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[Reciproca teoremei lui Stolz-Cessaro]

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RECIPROCA TEOREMEI LUI STOLZ-CESSARO

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{a_n-a_{n-1}}{b_n-b_{n-1}}\left(1-\frac{b_{n-1}}{b_n}\right)+\frac{a_{n-1}}{b_{n-1}}\cdot\frac{b_{n-1}}{b_n}$$

$$x=\lim_{n\to\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}}\cdot(1-u)+xu$$

$$(1-u)x=(1-u)\lim_{n\to\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}}$$ (dar $u\neq 1$, se poate împarti prin u)

$$\Rightarrow \lim_{n\to\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}}=x$$

în cazul în care $\lim_{n\to\infty}\frac{b_n}{b_{n+1}}$ nu exista sau este egala cu 1, nu ne putem pronunta asupra reciprocei teoremei Stoltz - Cesaro. Totusi, în acest caz, numit caz de dubiu reciproca poate fi adevarata.

Exempul :

Sa se calculeze :

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}-\sqrt[n-1]{(n-1)!}\right)$$ (Problema lui Traian Lalescu)

Vom folosi Teorema lui D'Alambert :

Daca $(u_n)_{n\in N^*}$ este un sir de numere reale stric pozitive si daca

$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=u>0$ atunci $\sqrt[n]{u_n}=u$

Fie $u_n=\frac{(n+1)^n}{n!}$ atunci :

$$\frac{u_{n+1}}{u_n}=\frac{(n+2)^{n+1}}{(n+1)!}\cdot\frac{n!}{(n+1)^n}=\left(\frac{n+2}{n+1}\right)^{n+1}=\left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\to\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}=e$$

Deci $\lim_{n\to\infty}\sqrt[n]{\frac{(n+1)^n}{n!}}=e\to\lim_{n\to\infty}\frac{n+1}{\sqrt[n]{n!}}=e$

Fie $u_n = \frac{n^n}{n!}$ atunci :

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Deci $\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}} = e \rightarrow \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$

Avem :

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n = \lim_{n\to\infty}\left(\sqrt[n(n+1)]{\frac{[(n+1)!]^n}{(n!)^{n+1}}}\right)^n =$$

$$\lim_{n\to\infty}\left(\sqrt[n(n+1)]{\frac{(n+1)^n}{n!}}\right)^n = \lim_{n\to\infty}\left(\sqrt[n+1]{\frac{n+1}{\sqrt[n]{n!}}}\right)^n =$$

$$\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n]{n!}}\right)^{\frac{n}{n+1}} = \lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n]{n!}}\right)^{\lim_{n\to\infty}\left(\frac{n}{n+1}\right)} \text{ (Dar } \lim_{n\to\infty}\frac{n}{n+1} = 1) \rightarrow$$

$$\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n]{n!}}\right)^{\lim_{n\to\infty}\left(\frac{n}{n+1}\right)} = \lim_{n\to\infty}\frac{n+1}{\sqrt[n]{n!}} = e$$

Prin urmare :

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n = e$$

$$e = \lim_{n\to\infty}\left(1+\frac{\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}}{\sqrt[n]{n!}}\right)^n =$$

$$\lim_{n\to\infty}\left(\left(1+\frac{\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}}{\sqrt[n]{n!}}\right)^{\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}}}\right)^{\frac{\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}}{\sqrt[n]{n!}}\cdot n}$$

$$\rightarrow e = e^{\lim_{n\to\infty}\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\frac{n}{\sqrt[n]{n!}}} \rightarrow$$

$$\lim_{n\to\infty} \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\frac{n}{\sqrt[n]{n!}} = 1 \to$$

$$\lim_{n\to\infty} (\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}) = \frac{1}{\lim\limits_{n\to\infty} \frac{\sqrt[n]{n!}}{n}} \to$$

$$\to \lim_{n\to\infty} (\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}) = \frac{1}{e}$$

(Rezolvarea lui Marcel Tena)

O alta rezolvare ar fi aplicând reciproca teoremei

Stoltz - Cesaro în cazul de dubiu :

$a_n = \sqrt[n]{n!}$ si $b_n = n \ \forall n \in N \left(\lim\limits_{n\to\infty} \frac{b_n}{b_{n+1}} = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1\right)$ si avem :

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{\sqrt[n]{n!} - \sqrt[n-1]{(n-1)!}}{n - n + 1}$$

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} (\sqrt[n]{n!} - \sqrt[n-1]{(n-1)!}) \Rightarrow$$

$$\lim_{n\to\infty} (\sqrt[n]{n!} - \sqrt[n-1]{(n-1)!}) = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$$

$\Rightarrow$

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} =$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} (\frac{n}{n+1})^n = \lim_{n\to\infty} (\frac{1}{\frac{n+1}{n}})^n =$$

$$= \lim_{n\to\infty} (\frac{1}{\frac{n+1}{n}})^n = \lim_{n\to\infty} (\frac{1}{1+\frac{1}{n}})^n = \frac{1}{e}$$

$$\Rightarrow \lim_{n\to\infty} (\sqrt[n]{n!} - \sqrt[n-1]{(n-1)!}) = \frac{1}{e}$$

Dupa cum se observa $\lim\limits_{n\to\infty} \frac{b_n}{b_{n+1}} = 1$ si totusi reciproca teoremei

Stoltz - Cesaro functioneaza. Aceasta a doua solutie nu poate fi

considerata decât o verificare a primei rezolvari si a cazului ca

în acest caz reciproca Stoltz - Cesaro este adevarata.

Reciproca a doua a teoremei Stoltz-Cesaro

Fie $(a_n)_{n\in N^*}$ si $(b_n)_{n\in N^*}$ doua siruri de numere reale a.î :

a) Exista $\frac{a_n}{b_n} = \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = x$ (finit sau $\pm\infty$)

Atunci :

b) Sirul $(b_n)_{n\in N^*}$ are proprietatea ca :

$0 < b_1 < b_2 < \ldots < b_n < \ldots$

si $\lim\limits_{n\to\infty} b_n = +\infty$

Nici reciproca a doua nu este în general adevarata, dupa cum arata urmatorul contraexemplu. Fie $(a_n)_{n\in N^*}$ si $(b_n)_{n\in N^*}$ cu $a_n = b_n = (-1)^n$

Este evident ca :

$\frac{a_n}{b_n} = \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = 1$,sirul $(b_n)_{n\in N}$ nu are proprietatile din concluzie.