots$ is found in [62–64]. Among several alternative representations, we mention $\varphi_c = 1 - \exp(-\pi\lambda_c) = 0.676339\ldots$ [65] and $\pi\lambda_c = 1.128057\ldots$ [66]. The latter is simply the normalized total area of all the disks, disregarding whether they overlap or not, whereas φ_c takes overlapping portions into account. Continuum percolation shares many mathematical properties with lattice percolation, yet in many ways it is a more accurate model of physical disorder. Interestingly, it has also recently been applied in pure mathematics itself, to the study of gaps in the set of Gaussian primes [67].

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5.19 Klarner's Polyomino Constant

A **domino** is a pair of adjacent squares. Generalizing, we say that a **polyomino** or **lattice animal** of order n is a connected set of n adjacent squares [1–7]. See Figures 5.17 and 5.18.

Define $A(n)$ to be the number of polyominoes of order n, where it is agreed that two polyominoes are distinct if and only if they have different shapes *or* different

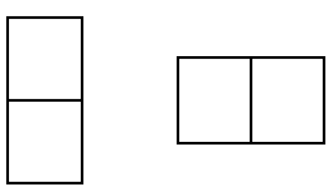

Figure 5.17. All dominoes (polyominoes of order 2); $A(2) = 2$.

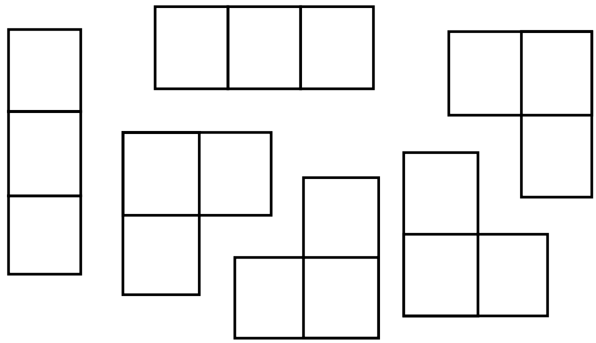

Figure 5.18. All polyominoes of order 3; $A(3) = 6$.

orientations:

$$A(1) = 1,\ A(2)=2,\ A(3)=6,\ A(4)=19,\ A(5)=63,$$
$$A(6) = 216,\ A(7)=760, \ldots .$$

There are different senses in which polyominoes are defined, for example, free versus fixed, bond versus site, simply-connected versus not necessarily so, and others. For brevity, we focus only on the fixed, site, possibly multiply-connected case.

Redelmeier [8] computed $A(n)$ up to $n = 24$, and Conway & Guttmann [9] found $A(25)$. In a recent flurry of activity, Oliveira e Silva [10] computed $A(n)$ up to $n = 28$, Jensen & Guttmann [11, 12] extended this to $A(46)$, and Knuth [13] found $A(47)$. Klarner [14, 15] proved that the limit

$$\alpha = \lim_{n\to\infty} A(n)^{\frac{1}{n}} = \sup_n A(n)^{\frac{1}{n}}$$

exists and is nonzero, although Eden [16] numerically investigated α several years earlier. The best-known bounds on α are $3.903184 \le \alpha \le 4.649551$, as discussed in [17–20]. Improvements are possible using the new value $A(47)$. The best-known estimate, obtained via series expansion analysis by differential approximants [11], is $\alpha = 4.062570\ldots$. A more precise asymptotic expression for $A(n)$ is

$$A(n) \sim \left(\frac{0.316915\ldots}{n} - \frac{0.276\ldots}{n^{3/2}} + \frac{0.335\ldots}{n^2} - \frac{0.25\ldots}{n^{5/2}} + O\left(\frac{1}{n^3}\right)\right)\alpha^n,$$

but such an empirical result is far from being rigorously proved.

Satterfield [5, 21] reported a lower bound of 3.91336 for α, using one of several algorithms he developed with Klarner and Shende. Details of their work unfortunately remain unpublished.

We mention that parallel analysis can be performed on the triangular and hexagonal lattices [7, 22].

Any self-avoiding polygon [5.10] determines a polyomino, but the converse is false since a polyomino can possess holes. A polyomino is **row-convex** if every (horizontal) row consists of a single strip of squares, and it is **convex** if this requirement is met for every column as well. Note that a convex polyomino does not generally determine a convex polygon in the usual sense. Counts of row-convex polyominoes obey a third-order linear recurrence [23–28], but counts $\tilde{A}(n)$ of convex polyominoes are somewhat more difficult to analyze [29, 30]:

$$\tilde{A}(1) = 1,\ \tilde{A}(2) = 2,\ \tilde{A}(3) = 6,\ \tilde{A}(4) = 19,\ \tilde{A}(5) = 59,$$
$$\tilde{A}(6) = 176,\ \tilde{A}(7) = 502, \ldots,$$
$$\tilde{A}(n) \sim (2.67564\ldots)\tilde{\alpha}^n,$$

where $\tilde{\alpha} = 2.3091385933\ldots = (0.4330619231\ldots)^{-1}$. Exact generating function formulation for $\tilde{A}(n)$ was discovered only recently [31–33] but is too complicated to include here. Bender [30] further analyzed the expected shape of convex polyominoes, finding that, when viewed from a distance, most convex polyominoes resemble rods tilted 45° from the vertical with horizontal (and vertical) thickness roughly equal to $2.37597\ldots$. More results like this are found in [34–36].

It turns out that the growth constant $\tilde{\alpha}$ for convex polyominoes is the same as the growth constant α' for **parallelogram polyominoes**, that is, polyominoes whose left and right boundaries both climb in a northeasterly direction:

$$A'(1) = 1,\ A'(2) = 2,\ A'(3) = 4,\ A'(4) = 9,\ A'(5) = 20,$$
$$A'(6) = 46,\ A'(7) = 105, \ldots.$$

These have the virtue of a simpler generating function $f(q)$. Let $(q)_0 = 1$ and $(q)_n = \prod_{j=1}^{n}(1 - q^j)$; then $f(q)$ is a ratio $J_1(q)/J_0(q)$ of q-analogs of Bessel functions:

$$J_0(q) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q)_n (q)_n}, \quad J_1(q) = -\sum_{n=1}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q)_{n-1} (q)_n},$$

which gives $\alpha' = \tilde{\alpha}$, but a different multiplicative constant $0.29745\ldots$.

There are many more counting problems of this sort than we can possibly summarize! Here is one more example, studied independently by Glasser, Privman & Svrakic [37] and Odlyzko & Wilf [38–40]. An ***n*-fountain** (Figure 5.19) is best pictured as a connected, self-supporting stacking of n coins in a triangular lattice array against a vertical wall.

Note that the bottom row cannot have gaps but the higher rows can; each coin in a higher row must touch two adjacent coins in the row below. Let $B(n)$ be the number of n-fountains. The generating function for $B(n)$ satisfies a beautiful identity involving

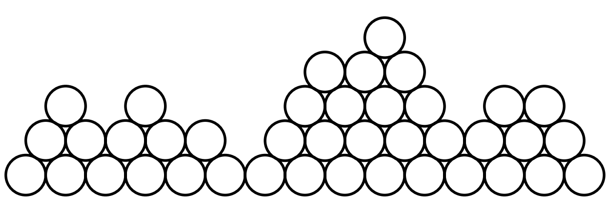

Figure 5.19. An example of an n-fountain.

Ramanujan's continued fraction:

$$1+\sum_{n=1}^{\infty} B(n)x^n = 1+x+x^2+2x^3+3x^4+5x^5+9x^6+15x^7+26x^8+45x^9+\cdots$$
$$= \frac{1|}{|1} - \frac{x|}{|1} - \frac{x^2|}{|1} - \frac{x^3|}{|1} - \frac{x^4|}{|1} - \frac{x^5|}{|1} - \cdots,$$

and the following growth estimates arise:

$$\lim_{n\to\infty} B(n)^{\frac{1}{n}} = \beta = 1.7356628245\ldots = (0.5761487691\ldots)^{-1},$$
$$B(n) = (0.3123633245\ldots)\beta^n + O\left(\left(\frac{5}{3}\right)^n\right).$$

See [41] for other related counting problems.

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5.20 Longest Subsequence Constants

5.20.1 Increasing Subsequences

Let π denote a random permutation on the symbols $1, 2, \ldots, N$. An **increasing subsequence** of π is a sequence $(\pi(j_1), \pi(j_2), \ldots, \pi(j_k))$ satisfying both $1 \le j_1 < j_2 < \ldots < j_k \le N$ and $\pi(j_1) < \pi(j_2) < \ldots < \pi(j_k)$. Define L_N to be the length of the longest increasing subsequence of π. For example, the permutation $\pi = (2, 7, 4, 1, 6, 3, 9, 5, 8)$ has longest increasing subsequences $(2, 4, 6, 9)$ and $(1, 3, 5, 8)$; hence $L_9 = 4$. What can be said about the probability distribution of L_N (e.g., its mean and variance) as $N \to \infty$?

This question has inspired an avalanche of research [1–4]. Vershik & Kerov [5] and Logan & Shepp [6] proved that

$$\lim_{N\to\infty} N^{-\frac{1}{2}}\,\mathrm{E}(L_N) = 2,$$

building upon earlier work in [7–10]. Odlyzko & Rains [11] conjectured in 1993 that both limits

$$\lim_{N\to\infty} N^{-\frac{1}{3}}\,\mathrm{Var}(L_N) = c_0, \quad \lim_{N\to\infty} N^{-\frac{1}{6}}(\mathrm{E}(L_N) - 2\sqrt{N}) = c_1$$

exist and are finite and nonzero; numerical approximations were computed via Monte Carlo simulation. In a showcase of analysis (using methods from mathematical physics), Baik, Deift & Johansson [12] obtained

$$c_0 = 0.81318\ldots \text{ (i.e., } \sqrt{c_0} = 0.90177\ldots), \quad c_1 = -1.77109\ldots,$$

confirming the predictions in [11]. These constants are defined exactly in terms of the solution to a Painlevé II equation. (Incidently, Painlevé III arises in [5.22] and Painlevé V arises in [2.15.3].) The derivation involves a relationship between random matrices and random permutations [13, 14]. More precisely, Tracy & Widom [15–17] derived a certain probability distribution function $F(x)$ characterizing the largest eigenvalue of a random Hermitian matrix, generated according to the Gaussian Unitary Ensemble (GUE) probability law. Baik, Deift & Johansson proved that the limiting distribution of L_N is Tracy & Widom's $F(x)$, and then obtained estimates of the constants c_0 and c_1 via moments quoted in [16].

Before presenting more details, we provide a generalization. A **2-increasing subsequence** of π is a union of two disjoint increasing subsequences of π. Define $\tilde{L}_N$ to be the length of the longest 2-increasing subsequence of π, minus L_N. For example, the permutation $\pi = (2, 4, 7, 9, 5, 1, 3, 6, 8)$ has longest increasing subsequence $(2, 4, 5, 6, 8)$ and longest 2-increasing subsequence $(2, 4, 7, 9) \cup (1, 3, 6, 8)$; hence $\tilde{L}_9 = 8 - 5 = 3$. As before, both

$$\lim_{N\to\infty} N^{-\frac{1}{3}}\,\mathrm{Var}(\tilde{L}_N) = \tilde{c}_0, \quad \lim_{N\to\infty} N^{-\frac{1}{6}}(\mathrm{E}(\tilde{L}_N) - 2\sqrt{N}) = \tilde{c}_1$$

exist and can be proved [18] to possess values

$$\tilde{c}_0 = 0.5405\ldots, \quad \tilde{c}_1 = -3.6754\ldots.$$

The corresponding distribution function $\tilde{F}(x)$ characterizes the second-largest eigenvalue of a random Hermitian matrix under GUE. Such proofs were extended to m-increasing subsequences, for arbitrary $m > 2$, and to the joint distribution of row lengths from random Young tableaux in [19–21].

Here are the promised details [12,18]. Fix $0 < t \le 1$. Let $q_t(x)$ be the solution of the Painlevé II differential equation

$$q_t''(x) = 2q_t(x)^3 + xq_t(x), \;\; q_t(x) \sim \frac{1}{2}\left(\frac{t}{\pi}\right)^{\frac{1}{2}} x^{-\frac{1}{4}} \exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right) \text{ as } x \to \infty,$$

and define

$$\Phi(x,t) = \exp\left[-\int_x^\infty (y-x)q_t(y)^2 dy\right].$$

The Tracy-Widom functions are

$$F(x) = \Phi(x,1), \quad \tilde{F}(x) = \Phi(x,1) - \left.\frac{\partial \Phi}{\partial t}(x,t)\right|_{t=1}$$

and hence

$$c_0 = \int_{-\infty}^{\infty} x^2 F'(x)dx - \left(\int_{-\infty}^{\infty} xF'(x)dx\right)^2, \quad c_1 = \int_{-\infty}^{\infty} xF'(x)dx,$$
$$\tilde{c}_0 = \int_{-\infty}^{\infty} x^2 \tilde{F}'(x)dx - \left(\int_{-\infty}^{\infty} x\tilde{F}'(x)dx\right)^2, \quad \tilde{c}_1 = \int_{-\infty}^{\infty} x\tilde{F}'(x)dx$$

are the required formulas. Note that the values of $c_0, c_1, \tilde{c_0}$, and $\tilde{c_1}$ appear in the caption of Figure 2 of [16]. Hence these arguably should be called Odlyzko–Rains–Tracy–Widom constants.

What makes this work especially exciting [1,22] is its connection with the common cardgame of solitaire (for which no successful analysis has yet been performed) and possibly with the unsolved Riemann hypothesis [1.6] from prime number theory. See [23,24] for other applications.

5.20.2 Common Subsequences

Let a and b be random sequences of length n, with terms a_i and b_j taking values from the alphabet $\{0, 1, \ldots, k-1\}$. A sequence c is a **common subsequence** of a and b if c is a subsequence of both a and b, meaning that c is obtained from a by deleting zero or more terms a_i and from b by deleting zero or more terms b_j. Define $\lambda_{n,k}$ to be the length of the longest common subsequence of a and b. For example, the sequences $a = (1, 0, 0, 2, 3, 2, 1, 1, 0, 2)$, $b = (0, 1, 1, 1, 3, 3, 3, 0, 2, 1)$ have longest common subsequence $c = (0, 1, 1, 0, 2)$ and $\lambda_{10,3} = 5$. What can be said about the mean of $\lambda_{n,k}$ as $n \to \infty$, as a function of k?

It can be proved that $\mathrm{E}(\lambda_{n,k})$ is superadditive with respect to n, that is, $\mathrm{E}(\lambda_{m,k}) + \mathrm{E}(\lambda_{n,k}) \le \mathrm{E}(\lambda_{m+n,k})$. Hence, by Fekete's theorem [25,26], the limit

$$\gamma_k = \lim_{n\to\infty} \frac{\mathrm{E}(\lambda_{n,k})}{n} = \sup_n \frac{\mathrm{E}(\lambda_{n,k})}{n}$$

Table 5.5. *Estimates for Ratios γ_k*

k	Lower Bound	Numerical Estimate	Upper Bound
2	0.77391	0.8118	0.83763
3	0.63376	0.7172	0.76581
4	0.55282	0.6537	0.70824
5	0.50952	0.6069	0.66443

exists. Beginning with Chvátal & Sankoff [27–30], a number of researchers [31–37] have investigated γ_k. Table 5.5 contains rigorous lower and upper bounds for γ_k, as well as the best numerical estimates of γ_k presently available [37].

It is known [27,31] that $1 \leq \gamma_k\sqrt{k} \leq e$ for all k and conjectured [38] that $\lim_{k\to\infty} \gamma_k\sqrt{k} = 2$. There is interest in the rate of convergence of the limiting ratio [39–41]

$$\gamma_k\, n - O(\sqrt{n\ln(n)}) \leq \mathrm{E}(\lambda_{n,k}) \leq \gamma_k\, n$$

as well as in $\mathrm{Var}(\lambda_{n,k})$, which is conjectured [39,41,42] to grow linearly with n.

A sequence c is a **common supersequence** of a and b if c is a supersequence of both a and b, meaning that both a and b are subsequences of c. The shortest common subsequence length $\Lambda_{n,k}$ of a and b can be shown [34,43,44] to satisfy

$$\lim_{n\to\infty} \frac{\mathrm{E}(\Lambda_{n,k})}{n} = 2 - \gamma_k.$$

Such nice duality as this fails, however, if we seek longest subsequences/shortest supersequences from a set of > 2 random sequences.

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5.21 k-Satisfiability Constants

Let $x_1, x_2, \ldots, x_n$ be Boolean variables. Choose k elements randomly from the set $\{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n\}$ under the restriction that x_j and $\neg x_j$ cannot both be selected. These k **literals** determine a **clause**, which is the disjunction ($\bigvee$, that is, "inclusive or") of the literals.

Perform this selection process m times. The m independent clauses determine a **formula**, which is the conjunction ($\bigwedge$, that is, "and") of the clauses. A sample formula, in the special case $n = 5$, $k = 3$, and $m = 4$, is

$$[x_1 \vee (\neg x_5) \vee (\neg x_2)] \wedge [(\neg x_3) \vee x_2 \vee (\neg x_1)] \wedge [x_5 \vee x_2 \vee x_4] \wedge [x_4 \vee (\neg x_3) \vee x_1].$$

A formula is **satisfiable** if there exists an assignment of 0s and 1s to the xs so that the formula is true (that is, has value 1). The design of efficient algorithms for discovering such an assignment, given a large formula, or for proving that the formula is **unsatisfiable**, is an important topic in theoretical computer science [1–3].

The k-satisfiability problem, or k-SAT, behaves differently for $k = 2$ and $k \geq 3$. For $k = 2$, the problem can be solved by a linear time algorithm, whereas for $k \geq 3$, the problem is NP-complete.

There is another distinction involving ideas from percolation theory [5.18]. As $m \to \infty$ and $n \to \infty$ with limiting ratio $m/n \to r$, empirical evidence suggests that the random k-SAT problem undergoes a phase transition at a critical value $r_c(k)$ of the parameter r. For $r < r_c$, a random formula is satisfiable with probability $\to 1$ as $m, n \to \infty$. For $r > r_c$, a random formula is likewise unsatisfiable almost surely. Away from the boundary, k-SAT is relatively easy to solve; computational difficulties appear to be maximized at the threshold $r = r_c$ itself. This observation may ultimately help in improving algorithms for solving the traveling salesman problem [8.5] and other combinatorial nightmares.

In the case of 2-SAT, it has been proved [4–6] that $r_c(2) = 1$. A rigorous understanding of 2-SAT from a statistical mechanical point-of-view was achieved in [7].

In the case of k-SAT, $k \geq 3$, comparatively little has been proved. Here is an inequality [4] valid for all $k \geq 3$:

$$\frac{3}{8}\frac{2^k}{k} \leq r_c(k) \leq \ln(2) \cdot \ln\left(\frac{2^k}{2^k - 1}\right)^{-1} \sim \ln(2) \cdot 2^k.$$

Many researchers have contributed to placing tight upper bounds [8–16] and lower bounds [17–20] on the 3-SAT threshold:

$$3.26 \leq r_c(3) \leq 4.506.$$

Large-scale computations [21–23] give an estimate $r_c(3) = 4.25\ldots$. Estimates for larger k [1] include $r_c(4) = 9.7\ldots$, $r_c(5) = 20.9\ldots$, and $r_c(6) = 43.2\ldots$, but these can be improved. Unlike 2-SAT, we do not yet possess a proof that $r_c(k)$ exists, for $k \geq 3$, but Friedgut [24] took an important step in this direction. Sharp phase transitions, corresponding to certain properties of random graphs, play an essential role in his paper. The possibility that $r_c(k)$ oscillates between the bounds $O(2^k/k)$ and $O(2^k)$ has not been completely ruled out, but this would be unexpected.

We mention a similar instance of threshold phenomena for random graphs. When $m \to \infty$ and $n \to \infty$ with limiting ratio $m/n \to s$, then in a random graph G on n vertices and with m edges, it appears that G is k-colorable with probability $\to 1$ for $s < s_c(k)$ and G is not k-colorable with probability $\to 1$ for $s > s_c(k)$. As before, the existence of $s_c(k)$ is only conjectured if $k \geq 3$, but we have bounds [25–33]

$$1.923 \leq s_c(3) \leq 2.495, \quad 2.879 \leq s_c(4) \leq 4.587,$$
$$3.974 \leq s_c(5) \leq 6.948, \quad 5.190 \leq s_c(6) \leq 9.539$$

and an estimate [34] $s_c(3) = 2.3$.

Consider also the discrete n-cube Q of vectors of the form $(\pm 1, \pm 1, \pm 1, \ldots, \pm 1)$. The **half cube** H_v generated by any $v \in Q$ is the set of all vectors $w \in Q$ having negative inner product with v. If a vector $u \in H_v$, it is natural to say that H_v **covers** u. Let $v_1, v_2, \ldots, v_m$ be drawn randomly from Q. When $m \to \infty$ and $n \to \infty$ with limiting ratio $m/n \to t$, it appears that $\bigcup_{k=1}^{m} H_{v_k}$ covers all of Q with probability $\to 1$ for $t > t_c$ but fails to do so with probability $\to 1$ for $t < t_c$. The existence of t_c was conjectured in [35] but a proof is not known. We have bounds [36, 37]

$$0.005 \leq t_c \leq 0.9963 = 1 - 0.0037$$

and an estimate [38, 39] $t_c = 0.82$. The motivation for studying this problem arises in binary neural networks.

Here is an interesting variation that encompasses both 2-SAT and 3-SAT. Fix a number $0 \leq p \leq 1$. When selecting m clauses at random, choose a 3-clause with probability p and a 2-clause with probability $1 - p$. This is known as $(2 + p)$-SAT and is useful in understanding the onset of complexity when moving from 2-SAT to 3-SAT [3, 40–42]. Clearly the critical value for this model satisfies

$$r_c(2 + p) \leq \min\left\{\frac{1}{1 - p}, \frac{1}{p} r_c(3)\right\}$$

for all p. Further [43], if $p \leq 2/5$, then with probability $\to 1$, a random $(2 + p)$-SAT formula is satisfiable if and only if its 2-SAT subformula is satisfiable. This is a remarkable result: A random mixture containing 60% 2-clauses and 40% 3-clauses behaves like 2-SAT! Evidence for a conjecture that the critical threshold $p_c = 2/5$ appears in [44]. See also [45].

Another variation involves replacing “inclusive or” when forming clauses by “exclusive or.” By way of contrast with k-SAT, $k \geq 3$, the XOR-SAT problem can be solved in polynomial time, and its transition from satisfiability to unsatisfiability is completely understood [46].

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5.22 Lenz–Ising Constants

The Ising model is concerned with the physics of phase transitions, for example, the tendency of a magnet to lose strength as it is heated, with total loss occurring above a certain finite critical temperature. This essay can barely introduce the subject. Unlike hard squares [5.12] and percolation clusters [5.18], a concise complete problem statement here is not possible. We are concerned with large arrays of 1s and -1s whose joint distribution passes through a singularity as a parameter T increases. The definition and characterization of the joint distribution is elaborate; our treatment is combinatorial and focuses on series expansions. See [1–10] for background.

Let L denote the regular d-dimensional cubic lattice with $N = n^d$ sites. For example, in two dimensions, L is the $n \times n$ square lattice with $N = n^2$. To eliminate boundary effects, L is wrapped around to form a d-dimensional torus so that, without exception, every site has $2d$ nearest neighbors. This convention leads to negligible error for large N.

5.22.1 Low-Temperature Series Expansions

Suppose that the N sites of L are colored black or white at random. The dN edges of L fall into three categories: black-black, black-white, and white-white. What can be said jointly about the relative numbers of these? Over all possible such colorings, let $A(p, q)$ be the number of colorings for which there are exactly p black sites and exactly q black-white edges. (See Figure 5.20.)

Then, for large enough N [11–14],

$$\begin{aligned}
&A(0, 0) = 1 && \text{(all white)},\\
&A(1, 2d) = N && \text{(one black)},\\
&A(2, 4d-2) = dN && \text{(two black, adjacent)},\\
&A(2, 4d) = \tfrac{1}{2}(N - 2d - 1)N && \text{(two black, not adjacent)},\\
&A(3, 6d-4) = (2d-1)dN && \text{(three black, adjacent)}.
\end{aligned}$$

Properties of this sequence can be studied via the bivariate generating function

$$a(x, y) = \sum_{p,q} A(p, q)x^p y^q$$

and the formal power series

$$\begin{aligned}
\alpha(x, y) &= \lim_{n\to\infty} \frac{1}{N} \ln(a(x, y))\\
&= xy^{2d} + dx^2y^{4d-2} - \frac{2d+1}{2}x^2y^{4d} + (2d-1)dx^3y^{6d-4} + \cdots
\end{aligned}$$

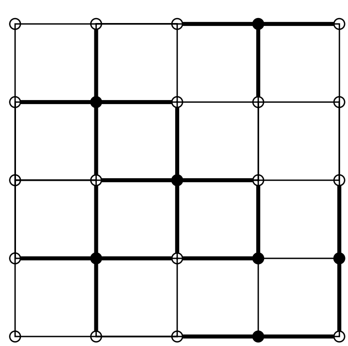

Figure 5.20. Sample coloring with $d = 2$, $N = 25$, $p = 7$, and $q = 21$ (ignoring wraparound).

obtained by merely collecting the coefficients that are linear in N. The latter is sometimes written as [15]

$$\exp(\alpha(x, y)) = 1 + xy^{2d} + dx^2y^{4d-2} - dx^2y^{4d} + (2d-1)dx^3y^{6d-4} + \cdots,$$

a series whose coefficients are integers only. This is what physicists call the **low-temperature series** for the **Ising free energy per site**. The letters x and y are not dummy variables but are related to temperature and magnetic field; the series $\alpha(x, y)$ is not merely a mathematical construct but is a thermodynamic function with properties that can be measured against physical experiment [16]. In the special case when $x = 1$, known as the **zero magnetic field case**, we write $\alpha(y) = \alpha(1, y)$ for convenience.

When $d = 2$, we have [11, 17]

$$\exp(\alpha(y)) = 1 + y^4 + 2y^6 + 5y^8 + 14y^{10} + 44y^{12} + 152y^{14} + 566y^{16} + \cdots.$$

Onsager [18–23] discovered an astonishing closed-form expression:

$$\alpha(y) = \frac{1}{2}\int_0^1\int_0^1 \ln\left[(1+y^2)^2 - 2y(1-y^2)(\cos(2\pi u) + \cos(2\pi v))\right] du\, dv$$

that permits computation of series coefficients to arbitrary order [24] and much more.

When $d = 3$, we have [11, 25–30]

$$\exp(\alpha(y)) = 1 + y^6 + 3y^{10} - 3y^{12} + 15y^{14} - 30y^{16} + 101y^{18} - 261y^{16} + - \cdots$$

No closed-form expression for this series has been found, and the required computations are much more involved than those for $d = 2$.

5.22.2 High-Temperature Series Expansions

The associated high-temperature series arises via a seemingly unrelated combinatorial problem. Let us assume that a nonempty *subgraph* of L is connected and contains at least one edge. Suppose that several subgraphs are drawn on L with the property that

- each edge of L is used at most once, and
- each site of L is used an *even* number of times (possibly zero).

Call such a configuration on L an **even polygonal drawing**. (See Figure 5.21.) An even polygonal drawing is the union of simple, closed, edge-disjoint polygons that need not be connected.

Let $B(r)$ be the number of even polygonal drawings for which there are exactly r edges. Then, for large enough N [4, 11, 31],

$$B(4) = \tfrac{1}{2}d(d-1)N \qquad \text{(square)},$$

$$B(6) = \tfrac{1}{3}d(d-1)(8d-13)N \qquad \text{(two squares, adjacent)},$$

$$B(8) = \tfrac{1}{8}d(d-1)\left(d(d-1)N + 216d^2 - 848d + 850\right)N \qquad \text{(many possibilities)}.$$

On the one hand, for $d \geq 3$, the drawings can intertwine and be knotted [32], so computing $B(r)$ for larger r is quite complicated! On the other hand, for $d = 2$, clearly

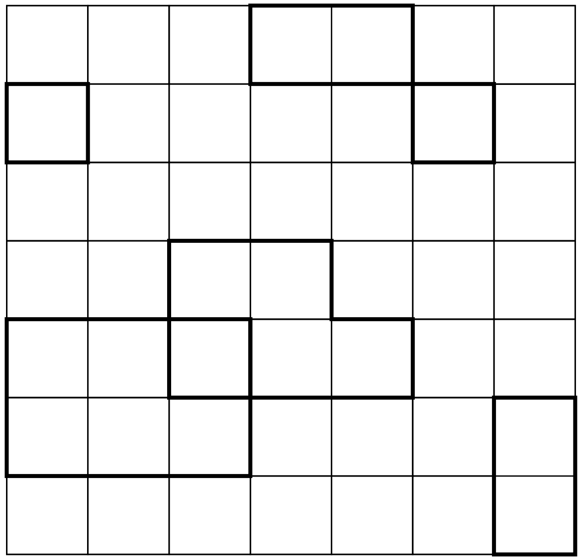

Figure 5.21. An even polygonal drawing for $d = 2$; other names include closed or Eulerian subgraph.

$B(q) = \sum_p A(p, q)$ always. As before, we define a (univariate) generating function

$$b(z) = 1 + \sum_r B(r) z^r$$

and a formal power series

$$\begin{aligned}\beta(z) &= \lim_{n\to\infty} \frac{1}{N} \ln(b(z)) \\ &= \frac{1}{2} d(d-1)z^4 + \frac{1}{3} d(d-1)(8d-13)z^6 + \frac{1}{4} d(d-1)(108d^2 - 424d + 425)z^8 \\ &\quad + \frac{2}{15} d(d-1)(2976d^3 - 19814d^2 + 44956d - 34419)z^{10} + \cdots\end{aligned}$$

called the **high-temperature zero-field series** for the **Ising free energy**. When $d = 3$ [11, 25, 29, 33–36],

$$\exp(\beta(z)) = 1 + 3z^4 + 22z^6 + 192z^8 + 2046z^{10} + 24853z^{12} + 329334z^{14} + \cdots,$$

but again our knowledge of the series coefficients is limited.

5.22.3 Phase Transitions in Ferromagnetic Models

The two major unsolved problems connected to the Ising model are [4, 31, 37]:

- Find a closed-form expression for $\alpha(x, y)$ when $d = 2$.
- Find a closed-form expression for $\beta(z)$ when $d = 3$.

Why are these so important? We discuss now the underlying physics, as well its relationship to the aforementioned combinatorial problems.

Place a bar of iron in an external magnetic field at constant absolute temperature T. The field will induce a certain amount of magnetization into the bar. If the external field is then slowly turned off, we empirically observe that, for small T, the bar retains some of its internal magnetization, but for large T, the bar's internal magnetization disappears completely.

There is a unique **critical temperature**, T_c, also called the **Curie point**, where this qualitative change in behavior occurs. The Ising model is a simple means for explaining the physical phenomena from a microscopic point of view.

At each site of the lattice L, define a "spin variable" $\sigma_i = 1$ if site i is "up" and $\sigma_i = -1$ if site i is "down." This is known as the **spin-1/2 model**. We study the **partition function**

$$Z(T) = \sum_{\sigma} \exp\left[\frac{1}{\kappa T}\left(\sum_{(i,j)} \xi \sigma_i \sigma_j + \sum_{k} \eta \sigma_k\right)\right],$$

where ξ is the coupling (or interaction) constant between nearest neighbor spin variables, $\eta \geq 0$ is the intensity constant of the external magnetic field, and $\kappa > 0$ is Boltzmann's constant.

The function $Z(T)$ captures all of the thermodynamic features of the physical system and acts as a kind of "denominator" when calculating state probabilities. Observe that the first summation is over all 2^N possible values of the vector $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ and the second summation is over all edges of the lattice (sites i and j are distinct and adjacent). Henceforth we will assume $\xi > 0$, which corresponds to the **ferromagnetic case**. A somewhat different theory emerges in the antiferromagnetic case ($\xi < 0$), which we will not discuss.

How is Z connected to the combinatorial problems discussed earlier? If we assign a spin 1 to the color white and a spin -1 to the color black, then

$$\sum_{(i,j)} \sigma_i \sigma_j = (dN - q) \cdot 1 + q \cdot (-1) = dN - 2q,$$

$$\sum_{k} \sigma_k = (N - p) \cdot 1 + p \cdot (-1) = N - 2p,$$

and therefore

$$Z = x^{-\frac{1}{2}N} y^{-\frac{d}{2}N} a(x, y),$$

where

$$x = \exp\left(-\frac{2\eta}{\kappa T}\right), \quad y = \exp\left(-\frac{2\xi}{\kappa T}\right).$$

Since small T gives small values of x and y, the phrase low-temperature series for $\alpha(x, y)$ is justified. (Observe that $T = \infty$ corresponds to the case when lattice site colorings are assigned equal probability, which is precisely the combinatorial problem described earlier. The range $0 < T < \infty$ corresponds to unequal weighting, accentuating the states with small p and q. The point $T = 0$ corresponds to an ideal case when all spins are aligned; heat introduces disorder into the system.)

For the high-temperature case, rewrite Z as

$$Z = \left(\frac{4}{(1-z^2)^d(1-w^2)} \right)^{\frac{N}{2}} \frac{1}{2^N} \sum_{\sigma} \left(\prod_{(i,j)} (1+\sigma_i\sigma_j z) \cdot \prod_k (1+\sigma_k w) \right),$$

where

$$z = \tanh\left(\frac{\xi}{\kappa T}\right), \quad w = \tanh\left(\frac{\eta}{\kappa T}\right).$$

In the zero-field scenario ($\eta = 0$), this expression simplifies to

$$Z = \left(\frac{4}{(1-z^2)^d} \right)^{\frac{N}{2}} b(z),$$

and since large T gives small z, the phraseology again makes sense.

5.22.4 Critical Temperature

We turn attention to some interesting constants. The radius of convergence y_c in the complex plane of the low-temperature series $\alpha(y) = \sum_{k=0}^{\infty} \alpha_k y^k$ is given by [29]

$$y_c = \lim_{k\to\infty} |\alpha_{2k}|^{-\frac{1}{2k}} = \begin{cases} \sqrt{2}-1 = 0.4142135623\ldots & \text{if } d = 2, \\ \sqrt{0.2853\ldots} = 0.5341\ldots & \text{if } d = 3; \end{cases}$$

hence, if $d = 2$, the ferromagnetic critical temperature T_c satisfies

$$K_c = \frac{\xi}{\kappa T_c} = \frac{1}{2}\ln\left(\frac{1}{y_c}\right) = \frac{1}{2}\ln(\sqrt{2}+1) = 0.4406867935\ldots.$$

The two-dimensional result is a famous outcome of work by Kramers & Wannier [38] and Onsager [18]. For $d = 3$, the singularity at $y^2 = -0.2853\ldots$ is nonphysical and thus is not relevant to ferromagnetism; a second singularity at $y^2 = 0.412048\ldots$ is what we want but it is difficult to compute directly [29,39]. To accurately obtain the critical temperature here, we examine instead the high-temperature series $\beta(z) = \sum_{k=0}^{\infty} \beta_k z^k$ and compute

$$z_c = \lim_{k\to\infty} {\beta_{2k}}^{-\frac{1}{2k}} = 0.218094\ldots, \quad K_c = \tfrac{1}{2}\ln\left(\frac{1+z_c}{1-z_c}\right) = 0.221654\ldots.$$

There is a huge literature of series and Monte Carlo analyses leading to this estimate [40–53]. (A conjectured exact expression for z_c in [54] appears to be false [55].) For $d > 3$, the following estimates are known [56–65]:

$$z_c = \begin{cases} 0.14855\ldots & \text{if } d = 4, \\ 0.1134\ldots & \text{if } d = 5, \\ 0.0920\ldots & \text{if } d = 6, \\ 0.0775\ldots & \text{if } d = 7, \end{cases} \qquad K_c = \begin{cases} 0.14966\ldots & \text{if } d = 4, \\ 0.1139\ldots & \text{if } d = 5, \\ 0.0923\ldots & \text{if } d = 6, \\ 0.0777\ldots & \text{if } d = 7. \end{cases}$$

An associated critical exponent γ will be discussed shortly.

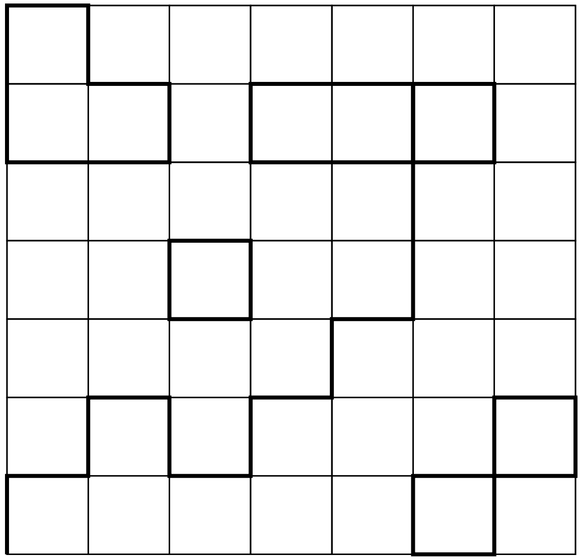

Figure 5.22. An odd polygonal drawing for $d = 2$.

5.22.5 Magnetic Susceptibility

Here is another combinatorial problem. Suppose that several subgraphs are drawn on L with the property that

- each edge of L is used at most once,
- all sites of L, except two, are even, and
- the two remaining sites are odd and must lie in the same (connected) subgraph.

Call this configuration an **odd polygonal drawing**. (See Figure 5.22.) Note that an odd polygonal drawing is the edge-disjoint union of an even polygonal drawing and an (undirected) self-avoiding walk [5.10] linking the two odd sites.

Let $C(r)$ be twice the number of odd polygonal drawings for which there are exactly r edges. Then, for large enough N [12, 66],

$$\begin{aligned}
&C(1) = 2dN && \text{(SAW)},\\
&C(2) = 2d(2d-1)N && \text{(SAW)},\\
&C(3) = 2d(2d-1)^2N && \text{(SAW)},\\
&C(4) = 2d\left(2d(2d-1)^3 - 2d(2d-2)\right)N && \text{(SAW)},\\
&C(5) = d^2(d-1)N^2 + 2d\left(16d^4 - 32d^3 + 16d^2 + 4d - 3\right)N && \text{(square and/or SAW)}.
\end{aligned}$$

As before, we may define a generating function and a formal power series

$$c(z) = N + \sum_r C(r)z^r, \quad \chi(z) = \lim_{n\to\infty} \frac{1}{N}\ln(c(z)) = \sum_{k=0}^{\infty} \chi_k z^k,$$

which is what physicists call the **high-temperature zero-field series** for the **Ising magnetic susceptibility per site**. The radius of convergence z_c of $\chi(z)$ is the same as that for $\beta(z)$ for $d > 1$. For example, when $d = 3$, analyzing the series [67–73]

$$\chi(z) = 1 + 6z + 30z^2 + 150z^3 + 726z^4 + 3510z^5 + 16710z^6 + \cdots$$

is the preferred way to obtain critical parameter estimates (being the best behaved of several available series). Further, the limit

$$\lim_{k\to\infty} \frac{\chi_k}{z_c^{-k} k^{\gamma-1}}$$

appears to exist and is nonzero for a certain positive constant γ depending on dimensionality. As an example, if $d = 2$, numerical evidence surrounding the series [67, 74, 75]

$$\chi(z) = 1 + 4z + 12z^2 + 36z^3 + 100z^4 + 276z^5 + 740z^6 + 1972z^7 + 5172z^8 + \cdots$$

suggests that the **critical susceptibility exponent** γ is $7/4$ and that γ is *universal* (in the sense that it is independent of the choice of lattice). No analogous exact expressions appear to be valid for γ when $d \geq 3$; for $d = 3$, the consensus is that $\gamma = 1.238\ldots$ [40, 44, 46, 49–52, 71, 73].

We finally make explicit the association of $\chi(z)$ with the Ising model [76]:

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{N} \ln(Z(z, w)) &= \ln(2) - \frac{d}{2} \ln(1 - z^2) - \frac{1}{2} \ln(1 - w^2) + \beta(z) \\ &\quad + \frac{1}{2} (\chi(z) - 1)\, w^2 + O(w^4),\end{aligned}$$

where the big O depends on z. Therefore $\chi(z)$ occurs when evaluating a second derivative with respect to w, specifically, when computing the variance of P (defined momentarily).

5.22.6 *Q* and *P* Moments

Let us return to the random coloring problem, suitably generalized to incorporate temperature. Let

$$Q = d - \frac{2}{N} q = \frac{1}{N} \sum_{(i,j)} \sigma_i \sigma_j, \quad P = 1 - \frac{2}{N} p = \frac{1}{N} \sum_k \sigma_k$$

for convenience and assume henceforth that $d = 2$. To study the asymptotic distribution of Q, define

$$F(z) = \lim_{n\to\infty} \frac{1}{N} \ln(Z(z)).$$

Then clearly

$$\lim_{n\to\infty} \mathrm{E}(Q) = (\kappa T)\frac{dF}{d\xi}, \quad \lim_{n\to\infty} N\,\mathrm{Var}(Q) = (\kappa T)^2 \frac{d^2 F}{d\xi^2}$$

via term-by-term differentiation of $\ln(Z)$. Exact expressions for both moments are possible using Onsager's formula:

$$F(z) = \ln\left(\frac{2}{1-z^2}\right) + \frac{1}{2}\int_0^1\int_0^1 \ln\left[(1+z^2)^2 - 2z(1-z^2)\left(\cos(2\pi u) + \cos(2\pi v)\right)\right] du\, dv,$$

but we give results at only two special temperatures. In the case $T = \infty$, for which states are assigned equal weighting, $\mathrm{E}(Q) \to 0$ and $N\,\mathrm{Var}(Q) \to 2$, confirming reasoning in [77]. In the case $T = T_c$, note that the singularity is fairly subtle since F and its first derivative are both well defined [11]:

$$F(z_c) = \frac{\ln(2)}{2} + \frac{2G}{\pi} = 0.9296953983\ldots = \frac{1}{2}\left(\ln(2) + 1.1662436161\ldots\right),$$

$$\lim_{n\to\infty} \mathrm{E}(Q) = \sqrt{2},$$

where G is Catalan's constant [1.7]. The second derivative of F, however, is unbounded in the vicinity of $z = z_c$ and, in fact [5],

$$\lim_{n\to\infty} N\,\mathrm{Var}(Q) \approx -\frac{8}{\pi}\left(\ln\left|\frac{T}{T_c} - 1\right| + g\right),$$

where g is the constant

$$g = 1 + \frac{\pi}{4} + \ln\left(\frac{\sqrt{2}}{4}\ln(\sqrt{2}+1)\right) = 0.6194036984\ldots.$$

This is related to what physicists call the **logarithmic divergence** of the **Ising specific heat**. (See Figure 5.23.)

As an aside, we mention that corresponding values of $F(z_c)$ on the triangular and hexagonal planar lattices are, respectively [11],

$$\ln(2) + \frac{\ln(3)}{4} + \frac{H}{2} = 0.8795853862\ldots,$$

$$\frac{3\ln(2)}{4} + \frac{\ln(3)}{2} + \frac{H}{4} = 1.0250590965\ldots.$$

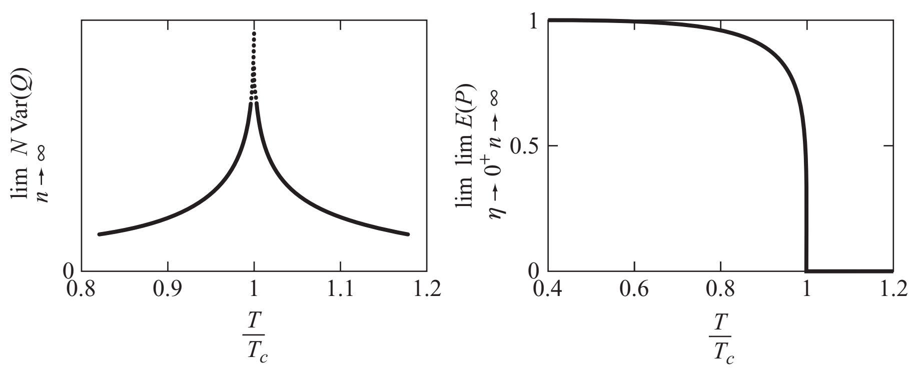

Figure 5.23. Graphs of Ising specific heat and spontaneous magnetization.

Both results feature a new constant [78, 79]:

$$H = \frac{5\sqrt{3}}{6\pi}\psi'\left(\frac{1}{3}\right) - \frac{5\sqrt{3}}{9}\pi - \ln(6) = \frac{\sqrt{3}}{6\pi}\psi'\left(\frac{1}{6}\right) - \frac{\sqrt{3}}{3}\pi - \ln(6)$$
$$= -0.1764297331\ldots,$$

where $\psi'(x)$ is the trigamma function (derivative of the digamma function $\psi(x)$ [1.5.4]). See [80–82] for other occurrences of H; note that the formula

$$\ln(2) + \ln(3) + H = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \ln\left[6 - 2\cos(\theta) - 2\cos(\varphi) - 2\cos(\theta+\varphi)\right] d\theta\, d\varphi$$
$$= \frac{3\sqrt{3}}{\pi}\left(1 - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{13^2} - \frac{1}{17^2} + - \cdots\right)$$
$$= 1.6153297360\ldots$$

parallels nicely similar results in [3.10] and [5.23].

A more difficult analysis allows us to compute the corresponding two moments of P and also to see more vividly the significance of magnetic susceptibility and critical exponents. Let

$$F(z, w) = \lim_{n\to\infty} \frac{1}{N}\ln(Z(z, w));$$

then clearly

$$\lim_{\eta\to 0^+}\lim_{n\to\infty} \mathrm{E}(P) = (\kappa T)\left.\frac{\partial F}{\partial \eta}\right|_{\eta=0}, \quad \lim_{\eta\to 0^+}\lim_{n\to\infty} N\,\mathrm{Var}(P) = (\kappa T)^2\left.\frac{\partial^2 F}{\partial \eta^2}\right|_{\eta=0}$$

as before. Of course, we do not know $F(z, w)$ exactly when $w \neq 0$. Its derivative at $w = 0$, however, has a simple expression valid for all z:

$$\lim_{\eta\to 0^+}\lim_{n\to\infty} \mathrm{E}(P) = \begin{cases} \left[1 - \sinh\left(\dfrac{2\xi}{\kappa T}\right)^{-4}\right]^{\frac{1}{8}} & \text{if } T < T_c, \\ 0 & \text{if } T > T_c, \end{cases}$$
$$= \begin{cases} (1+y^2)^{\frac{1}{4}}(1-6y^2+y^4)^{\frac{1}{8}}(1-y^2)^{-\frac{1}{2}} & \text{if } T < T_c, \\ 0 & \text{if } T > T_c \end{cases}$$

due to Onsager and Yang [83–85]. A rigorous justification is found in [86–88]. For the special temperature $T = \infty$, we have $\mathrm{E}(P) \to 0$ and $N\,\mathrm{Var}(P) \to 1$ since p is Binomial $(N, 1/2)$ distributed. At criticality, $\mathrm{E}(P) \to 0$ as well, but the second derivative exhibits fascinatingly complicated behavior:

$$\lim_{\eta\to 0^+}\lim_{n\to\infty} N\,\mathrm{Var}(P) = \chi(z) \approx c_0^+ t^{-\frac{7}{4}} + c_1^+ t^{-\frac{3}{4}} + d_0 + c_2^+ t^{\frac{1}{4}} + e_0\, t\ln(t) + d_1 t + c_3^+ t^{\frac{5}{4}},$$

where $0 < t = 1 - T_c/T$, $c_0^+ = 0.9625817323\ldots$, $d_0 = -0.1041332451\ldots$, $e_0 = 0.0403255003\ldots$, $d_1 = -0.14869\ldots$, and

$$c_1^+ = \tfrac{\sqrt{2}}{8}K_c c_0^+, \quad c_2^+ = \tfrac{151}{192}K_c^2 c_0^+, \quad c_3^+ = \tfrac{615\sqrt{2}}{512}K_c^3 c_0^+.$$

Wu, McCoy, Tracy & Barouch [89–99] determined exact expressions for these series coefficients in terms of the solution to a Painlevé III differential equation (described in the next section). Different numerical values of the coefficients apply for $T < T_c$, as well as for the antiferromagnetic case [100, 101]. For example, when $t < 0$, the corresponding leading coefficient is $c_0^- = 0.0255369745\ldots$. The study of magnetic susceptibility $\chi(z)$ is far more involved than the other thermodynamic functions mentioned in this essay, and there are still gaps in the rigorous line of thought [102]. Also, in a recent breakthrough [103, 104], the entire asymptotic structure of $\chi(z)$ has now largely been determined.

5.22.7 Painlevé III Equation

Let $f(x)$ be the solution of the Painlevé III differential equation [105]

$$\frac{f''(x)}{f(x)} = \left(\frac{f'(x)}{f(x)}\right)^2 - \frac{1}{x}\frac{f'(x)}{f(x)} + f(x)^2 - \frac{1}{f(x)^2}$$

satisfying the boundary conditions

$$f(x) \sim 1 - \frac{e^{-2x}}{\sqrt{\pi x}} \text{ as } x \to \infty, \quad f(x) \sim x\,(2\ln(2) - \gamma - \ln(x)) \text{ as } x \to 0^+,$$

where γ is Euler's constant [1.5]. Define

$$g(x) = \left[\frac{x f'(x)}{2 f(x)} + \frac{x^2}{4 f(x)^2}\left(\left(1 - f(x)^2\right)^2 - f'(x)^2\right)\right]\ln(x).$$

Then exact expressions for c_0^+ and c_0^- are

$$\begin{aligned} c_0^+ &= 2^{\frac{5}{8}}\pi \ln(\sqrt{2}+1)^{-\frac{7}{4}} \int_0^\infty y(1 - f(y)) \\ &\quad \times \exp\left[\int_y^\infty x\ln(x)\left(1 - f(x)^2\right)\,dx - g(y)\right] dy, \\ c_0^- &= 2^{\frac{5}{8}}\pi \ln(\sqrt{2}+1)^{-\frac{7}{4}} \int_0^\infty y \\ &\quad \times \left\{(1 + f(y))\exp\left[\int_y^\infty x\ln(x)\left(1 - f(x)^2\right)\,dx - g(y)\right] - 2\right\} dy. \end{aligned}$$

Painlevé II arises in our discussion of the longest increasing subsequence problem [5.20], and Painlevé V arises in connection with the GUE hypothesis [2.15.3].

Here is a slight variation of these results. Define

$$h(x) = -\ln\left(f\left(\frac{x}{c}\right)\right)$$

for any constant $c > 0$; then the function $h(x)$ satisfies what is known as the sinh-Gordon differential equation:

$$h''(x) + \frac{1}{x}h'(x) = \frac{2}{c^2}\sinh(2h(x)),$$

$$h(x) \sim \sqrt{\frac{c}{\pi x}}\exp\left(-\frac{2x}{c}\right) \text{ as } x \to \infty.$$

Finally, we mention a beautiful formula:

$$\int_0^\infty x\ln(x)\left(1 - f(x)^2\right)dx = \frac{1}{4} + \frac{7}{12}\ln(2) - 3\ln(A),$$

where A is Glaisher's constant [2.15]. Conceivably, c_0^+ and c_0^- may someday be related to A as well.

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5.23 Monomer-Dimer Constants

Let L be a graph [5.6]. A **dimer** consists of two adjacent vertices of L and the (non-oriented) bond connecting them. A **dimer arrangement** is a collection of disjoint dimers on L. Uncovered vertices are called **monomers**, so dimer arrangements are also known as **monomer-dimer coverings**. We will discuss such coverings only briefly at the beginning of the next section.

A **dimer covering** is a dimer arrangement whose union contains all the vertices of L. Dimer coverings and the closely-related topic of tilings will occupy the remainder of this essay.

5.23.1 2D Domino Tilings

Let a_n denote the number of distinct monomer-dimer coverings of an $n \times n$ square lattice L and $N = n^2$; then $a_1 = 1$, $a_2 = 7$, $a_3 = 131$, $a_4 = 10012$ [1,2], and asymptotically [3–6]

$$A = \lim_{n\to\infty} a_n^{\frac{1}{N}} = 1.940215351\ldots = (3.764435608\ldots)^{\frac{1}{2}}.$$

No exact expression for the constant A is known. Baxter's approach for estimating A was based on the corner transfer matrix variational approach, which also played a role in [5.12]. A natural way for physicists to discuss the monomer-dimer problem is to associate an activity z with each dimer; A thus corresponds to the case $z = 1$. The mean number ρ of dimers per vertex is 0 if $z = 0$ and $1/2$ if $z = \infty$; when $z = 1$, ρ is $0.3190615546\ldots$, for which again there is no closed-form expression [3]. Unlike other lattice models (see [5.12], [5.18], and [5.22]), monomer-dimer systems do not have a phase transition [7].

Computing a_n is equivalent to counting (not necessarily perfect) **matchings** in L, that is, to counting independent sets of edges in L. This is related to the difficult problem of computing permanents of certain binary incidence matrices [8–14]. Kenyon, Randall & Sinclair [15] gave a randomized polynomial-time approximation algorithm for computing the number of monomer-dimer coverings of L, assuming ρ to be given.

Let us turn our attention henceforth to the zero monomer density case, that is, $z = \infty$. If b_n is the number of distinct dimer coverings of L, then $b_n = 0$ if n is odd and

$$b_n = 2^{N/2} \prod_{j=1}^{n/2} \prod_{k=1}^{n/2} \left(\cos^2 \frac{j\pi}{n+1} + \cos^2 \frac{k\pi}{n+1} \right)$$

if n is even. This exact expression is due to Kastelyn [16] and Fisher & Temperley [17, 18]. Further,

$$\begin{aligned} \lim_{\substack{n\to\infty \\ n \text{ even}}} \frac{1}{N} \ln(b_n) &= \frac{1}{16\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \ln\left[4 + 2\cos(\theta) + 2\cos(\varphi)\right] \, d\theta \, d\varphi \\ &= \frac{G}{\pi} = 0.2915609040\ldots; \end{aligned}$$

that is,

$$B = \lim_{\substack{n\to\infty \\ n \text{ even}}} b_n^{\frac{1}{N}} = \exp\left(\frac{G}{\pi}\right) = 1.3385151519\ldots = (1.7916228120\ldots)^{\frac{1}{2}},$$

where G is Catalan's constant [1.7]. This is a remarkable solution, in graph theoretic terms, of the problem of counting **perfect matchings** on the square lattice. It is also an answer to the following question: What is the number of ways of tiling an $n \times n$ chessboard with 2×1 or 1×2 **dominoes**? See [19–26] for more details. The constant B^2 is called δ in [3.10] and appears in [1.8] too; the expression $4G/\pi$ arises in [5.22], $G/(\pi \ln(2))$ in [5.6], and $8G/\pi^2$ in [7.7].

If we wrap the square lattice around to form a torus, the counts b_n differ somewhat, but the limiting constant B remains the same [16, 27]. If, instead, we assume the chessboard to be shaped like an Aztec diamond [28], then the associated constant $B = 2^{1/4} = 1.189\ldots < 1.338\ldots = e^{G/\pi}$. Hence, even though the square chessboard has slightly less area than the diamond chessboard, the former possesses many more domino tilings [29]. Lattice boundary effects are thus seen to be nontrivial.

5.23.2 Lozenges and Bibones

The analog of $\exp(2G/\pi)$ for dimers on a hexagonal (honeycomb) lattice with wraparound is [30–32]

$$C^2 = \lim_{n\to\infty} c_n^{\frac{2}{N}} = \exp\left(\frac{1}{8\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \ln\left[3+2\cos(\theta)+2\cos(\varphi)+2\cos(\theta+\varphi)\right]\,d\theta\,d\varphi\right)$$
$$= 1.3813564445\ldots.$$

This constant is called β in [3.10] and can be expressed by other formulas too. It characterizes lozenge tilings on a chessboard with triangular cells satisfying periodic boundary conditions. See [33–38] as well.

If there is no wraparound, then the sequence [39]

$$c_n = \prod_{j=1}^{n}\prod_{k=1}^{n}\frac{n+j+k-1}{j+k-1}$$

emerges, and a different growth constant $3\sqrt{3}/4$ applies. We have assumed that the hexagonal grid is center-symmetric with sides n, n, and n (i.e., the simplest possible boundary conditions). The sequence further enumerates plane partitions contained within an $n \times n \times n$ box [40,41].

The corresponding analog for dimers on a triangular lattice with wraparound is [30,42,43]

$$D^2 = \lim_{n\to\infty} d_n^{\frac{2}{N}} = \exp\left(\frac{1}{8\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \ln\left[6+2\cos(\theta)+2\cos(\varphi)+2\cos(\theta+\varphi)\right]\,d\theta\,d\varphi\right)$$
$$= 2.3565273533\ldots.$$

The expression $4\ln(D)$ bears close similarity to a constant $\ln(6)+H$ described in [5.22]. It also characterizes bibone tilings on a chessboard with hexagonal cells satisfying periodic boundary conditions. The case of no wraparound [1,44,45] apparently remains open.

5.23.3 3D Domino Tilings

Let h_n denote the number of distinct dimer coverings of an $n \times n \times n$ cubic lattice L and $N = n^3$. Then $h_n = 0$ if n is odd, $h_2 = 9$, and $h_4 = 5051532105$ [46,47]. An important unsolved problem in solid-state chemistry is the estimation of

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} h_n^{\frac{1}{N}} = \exp(\lambda)$$

or, equivalently,

$$\lambda = \lim_{\substack{n\to\infty \\ n \text{ even}}} \frac{1}{N}\ln(h_n).$$

Hammersley [48] proved that λ exists and $\lambda \geq 0.29156$. Lower bounds were improved by Fisher [49] to 0.30187, Hammersley [50, 51] to 0.418347, and Priezzhev [52, 53] to 0.419989. In a review of [54], Minc pointed out that a conjecture due to Schrijver & Valiant on lower bounds for permanents of certain binary matrices would imply that $\lambda \geq 0.44007584$. Schrijver [55] proved this conjecture, and this is the best-known result.

Fowler & Rushbrooke [56] gave an upper bound of 0.54931 for λ over sixty years ago (assuming λ exists). Upper bounds have been improved by Minc [8, 57, 58] to 0.5482709, Ciucu [59] to 0.463107, and Lundow [60] to 0.457547.

A sequence of nonrigorous numerical estimates by Nagle [30], Gaunt [31], and Beichl & Sullivan [61] has culminated with $\lambda = 0.4466\ldots$. As with a_n, computing h_n for even small values of n is hard and matrix permanent approximation schemes offer the only hope. The field is treacherously difficult: Conjectured exact asymptotic formulas for h_n in [62, 63] are incorrect.

A related topic is the number, k_n, of dimer coverings of the n-dimensional unit cube, whose 2^n vertices consist of all n-tuples drawn from {0, 1} [47, 64]. The term $k_6 = 16332454526976$ was computed independently by Lundow [46] and Weidemann [65]. In this case, we know the asymptotic behavior of k_n rather precisely [44, 65, 66]:

$$\lim_{n\to\infty} \frac{1}{n} k_n^{2^{1-n}} = \frac{1}{e} = 0.3678794411\ldots,$$

where e is the natural logarithmic base [1.3].

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5.24 Lieb's Square Ice Constant

Let L denote the $n \times n$ planar square lattice with wraparound and let $N = n^2$. An **orientation** of L is an assignment of a direction (or arrow) to each edge of L. What is the number, f_n, of orientations of L such that at each vertex there are exactly two inward and two outward pointing edges? Such orientations are said to obey the **ice rule** and are also called **Eulerian orientations**. The sequence $\{f_n\}$ starts with the terms $f_1 = 4$, $f_2 = 18$, $f_3 = 148$, and $f_4 = 2970$ [1,2]. After intricate analysis, Lieb [3–5] proved that

$$\lim_{n\to\infty} f_n^{\frac{1}{N}} = \left(\frac{4}{3}\right)^{\frac{3}{2}} = \sqrt{\frac{64}{27}} = 1.5396007178\ldots.$$

This constant is known as the **residual entropy for square ice**. A brief discussion of the underlying physics appears in [5.24.3]. The model is also called a **six-vertex model** since, at each vertex, there are six possible configurations of arrows [6–9]. See Figure 5.24.

We turn to several related results. Let $\tilde{f}_n$ denote the number of orientations of L such that at each vertex there are an even number of edges pointing in and an even number pointing out. Clearly $\tilde{f}_n \geq f_n$ and the model is called an **eight-vertex model**. In this case, however, the analysis is not quite so intricate and we have $\tilde{f}_n = 2^{N+1}$ via elementary linear algebra. The corresponding expression for the **sixteen-vertex model** (with no restrictions on the arrows) is obviously 2^{2N}.

Let us focus instead on the planar triangular lattice L with N vertices. What is the number, g_n, of orientations of L such that at each vertex there are exactly three inward and three outward pointing edges? (The phrase *Eulerian orientation* applies here, but not *ice rule*.) Baxter [10] proved that this **twenty-vertex model** satisfies

$$\lim_{n\to\infty} g_n^{\frac{1}{N}} = \sqrt{\frac{27}{4}} = 2.5980762113\ldots.$$

The problem of computing f_n and g_n is the same as counting nowhere-zero flows modulo

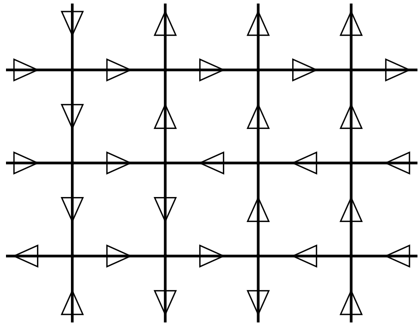


Figure 5.24. A sample planar configuration of arrows satisfying the ice rule.

3 on L [9, 11, 12]. Mihail & Winkler [13] studied related computational complexity issues.

One of several solutions of the famous alternating sign matrix conjecture [1, 14–16] is closely related to the square ice model. This achievement serves to underscore (once again) the commonality of combinatorial theory and statistical physics.

5.24.1 Coloring

Here is a fascinating topic that anticipates the next essay [5.25]. Let u_n denote the number of ways of coloring the vertices of the square lattice with three colors so that no two adjacent vertices are colored alike. Lenard [5] pointed out that $u_n = 3f_n$. In words, the number of 3-colorings of a square map is thrice the number of square ice configurations. We will return to u_n momentarily, with generalization in mind.

Replace the square lattice by the triangular lattice L and fix an integer $q \geq 4$. Let v_n denote the number of ways of coloring the vertices of L with q colors so that no two adjacent vertices are colored alike. Baxter [17, 18] proved that, if a parameter $-1 < x < 0$ is defined for $q > 4$ by $q = 2 - x - x^{-1}$, then

$$\lim_{n\to\infty} v_n^{\frac{1}{N}} = -\frac{1}{x}\prod_{j=1}^{\infty}\frac{(1-x^{6j-3})(1-x^{6j-2})^2(1-x^{6j-1})}{(1-x^{6j-5})(1-x^{6j-4})(1-x^{6j})(1-x^{6j+1})}.$$

In particular, letting $q \to 4^+$ (note that the formula makes sense for real q), we obtain

$$C^2 = \lim_{n\to\infty} v_n^{\frac{1}{N}} = \prod_{j=1}^{\infty}\frac{(3j-1)^2}{(3j-2)(3j)} = \frac{3}{4\pi^2}\Gamma\left(\tfrac{1}{3}\right)^3$$
$$= 1.4609984862\ldots = (1.2087177032\ldots)^2,$$

which we call **Baxter's 4-coloring constant** for a triangular lattice.

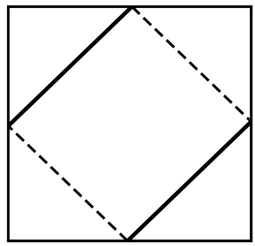
Define likewise u_n and w_n for the number of q-colorings of the square lattice and the hexagonal (honeycomb) lattice with N vertices, respectively. Analytical expressions for the corresponding limiting values are not available, but numerical assessment of certain series expansions provide the list in Table 5.6 [19–21]. The only known quantity in this table is Lieb's constant in the upper left corner. See [5.25] for related discussion on chromatic polynomials.

Table 5.6. *Limiting Values of Roots* $u_n^{1/N}$ *and* $w_n^{1/N}$

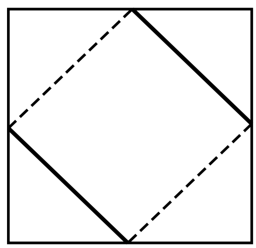
q	$\lim_{n\to\infty} u_n^{1/N}$	$\lim_{n\to\infty} w_n^{1/N}$
3	1.5396...	1.6600...
4	2.3360...	2.6034...
5	3.2504...	3.5795...
6	4.2001...	4.5651...
7	5.1667...	5.5553...

5.24.2 Folding

The *square-diagonal folding* problem may be translated into the following coloring problem. Cover the faces of the square lattice with either of the two following square tiles.



Tile 1



Tile 2

Tile 1: Alternating black and white segments join the centers of the consecutive edges around the square; west-to-north segment is black, north-to-east is white, east-to-south is black, and south-to-west is white. Tile 2: The opposite convention is adopted; west-to-north segment is white, north-to-east is black, east-to-south is white, and south-to-west is black.

There are $2N$ such coverings for a lattice made of N squares. Now, surrounding each vertex of the original lattice, there is a square *loop* formed from the four neighboring tiles. Count the number K_w of purely white loops and the number K_b of purely black loops, assuming wraparound. In the sample covering of Figure 5.25, both K_w and K_b are zero. Define

$$s = \lim_{n\to\infty} \frac{1}{4N} \ln\left(\sum_{\text{coverings}} 2^{K_w+K_b}\right)$$

to be the **entropy of folding** of the **square-diagonal lattice**, where the sum is over all 2^N tiling configurations. (This entropy is per triangle rather than per tile, which explains the additional factor of $1/4$.)

An obvious lower bound for s is

$$s \geq \lim_{n\to\infty} \frac{1}{4N} \ln(2^N + 2^N) = \lim_{n\to\infty} \frac{N+1}{4N} \ln(2) = \frac{1}{4}\ln(2) = 0.1732\ldots,$$

which is obtained by allowing the tiling configurations to alternate like a chessboard. There are two such possibilities (by simple exchanging of all tile 1s by tile 2s and all tile 2s by tile 1s). A more elaborate argument [22,23] gives $s = 0.2299\ldots.$

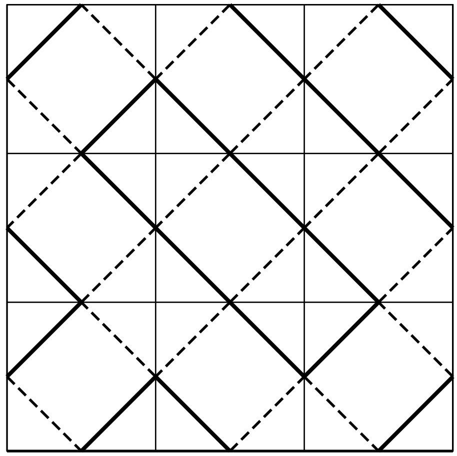


Figure 5.25. Sample covering of a lattice by tiles of both types.

The corresponding **entropy of folding** of the **triangular lattice** is $\ln(C) = 0.1895600483\ldots$ due to Baxter [17, 18] and possesses a simpler coloring interpretation, as already mentioned.

5.24.3 Atomic Arrangement in an Ice Crystal

Square ice is a two-dimensional idealization of water (H_2O) in its solid phase. The oxygen (O) atoms are pictured as the vertices of the square lattice, with outward pointing edges interpreted as the hydrogen (H) atoms. In actuality, however, there are several kinds of three-dimensional ice, depending on temperature and pressure [24, 25]. The residual entropies W for *ordinary hexagonal ice* Ice-Ih and for *cubic ice* Ice-Ic satisfy [3, 26–30]

$$1.5067 < W < 1.5070$$

and are equal within the limits of Nagle's estimation error. These complicated three-dimensional lattices are not the same as the simple models mathematicians tend to focus on.

It would be interesting to see the value of W for the customary $n \times n \times n$ cubic lattice, either with the ice rule in effect (two arrows point out, two arrows point in, and two null arrows) or with Eulerian orientation (three arrows point out and three arrows point in). No one appears to have done this.

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5.25 Tutte–Beraha Constants

Let G be a graph with n vertices v_j [5.6] and let λ be a positive integer. A **λ-coloring** of G is a function $\{v_1, v_2, \ldots, v_n\} \to \{1, 2, \ldots, \lambda\}$ with the property that adjacent vertices must be colored differently. Define $P(\lambda)$ to be the number of λ-colorings of G. Then $P(\lambda)$ is a polynomial of degree n, called the **chromatic polynomial** (or **chromial**) of G. For example, if G is a triangle (three vertices with each pair connected), then $P(\lambda) = \lambda(\lambda - 1)(\lambda - 2)$. Chromatic polynomials were first extensively studied by Birkhoff & Lewis [1]; see [2–6] for introductory material.

A graph G is **planar** if it can be drawn in the plane in such a way that no two edges cross except at a common vertex. The famous Four Color Theorem for geographic maps can be restated as follows: If G is a planar graph, then $P(4) > 0$. Among several restatements of the theorem, we mention Kauffman's combinatorial three-dimensional vector cross product result [7–9].

We can ask about the behavior of $P(\lambda)$ at other real values. Clearly $P(0) = 0$ and, if G is connected, then $P(1) = 0$ and $P(\lambda) \neq 0$ for $\lambda < 0$ or $0 < \lambda < 1$. Further, $P(\varphi + 1) \neq 0$, where φ is the Golden mean [1.2]; more concerning φ will be said shortly.

A connected planar graph G determines a subdivision of the 2-sphere (under stereographic projection) into simply connected regions (faces). If each such region is bounded by a simple closed curve made up of exactly three edges of G, then G is called a **spherical triangulation**. We henceforth assume that this condition is always met.

Clearly $P(2) = 0$ for any spherical triangulation G. Empirical studies of typical G suggest that $P(\lambda) \neq 0$ for $1 < \lambda < 2$, but a single zero is expected in the interval $2 < \lambda < 3$. Tutte [10, 11] proved that

$$0 < |P(\varphi + 1)| \leq \varphi^{5-n};$$

hence $\varphi + 1$, although not itself a zero of $P(\lambda)$, is arbitrarily close to being a zero for large enough n. For this reason, the constant $\varphi + 1$ is called the **golden root**.

It is known that $P(3) > 0$ if and only if G is Eulerian; that is, the number of edges incident with each vertex is even [5]. Hence for non-Eulerian triangulations, we have $P(3) = 0$.

Tutte [12–14] subsequently proved a remarkable identity:

$$P(\varphi + 2) = (\varphi + 2)\varphi^{3n-10} \left(P(\varphi + 1)\right)^2,$$

which implies that $P(\varphi + 2) > 0$. Note that $\varphi + 2 = \sqrt{5}\varphi = 3.6180339887\ldots$. As stated earlier, $P(4) > 0$, and $P(\lambda) > 0$ for $\lambda \geq 5$ [1]. It is natural to ask about the possible whereabouts of the next accumulation point for zeros (after $2.618\ldots$).

Rigorous theory fails us here, so numerical evidence must suffice [15–18]. In the following, fix a family $\{G_k\}$ of spherical triangulations, where n_k is the order of G_k and $n_k \to \infty$ as $k \to \infty$. Typically, the graph G_k is recursively constructed from G_{k-1} for each k, but this is not essential. Experimental results indicate that the next batch of chromatic zeros might cluster around the point

$$\psi = 2 + 2\cos\left(\frac{2\pi}{7}\right) = 4\cos\left(\frac{\pi}{7}\right)^2 = 3.2469796037\ldots,$$

that is, a solution of the cubic equation $\psi^3 - 5\psi^2 + 6\psi - 1 = 0$. The constant ψ is called the **silver root** by analogy with the golden root $\varphi + 1$.

Or the zeros might cluster around some other point $> \psi$, but ≤ 4. Beraha [19] observed a pattern in the potential accumulation points, independent of the choice of $\{G_k\}$. He conjectured that, for arbitrary $\{G_k\}$, if chromatic zeros z_k cluster around a real number x, then $x = B_N$ for some $N \geq 1$, where

$$B_N = 2 + 2\cos\left(\frac{2\pi}{N}\right) = 4\cos\left(\frac{\pi}{N}\right)^2.$$

In words, the limiting values x cannot fall outside of a certain countably infinite set. Note that the **Tutte–Beraha constants** B_N include all the roots already discussed:

$$\begin{array}{llll} B_2 = 0, & B_3 = 1, & B_4 = 2, & B_5 = \varphi + 1, \\ B_6 = 3, & B_7 = \psi, & B_{10} = \varphi + 2, & \lim_{N\to\infty} B_N = 4. \end{array}$$

Specific families $\{G_k\}$ have been constructed that can be proved to possess B_5, B_7, or B_{10} as accumulation points [20–23]. The marvel of Beraha's conjecture rests in its generality: It applies regardless of the configuration of G_k.

Beraha & Kahane also built a family $\{G_k\}$ possessing $B_1 = 4$ as an accumulation point. This is surprising since we know $P(4) > 0$ always, but $P_k(z_k) = 0$ for all k and $\lim_{k\to\infty} z_k = 4$. Hence the Four Color Theorem, although true, is nearly false [24].

The Tutte–Beraha constants also arise in mathematical physics [25–28] since evaluating $P(\lambda)$ over a lattice is equivalent to solving the λ-state zero-temperature antiferromagnetic Potts model. A heuristic explanation of the Beraha conjecture in [27] is insightful but is not a rigorous proof [8]. See [5.24] for related discussion on coloring and ice models. Other expressions containing $\cos(\pi/7)$ are mentioned in [2.23] and [8.2].

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6

Constants Associated with Functional Iteration

6.1 Gauss' Lemniscate Constant

In 1799, Gauss observed that the limiting value, M, of the following sequence:

$$a_0 = 1,\ \ b_0 = \sqrt{2},\ \ a_n = \frac{a_{n-1} + b_{n-1}}{2},\ \ b_n = \sqrt{a_{n-1}b_{n-1}} \text{ for } n \geq 1$$

must satisfy

$$\begin{aligned}\frac{1}{M} &= \lim_{n\to\infty} \frac{1}{a_n} = \lim_{n\to\infty} \frac{1}{b_n} = \frac{2}{\pi}\int\limits_0^1 \frac{dx}{\sqrt{1-x^4}} = 0.8346268416\ldots\\ &= \frac{1}{1.1981402347\ldots}.\end{aligned}$$

The recursive formulation is based on what is called the **arithmetic-geometric-mean (AGM) algorithm**. Gauss recognized this limit to be an extraordinary result and pointed out an interesting connection to geometry as well. The total arclength of the lemniscate $r^2 = \cos(2\theta)$ is given by $2L$, where

$$L = \int\limits_0^\pi \frac{d\theta}{\sqrt{1+\sin(\theta)^2}} = 2\int\limits_0^1 \frac{dx}{\sqrt{1-x^4}} = 2.6220575542\ldots$$

and thus $L = \pi/M$. The **lemniscate constant** L plays a role for the lemniscate analogous to what π plays for the circle, and the AGM algorithm provides a quadratically convergent method of computing it [1–5].

Other representations of L are

$$L = \sqrt{2}K\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2\pi}}\Gamma\left(\frac{1}{4}\right)^2 = \frac{\pi}{\sqrt{2}}\exp\left(\frac{1}{2}\left[\gamma - \frac{\beta'(1)}{\beta(1)}\right]\right),$$

where K denotes the complete elliptic integral of the first kind [1.4.6], $\Gamma(x)$ is the Euler gamma function [1.5.4], γ is the Euler–Mascheroni constant [1.5], and $\beta(x)$ is Dirichlet's beta function [1.7]. As stated in [2.10], clearly this is a meeting place for many ideas! Two rapidly convergent series are [4, 6]

$$\frac{1}{M} = \left[\sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi n^2}\right]^2 = 2^{\frac{5}{4}} e^{\frac{-\pi}{3}} \left[\sum_{n=-\infty}^{\infty} (-1)^n e^{-2\pi(3n+1)n}\right]^2.$$

A third series involving central binomial coefficients appears in [1.5.4].

Several authors [7, 8] identify $M/\sqrt{2} = 0.8472130848\ldots$ as the so-called "ubiquitous constant," and the value $L/\sqrt{2} = 1.8540746773\ldots$ is also given in [9]. The definite integrals

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{L}{2} = 1.3110287771\ldots, \quad \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} = \frac{M}{2} = 0.5990701173\ldots$$

are sometimes called, respectively, the first and second lemniscate constants [4, 10, 11].

Gauss correctly anticipated that his limiting result and others like it would ignite research for many years to come. The massive field of *elliptic modular functions*, associated with names such as Abel, Jacobi, Cayley, Klein, and Fricke, can be said to have started with Gauss' observation [1, 4]. Although the theory slipped into obscurity by the 1900s, it has recently enjoyed a renaissance. Two contributing factors in this renaissance are the widespread awakening to Ramanujan's achievements and the discovery of fast algorithms for computing π, based on AGM-like recursions.

The constant L was proved in 1937 to be transcendental by Schneider [12]. Let us now consider something slightly more complicated. The infinite product over all nonzero Gaussian integers

$$\sigma(z) = z \prod_{\omega \neq 0} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)$$

is called the **Weierstrass sigma function** [13, 14]. One has [15–17]

$$\sigma\left(\tfrac{1}{2}\right) = 2^{\frac{5}{4}} \pi^{\frac{1}{2}} e^{\frac{\pi}{8}} \Gamma\left(\tfrac{1}{4}\right)^{-2} = 2^{-\frac{1}{4}} e^{\frac{\pi}{8}} L^{-1} = 0.4749493799\ldots,$$

and this is transcendental, thanks to work by Nesterenko in 1996. Hence it took nearly sixty years for sufficient progress to be made to deal with the extra $\exp(\pi/8)$ factor in $\sigma(1/2)$! More results of this nature appear in [1.5.4].

Instead of the Gaussian integers ω, examine the lattice of points

$$\left\{\tilde{\omega} = j \cdot \left(\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}\right) + k \cdot \left(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right) : -\infty < j, k < \infty \text{ are integers}\right\},$$

and define $\tilde{\sigma}(z)$ analogously over all such nonzero $\tilde{\omega}$. We will need this function shortly [6.1.1].

Starting with work of Erdös, Herzog & Piranian [18], Borwein [19] studied an interesting question. Let $p(z)$ denote a monic polynomial of degree n. Consider the curve in the complex plane given by $|p(z)| = 1$. Is the total arclength of this curve no greater than that for $p(z) = z^n - 1$? In the special case when $n = 2$, this reduces to the lemniscate $r^2 = 2\cos(2\theta)$, which has arclength $2\sqrt{2}L$. See [20] for recent progress on answering this question.

The integral

$$\int_0^1 \sqrt{1-x^4}dx = \frac{L}{3} = 0.8740191847\ldots$$

occurs in our discussion of the Landau–Ramanujan constant [2.3], in connection with recent number theoretic work by Friedlander & Iwaniec. Also, from geometric probability, M arises in an expression for the expected perimeter of the convex hull of N random points in the unit square, as discussed in [8.1].

6.1.1 Weierstrass Pe Function

Given $\sigma(z)$ and $\tilde{\sigma}(z)$ as defined in the previous section, let

$$\wp(z) = -\frac{d^2}{dz^2}\ln(\sigma(z)), \quad \tilde{\wp}(z) = -\frac{d^2}{dz^2}\ln(\tilde{\sigma}(z)).$$

Like the Jacobi elliptic functions [1.4.6], both $\wp(z)$ and $\tilde{\wp}(z)$ are doubly periodic meromorphic functions. The real half-period r of $\wp(x)$ is $L/\sqrt{2} = 1.8540746773\ldots$, whereas the real half-period $\tilde{r}$ of $\tilde{\wp}(x)$ is [1,9,21]

$$\frac{\sqrt[3]{2}}{\sqrt[4]{3}}K\left(\frac{\sqrt{2-\sqrt{3}}}{2}\right) = \frac{1}{4\pi}\Gamma\left(\frac{1}{3}\right)^3 = 1.5299540370\ldots.$$

Further, for all $0 < x \le r$ and $0 < y \le \tilde{r}$, we have

$$x = \int_{\wp(x)}^{\infty} \frac{1}{\sqrt{(4t^2-1)t}}dt, \quad y = \int_{\tilde{\wp}(y)}^{\infty} \frac{1}{\sqrt{4t^3-1}}dt,$$

which suggest why $\wp(z)$ and $\tilde{\wp}(z)$ are important in elliptic curve theory [22]. The **Weierstrass pe function** is, in fact, a two-parameter family of functions and encompasses the two examples provided here.

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6.2 Euler–Gompertz Constant

The regular continued fraction

$$c_1 = 0 + \frac{1|}{|1} + \frac{1|}{|2} + \frac{1|}{|3} + \frac{1|}{|4} + \frac{1|}{|5} + \cdots$$

is convergent (hence it differs from the harmonic series in this regard). Its limiting value is [1–3]

$$\frac{I_1(2)}{I_0(2)} = c_1 = 0.6977746579\ldots,$$

where $I_0(x)$, $I_1(x)$ denote modified Bessel functions [3.6]. Using this formula, Siegel [4, 5] proved that c_1 is transcendental.

What happens if we reverse the patterns of the numerators and denominators prescribed in c_1? We obtain [6, 7]

$$\begin{aligned} C_1 &= 0 + \frac{1|}{|1} + \frac{1|}{|1} + \frac{2|}{|1} + \frac{3|}{|1} + \frac{4|}{|1} + \frac{5|}{|1} + \cdots = \sqrt{\frac{\pi e}{2}}\operatorname{erfc}\left(\frac{1}{\sqrt{2}}\right) \\ &= \int_1^\infty \exp\left[\tfrac{1}{2}(1-x^2)\right]dx = \sqrt{\frac{\pi e}{2}} - \tilde{C}_1 = 0.6556795424\ldots, \end{aligned}$$

where erfc is the complementary error function [4.6] and

$$\tilde{C}_1 = \sum_{n=1}^\infty \frac{1}{1\cdot 3\cdot 5\cdots(2n-1)} = \sqrt{\frac{\pi e}{2}}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) = 1.4106861346\ldots.$$

What happens if we additionally repeat each numerator? In this case, we obtain [6, 8]

$$\begin{aligned} C_2 &= 0 + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + \frac{2|}{|1} + \frac{2|}{|1} + \frac{3|}{|1} + \frac{3|}{|1} + \frac{4|}{|1} + \frac{4|}{|1} + \frac{5|}{|1} + \frac{5|}{|1} + \cdots \\ &= -e\,\mathrm{Ei}(-1) = \int_1^\infty \frac{\exp(1-x)}{x}dx = 0.5963473623\ldots, \end{aligned}$$

where Ei is the exponential integral [6.2.1]. More about the **Euler–Gompertz constant** C_2 appears shortly.

No one knows the exact outcome if we instead repeat each denominator in c_1, although numerically we find $c_2 = 0.5851972651\ldots$.

Euler [9–11] discovered that

$$0 + \frac{1|}{|1} + \frac{1^2|}{|1} + \frac{2^2|}{|1} + \frac{3^2|}{|1} + \frac{4^2|}{|1} + \frac{5^2|}{|1} + \cdots = \ln(2) = 0.6931471805\ldots$$

and Ramanujan [12, 13] discovered that

$$\begin{aligned} &0 + \frac{1|}{|1} + \frac{1^2|}{|1} + \frac{1^2|}{|1} + \frac{2^2|}{|1} + \frac{2^2|}{|1} + \frac{3^2|}{|1} + \frac{3^2|}{|1} + \frac{4^2|}{|1} + \frac{4^2|}{|1} + \frac{5^2|}{|1} + \frac{5^2|}{|1} + \cdots \\ &= 4\int_1^\infty \frac{x\exp(-\sqrt{5}x)}{\cosh(x)}dx = 0.5683000031\ldots. \end{aligned}$$

Again, however, no one knows the exact outcome if we reverse the patterns of the numerators and denominators, or if the exponents are chosen to be ≥ 3.

6.2.1 Exponential Integral

Let γ be the Euler–Mascheroni constant [1.5]. The **exponential integral** $\mathrm{Ei}(x)$ is defined by

$$\mathrm{Ei}(x) = \gamma + \ln|x| + \sum_{k=1}^\infty \frac{x^k}{k\cdot k!} = \begin{cases} \lim\limits_{\varepsilon\to 0^+}\left(\int\limits_{-\infty}^{-\varepsilon}\frac{e^t}{t}dt + \int\limits_{\varepsilon}^{x}\frac{e^t}{t}dt\right) & \text{if } x > 0, \\ \int\limits_{-\infty}^{x}\frac{e^t}{t}dt & \text{if } x < 0; \end{cases}$$

that is, $\mathrm{Ei}(x)$ is the Cauchy principal value of the improper integral. Sample applications of $\mathrm{Ei}(x)$ include evaluating the Raabe integrals [14–16]

$$A = \int_0^\infty \frac{\sin(x)}{1+x^2}dx = \tfrac{1}{2}\left(e^{-1}\,\mathrm{Ei}(1) - e\,\mathrm{Ei}(-1)\right),$$

$$B = \int_0^\infty \frac{x\cos(x)}{1+x^2}dx = -\tfrac{1}{2}\left(e^{-1}\,\mathrm{Ei}(1) + e\,\mathrm{Ei}(-1)\right),$$

which provide closure to an issue raised in [1.4.3].

6.2.2 Logarithmic Integral

Define the **logarithmic integral** for $0 < x \neq 1$ by the formula $\mathrm{Li}(x) = \mathrm{Ei}(\ln(x))$. There exists a unique number $\mu > 1$ satisfying $\mathrm{Li}(\mu) = 0$, and Ramanujan and Soldner [17–22] numerically calculated $\mu = 1.4513692348\ldots$ For example [23],

$$\mathrm{Li}(2) = \lim_{\varepsilon \to 0^+} \left(\int_0^{1-\varepsilon} \frac{1}{\ln(t)} dt + \int_{1+\varepsilon}^{2} \frac{1}{\ln(t)} dt \right) = \int_\mu^2 \frac{1}{\ln(t)} dt = 1.0451637801\ldots.$$

The famous Prime Number Theorem [2.1] is usually stated in terms of $\mathrm{Li}(x)$ or $\mathrm{li}(x) = \mathrm{Li}(x) - \mathrm{Li}(2)$. Since these are both $O(x/\ln(x))$ as $x \to \infty$, the difference $\mathrm{Li}(2)$ is regarded by analytic number theorists as (asymptotically) insignificant.

6.2.3 Divergent Series

What meaning can be given to the divergent alternating factorial series $0! - 1! + 2! - 3! + - \cdots$? Euler formally deduced that [24–28]

$$\sum_{n=0}^{\infty} (-1)^n n! = \sum_{n=0}^{\infty} \left((-1)^n \int_0^\infty x^n e^{-x} dx \right) = \int_0^\infty \frac{e^{-x}}{1+x} dx = C_2.$$

The even and odd parts of the series can be evaluated separately [29–31]:

$$\sum_{n=0}^{\infty} (2n)! = A = 0.6467611227\ldots, \quad \sum_{n=0}^{\infty} (2n+1)! = -B = 0.0504137604\ldots,$$

where A and B are the definite integrals defined earlier. Also, in the same extended sense [32, 33],

$$\sum_{n=1}^{\infty} (-1)^{n+1} (2n+1)!! = 1 \cdot 3 - 1 \cdot 3 \cdot 5 + 1 \cdot 3 \cdot 5 \cdot 7 - 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 + - \cdots = C_1.$$

6.2.4 Survival Analysis

Le Lionnais [34] called C_2 Gompertz's constant; it is interesting to attempt an explanation. Let the lifetime X of an individual be a random variable with cumulative distribution function $F(x) = \mathrm{P}(X \leq x)$ and probability density function $f(x) = F'(x)$. Then the probability that an individual, having survived to time x, will survive at most an additional time t, is

$$\mathrm{P}(X - x \leq t \mid X > x) = \frac{\mathrm{P}(x < X \leq x + t)}{\mathrm{P}(X > x)} = \frac{F(x+t) - F(x)}{1 - F(x)}.$$

This is related to what is known in actuarial science as the **force of mortality** or the **hazard function** [35, 36]. The conditional expectation of $X - x$, given $X > x$, is hence

$$\mathrm{E}(X - x \mid X > x) = \int_0^\infty \frac{t \cdot f(x+t)}{1 - F(x)} dt.$$

Consider the well-known Gompertz distribution [37]

$$F(x) = 1 - \exp\left[\frac{b}{a}(1 - e^{ax})\right], \quad x > 0, \ a > 0, \ b > 0,$$

and let $x = m$ be the mode of f, that is, the unique point at which $f'(m) = 0$. Then it is easily shown that [38], for all a and b,

$$\mathrm{E}(X - m \mid X > m) = \frac{C_2}{a},$$

which is a curious occurrence of Euler's original constant.

Similarly, if $\Phi(x) = \operatorname{erf}(x/\sqrt{2})$ and $\varphi(x) = \Phi'(x)$, that is, if X follows the half-normal (folded) distribution, then at the point of inflection $x = 1$,

$$\mathrm{E}(X - 1 \mid X > 1) = \sqrt{\frac{\pi}{2}}\left(\frac{1}{C_1} - 1\right), \quad \frac{1 - \Phi(1)}{\varphi(1)} = C_1.$$

In closing, here are two additional continued fraction expansions [6, 10, 39–41]:

$$\tilde{C}_1 = 0 + \frac{1|}{|1} - \frac{1|}{|3} + \frac{2|}{|5} - \frac{3|}{|7} + \frac{4|}{|9} - + \cdots,$$

$$C_2 = 0 + \frac{1|}{|2} - \frac{1^2|}{|4} - \frac{2^2|}{|6} - \frac{3^2|}{|8} - \frac{4^2|}{|10} - \cdots.$$

Note that $(1 - C_2)/e = 0.1484955067\ldots$ is connected with two-sided generalized Fibonacci sequences [42]. The Euler–Gompertz constant also appears in [5.6.2] with regard to increasing mobile trees.

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6.3 Kepler–Bouwkamp Constant

Draw a circle C_1 of unit radius and inscribe it with an equilateral triangle. Inscribe the triangle with another circle C_2 and inscribe C_2 with a square. Continue with a third circle C_3 inscribing the square and inscribe C_3 with a regular pentagon. Repeat this procedure indefinitely, each time increasing the number of sides of the regular polygon by one. The radius of the limiting circle C_∞ is given by [1–3]

$$\rho = \prod_{j=3}^{\infty} \cos\left(\frac{\pi}{j}\right) = 0.1149420448\ldots = (8.7000366252\ldots.)^{-1}.$$

This construction originated with Kepler [4, 5], who at one point believed that the orbits of Jupiter and Saturn around the sun might be approximated by the circumscribed and inscribed circles of an equilateral triangle, that is, by suitably scaled C_1 and C_2. Since the equilateral triangle is the first regular polygon, he thought that the orbit of Mars would thus correspond to C_3, the orbit of Earth would correspond to C_4, etc. (This model, however, could not explain the fact that there were only six known planets. Kepler subsequently replaced two-dimensional regular polygons by three-dimensional regular polyhedra, of which there are precisely five, and also obtained better agreement with astronomical data.)

Consider the same construction with the word “inscribe” replaced everywhere by “circumscribe.” The limiting radius is not a new constant, but simply ρ^{-1} [6]. Consider as well the infinite product

$$\sigma = \prod_{j=2}^{\infty} \frac{j}{\pi} \sin\left(\frac{\pi}{j}\right) = 0.3287096916\ldots = \frac{2}{\pi}(0.5163359762\ldots),$$

which has no apparent link with ρ. By way of contrast, the product

$$\prod_{j=3}^{\infty} \left(1 - \sin\left(\frac{\pi}{j}\right)\right)$$

diverges to zero.

Bouwkamp apparently was the first mathematician to exploit the more rapidly convergent formulas [7, 8]

$$\rho = \frac{2}{\pi} \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} \left(1 - \frac{1}{m^2 \left(n + \frac{1}{2}\right)^2}\right) = \frac{2}{\pi} \exp\left[-\sum_{k=1}^{\infty} \frac{\zeta(2k) 2^{2k} (\lambda(2k) - 1)}{k}\right],$$

$$\sigma = \prod_{m=1}^{\infty} \prod_{n=2}^{\infty} \left(1 - \frac{1}{m^2 n^2}\right) = \exp\left[-\sum_{k=1}^{\infty} \frac{\zeta(2k)(\zeta(2k) - 1)}{k}\right]$$

for computation’s sake. Here $\zeta(x)$ is defined in [1.6] and $\lambda(x)$ is defined in [1.7].

A recent result involves the function

$$f(x) = \prod_{j=1}^{\infty} \cos\left(\frac{x}{j}\right), \quad \lim_{x\to\pi} \frac{f(x)}{x-\pi} = \frac{\rho}{2},$$

for which it is known that [9]

$$\int_0^\infty f(x)\, dx = 0.7853805572\ldots < \frac{\pi}{4} = 0.7853981633\ldots.$$

The function

$$g(x) = \prod_{j=1}^{\infty} \frac{j}{x} \sin\left(\frac{x}{j}\right), \quad \lim_{x\to\pi} \frac{g(x)}{x-\pi} = -\frac{\sigma}{\pi},$$

can be similarly analyzed. See also [10–12] for an intriguing connection between $f(x)$, $g(x)$ and the divisor problem from number theory.

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6.4 Grossman's Constant

Grossman [1] defined a sequence of real numbers via the nonlinear recurrence

$$a_0 = 1, \qquad a_1 = y, \qquad a_{n+2} = \frac{a_n}{1 + a_{n+1}} \quad \text{for } n \geq 0.$$

On the basis of compelling numerical evidence, he conjectured that there is precisely one real value of $y = \eta$ for which this sequence converges, namely, $\eta = 0.7373383033\ldots$.

Janssen & Tjaden [2] succeeded in proving Grossman's conjecture. Nyerges [3] further demonstrated that existence and uniqueness of $y = F(x)$ holds, given an *arbitrary* starting point $a_0 = x \geq 0$. This gives rise to the functional equation

$$x = (1 + F(x))\, F(F(x)), \quad F : [0, \infty) \to [0, \infty) \text{ continuous},$$

and Grossman's constant is the special value $\eta = F(1)$. Other than this, there is no easily available description of η in terms of well-known constants or functions.

Ewing & Foias [4] examined the recurrence

$$b_1 = x > 0, \quad b_{n+1} = \left(1 + \frac{1}{b_n}\right)^n \text{ for } n \geq 1$$

and determined that there is exactly one value $x = \xi$ for which $b_n \to \infty$. In this case, $\xi = 1.1874523511\ldots$ thanks to a computation by Ross [4]. Again, there is a shortage of representations of ξ, as with η.

In [3.5] and [6.10], we observe other constants reminiscent of Grossman's constant.

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[4] J. Ewing and C. Foias, An interesting serendiptous real number, *Finite Versus Infinite: Contributions to an Eternal Dilemma*, ed. C. S. Calude and G. Paun, Springer-Verlag, 2000, pp. 119–126; MR 2001k:11267.

6.5 Plouffe's Constant

We start with a formula that is surprising at first glance:

$$\sum_{n=0}^{\infty} \frac{\rho(a_n)}{2^{n+1}} = \frac{1}{2\pi},$$

where

$$a_n = \sin(2^n) = \begin{cases} \sin(1) & \text{if } n = 0, \\ 2a_0\sqrt{1 - a_0^2} & \text{if } n = 1, \\ 2a_{n-1}\left(1 - 2a_{n-2}^2\right) & \text{if } n \geq 2, \end{cases}$$

and $\rho(x) = 1$ if $x < 0$ and $\rho(x) = 0$ if $x \geq 0$. In words, the binary expansion of $1/(2\pi)$ is completely determined by the sign pattern of the second-order recurrence $\{a_n\}$. The trivial proof uses the double-angle formulas for sine and cosine. One might believe that we have uncovered here a fast way of computing the binary expansion of $1/(2\pi)$, but this would be a mistake. The reason is that we would need sin(1) to high accuracy for initialization, but computing sin(1) is no easier than computing $1/(2\pi)$.

The double-angle formula for cosine gives rise to a simpler, first-order recurrence

$$b_n = \cos(2^n) = \begin{cases} \cos(1) & \text{if } n = 0, \\ 2b_{n-1}^2 - 1 & \text{if } n \geq 1, \end{cases}$$

but the sum

$$K = \sum_{n=0}^{\infty} \frac{\rho(b_n)}{2^{n+1}} = 0.4756260767\ldots$$

does not appear to have a closed-form expression. (We will revisit this question later.) The double-angle formula for tangent, however, gives rise to both a first-order recursion

$$c_n = \tan(2^n) = \begin{cases} \tan(1) & \text{if } n = 0, \\ \dfrac{2c_{n-1}}{1 - c_{n-1}^2} & \text{if } n \geq 1 \end{cases}$$

and a closed-form expression for the sum

$$\sum_{n=0}^{\infty} \frac{\rho(c_n)}{2^{n+1}} = \frac{1}{\pi}$$

by a trivial proof like before. Again, computing $\tan(1)$ is no easier than computing $1/\pi$.

We have observed so far that, for sine and tangent, certain irrational inputs yield recognizable irrational outputs. Plouffe [1–3] wondered if this process could be adjusted somewhat. He asked whether it was possible to initialize any of these three recurrences with *rational* values, such as $1/2$, and still obtain recognizable irrational binary expansions. Define

$$\alpha_n = \sin\left(2^n \arcsin(\tfrac{1}{2})\right) = \begin{cases} 1/2 & \text{if } n = 0, \\ \sqrt{3}/2 & \text{if } n = 1, \\ 2\alpha_{n-1}\left(1 - 2\alpha_{n-2}^2\right) & \text{if } n \geq 2, \end{cases}$$

$$\beta_n = \cos\left(2^n \arccos(\tfrac{1}{2})\right) = \begin{cases} 1/2 & \text{if } n = 0, \\ 2\beta_{n-1}^2 - 1 & \text{if } n \geq 1, \end{cases}$$

$$\gamma_n = \tan\left(2^n \arctan(\tfrac{1}{2})\right) = \begin{cases} 1/2 & \text{if } n = 0, \\ \dfrac{2\gamma_{n-1}}{1 - \gamma_{n-1}^2} & \text{if } n \geq 1; \end{cases}$$

then the first two sums

$$\sum_{n=0}^{\infty} \frac{\rho(\alpha_n)}{2^{n+1}} = \frac{1}{12}, \quad \sum_{n=0}^{\infty} \frac{\rho(\beta_n)}{2^{n+1}} = \frac{1}{2}$$

are rational, but the third sum

$$C = \sum_{n=0}^{\infty} \frac{\rho(\gamma_n)}{2^{n+1}} = 0.1475836176\ldots$$

is more mysterious. Plouffe numerically determined that

$$C = \frac{1}{\pi} \arctan(\tfrac{1}{2}),$$

but rigorous justification remained an open problem.

Borwein & Girgensohn [4] succeeded in proving Plouffe's formula for C and much more. They demonstrated that, given an arbitrary real value x, if

$$\xi_n = \tan\left(2^n \arctan(x)\right) = \begin{cases} x & \text{if } n = 0, \\ \dfrac{2\xi_{n-1}}{1-\xi_{n-1}^2} & \text{if } n \geq 1 \text{ and } |\xi_{n-1}| \neq 1, \\ -\infty & \text{if } n \geq 1 \text{ and } |\xi_{n-1}| = 1, \end{cases}$$

then

$$\sum_{n=0}^{\infty} \frac{\rho(\xi_n)}{2^{n+1}} = \begin{cases} \dfrac{\arctan(x)}{\pi} & \text{if } x \geq 0, \\ 1 + \dfrac{\arctan(x)}{\pi} & \text{if } x < 0, \end{cases}$$

which we call **Plouffe's recursion**.

This, however, was only one facet of their paper. It turns out to be crucial that the aforementioned sum, call it $f(x)$, satisfies the functional equation

$$2f(x) = f\left(\frac{2x}{1-x^2}\right) \ \text{if } x \geq 0, \ \ 2f(x) - 1 = f\left(\frac{2x}{1-x^2}\right) \ \text{if } x < 0.$$

A vastly more general functional equation gives rise to other interesting recurrences and binary expansions. We will not attempt to summarize these results except to remark that Plouffe's recursion appears to be the simplest example in the theory. Other examples, associated with logarithmic, hyperbolic, and elliptic integrals of the first kind, are presented in [4] as well.

A well-known theorem of Lehmer [5] gives that C is irrational. In fact, C is transcendental [6].

Chowdhury [7] recently observed that the constant K defined earlier can be expressed in binary as the bitwise XOR sum of $1/(2\pi)$ and $1/\pi$. That is,

$$\begin{aligned} & 0.00101000101111100110\ldots \\ \oplus\ & 0.01010001011111001100\ldots \\ =\ & 0.01111001110000101010\ldots \end{aligned}$$

and "addition exclusive or" is identical to addition modulo two without carries. Since $1/(2\pi)$ is simply a shifted version of $1/\pi$, the constant K is truly quite interesting! More generally, if $-1 \leq x \leq 1$, the bitwise XOR sum of $\arccos(x)/(2\pi)$ and $\arccos(x)/\pi$ is $\sum_{n=0}^{\infty} \rho(\eta_n) 2^{-n-1}$, where

$$\eta_n = \cos\left(2^n \arccos(x)\right) = \begin{cases} x & \text{if } n = 0, \\ 2\eta_{n-1}^2 - 1 & \text{if } n \geq 1. \end{cases}$$

This is a well-studied object: The sequence $\{1 - 2\eta_n\}$ is equal to iterates of the chaotic logistic map $y \mapsto 4y(1-y)$ defined in [1.9] with seed value $1 - 2x$. Unfortunately, this insight does not help us in more clearly identifying the constant K.

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6.6 Lehmer's Constant

Every irrational number x has a unique infinite continued fraction representation of the form

$$x = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots,$$

where each a_k is a positive integer for $k \ge 1$ and a_0 is an integer [1]. Conversely, every such expression is convergent. The Golden mean

$$\frac{1+\sqrt{5}}{2} = 1 + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + \cdots$$

can be said to be the case for which the convergence rate is slowest.

Lehmer [2, 3] discovered an interesting analog of continued fractions. Every positive irrational x has a unique infinite **continued cotangent representation** of the form

$$x = \cot\left(\sum_{k=0}^{\infty} (-1)^k \operatorname{arccot}(b_k)\right),$$

where each b_k is a nonnegative integer for $k \ge 0$ and $b_k \ge b_{k-1}^2 + b_{k-1} + 1$ for $k \ge 1$. Conversely, every such expression is convergent. Lehmer's constant, ξ, corresponds to the Golden mean under the analogy and

$$\begin{aligned} \xi &= \cot\left(\operatorname{arccot}(0) - \operatorname{arccot}(1) + \operatorname{arccot}(3) - \operatorname{arccot}(13) + - \cdots + (-1)^k c_k + \cdots\right) \\ &= 0.5926327182\ldots \end{aligned}$$

can be said to be the case for which the convergence rate is slowest. Here the k^{th} arccotangent argument is defined via the quadratic recurrence [4]

$$c_0 = 0, \quad c_k = c_{k-1}^2 + c_{k-1} + 1 \text{ for } k \ge 1,$$

which is itself an interesting object of study. Lehmer proved that ξ is not an algebraic number of degree < 4. When coupled with Roth's theorem [2.22], which Lehmer did not have available back in 1938, the argument implies the transcendence of ξ [5].

What inspired Lehmer to even begin examining continued cotangents? He observed that the iteration of simple two-variable functions such as

$$f(x, y) = x + y, \qquad g(x, y) = x + \frac{1}{y},$$
$$h(x, y) = \frac{xy + 1}{y - x} = \cot(\operatorname{arccot}(x) - \operatorname{arccot}(y))$$

give rise to

$$f(x_1, f(x_2, f(x_3, \ldots))) = \sum_{j=1}^{\infty} x_j,$$

$$g(x_1, g(x_2, g(x_3, \ldots))) = x_1 + \frac{1|}{|x_2} + \frac{1|}{|x_3} + \cdots,$$

$$h(x_1, h(x_2, h(x_3, \ldots))) = \cot\left(\sum_{j=1}^{\infty}(-1)^{j+1} \operatorname{arccot}(x_j)\right).$$

The first two results, infinite sums and infinite continued fractions, occur throughout mathematics. Lehmer's result and conceivably others might find applications in the future.

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[5] P. Borwein, Lehmer's constant is transcendental, unpublished note (1999).

6.7 Cahen's Constant

Here is a little known example of a **self-generating continued fraction**. Start with

$$\frac{0}{1} = 0, \quad \frac{1}{1} = 0 + \frac{1}{1}$$

and define $q_0 = 1$ and $q_1 = 1$, the denominators on the left-hand side. Continue with

$$\frac{p_2}{q_2} = 0 + \cfrac{1}{1 + \cfrac{1}{q_0}} = 0 + \frac{1|}{|1} + \frac{1|}{|q_0},$$

where $\gcd(p_2, q_2) = 1$, obtaining $q_2 = 2$. (Henceforth, whenever we write a fraction p/q, it is assumed, for simplicity, to be in lowest terms.) Continue with

$$\frac{p_3}{q_3} = 0 + \frac{1|}{|1} + \frac{1|}{|q_0} + \frac{1|}{|q_1},$$

obtaining $q_3 = 3$. Continue with

$$\frac{p_4}{q_4} = 0 + \frac{1|}{|1} + \frac{1|}{|q_0} + \frac{1|}{|q_1} + \frac{1|}{|q_2},$$

obtaining $q_4 = 8$. At each step in the process, the n^{th} partial denominator q_n is defined in terms of the finite continued fraction with partial quotients up to q_{n-2}. Maintaining this indefinitely, one finds that the sequence of qs

$$1, 1, 2, 3, 8, 27, 224, 6075, 1361024, 8268226875, 11253255215681024, \ldots$$

satisfies the quadratic recurrence $q_{n+2} = q_n(q_{n+1} + 1)$ and that the limiting value of the continued fraction coincides with the sum of a certain alternating infinite series:

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \sum_{j=0}^{\infty} \frac{(-1)^j}{q_j q_{j+1}} = 0.6294650204\ldots.$$

This constant was apparently first discussed by Davison & Shallit [1], who proved it is transcendental.

Let us now start over, but proceeding more generally. Let $w_0, w_1, w_2, \ldots$ be an infinite sequence of positive integers. From

$$\frac{0}{1} = 0, \quad \frac{1}{w_0} = 0 + \frac{1}{w_0}$$

define $q_0 = 1$ and $q_1 = w_0$. From

$$\frac{p_2}{q_2} = 0 + \frac{1|}{|w_0} + \frac{1|}{|w_1 q_0}$$

obtain $q_2 = q_0(w_1 q_1 + 1)$. From

$$\frac{p_3}{q_3} = 0 + \frac{1|}{|w_0} + \frac{1|}{|w_1 q_0} + \frac{1|}{|w_2 q_1}$$

obtain $q_3 = q_1(w_2 q_2 + 1)$. Maintaining this indefinitely, one finds that the sequence of qs satisfies the recurrence $q_{n+2} = q_n(w_{n+1} q_{n+1} + 1)$ and that the limiting value of the continued fraction coincides with the series

$$\xi(w) = \lim_{n\to\infty} \frac{p_n}{q_n} = \sum_{j=0}^{\infty} \frac{(-1)^j}{q_j q_{j+1}}.$$

It can be proved [1] that the number $\xi(w)$ is always transcendental, regardless of the choice of ws.

Let k be a positive integer. As a special case of the preceding, define $w_0 = 1$ and $w_{j+1} = q_j^{k-1}$ for all $j \ge 0$. Then the sequence of qs satisfies the recurrence $q_{n+2} = q_n(q_n^{k-1} q_{n+1} + 1)$ and the corresponding limiting value $\xi(w)$ is

$$0 + \frac{1|}{|1} + \frac{1|}{|q_0^k} + \frac{1|}{|q_1^k} + \frac{1|}{|q_2^k} + \cdots = \xi_k = \sum_{j=0}^{\infty} \frac{(-1)^j}{q_j q_{j+1}}.$$

The Davison–Shallit constant arises from the instance for which $k = 1$. The case $k = 2$ is often rewritten as $s_n = q_n q_{n+1} + 1$; hence

$$\xi_2 = c = \sum_{j=0}^{\infty} \frac{(-1)^j}{s_j - 1} = 0.6434105462\ldots,$$

where $s_0 = 2$, $s_{n+1} = s_n^2 - s_n + 1$ is **Sylvester's sequence**. This sequence is also discussed in [6.10]. Cahen [2] was the first to examine the constant c. Subsequent references include [3–6]. In the 1930s, Mahler partitioned the set of all transcendental numbers into three classes: S, T, and U, the classification being determined by how small a polynomial with bounded degree and height can be when evaluated at the point in question. Töpfner [7] succeeded in proving that c must fall in the class S. The case $k \geq 3$ has not been examined, as far as is known: $\xi_3 = 0.6539007091\ldots$, $\xi_4 = 0.6600049346\ldots$, and $\xi_5 = 0.6632657345\ldots$.

Some variations on Cahen's constant c are worth pointing out. The number $c' = \sum_{j=0}^{\infty}(-1)^j/s_j$ satisfies $2c = c' + 1$, and thus c' is also transcendental, whereas $\sum_{j=0}^{\infty} 1/s_j = 1$. What can be said about $\sum_{j=0}^{\infty} 1/(s_j - 1) = 1.6910302067\ldots$? Finally, what other kinds of self-generating continued fractions have appeared in the literature?

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6.8 Prouhet–Thue–Morse Constant

The Prouhet–Thue–Morse binary sequence $\{t_n\} = \{0, 1, 1, 0, 1, 0, 0, 1, 1, 0, \ldots\}$ has several equivalent definitions: [1]

- $t_0 = 0$, $t_{2n} = t_n$, and $t_{2n+1} = 1 - t_n$ for all $n \geq 0$;
- t_n is the number of ones, modulo two, in the binary expansion of n [2.16];
- $(-1)^{t_n}$ is the coefficient of x^n in the power series expansion of $\prod_{k=0}^{\infty}(1 - x^{2^k})$;
- $\{0, 0, 1, 0, 0, 1, 1 - t_0, 1 - t_1, 1 - t_2, 1 - t_3, \ldots\}$ is the lexicographically smallest overlap-free infinite binary word [5.17].

We begin with the constant

$$\tau = \sum_{n=0}^{\infty} \frac{t_n}{2^{n+1}} = 0.4124540336\ldots = \frac{1}{2}(0.8249080672\ldots),$$

sometimes called the **parity constant**, which is known to be transcendental [2–6]. Less "artificial" formulas include the infinite product [7, 8]

$$\prod_{k=0}^{\infty}\left(1 - \frac{1}{2^{2^k}}\right) = 2(1 - 2\tau)$$

and the continued fraction

$$2 - \frac{1|}{|4} - \frac{3|}{|16} - \frac{15|}{|256} - \frac{255|}{|65536} - \frac{65535|}{|4294967296} - \cdots = \frac{\tau}{3\tau - 1},$$

where the pattern is generated by 2^{2^n} and $2^{2^n} - 1$.

6.8.1 Probabilistic Counting

Woods & Robbins [9] proved that

$$\prod_{m=0}^{\infty}\left(\frac{2m+1}{2m+2}\right)^{(-1)^{t_m}} = \frac{1}{\sqrt{2}}.$$

Shallit [10] generalized this result and wrote a base-3 version. Other generalizations include [11–14]

$$\prod_{m=0}^{\infty}\left(\frac{(2m+1)^2}{(m+1)(4m+1)}\right)^{(-1)^{u_m}} = \frac{1}{\sqrt{2}},$$

where u_m is the Golay–Rudin–Shapiro sequence, which counts the number of (possibly overlapping) elevens in the binary expansion of m, modulo two.

Here is a problem involving n coins. For each $1 \le k \le n$, let X_k be the number of independent tosses of the k^{th} coin required for heads to appear, minus one. Define R_n to be the smallest nonnegative integer $\ne X_k$ for all k; then clearly $0 \le R_n \le n$. Flajolet & Martin [15] proved that

$$\mathrm{E}(R_n) = \frac{1}{\ln(2)} \ln(\psi n) + \delta(n) + o(1),$$

where

$$\psi = \frac{e^{\gamma}}{\sqrt{2}} \prod_{m=1}^{\infty}\left(\frac{2m+1}{2m}\right)^{(-1)^{t_m}} = 0.7735162909\ldots,$$

γ is the Euler–Mascheroni constant [1.5], and $\delta(n)$ is a "negligible" periodic function of small amplitude ($|\delta(n)| < 10^{-5}$) of the type mentioned in [5.14]. A more complicated expression for $\mathrm{Var}(R_n) \sim 1.257\ldots + \varepsilon(n)$ appears in [15–17]. The proof involves the analytic continuation of a function

$$F(z)=\sum_{k=1}^{\infty}\frac{(-1)^{t_k}}{k^z},\quad \mathrm{Re}(z)>1,$$

to the entire complex plane. This is useful in assessing **probabilistic counting algorithms** for data mining, and it is interesting how the sequence $\{t_n\}$ persists throughout. Plouffe [18] gave the following products:

$$\prod_{m=1}^{\infty}\left(\frac{m}{m+1}\right)^{(-1)^{t_{m-1}}}=0.8116869215\ldots,$$
$$\prod_{m=1}^{\infty}\left(\frac{2m}{2m+1}\right)^{(-1)^{t_{m-1}}}=0.8711570464\ldots,$$
$$\prod_{m=1}^{\infty}\left(\frac{2m}{2m+1}\right)^{(-1)^{t_m}}=1.6281601297\ldots,$$
$$\prod_{m=1}^{\infty}\left(\frac{m}{m+1}\right)^{(-1)^{t_m}}=2.3025661371\ldots,$$

due to Flajolet; the third is $2^{-1/2}e^{\gamma}\psi^{-1}$ of course. A finite expression for these in terms of more familiar constants is not known. This situation makes the Woods–Robbins formula and others all the more remarkable!

6.8.2 Non-Integer Bases

Fix q to be a real number satisfying $1<q\le 2$. Define a **q-development** to be a series

$$\sum_{n=1}^{\infty}\varepsilon_n q^{-n}=1,$$

where $\varepsilon_n=0$ or 1 for every n. The greedy algorithm shows that q-developments exist. If $q=2$, then $\varepsilon_n=1$ for all n and this is the unique 2-development. Do there exist other values of q, $1<q<2$, for which there is a unique q-development?

Intuitively, one would expect the answer to be no. Indeed, if we fix $1<q<\varphi$, where φ is the Golden mean [1.2], then there exist uncountably many q-developments. Also, if $q=\varphi$, then there exist a countably infinite number of q-developments [19–21].

If we fix $\varphi<q<2$, however, intuition fails. There is an uncountable, measure-zero subset of exceptional q-values, each with only one q-development. Moreover, the exceptional subset possesses a minimum element that can be characterized exactly [22]. This special q-value is the unique positive solution of the equation

$$\prod_{k=0}^{\infty}\left(1-\frac{1}{q^{2^k}}\right)=\left(1-\frac{1}{q}\right)^{-1}-2;$$

hence $q=1.7872316501\ldots$. The corresponding q-development satisfies $\varepsilon_n=t_n$ for all $n \geq 1$, an unexpected occurrence of the Prouhet–Thue–Morse sequence. Also, the **Komornik–Loreti constant** q is transcendental, as shown by Allouche & Cosnard [23].

6.8.3 External Arguments

Here is a connection between τ and the Myrberg constant $c_\infty = 1.4011551890\ldots$ from fractal geometry [1.9]. Imagine the Mandelbrot set M [6.10] to be electrically charged; thus it determines in the plane **equipotential curves** (which encircle M) and **field trajectories** (which are orthogonal to the equipotential curves). Seen from far away, M resembles a point charge and the field trajectories approach rays of the form $r\,\exp(2\pi i\theta)$ as $r \to \infty$. The **external arguments** θ_k corresponding to the bifurcation points c_k of $1 - cx^2$, given by [1.9]

$$c_2 = \tfrac{5}{4} = 1.25, \quad c_3 = 1.3680\ldots, \quad c_4 = 1.3940\ldots,$$

are (in binary)

$$\theta_2 = 0.\overline{01} = \tfrac{1}{3}, \quad \theta_3 = 0.\overline{0110} = \tfrac{2}{5}, \quad \theta_4 = 0.\overline{01101001} = \tfrac{7}{17},$$

with limiting value $\theta_\infty = \tau$. Unfortunately the details are too elaborate to explain further [24–26].

6.8.4 Fibonacci Word

Another "self-generating" constant is the so-called **rabbit constant**, which can be defined via recursive bit substitutions $0 \mapsto 1$, $1 \mapsto 10$ leading to the infinite binary Fibonacci word [27–32]. (The analogous substitution map for the Thue–Morse word is $0 \mapsto 01$, $1 \mapsto 10$.) A simpler definition is

$$\rho = \sum_{k=1}^{\infty} \frac{1}{2^{\lfloor k\varphi \rfloor}} = 0.7098034428\ldots,$$

where φ is the Golden mean [1.2]. It is known that [33–37]

$$\rho = 0 + \frac{1|}{|2^0} + \frac{1|}{|2^1} + \frac{1|}{|2^1} + \frac{1|}{|2^2} + \frac{1|}{|2^3} + \frac{1|}{|2^5} + \frac{1|}{|2^8} + \cdots,$$

where the exponents form none other than the classical Fibonacci sequence, and hence ρ is transcendental.

6.8.5 Paper Folding

Consider the act of folding a strip of paper in half, right over left [38]. Iterating this process gives a sequence of creases in the strip, appearing when unfolded as either valleys (1) or peaks (0). The **paper folding sequence** $\{s_n\} = \{1, 1, 0, 1, 1, 0, 0, 1, 1, 1, \ldots\}$ is defined by $s_{4n-3} = 1$, $s_{4n-1} = 0$, and $s_{2n} = s_n$ for all $n \geq 1$, or alternatively, by the word transformation $w \mapsto w1\tilde{w}$, where $\tilde{w}$ is the mirror image of w with 0s replaced by 1s and 1s by 0s. It can be shown that

$$\sigma = \sum_{n=1}^{\infty} \frac{s_n}{2^n} = 0.8507361882\ldots = \sum_{k=0}^{\infty} \frac{1}{2^{2^k}} \left(1 - \frac{1}{2^{2^{k+2}}}\right)^{-1}$$

and transcendentality of σ follows [5, 39].

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6.9 Minkowski–Bower Constant

Define a function $? : [0, 1] \to [0, 1]$ by

$$?\left(0 + \frac{1|}{|a} + \frac{1|}{|b} + \frac{1|}{|c} + \frac{1|}{|d} + \cdots\right) = 0.\overbrace{00\ldots0}^{a-1}\underbrace{11\ldots1}_{b}\overbrace{00\ldots0}^{c}\underbrace{11\ldots1}_{d}\,00\ldots,$$

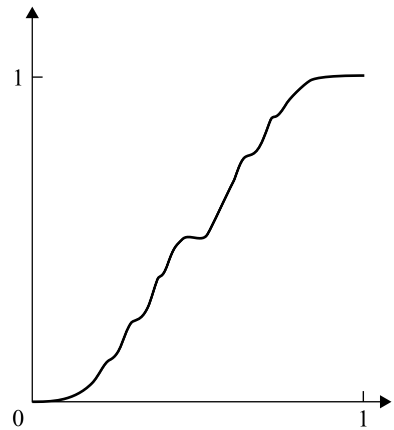

Figure 6.1. A graph of Minkowski's question mark function.

where the input is a regular continued fraction and the output is written in binary [1–3]. This is known as **Minkowski's question mark function** (see Figure 6.1). It is continuous, strictly increasing, but fractal-like. In fact, it is *singular* in the sense that its derivative is zero almost everywhere (except on a set of Lebesgue measure zero). Special values include

$$?\left(\tfrac{-1+\sqrt{5}}{2}\right)=\tfrac{2}{3},\quad ?\left(-1+\sqrt{2}\right)=\tfrac{2}{5},\quad ?\left(\tfrac{-1+\sqrt{3}}{2}\right)=\tfrac{2}{7}.$$

Bower [4, 5] asked about the fixed points of ? other than 0, $1/2$, and 1. There appear to be at least two more, arranged symmetrically around the center point. Are there exactly two? He computed the lesser value to be 0.4203723394... (in decimal). Does this constant have a closed-form expression? Is it algebraic? A definition of ? in terms of Farey fractions is also possible.

While on the subject of artificial constants, let us mention the **Champernowne number** [6]

$$C = 0.1234567891011121314151617181920212223242 5\ldots,$$

which is constructed by concatenating the digits of all positive integers, and the **Copeland–Erdös number** [7]

$$0.2357111317192329313741434753596167717379\ldots,$$

which is likewise constructed by concatenating the digits of all primes. Both are known to be irrational; see [8–10] for recent proofs. Mahler [11] was the first to prove that C is transcendental. His theorem is consistent with the observation that relatively "short" rational numbers (e.g., 10/81 or 60499999499/490050000000) yield excellent approximations of C. This observation, in turn, implies the existence of extraordinarily large partial denominators in the regular continued fraction expansion for C (e.g., the 1709th partial denominator is $\approx 10^{4911098}$, due to Sofroniou & Spaletta [12]).

We also mention **Trott's constant** E, defined to be the (apparently unique) number with decimal digits $\{\varepsilon_k\}$ that coincide with its partial fraction denominators [12]:

$$E = 0.\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\ldots = 0 + \frac{1|}{|\varepsilon_1} + \frac{1|}{|\varepsilon_2} + \frac{1|}{|\varepsilon_3} + \frac{1|}{|\varepsilon_4} + \cdots, \quad 0 \leq \varepsilon_k \leq 9 \text{ for all } k,$$

and this turns out to be $0.1084101512\ldots$. Is E transcendental? Are alternative expressions for E possible?

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6.10 Quadratic Recurrence Constants

Linear recurrences include the Fibonacci sequence, which is discussed in [1.2]. Quadratic recurrences are far less understood and far more mysterious than linear recurrences. The simplest example is

$$a_0 = 2, \qquad a_n = a_{n-1}^2 \quad \text{for } n \geq 1,$$

with solution $a_n = 2^{2^n}$. A more challenging example is the total number of strongly binary trees [5.6] of height at most n:

$$b_0 = 1, \qquad b_n = b_{n-1}^2 + 1 \quad \text{for } n \geq 1.$$

(See Figure 6.2.) Aho & Sloane [1,2] showed that this quadratic recurrence likewise has a doubly exponential solution $b_n = \lfloor \beta^{2^n} \rfloor$, but β is not precisely known and, in fact,

$$\beta = \exp\left[\sum_{j=0}^{\infty} 2^{-j-1} \ln\left(1 + b_j^{-2}\right)\right] = 1.5028368010\ldots.$$

If one could find an expression for β independent of $\{b_n\}$, this would be very surprising.

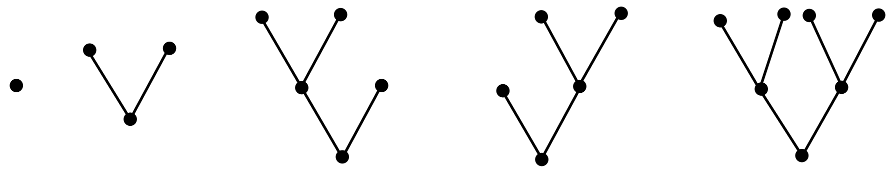

Figure 6.2. There are five strongly binary trees of height at most 2.

Another example is the closest strict under-approximation $C_n = \sum_{i=1}^{n} 1/c_i$ of the number 1, where $1 < c_1 < c_2 < \ldots < c_n$ are integers. This is given by the quadratic recurrence [6.7]

$$c_1 = 2, \qquad c_n = c_{n-1}^2 - c_{n-1} + 1 \quad \text{for } n \geq 2,$$

known as **Sylvester's sequence**. Further, $C_n = 1 - 1/(c_{n+1} - 1)$, which implies that C_n is formed by the greedy algorithm, equivalently, by choosing for the next term the largest feasible unit fraction [3–13]. Here, Aho & Sloane determined $c_n = \left\lfloor \chi^{2^n} + 1/2 \right\rfloor$, where

$$\chi = \frac{\sqrt{6}}{2} \exp\left[\sum_{j=1}^{\infty} 2^{-j-1} \ln\left(1 + (2c_j - 1)^{-2}\right)\right] = 1.2640847353\ldots.$$

Again, an independent expression for χ would be very surprising. We have encountered such doubly exponential functions elsewhere in [2.13], [5.7], and [5.16].

A well-known example is the Lucas recurrence [14–21]

$$u_n = u_{n-1}^2 - 2,$$

which has been studied extensively because of its connection with Mersenne prime theory when $|u_0| > 2$. In this case we have

$$u_n = \left(\frac{1}{2}u_0 + \frac{1}{2}\sqrt{u_0^2 - 4}\right)^{2^n} + \left(\frac{1}{2}u_0 - \frac{1}{2}\sqrt{u_0^2 - 4}\right)^{2^n},$$

so divergence always occurs in this regime. For $|u_0| < 2$ the long-term behavior is more intricate and interesting to dynamical system theorists. See [1.9] for a related discussion of the recurrence

$$0 \leq x_0 \leq 1, \;\; x_n = a\, x_{n-1}(1 - x_{n-1}) \;\; \text{for } n \geq 1, \;\; 0 \leq a \leq 4,$$

with its cycle structure and period-doubling bifurcations.

Another well-known example is the Lehmer recurrence

$$v_0 = 1, \qquad v_n = v_{n-1}^2 + v_{n-1} + 1 \quad \text{for } n \geq 1,$$

which generates the coefficients of the least rapidly convergent continued cotangent [6.6].

Quadratic recurrences arise in tree-related contexts in other ways [5.6]: in the extinction probabilities associated with Galton–Watson branching processes,

$$y_0 = 0, \;\; y_n = (1 - p) + p\, y_{n-1}^2 \;\; \text{for } n \geq 1, \;\; 0 < p < 1,$$

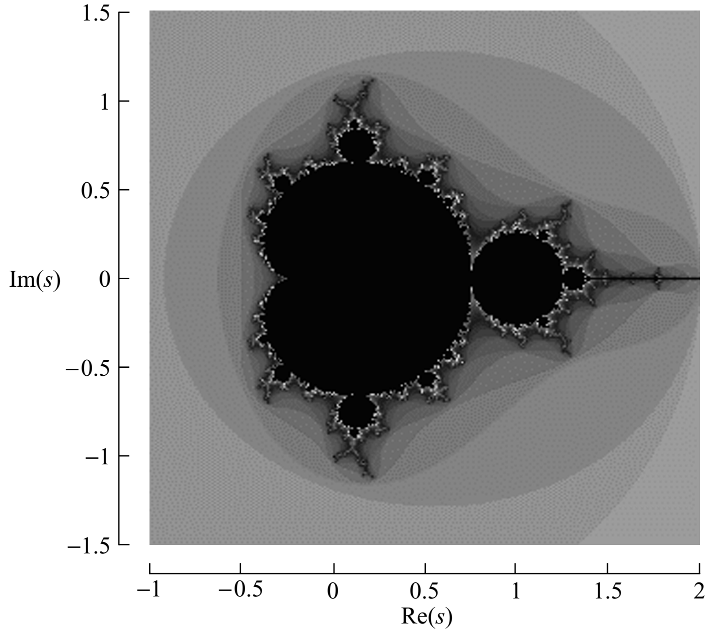

Figure 6.3. The Mandelbrot set is the black cardiod-shaped region and is entirely contained within the indicated rectangle. Its intersection with the real line is the interval $[-1/4, 2]$.

and in the asymptotics of non-isomorphic binary trees,

$$w_0 = 2, \qquad w_n = w_{n-1}^2 + 2 \quad \text{for } n \geq 1.$$

In the study of 1-additive sequences, the ternary quadratic recurrence

$$t_n = 2\left(t_{n-2}(t_{n-2}+1) + t_{n-4k-3}(t_{n-4k-3}+1) + t_{n-8k-4}(t_{n-8k-4}+1)\right) \bmod 3,$$

with initial data $(t_1, t_2, \ldots, t_{8k+3}, t_{8k+4}) = (0, 0, \ldots, 0, 1)$, turns out to be crucial [22] and is related to the Stolarsky–Harborth constant [2.16].

The most famous quadratic recurrence, however, is

$$s_0 = 0, \qquad s_n = s_{n-1}^2 - \mu \quad \text{for } n \geq 1,$$

where μ may be any complex number. The **Mandelbrot set** M is defined to be the set of all such μ for which $s_n \not\to \infty$ (see Figure 6.3). Since the boundary of M is a fractal of Hausdorff dimension 2 [8.20], it has infinite length [23]. However, the area of M has been rigorously bounded between 1.506302 and 1.561303 and has been heuristically estimated as 1.50659177. See [24–29] for details. No one has dared to conjecture an exact formula for the area of M.

Davison & Shallit [30] studied the second-order quadratic recurrence [6.7]

$$q_0 = q_1 = 1, \quad q_{n+2} = q_n(q_{n+1} + 1) \quad \text{for } n \geq 0$$

and determined that $q_n = \left\lfloor \xi^{\varphi^n} \eta^{(1-\varphi)^n} \right\rfloor$, where φ is the Golden mean [1.2],

$\xi = 1.3505061\ldots$, and $\eta = 1.4298155\ldots$. Another such recurrence [31]

$$r_0 = 0, \;\; r_1 = 1, \;\; r_{n+2} = r_{n+1} + r_n^2 \;\; \text{for } n \geq 0$$

satisfies

$$r_{2n} \sim (1.436\ldots)^{\sqrt{2}^{2n}}, \;\; r_{2n+1} \sim (1.451\ldots)^{\sqrt{2}^{2n+1}}.$$

The dependence of the growth on the subscript parity is intriguing.

Greenfield & Nussbaum [32] considered the possibility of a bi-infinite sequence $\{z_n : n = \ldots, -2, -1, 0, 1, 2, \ldots\}$ of positive reals satisfying the recurrence

$$z_0 = 1, \;\; z_n = z_{n-1} + z_{n-2}^2 \;\; \text{for all } n.$$

It turns out that there is exactly one value $z_1 = 1.5078747554\ldots$ for which this happens.

Stein & Everett [33] and Wright [34] studied the recurrence

$$d_1 = 1, \;\; d_{n+1} = (n+\delta)\sum_{k=1}^{n} d_k d_{n-k+1} \;\; \text{for } n \geq 1$$

for various values of δ. For $\delta = 0$ and $\delta = -1/3$, they obtained

$$d_n \sim \frac{1}{e}\prod_{j=2}^{n}(2j-1), \;\; d_n \sim (0.35129898\ldots)\prod_{j=2}^{n}(2j-2),$$

respectively, where e is the natural logarithmic base [1.3]. Both cases possess combinatorial interpretations.

Lenstra [12] and Zagier [35] examined **Göbel's sequence**

$$f_0 = 1, \;\; f_n = \frac{1}{n}\left(1 + \sum_{k=0}^{n-1} f_k^2\right) \;\; \text{for } n \geq 1$$

and determined that the first non-integer term is $f_{43} > 10^{178485291567}$; further,

$$f_n \sim (1.0478314475\ldots)^{2^n}\left(n + 2 - n^{-1} + 4n^{-2} - 21n^{-3} + 137n^{-4} - + \cdots\right).$$

Somos [36] examined a related sequence

$$g_0 = 1, \;\; g_n = n g_{n-1}^2 \;\; \text{for } n \geq 1$$

and found that

$$g_n \sim \gamma^{2^n}\left(n + 2 - n^{-1} + 4n^{-2} - 21n^{-3} + 137n^{-4} - + \cdots\right)^{-1},$$

where the constant γ has an infinite radical expansion

$$\gamma = 1.6616879496\ldots = \sqrt{1\cdot\sqrt{2\cdot\sqrt{3\cdot\sqrt{4\cdots}}}} = \prod_{j=1}^{\infty} j^{2^{-j}}.$$

Another Somos constant $\lambda = 0.3995246670\ldots$ arises as follows: If $\kappa < \lambda$, then the sequence

$$h_0 = 0, \;\; h_1 = \kappa, \;\; h_n = h_{n-1}(1 + h_{n-1} - h_{n-2}) \;\; \text{for } n \geq 2$$

converges to a limit less than 1; if $\kappa > \lambda$, then the sequence diverges to infinity. This is similar to Grossman's constant [6.4].

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6.11 Iterated Exponential Constants

Given $y > 0$, what numbers $x > 0$ satisfy $y = x^x$? The answer is more complicated than one might expect. For example,

- $x = 3$ is the unique solution of $x^x = 27$,
- $x = 2$ is the unique solution of $x^x = 4$,
- $x = 1/2$ and $x = 1/4$ are both solutions of $x^x = 2^{-1/2}$, and there are no others.

More generally [1–3],

- $x = \left(\cdots \log_{\frac{1}{y}} \log_{\frac{1}{y}} \frac{1}{e}\right)^{-1}$ is the unique solution of $x^x = y$ for $y \geq e^e = 15.154\ldots$,
- $x = y^{\frac{1}{y}^{\frac{1}{y}^{\cdot^{\cdot^{\cdot}}}}}$ is the unique solution of $x^x = y$ for $1 \leq y \leq e^e$,
- $x = y^{\frac{1}{y}^{\frac{1}{y}^{\cdot^{\cdot^{\cdot}}}}}$ and $x = \left(\cdots \log_{\frac{1}{y}} \log_{\frac{1}{y}} e\right)^{-1}$ are both solutions of $x^x = y$ for $0.692\ldots = e^{-1/e} \leq y < 1$, and there are no others.

This is a consequence, in part, of the fact that the **iterated exponential** $\xi^{\xi^{\xi^{\cdot^{\cdot^{\cdot}}}}}$ converges for $0.065\ldots = e^{-e} \leq \xi \leq e^{1/e} = 1.444\ldots$ and diverges for positive ξ outside this interval. Other phrases for the same type of function include **hyperpower sequence** and **tower of exponents**.

An alternative representation of x as a function of y is $\exp(W(\ln(y)))$, where the **Lambert W function** [3,4] is

$$W(\eta) = \begin{cases} -\ln\left(\cdots \log_{e^{-\eta}} \log_{e^{-\eta}} \frac{1}{e}\right) & \text{if } \eta \geq e = 2.718\ldots, \\ \eta\,(e^{-\eta})^{(e^{-\eta})^{\cdot^{\cdot^{\cdot}}}} & \text{if } -0.367\ldots = -e^{-1} \leq \eta \leq e \end{cases}$$

and satisfies $W(\eta)\exp(W(\eta)) = \eta$. In particular, $W(\ln(27)) = \ln(3)$, $W(\ln(4)) = \ln(2)$, and $W(-\ln(2)/2) = -\ln(2)$. We will refer to Lambert's function throughout the remainder of this essay.

Consider the equation $x^2 = 2^x$, which has three real roots including 2 and 4. The third root can be written as

$$x = -1 \cdot 2^{-\frac{1}{2}\cdot 2^{-\frac{1}{2}\cdot 2^{\cdot^{\cdot^{\cdot}}}}} = -\frac{2}{\ln(2)} W\left(\frac{\ln(2)}{2}\right) = -0.7666646959\ldots$$

and is known to be transcendental [5]. It is interesting that $W(-\ln(2)/2)$ is elementary but $W(\ln(2)/2)$ is not. Consider instead the equation $x + e^x = 0$, which possesses a unique real root:

$$x = -1 \cdot e^{-1\cdot e^{-1\cdot e^{\cdot^{\cdot^{\cdot}}}}} = -W(1) = -0.5671432904\ldots = -\ln(1.7632228343\ldots).$$

Other examples suggest themselves.

The hyperpower analog of the harmonic series

$$H_n = \left(\frac{1}{2}\right)^{\left(\frac{1}{3}\right)^{\cdot^{\cdot^{\cdot^{\left(\frac{1}{n}\right)}}}}}$$

is divergent in the sense that even and odd partial exponentials converge to distinct limits [6–8]:

$$\lim_{n\to\infty} H_{2n} = 0.6583655992\ldots < 0.6903471261\ldots = \lim_{n\to\infty} H_{2n+1}.$$

No alternative expressions for these constants are known.

Let i denote the imaginary unit; then the multivalued expression i^i is always real:

$$i^i = \exp\left(-\frac{\pi}{2}(4n+1)\right),$$

which, when $n = 0$, gives $i^i = \exp(-\pi/2) = 0.2078795764\ldots$. If we restrict attention to the principal branch of the logarithm ($n = 0$), iterating the exponential can be proved [9–13] to converge to

$$\frac{2}{\pi} i\, W\left(-\frac{\pi}{2} i\right) = 0.4382829367\ldots + (0.3605924718\ldots)i.$$

Here are two striking integrals: [14–16]

$$\int_0^1 x^x\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^n} = 0.7834305107\ldots,$$

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{n=1}^{\infty} \frac{1}{n^n} = 1.2912859970\ldots.$$

These are easily proved via term-by-term integration of Maclaurin series expansions. A more difficult evaluation concerns the series [17]

$$\lim_{N\to\infty}\sum_{n=1}^{2N}(-1)^n n^{\frac{1}{n}} = \sum_{k=1}^{\infty}\left((2k)^{\frac{1}{2k}} - (2k-1)^{\frac{1}{2k-1}}\right) = \sum_{m=1}^{\infty}(-1)^m\left(m^{\frac{1}{m}} - 1\right)$$

$$= 1 + \lim_{N\to\infty}\sum_{n=1}^{2N+1}(-1)^n n^{\frac{1}{n}} = 0.1878596424\ldots,$$

which is slowly convergent. No exact formulas are known, although the series bear some resemblance to expressions mentioned in [2.15]. Cesàro summation and Cohen–Villegas–Zagier acceleration [18] are two techniques available to compute the sum.

Long ago, Poisson [19] discovered a remarkable identity:

$$-\frac{\pi}{2}W(-x) = \int_0^{\pi}\frac{\sin(\frac{3}{2}\theta) - x\,e^{\cos(\theta)}\sin(\frac{5}{2}\theta - \sin(\theta))}{1 - 2x\,e^{\cos(\theta)}\cos(\theta - \sin(\theta)) + x^2e^{2\cos(\theta)}}\sin(\tfrac{1}{2}\theta)\,d\theta,$$

valid for $|x| < e^{-1}$. We wonder if his theory might someday lead to the solution, in terms of a "compact" definite integral, of other transcendental equations (e.g., Kepler's equation [4.8]).

6.11.1 Exponential Recurrences

There is not as much to say about exponential recurrences as about quadratic recurrences [6.10]. The simplest example is [20]

$$c_0 = 0, \qquad c_n = 2^{c_{n-1}} \quad \text{for } n \geq 1.$$

If $\emptyset$ denotes the empty set, then $c_1 = 1$ is the cardinality of the power set $P(\emptyset)$ of $\emptyset$, $c_2 = 2$ is the cardinality of $P(P(\emptyset))$, $c_3 = 4$ is the cardinality of $P(P(P(\emptyset)))$, etc. The Ackermann-like growth of $\{c_n\}$ greatly exceeds that of any exponential function.

Another occurrence of $\{c_n\}$ is as follows. A **rooted identity tree** is a rooted tree for which the only automorphism fixing the root is the identity map [5.6]. Fix an integer $h > 0$. An identity tree of height h consists of a root, a nonempty set of identity trees (all different) of height $h-1$, and a (possibly empty) set of identity trees (all different) of height $< h-1$. (See Figure 6.4.) The cardinality of all such identity trees is therefore

$$\left(2^{c_h - c_{h-1}} - 1\right)2^{c_{h-1}} = c_{h+1} - c_h$$

since repetitions are not allowed. These are equivalent to what are called *ranked sets* in set theory.

A variation of this,

$$\gamma_0 = 0.1490279983\ldots, \qquad \gamma_n = 2^{\gamma_{n-1}} \quad \text{for } n \geq 1,$$

arises in combinatorial game theory [21,22]. The number of *impartial misère games* at day n is $g_n = \lceil \gamma_n \rceil$, and each such game can be thought of as a rooted identity tree t satisfying special conditions. Let $S(t)$ denote the set of (distinct) identity subtrees

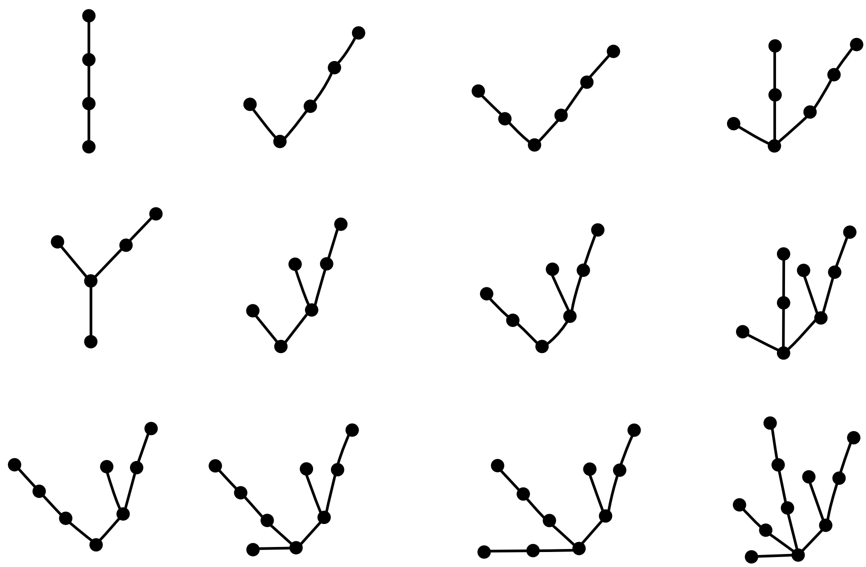

Figure 6.4. There exist twelve rooted identity trees of height 3.

of t with roots adjacent to the root of t. The **outcome** $O(t)$ of t is N if $O(s) = P$ for some $s \in S(t)$ or if t is a single vertex; otherwise $O(t) = P$. A tree t is **reversible** if, for some tree u, $S(u)$ is a proper subset of $S(t)$ and, if $v \in S(t) - S(u)$, then $u \in S(v)$; further, if u is a single vertex, then $O(w) = P$ for some $w \in S(t)$. Finally, a tree t is **canonical** if t is not reversible and if each $s \in S(t)$ is canonical. The number g_n of canonical trees of height $\leq n$ is 1, 2, 3, 5, 22, and 4171780 for $0 \leq n \leq 5$; a corrected value of $g_6 \approx 2^{4171780}$ appears in [23–25]. Conway [21] claimed that constructing an existence proof for the constant γ_0, valid for all n, is not difficult.

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6.12 Conway's Constant

Suppose we start with a string of digits, for example, 13. We might describe this as "one one, one three" and thus write the **derived string**, 1113. This in turn we describe as "three ones, one three," giving 3113. Continuing, the following sequence of strings are obtained [1]:

132113,

1113122113,

311311222113,

13211321322113,

1113122113121113222113,

31131122211311123113322113,

132113213221133112132123222113,

11131221131211132221232112111312111213322113,

31131122211311123113321112131221123113111231121123222113.

We have given the first twelve strings of this sequence ($k = 1$ to $k = 12$). It can be proved that only the digits 1, 2, and 3 appear at any step, so the process can be continued indefinitely. What can be said about the length of the k^{th} string? Its growth appears to be exponential and at first glance one would anticipate this to be impossibly difficult to characterize more precisely. Conway [2–5], defying expectation, proved that the growth is asymptotic to $C\lambda^k$, where $\lambda = 1.3035772690\ldots = (0.7671198507\ldots)^{-1}$ is the largest zero of the polynomial

$$\begin{aligned}
&x^{71} - x^{69} - 2x^{68} - x^{67} + 2x^{66} + 2x^{65} + x^{64} - x^{63} - x^{62} - x^{61} - x^{60}\\
&\quad - x^{59} + 2x^{58} + 5x^{57} + 3x^{56} - 2x^{55} - 10x^{54} - 3x^{53} - 2x^{52} + 6x^{51} + 6x^{50}\\
&\quad + x^{49} + 9x^{48} - 3x^{47} - 7x^{46} - 8x^{45} - 8x^{44} + 10x^{43} + 6x^{42} + 8x^{41} - 5x^{40}\\
&\quad - 12x^{39} + 7x^{38} - 7x^{37} + 7x^{36} + x^{35} - 3x^{34} + 10x^{33} + x^{32} - 6x^{31} - 2x^{30}\\
&\quad - 10x^{29} - 3x^{28} + 2x^{27} + 9x^{26} - 3x^{25} + 14x^{24} - 8x^{23} - 7x^{21} + 9x^{20}\\
&\quad + 3x^{19} - 4x^{18} - 10x^{17} - 7x^{16} + 12x^{15} + 7x^{14} + 2x^{13} - 12x^{12} - 4x^{11}\\
&\quad -2x^{10} + 5x^{9} + x^{7} - 7x^{6} + 7x^{5} - 4x^{4} + 12x^{3} - 6x^{2} + 3x - 6.
\end{aligned}$$

This polynomial and λ were first computed by Atkin; Vardi [6] noticed a typographical error (the x^{35} term was off by a sign in [4]).

Moreover, the same constant λ applies to the growth rates of *all* such sequences, regardless of the starting string, with two trivial exceptions. We started with the string 13 earlier; the constant λ is *universally applicable* except for the empty initial string and the string 22. This astonishing fact is a consequence of what is known as the **Cosmological Theorem**, the proof of which was lost until recently [7]. Ekhad & Zeilberger's tour-de-force is a splendid illustration of the use of software in proving theorems.

Even more can be said. Sometimes a string factors as the concatenation of two strings L and R whose descendents never interfere with each other. We say that the string LR **splits** as $L.R$ and LR is called a **compound**. A string with no nontrivial splittings is called an **element** or **atom**. It turns out that there are ninety-two special atoms (named after the chemical elements Hydrogen, Helium, ..., Uranium). *Every* string of 1s, 2s, and/or 3s eventually decays into a compound of these elements. Additionally, the relative abundances of the elements approach fixed positive limits, independent of the initial string. Thus, of every million atoms, about 91790 on average will be of Hydrogen (the most common) whereas only about 27 will be of Arsenic (the least common).

Conway's Periodic Table of Elements [3, 4] traces the evolution of the string 13 as previously, but indicates the evolution in terms of elements rather than long ternary strings. For example, when $k = 1$ to $k = 6$, the strings are the elements Pa, Th, Ac, Ra, Fr, and Rn, but when $k = 7$, the first compound emerges: 13211321322113, which may be rewritten as Ho.At because Ho is 1321132 and At is 1322113. As another example, when $k = 91$, Helium derives to the compound Hf.Pa.H.Ca.Li because H is 22.

Let us illustrate further. If we start with 11, we obtain 21, then 1211, and then

$$111221,$$

$$312211,$$

$$13112221,$$

$$1113213211 = 11132.13211 = \text{Hf.Sn}.$$

If we start with 12, the first string is already an element: $12 = \text{Ca}$, while starting with 32 or 23 gives

$$1312, \quad 1213,$$

$$11131112 = 1113.1112 = \text{Th.K}, \quad 11121113 = 1112.1113 = \text{K.Th}.$$

There is also the more general case of strings containing digits other than 1, 2, or 3. If we start with, say, 14 or 55, the theorem regarding relative abundances still applies, but we allow just two additional elements (**isotopes** of Plutonium and Neptunium)

$$\text{Pu}_4 = 31221132221222112112322211 4,$$
$$\text{Np}_4 = 1311222113321132211221121332211 4,$$

$$\text{Pu}_5 = 31221132221222112112322211 5,$$
$$\text{Np}_5 = 1311222113321132211221121332211 5,$$

the relative abundances of which tend to 0. This is true for strings with digits 6, 7, 8, 9, ... as well.

We return finally to Conway's constant λ. It is the (unique) largest eigenvalue (in modulus) of the 92×92 transition matrix M whose $(i, j)^{\text{th}}$ element is the number of atoms of element j resulting from the decay of one atom of element i. The relative abundances also arise in a careful eigenanalysis of M. We know that Conway's 71^{st} degree polynomial has Galois group S_{71}, and hence λ cannot be expressed in terms of radicals [8]. See [9–12] as well.

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7

Constants Associated with Complex Analysis

7.1 Bloch–Landau Constants

Let F denote the set of all complex analytic functions f defined on the open unit disk D, centered at the origin, and satisfying $f(0) = 0$, $f'(0) = 1$.

For each $f \in F$, let $b(f)$ be the supremum of all numbers r such that there is a subregion S of D on which f is one-to-one and such that $f(S)$ contains a disk of radius r. Bloch [1–7] showed that $b(f)$ is at least $1/12$. **Bloch's constant** B is defined to be $\inf\{b(f) : f \in F\}$. The precise value of B is unknown, but the following bounds were established by Ahlfors & Grunsky [8] and Heins [9]:

$$0.433 < \frac{\sqrt{3}}{4} < B \leq \frac{1}{\sqrt{1+\sqrt{3}}} \frac{\Gamma(\frac{1}{3})\Gamma(\frac{11}{12})}{\Gamma(\frac{1}{4})} = 0.4718616534\ldots.$$

Ahlfors & Grunsky further conjectured that B is equal to its upper bound.

A related constant is defined as follows: For each $f \in F$, let $l(f)$ be the supremum of all numbers r such that $f(D)$ contains a disk of radius r. **Landau's constant** L [3, 5, 7, 10] is defined to be $\inf\{l(f) : f \in F\}$. It is clear that L is at least as large as B. Like B, we do not know the value of L exactly. The following bounds were determined by Robinson [11] and, independently, by Rademacher [12]:

$$0.5 = \frac{1}{2} < L \leq \frac{\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{6})} = 0.5432589653\ldots.$$

Rademacher also conjectured that L is equal to its upper bound.

Both of these conjectures remain unproven to this day [13–17]. The form of the conjectured exact expressions, ratios of gamma function values [1.5.4], are fascinating.

Bonk [18] proved in 1990 that a lower bound for B is $\sqrt{3}/4 + 10^{-14}$, which Minda [19] called the first quantitative improvement in estimating B in a half century. Chen & Gauthier [20, 21] adapted Bonk's method to replace 10^{-14} by 2×10^{-4}, and Yanagihara [22] improved the lower bound for L to $1/2 + 10^{-335}$.

Let G denote the subset of F consisting of one-to-one functions. Such functions are said to be **univalent** (or **schlicht**). Over G, the notions of Bloch constant and Landau constant obviously coincide. Define the **univalent Bloch–Landau constant** K (or **schlicht Bloch–Landau constant**) to be $\inf\{l(f) : f \in G\}$. The most current bounds on K are $0.57088 < K < 0.6564155$ [23–28]. No one has yet hypothesized an exact expression for K.

There are various extensions of these ideas, for example, to a domain D that is not a disk but an annulus [29], or to functions f of not one but several complex variables [30, 31]. To discuss these would take us far afield.

MacGregor [32] raised some interesting questions concerning other geometrical properties of $f(D)$. If $f \in F$, it can be shown that the **diameter** of $f(D)$, defined to be $\sup\{|f(z) - f(w)| : z, w \in D\}$, is at least 2. See [33] for a proof. What else can be said? If $f \in G$, let $a(f)$ denote the area of the intersection of $f(D)$ with the unit disk. The work of Goodman, Jenkins & Reich [34–36] yields that $0.62\pi < A = \inf\{a(f) : f \in G\} < 0.7728\pi$. What is the precise value of the constant A? Also, Strohhäcker [37] showed that, given $f \in G$, there is a line segment in $f(D)$ with one endpoint at the origin and possessing length greater than 0.73. What is the largest number 0.73 can be replaced by? Conceivably this question is related to what is known as the Hayman-Wu constant [7.5]. See also [8.19] for other relevant material.

For each $f \in G$, let $m(f)$ be the supremum of all numbers r such that $f(D)$ contains the disk of radius r, centered at the origin. Note the final hypothesis. Define the **Koebe constant** M [38, 39] to be $M = \inf\{m(f) : f \in G\}$. Koebe [40] proved the existence of M and Bieberbach [41] established Koebe's conjecture that $M = 1/4$. The extremal functions consist of precisely the mapping

$$f(z) = \frac{z}{(1 - z)^2}$$

and its rotations. Observe that there is no nonzero analog of M for the set F. For arbitrarily large integer n, $f(z) = (\exp(nz) - 1)/n$ is in F, but it omits the value $-1/n$ since the exponential function is never zero. Hence no disk, centered at the origin, is contained in $f(D)$ for suitably large n.

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7.2 Masser–Gramain Constant

Suppose $f(z)$ is an entire function such that $f(n)$ is an integer for each positive integer n. Under what circumstances can we conclude that f is a polynomial? Pólya [1] proved that if

$$\limsup_{r\to\infty} \frac{\ln(M_r)}{r} < \ln(2) = 0.6931471805\ldots, \quad \text{where } M_r = \sup_{|z|\le r} |f(z)|,$$

then the conclusion follows. Moreover, the special case $f(z) = 2^z$ demonstrates that ln(2) is the largest constant (or "best constant") for which this line of reasoning holds [2–4].

Here is a more difficult but related problem. It involves the Gaussian integers, which constitute the set of all complex numbers with integer real parts and integer imaginary parts. Suppose $f(z)$ is an entire function such that $f(n)$ is a Gaussian integer for each Gaussian integer n. Under what circumstances, again, can we conclude that f is a polynomial? Gel'fond [5], building upon the work of Fukasawa [6], proved that there exists a positive constant α such that

$$\limsup_{r\to\infty} \frac{\ln(M_r)}{r^2} < \alpha$$

implies the conclusion. Not surprisingly, a stronger limiting condition (involving r^2 in the denominator instead of r) is needed to force f to be a polynomial. We will discuss the best constant α later. Our focus is on a different constant δ that arose in one attempt to identify α.

Masser [7] proved that α could be no larger than $\pi/(2e) = 0.5778636748\ldots$ and believed α to be equal to $\pi/(2e)$. He also proved the following weaker result: f must be a polynomial if the following holds:

$$\limsup_{r\to\infty} \frac{\ln(M_r)}{r^2} < \alpha_0 = \frac{1}{2}\exp\left(-\delta + \frac{4c}{\pi}\right),$$

where

$$c = \gamma\,\beta(1) + \beta'(1) = \frac{\pi}{4}\left(-\ln(2) + 2\ln(\pi) + 2\gamma - 2\ln(L)\right) = 0.6462454398\ldots,$$

γ is the Euler–Mascheroni constant [1.5], $\beta(x)$ is the Dirichlet beta function [1.7], L is Gauss' lemniscate constant [6.1], and δ will be defined shortly. Expressions similar to this appear in our essays on the Landau–Ramanujan constant [2.3] and Sierpinski's constant S [2.10]; in fact, $c = \pi S/4$.

Define δ as a natural two-dimensional generalization of the Euler–Mascheroni constant:

$$\delta = \lim_{n\to\infty}\left(\sum_{k=2}^{n}\frac{1}{\pi\, r_k^2} - \ln(n)\right),$$

where r_k is the minimum over all $r \geq 0$ such that there exists a complex number z for which the closed disk with center z and radius r contains at least k distinct Gaussian integers.

The computation of δ is exceedingly difficult. Gramain & Weber [8] determined bounds $1.811447299 < \delta < 1.897327117$, which imply that $0.1707339 < \alpha_0 < 0.1860446$. It turns out that α_0 is the largest constant that Gel'fond's technique (known as the method of series interpolation) can give. Certainly α_0 is far away from the conjectured best constant $\pi/(2e)$, but it is interesting that α_0 is close to $1/(2e) = 0.1839397205\ldots$. Gramain [9, 10] conjectured that $\alpha_0 = 1/(2e)$, which would imply $\delta = 1 + 4c/\pi = 1.8228252496\ldots$, but no one knows whether this is true.

How would one calculate the Masser–Gramain constant δ to, for example, four decimal places? No formula for r_k is known, so Gramain & Weber [8] had no choice but to evaluate r_k for large k via its definition. One has, for example [7], $r_2 = 1/2$, $r_3 = r_4 = 1/\sqrt{2}$, and bounds [9]

$$\frac{\sqrt{\pi\,(k-1)+4}-2}{\pi} < r_k < \sqrt{\frac{k-1}{\pi}}.$$

The upper bound is quite good, but the lower bound must be improved for the sake of accurate estimation of δ. One has

$$\frac{\sqrt{\pi\,(k-6)+2}-\sqrt{2}}{\pi} \leq r_k$$

for $k \geq 6$, but required further improvements [9, 10] are too complicated to present here. To obtain δ to four decimal places would necessitate computing r_k for k up to 5×10^{13} according to [8]. Unless the algorithm for calculating r_k is made more efficient, the bounds for r_k are improved, another procedure for computing δ is found, or a breakthrough in computer hardware occurs, the identity of δ will remain unknown.

A completely different n-dimensional lattice sum generalization of Euler's constant is discussed in [1.10.1].

Finally, let us resolve a remaining issue. Gramain [9, 10], building upon the work of Gruman [11], proved Masser's conjecture that the best constant α is $\pi/(2e)$. This achievement does not, however, shed any light on the value of δ or α_0.

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7.3 Whittaker–Goncharov Constants

Suppose f is an entire function such that f and its derivatives $f^{(n)}, n = 1, 2, 3, \ldots$, each have at least one zero z_n in the unit disk. Under what circumstances can we conclude that f is identically zero? It is not difficult [1–5] to prove that if

$$\limsup_{r\to\infty} \frac{\ln(M_r)}{r} < \ln(2), \quad \text{where } M_r = \sup_{|z|\le r} |f(z)|,$$

then the conclusion $f = 0$ follows. This bound is not the best possible. Define **Whittaker's constant** W to be the largest number for which

$$\limsup_{r\to\infty} \frac{\ln(M_r)}{r} < W \quad \text{implies} \quad f = 0.$$

Then the previous result plus the example $f(z) = \sin(z) + \cos(z)$ show that $\ln(2) = 0.693\ldots \le W \le 0.785\ldots = \pi/4$. We alternatively have the identity

$$\limsup_{n\to\infty} \left| f^{(n)}(z) \right|^{\frac{1}{n}} = \limsup_{r\to\infty} \frac{\ln(M_r)}{r}$$

for any choice of complex number z. In words, the asymptotic local behavior of $f^{(n)}$ is governed by the global nature of the maximum modulus function M.

Other formulations exist for W in terms of Maclaurin series coefficients, as well as conditions involving the behavior of the sequence $\{z_n\}$ or the possible univalence of f. We do not discuss these except to mention that **Goncharov's constant** G arises in a such a way [6, 7] and $W = G$ was later proved by Buckholtz [8, 9]. A formulation involving what are known as Goncharov polynomials is discussed later [7.3.1].

The best-known rigorous bounds on W are due to Macintyre [10–12]:

$$0.7259\ldots < W < 0.7378\ldots,$$

building upon earlier work by Pólya, Boas, and Levinson. The upper bound arises from a study of entire solutions of the functional differential equation

$$\frac{d}{dz}\varphi(z,q) = \varphi(q\,z,q),$$

that is,

$$\varphi(z,q) = \sum_{n=0}^{\infty} \frac{1}{n!} q^{\frac{n(n-1)}{2}} z^n, \quad |q| \le 1.$$

More precisely, W is no greater than the smallest moduli of zeros of $\varphi(z,q)$, considered over all q. The lower bound for W comes about in a different way.

Numerical heuristics allowed Varga & Wang [12, 13] to deduce that $0.7360 < W$, hence disproving Boas' conjecture [14, 15] that $W = 2/e$. More computations led Waldvogel [16] to deduce that $0.73775075 < W$, but we emphasize that rigorous theoretical support for this work has not been finalized. However, refined estimates of Macintyre's upper bound [12, 13, 16] give $W < 0.7377507574\ldots$. Thus Varga & Waldvogel have conjectured that W is equal to its upper bound. No amount of floating point calculations will suffice to prove an exact equality as such!

Some generalizations of W were defined in [4, 17–22]. Oskolkov [23] claimed to possess a new method for computing an arbitrarily close lower bound to W.

Here is a related topic. Differentiating a power series

$$\sum_{n=0}^{\infty} a_n z^n \to \sum_{n=1}^{\infty} n\, a_n z^{n-1}$$

and *shifting* a power series (i.e., forming a normalized remainder)

$$\sum_{n=0}^{\infty} a_n z^n \to \sum_{n=1}^{\infty} a_n z^{n-1}$$

are somewhat similar operations. The aforementioned theory involving W, Goncharov polynomials, and differentiation has an analog for shifting. We will take an alternative viewpoint, for the sake of both simplicity and variety.

Let f be an analytic function whose Maclaurin series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

has radius of convergence exactly equal to 1. Let

$$S_n(z,f) = \sum_{k=0}^{n} a_k z^k, \quad n = 1, 2, 3, \ldots,$$

be the n^{th} partial sum of f and define $\rho_n(f)$ to be the largest moduli of the zeros of the polynomial S_n. Let

$$\rho(f) = \liminf_{n\to\infty} \rho_n(f)$$

and define the **power series constant**

$$P = \sup_f \rho(f).$$

Porter [24] and Kakeya [25, 26] showed that $P \leq 2$. Clunie & Erdös [25] demonstrated that $\sqrt{2} < P < 2$. Buckholtz [27] improved this to $1.7 < P < 1.862$ and Frank [27] improved this to $1.7818 < P < 1.82$. Independent work in estimating $1/P$ was done by Pommiez [28–30]. Just as Whittaker's constant W has formulation in terms of Goncharov polynomials, the power series constant P has formulation in terms of what are called remainder polynomials [7.3.2].

In this case, we consider not a functional differential equation, but rather a functional equation involving shifting. The zeros of the solution

$$\psi(z,q) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} z^n, \quad |q| \leq 1,$$

are again studied, yielding a lower bound $P \geq 1.7818046151\ldots$. Waldvogel [16] conjectured that the lower bound is, in fact, the true value of P. This is analogous to before, although the analysis is more complicated.

A third constant, examined in [16], involves certain Padé approximants. Relevant material includes [31–33].

7.3.1 Goncharov Polynomials

Bounds for the Whittaker–Goncharov constant W can theoretically be determined via the **Goncharov polynomials** [7]:

$$G_0(z) = 1, \quad G_n(z, z_0, z_1, \ldots, z_{n-1}) = \int_{z_0}^{z} \int_{z_1}^{t_1} \cdots \int_{z_{n-2}}^{t_{n-2}} \int_{z_{n-1}}^{t_{n-1}} 1 \, dt_n \, dt_{n-1} \ldots dt_2 \, dt_1$$

for $n \geq 1$. An equivalent recursive definition is

$$G_n(z, z_0, z_1, \ldots, z_{n-1}) = \frac{z^n}{n!} - \sum_{k=0}^{n-1} \frac{z_k^{n-k}}{(n-k)!} G_k(z, z_0, z_1, \ldots, z_{k-1}).$$

Evgrafov [34] proved that

$$\left(\limsup_{n\to\infty} g_n^{\frac{1}{n}} \right)^{-1} = W,$$

where

$$g_n = \max_{\substack{|z_k|=1 \\ 0 \leq k \leq n-1}} \left| G_n(0, z_0, z_1, \ldots, z_{n-1}) \right|.$$

Buckholtz [35] further showed that

$$\left(\frac{2}{5}\right)^{\frac{1}{n}} g_n^{-\frac{1}{n}} < W \leq g_n^{-\frac{1}{n}},$$

and hence the limit superior can be replaced by a limit. Unfortunately, the convergence rate using these formulas is much too slow for accurate estimation of W [12]. Other techniques must be used.

7.3.2 Remainder Polynomials

A lower bound for the power series constant P can theoretically be determined via the **remainder polynomials** [30, 36, 37]:

$$B_0(z) = 1, \quad B_n(z, z_0, z_1, \ldots, z_{n-1}) = z^n - \sum_{k=0}^{n-1} z_k^{n-k}\, B_k(z, z_0, z_1, \ldots, z_{k-1})$$

for $n \geq 1$. Buckholtz [36] proved that

$$\lim_{n\to\infty} b_n^{\frac{1}{n}} = P,$$

where

$$b_n = \max_{\substack{|z_k|=1 \\ 0\leq k\leq n-1}} \left| B_n(0, z_0, z_1, \ldots, z_{n-1}) \right|.$$

Unfortunately, as with the Goncharov polynomials, the convergence rate using these formulas is much too slow for accurate estimation of P.

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7.4 John Constant

Let X and Y be real Banach spaces (for example, X and Y may be taken to be finite-dimensional Euclidean spaces) and let D be an open subset of X. Suppose two numbers m, M are given with $0 < m \leq M < \infty$. Define a mapping $f : D \to Y$ to be an

(m, M)-**isometry** if it is continuous, open, locally one-to-one, and additionally satisfies

$$m \le \liminf_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}, \quad \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} \le M$$

for all $x \in D$.

What does the last part of this definition mean? If we picture f as deforming the domain D, then it does so in such a manner that lengths of line elements in D are altered by factors constrained to lie between m and M. Such a mapping f is also known as a **quasi-isometry** or a **bi-Lipschitz map**.

John [1–3] proved that, if $m = M$, then f must obey $|f(y) - f(x)|/|y - x| = m$ for all $x, y \in D$ and thus f is a rigid motion, scaled by m. In particular, f is (globally) one-to-one on D.

With this result in mind, it is natural to ask for the largest number $\mu = \mu(D)$ with the property that $M/m < \mu$ implies that all (m, M)-isometries of D are one-to-one. Henceforth assume D is an open ball in X. Gevirtz [4] proved that $\mu \ge r = 1.114305\ldots$, where r is the unique real root of the equation

$$r = \frac{r + \sqrt{25r^2 - 8r}}{2r(3r - 1)}.$$

A numerically sharp lower bound is not known. A few words about upper bounds for μ appear at the end of this essay.

If X is, moreover, a Hilbert space (hence angles can be measured in X), then the additional structure permits improved bounds. Gevirtz [4, 5], extending a result by John [3], showed that $\mu \ge \sqrt{2} = 1.414213\ldots$. If *both* X and Y are Hilbert spaces, then Gevirtz [4], sharpening John [3], demonstrated that $\mu \ge \sqrt{1 + \sqrt{2}} = 1.553773\ldots$ and in [5] showed that in fact $\mu \ge s = 1.65743\ldots$, where s is the minimum value for $t > 0$ of the function

$$s = s(t) = \frac{\pi + 2\sqrt{1 + t^2}}{1 + \frac{\pi}{2} + t}.$$

The proofs of these lower bounds entail fairly complicated arguments that use the basic principles for quasi-isometries established by John. Such lines of attack, however, are not powerful enough to produce numerically sharp results.

John [6] considered the special case in which the mapping is effected by an analytic function of one complex variable. That is, he considered analytic functions f defined on the unit disk D in the z-plane that satisfy $m \le |f'(z)| \le M$ at all points $z \in D$. As before, what is the largest number γ such that $M/m \le \gamma$ implies that f is univalent (in the disk)? The value γ is called the **John constant** for D. Since this is a special case of the preceding, we may expect γ to be larger than μ.

Several researchers, including Avhadiev & Aksentev [7], John [6], Yamashita [8], and Gevirtz [9, 10], worked to determine γ. The best-known bounds [6, 9] are

$$4.810477\ldots \le \exp(\tfrac{1}{2}\pi) \le \gamma \le \exp(\lambda\pi) = 7.1879033516\ldots,$$

where $\lambda = 0.6278342677\ldots$ satisfies the transcendental equation

$$\frac{\pi}{\exp(2\pi\lambda) - 1} = \sum_{k=1}^{\infty} \frac{k}{k^2 + \lambda^2} \exp\left(-\frac{k\pi}{2\lambda}\right).$$

Gevirtz [10] conjectured that, in fact, $\gamma = \exp(\lambda\pi)$ and gave compelling reasons for why this equality might hold. A rigorous proof is not known.

Again, if we picture f as deforming the disk D, a helpful physical interpretation emerges. If D is made of a hypothetical material that offers no resistance to infinitesimal contractions and stretchings by factors between m and M, and infinite resistance beyond these bounds, then how large must the ratio M/m be for one to bend D in such a way to make D touch itself? For analytic functions f, the answer would appear to be $7.1879033516\ldots$.

John constants can be defined for domains D in the complex plane other than the unit disk. A variational approach initiated in this setting [10, 11] provides evidence for the truth of Gevirtz's conjecture.

As a postlude, let us return to the more general conditions of the beginning. If $X = Y$ and X is the one-dimensional real line, then $\mu = \infty$ since a real-valued local homeomorphism of an interval must be a global homeomorphism (since it is monotonic). If $X = Y$ and the dimension of X is at least two, then upper bounds can be placed on μ. For example, if X is a Hilbert space, then $2 \geq \mu \geq 1.65743\ldots$. This is an outgrowth of a simple two-dimensional example by John [3]. If X is only a Banach space, then all that can be said is $64 \geq \mu \geq 1.114305\ldots$. The proof of these bounds appears in [5].

This essay is partly based on a letter from Julian Gevirtz. He also mentioned his long personal association with Fritz John. For this reason, we offer this essay as a small tribute to John's memory [12].

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7.5 Hayman Constants

7.5.1 Hayman–Kjellberg

Let f be a transcendental entire function. That is, f is analytic on the whole complex plane but is not a polynomial. For each $r > 0$, define

$$M(r) = \max_{|z|=r} |f(z)|,$$

the maximum modulus of f over the circle of radius r centered at the origin. Consider the function

$$a(r) = \frac{d^2}{d\ \ln(r)^2} \ln(M(r)) = \left(r\frac{d}{dr}\right)^2 \ln(M(r)),$$

which exists and is continuous except at isolated points. Hadamard's three circles theorem [1] asserts that $a(r) \geq 0$. What else can be said about $a(r)$?

Hayman [2] proved that there is a constant $A > 0.18$ such that

$$\limsup_{r\to\infty} a(r) \geq A$$

for all f. He conjectured that $A = 1/4$, but this was disproved by Kjellberg [3], who demonstrated that $0.24 < A < 0.25$. Kjellberg mentioned that Richardson might have a proof that $A < 0.245$. More accurate, computer-based estimates of A are still unknown.

7.5.2 Hayman–Korenblum

Let p be a real number with $p \geq 1$. Define $c(p)$ to be the largest real number < 1 so that the following holds: For any functions f and g analytic on the unit disk, if $|f(z)| \leq |g(z)|$ for all z satisfying $c(p) < |z| < 1$, then

$$\int\limits_{|z|\leq 1} |f(z)|^p dx\, dy \leq \int\limits_{|z|\leq 1} |g(z)|^p dx\, dy,$$

where $z = x + i y$.

Hayman [4] proved that $c(2)$ exists and $0.04 = 1/25 \leq c(2) \leq 1/\sqrt{2} = 0.7071\ldots$, confirming a conjecture of Korenblum [5]. (More precisely, Korenblum conjectured the existence of $c(2)$ and conditionally demonstrated that the upper bound holds.) In a significant extension, Hinkkanen [6] proved that $c(p)$ exists and $0.15724 \leq c(p)$, and he asked whether $c(p) \to 1$ as $p \to \infty$. No conjectures have been made about the exact value of $c(2)$, let alone $c(p)$.

7.5.3 Hayman–Stewart

Let f be a meromorphic function. That is, f is analytic on the whole complex plane except for (isolated) poles. It can be proved that f is a quotient of two entire functions. One customarily views f as a map to the Riemann sphere S, because where f has poles it can be considered to take the value ∞. For every $r > 0$ and every point $a \in S$, define

$$n(r, a) = \text{the number of roots of the equation } f(z) = a \text{ in the disk } |z| \le r\text{, with due count of multiplicity,}$$

the counting function of a-points of f. Now define two related quantities:

$$n(r) = \max_{a \in S} n(r, a),$$

$$A(r) = \operatorname*{mean}_{a \in S} n(r, a) = \frac{1}{\pi} \int_S n(r, a)\, da = \frac{1}{\pi} \int_{|z| \le r} \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2}\, dx\, dy,$$

where $z = x + i\,y$. It is natural to compare these quantities as $r \to \infty$. Both $A(r) \to \infty$ and $n(r) \to \infty$, except in the case where f is a rational function (quotient of two polynomials), which does not interest us.

Clearly $n(r) \ge A(r)$ for all r since a maximum always exceeds an average. Certain meromorphic functions f can be constructed for which $\limsup_{r \to \infty} n(r)/A(r) = \infty$. Hence we turn attention to the ratio

$$H(f) = \liminf_{r \to \infty} \frac{n(r)}{A(r)}.$$

Hayman & Stewart [7–9] proved that $1 \le H(f) \le e$ for all f. The first example of a meromorphic function with $H(f) > 1$ was constructed by Toppila [10]; in fact, in his example $H(f)$ is at least $80/79$. However, Miles [11] proved that $H(f)$ is no larger than $e - 10^{-28}$ for all f. Thus if we define a constant $h = \sup_f H(f)$, where the supremum is over all nonconstant meromorphic functions f, we have $80/79 \le h \le e - 10^{-28}$.

Here is an interesting variation. Define

$$n_T(r) = \max_{a \in T} n(r, a)$$

for each finite subset T of S. For fixed T, clearly $n_T(r) \le n(r)$. Gary [12] proved that

$$\liminf_{r \to \infty} \frac{n_T(r)}{A(r)} \le 2.65$$

for all f, which contrasts nicely against Miles' more elaborate result. Dare we hope for greater accuracy in estimating any of these constants any time soon?

In a letter, Alexandre Eremenko wrote: "Hayman's constants are all defined as solutions of some complicated extremal problems (extremum over a class of meromorphic functions). It seems that none of these extremal problems has a nice symmetric solution. So one cannot hope for more than finding good numerical bounds for them. Another constant of this type is the univalent Bloch–Landau constant [7.1] By contrast, the ordinary Bloch–Landau constants are (presumably) of a different nature: They are related to some beautiful symmetric extremal configuration (if the conjectured values are correct). Carleson & Jones, by conjecturing that the Clunie–Pommerenke constant β is $1/4$ [7.6], believe that β is of this second kind. Of course, $\beta = 1/4$ cannot happen by accident: Some hidden symmetry should be responsible for this."

7.5.4 Hayman–Wu

Hayman & Wu [13] proved that there is a constant C such that if $f(z)$ is univalent on the open unit disk and L is any line in the plane, then the preimage $f^{-1}(L)$ has length $|f^{-1}(L)| \leq C$. Øyma [14, 15] has proved that the least possible value of C satisfies $\pi^2 \leq C < 4\pi$ and further conjectured that C is equal to the lower limit here.

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7.6 Littlewood–Clunie–Pommerenke Constants

7.6.1 Alpha

Let $p(z)$ be a polynomial of degree n. The expression $|p'(z)|/(1 + |p(z)|^2)$ is called the **spherical derivative** of $p(z)$, in the sense that it measures how p changes with z, regarded as a map into the Riemann sphere [1]. Define

$$P(p) = \int\limits_{|z|\le 1} \frac{|p'(z)|}{1+|p(z)|^2}\,dx\,dy,$$

where $z = x + i\,y$. This double integral is proportional to the mean spherical derivative of $p(z)$ over the unit disk. We ask about the maximal value

$$F(n) = \sup\{P(p) : \; p \text{ is a polynomial of degree } n\}$$

and the superior limit

$$\alpha = \limsup_{n\to\infty} \frac{\ln(F(n))}{\ln(n)}.$$

Littlewood [2] proved that $F(n)$ is finite and $F(n) \le \pi\sqrt{n}$, that is, $\alpha \le 1/2$. He conjectured that $\alpha < 1/2$. Eremenko & Sodin [3,4] proved that $F(n) = o(\sqrt{n})$ as $n \to \infty$. Soon afterward, Lewis & Wu [5] proved that $\alpha \le 1/2 - 2^{-264}$, thus confirming Littlewood's conjecture. However, Eremenko [6] demonstrated that $\alpha > 0$ and Baker & Stallard [7] improved this to $\alpha \ge 1.11 \times 10^{-5}$.

For rational functions (as opposed to polynomials), the analog of α has value 1/2 [2,8,9]. Littlewood [2] also provided several alternative definitions of α not involving the spherical derivative. The definition of α as given here was provided by Eremenko [10].

7.6.2 Beta and Gamma

A complex analytic function f defined on an open planar region is **univalent** (or **schlicht**) if f is one-to-one; that is, $f(z) = f(w)$ if and only if $z = w$. Let

$$D = \{z : |z| < 1\} \text{ (the open disk)}, \quad E = \{z : |z| > 1\} \text{ (an open annulus)},$$

$$S = \left\{\text{univalent } f \text{ on } D \text{ with } f(z) = z + \sum_{n=2}^{\infty} c_n z^n\right\},$$

$$S_1 = \left\{\text{bounded univalent } f \text{ on } D \text{ with } f(z) = \sum_{n=1}^{\infty} a_n z^n \text{ and } \sup_{z\in D} |f(z)| \le 1\right\},$$

$$S_2 = \left\{\text{univalent } f \text{ on } E \text{ with } f(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}\right\}.$$

For the class S, de Branges [11,12] proved that $|c_n| \le n$, confirming Bieberbach's famous conjecture [13]. This inequality is sharp. For S_1 and S_2, analogous sharp inequalities are unknown. It turns out that estimating coefficient decay rates for S_1 and

S_2 are closely related: Let

$$A_n = \sup_{f \in S_1} |a_n|, \qquad B_n = \sup_{f \in S_2} |b_n|,$$
$$-\gamma_1 = \lim_{n \to \infty} \frac{\ln(A_n)}{\ln(n)}, \quad -\gamma_2 = \lim_{n \to \infty} \frac{\ln(B_n)}{\ln(n)}.$$

For each $k = 1, 2$, we have relatively simple bounds $1/2 \le \gamma_k \le 1$. Building upon earlier work by Littlewood [14], Clunie & Pommerenke [15–18] showed that

$$0.503125 = \frac{1}{2} + \frac{1}{320} < \gamma_k < 0.803,$$

and Carleson & Jones [19] improved the upper bound to $\gamma_k < 0.76$.

Here is an alternative, more geometric formulation. For $\varepsilon > 0$ and $f \in S_k$, consider the arclength of the image of the circle $|z| = \exp((-1)^k \varepsilon)$ under the map f. Let

$$L_\varepsilon = \sup_{f \in S_1} |\{f(z) : |z| = \exp(-\varepsilon)\}|, \quad M_\varepsilon = \sup_{f \in S_2} |\{f(z) : |z| = \exp(\varepsilon)\}|,$$
$$-\beta_1 = \lim_{\varepsilon \to 0^+} \frac{\ln(L_\varepsilon)}{\ln(\varepsilon)}, \qquad -\beta_2 = \lim_{\varepsilon \to 0^+} \frac{\ln(M_\varepsilon)}{\ln(\varepsilon)}.$$

Carleson & Jones' arguments show that $0.503 < \gamma_1 = \gamma_2 = 1 - \beta_1 = 1 - \beta_2 < 0.76$ (in fact, they proved more.) The relation $\beta + \gamma = 1$ between power series coefficients and circular image arclengths seemed to be anticipated in earlier papers, but Carleson & Jones proved it explicitly and precisely for the first time.

Eremenko [10] provided a third formulation for these constants in terms of arclengths of Green's function level curves.

7.6.3 Conjectural Relations

Carleson & Jones [19] conjectured that $\gamma = 3/4$ (and hence $\beta = 1/4$) on the basis of numerical experimentation. There may be some skepticism about this belief, but there are no reliable means to confirm it yet.

Eremenko [6, 10] conjectured that $\alpha = \beta$ and further remarked that this can be proved (or disproved) without actual knowledge of α or β. The problem of whether $\alpha = \beta$ is perhaps easier than establishing their actual values.

We close with an unrelated problem. Consider the set of real numbers λ for which

$$\int_{|z| \le 1} |f'(z)|^\lambda dx\, dy < \infty$$

is true for all $f \in S$. Brennan [20–22] proved that the integral is finite for $-1 - \delta < \lambda < 2/3$ for some $\delta > 0$, but the integral is infinite if $\lambda = 2/3$ or $\lambda = -2$. He conjectured that the integral is finite for $-2 < \lambda < 2/3$, that is, one may take $\delta = 1$. The best value of δ remains an open question.

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7.7 Riesz–Kolmogorov Constants

Let $F(z) = f(z) + i\tilde{f}(z)$ be an analytic function defined on the closed unit disk, with the property that its imaginary part satisfies $\tilde{f}(0) = 0$. Define the p-**Hardy norm** [1,2]

$$||f||_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right)^{\frac{1}{p}}, \quad 0 < p < \infty.$$

What can be said about the relative sizes of the conjugate functions f and $\tilde{f}$? Riesz [3] proved that

$$||\tilde{f}||_p \le C_p \cdot ||f||_p, \ \ 1 < p < \infty,$$

and Pichorides [4] and Cole [5] determined the best constant in this inequality to be [6–8]

$$C_p = \begin{cases} \tan\left(\frac{\pi}{2p}\right) & 1 < p \le 2, \\ \cot\left(\frac{\pi}{2p}\right) & 2 < p < \infty. \end{cases}$$

If $p = 1$, there exist functions F for which $||f||_1 < \infty$ but $||\tilde{f}||_1 = \infty$. Hence a revised sense of "relative size" becomes necessary in this case.

If S is a measurable subset of the unit circle, let $|S|$ denote its Lebesgue measure, divided by 2π. For $t \ge 0$, define the set

$$S_t(f) = \{z : |f(z)| \ge t \text{ and } |z| = 1\}.$$

Kolmogorov [9] proved the **weak type 1-1 inequality**

$$|S_t(\tilde{f})| \le C_1 \cdot \frac{1}{t} \cdot ||f||_1 \ \text{ for all } t > 0$$

and Davis [10] determined the best constant to be

$$C_1 = \frac{\pi^2}{8G} = 1.3468852519\ldots = (0.7424537454\ldots)^{-1},$$

where G is Catalan's constant [1.7]. A corollary of Kolmogorov's theorem is

$$||\tilde{f}||_p \le C_p \cdot ||f||_1, \ \ 0 < p < 1.$$

Davis [11, 12] identified the best constants here to be

$$C_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |\csc(\theta)|^p d\theta\right)^{\frac{1}{p}} = \left(\frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1-p}{2}\right)}{\Gamma\left(\frac{2-p}{2}\right)}\right)^{\frac{1}{p}},$$

where $\Gamma(x)$ is the gamma function [1.5.4]. There is a related issue of the relative sizes of F and f, which we will not discuss. See also [13–17].

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7.8 Grötzsch Ring Constants

Let R be a planar ring, that is, an open connected subset of the complex plane $\mathbb{C}$. Two regions R_1 and R_2 are **conformally equivalent** if there is an analytic function $f : R_1 \to R_2$ such that f is one-to-one and onto. Clearly this is an equivalence relation. The famous Riemann mapping theorem implies the following:

- Among the simply connected regions, there are exactly two equivalence classes: one consisting of $\mathbb{C}$ alone and the other containing the unit disk (and much more).
- Among the doubly connected regions, there are uncountably many equivalence classes, each containing a circular annulus $A(1, r) = \{z : 1 < |z| < r\}$ for some unique real $r > 1$ (and much more).

In particular, two annuli $A(s, t)$ and $A(u, v)$ are conformally equivalent if and only if $t/s = v/u$, that is, the ratio of outer radius and inner radius is a conformal invariant [1,2].

Let us change the subject slightly for a moment. By a **ring** R in n-dimensional Euclidean space, we mean a region whose complement consists of two components C_0 and C_1, where C_0 is bounded and C_1 is unbounded. Let B_0 and B_1 be the boundary components of R. The **conformal capacity** of R is

$$\operatorname{cap}(R) = \inf_{\varphi} \int_R |\nabla \varphi|^n \, dx,$$

where the infimum is over all real continuously differentiable functions φ on R with values 0 on B_0 and 1 on B_1. The **modulus** of R is

$$\operatorname{mod}(R) = \left(\frac{\sigma}{\operatorname{cap}(R)}\right)^{\frac{1}{n-1}},$$

where $\sigma = n\pi^{n/2}\Gamma(1+n/2)^{-1}$ is the surface area of the sphere of radius 1 in n-dimensional space. For an n-dimensional spherical annulus $A(s,t)$, we find that [3–8]

$$\operatorname{mod}(A(s,t)) = \ln\left(\frac{t}{s}\right).$$

Therefore, in the case $n = 2$, the modulus of a ring is a conformal invariant. For $n \geq 3$, we lose this nice geometric interpretation since the Riemann mapping theorem no longer applies: The unit n-dimensional ball is conformally equivalent only to another ball or to a half-space. Nevertheless, the modulus is important in other ways (e.g., in distortion theorems associated with quasiconformal mappings).

Let $G(n,a)$ denote the n-dimensional **Grötzsch ring**, that is, the ring whose complementary components are

$$C_0 = \{(x,0,0,\ldots,0) : 0 \leq x \leq a\}, \text{ where } 0 < a < 1;$$

$$C_1 = \left\{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i^2 \geq 1\right\}.$$

In words, $G(n,a)$ is the unit n-ball, slit from 0 to a along a radial vector. It is known that the following limit exists and is finite [7–15]:

$$\ln(\lambda_n) = \lim_{a \to 0^+} \left(\operatorname{mod}(G(n,a)) + \ln(a)\right);$$

that is, $\operatorname{mod}(G(n,a))$ experiences logarithmic growth as a decreases to 0. In the special case $n = 2$, we have [4, 13, 14]

$$\operatorname{mod}(G(2,a)) = \frac{\pi}{2}\frac{K(\sqrt{1-a^2})}{K(a)}$$

and hence $\lambda_2 = 4$. K is the complete elliptic integral of the first kind; a similar expression appeared in [4.5]. We also have the interesting asymptotic result [9]

$$\lim_{n \to \infty} \lambda_n^{\frac{1}{n}} = e,$$

where e is the natural logarithmic base [1.3].

No such exact formulas have been found for λ_3 or λ_4. Rigorous lower and upper bounds for λ_n, plus the best-known numerical estimates, are given in Table 7.1 [12, 15]. A table of bounds for $\lambda_n \exp(-n)$ for $3 \leq n \leq 22$ appears in [14], along with a simple inequality

$$2\exp(0.76(n-1)) \leq \lambda_n \leq 2\exp(n-1).$$

We conclude by returning to the case $n = 2$. What is the formula for the conformal function f that maps $A(1,r)$ onto $G(2,a(r))$, where the slit length $a(r)$ is defined below?

Table 7.1. *Estimates for Parameters λ_n*

n	Lower bound	Best estimate for λ_n	Upper bound
3	9.341	9.37 ± 0.02	9.9002
4	21.85	22.6 ± 0.2	26.046

The mapping turns out to involve the Jacobi elliptic sine function sn [1.4.6]. Higher transcendental functions often occur in this study: The appropriate generalizations for $n \geq 3$ await discovery.

7.8.1 Formula for $a(r)$

The annulus $A(1, r)$ and the Grötzsch ring $G(2, a)$ are conformally equivalent if and only if

$$\ln(r) = \mathrm{mod}(A(1, r)) = \mathrm{mod}(G(2, a)) = \frac{\pi}{2}\frac{K(\sqrt{1-a^2})}{K(a)}.$$

We wish to solve for a as a function of r. It turns out that $a(r)$ can be written in terms of an infinite product [14, 16]:

$$a(r) = \frac{2b(r)}{1 + b(r)^2}, \text{ where } b(r) = \frac{2}{r}\prod_{j=1}^{\infty}\left(\frac{1 + r^{-8j}}{1 + r^{-8j+4}}\right)^2.$$

Consider the ring $H(n, b)$ whose complementary components are

$$D_0 = \{(x, 0, 0, \ldots, 0) : -b \leq x \leq b\}, \text{ where } 0 < b < 1;$$

$$D_1 = \left\{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i^2 \geq 1\right\}.$$

In words, $H(n, b)$ is the unit n-ball, slit symmetrically from $-b$ to b through the origin. Then $H(2, b(r))$, $A(1, r)$, and $G(2, a(r))$ are conformally equivalent. Results for $\mathrm{mod}(G(n, a))$ conceivably have analogs for $\mathrm{mod}(H(n, b))$. See also [17, 18].

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8

Constants Associated with Geometry

8.1 Geometric Probability Constants

We will only briefly touch the large subject of geometric probability [1] but enough to introduce a few questions.

Suppose a point is randomly selected from the n-dimensional unit cube. The expected Euclidean distance to the cube center, $\delta(n)$, has the following closed-form expressions [2–7]:

$$\delta(1) = \frac{1}{4}, \quad \delta(2) = \frac{1}{6}\left(\sqrt{2} + \ln(1+\sqrt{2})\right) = 0.3825978582\ldots,$$

$$\delta(3) = \frac{1}{48}\left(6\sqrt{3} + 12\ln(2+\sqrt{3}) - \pi\right) = 0.4802959782\ldots.$$

It possesses the following bounds (for all n) and asymptotics:

$$\frac{1}{4}n^{\frac{1}{2}} \le \delta(n) \le \frac{1}{2}\left(\frac{n}{3}\right)^{\frac{1}{2}}, \quad \delta(n) \sim \frac{1}{2}\left(\frac{n}{3}\right)^{\frac{1}{2}}$$

(in particular, $\delta(n)$ is unbounded). Are closed-form expressions for $\delta(4) = 0.5609498093\ldots$ and $\delta(5) = 0.6312033175\ldots$ possible? Incidently, $2\delta(n)$ is the mean distance from the point to an arbitrary corner of the n-cube. If we examine the analogous problem corresponding to the n-dimensional unit ball [8–14], the expected Euclidean distance is $n/(n+1)$ (which is bounded, of course).

Suppose two points are independently and uniformly chosen from the unit n-cube. The expected Euclidean distance between them, $\Delta(n)$, is

$$\Delta(1) = \frac{1}{3}, \quad \Delta(2) = \frac{1}{15}\left(\sqrt{2} + 2 + 5\ln(1+\sqrt{2})\right) = 0.5214054331\ldots,$$

$$\Delta(3) = \frac{1}{105}\left(4 + 17\sqrt{2} - 6\sqrt{3} + 21\ln(1+\sqrt{2}) + 42\ln(2+\sqrt{3}) - 7\pi\right)$$
$$= 0.6617071822\ldots$$

and has corresponding bounds and asymptotics:

$$\frac{1}{3}n^{\frac{1}{2}} \le \Delta(n) \le \left(\frac{n}{6}\right)^{\frac{1}{2}}, \quad \Delta(n) \sim \left(\frac{n}{6}\right)^{\frac{1}{2}}.$$

Are closed-form expressions for $\Delta(4) = 0.7776656535\ldots$ and $\Delta(5) = 0.8785309152\ldots$ possible? Much more is known for the unit n-ball analog of this problem: The mean distance in this scenario is a ratio of gamma function values and tends to $\sqrt{2}$ as $n \to \infty$. The fact that, as n grows, the limiting $\Delta(n)$ is finite for n-balls but infinite for n-cubes is very interesting! Additionally, the variance of the distance separating the points in the n-ball tends to zero. Thus, for large n, the separation between two random points is almost always equal to the distance between the extremities of two orthogonal radii [1].

We mention that the expected reciprocal Euclidean distance between two random points in the unit 3-cube is [15, 16]

$$2\left(\frac{\sqrt{2}+1-2\sqrt{3}}{5} - \frac{\pi}{3} - \ln\left[(\sqrt{2}-1)(2-\sqrt{3})\right]\right) = 1.8823126444\ldots,$$

and clearly generalization is possible.

Suppose instead that three (rather than two) points are randomly selected in the unit n-cube. What is the probability, $\Pi(n)$, that the three points form an obtuse triangle? Langford [17, 18] proved that

$$\Pi(2) = \frac{97}{150} + \frac{\pi}{40} = 0.7252064830\ldots,$$

but no one has performed a similar calculation for $\Pi(n)$, $n > 2$. Again, much more is known for the n-ball analog of this problem [19, 20]. Random triangles in the n-ball tend to be acute for large n since most of the volume of the n-ball is near its surface [21]. In fact, such random triangles tend to be approximately equilateral and thus have small probability of being obtuse. See [22–28] for related discussion.

Suppose instead that N points $p_1, p_2, \ldots, p_N$ are randomly selected in the unit n-cube. Let C denote the convex hull of $p_1, p_2, \ldots, p_N$; that is,

$$C = \left\{\sum_{j=1}^{N} \lambda_j p_j : \lambda_j \ge 0 \text{ for all } j \text{ and } \sum_{j=1}^{N} \lambda_j = 1\right\}$$

is the intersection of all convex sets containing $p_1, p_2, \ldots, p_N$. Then,

- the expected n-dimensional volume, $\mathrm{E}(V_n(N))$, of C,
- the expected $(n-1)$-dimensional surface area, $\mathrm{E}(S_n(N))$, of C, and
- the expected number of vertices, $\mathrm{E}(P_n(N))$, on the (polygonal) boundary of C

satisfy

$$\lim_{N\to\infty} \frac{N}{\ln(N)}(1 - \mathrm{E}(V_2(N))) = \frac{8}{3},$$

$$\lim_{N\to\infty} \sqrt{N}(4 - \mathrm{E}(S_2(N))) = 2\sqrt{\pi}M = 4.2472965459\ldots,$$

$$\lim_{N\to\infty} \mathrm{E}(P_2(N)) - \frac{8}{3}\ln(N) = \frac{8}{3}(\gamma - \ln(2)) = -0.3091507084\ldots$$

according to Rényi & Sulanke [29–39], where γ denotes the Euler–Mascheroni constant [1.5] and M is Gauss' lemniscate constant [6.1]. Affentranger & Wieacker [40,41] obtained asymptotics for $V_n(N)$ and $P_n(N)$ for $n \geq 3$. Cabo & Groeneboom [42–45] demonstrated that

$$\lim_{N\to\infty} N\,\mathrm{Var}(S_2(N)) = 4(J - I^2) = 0.9932\ldots,$$

where

$$I = \sqrt{\frac{\pi}{8}}\left[2 - \int_1^\infty \left(\sqrt{1+s^2} - s\right) s^{-3/2} ds\right] = \frac{\sqrt{\pi}}{2}M = 1.0618241364\ldots,$$

$$\begin{aligned} J = {} & 2 - 4\int_1^\infty \left(\sqrt{1+s^2} - s\right)\varphi(s-1)\,ds + \frac{4}{5}\int_1^\infty \left(\sqrt{1+s^2} - s\right)^2 s^{-2} ds \\ & + \frac{1}{4}\int_1^\infty \int_1^t \left(\sqrt{1+s^2} - s\right)\left(\sqrt{1+t^2} - t\right)\psi\left(\frac{t}{s} - 1\right) s^{-3} ds\,dt \\ & + \frac{1}{8}\int_1^\infty \int_1^\infty \left(\sqrt{1+s^2} - s\right)\left(\sqrt{1+t^2} - t\right)\psi\left(s\,t - 1\right)\,ds\,dt \\ = {} & 1.37575\ldots, \end{aligned}$$

and

$$\varphi(s) = \frac{1}{2(s+1)^2} - \frac{1}{4s(s+1)} + \frac{1}{4s}\frac{\arctan(\sqrt{s})}{\sqrt{s}},$$

$$\psi(s) = \frac{15}{s^3} + \frac{1}{s^2} - \left(\frac{15}{s^3} + \frac{6}{s^2} - \frac{1}{s}\right)\frac{\arctan(\sqrt{s})}{\sqrt{s}}.$$

No higher-dimensional analog of this result is known.

Suppose instead that N *lines* are randomly drawn in the square [46,47]. The average number of regions into which the lines divide the square is given by [48,49]

$$\frac{N(N-1)\pi}{16} + N + 1,$$

which is another fascinating occurrence of Archimedes' constant π in geometry. The average number of regions into which N random *planes* divide the cube is

$$\frac{(2N+23)N(N-1)\pi}{324}+N+1.$$

What are the higher dimensional analogs of these results? Related material on the maximum possible number of regions appears in [50–52].

We close with a different type of problem (not actually from geometric probability). Here the issue is existence. Is there a positive constant c such that any measurable plane set of area c must contain the vertices of a triangle of area exactly equal to 1? Erdös [53, 54] wondered if c might be as small as $4\pi/(3\sqrt{3})$ but no progress has been made on determining whether c is even finite. A related question, concerning whether every convex region in the Euclidean plane with area 1 can be inscribed in a triangle of area at most equal to 2, was answered long ago [55, 56]. The three-dimensional analog remains unsolved [57].

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8.2 Circular Coverage Constants

The problem of completely covering the unit interval $[0, 1]$ by N smaller equal subintervals is trivial: Tile the interval with subintervals of length $1/N$. The only necessary overlap occurs at boundary points of the tiling.

The problem of completely covering the planar unit disk D by N smaller equal subdisks is harder. Here overlap is substantial and contributes to the difficulty of solution. Let $r(N)$ denote the minimum radius for which there exists a covering. If D is covered, then in particular its boundary C (the unit circle) must be covered. To cover a unit circular subarc of length $2\pi/N$ requires a disk of radius at least $\sin(\pi/N)$; therefore we have the bound $r(N) \geq \sin(\pi/N)$. Equality occurs, in fact, for $N = 2$, 3, and 4 (see Table 8.1). The case for $N = 7$ is also straightforward: A regular hexagon inscribed in C has edges of length 1, so at least six disks of radius $1/2$ are needed to cover C. A seventh disk of radius $1/2$ is then sufficient to cover the remaining central portion of D.

The case for $N = 5$ is the first nontrivial case. Neville [1,2] provided the first known published solution (see Figure 8.1), although in the last step the value $r(5)$ was given incorrectly. Early editions of [3] repeated his error. One correctly obtains $r(5) = 0.6093828640\ldots$ as the value of $\cos(\theta + \varphi/2)$, where θ and φ are solutions of

Table 8.1. *Minimum Common Radius $r(N)$ of N Subdisks Covering the Unit Disk*

N	1	2	3	4	5	6
$r(N)$	1	1	$\frac{\sqrt{3}}{2} = 0.866025\ldots$	$\frac{\sqrt{2}}{2} = 0.707106\ldots$	$0.609382\ldots$	$0.555905\ldots$
N	7	8	9	10	11	12
$r(N)$	$\frac{1}{2} = 0.5$	$0.445041\ldots$	$0.414213\ldots$	$0.394930\ldots$	$0.380006\ldots$	$0.361103\ldots$

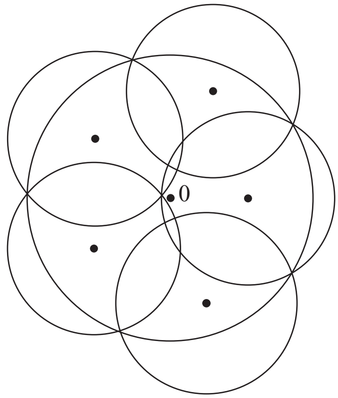

Figure 8.1. Neville's minimal configuration of five circles. It is asymmetric since three disks pass near 0 but two do not.

the following nonlinear system of four equations in four unknowns:

$$2\sin(\theta) - \sin(\theta + \tfrac{1}{2}\varphi + \psi) - \sin(\psi - \theta - \tfrac{1}{2}\varphi) = 0,$$

$$2\sin(\varphi) - \sin(\theta + \tfrac{1}{2}\varphi + \chi) - \sin(\chi - \theta - \tfrac{1}{2}\varphi) = 0,$$

$$\begin{aligned}2\sin(\theta) &+ \sin(\chi + \theta) - \sin(\chi - \theta) - \sin(\psi + \varphi) - \sin(\psi - \varphi)\\ &- 2\sin(\psi - 2\theta) = 0,\end{aligned}$$

$$\begin{aligned}\cos(2\psi - \chi + \varphi) &- \cos(2\psi + \chi - \varphi) - 2\cos(\chi) + \cos(2\psi + \chi - 2\theta)\\ &+ \cos(2\psi - \chi - 2\theta) = 0.\end{aligned}$$

A different characterization was provided by Bezdek [4–6]: $r(5)^{-1}$ is the largest real zero of the polynomial

$$a(y)x^6 - b(y)x^5 + c(y)x^4 - d(y)x^3 + e(y)x^2 - f(y)x + g(y)$$

maximized over all y, subject to the constraints $\sqrt{2} < x < 2y + 1$, $-1 < y < 1$, where

$$a(y) = 80y^2 + 64y, \quad b(y) = 416y^3 + 384y^2 + 64y,$$

$$c(y) = 848y^4 + 928y^3 + 352y^2 + 32y,$$

$$d(y) = 768y^5 + 992y^4 + 736y^3 + 288y^2 + 96y,$$

$$e(y) = 256y^6 + 384y^5 + 592y^4 + 480y^3 + 336y^2 + 96y + 16,$$

$$f(y) = 128y^5 + 192y^4 + 256y^3 + 160y^2 + 96y + 32, \quad g(y) = 64y^2 + 64y + 16.$$

Neville [2] knew that $r(5)$ is an algebraic number, for he wrote the following sentence about his system of equations: "It is evident that these particular equations are algebraic and even rational in the tangents of the angles $\theta/2$, $\varphi/4$, $\psi/2$, $\chi/2$, so that an algebraic equation can be found for $\cos(\theta + \varphi/2) \ldots$." Melissen [7] and Zimmermann [8] independently obtained the minimal polynomial of $r(5)$:

$$1296x^8 + 2112x^7 - 3480x^6 + 1360x^5 + 1665x^4 - 1776x^3 + 22x^2 - 800x + 625;$$

however, they may not have been the first to achieve this.

Zahn [9] computed $r(N)$ for $N = 6$ and $8 \le N \le 10$ by computer experimentation. Bezdek [10] numerically obtained $r(6) = 0.5559052114\ldots$ as reported in [5–7]; conceivably he may have found a polynomial optimization characterization of $r(6)$ analogous to $r(5)$. Nagy [11] and Krotoszynski [12] conjectured that, for $8 \le N \le 10$,

$$r(N) = \left(1 + 2\cos\left(\frac{2\pi}{N-1}\right)\right)^{-1} = \begin{cases} 0.4450418679\ldots & \text{if } N = 8, \\ \sqrt{2} - 1 = 0.4142135623\ldots & \text{if } N = 9, \\ 0.3949308436\ldots & \text{if } N = 10, \end{cases}$$

and Fejes Tóth [13] succeeded in proving the formulas for $r(8)$ and $r(9)$. Evidence for the $r(10)$ formula was given by Melissen [7], who also provided an excellent survey of the subject. More recently, Faugère & Zimmermann [14] discovered the minimal polynomial for $r(6)$:

$$\begin{aligned} &7841367x^{18} - 3344997x^{16} + 62607492x^{14} - 63156942x^{12} + 41451480x^{10} \\ &\quad - 19376280x^8 + 5156603x^6 - 746832x^4 + 54016x^2 + 3072. \end{aligned}$$

All cases $r(N)$ for $N \ge 10$ remain open; we mention that $r(11) < (1 + 2\cos(\pi/5))^{-1}$ and also the conjecture

$$r(12) = \frac{1}{3}\left(1 + (1 + 3\sqrt{57})^{\frac{1}{3}} - 8(1 + 3\sqrt{57})^{-\frac{1}{3}}\right) = 0.3611030805\ldots$$

due to Melissen & Schuur [7].

There are some interesting "inverse" results due to Kerschner [15] and Verblunsky [16]. For example, if we let $N(\varepsilon)$ denote the smallest number of disks of radius ε needed to cover D, the limit of the ratio of the area of D to the total area of the disks,

$$\lim_{\varepsilon \to 0^+} \frac{\pi}{(\pi\varepsilon^2)N(\varepsilon)} = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^2 N(\varepsilon)} = \frac{3\sqrt{3}}{2\pi} = 0.8269933431\ldots,$$

can be thought of as measuring the *asymptotic efficiency* of the covering. If one replaces the unit disk D by a square, one can be even more precise.

Here is a related problem. We can cover the unit interval by intervals of length $1/2$, $1/4, 1/8, 1/16, 1/32, \ldots$ in the natural way. Moreover, the common ratio $1/2$ cannot be made smaller. What is the two-dimensional analog of this result? Eppstein [17] found that D could be covered by smaller disks of radii s^k, $k = 1, 2, 3, \ldots$, for $s = 0.77$ but evidently not for $s = 0.765$. A more precise estimate of the smallest $s \le 0.77$ would be good to see.

The problem of covering a unit square by N smaller equal disks is surveyed in [7, 18]. The dual problem of *packing* disks in a unit disk [1, 7, 19–22] or square [1, 7, 23–28] has attracted much attention, but we will say only a few words. Let $t(N)$ denote the greatest

Table 8.2. *Maximum Common Radius $t(N)$ of N Subdisks Packing the Unit Square*

N	2	3	4	5
$t(N)$	$\sqrt{2} = 1.414213\ldots$	$\sqrt{6}-\sqrt{2} = 1.035276\ldots$	1	$\frac{\sqrt{2}}{2} = 0.707106\ldots$
N	6	7	8	9
$t(N)$	$\frac{\sqrt{13}}{6} = 0.600925\ldots$	$2(2-\sqrt{3}) = 0.535898\ldots$	$\frac{\sqrt{6}-\sqrt{2}}{2} = 0.517638\ldots$	$\frac{1}{2} = 0.5$

possible minimum distance between N points in the square (see Table 8.2). Computing $t(10) = 0.4212795439\ldots$ was a major obstacle until recently: Schlüter's conjecture [29, 30] has been proven true [31] and here is the minimal polynomial for $t(10)$:

$$\begin{aligned}&1180129x^{18} - 11436428x^{17} + 98015844x^{16} - 462103584x^{15} + 1145811528x^{14}\\&\quad - 1398966480x^{13} + 227573920x^{12} + 1526909568x^{11} - 1038261808x^{10}\\&\quad - 2960321792x^{9} + 7803109440x^{8} - 9722063488x^{7} + 7918461504x^{6}\\&\quad - 4564076288x^{5} + 1899131648x^{4} - 563649536x^{3} + 114038784x^{2}\\&\quad - 14172160x + 819200.\end{aligned}$$

Here also, as an aside, are two elementary problems involving just two circles.

Imagine two overlapping circles, each of radius 1. If the area A of the inner overlap region is equal to the sum of the areas of the two outer crescents, then clearly $A = 2\pi/3$. What is the distance $2u$ between the centers of the two circles? It can be shown that $u = 0.2649320846\ldots$ is the unique root of the equation

$$u\sqrt{1-u^2} + \arcsin(u) = \frac{\pi}{6}$$

in the interval [0, 1]. Is u transcendental? This is called *Mrs. Miniver's problem* [32, 33].

The second problem is called the *grazing goat problem* [34, 35]. A goat is tethered to a post on the perimeter of a circular field of radius 1. How long should the rope be so that the goat can eat exactly half of the grass in the field? One shows that the length, v, of the rope satisfies

$$v\sqrt{4-v^2} - 2(v^2-2)\arccos\left(\frac{v}{2}\right) = \pi,$$

and hence $v = 1.1587284730\ldots$. Is v transcendental? Are v and u algebraically independent?

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8.3 Universal Coverage Constants

Let U denote the class of all sets in the plane of unit diameter. A planar region R is called a **displacement cover** (or **universal cover**) **for** U if it contains a congruent copy of every set in U. That is, each set of unit diameter can be covered by R after suitable translation and rotation [1–6].

Let S denote a class of specified regions in the plane (e.g., the class of all circular disks). Does there exist an element of S that is both a displacement cover for U and has area as small as possible? If yes, define $A(S)$ to be the area of such an element.

For example, if we focus on the class of all circular disks [3, 4], then

$$A(\text{circles}) = \frac{\pi}{3} = 1.0471975511\ldots,$$

the area of a circle of radius $1/\sqrt{3}$. A similar line of reasoning gives that a square region of side 1 will also suffice:

$$A(\text{squares}) = 1.$$

Better still is the class of regular hexagonal regions:

$$A(\text{regular hexagons}) = \frac{\sqrt{3}}{2} = 0.8660254037\ldots.$$

Consider now the class C of all *convex* planar regions. Lebesgue [7] asked about the value of $\mu = A(C)$, that is, the area of the smallest possible convex blanket that covers all sets of unit diameter. The best-known bounds are

$$0.8257117836\ldots = \frac{\pi}{8} + \frac{\sqrt{3}}{4} \le \mu \le \frac{\sqrt{3}}{2} - 2\varepsilon_P - \varepsilon_S - \varepsilon_H \le 0.84413770$$

due to Pál [7], Sprague [8], and Hansen [9, 10]. The lower bound is the area of the convex hull of a circle and an equilateral triangle, both of unit diameter, with the circle centered at the triangle centroid. The upper bound estimates, incrementally improving on each other, are based on cutting corners off Pál's original regular hexagonal cover:

$$\varepsilon_P = \frac{7\sqrt{3}}{12} - 1 \sim 10^{-2}, \quad \varepsilon_S \sim 10^{-3}.$$

Hansen's two improvements on Sprague's upper bound estimate are tiny: $\sim 10^{-19}$ and 10^{-11}. A more dramatic improvement, in [11], from 0.8441 to 0.8430, was conjectural only. One interesting aspect about Hansen's work is his use of computer simulation. For example, he ruled out certain types of configurations by simulation in [10]; it is not clear whether he has withdrawn his 1981 conjecture. As Klee & Wagon [5] wrote, "Progress on this problem, which has been painfully slow in the past, may be even more painfully slow in the future."

For nonconvex covers, Duff [12, 13] constructed a region with area $0.84413570\ldots$, which is smaller than all known convex examples. It is not surprising that nonconvexity can improve matters; see the related discussion in [8.4] and [8.17].

There are many variations on these problems. If we restrict the meaning of *cover* to encompass only translations (rather than displacements, i.e., translations and rotations,

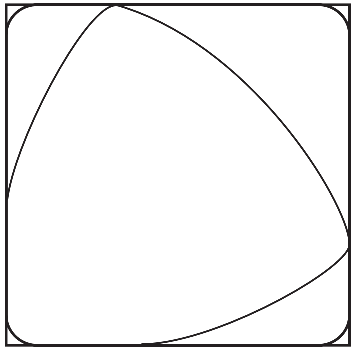

Figure 8.2. A truncated unit square obtained by revolving an inscribed Reuleaux triangle completely within the square and removing the four corner sets not touched.

as we have assumed so far), then the various outcomes are given in [8.3.1]. Also, one can minimize the cover perimeter or mean width rather than area [14–16]. A different sense of minimality – namely, a cover for which no proper subset is a cover – was studied by Eggleston [17] in n dimensions.

There is a discrepancy in the reporting of the upper bound estimate for μ. Meschkowski [2] and Hansen [9] reported Sprague's estimate to be 0.844144, whereas Duff [12] and Klee & Wagon [5] reported the estimate to be 0.84413770. No explanation can be found for this discrepancy.

Finally, we note that early papers on this subject often mistakenly refer to this as *Besicovitch's problem* [18–20].

8.3.1 Translation Covers

A planar region R is called a **translation cover** (or **strong universal cover**) **for** U if each set of unit diameter can be covered by R after suitable translation [5, 14, 21]. No rotations are allowed. Using notation similar to before, $\tilde{A}(\text{circles}) = \pi/3$ by the obvious rotational symmetry of a disk and $\tilde{A}(\text{squares}) = 1$, but the regular hexagon of Pál is *not* a translation cover [21, 22]. What, therefore, is the value of $\tilde{A}(\text{regular hexagons})$?

If C denotes the class of all convex planar regions, then there is a conjecture [5, 15, 16] that

$$\tilde{A}(C) = \frac{\pi}{6} + 2\sqrt{3} - 3 = 0.9877003907\ldots,$$

which is the area of the truncated unit square in Figure 8.2. No rigorous tight bounds on $\tilde{A}(C)$ seem to appear in the literature. See [23] for a curious connection to the Watts square drill bit. More constants associated with Reuleaux triangles are found in [8.10].

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8.4 Moser's Worm Constant

A **worm** is a continuous rectifiable arc of unit length contained in the plane. Let W denote the class of all worms. A planar region R is called a **displacement cover** (or **universal cover**) for W if it contains a congruent copy of every worm in W. That is, each arc of unit length can be covered by R after suitable translation and rotation [1,2].

Let S denote a class of specified regions in the plane (e.g., the class of all circular disks). Does there exist an element of S that is both a displacement cover for W and has area as small as possible? If yes, define $A(S)$ to be the area of such an element.

For example, if we focus on the class of all circular disks [3], then

$$A(\text{circles}) = \frac{\pi}{4} = 0.7853981633\ldots,$$

the area of a circle of diameter 1. It is somewhat more difficult to prove [4, 5] that a square region of diagonal 1 will also suffice:

$$A(\text{squares}) = \frac{1}{2} = 0.5.$$

Over the larger class of rectangular regions [4, 5],

$$A(\text{rectangles}) = \beta\sqrt{1-\beta^2} = 0.3943847688\ldots,$$

the area of a rectangle with sides β and $\sqrt{1-\beta^2}$, where β arises with regard to the broadest curve of unit length [8.4.1]. Better still is the class of semicircular regions [6]:

$$A(\text{semicircles}) = \frac{\pi}{8} = 0.3926990816\ldots,$$

as proved by Meir. Interestingly, the class of equilateral triangular regions remains a mystery. Besicovitch [7] proved that

$$A(\text{equilateral triangles}) \geq \frac{7\sqrt{3}}{27} = 0.4490502094\ldots,$$

the area of the triangle with side $2\sqrt{21}/9$, and thought it likely that equality holds. The conjectured exact expression for A was found by Knox [8]. Any counterexample to this claim, if such a worm exists, must be **zig-zag** in the sense that the worm meets the line segment joining its two endpoints at a third point (possibly more) [9].

Consider now the class C of all *convex* planar regions. **Moser's worm constant** μ is defined to be the value of $A(C)$, that is, the area of the smallest possible convex blanket that covers all worms. The best-known bounds are

$$0.2194626846\ldots = \frac{\beta}{2} \leq \mu \leq 0.27524\ldots$$

as found by Schaer & Wetzel [5, 6] and Poole, Gerriets, Norwood & Laidacker [10–12]. The upper bound is the area of a certain rhombus with portions of two adjacent sides replaced by a circular arc. Some recent unsuccessful attempts have been made to improve the upper bound [13, 14]. Of many conjectures, we mention one in [6, 11, 12]: The circular sector of radius 1 and angle $\pi/6$ covers all possible worms. If true, this would reduce the upper bound on μ to $\pi/12 = 0.261799\ldots$.

There are many variations on these problems. If we restrict worms to be *closed*, that is, with initial point coincident with terminal point, then the results are given in [8.4.2]. If we restrict the meaning of *cover* to encompass only translations (rather than displacements, i.e., translations and rotations, as we have assumed so far), then the various outcomes are given in [8.4.3]. One can minimize the cover perimeter rather than area [15]. Also, one can ask how efficient the cover is, for example, whether the worm is necessarily close to the boundary of the cover, and such an inquiry leads naturally to Bellman's "lost in a forest" problem [16–19].

Here is a related problem [20]: Prove that any worm can be covered by some rectangular blanket of area $1/4$, and that this is the best possible. The question (given a worm, find an element of S that covers it) is similar to the foregoing (find an element of S that covers all worms) but has not received the same amount of attention. Another problem is as follows: Given a worm, show that the maximum possible area of its smallest convex cover is $1/(2\pi) = 0.159154\ldots$. This is attained for a semicircle of unit length [21]. What is the three dimensional analog of this result?

Interesting things happen if we drop the convexity requirement [1,2]. Hansen [22] proposed (without proof) a nonconvex universal cover of area $0.246\ldots$, which is less than the best-known convex cover, but his claim remains unconfirmed. The smallest provable upper bound in this case is $0.26044\ldots$ [23]. Davies [24] constructed nonconvex sets of measure zero that are translation covers for the class of all polygonal arcs in the plane. This is closely allied with the Kakeya–Besicovitch problem [8.17]. Marstrand [25,26], however, proved that any displacement cover for the class of all rectifiable arcs must have positive measure.

8.4.1 Broadest Curve of Unit Length

What is the minimum width of an infinitely long planar strip that contains a congruent copy of every worm in W? Equivalently, fix a worm w for consideration and, for $0 \le \theta \le \pi$, let $d(w,\theta)$ denote the distance between supporting parallel lines at angle θ to the x-axis. Define the **breadth** of w to be the minimum value of $d(w,\theta)$ taken over all θ. Our question becomes: What is the worm of largest breadth?

The answer is a **broadworm** or **caliper**, as first discovered by Zalgaller [17,27,28]. See Figure 8.3. This curve has breadth given exactly by

$$\begin{aligned}\beta = \sup_{w}\min_{\theta} d(w,\theta) &= \frac{1}{2}\left(\frac{\pi}{2} - \varphi - 2\psi + \tan(\varphi) + \tan(\psi)\right)^{-1} \\ &= 0.4389253692\ldots = (2.2782916414\ldots)^{-1},\end{aligned}$$

where the angles φ and ψ are defined by

$$\varphi = \arcsin\left[\tfrac{1}{6} + \tfrac{4}{3}\sin\left(\tfrac{1}{3}\arcsin(\tfrac{17}{64})\right)\right], \quad \psi = \arctan\left(\tfrac{1}{2}\sec(\varphi)\right).$$

It follows immediately that any universal rectangular cover must have both sides $\ge \beta$ (to accommodate the caliper) and diagonal ≥ 1 (to accommodate the unit line segment); proving that the $\beta \times \sqrt{1-\beta^2}$ rectangle is indeed universal requires more work.

Zalgaller [29] also examined the three-dimensional analog of this problem and conjectured that the broadest curve in three-space of unit length has breadth $1/3.921545\ldots = 0.255001\ldots$.

8.4.2 Closed Worms

A **closed worm** is a continuous rectifiable closed curve (with initial point coincident with terminal point) of unit length contained in the plane. As before, we are interested

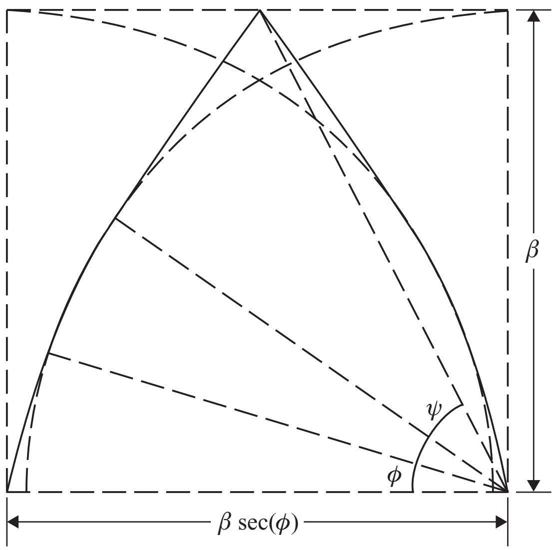

Figure 8.3. A caliper consists of two circular arcs with four tangent segments, configured in a very precise fashion.

in displacement covers of least area. In this more restrictive scenario, we have

$$A'(\text{circles}) = \frac{\pi}{16} = 0.1963495408\ldots,$$

the area of a circle [3, 30, 31] of diameter $1/2$,

$$A'(\text{squares}) = \frac{1}{8} = 0.125,$$

the area of a square [5, 31] of diagonal $1/2$,

$$A'(\text{rectangles}) = \frac{\sqrt{\pi^2 - 4}}{2\pi^2} = 0.1227367657\ldots,$$

the area of a rectangle [5, 31] of sides $1/\pi$ and $\sqrt{\pi^2 - 4}/(2\pi)$, and

$$A'(\text{general triangles}) = \frac{3\sqrt{3}}{4\pi^2} = 0.1316200785\ldots,$$

the area of an equilateral triangle [32, 33] with side $\sqrt{3}/\pi$.

It is curious that so much more is known about the general triangular case for covering closed worms than for covering arbitrary worms (arcs). Here is a related result [34]. The smallest equilateral triangle that can cover every triangle of perimeter 2 has side not 1, but $s = 2/y = 1.0028514266\ldots$, where y is the global minimum of the trigonometric function

$$f(x) = \sqrt{3}\left(1 + \sin\left(\frac{x}{2}\right)\right)\sec\left(\frac{\pi}{6} - x\right)$$

on the interval $[0, \pi/6]$. The constant s also appears in [35] in connection with a more expansive problem.

What can be said about the analog of Moser's worm constant here, that is, the area $\mu' = A'(C)$ of the smallest possible convex blanket that covers all closed worms? Schaer & Wetzel [5] and Chakerian & Klamkin [31] proposed a lower bound equal to the area of the convex hull of a circle of circumference 1 and a line segment of length $1/2$ with midpoint at the circle center:

$$\mu' \geq \frac{1}{4\pi^2}\left(\sqrt{\pi^2 - 4} + \pi - 2\arccos\left(\frac{2}{\pi}\right)\right) = 0.0963296165\ldots.$$

More recently, Füredi & Wetzel [35] gave improved bounds $0.09666 \leq \mu' \leq 0.11754$, where the upper bound comes from the area of the best rectangle (mentioned earlier) with one small corner clipped off.

Here is a related problem from [31] due to Schaer: Prove that any closed worm can be covered by some rectangular blanket of area $1/\pi^2$, and that this is the best possible. The question (given a worm, find an element of S that covers it) is similar to the foregoing (find an element of S that covers all worms).

8.4.3 Translation Covers

A planar region R is called a **translation cover** (or **strong universal cover**) for W if each worm in W can be covered by R after suitable translation. No rotations are allowed. Since there are two types of worms, we study these separately. For arbitrary worms (arcs), let us consider only the class C of all convex planar regions. In this scenario, we have a complete solution due to Pál [36]:

$$\tilde{\mu} = \tilde{A}(C) = \frac{\sqrt{3}}{3} = 0.5773502692\ldots,$$

the area of an equilateral triangle of height 1. This scenario is perhaps the simplest of all.

For closed worms, we have

$$\hat{A}(\text{circles}) = \frac{\pi}{16} = 0.1963495408\ldots$$

by the obvious rotational symmetry of a disk [3, 30, 31],

$$\hat{A}(\text{general triangles}) = \frac{\sqrt{3}}{9} = 0.1924500897\ldots,$$

the area of an equilateral triangle [32, 33] with side $2/3$,

$$\hat{A}(\text{rectangles}) = \frac{1}{4} = 0.25,$$

the area of a square with side $1/2$, and

$$0.1554479088\ldots \leq \hat{\mu} = \hat{A}(C) \leq 0.16526\ldots,$$

for the convex case, owing to Wetzel [6] and Bezdek & Connelly [15].

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8.5 Traveling Salesman Constants

Consider n distinct points in the d-dimensional unit cube. Of all $(n-1)!/2$ closed paths (or **tours**) passing through each point precisely once, what is the length $L_d(n)$ of the shortest such path?

Determining $L_d(n)$, the minimum tour-length, is known as the **traveling salesman problem** (TSP). This is one of the best-known combinatorial optimization problems, dominating fields such as operations research, algorithm development, and complexity theory. Its solution is difficult because it cannot be computed in polynomial time, that is, the problem is NP-hard.

We nevertheless encounter some interesting asymptotics: There is a smallest constant α_d such that

$$\limsup_{n\to\infty} \frac{L_d(n)}{n^{(d-1)/d}} \le \alpha_d, \quad \alpha'_d = \frac{\alpha_d}{\sqrt{d}}$$

for all optimal tours in the cube, and there is another constant β_d such that

$$\lim_{n\to\infty} \frac{L_d(n)}{n^{(d-1)/d}} = \beta_d, \quad \beta'_d = \frac{\beta_d}{\sqrt{d}}$$

for *almost all* optimal tours in the cube, in the sense that the limit fails only for a negligible (measure-zero) subset of the tours. These constants were first examined by Beardwood, Halton & Hammersley [1,2]. Rigorous bounds are listed in Table 8.3 [3–9].

It is known that [10–14]

$$\lim_{d\to\infty} \beta'_d = \frac{1}{\sqrt{2\pi e}} = 0.2419707245\ldots,$$

$$\frac{1}{\sqrt{2\pi e}} \le \lim_{d\to\infty} \alpha'_d \le \frac{2(3-\sqrt{3})\theta}{\sqrt{2\pi e}} = 0.40509\ldots,$$

where

$$\frac{1}{2} \le \theta = \lim_{d\to\infty} \theta_d^{\frac{1}{d}} \le 0.66019$$

Table 8.3. *Bounds on Traveling Salesman Constants α'_d and β'_d*

d	Lower Bound for β'_d	Upper Bound for β'_d	Lower Bound for α'_d	Upper Bound for α'_d
2	0.44194	0.6508	0.75983	0.98398
3	0.37313	0.61772	0.64805	0.90422
4	0.34207	0.55696	0.5946	0.8364

and θ_d is the best sphere packing density in d-space [8.7]. Even if someday the upper bound $2^{-0.59905d+o(d)}$ for θ_d is improved to $2^{-d+o(d)}$, as is believed to be true [15], the upper bound for $\lim_{d\to\infty} \alpha'_d$ will be reduced only to 0.30681. New insights will be required to evaluate this limit exactly [13].

Nonrigorous numerical estimates of β_d, due to Johnson, McGeoch & Rothberg [16] and Percus & Martin [17, 18], give

$$\beta_2 = 0.7124\ldots, \quad \beta_3 = 0.6979\ldots, \quad \beta_4 = 0.7234\ldots.$$

The fact that earlier estimates of β_2 do not agree well may be connected with finite size effects associated with the different experimental methods of computation. Another recent estimate of β_2 is $0.714\ldots$, due to Applegate, Cook & Rohe [19]. This might indicate that Norman & Moscato's [20] conjectured expression for β_2 (based on a fractal space-filling curve),

$$\beta_2 = \frac{4(1+2\sqrt{2})\sqrt{51}}{153} = 0.7147827007\ldots,$$

is justified; it surely indicates the need to assess the quality of random generations underlying TSP simulations.

If the n points are independently and uniformly distributed in the unit square, then the length $\Lambda_2(n)$ of a *random* (not necessarily optimal) tour satisfies [21]

$$\lim_{n\to\infty} \frac{\mathrm{E}(\Lambda_2(n))}{n} = \frac{1}{15}\left(\sqrt{2}+2+5\ln(1+\sqrt{2})\right) = 0.521405433\ldots,$$

where E denotes both the average over all tours and the average over all point sets. The exact expression for $0.5214\ldots$ is due to Ghosh [22] and is discussed further in [8.1]. Note that $\mathrm{E}(\Lambda_2(n))$ increases on the order of n whereas $L_2(n)$ typically increases on the order of $\sqrt{n}$.

A more precise version of $\lim_{d\to\infty} \beta'_d$ has been conjectured [18]:

$$\beta_d = \sqrt{\frac{d}{2\pi e}}(\pi d)^{\frac{1}{2d}}\left[1 + \frac{2-\ln(2)-2\gamma}{d} + O\left(\frac{1}{d^2}\right)\right],$$

where γ denotes the Euler–Mascheroni constant [1.5]. The basis for this formula is known as the random links TSP, a special case of which we will discuss momentarily.

8.5.1 Random Links TSP

Let K_n be the complete graph on n vertices, that is, every pair of distinct vertices determines an edge. We have removed the ambient d-dimensional space and hence any metric from this setting. Assign independently to each edge a Uniform [0, 1] random variable called a **length**. Observe that lengths are not distances in the usual sense since the triangle inequality is not satisfied. Of all $(n-1)!/2$ tours passing through each vertex precisely once, we can determine the shortest such path, with minimum sum of lengths $L(n)$, and define

$$\lim_{n\to\infty} L(n) = \beta \quad \text{with probability } 1.$$

Krauth & Mézard [23] nonrigorously obtained an analytical expression for β via the cavity method:

$$\beta = \frac{1}{2}\int_{-\infty}^{\infty} f(x)(1+f(x))\exp(-f(x))\,dx = 2.0415\ldots = 2(1.0208\ldots),$$

where $f(x)$ is the solution of the integral equation

$$f(x) = \int_{-x}^{\infty} (1+f(y))\exp(-f(y))\,dy.$$

In actuality, this is just one scenario (corresponding to $d = 1$) of a d-parametrized family of random link approximations to the d-dimensional Euclidean TSP [16–18, 24].

8.5.2 Minimum Spanning Trees

Let us return to the familiar setting of n distinct points in the unit d-cube. Denote the set of points by V. A **minimum spanning tree** (MST) is a connected graph [5.6] with vertex-set V that has smallest possible length $L_d(n)$ (meaning the sum of edge-lengths in the usual Euclidean sense). Define

$$\lim_{n\to\infty} \frac{L_d(n)}{n^{(d-1)/d}} = \beta_d \quad \text{with probability 1.}$$

Numerical estimates [25, 26] and theoretical results [11] include

$$\beta_2 = 0.6331\ldots, \quad \beta_3 = 0.6232\ldots, \quad \beta_d \sim \sqrt{\frac{d}{2\pi e}} \text{ as } d \to \infty.$$

It is remarkable that an exact (but complicated) expression for β_d exists [27, 28]. We give the formula only for the case $d = 2$. Let Δ_i denote the set of all points $\{x_1, x_2, \ldots, x_{i-1}\}$ in the plane such that the disks D_j of center x_j and radius $1/2$, $0 \le j \le i-1$, form a connected set, where $x_0 = 0$. Define $g_i(x_1, x_2, \ldots, x_{i-1})$ to be the area of $\bigcup_{j=0}^{i-1} D_j$; then

$$\beta_2 = \frac{1}{2} + \frac{1}{2}\sum_{i=2}^{\infty} \frac{\Gamma(i-\frac{1}{2})}{i!} \int_{\Delta_i} g_i(x_1, x_2, \ldots, x_{i-1})^{-i+\frac{1}{2}}\, dx_1\, dx_2 \ldots dx_{i-1}.$$

Using the first five terms of this series, we can obtain a rigorous lower bound $\beta_2 \ge 0.600822$ [27].

Given a minimum spanning tree, we can study characteristics other than $L_d(n)$. Consider as an example $\tilde{L}_d(n)$, the sum of squared edge-lengths, and define

$$\lim_{n\to\infty} \frac{\tilde{L}_d(n)}{n^{(d-1)/d}} = \tilde{\beta}_d \quad \text{with probability 1.}$$

The existence of $\tilde{\beta}_d$ was proved by Aldous & Steele [29, 30]; numerical estimates include $\tilde{\beta}_2 = 0.4769\ldots$ (which is often called **Bland's constant**) and $\tilde{\beta}_3 = 0.4194\ldots$

[26]. An exact expression for $\tilde{\beta}_d$ can be found as previously [27], with a rigorous lower bound $\tilde{\beta}_2 \geq 0.401\ldots$.

The sum of squared edge-lengths parameter $\tilde{L}_d(n)$ is also interesting for TSP, given an optimal tour. Although an existence proof for $\tilde{\beta}_d$ is not known, specific point configurations can be constructed so that [31–33]

$$\frac{\tilde{L}_d(n)}{n^{(d-1)/d}} > c_d \ln(n)$$

as $n \to \infty$ for some $c_d > 0$; hence $\tilde{\alpha}_d$ definitely does *not* exist. Other variations abound. If we minimize $\tilde{L}_d(n)$ rather than $L_d(n)$ when computing optimal tours, a different path is often determined (because of the power weighting) and the worst-case constant [34–37]

$$\limsup_{n\to\infty} \frac{\tilde{L}_d(n)}{n^{(d-2)/d}} \leq \hat{\alpha}_d$$

is 4 when $d = 2$. Yukich [38, 39] proved that the corresponding average-case constant $\hat{\beta}_d$ exists as well, but the value of $\hat{\beta}_2$ is open.

For K_n, the complete graph on n vertices with independent Uniform [0, 1] random edge-lengths, consider the MST with sum of lengths $L(n)$. Frieze [40–42] demonstrated that

$$\lim_{n\to\infty} L(n) = \zeta(3) = 1.2020569031\ldots \quad \text{in probability}$$

where $\zeta(3)$ is Apéry's constant [1.6], a beautiful result! Janson [43] showed that $\sqrt{n}(L(n) - \zeta(3))$ is asymptotically Normal $(0, \sigma^2)$ with

$$\sigma^2 = \frac{\pi^4}{45} - 2\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{(i+k-1)!k^k(i+j)^{i-2}j}{i!k!(i+j+k)^{i+k+2}} = 1.6857\ldots$$

but no simplification of this constant seems possible. Another relevant occurrence of $\zeta(3)$ is in [44].

8.5.3 Minimum Matching

Again, we consider n distinct points in the unit d-cube, with the additional assumption that n is even. A **matching** is a (disconnected) graph consisting of $n/2$ edges such that each of the n points is met by exactly one edge. A **minimum matching** (MM) is a matching of smallest possible length $L_d(n)$ (meaning the sum of edge-lengths in the usual Euclidean sense). Define β_d as before; the planar case β_2 is often called **Papadimitriou's constant** [45, 46]. Numerical estimates [47–54] and theoretical results [11] include

$$\beta_2 = 0.3104\ldots, \quad \beta_3 = 0.3172\ldots, \quad \beta_d \sim \frac{1}{2}\sqrt{\frac{d}{2\pi e}} \text{ as } d \to \infty.$$

The corresponding worst-case constant α_2 satisfies $0.537 \leq \alpha_2 \leq 0.707$ [47].

For K_n, the complete graph on n vertices with independent Uniform [0, 1] random edge-lengths, consider the MM with sum of lengths $L(n)$. Mézard & Parisi [55–57] identified $\beta = \pi^2/12 = 0.8224670334\ldots$ via the replica method (plus an integral equation simpler than that for $f(x)$ earlier), and Aldous [58] found a rigorous proof. Experimental verification appears in [53, 54]. As before, this is just one scenario (corresponding to $d = 1$) of a d-parametrized family of random link approximations to the d-dimensional Euclidean MM problem.

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8.6 Steiner Tree Constants

Let P denote a set of n points in d-dimensional space. Define

- the **Steiner minimal tree** (SMT) of P to be the shortest connected graph [5.6] that connects P, and
- the **minimum spanning tree** (MST) of P to be the shortest connected graph with vertex-set P that connects P.

Let P_n denote the n vertices of a regular planar polygon with n sides. Figures 8.4 and 8.5 show that, for MSTs, only inter-vertex line segments are permitted, whereas for SMTs,

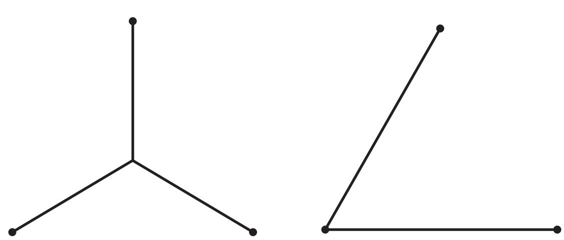

Figure 8.4. The SMT and MST of P_3.

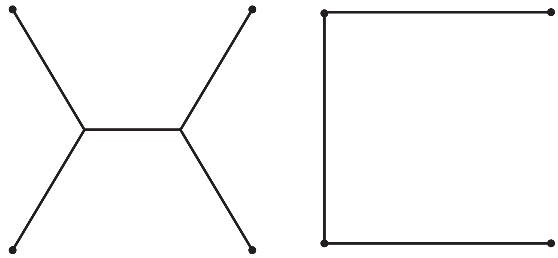

Figure 8.5. The SMT and MST of P_4.

additional vertices can be added to optimize the tree (hence the latter are more difficult to compute, because infinitely many vertex locations are available). If $|G|$ denotes the total edge length of a graph G, then clearly

$$\frac{|\,\mathrm{SMT}(P_3)|}{|\,\mathrm{MST}(P_3)|} = \frac{\sqrt{3}}{2} = 0.866\ldots, \quad \frac{|\,\mathrm{SMT}(P_4)|}{|\,\mathrm{MST}(P_4)|} = \frac{1+\sqrt{3}}{3} = 0.910\ldots.$$

Incidently, SMT(P_5) similarly consists of three additional vertices (called **Torricelli** or **Steiner** points), each the intersection of three edges meeting at 120°, but SMT(P_n) = MST(P_n) for $n \geq 6$. This can be confirmed by soap-film experiments as with minimum area solutions of *Plateau's problem* [1].

For an arbitrary set P, it is relatively easy to determine MST(P); therefore we are interested in the value

$$\rho_d = \inf_{n,\,P} \frac{|\,\mathrm{SMT}(P)|}{|\,\mathrm{MST}(P)|},$$

the infimum being over all n-point sets in d-dimensional space, over all positive integers n. The **Steiner ratio** ρ_d indicates how much the total length of an MST can be decreased by allowing Steiner points. Point sets achieving this infimum may be regarded as "possessing the most shortcuts" [2–5].

Du & Hwang [6] proved Gilbert & Pollak's [7] conjecture that

$$\rho_2 = \frac{\sqrt{3}}{2} = 0.8660254037\ldots,$$

and Smith & Smith [8, 9] proved that

$$\rho_3 \leq s_3 = \frac{3\sqrt{3}+\sqrt{7}}{10} = 0.7841903733\ldots$$

by use of a set P called the **3-sausage** (whose points are evenly spaced along a circular helix; see [10–13]). They provided extensive heuristic evidence that $\rho_3 = s_3$, but a rigorous proof is not known. The best lower bound for ρ_3, in fact, for any ρ_d, is [14, 15]

$$\rho_d \geq \frac{2+x-\sqrt{x^2+x+1}}{\sqrt{3}} = 0.6158277481\ldots,$$

where x is the unique positive root of

$$128x^6 + 456x^5 + 783x^4 + 764x^3 + 408x^2 + 108x - 28 = 0.$$

Let us discuss upper bounds in more detail. Define the d-**simplex** to be the natural generalization of the equilateral triangle for $d = 2$ and the regular tetrahedron for $d = 3$. Chung & Gilbert [16] computed bounds for the Steiner ratio r_d in this case and showed that

$$\limsup_{d\to\infty} r_d \le \frac{\sqrt{3}}{4-\sqrt{2}} = 0.6698352124\ldots.$$

Smith [15, 17] conjectured that *limit supremum* can be replaced here by *limit* and that this inequality is in fact equality. It is known that $r_d > \rho_d$ if $d \ge 3$ and, for example,

$$r_3 = \frac{1+\sqrt{6}}{3\sqrt{2}} = 0.813053\ldots, \quad r_4 = \frac{\sqrt{3}+\sqrt{5}+2\sqrt{6}}{8\sqrt{2}} = 0.783748\ldots.$$

The Steiner ratios s_d corresponding to analogous higher-dimensional d-**sausages** are also known to satisfy $s_d < r_d$ for $d \ge 3$. s_d is strictly decreasing as a function of d. We do not, however, know the numerical value of $\lim_{d\to\infty} s_d$ nor whether $\rho_d = s_d$ for any $d \ge 3$. Du & Smith [15] thought that equality might possibly be true for small d but not for large $d \ge 15$. For example, $s_4 = 0.7439856178\ldots$ has the minimal polynomial [18]

$$900s^8 - 1863s^6 + 2950s^4 - 1511s^2 + 164,$$

and similar progress on evaluating $s_5 = 0.7218106748\ldots$ is perhaps not faraway.

Here is a different viewpoint (similar to our discussion of MSTs in [8.5]). If the n points of P are all constrained to fall within the unit square, then there exist constants c and C for which

$$0.930\sqrt{n} + c < \left(\tfrac{3}{4}\right)^{\frac{1}{4}}\sqrt{n} + c \le |\,\mathrm{SMT}(P)| < 0.995\sqrt{n} + C$$

for all n, as found by Chung & Graham [19–21]. If the points are instead constrained to fall within the unit d-cube, then

$$|\,\mathrm{SMT}(P)| \le \sqrt{\frac{d}{2\pi e}}\, n^{1-\frac{1}{d}}$$

as $n \to \infty$, where d is sufficiently large [2, 22]. Improvements of both asymptotic results would be good to see.

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8.7 Hermite's Constants

What is the densest (lattice or non-lattice) packing of equal, non-overlapping spheres in n-dimensional space [1,2]? For $n = 1$, this corresponds to tiling the line with segments of equal length; hence the maximum density Δ_n clearly satisfies $\Delta_1 = 1$. For $n = 2$, the hexagonal lattice packing of circles in the plane gives $\Delta_2 = \pi/\sqrt{12} = 0.9068996821\ldots$, which was first proved by Thue [3,4]. Subsequent proofs were found by Fejes Tóth [5,6] and Segre & Mahler [7]. For $n = 3$, the face-centered cubic packing of spheres in 3-space gives $\Delta_3 = \pi/\sqrt{18} = 0.7404804896\ldots$. This was a well-known conjecture, attributed to Kepler, until it was first proved by Hales [8–10]. What can be said about Δ_n for $n \geq 4$? Can non-lattice packings in 4-space improve upon lattice packings?

If we restrict attention to only lattice packings, then the maximum density δ_n is known for all $n \leq 8$. Let $\omega_n = \pi^{n/2}\Gamma(n/2+1)^{-1}$ be the volume of the unit sphere in n-dimensional space and let

$$\gamma_n = 4\left(\frac{\delta_n}{\omega_n}\right)^{\frac{2}{n}}$$

Table 8.4. *Hermite's Constants δ_n and γ_n^n*

n	δ_n	γ_n^n
1	1	1
2	$\frac{\pi}{2\sqrt{3}} = 0.9068996821\ldots$	$\frac{4}{3}$
3	$\frac{\pi}{3\sqrt{2}} = 0.7404804896\ldots$	2
4	$\frac{\pi^2}{16} = 0.6168502750\ldots$	4
5	$\frac{\pi^2}{15\sqrt{2}} = 0.4652576133\ldots$	8
6	$\frac{\pi^3}{48\sqrt{3}} = 0.3729475455\ldots$	$\frac{64}{3}$
7	$\frac{\pi^3}{105} = 0.2952978731\ldots$	64
8	$\frac{\pi^4}{384} = 0.2536695079\ldots$	256

denote **Hermite's constant** of order n. Table 8.4 summarizes what is known for small n [3, 11, 12]. Also, for sufficiently large n, it can be proved that

$$-1 \leq \frac{\log_2(\delta_n)}{n} \leq \frac{\log_2(\Delta_n)}{n} \leq -0.59905\ldots, \quad \frac{1}{2\pi e} \leq \frac{\gamma_n}{n} \leq \frac{1.74338\ldots}{2\pi e}.$$

The expressions for the bounds $c = -0.59905\ldots$ and $4^{1+c} = 1.74338\ldots$ are complicated and are due to Kabatyanskii & Levenshtein [1, 13, 14]. It is believed that $c = -1$ [15], which would imply that $\gamma_n/n \to 1/(2\pi e)$ as $n \to \infty$, but we do not even know whether the limit exists [11]. Need γ_n^n be rational for all n? The Hermite constants γ_n are important as well in the study of quadratic forms and in coding theory.

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8.8 Tammes' Constants

Let $S = \{(u, v, w) : u^2 + v^2 + w^2 = 1\}$ denote the unit sphere in three-dimensional space and $|p - q|$ denote Euclidean distance between two points p and q. Let $N \geq 2$ be an integer and α be a real number. The **α-energy** associated with a finite subset $\omega_N = \{x_1, x_2, \ldots, x_N\}$ of points on S is

$$\varepsilon(\alpha, \omega_N) = \begin{cases} \sum\limits_{i<j} |x_i - x_j|^\alpha & \text{if } \alpha \neq 0, \\ \sum\limits_{i<j} \ln\left(\dfrac{1}{|x_i - x_j|}\right) & \text{if } \alpha = 0. \end{cases}$$

Define the **extremal energy** for N points on S by

$$E(\alpha, N) = \begin{cases} \min\limits_{\omega_N \subseteq S} \varepsilon(\alpha, \omega_N) & \text{if } \alpha \leq 0, \\ \max\limits_{\omega_N \subseteq S} \varepsilon(\alpha, \omega_N) & \text{if } \alpha > 0. \end{cases}$$

There is tremendous interest in the value of $E(\alpha, N)$ and a representative configuration of points ω_N at which the minimum or maximum energy occurs. The applications include coding theory, electrostatics, crystallography, botany, geometry, and computational complexity. We will mention only a few results here.

Maximizing 1-energy is the same as maximizing the average distance between all pairs of points [1–5]. One can prove that

$$\lim_{N\to\infty} \frac{1}{N^2} E(1, N) = \frac{2}{3},$$

and it is known that

$$\lim_{N\to\infty} \frac{E(1, N) - \frac{2}{3}N^2}{N^{1/2}} = \lambda,$$

where we have rigorous bounds $-2.5066282746\ldots = -\sqrt{2\pi} \leq \lambda < 0$ and an estimate $\lambda = -0.40096\ldots$ [6,7].

Determining $E(-1, N)$ corresponds to locating identical point electrical charges on the sphere so that they are in equilibrium (assuming the particles repel each other according to the *Coulomb potential*). This is known as **Thomson's electron problem** and the optimizing point configurations are called **Fekete points** [8–13]. One can prove that

$$\lim_{N\to\infty} \frac{1}{N^2} E(-1, N) = \frac{1}{2},$$

and, building upon the work of Wagner [14, 15], Kuijlaars & Saff [16, 17] conjectured that

$$\lim_{N\to\infty} \frac{E(-1,N)-\frac{1}{2}N^2}{N^{3/2}} = \sqrt{3}\left(\frac{\sqrt{3}}{8\pi}\right)^{1/2} \zeta\left(\frac{1}{2}\right)\left(\zeta\left(\frac{1}{2},\frac{1}{3}\right)-\zeta\left(\frac{1}{2},\frac{2}{3}\right)\right)$$
$$= -0.5530512933\ldots,$$

where $\zeta(s)$ is the usual Riemann zeta function and

$$\zeta(s,a) = \sum_{\substack{k=0\\ k+a\neq 0}}^{\infty} \frac{1}{(k+a)^s}$$

is the Hurwitz zeta function (with analytic continuation). There is considerable theoretical and empirical evidence that this conjecture is true.

Minimizing 0-energy is equivalent to maximizing the product of distances $\prod_{i<j}|x_i - x_j|$, and it is known that

$$\lim_{N\to\infty} \frac{E(0,N)-\left(-\frac{1}{4}\ln(\frac{4}{e})N^2 - \frac{1}{4}N\ln(N)\right)}{N} = \mu,$$

where we have rigorous bounds $-0.1127687700\ldots \le \mu \le -0.0234972918\ldots$ and an estimate $\mu = -0.026422\ldots$ [6, 7, 18, 19].

As $\alpha \to -\infty$, the α-energy is increasingly dominated by the term involving the smallest of the distances, that is,

$$\lim_{\alpha\to-\infty} \varepsilon(\alpha,\omega_N)^{\frac{1}{\alpha}} = \min_{i<j}|x_i - x_j|.$$

Therefore, the minimal energy problem reduces to calculating

$$d_N = \max_{\omega_N}\min_{i<j}|x_i - x_j|,$$

which is the answer to Tammes' 1930 question about pollen grains [8, 21–28]. Equivalently, what is the largest diameter of N congruent circles that can be packed on S (without overlap)? It is known that

$$d_N = \left(\frac{8\pi}{\sqrt{3}}\right)^{\frac{1}{2}} N^{-\frac{1}{2}} + O\left(N^{-\frac{2}{3}}\right)$$

as $N \to \infty$. A more precise estimate of the error term evidently has not been made. Bounds were determined by Fejes Tóth [26, 29, 30] and van der Waerden [26, 31]:

$$2\left[\frac{\sqrt{3}}{2\pi}N + 3\left(\frac{N}{4\pi}\right)^{\frac{2}{3}} + 3\left(\frac{N}{4\pi}\right)^{\frac{1}{3}}\right]^{-\frac{1}{2}} \le d_N \le \left[4 - \csc\left(\frac{\pi}{6}\frac{N}{N-2}\right)^2\right]^{\frac{1}{2}}.$$

Related questions ask for the smallest diameter of N congruent circles that can *cover* S [32] and for N-point charge configurations on the unit *disk* that achieve equilibrium [33].

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8.9 Hyperbolic Volume Constants

We first describe a certain enumeration problem. Let n be a positive integer. An n-**simplex** is the convex hull of $n+1$ points in n-dimensional Euclidean space, which are assumed to be in general position. For example, a 1-simplex is a line segment, a 2-simplex is a triangle (with its interior), and a 3-simplex is a tetrahedron (with its interior).

An n-cube is **triangulated** (or, more precisely, **face-to-face vertex triangulated**) if it is partitioned into finitely many n-simplices with disjoint interiors, subject to the constraints that

- the vertices of any n-simplex are also vertices of the cube, and
- the intersection of any two n-simplices is a face of each of them.

Define the **simplexity** $f(n)$ of the n-cube to be the minimum number of n-simplices required to triangulate it (see Figure 8.6). An enormous amount of computation leads to the values of $f(n)$ listed in Table 8.5 and bounds for $f(n)$ listed in Table 8.6 [1–7]. An unsolved problem is to determine a tight lower bound for $f(n)$, valid for all n. We will describe an attempt to do this shortly.

The **standard** n**-simplex** S_n is the regular n-simplex inscribed in the unit n-sphere (e.g., S_2 is the equilateral triangle of area $3\sqrt{3}/4$). The **standard** n**-cube** C_n is the n-cube of side $2/\sqrt{n}$, centered at the origin. Clearly

$$\text{volume of } S_n = \frac{\sqrt{n+1}}{n!}\left(1+\frac{1}{n}\right)^{\frac{n}{2}}, \quad \text{volume of } C_n = \left(\frac{4}{n}\right)^{\frac{n}{2}}.$$

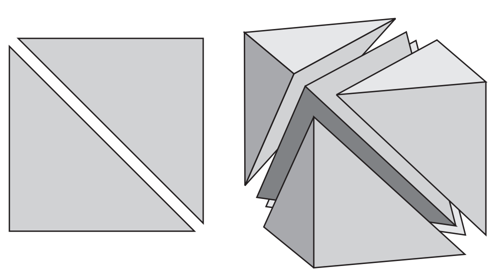

Figure 8.6. Triangulation of the n-cube: $f(2) = 2$ and $f(3) = 5$.

Table 8.5. *Simplexity Values*

n	1	2	3	4	5	6	7
$f(n)$	1	2	5	16	67	308	1493

The best-known attempt to minorize $f(n)$ involves the integrals

$$\xi_n = \text{volume of ideal hyperbolic } n\text{-cube} = \int\limits_{C_n} \left(1 - \sum_{k=1}^{n} x_k^2\right)^{-\frac{n+1}{2}} dx_1\, dx_2 \, \ldots \, dx_n,$$

$$\begin{aligned}\eta_n &= \text{volume of regular ideal hyperbolic } n\text{-simplex} \\ &= \int\limits_{S_n} \left(1 - \sum_{k=1}^{n} x_k^2\right)^{-\frac{n+1}{2}} dx_1\, dx_2 \, \ldots \, dx_n.\end{aligned}$$

More precisely,

$$f(n) \geq \frac{\xi_n}{\eta_n} \geq \frac{1}{2} 6^{\frac{n}{2}} (n+1)^{-\frac{n+1}{2}} n!,$$

as shown by Smith [8] and, independently, Marshall. There is considerable room for improvement – the gap between $f(n)$ and its bounds is huge – but the occurrence of the constants ξ_n and η_n is interesting to us.

It can be demonstrated that $\eta_2 = \pi$, $\eta_3 = \pi \ln(\beta) = 1.0149416064\ldots$, where β is defined in [3.10], and [9–11]

$$\eta_4 = \frac{10\pi}{3} \arcsin\left(\frac{1}{3}\right) - \frac{\pi^2}{3} = 0.2688956601\ldots, \quad \eta_5 = 0.05756\ldots.$$

Also, $\xi_2 = 2\pi$, $\xi_3 = 5\eta_3 = 5.0747080320\ldots$, $\xi_4 = 3.92259368\ldots$, and $\xi_5 = 2.75861972\ldots$ [11, 12]. Asymptotically, we have [8, 9, 12]

$$\eta_n \sim e\frac{\sqrt{n}}{n!}, \quad \xi_n \sim 2\sqrt{\pi}\frac{c^n}{\Gamma\left(\frac{n+1}{2}\right)}$$

as $n \to \infty$, where e is the natural logarithmic base [1.3] and $c = 1.0820884492\ldots$ is twice the maximum of **Dawson's integral** [13, 14]:

$$D(x) = \exp(-x^2) \int\limits_0^x \exp(t^2)\, dt, \quad \frac{c}{2} = 0.5410442246\ldots = \frac{1.2615225101\ldots}{\sqrt{2e}},$$

which occurs uniquely when $x = 0.9241388730\ldots = 1/c$.

Table 8.6. *Bounds for Simplexity* $f(n)$

n	8	9	10
Lower Bound	5522	26593	131269
Upper Bound	13136	105341	928780

In spite of this detailed asymptotic information, it remains open whether $f(n) \geq \gamma^n n!$ for some constant $\gamma > 0$ [15].

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8.10 Reuleaux Triangle Constants

Of all planar sets of constant width 1, the **Reuleaux triangle** (see Figure 8.7) possesses the least area [1–11] and is the most asymmetric [12–15]. Let us examine certain key phrases in the statement of this theorem more carefully, so that we may introduce several related constants.

A compact convex set $C \subseteq \mathbb{R}^2$ is of **constant width** w if all orthogonal projections of C onto lines have the same length w. More generally, for $C \subseteq \mathbb{R}^d$, $d > 2$, the required condition becomes that every pair of parallel supporting $(d-1)$-dimensional planes are at the same distance w apart. (The word *breadth* was used in [8.4.1] for reasons of convention.) For simplicity, set $w = 1$. The first part of the theorem is that the area, $\mu(C)$, of $C \subseteq \mathbb{R}^2$ satisfies

$$\mu(C) \geq \frac{\pi - \sqrt{3}}{2} = 0.7047709230\ldots.$$

It is believed that the volume, $\mu(C)$, of $C \subseteq \mathbb{R}^3$ satisfies

$$\mu(C) \geq \left(\frac{2}{3} - \frac{\sqrt{3}}{4}\arccos\left(\frac{1}{3}\right)\right)\pi = 0.4198600459\ldots,$$

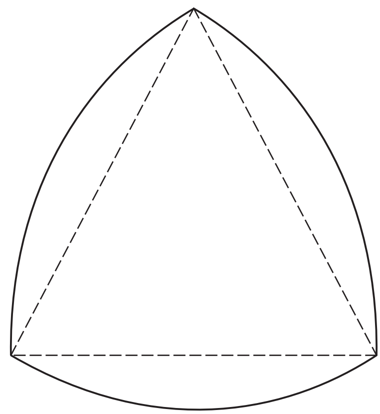

Figure 8.7. The Reuleaux triangle (solid curves) consists of the vertices of an equilateral triangle (dotted lines) together with three arcs of circles, each circle having a center at one of the vertices and endpoints at the other two vertices.

which corresponds to Meisser's tetrahedral analog of the Reuleaux triangle [1, 16]. The best-known lower bound thus far is $(3\sqrt{6}-7)\pi/3 = 0.3649161225\ldots$; hence there is considerable room for improvement [8, 11].

Asymmetry is more difficult to define, primarily because there are competing notions of it! We focus on just two measures of symmetry, called the Kovner–Besicovitch (inner) and Estermann (outer) measures, respectively [14]:

$$\sigma(C) = \frac{\mu(A)}{\mu(C)}, \quad \tau(C) = \frac{\mu(C)}{\mu(B)},$$

where A is the largest convex centrally symmetric subset of C and B is the smallest convex centrally symmetric superset of C. The second part of the theorem is that, for $C \subseteq \mathbb{R}^2$ [8, 12],

$$\begin{aligned}\sigma(C) \geq \frac{6\arccos\left(\frac{5+\sqrt{33}}{12}\right)+\sqrt{3}-\sqrt{11}}{\pi-\sqrt{3}} &= 0.8403426028\ldots \\ &= 1 - 0.1596573971\ldots,\end{aligned}$$

$$\tau(C) \geq \frac{\pi-\sqrt{3}}{\sqrt{3}} = 0.8137993642 = 1 - 0.1862006357\ldots.$$

The corresponding superset B is a regular hexagon circumscribed about the minimizing Reuleaux triangle C; the subset A is a circular hexagon obtained by reflecting C across its center, calling this new subset C', and then forming $C \cap C'$. A higher-dimensional analog of this bound is not known.

Here is one more result. What is the set $C \subseteq \mathbb{R}^2$ of maximal constant width w that avoids all vertices of the integer square lattice? The answer is a Reuleaux triangle, oriented so that one axis of symmetry lies midway between two parallel lattice edges.

Its width $w = 1.5449417003\ldots$ has minimal polynomial [9]

$$4x^6 - 12x^5 + x^4 + 22x^3 - 14x^2 - 4x + 4.$$

We mention that the Reuleaux triangle also appears in conjectures surrounding planar convex translations [8.3.1], maximal planar rendezvous constants [8.21], and exact values of the Bloch–Landau constants [7.1].

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8.11 Beam Detection Constant

A **beam detector** for the unit circle C is a set of points that intercepts all lines (i.e., **beams**) crossing C. Clearly C is itself a beam detector for C, although it is inefficient. There exist shorter curves that meet the required condition [1, 2]. We need to explain what we mean by *curve* before continuing.

A **path** is a continuous image of an interval in the plane, and an **arc** is a path with no self-intersections. If this is the sense in which we interpret the word *curve*, then there is a complete solution. Joris [3] and Faber, Mycielski & Pedersen [4, 5] proved that a bow-shaped arc (see Figure 8.8) is the shortest path that meets all lines meeting the unit circle.

If we loosen the notion of *curve* then the length can be reduced substantially. An n**-arc** is a union of n (possibly disconnected) arcs. Makai [6, 7] found the 2-arc of

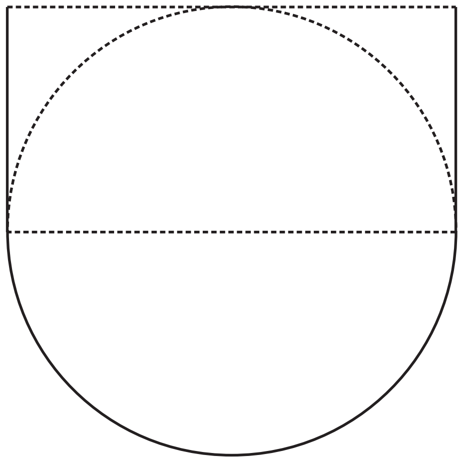

Figure 8.8. Bow-shaped arc of length $\pi + 2 = 5.1415926535\ldots$.

smallest known length, called a **bow-and-arrow** configuration by Thurston [1] (see Figure 8.9). Faber & Mycielski [5] improved on this and found the 3-arc of smallest known length (see Figure 8.10). These examples were rediscovered by Day [8]. For the 2-arc case, the solutions of the simultaneous equations

$$2\cos(\theta_1) - \sin(\tfrac{\theta_2}{2}) = 0, \quad \tan(\tfrac{\theta_1}{2})\cos(\tfrac{\theta_2}{2}) + \sin(\tfrac{\theta_2}{2})\left(\sec(\tfrac{\theta_2}{2})^2 + 1\right) = 2, \quad \theta_3 = \theta_1,$$

give the angles

$$\theta_1 = \theta_3 = 1.2865112676\ldots \approx 73.71^\circ, \quad \theta_2 = 1.1910478286\ldots \approx 68.24^\circ,$$

yielding an upper bound on length for 2-arcs:

$$\begin{aligned} L_2 &\le 2\pi - 2\theta_1 - \theta_2 + 2\tan(\tfrac{\theta_1}{2}) + \sec(\tfrac{\theta_2}{2}) - \cos(\tfrac{\theta_2}{2}) + \tan(\tfrac{\theta_1}{2})\sin(\tfrac{\theta_2}{2}) \\ &= 4.8189264563\ldots. \end{aligned}$$

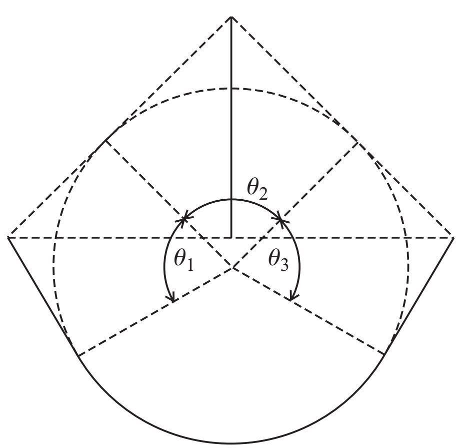

Figure 8.9. The length of a 2-arc is $4.8189264563\ldots$, where $\theta_1 = 1.2865\ldots$, $\theta_2 = 1.1910\ldots$, and $\theta_3 = 1.2865\ldots$; the name "bow-and-arrow" is well justified.

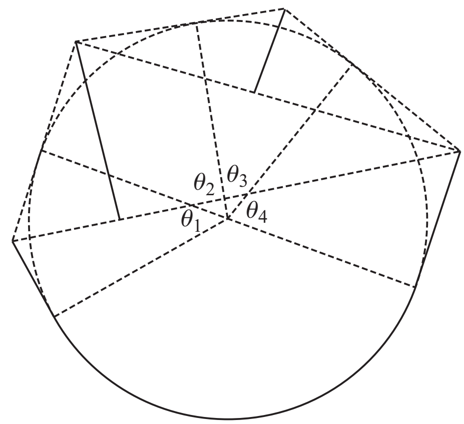

Figure 8.10. The length of a 3-arc is 4.799891547..., where $\theta_1 = 0.96\ldots$, $\theta_2 = 1.04\ldots$, $\theta_3 = 0.7\ldots$, and $\theta_4 = 1.2\ldots$.

Similar equations give rise to an upper bound on length for 3-arcs:

$$L_3 \le 4.799891547\ldots$$

but nothing is known corresponding to 4-arcs or 5-arcs. Define the **beam detection constant** to be

$$L = \inf_{n \ge 1} L_n \ge \pi,$$

where the lower bound is due to Croft [9] and Thurston [1]. Some people presume that the sequence $\{L_n\}$ is strictly decreasing, but others believe that n-arcs, $n \ge 4$, cannot improve on 3-arcs.

One could equally well call L the **trench diggers' constant**. Suppose a straight cable of unknown direction is buried underground and all we know is that the cable passes within one unit of a given marker. There is a strategy for digging (highly disconnected) trenches, guaranteed to locate the cable, of total length $L + \varepsilon$ for any $\varepsilon > 0$. Related strategies include escape trajectories for a hunter lost in a dense jungle or a swimmer at sea in a thick fog, who know they are within one unit of a straight boundary [5]. These are special cases of what is known as the "lost in a forest" problem [3, 10–12].

A different generalization of *path* is possible. Instead of the continuous image of an interval, consider any connected closed set in the plane. Instead of ordinary length, consider one-dimensional Hausdorff measure. Eggleston [9, 13] determined that, even for this extended class of curves, the optimal beam detector for C is the bow-shaped arc of length $\pi + 2$. Curiously, if we replace the unit circle C by an equilateral triangle or a square, the optimal known connected beam detector is tree-like, with several branches, called the *Steiner span* of the vertices [8.6]. For the square, as for the circle, we do better still if we discard connectivity [5, 14–19]. The conjectured optimal beam detector for the unit square has two components (as shown in Figure 8.11) and length $(2 + \sqrt{3})/\sqrt{2} = 2.6389584337\ldots = 4(0.6597396084\ldots)$.

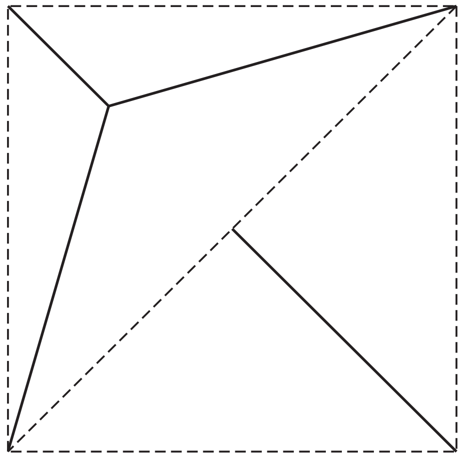

Figure 8.11. The length of the conjectured shortest opaque square fence is 2.6389584337 . . . .

Eppstein [1] pointed out an interesting connection with the design of algorithms for computing a minimal *opaque forest* of a convex polygon [20–22]. Other variations of beam detection appear in [23–25].

Zalgaller [26] reformulated the first problem as follows: What is the shortest connected curve in the plane outside an open unit disk such that, moving along this curve, we can see all points of the unit circle C? He then examined the three-dimensional analog: What is the shortest connected curve in 3-space outside an open unit ball such that, moving along this curve, we can see all the points of the unit sphere S? By a nonrigorous argument, Zalgaller obtained an *inspection trajectory* of length 9.576778 . . . .

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8.12 Moving Sofa Constant

What is the longest ladder L that can be moved around a right-angled corner in a hallway of unit width? We assume that the ladder is straight and rigid, and that it must remain entirely within the hallway as it is passed through the turn. (All discussion throughout this essay will be constrained to the two-dimensional setting; see Figure 8.12.) The answer to the question is easy: L has the same length as the shortest line segment ab intersecting the point c, which is clearly $2\sqrt{2}$ [1].

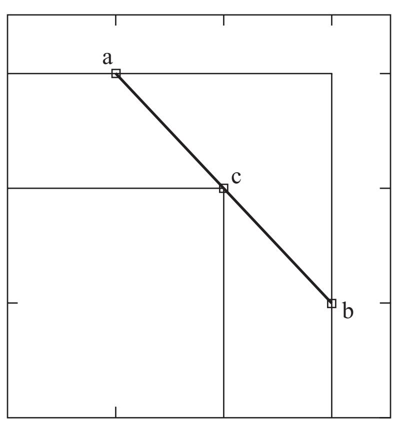

Figure 8.12. This is the optimal ladder passing around the hallway corner.

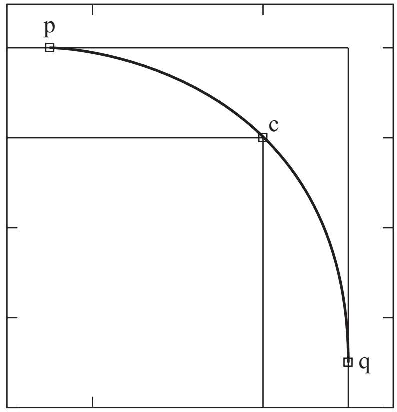

Figure 8.13. This is the optimal wire passing around the hallway corner.

Here is another question: If W is a connected, rigid piece of wire that can be moved around the corner, how large can the diameter of W be? The **diameter** of any continuously differentiable curve is defined to be the maximum of all distances $|x - y|$ between points x and y on the curve. If W is not at all bent, then this reduces to the ladder problem. The largest diameter turns out to be $2(1 + \sqrt{2})$ (see Figure 8.13). The best curve W is the unique quarter-circle pq intersecting the point c [2].

Here is a more difficult problem: What is the greatest possible area for a sofa S that can be moved around the corner [3–5]? We assume only that S is a connected region of the plane. Hammersley [6] showed that the largest area is at least $\pi/2 + 2/\pi = 2.2074\ldots$ (see Figure 8.14) but, contrary to intuition, his region is not optimal.

Gerver (and, independently, Logan) constructed a certain sofa, with complicated boundaries, that possesses a larger area than any other so far examined [7,8]. (See

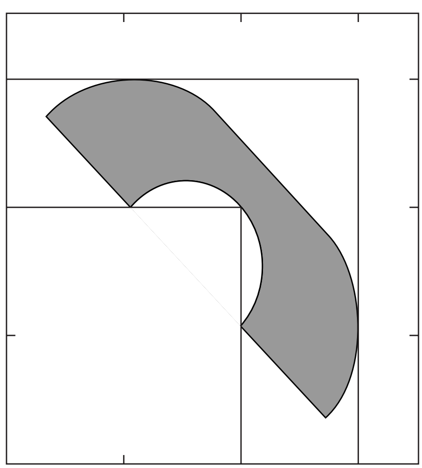

Figure 8.14. Hammersley's sofa consists of two quarter-circles on either side of a $1 \times 4/\pi$ rectangle from which a semicircle of radius $2/\pi$ has been removed.

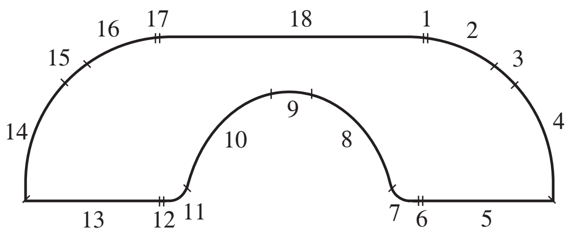

Figure 8.15. The boundary of Gerver's conjectured optimal sofa has eighteen separate pieces.

Figure 8.15.) Further, his sofa is *provably* optimal within the class Σ of all sofas S that

- rotate 90° as S moves around the corner, and
- touch the wall first at two points as S starts to rotate, then at four points, then at three points (when S has rotated 45°), then at four points again, and then at two points again as S finishes rotating.

It would be very surprising if a larger sofa could be found, because it could not be in Σ.

What is the area of Gerver's sofa? To answer this question, first compute constants A, B, φ, and θ via the simultaneous set of four equations

$$\begin{aligned}&A\,(\cos(\theta)-\cos(\varphi))-2B\sin(\varphi)+(\theta-\varphi-1)\cos(\theta)-\sin(\theta)\\&\quad+\cos(\varphi)+\sin(\varphi)=0,\end{aligned}$$

$$\begin{aligned}&A\,(3\sin(\theta)+\sin(\varphi))-2B\cos(\varphi)+3\,(\theta-\varphi-1)\sin(\theta)+3\cos(\theta)\\&\quad-\sin(\varphi)+\cos(\varphi)=0,\end{aligned}$$

$$A\cos(\varphi)-\left(\sin(\varphi)+\frac{1}{2}-\frac{1}{2}\cos(\varphi)+B\sin(\varphi)\right)=0,$$

$$\left(A+\frac{\pi}{2}-\varphi-\theta\right)-\left(B-\frac{1}{2}(\theta-\varphi)(1+A)-\frac{1}{4}(\theta-\varphi)^2\right)=0,$$

obtaining $A=0.0944265608\ldots$, $B=1.3992037273\ldots$, $\varphi=0.0391773647\ldots$, and $\theta=0.6813015093\ldots$. Next, let

$$r(\alpha)=\begin{cases}\frac{1}{2} & \text{if } 0\le\alpha<\varphi,\\ \frac{1}{2}(1+A+\alpha-\varphi) & \text{if } \varphi\le\alpha<\theta,\\ A+\alpha-\varphi & \text{if } \theta\le\alpha<\frac{\pi}{2}-\theta,\\ B-\frac{1}{2}(\frac{\pi}{2}-\alpha-\varphi)(1+A)-\frac{1}{4}(\frac{\pi}{2}-\alpha-\varphi)^2 & \text{if } \frac{\pi}{2}-\theta\le\alpha<\frac{\pi}{2}-\varphi,\end{cases}$$

$$s(\alpha)=1-r(\alpha),$$

$$u(\alpha)=\begin{cases}B-\frac{1}{2}(\alpha-\varphi)(1+A)-\frac{1}{4}(\alpha-\varphi)^2 & \text{if } \varphi\le\alpha<\theta,\\ A+\frac{\pi}{2}-\varphi-\alpha & \text{if } \theta\le\alpha<\frac{\pi}{4},\end{cases}$$

and let u' denote the derivative of u. Define three functions y_1, y_2, y_3 by

$$y_1(\alpha) = 1 - \int_0^{\alpha} r(t)\sin(t)\,dt, \quad y_2(\alpha) = 1 - \int_0^{\alpha} s(t)\sin(t)\,dt,$$

$$y_3(\alpha) = y_2(\alpha) - u(\alpha)\sin(\alpha).$$

Then the area of the optimal sofa is $2.2195316688\ldots$, that is,

$$2\int_0^{\frac{\pi}{2}-\varphi} y_1(\alpha)r(\alpha)\cos(\alpha)\,d\alpha + 2\int_0^{\theta} y_2(\alpha)s(\alpha)\cos(\alpha)\,d\alpha$$
$$+2\int_{\varphi}^{\frac{\pi}{4}} y_3(\alpha)\left(u(\alpha)\sin(\alpha) - u'(\alpha)\cos(\alpha) - s(\alpha)\cos(\alpha)\right)\,d\alpha.$$

The three integrals represent, respectively, the area under the convex part of the outer boundary, the area over the convex part of the inside boundary, and the area over the concave part of the inside boundary (where the corner of the hallway scrapes against the sofa).

Sommers [9] examined the problem with the additional condition that S is convex, and he numerically determined the optimal area to be $\geq 1.644703\ldots$. Much more is known if S is rectangular, even if the hallway corner is not right-angled and the two corridors are of different widths [10].

This subject is related to motion planning in robotics, specifically, what is known as the piano mover's problem [11]. Given an open subset U in n-dimensional space and two compact subsets C_0 and C_1 of U, where C_1 is derived from C_0 by a continuous motion, is it possible to move C_0 to C_1 while remaining entirely inside U [12–15]?

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8.13 Calabi's Triangle Constant

Let T denote an equilateral triangle. There are clearly three congruent largest squares that can be wedged within T (see Figure 8.16). Do there exist non-equilateral triangles with this property? One would at first expect the answer to be no; for example, a right triangle U always has a unique largest square wedged within U, namely, the square with sides aligned with the perpendicular legs of U.

Calabi examined the question and found an answer defying expectation [1,2]: A non-equilateral triangle with three congruent largest squares *does* exist and is unique (see Figure 8.17). It is an isosceles triangle and, if AB is the triangular base and $AC = BC$, then the ratio

$$\frac{AB}{AC} = 2\cos(\alpha) = 1.5513875245\ldots$$

is algebraic with minimal polynomial $2x^3 - 2x^2 - 3x + 2$. Also, the angle α at vertex A is given by

$$\alpha = 0.6829826991\ldots \sim 39.13^\circ.$$

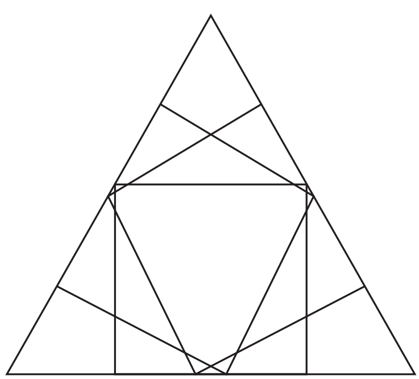

Figure 8.16. An equilateral triangle with three distinct inscribed squares of maximal size.

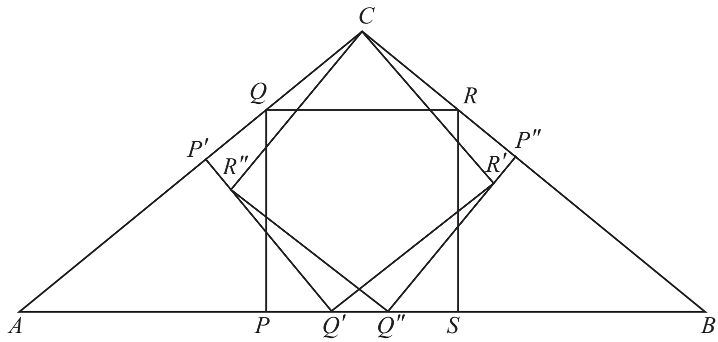

Figure 8.17. A non-equilateral triangle with three distinct inscribed squares of maximal size.

Further related research was conducted by Wetzel [3,4]. Here is an unresolved issue: What is the three-dimensional tetrahedral analog of this result?

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8.14 DeVicci's Tesseract Constant

How large a square can be inscribed within a unit cube? This is known as **Prince Rupert's problem**. More generally, how large an m-dimensional cube can be inscribed within a unit n-dimensional cube, where $m < n$?

Let $f(m, n)$ be the edge-length of the optimal m-cube. Clearly $f(1, n) = \sqrt{n}$ for all n. Figure 8.18 suggests that

$$f(2,3) = \frac{3}{4}\sqrt{2} = 1.0606601717\ldots,$$

and this result has been known for a long time to be true [1–4].

DeVicci [5] proved that

$$f(m,n) = \sqrt{\frac{n}{m}} \quad \text{if } m \text{ divides } n, \quad f(2,n) = \begin{cases} \sqrt{\dfrac{n}{2}} & \text{if } n \text{ is even,} \\ \sqrt{\dfrac{4n-3}{8}} & \text{if } n \text{ is odd.} \end{cases}$$

An elaborate argument gives that [5]

$$f(3,4) = 1.0074347569\ldots,$$

which has minimal polynomial $4x^8 - 28x^6 - 7x^4 + 16x^2 + 16$. In fact, $f(3, 4)$ is solvable in radicals. Since the name *tesseract* is often used [7] to refer to the 4-cube, we call

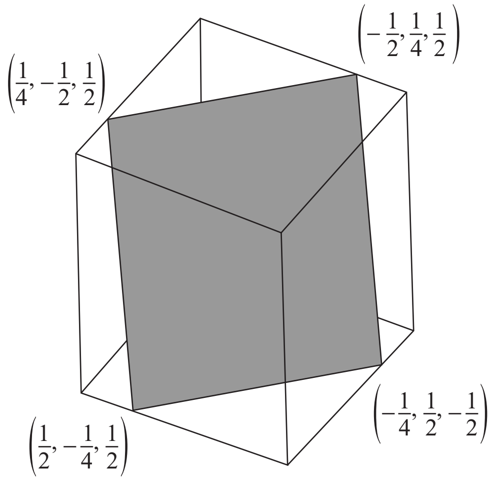

Figure 8.18. The 3-cube with corners at $(\pm 1/2, \pm 1/2, \pm 1/2)$, along with the largest inscribed square.

$f(3, 4)$ **DeVicci's tesseract constant**. According to Gardner [8, 9], the list of people who numerically anticipated this result includes Baer, Bosch, and de Josselin de Jong.

Huber [10] determined more exact evaluations of $f(m, n)$, for example,

$$f(3, 5) = \sqrt{11 - 4\sqrt{6}} = 1.0963763171\ldots.$$

It is known that $f(m, n)$ is always an algebraic number [6]. Might the degree of the corresponding minimal polynomial follow some recognizable function of m and n?

The same problem for maximal rectangles with fixed aspect ratio (instead of squares) in cubes has been comparatively neglected until recently [11].

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8.15 Graham's Hexagon Constant

Let P denote an n-sided convex polygon in the plane. Assume that P is of unit diameter, equivalently, that the maximum distance between any two vertices of P is 1. What is the largest possible area, F_n, enclosed by P?

Clearly $F_3 = \sqrt{3}/4 = 0.4330127018\ldots$ and this is achieved uniquely by the equilateral triangle with unit sides. More generally, we have upper and lower bounds

$$\frac{n}{8}\sin\left(\frac{2\pi}{n}\right) \le F_n \le \frac{n}{2}\cos\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{2n}\right)$$

valid for all n. Reinhardt [1] proved that the right-hand inequality becomes equality for all odd n, and that this is achieved uniquely by the regular n-gon of unit diameter. One would naively expect the left-hand inequality to become equality for even n, with a similar uniqueness result.

If $n = 4$, the left-hand inequality becomes equality. In all other respects, the situation for even n is unexpected. $F_4 = 1/2$ is achieved not only by the unit square, but by an infinite family of quadrilaterals of unit diameter. So uniqueness fails for $n = 4$. Interestingly, uniqueness holds for $n = 6$. It is not known whether uniqueness also holds for $n = 8, 10, 12, \ldots$.

Let us focus on the case $n = 6$. The regular hexagon of unit diameter has area

$$\left.\frac{n}{8}\sin\left(\frac{2\pi}{n}\right)\right|_{n=6} = \frac{3\sqrt{3}}{8} = 0.6495190528\ldots.$$

Graham [2–5] proved the surprising result that this is *not* optimal. He constructed a hexagon of unit diameter that has area $F_6 = 0.6749814429\ldots$, an algebraic number with minimal polynomial

$$\begin{aligned}&4096x^{10} + 8192x^9 - 3008x^8 - 30848x^7 + 21056x^6 + 146496x^5\\&\quad - 221360x^4 + 1232x^3 + 144464x^2 - 78488x + 11993\end{aligned}$$

(see Figure 8.19).

What can be said about the maximum area for a unit-diameter octagon ($n = 8$)? Briggs, Prieto, Vanderbei, Wright, Gay, and others obtained $F_8 = 0.726868\ldots$ via numerical global optimization techniques. More recently, Audet et al. [6] proved a conjecture of Graham's on the shape of the optimal octagon via a quadratic programming scheme; the corresponding minimal polynomial still remains an open question.

No exact results are known for the decagon ($n = 10$) or the dodecagon ($n = 12$), but numerical estimates are $F_{10} = 0.749137\ldots$ and $F_{12} = 0.760729\ldots$, respectively. Perimeters can be maximized rather than areas [7]. Not much is known about higher dimensions: We know the largest volumes of d-dimensional convex polyhedra with $d + 2$ vertices [8], but cases involving $> d + 2$ vertices evidently remain unsolved.

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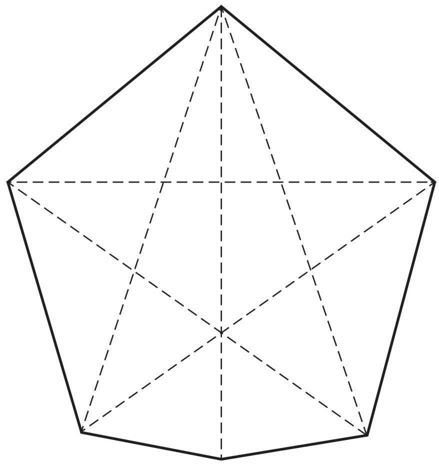

Figure 8.19. Graham's hexagon is the optimal hexagon (meaning it has maximum area) of unit diameter.

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8.16 Heilbronn Triangle Constants

The n^{th} **Heilbronn triangle constant** is the infimum of all numbers H_n for which the following holds [1]: Given any arrangement of n points in the unit square, the smallest triangle formed by any three of the points has area $\leq H_n$.

Goldberg [2] considered the exact values of the first several Heilbronn constants, including $H_3 = H_4 = 1/2 = 0.5$ and made several conjectures. Yang, Zhang & Zeng [3,4] disproved one of the conjectures by showing that $H_5 = \sqrt{3}/9 = 0.1924500897\ldots$ but confirmed Goldberg's assertion that $H_6 = 1/8 = 0.125$. See Figures 8.20 and 8.21. It is also known that $H_7 \geq 0.0838590090\ldots$, where the lower bound has minimal polynomial $152x^3 + 12x^2 - 14x + 1 = 0$, and [5]

$$H_8 \geq \frac{\sqrt{13}-1}{36} = 0.0723764243\ldots, \quad H_9 \geq \frac{9\sqrt{65}-55}{320} = 0.0548759991\ldots.$$

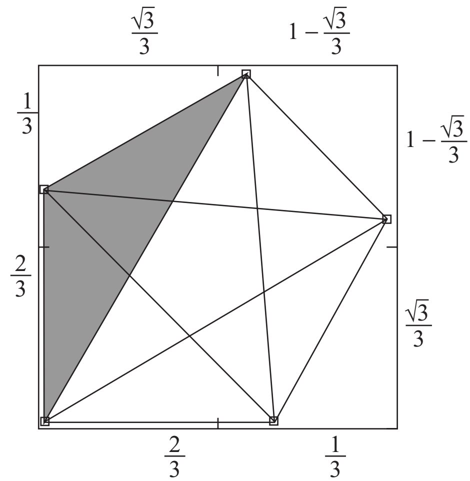

Figure 8.20. The best arrangement of points corresponding to $n = 5$.

Comellas & Yebra [5] expressed confidence that these bounds very likely are optimal, but acknowledged that there is (as yet) no proof of this.

What can be said about the asymptotics of H_n? Heilbronn conjectured in 1950 that $H_n = O(n^{-2})$ as $n \to \infty$. Roth, Schmidt, and others made progress toward proving this by showing that [6, 7]

$$H_n = O(n^{-\frac{8}{7}+\varepsilon})$$

for all sufficiently large n, for any $\varepsilon > 0$. Komlós, Pintz & Szemerédi, however, disproved Heilbronn's conjecture by demonstrating that there exists a constant $c > 0$ for

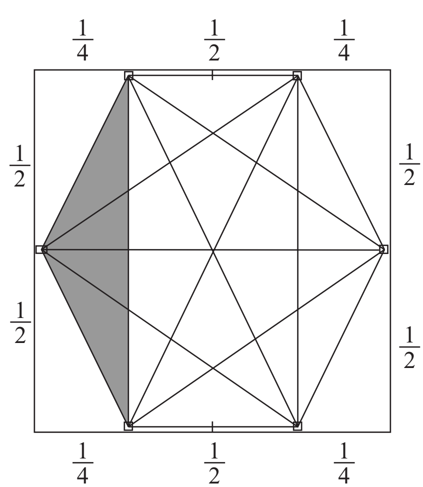

Figure 8.21. The best arrangement of points corresponding to $n = 6$.

which [8]

$$\frac{c \ln(n)}{n^2} \leq H_n$$

for large enough n. Their proof was highly nonconstructive. A recent alternative proof of the lower bound [9] gives a polynomial-time algorithm for finding a configuration of n points where all triangles have area $\geq c \ln(n)/n^2$, for each n. With regard to the upper bound, can the exponent 8/7 be replaced by 2? This is a difficult question and no one expects a complete answer soon.

Jiang, Li & Vitányi [10, 11] analyzed the average-case scenario (rather than the worst-case one), given n uniformly distributed points in the unit square, and found that the smallest triangle has expected area between $a\,n^{-3}$ and $b\,n^{-3}$ for some constants $0 < a < b$. A study of a higher dimensional analog of Heilbronn's problem was undertaken in [12, 13].

If we replace the unit square in the definition of H_n by an equilateral triangle of unit area, then $\tilde{H}_3 = 1$, $\tilde{H}_4 = 1/3$, $\tilde{H}_5 = 3 - 2\sqrt{2}$, and $\tilde{H}_6 = 1/8$ [14]. In fact, we need not specify that the domain be equilateral, since $\tilde{H}_n$ is independent of the shape of the unit triangle under consideration [6]. Moreover, the asymptotics discussed earlier actually apply (within a constant factor) to the general case of n points sitting in a compact convex domain in the plane.

Here is a vaguely related problem. Suppose the unit square is partitioned into m connected sets. Let d be the maximum of the diameters of the m sets. What is the minimum possible value of d [15–19]? For example, if $m = 3$, then $d = \sqrt{65}/8 = 1.0077822185\ldots$.

Another problem is reminiscent of Dirichlet–Voronoi cells and other geometric close-proximity questions. How should k points be arranged inside a unit square to minimize the average distance in the square to the nearest of the k points [20–26]? As $k \to \infty$, the k points approach the vertices of a regular hexagonal lattice. There are many variations. We mention finally that the problem of packing l disks in a unit square is the same as determining the greatest possible minimum distance between l points in the square [8.2].

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8.17 Kakeya–Besicovitch Constants

A region R in the plane is a **Kakeya region** if, inside R, a line segment of unit length can be **reversed**, that is, maneuvered continuously and without leaving R to reach its original position but rotated through 180°. Kakeya [1] asked what the least possible area of such a region R might be.

Let

$$K = \inf_{R \text{ Kakeya}} \operatorname{area}(R),$$

where the infimum extends over all Kakeya regions. Besicovitch [2,3] proved the astonishing result that $K = 0$, which is to say that unit line segments can be reversed within regions of arbitrarily small area. His proof used highly multiply connected regions (i.e., with many holes) that are unbounded (i.e., with large diameters). People wondered if such complicated regions were truly necessary and what the effect of further restrictions on R might be [4–7].

Van Alphen [8] proved that $K = 0$ if R is restricted to fall within a circle of radius $2 + \varepsilon$, for any $\varepsilon > 0$. So boundedness is not an issue. Later, Cunningham [9] proved that $K = 0$ even if R is simply connected (i.e., with no holes) and falls within a circle of radius 1. So even the absence of holes is not an issue. These are remarkably intricate results and explanations of their significance outside geometry may be found in [10–13].

Different restrictions give rise to different results. Let

$$K_c = \inf_{\substack{R \text{ convex} \\ \text{Kakeya}}} \operatorname{area}(R)$$

(meaning that, for any two points P, $Q \in R$, the line segment $PQ \subseteq R$) and

$$K_s = \inf_{\substack{R \text{ star-shaped} \\ \text{Kakeya}}} \operatorname{area}(R)$$

(meaning that there is a point $O \in R$ such that, for any point $P \in R$, the line segment $OP \subseteq R$). Pál [14] proved that

$$K_c = \frac{\sqrt{3}}{3} = 0.5773502691\ldots,$$

which corresponds to the equilateral triangle of height 1.

In contrast, Bloom, Schoenberg & Cunningham [6,9,15] proved that

$$\begin{aligned} 0.0290888208\ldots = \frac{\pi}{108} \le K_s \le \frac{5 - 2\sqrt{2}}{24}\pi &= 0.2842582246\ldots \\ &= (0.0904822031\ldots)\pi, \end{aligned}$$

and Schoenberg further conjectured that K_s is equal to its upper bound. This evidently remains an open problem.

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8.18 Rectilinear Crossing Constant

Let G be a graph [5.6]. A **rectilinear drawing** is a mapping of G into the plane with the property that vertices go to distinct points and edges go to straight line segments. Over all possible such drawings of G, determine one with the minimum number, $\bar{\nu}(G)$, of crossings of edges in the plane. Call $\bar{\nu}(G)$ the **rectilinear crossing number** of G [1–4].

For the complete graph K_n, with n vertices and all $\binom{n}{2}$ possible edges, the known values of and bounds on $\bar{\nu}(K_n)$ are listed in Tables 8.7 and 8.8 [5–8].

Asymptotically, we have [8, 9]

$$0.311507 < \rho = \lim_{n\to\infty} \frac{\bar{\nu}(K_n)}{\binom{n}{4}} = \sup_n \frac{\bar{\nu}(K_n)}{\binom{n}{4}} \leq \frac{6467}{16848} < 0.383844.$$

An exact value for ρ is unknown.

Here is a seemingly unconnected problem, due to Sylvester [10], from geometric probability. Let R be an open convex set in the plane with finite area. Randomly choose four points independently and uniformly in R. With probability 1, no three of the points are collinear, so the convex hull of the four points is either a triangle (one point in the convex hull of the other three) or a quadrilateral. Let $q(R)$ denote the probability that the convex hull is a quadrilateral. Sylvester asked for the minimum and maximum values of $q(R)$ over all convex sets R in the plane.

Table 8.7. *Values of* $\bar{\nu}(K_n)$

n	4	5	6	7	8	9	10	11	12
$\bar{\nu}(K_n)$	0	1	3	9	19	36	62	102	153

Table 8.8. *Bounds on* $\bar{\nu}(K_n)$

n	13	14	15
Upper Bound	229	324	447
Lower Bound	221	310	423

Blaschke [11, 12] proved that the maximum of $q(R)$ is

$$1 - \frac{35}{12\pi^2} = 0.7044798810\ldots,$$

which is achieved when R is an ellipse, and the minimum is $2/3$, attained when R is a triangle. See [13–21] for details and related problems.

If we relax the conditions on R, what corresponding results hold? Let R be an open set in the plane with finite area (i.e., convexity is no longer required). Define $q(R)$ as before. Then clearly $\sup_R q(R) = 1$ since we may take R to be a very thin annulus, in which four randomly selected points will almost surely span a quadrilateral.

The infimum of $q(R)$ is more difficult to study. Scheinerman & Wilf [22, 23] proved the remarkable fact that

$$\inf_R q(R) = \rho,$$

thus relating two seemingly unconnected constants. With its heightened status, ρ perhaps will attract the attention necessary for it someday to be computed.

We have discussed rectilinear drawings; by way of contrast, **ordinary drawings** permit curved edges that lead to the **ordinary crossing number** $\nu(G)$. In this case, Guy [1] conjectured that

$$\nu(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor,$$

and this has been confirmed for $n \le 12$ [24]. No analogous conjectured formula is known for $\bar{\nu}(K_n)$. It is believed that $\bar{\nu}(K_n) > \nu(K_n)$ for sufficiently large n [25].

There are several related notions of the *thickness* of a graph; see [25, 26] for definitions and references. Many fundamental constants like ρ apparently exist in geometric probability (in the older literature, under what was once called *integral geometry*), yet are extremely difficult to calculate.

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8.19 Circumradius-Inradius Constants

The **circumradius** $R(K)$ of a planar compact convex set K is the radius of the smallest disk that contains K, and the **inradius** $r(K)$ is the radius of the largest disk contained by K. Formulas for R and r corresponding to well-known sets appear in [1–3]. Interesting constants involving R or r emerge in various geometric optimization problems over families of sets; we will give three examples out of potentially many.

Consider all triangles Δ that lie in a compact convex set F of width 1. (The **width** of F is the minimum over lengths of all orthogonal projections of F onto lines.) Let us examine the maximum inradius $a(F) = \max_\Delta r(\Delta)$ over all such triangles for several special sets F:

- If F_4 is the square of width 1 (i.e., of side 1), then [4, 5]
$$a(F_4) = \frac{-1+\sqrt{5}}{4} = 0.3090169943\ldots.$$
- If F_5 is the regular pentagon of width 1 (i.e., of side $2\cot(2\pi)/5$), then
$$a(F_5) = 0.2440155280\ldots,$$
which has minimal polynomial [6, 7]
$$5x^9 - 170x^8 + 436x^7 - 205x^6 - 96x^5 + 440x^4 - 120x^3 + 64x^2 - 80x + 16.$$
- If F_6 is the regular hexagon of width 1 (i.e., of side $1/\sqrt{3}$), then
$$a(F_6) = \frac{1}{4} = 0.25.$$

Note that $a(F_5) < \min\{a(F_4), a(F_6)\}$. In fact, it is known that [8]
$$0.166 < \frac{1}{6} \le \inf_F a(F) \le a(F_5),$$
where the infimum is taken over arbitrary F. Might this infimum actually be equal to its upper bound? This is an unsolved problem.

For the following, we require some notation. Let S denote the square with vertices $(\pm 1, \pm 1)$ and let $h_1, h_2, \ldots, h_8$ denote its half-edges (proceeding counterclockwise). Given a nonvertical line L passing through (0, 0), let L^+ denote the half-line in the right half-plane and let L^- denote the half-line in the left half-plane. Let us agree that L^+ intercepts h_i and L^- intercepts h_j, where $i \equiv j \bmod 4$. Define M^+ to be a third half-line passing through (0, 0) that intercepts h_k, where $k \ne i$ and $k \ne j$; we say that M^+ is **suitably distinct** from L. Finally, let Z denote the standard integer lattice in the plane, that is, with basis vectors (1, 0) and (0, 1).

Consider all compact convex sets G whose interiors contain the origin but no other lattice points. (In the language of [2.23], G is Z**-allowable**.) Assume further that the circumcenter of G is at the origin, that its corresponding circumcircle is C, and that for any line L passing through (0, 0), we cannot have both $G \cap L^+ \cap C \ne \emptyset$ and $G \cap L^- \cap C \ne \emptyset$ unless there exists a suitably distinct half-line M^+ for which $G \cap M^+ \cap C \ne \emptyset$. (In words, G does not protrude outside S simultaneously in opposite directions unless it protrudes significantly elsewhere too.) Then [9]
$$\sup_G R(G) = 1.6847127097\ldots,$$

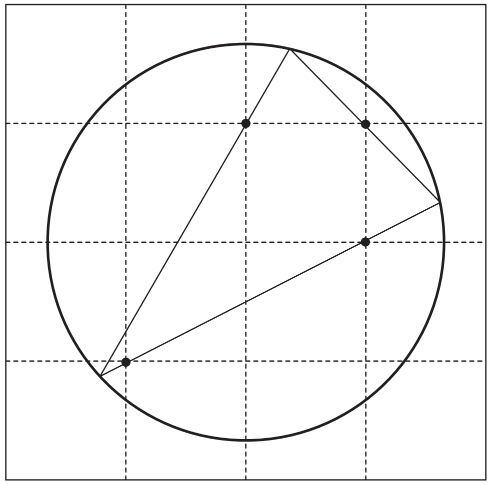

Figure 8.22. This Z-allowable triangle T has maximal circumradius $R(T) = 1.6847127097\ldots$.

which has minimal polynomial $5x^6 - 15x^4 + 3x^2 - 2$. A set with maximal circumradius is the non-isosceles triangle T shown in Figure 8.22. If we did not impose the technical condition regarding L^+, L^-, and M^+, then the supremum would be infinite (imagine a thin plank of width ε and length $1/\varepsilon$, passing through the origin and avoiding all nonzero lattice points).

Here is a result that relates the circumcenter of a compact convex set K with its centroid (i.e., center of gravity). Let $b(K)$ denote the distance between the circumcenter and the centroid, divided by $R(K)$. Clearly $\inf_K b(K) = 0$, for consider a disk or an equilateral triangle. It is known that [10]

$$\sup_K b(K) = \frac{2}{3}x = 0.4278733971\ldots,$$

where x is the unique solution of the transcendental equation

$$x^2 + 2\sqrt{1 - x^2} = 2x(x + \arccos(x)), \quad -1 \le x \le 1.$$

The extremal set is, in this case, a certain symmetric trapezoid with one of its parallel edges replaced by a circular arc.

Inradii are involved in the formulation of certain problems far removed from geometry, for example, Bloch–Landau constants [7.1] and the eigenanalysis of vibrating membranes [11–13].

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8.20 Apollonian Packing Constant

Consider the two pictures in Figure 8.23. The left starts with a large circular boundary and three inner disks; the right starts with a curvilinear triangular boundary and a single disk. Both packings are obtained by inscribing a disk D_i of maximal radius in each gap left uncovered by previous iterations. Every new disk is tangent to all existing disks it touches and, clearly, the resulting configuration has three-fold rotational symmetry.

What can be said about the **residual set** E of the packing, that is, the points not covered by a disk? The set E can be shown to be of Lebesgue measure zero. One important quantity is the **packing exponent** ε, defined to be the infimum value of e for which [1,2]

$$\sum_{i=1}^{\infty} |D_i|^e < \infty,$$

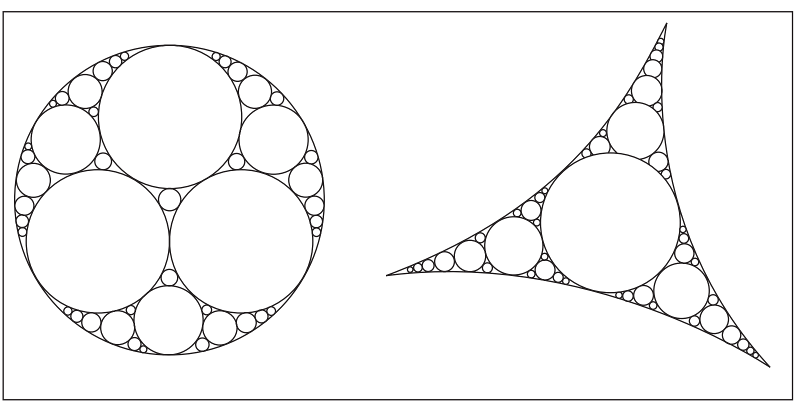

Figure 8.23. Apollonian packing illustrated with initial circle and inital curvilinear triangle.

Table 8.9. *Estimates of Packing Constant* ε

Estimate	Source
1.306951	Melzak [2]
1.3058	Boyd, as reported by Mandelbrot [11]
1.305636	Boyd [12]
1.305684	Manna & Herrmann [13]
1.305686729	Thomas & Dhar [14]
1.305688	McMullen [15]

where $|D|$ denotes the diameter of D. Another important quantity is the **Hausdorff dimension** $\dim(E)$, defined to be the unique value for which [1,3]

$$\sup_{\delta>0} \inf_{\substack{\text{countable} \\ \delta\text{-covers} \\ U_i}} \sum_{i=1}^{\infty} |U_i|^s = \begin{cases} \infty & \text{if } 0 \le s < \dim(E), \\ 0 & \text{if } s > \dim(E), \end{cases}$$

where, by a δ**-cover** U_i, we mean $E \subseteq \bigcup_{i=1}^{\infty} U_i$, where each U_i is an open set and $0 < |U_i| \le \delta$ for all i. It turns out that

$$\varepsilon = \dim(E),$$

as shown by Larman [4] and Boyd [5–7]. Further work by Boyd [8–10] and others yielded rigorous bounds

$$1.300197 < \varepsilon < 1.314534.$$

We also have numerical estimates from various sources (see Table 8.9).

Is $\dim(E)$ minimal, considered against all other disk packing strategies? Boyd [6] answered that this is a difficult question. Whether any progress has been made in resolving this is not known. See [2.16], which makes reference to Sierpinski's gasket, that is, to the packing of similarly-oriented equilateral triangles in an oppositely-oriented triangle (for which $\dim(E)$ is known to be exactly $\ln(3)/\ln(2)$). The subject has also recently become interesting to number theorists [16].

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8.21 Rendezvous Constants

Let E denote a compact, connected subset of d-dimensional Euclidean space. Gross [1] and Stadje [2] independently proved the following: There is a unique real number $a(E)$ such that, for all (not necessarily distinct) points $x_1, x_2, \ldots, x_n \in E$, there exists $y \in E$ with

$$\frac{1}{n}\sum_{i=1}^{n}|x_i - y| = a(E).$$

In words, there is a point $y \in E$ such that the average distance from y to $x_1, x_2, \ldots, x_n$ is $a(E)$. The constant $a(E)$ works for all collections of n points, for any positive integer n. Moreover, no other constant will work, which is most surprising!

For example, if E is convex, then $a(E)$ is the circumradius of E. We henceforth will focus on nonconvex sets E. If C is a circle of diameter 1, then $a(C) = 2/\pi = 0.6366197723\ldots$ [3,4]. If Δ is an isosceles triangle with baselength 2 and perimeter $2\lambda + 2$, then [5]

$$a(\Delta) = \begin{cases} \frac{\lambda^2+2\lambda-\sqrt{\lambda^2-1}-2\sqrt{(\lambda-\sqrt{\lambda^2-1})\lambda(\lambda+1)}}{\lambda^2+3\lambda-1-\lambda\sqrt{\lambda^2-1}-2\sqrt{(\lambda-\sqrt{\lambda^2-1})\lambda(\lambda+1)}} & \text{for } \sqrt{2} \le \lambda \le \xi, \\ \frac{\lambda^2+1}{2\lambda} & \text{for } \lambda \ge \xi, \end{cases}$$

where $\xi = 2.3212850380\ldots$ has minimal polynomial $2x^5 - 4x^4 - 5x^2 + 4x - 1$. No one has yet found a closed-form expression for $a(E)$ if E is an arbitrary ellipse or acute triangle.

Two alternative definitions of $a(E)$ are as follows:

$$a(E) = \sup_{n\ge 1}\ \sup_{x_1,x_2,\ldots,x_n\in E}\ \min_{y\in E}\frac{1}{n}\sum_{i=1}^{n}|x_i - y| = \inf_{n\ge 1}\ \inf_{x_1,x_2,\ldots,x_n\in E}\ \max_{y\in E}\frac{1}{n}\sum_{i=1}^{n}|x_i - y|,$$

and its association with the minimax theorem of game theory becomes obvious [1,3,6].

Define the **rendezvous constant**, $r(E)$, of E to be the normalized ratio

$$r(E) = \frac{a(E)}{\mathrm{diam}(E)}, \quad \mathrm{diam}(E) = \max_{u,v \in E} |u - v|.$$

For this to make sense, E cannot be a single point p (for the diameter to be nonzero) and cannot be a finite set (by connectedness). With these restrictions, Gross and Stadje proved that $1/2 \le r(E) < 1$. What is the maximum value, R_d, of $r(E)$ considered over all sets E in d-dimensional Euclidean space? Clearly $R_1 = 1/2$. When $d = 2$, it seems likely that the Reuleaux triangle T provides the answer [8.10]. Nickolas & Yost [7] and Wolf [8] rigorously established the bounds

$$\max\left\{\frac{2}{3}, r(T)\right\} \le R_2 \le \frac{1}{2} + \frac{\pi}{16} < 0.69634955,$$

and the best-known numerical estimate of $r(T)$ is $0.6675277360\ldots$ [9]. No closed-form expression for $r(T)$ has been discovered. The conjecture $R_2 = r(T)$ deserves more attention!

For $d > 2$, we have bounds [7]

$$\frac{d}{d+1} \le R_d \le \frac{\Gamma(\frac{d}{2})^2 2^{d-2}\sqrt{2d}}{\Gamma(d - \frac{1}{2})\sqrt{\pi(d+1)}} < \sqrt{\frac{d}{d+1}},$$

where $\Gamma(x)$ is the gamma function [1.5.4]. These bounds are less precise than those for $d = 2$. No one has attempted to guess the higher dimensional shapes that maximize the rendezvous constant, as far as is known.

A second relevant conjecture is that $R_2 = S_2$, where [8–11]

$$S_d = \sup_{n \ge 1} \sup_{\substack{x_1, x_2, \ldots, x_n \\ |x_i - x_j| \le 1}} \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j|.$$

In words, S_d is the average pairwise distance of arbitrary points $x_1, x_2, \ldots, x_n$ in d-dimensional space, where no pair x_i, x_j has separation exceeding 1.

We have more bounds $a(E) \le b(E)$, where [8]

$$b(E) = \sup_{n \ge 1} \sup_{x_1, x_2, \ldots, x_n \in E} \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j|.$$

The study of $b(E)$ begins with a generalization: replacing the summations by integrals and the point masses x_i by a probability density, then applying potential theory [12–16]. A third conjecture is that $a(T) = b(T)$ [9]. Another special case, when E is the two-dimensional sphere, was discussed in [8.8].

The preceding material can be generalized: E may be any compact, connected metric space. In fact, E need not even have a metric: Stadje [2] proved that E need only be a compact, connected Hausdorff space possessing a real-valued continuous symmetric function $f(x, y)$ for $x, y \in E$ (a kind of "weak metric").

Finally, let E be the ellipse with semimajor axis 2 and semiminor axis 1. It is numerically known that $a(E) = 2.1080540666\ldots$ [9]. Although there is no precise formula for $a(E)$, as stated earlier, it would be good nevertheless someday to better understand the nature of this constant.

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Table of Constants

0	Zero; conjectured value of de Bruijn–Newman constant [2.32]
0.0001111582...	With Stieltjes constants [2.21]
0.00016...	One of Cameron's sum-free set constants [2.25]
0.0002206747...	6th Du Bois Reymond constant [3.12]
0.0005367882...	Hensley's constant [2.18]
0.0007933238...	5th Stieltjes constant [2.21]
0.0013176411...	Heath–Brown–Moroz constant, with Artin's constant [2.4]
0.0017486751...	λ_7; with Gauss–Kuzmin–Wirsing constant [2.17]
0.0019977469...	With Sobolev isoperimetric constants [3.6]
0.0020538344...	3rd Stieltjes constant [2.21]
0.0023253700...	4th Stieltjes constant [2.21]
0.0031816877...	Melzak's constant, with Sobolev isoperimetric constants [3.6]
0.0044939231...	With Golomb–Dickman constant [5.4]
0.0047177775...	$-\lambda_6$; with Gauss–Kuzmin–Wirsing constant [2.17]
0.0052407047...	4th Du Bois Reymond constant [3.12]
0.0063564559...	With Stieltjes constants [2.21]
0.0072973525...	Fine structure constant, with Feigenbaum–Coullet–Tresser [1.9]
0.0095819302...	With Sobolev isoperimetric constants [3.6]
0.0096903631...	Negative of 2nd Stieltjes constant [2.21]
0.0102781647...	p_3; with Vallée's constant [2.19]
0.0125537906...	With Golomb–Dickman constant [5.4]
0.0128437903...	λ_5; with Gauss–Kuzmin–Wirsing constant [2.17]
0.0173271405...	b_3; with Du Bois Reymond constants [3.12]
0.0176255...	With percolation cluster density constants [5.18]
0.0177881056...	$-41/32 + 3\sqrt{3}/4$; with percolation cluster density [5.18]
0.0183156388...	e^{-4}; one of Rényi's parking constants [5.3]
0.0186202233...	One of Pólya's random walk constants [5.9]
0.0219875218...	Gauchman's constant, with Shapiro–Drinfeld [3.1]
0.0230957089...	With Apéry [1.6], Stieltjes [2.21], de Bruijn–Newman [2.32]
0.0231686908...	Hensley's constant [2.18]

0.0255369745...	c_0^-; with Lenz–Ising constants [5.22]
0.0261074464...	4^{th} Matthews constant [2.4]
0.026422...	With Tammes' constants [8.8]
0.0275981...	$\kappa_S(p_c)$; with percolation cluster density constants [5.18]
0.0282517642...	3^{rd} Du Bois Reymond constant [3.12]
0.0333810598...	One of the Feigenbaum–Coullet–Tresser constants [1.9]
0.0354961590...	$-\lambda_4$; with Gauss–Kuzmin–Wirsing constant [2.17]
0.0355762113...	$-41/16 + 3\sqrt{3}/2$; with percolation cluster density [5.18]
0.0369078300...	With Golomb–Dickman constant [5.4]
0.0370072165...	With Golomb–Dickman constant [5.4]
0.0381563991...	One of Rényi's parking constants [5.3]
0.0403255003...	e_0; with Lenz–Ising constants [5.22]
0.0461543172...	With Stieltjes constants [2.21]
0.0482392690...	$m_{3,4}$; with Meissel–Mertens constants [2.2]
0.0484808014...	p_2; with Vallée's constant [2.19]
0.0494698522...	With Golomb–Dickman constant [5.4]
0.0504137604...	With Euler–Gompertz constant [6.2]
0.0548759991...	Conjectured value of H_9, Heilbronn triangle constants [8.16]
0.05756...	With hyperbolic volume constants [8.9]
0.0585498315...	$1/(2\pi e)$; with Hermite's constants [8.7]
0.0605742294...	One of the Euler totient constants [2.7]
0.0608216553...	3^{rd} Matthews constant [2.4]
0.0648447153...	One of Pólya's random walk constants [5.9]
0.0653514259...	Norton's constant [2.18]
0.0653725925...	With Stieltjes constants [2.21]
0.065770...	$\kappa_S(1/2)$; with percolation cluster density constants [5.18]
0.0657838882...	With Gibbs–Wilbraham constant [4.1]
0.0659880358...	e^{-e}; one of the iterated exponential constants [6.11]
0.0723764243...	Conjectured value of H_8, Heilbronn triangle constants [8.16]
0.0728158454...	Negative of 1^{st} Stieltjes constant [2.21]
0.0729126499...	One of Pólya's random walk constants [5.9]
0.0757395140...	With Vallée's constant [2.19]
0.0773853773...	With Vallée's constant [2.19]
0.0810614667...	With Stieltjes constants [2.21]
0.0838590090...	Conjectured value of H_7, Heilbronn triangle constants [8.16]
0.0858449341...	One of Pólya's random walk constants [5.9]
0.0883160988...	With Golomb–Dickman constant [5.4]
0.0894898722...	$G/\pi - 1/2$; with Gibbs–Wilbraham constant [4.1]
0.0904822031...	Conjectured value of K_s/π, with Kakeya–Besicovitch [8.17]
0.0923457352...	With de Bruijn–Newman constant [2.32]
0.0931878229...	$\exp(-\pi^2/(6\ln(2)))$; with Khintchine–Lévy constants [1.8]
0.0946198928...	With Meissel–Mertens constants [2.2]
0.0948154165...	With Otter's tree enumeration constants [5.6]
0.097...	Base-10 self-numbers density constant [2.24]

0.0980762113 ...	$(3\sqrt{3}-5)/2$; with percolation cluster density [5.18]
0.1008845092 ...	λ_3; with Gauss–Kuzmin–Wirsing constant [2.17]
0.1013211836 ...	$1/\pi^2$; with Sobolev isoperimetric constants [3.6]
0.1041332451 ...	$-d_0$; with Lenz–Ising constants [5.22]
0.1047154956 ...	One of Pólya's random walk constants [5.9]
0.1076539192 ...	"One-ninth" constant [4.5]
0.1084101512 ...	Trott's constant, with Minkowski–Bower [6.9]
0.1118442752 ...	With percolation cluster density constants [5.18]
0.1149420448 ...	Kepler–Bouwkamp constant [6.3]
0.1227367657 ...	With Moser's worm constant [8.4]
0.125	$1/8$; with Moser's worm [8.4], Heilbronn's triangle [8.16]
0.1316200785 ...	With Moser's worm constant [8.4]
0.1351786098 ...	One of Pólya's random walk constants [5.9]
0.14026 ...	One of the self-avoiding walk constants [5.10]
0.1433737142 ...	With Hafner–Sarnak–McCurley constant [2.5]
0.1473494003 ...	2^{nd} Matthews constant, with Artin's constant [2.4]
0.1475836176 ...	$\arctan(1/2)/\pi$; Plouffe's constant [6.5]
0.1484955067 ...	With Euler–Gompertz constant [6.2]
0.14855 ...	4D critical point, with Lenz–Ising constants [5.22]
0.14869 ...	$-d_1$; with Lenz–Ising constants [5.22]
0.1490279983 ...	Conway's impartial misère games constant [6.11]
0.14966 ...	4D inverse critical temperature, with Lenz–Ising [5.22]
0.1544313298 ...	$2\gamma-1$; with Euler–Mascheroni constant [1.5]
0.1596573971 ...	With Reuleaux triangle constants [8.10]
0.1598689037 ...	With Stieltjes constants [2.21]
0.1599 ...	One of the self-avoiding walk constants [5.10]
0.1624329213 ...	With hard square entropy constant [5.12]
0.164 ...	With percolation cluster density constants [5.18]
0.1709096198 ...	With Golomb–Dickman constant [5.4]
0.1715004931 ...	δ_0; Hall–Montgomery constant [2.33]
0.1715728753 ...	$3-2\sqrt{2}$; value of $\tilde{H}_5$, Heilbronn triangle constant [8.16]
0.1724297877 ...	One of Pólya's random walk constants [5.9]
0.1729150690 ...	With Gauss–Kuzmin–Wirsing constant [2.17]
0.1763470368 ...	One of Rényi's parking constants [5.3]
0.1764297331 ...	$-G_2$; with Lenz–Ising constants [5.22]
0.1770995223 ...	With Otter's tree enumeration constants [5.6]
0.1789797444 ...	$2G/\pi-1$; with Gibbs–Wilbraham constant [4.1]
0.1807171047 ...	Zagier's constant, with Freiman's constant [2.31]
0.1824878875 ...	With Shapiro–Drinfeld constant [3.1]
0.183 ...	With percolation cluster density constants [5.18]
0.1839397205 ...	$1/(2e)$; with Masser–Gramain constant [7.2]
0.1862006357 ...	With Reuleaux triangle constants [8.10]
0.1866142973 ...	C_6; one of the Hardy–Littlewood constants [2.1]
0.186985 ...	One of Rényi's parking constants [5.3]

0.1878596424...	With iterated exponential constants [6.11]
0.1895600483...	Triangular entropy of folding, with Lieb's square ice [5.24]
0.1924225474...	With Otter's tree enumeration constants [5.6]
0.1924500897...	$\sqrt{3}/9$; value of H_5, Heilbronn triangle constant [8.16]
0.1932016732...	One of Pólya's random walk constants [5.9]
0.1945280495...	2nd Du Bois Reymond constant [3.12]
0.1994588183...	Vallée's constant [2.19]
0.1996805161...	Conjectured value of weakly triple-free set constant [2.26]
0.2007557220...	With Meissel–Mertens constants [2.2]
0.2076389205...	With de Bruijn–Newman constant [2.32]
0.2078795764...	$i^i = \exp(-\pi/2)$; with iterated exponential constants [6.11]
0.209...	Base-4 self-numbers density constant [2.24]
0.2095808742...	With Golomb–Dickman constant [5.4]
0.21...	One of Cameron's sum-free set constants [2.25]
0.2173242870...	Lochs' constant, with Porter–Hensley constants [2.18]
0.218094...	3D critical point, with Lenz–Ising constants [5.22]
0.2183801414...	One of Pólya's random walk constants [5.9]
0.2192505830...	With Glaisher–Kinkelin constant [2.15]
0.221654...	3D inverse critical temperature, with Lenz–Ising [5.22]
0.2221510651...	With Otter's tree enumeration constants [5.6]
0.2265708154...	With hard square entropy constant [5.12]
0.2299...	Square-diagonal entropy of folding, with Lieb's square ice [5.24]
0.2351252848...	Conway–Guy constant, with Erdös' sum-distinct set constant [2.28]
0.2387401436...	With Otter's tree enumeration constants [5.6]
0.24...	One of the Hayman constants [7.5]
0.2419707245...	$1/\sqrt{2\pi e}$; Sobolev isoperimetric [3.6], traveling salesman [8.5]
0.2424079763...	With hard square entropy constant [5.12]
0.2440155280...	One of the circumradius-inradius constants [8.19]
0.247...	Abundant numbers density constant [2.11]
0.25	$1/4$; Koebe's constant, with Bloch–Landau constants [7.1]
0.2503634293...	With Otter's tree enumeration constants [5.6]
0.2526602590...	Binary self-numbers density constant [2.24]
0.2536695079...	δ_8; with Hermite's constants [8.7]
0.2545055235...	With Kalmár's composition constant [5.5]
0.255001...	With Moser's worm constants [8.4]
0.2614972128...	M; one of the Meissel–Mertens constants [2.2]
0.2649320846...	Mrs. Miniver's constant, with circular coverage constants [8.2]
0.2665042887...	With Otter's tree enumeration constants [5.6]
0.2677868402...	Unforgeable word constant, with pattern-free words [5.17]
0.2688956601...	With hyperbolic volume constants [8.9]
0.2696063519...	With Meissel–Mertens constants [2.2]
0.2697318462...	One of the Pell–Stevenhagen constants [2.8]

0.272190 ...	Cassaigne–Finch constant, with Stolarsky–Harborth [2.16]
0.2731707223 ...	With Hafner–Sarnak–McCurley constant [2.5]
0.2746 ...	Tarannikov's constant, with pattern-free words [5.17]
0.2749334633 ...	With Meissel–Mertens constants [2.2]
0.2763932022 ...	$(5-\sqrt{5})/10$; with hard square entropy constant [5.12]
0.2801694990 ...	Bernstein's constant [4.4]
0.28136 ...	One of the Pell–Stevenhagen constants [2.8]
0.2842582246 ...	Conjectured value of a Kakeya–Besicovitch constant [8.17]
0.2853 ...	With Lenz–Ising constants [5.22]
0.2857142857 ...	$2/7$; conjectured value of 2^{nd} Diophantine approximation [2.23]
0.2867420562 ...	$-m_{1,4}$; with Meissel–Mertens constants [2.2]
0.2867474284 ...	Strongly carefree constant, with Hafner–Sarnak–McCurley [2.5]
0.2887880950 ...	Q; with digital search tree constants [5.14], Lengyel's [5.7]
0.2898681336 ...	$p_1 = \pi^2/3 - 3$; with Vallée's constant [2.19]
0.29 ...	One of Pólya's random walk constants [5.9]
0.2915609040 ...	G/π; 2D dimer constant [5.23]
0.2952978731 ...	δ_7; with Hermite's constants [8.7]
0.29745 ...	With Klarner's polyomino constant [5.18]
0.2974615529 ...	One of the Pythagorean triple constants [5.2]
0.2979521902 ...	One of Pólya's random walk constants [5.9]
0.2993882877 ...	With Otter's tree enumeration constants [5.6]
0.3036552633 ...	With Kalmár's composition constant [5.5]
0.3036630028 ...	Gauss–Kuzmin–Wirsing constant [2.17]
0.3042184090 ...	With Otter's tree enumeration constants [5.6]
0.3061875165 ...	With Otter's tree enumeration constants [5.6]
0.3074948787 ...	$C_4 = 2E/27$; one of the Hardy–Littlewood constants [2.1]
0.3084437795 ...	Zygmund's constant, with Young–Fejér–Jackson [3.14]
0.3091507084 ...	$(8/3)(\ln(2)-\gamma)$; one of the geometric probability constants [8.1]
0.3104 ...	Papadimitriou's constant, with traveling salesman constants [8.5]
0.3110788667 ...	Zolotarev–Schur constant [3.9]
0.312 ...	τ; with percolation cluster density constants [5.18]
0.3123633245 ...	With Klarner's lattice animal constant [5.18]
0.3148702313 ...	One of Pólya's random walk constants [5.9]
0.3157184521 ...	$\gamma - M$; with Meissel–Mertens constants [2.2]
0.3166841737 ...	Atkinson–Negro–Santoro constant [2.28]
0.3172 ...	With traveling salesman constants [8.5]
0.3181736521 ...	Kalmár's composition constant [5.5]
0.3187590609 ...	With Sobolev isoperimetric constants [3.6]
0.3187766258 ...	With Otter's tree enumeration constants [5.6]
0.3190615546 ...	With monomer-dimer constants [5.23]
0.3230659472 ...	$\ln(\beta)$; with Kneser–Mahler polynomial constants [3.10]

0.3271293669...	With Landau–Ramanujan constant [2.3]
0.3287096916...	With Kepler–Bouwkamp constant [6.3]
0.3289868133...	$30/\pi^2$; with Hafner–Sarnak–McCurley constant [2.5]
0.3332427219...	Hard hexagon entropy constant, with hard square [5.12]
0.3333333333...	$1/3$; with Rényi's parking constants [5.3]
0.3349813253...	With Meissel–Mertens constants [2.2]
0.3383218568...	With Otter's tree enumeration constants [5.6]
0.3405373295...	One of Pólya's random walk constants [5.9]
0.3472963553...	$2\sin(\pi/18)$; with percolation cluster density constants [5.18]
0.35129898...	One of the quadratic recurrence constants [6.10]
0.3522211004...	With Hafner–Sarnak–McCurley constant [2.5]
0.3529622229...	With Otter's tree enumeration constants [5.6]
0.3532363719...	Hafner–Sarnak–McCurley constant [2.5]
0.3551817423...	With Otter's tree enumeration constants [5.6]
0.359072...	With percolation cluster density constants [5.18]
0.3605924718...	$\operatorname{Im}(i^{i^{i^{\cdots}}})$; with iterated exponential constants [6.11]
0.3607140971...	With Otter's tree enumeration constants [5.6]
0.3611030805...	$r(12)$ conjectured value; with circular coverage constants [8.2]
0.3625364234...	One of Otter's tree enumeration constants [5.6]
0.364132...	One of Rényi's parking constants [5.3]
0.3678794411...	$1/e$; natural logarithmic base [1.3], iterated exponentials [6.11]
0.368...	With hard square entropy constant [5.12]
0.3694375103...	C_7; one of the Hardy–Littlewood constants [2.1]
0.3720486812...	With digital search tree constants [5.14]
0.3728971438...	One of the extreme value constants [5.16]
0.3729475455...	δ_6; with Hermite's constants [8.7]
0.3733646177...	One of the binary search tree constants [5.13]
0.3739558136...	Artin's constant [2.4]
0.3790522777...	One of the self-avoiding walk constants [5.10]
0.380006...	$r(11)$ conjectured value; with circular coverage constants [8.2]
0.3825978582...	One of the geometric probability constants [8.1]
0.3919177761...	One of the extreme value constants [5.16]
0.3926990816...	$\pi/8$; with Moser's worm constants [8.4]
0.3943847688...	One of Moser's worm constants [8.4]
0.3949308436...	$r(10)$ conjectured value; with circular coverage constants [8.2]
0.3972130965...	With Otter's tree enumeration constants [5.6]
0.3995246670...	One of the quadratic recurrence constants [6.10]
0.3995352805...	α^{-1}; one of the Feigenbaum–Coullet–Tresser constants [1.9]
0.40096...	With Tammes' constants [8.8]
0.402...	With percolation cluster density constants [5.18]
0.4026975036...	With Otter's tree enumeration constants [5.6]
0.4074951009...	Hard square entropy constant [5.12]
0.4080301397...	$(2 - e^{-1})/4$; one of Rényi's parking constants [5.3]
0.4097321837...	Conjectured value of Berry–Esseen constant [4.7]

0.4098748850 ...	C_5; one of the Hardy–Littlewood constants [2.1]
0.412048 ...	With Lenz–Ising constants [5.22]
0.4124540336 ...	Prouhet–Thue–Morse constant [6.8]
0.4127732370 ...	With Kalmár's composition constant [5.5]
0.4142135623 ...	$\sqrt{2}-1$; with circular coverage [8.2], Lenz–Ising constants [5.22]
0.4159271089 ...	One of the extreme value constants [5.16]
0.4194 ...	One of the traveling salesman constants [8.5]
0.4198600459 ...	With Reuleaux triangle constants [8.10]
0.4203723394 ...	Minkowski–Bower constant [6.9]
0.4207263771 ...	Conjectured value of integer Chebyshev constant [4.9]
0.4212795439 ...	Schlüter's constant $t(10)$; with circular coverage constants [8.2]
0.4213829566 ...	$(6\ln(2))/\pi^2$; Lévy's constant [1.8]
0.4217993614 ...	With Pisot–Vijayaraghavan–Salem constants [2.30]
0.422 ...	With hard square entropy constant [5.12]
0.4227843351 ...	$1-\gamma$; with Euler–Mascheroni [1.5], Stieltjes constants [2.21]
0.4278733971 ...	With circumradius-inradius constants [8.19]
0.4281657248 ...	With Euler–Mascheroni constant[1.5]
0.4282495056 ...	Carefree constant, with Hafner–Sarnak–McCurley constant [2.5]
0.4302966531 ...	With Young–Fejér–Jackson constants [3.14]
0.4323323583 ...	$(1-e^{-2})/2$; one of Rényi's parking constants [5.3]
0.4330619231 ...	With Klarner's polyomino constant [5.19]
0.434 ...	With Hardy–Littlewood constants [2.1]
0.4381562356 ...	With Otter's tree enumeration constants [5.6]
0.4382829367 ...	$\mathrm{Re}(i^{i^{i^{\cdots}}})$; with iterated exponential constants [6.11]
0.4389253692 ...	One of Moser's worm constants [8.4]
0.43961 ...	One of the self-avoiding walk constants [5.10]
0.4399240125 ...	One of Otter's tree enumeration constants [5.6]
0.4406867935 ...	$\ln(\sqrt{2}+1)/2$; with Lenz–Ising constants [5.22]
0.4428767697 ...	With Otter's tree enumeration constants [5.6]
0.4450418679 ...	$r(8)$; with circular coverage constants [8.2]
0.4466 ...	3D dimer constant [5.23]
0.4472135955 ...	$1/\sqrt{5}$; 1st Diophantine approximation constant [2.23]
0.4490502094 ...	One of Moser's worm constants [8.4]
0.4522474200 ...	One of the Meissel–Mertens constants [2.2]
0.4545121805 ...	With Alladi–Grinstead constant [2.9]
0.4567332095 ...	With Otter's tree enumeration constants [5.6]
0.461543 ...	With Stieltjes constants [2.21]
0.4645922709 ...	With Landau–Ramanujan constant [2.3]
0.4652576133 ...	δ_5; with Hermite's constants [8.7]
0.4656386467 ...	With Otter's tree enumeration constants [5.6]
0.4702505696 ...	2·(Conway–Guy constant), with Erdös' sum-distinct set [2.28]
0.4718616534 ...	Conjectured value of Bloch's constant [7.1]

0.4749493799...	Weierstrass constant, with Gauss' lemniscate constant [6.1]
0.4756260767...	With Plouffe's constant [6.5]
0.4769...	Bland's constant, with traveling salesman constants [8.5]
0.4802959782...	One of the geometric probability constants [8.1]
0.4834983471...	With Golomb–Dickman constant [5.4]
0.4865198884...	With Landau–Ramanujan constant [2.3]
0.4876227781...	Gaussian twin prime constant, with Hardy–Littlewood constants [2.1]
0.4906940504...	With Landau–Ramanujan constant [2.3]
0.4945668172...	Shapiro–Drinfeld constant [3.1]
0.4956001805...	$1 - \gamma_0 - \gamma_1$; with [2.21] Stieltjes constants
0.5	$1/2$; with percolation cluster density [5.18], Landau–Ramanujan [2.3]
0.5163359762...	With Kepler–Bouwkamp constant [6.3]
0.5178759064...	With Otter's tree enumeration constants [5.6]
0.5212516264...	With Lebesgue constants [4.2]
0.5214054331...	Ghosh's constant, with geometric probability [8.1], traveling salesman [8.5]
0.5235987755...	$\pi/6$; with Archimedes [1.4], Madelung's constant [1.10]
0.531280...	With Gauss–Kuzmin–Wirsing constant [2.17]
0.5313399499...	One of the Pythagorean triple constants [5.2]
0.5341...	With Lenz–Ising constants [5.22]
0.5349496061...	One of Otter's tree enumeration constants [5.6]
0.5351070126...	With Artin's constant [2.4]
0.5392381750...	One of Pólya's random walk constants [5.9]
0.5396454911...	$\zeta(1/2) + 2$; with Euler–Mascheroni constant [1.5]
0.5405...	One of the longest subsequence constants [5.20]
0.5410442246...	With hyperbolic volume constants [8.9]
0.5432589653...	Conjectured value of Landau's constant [7.1]
0.5530512933...	Kuijlaars–Saff constant, with Tammes' constants [8.8]
0.5559052114...	Bezdek's constant $r(6)$; with circular coverage constants [8.2]
0.5598656169...	With Alladi–Grinstead constant [2.9]
0.5609498093...	One of the geometric probability constants [8.1]
0.5614594835...	$e^{-\gamma}$; Euler's constant [1.5], totient [2.7], Golomb–Dickman [5.4]
0.562009...	With Rényi's parking constant [5.3]
0.5671432904...	$W(1)$; solution of $xe^x = 1$, with iterated exponential constants [6.11]
0.5682854937...	With the abundant numbers density constant [2.11]
0.5683000031...	With Euler–Gompertz constant [6.2]
0.5697515829...	Weakly carefree constant, with Hafner–Sarnak–McCurley constant [2.5]
0.5731677401...	(3/4)·(Landau–Ramanujan constant) [2.3]
0.57339...	One of the Pell–Stevenhagen constants [2.8]

0.5743623733...	With Otter's tree enumeration constants [5.6]
0.5759599688...	Stephens' constant, with Artin's constant [2.4]
0.5761487691...	With Klarner's polyomino constant [5.18]
0.5767761224...	With Landau–Ramanujan constant [2.3]
0.5772156649...	Euler–Mascheroni constant, γ [1.5]; also Stieltjes constants [2.21]
0.5773502691...	$1/\sqrt{3}$; with Kakeya–Besicovitch constants [8.17]
0.5778636748...	$\pi/(2e)$; with Masser–Gramain constant [7.2]
0.5784167628...	$(8/7)\cos(2\pi/7)\cos(\pi/7)^2$; with Diophantine approximation constants [2.23]
0.5801642239...	One of the optimal stopping constants [5.15]
0.5805775582...	Pell constant [2.8]
0.5817480456...	With Madelung's constant [1.10]
0.5819486593...	With Landau–Ramanujan constant [2.3]
0.5825971579...	ρ; one of Pólya's random walk constants [5.9]
0.5831218080...	$2G/\pi$; 2D dimer constant [5.23]; also Kneser–Mahler [3.10]
0.5851972651...	With Euler–Gompertz constant [6.2]
0.5877...	One of the self-avoiding walk constants [5.10]
0.5878911617...	With hard square entropy constant [5.12]
0.59...	One of the optimal stopping constants [5.15]
0.5926327182...	Lehmer's constant [6.6]
0.5927460...	p_c; with percolation cluster density constants [5.18]
0.5947539639...	With Otter's tree enumeration constants [5.6]
0.5963473623...	Euler–Gompertz constant [6.2]
0.5990701173...	$M/2$; with Gauss' lemniscate constant [6.1]
0.6069...	One of the longest subsequence constants [5.20]
0.6079271018...	$6/\pi^2$; with Archimedes [1.4], Hafner–Sarnak–McCurley [2.5]
0.6083817178...	One of the Euler totient constants [2.7]
0.6093828640...	Neville's constant $r(5)$, with circular coverage constants [8.2]
0.6134752692...	Strongly triple-free set constant [2.26]
0.6168502750...	δ_4; with Hermite's constants [8.7]
0.6168878482...	With Otter's tree enumeration constants [5.6]
0.6180339887...	$\phi - 1$; with [1.2] Golden Mean
0.6194036984...	One of the Lenz–Ising constants [5.22]
0.6223065745...	Backhouse's constant, with Kalmár's constant [5.5]
0.6231198963...	With Otter's tree enumeration constants [5.6]
0.6232...	One of the traveling salesman constants [8.5]
0.6243299885...	Golomb–Dickman constant [5.4]
0.6257358072...	With Glaisher–Kinkelin constant [2.15]
0.6278342677...	With John constant [7.4]
0.6294650204...	Davison–Shallit constant ξ_1; with Cahen's constant [6.7]
0.6312033175...	One of the geometric probability constants [8.1]
0.6321205588...	$1 - 1/e$; with natural logarithmic base [1.3]
0.6331...	One of the traveling salesman constants [8.5]

0.6333683473 ...	2·(Atkinson–Negro–Santoro constant), with Erdös' sum-distinct set [2.28]
0.6351663546 ...	$C_3 = 2D/9$; one of the Hardy–Littlewood constants [2.1]
0.6366197723 ...	$2/\pi$; with Archimedes [1.4], rendezvous constants [8.21]
0.6389094054 ...	With Landau–Ramanujan constant [2.3]
0.6419448385 ...	With Meissel–Mertens constants [2.2]
0.6434105462 ...	Cahen's constant ξ_2 [6.7]
0.6462454398 ...	With Masser–Gramain constant [7.2]
0.6467611227 ...	With Euler–Gompertz constant [6.2]
0.6537 ...	One of the longest subsequence constants [5.20]
0.6539007091 ...	ξ_3; with Cahen's constant [6.7]
0.6556795424 ...	With Euler–Gompertz constant [6.2]
0.6563186958 ...	With Otter's tree enumeration constants [5.6]
0.6569990137 ...	$-\delta_0$; Hall–Montgomery constant [2.33]
0.6583655992 ...	One of the iterated exponential constants [6.11]
0.6594626704 ...	One of Pólya's random walk constants [5.9]
0.6597396084 ...	$(2+\sqrt{3})/(4\sqrt{2})$; with beam detection constant [8.11]
0.6600049346 ...	ξ_4; with Cahen's constant [6.7]
0.6601618158 ...	Twin prime constant, with Hardy–Littlewood constants [2.1]
0.6613170494 ...	Feller–Tornier constant, with Artin's constant [2.4]
0.6617071822 ...	One of the geometric probability constants [8.1]
0.6627434193 ...	Laplace limit constant [4.8]
0.6632657345 ...	ξ_5; with Cahen's constant [6.7]
0.6672538227 ...	With Feller's coin tossing constants [5.11]
0.6675277360 ...	With rendezvous constants [8.21]
0.6697409699 ...	Shanks' constant, with Hardy–Littlewood constants [2.1]
0.67 ...	Erdös–Lebensold constant [2.27]
0.6709083078 ...	With Madelung's constant [1.10]
0.6749814429 ...	Graham's hexagon constant [8.15]
0.676339 ...	One of the percolation cluster density constants [5.18]
0.6774017761 ...	With Kalmár's constant [5.5]
0.6821555671 ...	With Otter's tree enumeration constants [5.6]
0.6829826991 ...	With Calabi's triangle constant [8.13]
0.6844472720 ...	With Otter's tree enumeration constants [5.6]
0.6864067314 ...	C_{quad}; one of the Hardy–Littlewood constants [2.1]
0.6867778344 ...	With Kalmár's constant [5.5]
0.6903471261 ...	One of the iterated exponential constants [6.11]
0.6922006276 ...	$e^{-1/e}$; one of the iterated exponential constants [6.11]
0.6931471805 ...	$\ln(2)$; with natural logarithmic base [1.3]
0.6962 ...	One of the percolation cluster density constants [5.18]
0.6975013584 ...	2nd Pappalardi constant, with Artin's constant [2.4]
0.6977746579 ...	$I_1(2)/I_0(2)$; with Euler–Gompertz constant [6.2]
0.6979 ...	One of the traveling salesman constants [8.5]
0.6995388700 ...	One of Otter's tree enumeration constants [5.6]

0.70258 ...	Embree–Trefethen constant, with Golden mean [1.2]
0.7041699604 ...	With Fransén–Robinson constant [4.6]
0.7044798810 ...	$1 - 35/(12\pi^2)$; with rectilinear crossing constant [8.18]
0.7047534517 ...	With Landau–Ramanujan constant [2.3]
0.7047709230 ...	$(\pi - \sqrt{3})/2$; with Reuleaux triangle constants [8.10]
0.7059712461 ...	With Porter–Hensley constants [2.18]
0.708 ...	One of Pólya's random walk constants [5.9]
0.7098034428 ...	Rabbit constant, with Prouhet–Thue–Morse constant [6.8]
0.7124 ...	One of the traveling salesman constants [8.5]
0.7147827007 ...	Conjectured value, one of the traveling salesman constants [8.5]
0.7172 ...	One of the longest subsequence constants [5.20]
0.7213475204 ...	$1/(2\ln(2))$; with Lengyel's constant [5.7], Feller's coin tossing [5.11]
0.7218106748 ...	5D Steiner ratio, with Steiner tree constants [8.6]
0.7234 ...	One of the traveling salesman constants [8.5]
0.7235565167 ...	One of Pólya's random walk constants [5.9]
0.7236067977 ...	$(1/2)(1 + 1/\sqrt{5})$; with Diophantine approximation constants [2.23]
0.7252064830 ...	$97/150 + \pi/40$; Langford's constant, with geometric probability [8.1]
0.7266432468 ...	With van der Corput's constant [3.15]
0.726868 ...	With Graham's hexagon constant [8.15]
0.7322131597 ...	Unforgeable word constant, with pattern-free words [5.17]
0.7326498193 ...	With Landau–Ramanujan constant [2.3]
0.7373383033 ...	Grossman's constant [6.4]
0.7377507574 ...	Conjectured value of Whittaker–Goncharov constant [7.3]
0.7404804896 ...	$\pi/\sqrt{18}$; densest sphere packing, with Hermite's constants [8.7]
0.7424537454 ...	One of the Riesz–Kolmogorov constants [7.7]
0.7439711933 ...	Sarnak's constant, with Artin's constant [2.4]
0.7439856178 ...	4D Steiner ratio, with Steiner tree constants [8.6]
0.7475979202 ...	One of Rényi's parking constants [5.3]
0.749137 ...	With Graham's hexagon constant [8.15]
0.7493060013 ...	With Kneser–Mahler polynomial constants [3.10]
0.75	$3/4$; one of the self-avoiding walk constants [5.10]
0.7520107423 ...	One of the abelian group enumeration constants [5.1]
0.7578230112 ...	Flajolet–Odlyzko constant, with Golomb–Dickman [5.4]
0.760729 ...	With Graham's hexagon constant [8.15]
0.7608657675 ...	$(1/2)\cdot$(Bateman–Stemmler constant), with Hardy–Littlewood [2.1]
0.7642236535 ...	Landau–Ramanujan constant [2.3]
0.7647848097 ...	With Meissel–Mertens constants [2.2]
0.7656250596 ...	With Liouville–Roth constants [2.22]
0.7666646959 ...	With iterated exponential constants [6.11]

0.7669444905...	With Niven's constant [2.6]
0.7671198507...	Conway's constant [6.12]
0.77100...	One of the self-avoiding walk constants [5.10]
0.7711255236...	With Gauss–Kuzmin–Wirsing constant [2.17]
0.7735162909...	Flajolet–Martin constant, with Prouhet–Thue–Morse [6.8]
0.7759021363...	Bender's constant, with Lengyel's constant [5.7]
0.7776656535...	One of the geometric probability constants [8.1]
0.7824816009...	With Golomb–Dickman constant [5.4]
0.7834305107...	One of the iterated exponential constants [6.11]
0.7841903733...	3D Steiner ratio, with Steiner tree constants [8.6]
0.7853805572...	With Kepler–Bouwkamp constant [6.3]
0.7853981633...	$\pi/4$; with Kepler–Bouwkamp [6.3], Moser's worm [8.4]
0.7885305659...	Lüroth analog of Khintchine's constant [1.8]
0.79...	One of the optimal stopping constants [5.15]
0.7916031835...	One of Otter's tree enumeration constants [5.6]
0.7922082381...	Lal's constant, with Hardy–Littlewood constants [2.1]
0.8003194838...	Conjectured value, weakly triple-free set constant [2.26]
0.8008134543...	Bender's constant, with Lengyel's constant [5.7]
0.8019254372...	With Euler–Mascheroni constant [1.5]
0.8043522628...	One of the optimal stopping constants [5.15]
0.8086525183...	Solomon's parking constant, with Rényi's parking [5.3]
0.8093940205...	Alladi–Grinstead constant [2.9]
0.8116869215...	One of Flajolet's constants, with Thue–Morse [6.8]
0.8118...	One of the longest subsequence constants [5.20]
0.8125565590...	Stolarsky–Harborth constant [2.16]
0.8128252421...	With Young–Fejér–Jackson constants [3.14]
0.81318...	c_0; one of the longest subsequence constants [5.20]
0.8137993642...	With Reuleaux triangle constants [8.10]
0.8175121124...	With Shapiro–Drinfeld constant [3.1]
0.82...	With k-satisfiability constants [5.21]
0.822...	One of Pólya's random walk constants [5.9]
0.8224670334...	$\pi^2/12$; with traveling salesman constants [8.5]
0.8247830309...	$(\sqrt{5}-1)/\sqrt{2}$; one of Turán's power sum constants [3.16]
0.8249080672...	$2\cdot$(Prouhet–Thue–Morse constant) [6.8]
0.8269933431...	$3\sqrt{3}/(2\pi)$; with circular coverage constants [8.2]
0.8319073725...	$1/\zeta(3)$; with Apéry's constant [1.6]
0.8324290656...	Rosser's constant, with Hardy–Littlewood constants [2.1]
0.8346268416...	$1/M$; with Gauss' lemniscate constant [6.1]
0.8351076361...	With Hall–Montgomery constant [2.33]
0.8371132125...	A_3'; with Brun's constant [2.14]
0.8403426028...	With Reuleaux triangle constants [8.10]
0.8427659133...	$(12\ln(2))/\pi^2$; Lévy's constant [1.8]
0.8472130848...	$3M/\sqrt{2}$; ubiquitous constant, with Gauss' lemniscate [6.1]
0.8507361882...	Paper folding constant, with Prouhet–Thue–Morse [6.8]

0.8561089817 . . .	With Landau–Ramanujan constant [2.3]
0.8565404448 . . .	3rd Pappalardi constant, with Artin's constant [2.4]
0.8621470373 . . .	With Gauss–Kuzmin–Wirsing constant [2.17]
0.8636049963 . . .	With Stolarsky–Harborth constant [2.16]
0.8657725922 . . .	Conjectured value of integer Chebyshev constant [4.9]
0.8660254037 . . .	$\sqrt{3}/2$; 2D Steiner ratio [8.6], universal coverage [8.3]
0.8689277682 . . .	With Landau–Ramanujan constant [2.3]
0.8705112052 . . .	With Otter's tree enumeration constants [5.6]
0.8705883800 . . .	A_4; with Brun's constant [2.14]
0.8711570464 . . .	One of Flajolet's constants, with Thue–Morse [6.8]
0.8728875581 . . .	With Landau–Ramanujan constant [2.3]
0.8740191847 . . .	$L/3$; with Landau–Ramanujan [2.3], Gauss' lemniscate [6.1]
0.8740320488 . . .	One of Turán's power sum constants [3.16]
0.8744643684 . . .	With Niven's constant [2.6]
0.8785309152 . . .	One of the geometric probability constants [8.1]
0.8795853862 . . .	With Lenz–Ising constants [5.22]
0.8815138397 . . .	Average class number, with Artin's constant [2.4]
0.8856031944 . . .	Minimum of $\Gamma(x)$, with Euler–Mascheroni constant [1.5.4]
0.8905362089 . . .	$e^{\gamma}/2$; with Hardy–Littlewood constants [2.1]
0.8928945714 . . .	With Niven's constant [2.6]
0.8948412245 . . .	With Landau–Ramanujan constant [2.3]
0.90177 . . .	$\sqrt{c_0}$; one of the longest subsequence constants [5.20]
0.90682 . . .	One of Rényi's parking constants [5.3]
0.9068996821 . . .	$\pi/\sqrt{12}$; densest circle packing, with Hermite's constants [8.7]
0.9089085575 . . .	With "one-ninth" constant [4.5]
0.91556671 . . .	One of Rényi's parking constants [5.3]
0.9159655941 . . .	Catalan's constant, G [1.7]
0.9241388730 . . .	With hyperbolic volume constants [8.9]
0.9285187329 . . .	With Gauss–Kuzmin–Wirsing constant [2.17]
0.9296953983 . . .	$\ln(2)/2 + 2G/\pi$; with Lenz–Ising constants [5.22]
0.9312651841 . . .	4th Pappalardi constant, with Artin's constant [2.4]
0.9375482543 . . .	$-\zeta'(2)$; with Porter's constant [2.18]
0.9468064072 . . .	With Landau–Ramanujan constant [2.3]
0.9625228267 . . .	With Lebesgue constants [4.2]
0.9625817323 . . .	c_0^+; with [5.22] Lenz–Ising constants
0.9730397768 . . .	With Landau–Ramanujan constant [2.3]
0.9780124781 . . .	Elbert's constant, with Shapiro–Drinfeld [3.1]
0.9795555269 . . .	3rd Bendersky constant, with Glaisher–Kinkelin [2.15]
0.9848712825 . . .	One of Rényi's parking constants [5.3]
0.9852475810 . . .	With Landau–Ramanujan constant [2.3]
0.9877003907 . . .	With universal coverage constants [8.3]
0.9878490568 . . .	ln(Khintchine's constant) [1.8]
0.9891336344 . . .	2·(Shapiro–Drinfeld constant) [3.1]
0.9894312738 . . .	With Lebesgue constants [4.2]

0.9920479745 ...	4^{th} Bendersky constant, with Glaisher–Kinkelin [2.15]
0.9932 ...	With geometric probability constants [8.1]
1	One; conjectured value of Linnik's constant, Baker's constant [2.12]
1.0028514266 ...	With Moser's worm constants [8.4]
1.0031782279 ...	Generalized Stirling constant, with Stieltjes constants [2.21]
1.0074347569 ...	DeVicci's tesseract constant [8.14]
1.0077822185 ...	$\sqrt{65}/8$; with Heilbronn triangle constants [8.16]
1.0096803872 ...	5^{th} Bendersky constant, with Glaisher–Kinkelin [2.15]
1.0149416064 ...	$\pi \ln(\beta)$; Gieseking's constant, with Kneser–Mahler [3.10]
1.0174087975 ...	h_3; with Euler–Mascheroni constant [1.5.4]
1.0185012157 ...	With Porter–Hensley constant [2.18]
1.0208 ...	One of the traveling salesman constants [8.5]
1.0250590965 ...	With Lenz–Ising constants [5.22]
1.0306408341 ...	$\pi^2/(6\ln(2)\ln(10))$; Lévy's constant [1.8]
1.0309167521 ...	2^{nd} Bendersky constant, with Glaisher–Kinkelin [2.15]
1.0346538818 ...	One of the Meissel–Mertens constants [2.2]
1.0451637801 ...	Li(2); with Euler–Gompertz constant [6.2]
1.0471975511 ...	$\pi/3$; with universal coverage constants [8.3]
1.0478314475 ...	One of the quadratic recurrence constants [6.10]
1.0544399448 ...	With Landau–Ramanujan constant [2.3]
1.0547001962 ...	One of the self-avoiding walk constants [5.10]
1.0606601717 ...	With DeVicci's tesseract constant [8.14]
1.0662758532 ...	With Lebesgue constants [4.2]
1.0693411205 ...	One of Pólya's random walk constants [5.9]
1.0786470120 ...	One of Pólya's random walk constants [5.9]
1.0786902162 ...	With Sobolev isoperimetric constants [3.6]
1.0820884492 ...	With hyperbolic volume constants [8.9]
1.0873780254 ...	One of Feller's coin tossing constants [5.11]
1.0892214740 ...	With Vallée's constant [2.19]
1.0894898722 ...	$1/2 + G/\pi$; with Wilbraham–Gibbs constant [4.1]
1.0939063155 ...	One of Pólya's random walk constants [5.9]
1.0963763171 ...	With DeVicci's tesseract constant [8.14]
1.0978510391 ...	A_3; with Brun's constant [2.14]
1.0986419643 ...	Paris' constant, with Golden mean [1.2]
1.0986858055 ...	Lengyel's constant [5.7]
1.1009181908 ...	With digital search tree constants [5.14]
1.1038396536 ...	With Gauss–Kuzmin–Wirsing constant [2.17]
1.1061028674 ...	One of Pólya's random walk constants [5.9]
1.1064957714 ...	One of the Copson–de Bruijn constants [3.5]
1.1128357889 ...	$(4L)/(3\pi)$; with Landau–Ramanujan constant [2.3]
1.1169633732 ...	One of Pólya's random walk constants [5.9]
1.1178641511 ...	Goh-Schmutz constant, with Golomb–Dickman [5.4]
1.1180339887 ...	$\sqrt{5}/2$; one of the Steinitz constants [3.13]

1.128057 ...	One of the percolation cluster density constants [5.18]
1.1289822228 ...	With Otter's tree enumeration constants [5.6]
1.13198824 ...	Viswanath's constant, with Golden mean [1.2]
1.1365599187 ...	With Otter's tree enumeration constants [5.6]
1.1373387363 ...	One of the digital search tree constants [5.14]
1.1481508398 ...	With Porter's constant [2.18]
1.1504807723 ...	Goldbach–Vinogradov constant, with Hardy–Littlewood [2.1]
1.1530805616 ...	With Landau–Ramanujan constant [2.3]
1.1563081248 ...	One of Pólya's random walk constants [5.9]
1.1574198038 ...	With Otter's tree enumeration constants [5.6]
1.1575 ...	One of the self-avoiding walk constants [5.10]
1.1587284730 ...	Grazing goat constant, with circular coverage constants [8.2]
1.159 ...	One of the self-avoiding walk constants [5.10]
1.1662436161 ...	$4G/\pi$; with Lenz–Ising constants [5.22]
1.1762808182 ...	Salem constant [2.30]
1.177043 ...	One of the self-avoiding walk constants [5.10]
1.1789797444 ...	$2G/\pi$; with Wilbraham–Gibbs constant [4.1]
1.1803405990 ...	h_1; with Euler–Mascheroni constant [1.5.4]
1.1865691104 ...	$\pi^2/(12\ln(2))$; Lévy's constant [1.8]
1.1874523511 ...	Foias' constant, with Grossman's constant [6.4]
1.1981402347 ...	M; Gauss' lemniscate constant [6.1]
1.1996786402 ...	With Laplace limit constant [4.8]
1.2013035599 ...	Rosser's constant, with Hardy–Littlewood constants [2.1]
1.2020569031 ...	$\zeta(3)$; Apéry's constant [1.6]
1.205 ...	One of the self-avoiding walk constants [5.10]
1.2087177032 ...	Baxter's constant, with Lieb's square ice constant [5.24]
1.2160045618 ...	One of Otter's tree enumeration constants [5.6]
1.21667 ...	One of the self-avoiding walk constants [5.10]
1.2241663491 ...	One of Otter's tree enumeration constants [5.6]
1.2267420107 ...	Fibonacci factorial constant, with Golden mean [1.2]
1.2368398446..	One of Feller's coin tossing constants [5.11]
1.238 ...	With Lenz–Ising constants [5.22]
1.2394671218 ...	One of Pólya's random walk constants [5.9]
1.257 ...	With Prouhet–Thue–Morse constant [6.8]
1.2577468869 ...	With Alladi–Grinstead [2.9], Khintchine–Lévy [1.8]
1.2599210498 ...	$\sqrt[3]{2}$; with Pythagoras' constant [1.1]
1.2610704868 ...	With binary search tree constants [5.13]
1.2615225101 ...	With hyperbolic volume constants [8.9]
1.2640847353 ...	One of the quadratic recurrence constants [6.10]
1.2672063606 ...	μ_6; one of the extreme value constants [5.16]
1.272 ...	One of the self-avoiding walk constants [5.10]
1.275 ...	One of the self-avoiding walk constants [5.10]
1.2824271291 ...	Glaisher–Kinkelin constant [2.15]
1.2885745539 ...	With Feigenbaum–Coullet–Tresser constants [1.9]

1.2910603681 ...	With Vallée's constant [2.19]
1.2912859970 ...	One of the iterated exponential constants [6.11]
1.2923041571 ...	With Landau–Ramanujan constant [2.3]
1.2940 ...	One of the self-avoiding walk constants [5.10]
1.2985395575 ...	Bateman's A constant, with Hardy–Littlewood [2.1]
1.302 ...	Square-free word constant [5.17]
1.3035772690 ...	Conway's constant [6.12]
1.30568 ...	Apollonian packing constant [8.20]
1.3063778838 ...	Mills' constant [2.13]
1.3110287771 ...	Quarter-lemniscate arclength $L/2$, Gauss' constant [6.1]
1.3135070786 ...	K_{-3}; with Khintchine's constant [1.8]
1.3203236316 ...	$2C_{\text{twin}}$; one of the Hardy–Littlewood constants [2.1]
1.3247179572 ...	With Golden mean [1.2], Pisot–Vijayaraghavan constants [2.30]
1.3325822757 ...	With Meissel–Mertens [2.2], totient constants [2.7]
1.3385151519 ...	$\exp(G/\pi)$; 2D dimer constant [5.23]
1.3426439511 ...	With hard square entropy constant [5.12]
1.34375	$43/32$; one of the self-avoiding walk constants [5.10]
1.3468852519 ...	One of the Riesz–Kolmogorov constants [7.7]
1.3505061 ...	One of the quadratic recurrence constants [6.10]
1.3511315744 ...	With Vallée's constant [2.19]
1.3521783756 ...	μ_7; one of the extreme value constants [5.16]
1.3531302722 ...	With optimal stopping constants [5.15]
1.3694514039 ...	Shallit's constant, with Shapiro–Drinfeld constant [3.1]
1.3728134628 ...	$2C_{\text{quad}}$; one of the Hardy–Littlewood constants [2.1]
1.3750649947 ...	One of the Meissel–Mertens constants [2.2]
1.37575 ...	With geometric probability constants [8.1]
1.3813564445 ...	β; with Kneser–Mahler polynomial constants [3.10]
1.3905439387 ...	Bateman's B constant, with Hardy–Littlewood [2.1]
1.3932039296 ...	One of Pólya's random walk constants [5.9]
1.3954859724 ...	Hard hexagon entropy constant, with hard square [5.12]
1.3994333287 ...	With Kalmár's composition constant [5.5]
1.4011551890 ...	Myrberg's constant, with Feigenbaum–Coullet–Tresser [1.9]
1.4045759346 ...	Conjectured value of complex Grothendieck constant [3.11]
1.4092203477 ...	With Stolarsky–Harborth constant [2.16]
1.4106861346 ...	With Euler–Gompertz constant [6.2]
1.4142135623 ...	$\sqrt{2}$; Pythagoras' constant [1.1]
1.4236003060 ...	μ_8; one of the extreme value constants [5.16]
1.4298155 ...	One of the quadratic recurrence constants [6.10]
1.4359911241	$1/3 + 2\sqrt{3}/\pi$; 1^{st} Lebesgue constant [4.2]
1.4426950408 ...	$\ln(2)^{-1}$; with Porter–Hensley constants [2.18]
1.4446678610 ...	$e^{1/e}$; one of the iterated exponential constants [6.11]
1.4503403284 ...	K_{-2}; with Khintchine's constant [1.8]
1.4513692348 ...	Ramanujan–Soldner constant, with Euler–Gompertz [6.2]

1.4560749485 ...	Backhouse's constant, with [5.5]
1.457 ...	Cube-free word constant [5.17]
1.4603545088 ...	$-\zeta(1/2)$; with Apéry's constant [1.6]
1.4609984862 ...	Baxter's constant, with Lieb's square ice [5.24]
1.4616321449 ...	x minimizing $\Gamma(x)$, with Euler–Mascheroni constant [1.5.4]
1.4655712318 ...	Moore's constant, with the Golden mean [1.2]
1.4670780794 ...	Porter's constant [2.18]
1.4677424503 ...	One of the Feigenbaum–Coullet–Tresser constants [1.9]
1.4681911223 ...	With Alladi–Grinstead constant [2.9]
1.4741726868 ...	One of Otter's tree enumeration constants [5.6]
1.4762287836 ...	With Kalmár's composition constant [5.5]
1.4767 ...	One of the self-avoiding walk constants [5.10]
1.4879506635 ...	$-\zeta(2/3)/\zeta(2)$; with Niven's constant [2.6]
1.4880785456 ...	One of Otter's tree enumeration constants [5.6]
1.5028368010 ...	One of the quadratic recurrence constants [6.10]
1.5030480824 ...	Hard square entropy constant [5.12]
1.50659177 ...	Area of Mandelbrot set, quadratic recurrence [6.10]
1.50685 ...	Nagle's constant, with Lieb's square ice constant [5.24]
1.5078747554 ...	Greenfield–Nussbaum constant, quadratic recurrence [6.10]
1.5163860591 ...	One of Pólya's random walk constants [5.9]
1.5217315350 ...	Bateman–Stemmler constant, Hardy–Littlewood [2.1]
1.5299540370 ...	With Gauss' lemniscate constant [6.1]
1.5353705088 ...	With digital search tree constants [5.14]
1.5396007178 ...	$(4/3)^{3/2}$; Lieb's square ice constant [5.24]
1.5422197217 ...	Madelung constant for planar hexagonal lattice [1.10]
1.5449417003 ...	With Reuleaux triangle constants [8.10]
1.5464407087 ...	With hard square entropy constant [5.12]
1.5513875245 ...	Calabi's triangle constant [8.13]
1.5557712501 ...	One of the Feigenbaum–Coullet–Tresser constants [1.9]
1.5707963267 ...	$\pi/2$; with Archimedes' constant [1.4]
1.5849625007 ...	$\ln(3)/\ln(2)$; with Stolarsky–Harborth constant [2.16]
1.5868266790 ...	With Feigenbaum–Coullet–Tresser constants [1.9]
1.6066951524 ...	One of the digital search tree constants [5.14]
1.6153297360 ...	With Lenz–Ising constants [5.22]
1.6155426267 ...	Negative of 2D NaCl Madelung constant [1.10]
1.6180339887 ...	Golden mean, φ [1.2]
1.6222705028 ...	Odlyzko–Wilf constant, with Mills' constant [2.13]
1.6281601297 ...	Flajolet–Martin constant, with Prouhet–Thue–Morse [6.8]
1.6366163233 ...	With Erdös–Lebensold constant [2.27]
1.6421884352 ...	2nd Lebesgue constant [4.2]
1.644703 ...	With moving sofa constant [8.12]
1.6449340668 ...	$\pi^2/6$; with Apéry [1.6], Hafner–Sarnak–McCurley [2.5]
1.6467602581 ...	With digital search tree constants [5.14]
1.6600 ...	With Lieb's square ice constant [5.24]

1.6616879496...	One of the quadratic recurrence constants [6.10]
1.6813675244...	One of Otter's tree enumeration constants [5.6]
1.6824415102...	Bateman–Grosswald c_{23} constant, with Niven's constant [2.6]
1.6847127097...	One of the circumradius-inradius constants [8.19]
1.6857...	One of the traveling salesman constants [8.5]
1.6903029714...	One of the Feigenbaum–Coullet–Tresser constants [1.9]
1.6910302067...	With Cahen's constant [6.7]
1.6964441175...	With Khintchine–Lévy constants [1.8]
1.7052111401...	Niven's constant [2.6]
1.7091579853...	$\pi^2/(12\ln(\varphi))$; with Khintchine–Lévy constants [1.8]
1.7286472389...	Kalmár's composition constant [5.5]
1.7356628245...	With Klarner's polyomino constant [5.19]
1.7374623212...	With Golomb–Dickman constant [5.4]
1.7410611252...	One of Otter's tree enumeration constants [5.6]
1.7454056624...	K_{-1}; with Khintchine's constant [1.8]
1.7475645946...	Negative of 3D NaCl Madelung constant [1.10]
1.75	$7/4$; with Lenz–Ising constants [5.22]
1.7548776662...	One of Feller's coin tossing constants [5.11]
1.7555101394...	One of Otter's tree enumeration constants [5.6]
1.756...	One of the self-avoiding walk constants [5.10]
1.7579327566...	Infinite nested radical, with Golden mean [1.2]
1.7587436279...	With Alladi–Grinstead constant [2.9]
1.7632228343...	With iterated exponential constants [6.11]
1.76799378...	With optimal stopping constants [5.15]
1.77109...	$-c_1$; one of the longest subsequence constants [5.20]
1.7724538509...	$\sqrt{\pi}$; with Euler's constant [1.5.4], Carlson–Levin [3.2]
1.7783228615...	3rd Lebesgue constant [4.2]
1.7810724179...	e^{γ}; with Euler's constant [1.5], Erdös–Lebensold [2.27]
1.7818046151...	Conjectured value of power series constant [7.3]
1.7822139781...	Conjectured value of real Grothendieck constant [3.11]
1.7872316501...	Komornik–Loreti constant, with Prouhet–Thue–Morse [6.8]
1.7916228120...	$\exp(2G/\pi)$; 2D dimer constant [5.23]
1.7941471875...	With Kalmár's composition constant [5.5]
1.8173540210...	One of the self-avoiding walk constants [5.10]
1.8228252496...	Conjectured value of Masser–Gramain constant [7.2]
1.8356842740...	With Meissel–Mertens constants [2.2]
1.8392867552...	Associated with Tribonacci sequence and Golden mean [1.2]
1.8393990840...	Negative of 4D NaCl Madelung constant [1.10]
1.8442049806...	One of the Landau–Kolmogorov constants [3.3]
1.8477590650...	$\sqrt{2+\sqrt{2}}$; conjectured value of self-avoiding walk constant [5.10]
1.8519370519...	Wilbraham–Gibbs constant [4.1]
1.8540746773...	$L/\sqrt{2}$; with Gauss' lemniscate constant [6.1]
1.8823126444...	One of the geometric probability constants [8.1]

1.9021605831 ...	Brun's constant [2.14]
1.9081456268 ...	β^2 ; with Kneser–Mahler polynomial constants [3.10]
1.9093378156 ...	Negative of 5D NaCl Madelung constant [1.10]
1.9126258077 ...	One of Otter's tree enumeration constants [5.6]
1.9276909638 ...	One of the Feigenbaum–Coullet–Tresser constants [1.9]
1.9287800 ...	Wright's constant, with Mills' constant [2.13]
1.940215351 ...	2D monomer-dimer constant [5.23]
1.9435964368 ...	One of the Euler totient constants [2.7]
1.9484547890 ...	c_4; with Kneser–Mahler polynomial constants [3.10]
1.9504911124 ...	4·(Gaussian twin prime constant), with Hardy–Littlewood [2.1]
1.9655570390 ...	Negative of 6D NaCl Madelung constant [1.10]
1.9670449011 ...	c_5; with Kneser–Mahler polynomial constants [3.10]
1.9771268308 ...	c_6; with Kneser–Mahler polynomial constants [3.10]
1.9954559575 ...	With Fransén–Robinson constant [4.6]
2	Two; conjectured value of fast matrix multiplication constant [2.29]
2.006 ...	With Erdös–Lebensold constant [2.27]
2.0124059897 ...	Negative of 7D NaCl Madelung constant [1.10]
2.0287578381 ...	With Du Bois Reymond constants [3.12]
2.0415 ...	One of the traveling salesman constants [8.5]
2.0462774528 ...	Lüroth analog of Lévy's constant [1.8]
2.05003 ...	One of the Whitney–Mikhlin extension constants [3.8]
2.0524668272 ...	Negative of 8D NaCl Madelung constant [1.10]
2.0531987328 ...	With self-avoiding walk constants [5.10]
2.0780869212 ...	$\ln(\varphi)^{-1}$; with Porter–Hensley constants [2.18]
2.1080540666 ...	With rendezvous constants [8.21]
2.1102339661 ...	Brown–Wang constant, from Young–Fejér–Jackson [3.14]
2.158 ...	Mian–Chowla constant, with Erdös' reciprocal sum [2.20]
2.1732543125 ...	$\zeta(3/2)/\zeta(3)$; with Niven's constant [2.6]
2.1760161352 ...	With Kneser–Mahler polynomial constants [3.10]
2.1894619856 ...	One of Otter's tree enumeration constants [5.6]
2.1918374031 ...	One of Otter's tree enumeration constants [5.6]
2.2001610580 ...	Lüroth analog of Khintchine's constant [1.8]
2.2038565964 ...	One of the Euler totient constants [2.7]
2.2195316688 ...	Moving sofa constant [8.12]
2.2247514809 ...	Robinson's C constant, with Khintchine's constant [1.8]
2.2394331040 ...	Takeuchi–Prellberg constant [5.8]
2.2665345077 ...	With Fransén–Robinson constant [4.6]
2.2782916414 ...	One of Moser's worm constants [8.4]
2.2948565916 ...	One of the abelian group enumeration constants [5.1]
2.3 ...	Estimate of $s_c(3)$, with k-satisfiability constants [5.21]
2.3025661371 ...	One of Flajolet's constants, with Thue–Morse [6.8]
2.3038421962 ...	Robinson's A constant, with Khintchine's constant [1.8]
2.3091385933 ...	With Klarner's polyomino constant [5.19]

2.3136987039 ...	With Madelung's constant [1.10]
2.3212850380 ...	With rendezvous constants [8.21]
2.3360 ...	With Lieb's square ice constant [5.24]
2.3507 ...	One of the Landau–Kolmogorov constants [3.3]
2.3565273533 ...	One of the monomer-dimer constants [5.23]
2.3731382208 ...	$\pi^2/(6\ln(2))$; Lévy's constant [1.8]
2.37597 ...	With Klarner's polyomino constant [5.19]
2.3768417063 ...	Conjectured value of integer Chebyshev constant [4.9]
2.3979455861 ...	With Du Bois Reymond constants [3.12]
2.4048255576 ...	First zero of $J_0(x)$, with Sobolev isoperimetric constants [3.6]
2.4149010237 ...	With Golomb–Dickman constant [5.4]
2.4413238136 ...	With Lebesgue constants [4.2]
2.4725480752 ...	With Sobolev isoperimetric constants [3.6]
2.4832535361 ...	One of Otter's tree enumeration constants [5.6]
2.4996161129 ...	One of the abelian group enumeration constants [5.1]
2.5029078750 ...	α; one of the Feigenbaum–Coullet–Tresser constants [1.9]
2.5066282746 ...	$\sqrt{2\pi}$; Stirling's constant; with Archimedes [1.4], Glaisher–Kinkelin [2.15]
2.5175403550 ...	One of Otter's tree enumeration constants [5.6]
2.5193561520 ...	With Madelung's constant [1.10]
2.5695443449 ...	$e^\gamma / \ln(2)$; with Euler–Mascheroni constant [1.5]
2.5849817595 ...	Sierpinski's constant [2.10]
2.5980762113 ...	$\sqrt{27/4}$; with Lieb's square ice constant [5.24]
2.6034 ...	With Lieb's square ice constant [5.24]
2.6180339887 ...	Golden root $\varphi + 1$, with Tutte–Beraha [5.25], Gauss–Kuzmin–Wirsing [2.17]
2.6220575542 ...	Half-lemniscate arclength L, Gauss' constant [6.1]
2.6381585303 ...	Estimate of 2D self-avoiding walk constant [5.10]
2.6389584337 ...	$(2+\sqrt{3})/\sqrt{2}$; with beam detection constant [8.11]
2.67564 ...	With Klarner's polyomino constant [5.19]
2.6789385347 ...	$\Gamma(1/3)$; with Euler–Mascheroni constant [1.5.4]
2.6789638796 ...	4·(Shanks' constant), with Hardy–Littlewood constants [2.1]
2.6811281472 ...	One of Otter's tree enumeration constants [5.6]
2.6854520010 ...	Khintchine's constant [1.8]
2.7182818284 ...	Natural logarithmic base, e [1.3]
2.72062 ...	One of the self-avoiding walk constants [5.10]
2.7494879027 ...	One of Otter's tree enumeration constants [5.6]
2.75861972 ...	With hyperbolic volume constants [8.9]
2.7865848321 ...	With Fransén–Robinson constant [4.6]
2.8077702420 ...	Fransén–Robinson constant [4.6]
2.8154600332 ...	One of Otter's tree enumeration constants [5.6]
2.8264199970 ...	Murata's constant, with Artin [2.4], totient [2.7]
2.8336106558 ...	One of the Feigenbaum–Coullet–Tresser constants [1.9]
2.8372974794 ...	With Madelung's constant [1.10]

2.8582485957 . . .	One of the Hardy–Littlewood constants [2.1]
2.9409823408 . . .	$c_o/\sqrt{2\pi}$; with Lengyel's constant [5.7]
2.9409900447 . . .	$c_e/\sqrt{2\pi}$; with Lengyel's constant [5.7]
2.9557652856 . . .	One of Otter's tree enumeration constants [5.6]
2.9904703993 . . .	Goh–Schmutz constant, with Golomb–Dickman [5.4]
3	Three; with Tutte–Beraha constants [5.25]
3.0079 . . .	With Erdös' reciprocal sum constants [2.20]
3.01 . . .	With Erdös' reciprocal sum constants [2.20]
3.1415926535 . . .	Archimedes' constant, π [1.4]
3.1477551485 . . .	Quadratic residues constant, with Meissel–Mertens [2.2]
3.1704593421 . . .	One of the Euler totient constants [2.7]
3.1962206165 . . .	"Plate" constant, with Sobolev isoperimetric constants [3.6]
3.2099123007 . . .	$\exp(4G/\pi)$; 2D dimer constant [5.23]; also Kneser–Mahler [3.10]
3.2469796037 . . .	Silver root, one of the Tutte–Beraha constants [5.25]
3.2504 . . .	With Lieb's square ice constant [5.24]
3.2659724710 . . .	One of Otter's tree enumeration constants [5.6]
3.2758229187 . . .	$\exp(\pi^2)/(12\ln(2))$; Lévy's constant [1.8]
3.2871120555 . . .	One of Otter's tree enumeration constants [5.6]
3.2907434386 . . .	One of Otter's tree enumeration constants [5.6]
3.3038421963 . . .	Robinson's B constant, with Khintchine's constant [1.8]
3.33437 . . .	Bumby's constant, with Freiman's constant [2.31]
3.3412669407 . . .	With Otter's tree enumeration constants [5.6]
3.3598856662 . . .	With digital search tree constants [5.14]
3.3643175781 . . .	Van der Corput's constant [3.15]
3.4070691656 . . .	Magata's constant, with Kalmár's composition constant [5.5]
3.4201328816 . . .	With self-avoiding walk constants [5.10]
3.4493588902 . . .	Robinson's D constant, with Khintchine's constant [1.8]
3.4627466194 . . .	Q^{-1}; with digital search tree constants [5.14], Lengyel [5.7]
3.501838 . . .	With self-avoiding walk constants [5.10]
3.5070480758 . . .	With Feller's coin tossing constants [5.11]
3.5795 . . .	With Lieb's square ice constant [5.24]
3.6096567319 . . .	Conjectured value of ρ_2, Diophantine approximation [2.23]
3.6180339887 . . .	$\varphi + 2$; one of the Tutte–Beraha constants [5.25]
3.6256099082 . . .	$\Gamma(1/4)$; with Euler–Mascheroni constant [1.5.4]
3.63600703 . . .	One of the Feigenbaum–Coullet–Tresser constants [1.9]
3.6746439660 . . .	Quadratic residues constant, with Meissel–Mertens [2.2]
3.6754 . . .	One of the longest subsequence constants [5.20]
3.7038741039 . . .	2·(Wilbraham–Gibbs constant) [4.1]
3.764435608 . . .	2D monomer-dimer constant [5.23]
3.7962 . . .	z_c; with hard square entropy constant [5.12]
3.8264199970 . . .	Murata's constant + 1, with Artin [2.4], totient [2.7]
3.8695192413 . . .	With optimal stopping constants [5.15]
3.9002649200 . . .	With Madelung's constant [1.10]

3.921545 ...	With Moser's worm constants [8.4]
3.92259368 ...	With hyperbolic volume constants [8.9]
4	Four; Tutte–Beraha [5.25], 2D Grötzsch ring constant [7.8]
4.0180767046 ...	One of the Feigenbaum–Coullet–Tresser constants [1.9]
4.062570 ...	Klarner's polyomino constant [5.18]
4.121326 ...	One of the Feigenbaum–Coullet–Tresser constants [1.9]
4.1327313541 ...	$\sqrt{2\pi e}$; Sobolev isoperimetric [3.6], traveling salesman [8.5]
4.1507951 ...	One of the self-avoiding walk constants [5.10]
4.1511808632 ...	One of the Hardy–Littlewood constants [2.1]
4.2001 ...	With Lieb's square ice constant [5.24]
4.2472965459 ...	One of the geometric probability constants [8.1]
4.25 ...	Estimate of $r_c(3)$, with k-satisfiability constants [5.21]
4.3076923076 ...	$56/13$; Korn constant for 3D ball [3.7]
4.3110704070 ...	One of the binary search tree constants [5.13]
4.5278295661 ...	Freiman's constant [2.31]
4.5651 ...	With Lieb's square ice constant [5.24]
4.5678018826 ...	Gasper's constant, with Young–Fejér–Jackson [3.14]
4.5860790989 ...	One of Pólya's random walk constants [5.9]
4.5908437119 ...	$\Gamma(1/5)$; with Euler–Mascheroni constant [1.5.4]
4.6592661225 ...	Bateman–Grosswald c_{03} constant, with Niven's constant [2.6]
4.6692016091 ...	δ; one of the Feigenbaum–Coullet–Tresser constants [1.9]
4.68404 ...	Estimate of 3D self-avoiding walk constant [5.10]
4.7300407448 ...	"Rod" constant, with Sobolev isoperimetric [3.6]
4.799891547 ...	Three-arc approximation of beam detection constant [8.11]
4.8189264563 ...	Two-arc approximation of beam detection constant [8.11]
4.8426 ...	One of the self-avoiding walk constants [5.10]
4.9264 ...	With Artin's constant [2.4]
5.0747080320 ...	With hyperbolic volume constants [8.9]
5.1387801326 ...	With Sobolev isoperimetric constants [3.6]
5.1667 ...	With Lieb's square ice constant [5.24]
5.2441151086 ...	Lemniscate arclength $2L$, Gauss' constant [6.1]
5.2569464048 ...	With Euler–Mascheroni constant [1.5.4]
5.4545172445 ...	With Khintchine–Lévy constants [1.8]
5.5243079702 ...	With Khintchine–Lévy constants [1.8]
5.5553 ...	With Lieb's square ice constant [5.24]
5.5663160017 ...	$\Gamma(1/6)$; with Euler–Mascheroni constant [1.5.4]
5.6465426162 ...	One of Otter's tree enumeration constants [5.6]
5.6493764966 ...	Conjectured value of integer Chebyshev constant [4.9]
5.7831859629 ...	With Sobolev isoperimetric constants [3.6]
5.8726188208 ...	Bateman–Grosswald $-c_{13}$ constant, with Niven's constant [2.6]
5.9087 ...	With Artin's constant [2.4]
5.9679687038 ...	One of the Feigenbaum–Coullet–Tresser constants [1.9]

6.0 . . .	One of Cameron's sum-free set constants [2.25]
6.2831853071 . . .	2π; with Archimedes' constant [1.4]
6.3800420942 . . .	One of Otter's tree enumeration constants [5.6]
6.77404 . . .	Estimate of 4D self-avoiding walk constant [5.10]
6.7992251609 . . .	One of the Feigenbaum–Coullet–Tresser constants [1.9]
6.8 . . .	One of Cameron's sum-free set constants [2.25]
7.1879033516 . . .	Conjectured value of John constant [7.4]
7.2569464048 . . .	With Euler–Mascheroni constant [1.5.4]
7.2846862171 . . .	One of the Feigenbaum–Coullet–Tresser constants [1.9]
7.3719494907 . . .	c_o; with Lengyel's constant [5.7]
7.3719688014 . . .	c_e; with Lengyel's constant [5.7]
7.7431319855 . . .	One of the digital search tree constants [5.14]
7.7581602911 . . .	One of Otter's tree enumeration constants [5.6]
8.3494991320 . . .	One of the Feigenbaum–Coullet–Tresser constants [1.9]
8.7000366252 . . .	Kepler–Bouwkamp constant [6.3]
8.7210972 . . .	One of the Feigenbaum–Coullet–Tresser constants [1.9]
8.83854 . . .	Estimate of 5D self-avoiding walk constant [5.10]
9.0803731646 . . .	Hensley's constant [2.18]
9.27738 . . .	One of the Feigenbaum–Coullet–Tresser constants [1.9]
9.2890254919 . . .	Reciprocal of "one-ninth" constant [4.5]
9.2962468327 . . .	One of the Feigenbaum–Coullet–Tresser constants [1.9]
9.37 . . .	3D Grötzsch ring constant [7.8]
9.576778 . . .	With beam detection constant [8.11]
9.6694754843 . . .	Bateman–Grosswald c_{04} constant, with Niven's constant [2.6]
9.7 . . .	Estimate of $r_c(4)$, with k-satisfiability constants [5.21]
10.5101504239 . . .	Zagier's constant, with Freiman's constant [2.31]
10.7310157948 . . .	$\exp(\pi^2/(6\ln(2)))$; Lévy's constant [1.8]
10.87809 . . .	Estimate of 6D self-avoiding walk constant [5.10]
11.0901699437 . . .	$(11+5\sqrt{5})/2$; with hard square entropy constant [5.12]
12.262874 . . .	With self-avoiding walk constants [5.10]
12.6753318106 . . .	16·(Lal's constant), with Hardy–Littlewood constants [2.1]
14.1347251417 . . .	1st zeta function zero, with Glaisher–Kinkelin constant [2.15]
14.6475663016 . . .	One of the abelian group enumeration constants [5.1]
15.1542622415 . . .	e^e; one of the iterated exponential constants [6.11]
16.3638968792 . . .	β; one of the Feigenbaum–Coullet–Tresser constants [1.9]
16.9787814834 . . .	Bateman–Grosswald c_{14} constant, with Niven's constant [2.6]
19.4455760839 . . .	Bateman–Grosswald c_{05} constant, with Niven's constant [2.6]
20.9 . . .	Estimate of $r_c(5)$, with k-satisfiability constants [5.21]
21.0220396387 . . .	2nd zeta function zero, with Glaisher–Kinkelin constant [2.15]
22.6 . . .	4D Grötzsch ring constant [7.8]
25.0108575801 . . .	3rd zeta function zero, with Glaisher–Kinkelin constant [2.15]

29.576303 ...	One of the Feigenbaum–Coullet–Tresser constants [1.9]
39.1320261423 ...	With Calabi's triangle constant [8.13]
43.2 ...	Estimate of $r_c(6)$, with k-satisfiability constants [5.21]
55.247 ...	One of the Feigenbaum–Coullet–Tresser constants [1.9]
118.6924619727 ...	One of the abelian group enumeration constants [5.1]
137.0359 ...	Inverse fine structure constant, with Feigenbaum–Coullet–Tresser [1.9]

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The following results are too beautiful to be overlooked. The **Gaussian integers** $a + bi$, where a, b are integers and $i^2 = -1$, form a unique factorization domain with units $\{\pm 1, \pm i\}$. Suppose two Gaussian integers are chosen at random. The probability that they are coprime, in the limit over large disks, is [1,2]

$$\frac{6}{\pi^2 G} = 0.6637008046\ldots$$

where G is Catalan's constant [1.7]. This is slightly greater than the corresponding probability that two ordinary integers are coprime [1.4].

In the same way, the **Eisenstein–Jacobi integers** $a + b\omega$, where a, b are integers and $\omega = (-1 + i\sqrt{3})/2$, form a unique factorization domain with units $\{\pm 1, \pm i, \pm\omega\}$. The probability that two such randomly chosen integers are coprime, in the limit over large disks, is [1,3]

$$\frac{6}{\pi^2 H} = 0.7780944891\ldots$$

where

$$H = \frac{4\pi}{3\sqrt{3}} \ln(\beta) = \sum_{k=0}^{\infty} \left(\frac{1}{(3k+1)^2} - \frac{1}{(3k+2)^2} \right) = 0.7813024128\ldots$$

and β is discussed extensively in [3.10].

The constants $6/(\pi^2 G)$ and $6/(\pi^2 H)$ are also, respectively, the probabilities that a random Gaussian integer is square-free and a random Eisenstein–Jacobi integer is square-free. As in [2.5], there are related notions of *carefreeness* but the corresponding constants are not yet known.

Incidently, the pairwise coprimality result conjectured at the end of [2.5] has been proved to be true [4].

And, as this book goes to press, it is unclear [5] whether the prime limit infimum problem given at the conclusion of [2.13] is solved (or nearly so).

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