Sample Calculus Problems

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Part 1: Single Variable Functions

1. Evaluate the limit $\lim\limits_{x\to 5} \dfrac{4-\sqrt{3x+1}}{x^2-7x+10}$. (*Do not use L'Hôpital's Rule.*)

Solution:

$$\begin{aligned}
\lim_{x\to 5} \frac{4-\sqrt{3x+1}}{x^2-7x+10} &= \lim_{x\to 5} \frac{(4-\sqrt{3x+1})(4+\sqrt{3x+1})}{(x^2-7x+10)(4+\sqrt{3x+1})} \\
&= \lim_{x\to 5} \frac{16-(3x+1)}{(x^2-7x+10)(4+\sqrt{3x+1})} \\
&= \lim_{x\to 5} \frac{15-3x}{(x-5)(x-2)(4+\sqrt{3x+1})} \\
&= \lim_{x\to 5} \frac{3(5-x)}{(x-5)(x-2)(4+\sqrt{3x+1})} \\
&= \lim_{x\to 5} \frac{-3}{(x-2)(4+\sqrt{3x+1})} \\
&= \frac{-3}{(5-2)(4+\sqrt{3\cdot 5+1})} \\
&= \frac{-3}{3\cdot 8} \\
&= -\frac{1}{8}
\end{aligned}$$

$\square$

Remark: "=" is the most frequently used verb in mathematics. It was introduced in 1557 by Robert Recorde "to avoid the tedious repetition of the words 'is equal to'." It is important to use the equal sign correctly.

- To introduce " ", the *phantom equal sign*, to avoid the tedious repetition of the symbol "=" is not a good idea. The solution above should not go like:

$$\begin{aligned}
\lim_{x\to 5} \frac{4-\sqrt{3x+1}}{x^2-7x+10} &\quad \lim_{x\to 5} \frac{(4-\sqrt{3x+1})(4+\sqrt{3x+1})}{(x^2-7x+10)(4+\sqrt{3x+1})} \\
&\quad \lim_{x\to 5} \frac{15-3x}{(x-5)(x-2)(4+\sqrt{3x+1})} \\
&\quad \lim_{x\to 5} \frac{3(5-x)}{(x-5)(x-2)(4+\sqrt{3x+1})}
\end{aligned}$$

- One must also not use other symbols, which have completely different meanings, in place of "=". The solution above should not go like:

$$\begin{aligned}\lim_{x\to 5}\frac{4-\sqrt{3x+1}}{x^2-7x+10} &\Rightarrow \lim_{x\to 5}\frac{(4-\sqrt{3x+1})(4+\sqrt{3x+1})}{(x^2-7x+10)(4+\sqrt{3x+1})}\\ &\to \lim_{x\to 5}\frac{15-3x}{(x-5)(x-2)(4+\sqrt{3x+1})}\\ &☞ \lim_{x\to 5}\frac{3(5-x)}{(x-5)(x-2)(4+\sqrt{3x+1})}\end{aligned}$$

- The equal sign always stands between two things, although sometimes one of these things are at the end of the previous line or at the beginning of the next line. The solution above should not start like:

$$= \lim_{x\to 5}\frac{(4-\sqrt{3x+1})(4+\sqrt{3x+1})}{(x^2-7x+10)(4+\sqrt{3x+1})}$$

This begs the question: *What* is equal to $\lim_{x\to 5}\frac{(4-\sqrt{3x+1})(4+\sqrt{3x+1})}{(x^2-7x+10)(4+\sqrt{3x+1})}$?

- The equal sign can be used between two functions when we deal with *identities*, like

$$\frac{x^2-1}{x-1} = x+1 \;\; \textit{for all} \;\; x \neq 1$$

or when we deal with *equations*, like

$$\textit{Find all } x \textit{ such that } x^2 = 4.$$

Therefore we can not just drop some of the limit signs in the solution above to make it look like:

$$\begin{aligned}\lim_{x\to 5}\frac{4-\sqrt{3x+1}}{x^2-7x+10} &= \frac{(4-\sqrt{3x+1})(4+\sqrt{3x+1})}{(x^2-7x+10)(4+\sqrt{3x+1})} && ✗\\ &\vdots\\ &= \frac{-3}{(x-2)(4+\sqrt{3x+1})}\\ &= \frac{-3}{(5-2)(4+\sqrt{3\cdot 5+1})} && ✗\\ &= \frac{-3}{3\cdot 8}\\ &= -\frac{1}{8}\end{aligned}$$

The equalities on the lines marked with ✗ are not correct. $\frac{-3}{(x-2)(4+\sqrt{3x+1})}$ is not equal to $-\frac{1}{8}$ because, for instance, if we let $x = 1$ then $\frac{-3}{(x-2)(4+\sqrt{3x+1})} = \frac{-3}{(1-2)(4+\sqrt{3\cdot 1+1})} = \frac{1}{2} \neq -\frac{1}{8}$.

2. Let m be the slope of the tangent line to the graph of $y = \dfrac{x^2}{x+2}$ at the point $(-3,-9)$. Express m as a limit. (*Do not compute* m.)

Solution: The slope m of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ is given by the limit

$$m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

or equivalently by the limit

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} .$$

Therefore two possible answers are

$$m = \lim_{x \to -3} \frac{\dfrac{x^2}{x+2} - (-9)}{x - (-3)} = \lim_{h \to 0} \frac{\dfrac{(-3+h)^2}{-3+h+2} - (-9)}{h} .$$

□

3. Suppose that $\lim_{x \to c} f(x) \neq 0$ and $\lim_{x \to c} g(x) = 0$. Show that $\lim_{x \to c} \dfrac{f(x)}{g(x)}$ does not exist.

Solution: Assume that $\lim_{x \to c} \dfrac{f(x)}{g(x)}$ exists, and let $L = \lim_{x \to c} \dfrac{f(x)}{g(x)}$. Then by the product rule for limits we obtain

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left(\frac{f(x)}{g(x)} \cdot g(x) \right) = \lim_{x \to c} \frac{f(x)}{g(x)} \cdot \lim_{x \to c} g(x) = L \cdot 0 = 0 .$$

This contradicts the fact that $\lim_{x \to c} f(x) \neq 0$. Therefore our assumption cannot be true: $\lim_{x \to c} \dfrac{f(x)}{g(x)}$ does not exist. □

4. Suppose that $\lim_{x \to c} f(x) = 0$ and there exists a constant K such that $|g(x)| \leq K$ for all $x \neq c$ in some open interval containing c. Show that $\lim_{x \to c} (f(x)g(x)) = 0$.

Solution: We have $|f(x)g(x)| = |f(x)| \cdot |g(x)| \leq |f(x)| \cdot K$ in some open interval around c. Therefore

$$-K|f(x)| \leq f(x)g(x) \leq K|f(x)| .$$

Now applying the Sandwich Theorem and using the fact that $\lim_{x \to c} |f(x)| = 0$ we obtain the result. □

5. Determine the following limits if $\lim_{x\to 0^+} f(x) = A$ and $\lim_{x\to 0^-} f(x) = B$.

a. $\lim_{x\to 0^-} f(x^2 - x)$ **b.** $\lim_{x\to 0^-} \big(f(x^2) - f(x)\big)$ **c.** $\lim_{x\to 0^+} f(x^3 - x)$

d. $\lim_{x\to 0^-} \big(f(x^3) - f(x)\big)$ **e.** $\lim_{x\to 1^-} f(x^2 - x)$

Solution: a. If $x < 0$, then $x^2 > 0$ and $-x > 0$. Therefore $x^2 - x > 0$ for $x < 0$, and $x^2 - x$ approaches 0 from the right as x approaches 0 from the left. $\lim_{x\to 0^-} f(x^2 - x) = A$.

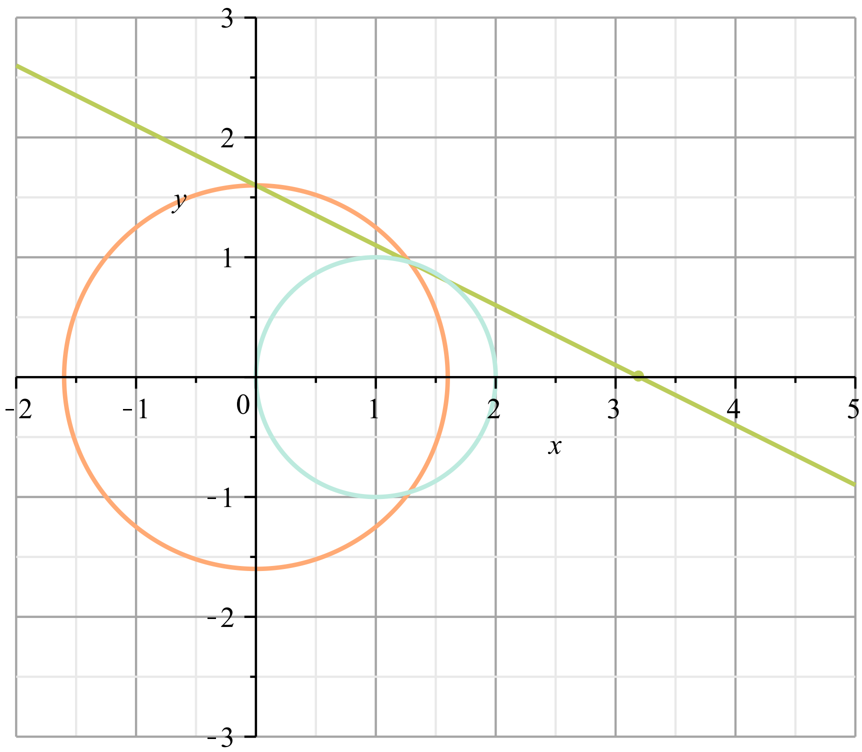
b. Since $x^2 > 0$ for $x < 0$, x^2 approaches 0 from the right as x approaches 0 from the left. Hence $\lim_{x\to 0^-} \big(f(x^2) - f(x)\big) = \lim_{x\to 0^-} f(x^2) - \lim_{x\to 0^-} f(x) = A - B$.

c. For $0 < x < 1$, we have $x^3 < x$ and $x^3 - x < 0$. So $x^3 - x$ approaches 0 from the left as x approaches 0 from the right. Therefore $\lim_{x\to 0^+} f(x^3 - x) = B$.

d. Since $x^3 < 0$ for $x < 0$, x^3 approaches 0 from the left as x approaches 0 from the left. Hence $\lim_{x\to 0^-} \big(f(x^3) - f(x)\big) = \lim_{x\to 0^-} f(x^3) - \lim_{x\to 0^-} f(x) = B - B = 0$.

e. For $0 < x < 1$ we have $x^2 < x$ and $x^2 - x < 0$. $x^2 - x$ approaches 0 from the left as x approaches 1 from the left. Hence $\lim_{x\to 1^-} f(x^2 - x) = B$. □

6. Let Q be the point of intersection in the first quadrant of the circle C_1 with equation $(x-1)^2 + y^2 = 1$ and the circle C_2 with equation $x^2 + y^2 = r^2$. Let R be the point where the line passing through the points $P(0, r)$ and Q intersects the x-axis. Determine what happens to R as $r \to 0^+$.



Solution: Subtracting $x^2 + y^2 = r^2$ from $(x-1)^2 + y^2 = 1$ we obtain $x = r^2/2$, and substituting this back in $x^2 + y^2 = r^2$ gives us $Q(r^2/2, \sqrt{r^2 - r^4/4})$.

Let $R(a, 0)$ be the coordinates of R and let S be the foot of the perpendicular from Q to the x-axis. Since the triangles RSQ and ROP are similar we have

$$\frac{a - r^2/2}{\sqrt{r^2 - r^4/4}} = \frac{a}{r}$$

and hence

$$a = \frac{r^3/2}{r - \sqrt{r^2 - r^4/4}} \, .$$

Then

$$\begin{aligned}
\lim_{r\to 0^+} a &= \lim_{r\to 0^+} \frac{r^3/2}{r - \sqrt{r^2 - r^4/4}} \\
&= \lim_{r\to 0^+} \left(\frac{r^3/2}{r^2 - (r^2 - r^4/4)} \cdot (r + \sqrt{r^2 - r^4/4}) \right) \\
&= 2 \lim_{r\to 0^+} (1 + \sqrt{1 - r^2/4}) \\
&= 2 \cdot (1 + \sqrt{1 - 0^2/4}) = 4 \, .
\end{aligned}$$

Therefore R approaches the point $(4, 0)$ as $r \to 0^+$. □

7.* Use the formal definition of the limit to show that $\lim_{x\to 1/2} \frac{1}{x} = 2$.

Solution: Given $\varepsilon > 0$ we want to find $\delta > 0$ such that

$$0 < \left| x - \frac{1}{2} \right| < \delta \Longrightarrow \left| \frac{1}{x} - 2 \right| < \varepsilon \, . \qquad (\star)$$

We will do this in two different ways.

"Solve the Inequality" Method: First we solve $\left| \frac{1}{x} - 2 \right| < \varepsilon$ for x.

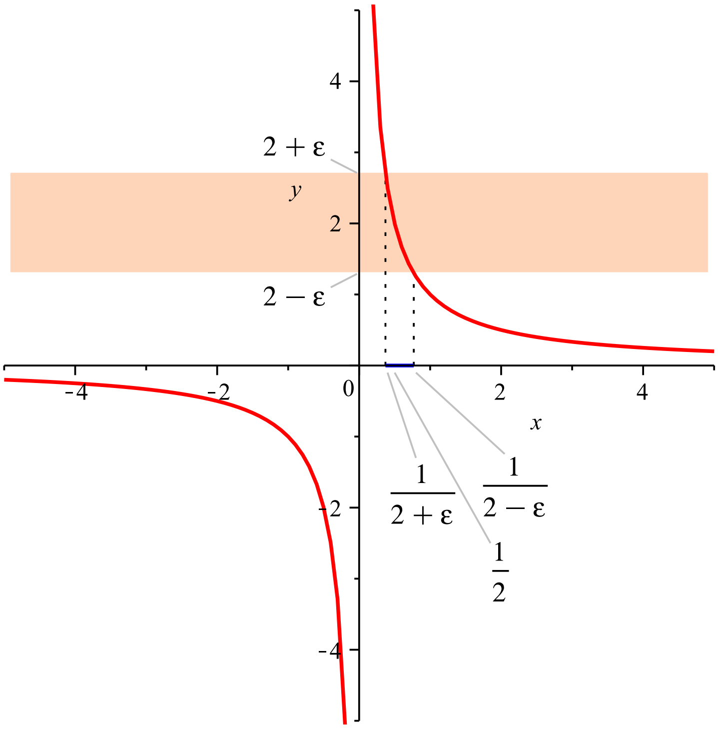
$$\left| \frac{1}{x} - 2 \right| < \varepsilon \Longleftrightarrow 2 - \varepsilon < \frac{1}{x} < 2 + \varepsilon$$

The next step depends on whether $2 - \varepsilon$ is positive, zero or negative.

If $2 - \varepsilon > 0$, that is if $\varepsilon < 2$, then

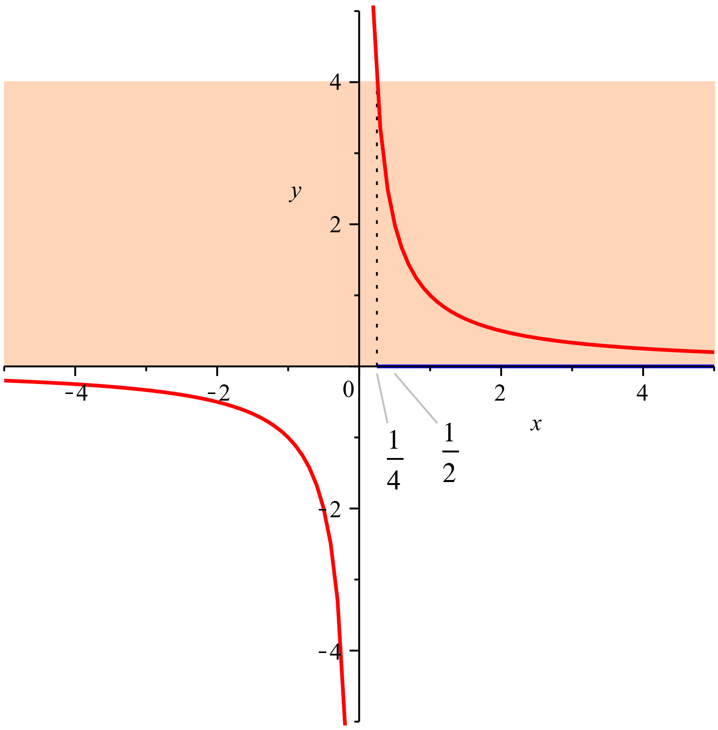
$$2 - \varepsilon < \frac{1}{x} < 2 + \varepsilon \Longleftrightarrow \frac{1}{2 - \varepsilon} > x > \frac{1}{2 + \varepsilon} \, .$$

*Examples marked red are not part of the Fall 2016 Syllabus.



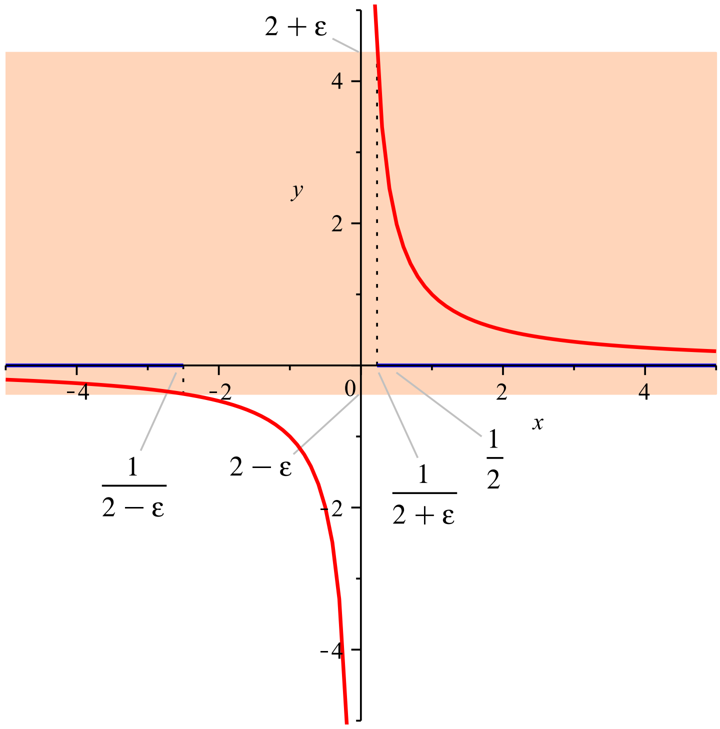
If $2 - \varepsilon = 0$, that is if $\varepsilon = 2$, then

$$2 - \varepsilon < \frac{1}{x} < 2 + \varepsilon \Longleftrightarrow x > \frac{1}{2 + \varepsilon} .$$



If $2 - \varepsilon < 0$, that is if $\varepsilon > 2$, then

$$2 - \varepsilon < \frac{1}{x} < 2 + \varepsilon \Longleftrightarrow x > \frac{1}{2 + \varepsilon} \quad \text{or} \quad -\frac{1}{\varepsilon - 2} > x .$$



Next we choose δ in such a way that every x satisfying the condition $0 < \left|x - \frac{1}{2}\right| < \delta$ lies in the solution set of $\left|\frac{1}{x} - 2\right| < \varepsilon$, and therefore the implication in $(\star)$ holds. In all three cases choosing a delta such that $0 < \delta \leq 1/2 - 1/(2+\varepsilon) = \varepsilon/(4+2\varepsilon)$ achieves this.

The Estimation Method: Suppose $0 < |x - 1/2| < \delta$ for some $\delta > 0$. Then

$$\left|\frac{1}{x} - 2\right| = \frac{2|x - 1/2|}{|x|} < \frac{2\delta}{|x|}\,.$$

At this point let us also decide to choose δ to satisfy $\delta \leq 1/4$. (Why 1/4?) Then $0 < |x - 1/2| < \delta \Longrightarrow 1/2 - \delta < x < 1/2 + \delta \Longrightarrow 1/4 < x < 3/4 \Longrightarrow 4 > 1/x > 4/3 \Longrightarrow 1/|x| < 4$ and therefore

$$\left|\frac{1}{x} - 2\right| < \frac{2\delta}{|x|} < 8\delta\,.$$

Hence for a given $\varepsilon > 0$ if we choose δ to satisfy $\delta \leq \varepsilon/8$ (as well as $\delta \leq 1/4$) then we will have $\left|\frac{1}{x} - 2\right| < 8\delta \leq 8 \cdot \frac{\varepsilon}{8} = \varepsilon$ and $(\star)$ will hold. In conclusion, any choice of δ satisfying $0 < \delta \leq \min\{\varepsilon/8, 1/4\}$ works. □

8.* Show that $\lim\limits_{x\to-3}(x^4 + 7x - 17) = 43$ using the formal definition of the limit.

Solution: For any given $\varepsilon > 0$ we have to find a $\delta > 0$ so that for all x we have

$$0 < |x - (-3)| < \delta \Longrightarrow |x^4 + 7x - 17 - 43| < \varepsilon\,.$$

*Examples marked red are not part of the Fall 2016 Syllabus.

We have $(x^4+7x-17)-43 = x^4+7x-60 = (x+3)(x^3-3x^2+9x-20)$. Suppose that $0<|x-(-3)|<\delta$ and $\delta \le 1$. Then $-4 \le -3-\delta < x < -3+\delta \le -2$. In particular, $|x| < 4$. Therefore, using the Triangle Inequality, we obtain $|x^3-3x^2+9x-20| \le |x|^3+3|x|^2+9|x|+20 < 4^3+3\cdot 4^2+9\cdot 4+20 = 168$. Now if we choose δ to satisfy $0<\delta\le\min\{\varepsilon/168, 1\}$, then we have

$$|x^4+7x-17-43| = |x^4+7x-60| = |x+3|\cdot|x^3-3x^2+9x-20| < \delta\cdot 168 \le \frac{\varepsilon}{168}\cdot 168 = \varepsilon$$

whenever $0<|x-(-3)|<\delta$. We are done. □

9.* Suppose that for all $0<\varepsilon<1$,

$$|x-1| < \frac{\varepsilon^2}{4} \Longrightarrow |f(x)-3| < \varepsilon \tag{$\star$}$$

and

$$|x-1| < \frac{\varepsilon}{35} \Longrightarrow |g(x)-4| < \varepsilon\,. \tag{$\star\star$}$$

Find a real number $\delta>0$ such that

$$|x-1|<\delta \Longrightarrow |f(x)+g(x)-7| < \frac{1}{5}\,.$$

Solution: If we take $\varepsilon = \dfrac{1}{10}$ in $(\star)$ we get

$$|x-1| < \frac{1}{400} \Longrightarrow |f(x)-3| < \frac{1}{10}\,.$$

If we take $\varepsilon = \dfrac{1}{10}$ in $(\star\star)$ we get

$$|x-1| < \frac{1}{350} \Longrightarrow |g(x)-4| < \frac{1}{10}\,.$$

Therefore if $|x-1| < \dfrac{1}{400}$, then we have

$$|f(x)+g(x)-7| = |f(x)-3+g(x)-4| \le |f(x)-3|+|g(x)-4| < \frac{1}{10}+\frac{1}{10} = \frac{1}{5}$$

by the Triangle Inequality. Hence we can take $\delta = \dfrac{1}{400}$. □

Remark: In fact, any $\delta \le \dfrac{9-4\sqrt{2}}{1225}$ works.

10.* Let $f(x) = \begin{cases} 1 & \text{if } x = \dfrac{1}{n} \text{ where } n \text{ is a positive integer,} \\ 0 & \text{otherwise.} \end{cases}$

a. Show that if $c \neq 0$ then $\lim\limits_{x\to c} f(x) = 0$.

b. Show that $\lim\limits_{x\to 0} f(x)$ does not exist.

*Examples marked red are not part of the Fall 2016 Syllabus.

Solution: **a.** Assume $c > 0$. Then there is a positive integer m such that $1/m$ is the closest to c among all real numbers different from c and of the form $1/n$ where n is a positive integer. (Why?) Let $\delta = \left| c - \dfrac{1}{m} \right| > 0$. Then for any $\varepsilon > 0$, we have

$$0 < |x - c| < \delta \Longrightarrow x \neq \frac{1}{n} \text{ for any positive integer } n$$
$$\Longrightarrow f(x) = 0 \Longrightarrow |f(x) - 0| = |0 - 0| = 0 < \varepsilon .$$

Therefore $\lim_{x \to c} f(x) = 0$.

Assume $c < 0$. Take $\delta = |c|$. Then for any $\varepsilon > 0$, we have

$$0 < |x - c| < \delta \Longrightarrow x < 0 \Longrightarrow x \neq \frac{1}{n} \text{ for any positive integer } n$$
$$\Longrightarrow f(x) = 0 \Longrightarrow |f(x) - 0| = |0 - 0| = 0 < \varepsilon .$$

Therefore $\lim_{x \to c} f(x) = 0$.

b. Let L be a real number and assume that $\lim_{x \to 0} f(x) = L$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x,

$$0 < |x - 0| < \delta \Longrightarrow |f(x) - L| < \varepsilon .$$

If L is not 0, let $\varepsilon = |L|/2 > 0$. Then there is a $\delta > 0$ such that

$$0 < |x| < \delta \Longrightarrow |f(x) - L| < |L|/2 .$$

Take $x = -\delta/2$. Then $0 < |x| < \delta$ is true, but $|L| = |0 - L| = |f(x) - L| < |L|/2$ is not true. We have a contradiction.

On the other hand, if $L = 0$, let $\varepsilon = 1/2$. Then there is a $\delta > 0$ such that

$$0 < |x| < \delta \Longrightarrow |f(x)| < 1/2 .$$

If n is a positive integer satisfying $n > 1/\delta$, take $x = 1/n$. Then $0 < |x| < \delta$ is true, but $1 = |1| = |f(x)| < 1/2$ is not true. Again we have a contradiction.

Hence $\lim_{x \to 0} f(x)$ cannot exist. $\square$

11. Show that the equation $x^2 - 10 = x \sin x$ has a real solution.

Solution: Consider the function $f(x) = x^2 - 10 - x \sin x$. Then $f(0) = -10 < 0$ and $f(10) = 10^2 - 10 - 10\sin(10) = 90 - 10\sin(10) \geq 90 - 10 = 80 > 0$. Note that f is continuous on $[0, 10]$. Therefore we can apply the Intermediate Value Theorem to the function f on the interval $[0, 10]$ for the value 0 to conclude that there is a point c in $(0, 10)$ such that $f(c) = 0$. This c is also a solution of the given equation. $\square$

12. Consider the equation $1 - \dfrac{x^2}{4} = \cos x$.

a. Show that this equation has at least one real solution.

b. Show that this equation has at least two real solutions.

c. Show that this equation has at least three real solutions.

Solution: Let $f(x) = 1 - \dfrac{x^2}{4} - \cos x$. The solutions of the equation $1 - \dfrac{x^2}{4} = \cos x$ correspond to the zeros of f.

As $f(0) = 0$, $x = 0$ is one zero.

Now observe that $f(\pi/2) = 1 - \pi^2/16 > 0$ and $f(\pi) = 2 - \pi^2/4 < 0$ as $4 > \pi > 3$. As f is continuous on the entire real line, applying the Intermediate Value Theorem to the function f on the interval $[\pi/2, \pi]$ we conclude that there is a point c in this interval such that $f(c) = 0$. This is our second zero.

Finally, as the function f is even, we have $f(-c) = f(c) = 0$, and $x = -c$ is our third zero. $\square$

13. Show that at any moment there are two antipodal points on the equator of the Earth with the same temperature.

Solution: First we will make a mathematical model of the problem. We will consider the equator as a circle, and use the longitude as our coordinate θ. We choose the positive direction for θ to correspond to the East, measure θ in radians, and let it take any real value. So 45°W corresponds to $\theta = -\pi/4$, $\theta = 7\pi/4$, and $\theta = 15\pi/4$, among other values. Note that $\theta + \pi$ corresponds to the antipode of the point corresponding to θ. We let $T(\theta)$ denote the temperature at θ. We have $T(\theta + 2\pi) = T(\theta)$ for all θ. We will assume that T is a continuous function. We want to show that there is a c such that $T(c + \pi) = T(c)$.

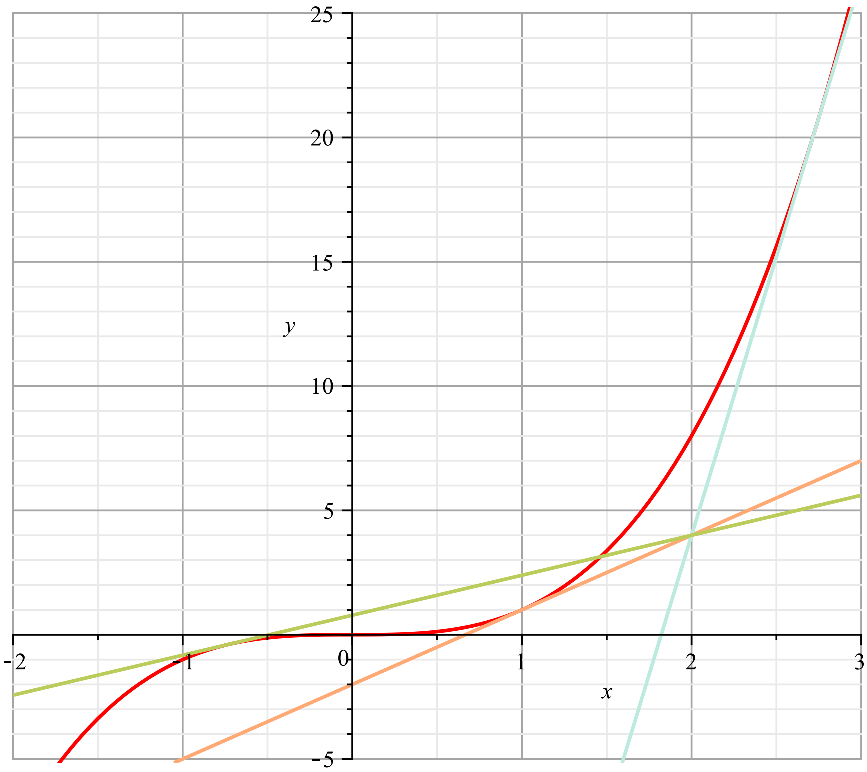
Consider the function $f(\theta) = T(\theta + \pi) - T(\theta)$. Note that since T is continuous, f is continuous. Our quest to find a c such that $T(c + \pi) = T(c)$ is equivalent to find a c such that $f(c) = 0$. If $f(0) = 0$, then we let $c = 0$ and we are done. Suppose that $f(0) \neq 0$. Observe that $f(0) = T(\pi) - T(0) = T(\pi) - T(2\pi) = -f(\pi)$. In other words, $f(0)$ and $f(\pi)$ have opposite signs. Now we apply the Intermediate Value Theorem to f on the interval $[0, \pi]$ for the value 0, and conclude that there is a point c in $[0, \pi]$ such that $f(c) = 0$. We are done. $\square$

Remark: It is possible to show that at any moment there are two antipodal points on Earth with the same temperature and the same pressure.

14. Find all tangent lines to the graph of $y = x^3$ that pass through the point $(2, 4)$.

Solution: As $dy/dx = d(x^3)/dx = 3x^2$, the equation of the tangent line through a point (x_0, x_0^3) on the graph is $y - x_0^3 = 3x_0^2(x - x_0)$. This line passes through $(2, 4)$ exactly when $4 - x_0^3 = 3x_0^2(2 - x_0)$, or in other words, $x_0^3 - 3x_0^2 + 2 = 0$. We observe that $x_0 = 1$ is a root of this polynomial. Therefore we have the factorization $x_0^3 - 3x_0^2 + 2 = (x_0 - 1)(x_0^2 - 2x_0 - 2)$. The roots of the quadratic factor are $x_0 = 1 \pm \sqrt{3}$.

Therefore the tangent lines to $y = x^3$ at the points $(1, 1)$, $(1 + \sqrt{3}, 10 + 6\sqrt{3})$, and $(1 - \sqrt{3}, 10 - 6\sqrt{3})$ pass through $(2, 4)$. The equations of these lines are $y = 3x - 2$, $y = (12 + 6\sqrt{3})x - (20 + 12\sqrt{3})$, and $y = (12 - 6\sqrt{3})x - (20 - 12\sqrt{3})$, respectively. $\square$



15. Evaluate the limit $\lim_{x \to 0} \dfrac{\sqrt{1 + \sin^2 x^2} - \cos^3 x^2}{x^3 \tan x}$.

Solution:

$$\lim_{x\to 0} \frac{\sqrt{1+\sin^2 x^2} - \cos^3 x^2}{x^3 \tan x} = \lim_{x \to 0}\left(\left(\frac{\sqrt{1+\sin^2 x^2} - 1}{x^4} + \frac{1 - \cos^3 x^2}{x^4}\right) \cdot \frac{x}{\tan x}\right)$$

$$= \lim_{x\to 0}\left(\left(\left(\frac{\sin x^2}{x^2}\right)^2 \cdot \frac{1}{\sqrt{1+\sin^2 x^2}+1} + \frac{1-\cos^3 x^2}{x^4}\right) \cdot \frac{x}{\tan x}\right)$$

Now we observe that:

$$\lim_{x\to 0} \frac{\sin x^2}{x^2} = 1$$

$$\lim_{x \to 0} \frac{1}{\sqrt{1+\sin^2 x^2}+1} = \frac{1}{\sqrt{1+\sin^2 0^2}+1} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{x}{\tan x} = 1$$

and

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos^3 x^2}{x^4} &= \lim_{x\to 0}\left(\frac{1-\cos x^2}{x^4}\cdot(1+\cos x^2+\cos^2 x^2)\right)\\
&= \lim_{x\to 0}\left(\frac{2\sin^2(x^2/2)}{x^4}\cdot(1+\cos x^2+\cos^2 x^2)\right)\\
&= \frac{1}{2}\cdot\left(\lim_{x\to 0}\frac{\sin(x^2/2)}{x^2/2}\right)^2\cdot\lim_{x\to 0}(1+\cos x^2+\cos^2 x^2)\\
&= \frac{1}{2}\cdot 1^2\cdot 3\\
&= \frac{3}{2}\,.
\end{aligned}$$

Therefore:

$$\lim_{x\to 0}\frac{\sqrt{1+\sin^2 x^2}-\cos^3 x^2}{x^3\tan x} = \left(1\cdot\frac{1}{2}+\frac{3}{2}\right)\cdot 1 = 2\,.$$

□

16. Find the equation of the tangent line to the graph of $y = \sin^2(\pi x^3/6)$ at the point with $x = 1$.

Solution: $y = \sin^2(\pi x^3/6) \Rightarrow \dfrac{dy}{dx} = 2\sin(\pi x^3/6)\cdot\cos(\pi x^3/6)\cdot 3\pi x^2/6$. Therefore, $\left.\dfrac{dy}{dx}\right|_{x=1} = 2\sin(\pi/6)\cdot\cos(\pi/6)\cdot\pi/2 = \dfrac{\sqrt{3}\,\pi}{4}$.

Since $y|_{x=1} = 1/4$, using the point-slope formula we find the equation of the tangent line as

$$y - \frac{1}{4} = \frac{\sqrt{3}\,\pi}{4}\,(x-1)$$

or, after some reorganization,

$$y = \frac{\sqrt{3}\,\pi}{4}\,x + \frac{1-\sqrt{3}\,\pi}{4}\,.$$

□

17. Let

$$f(x) = \begin{cases} 2x + x^2\sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0,\\[2ex] 0 & \text{if } x = 0.\end{cases}$$

a. Find $f'(x)$ for all x.

b. Show that f' is not continuous at 0.

Solution: a. For $x \neq 0$ we compute the derivative using the rules of differentiation:

$$f'(x) = \frac{d}{dx}\left(2x + x^2 \sin(1/x)\right) = 2 + 2x\sin(1/x) + x^2\cos(1/x)\cdot(-1/x^2)$$

for $x \neq 0$.

For $x = 0$ we must use the definition of the derivative:

$$\begin{aligned} f'(0) &= \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{2h + h^2\sin(1/h)}{h} \\ &= \lim_{h\to 0} 2 + \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 2 + 0 = 2 \end{aligned}$$

Here we used the fact that $\lim\limits_{h\to 0} h\,\sin(1/h) = 0$ whose proof uses the Sandwich (or Squeeze) Theorem. Here is a recap of the proof: Since $|\sin(1/h)| \leq 1$ for all $h \neq 0$ we have $|h\sin(1/h)| = |h|\cdot|\sin(1/h)| \leq |h|$ for all $h \neq 0$. Therefore

$$-|h| \leq h\sin\left(\frac{1}{h}\right) \leq |h| \quad \text{for all} \quad h \neq 0\,.$$

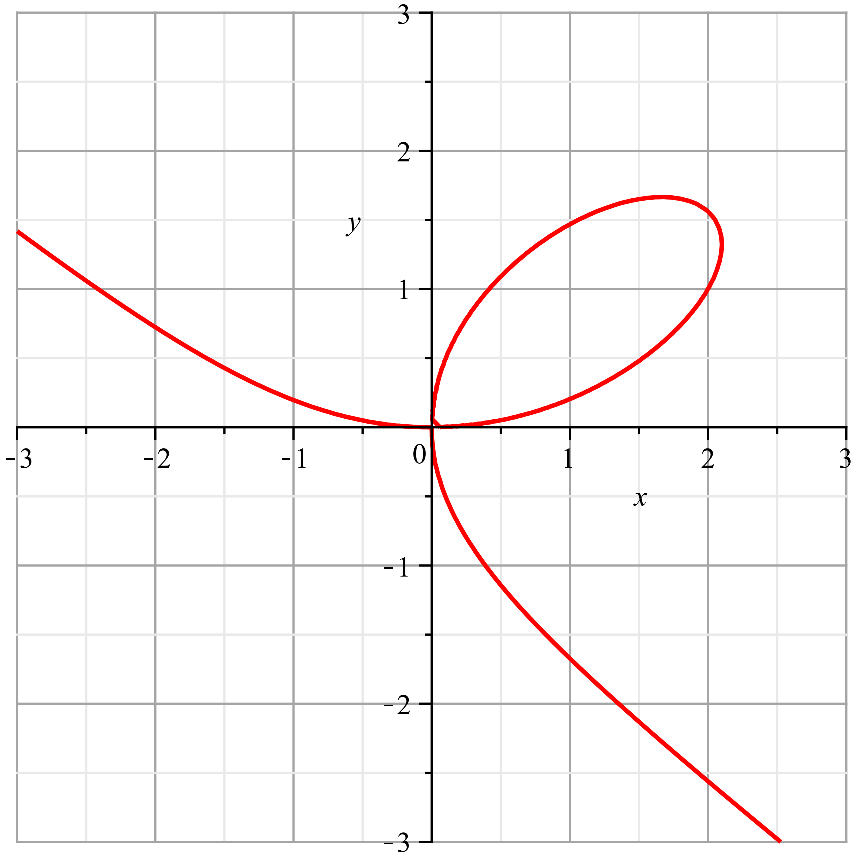
As $\lim\limits_{h\to 0}|h| = 0 = \lim\limits_{h\to 0}(-|h|)$, it follows by the Sandwich Theorem that $\lim\limits_{h\to 0} h\,\sin(1/h) = 0$.

To summarize:

$$f'(x) = \begin{cases} 2 + 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ \\ 2 & \text{if } x = 0. \end{cases}$$

b. Consider $\lim\limits_{x\to 0} f'(x)$. We have $\lim\limits_{x\to 0} 2 = 2$ and $\lim\limits_{x\to 0} 2x\,\sin(1/x) = 0$ as in part **(a)**. However $\lim\limits_{x\to 0}\cos(1/x)$ does not exist. It follows that $\lim\limits_{x\to 0} f'(x)$ does not exist and hence f' is not continuous at 0. $\square$

18. Find $\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(2,1)}$ if y is a differentiable function of x satisfying the equation $x^3+2y^3 = 5xy$.



Solution:

$$
\begin{aligned}
&x^3 + 2y^3 = 5xy \\
&\quad \Big\Downarrow \; d/dx \\
&3x^2 + 6y^2 \frac{dy}{dx} = 5y + 5x \frac{dy}{dx} \qquad (\star) \\
&\quad \Big\Downarrow \; x = 2, y = 1 \\
&12 + 6 \frac{dy}{dx} = 5 + 10 \frac{dy}{dx} \\
&\quad \Big\Downarrow \\
&4 \frac{dy}{dx} = 7 \\
&\quad \Big\Downarrow \\
&\frac{dy}{dx} = \frac{7}{4} \ \text{ at } \ (x,y) = (2,1)
\end{aligned}
$$

Now we differentiate the equation marked $(\star)$ with respect to x to find the second derivative.

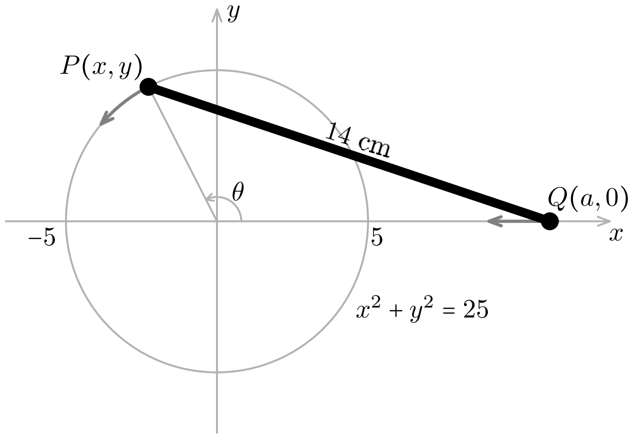
$$
\begin{gathered}
3x^2 + 6y^2\,\frac{dy}{dx} = 5y + 5x\,\frac{dy}{dx} \\
\Big\Downarrow {\scriptstyle d/dx} \\
6x + 12y\left(\frac{dy}{dx}\right)^2 + 6y^2\,\frac{d^2y}{dx^2} = 5\,\frac{dy}{dx} + 5\,\frac{dy}{dx} + 5x\,\frac{d^2y}{dx^2} \\
\Big\Downarrow {\scriptstyle x=2,\,y=1,\,\frac{dy}{dx}=\frac{7}{4}} \\
12 + 12\left(\frac{7}{4}\right)^2 + 6\,\frac{d^2y}{dx^2} = 10\cdot\frac{7}{4} + 10\,\frac{d^2y}{dx^2} \\
\Big\Downarrow \\
\frac{d^2y}{dx^2} = \frac{125}{16} \quad \text{at} \quad (x,y) = (2,1)
\end{gathered}
$$

□

Remark: An alternative approach is to solve y' from $(\star)$, viz. $y' = \dfrac{5y - 3x^2}{6y^2 - 5x}$, and then differentiate this with respect to x to find y''.

19. A piston is connected by a rod of length 14 cm to a crankshaft at a point 5 cm away from the axis of rotation of the crankshaft. Determine how fast the crankshaft is rotating when the piston is 11 cm away from the axis of rotation and is moving toward it at a speed of 1200 cm/sec.

Solution: Let $P(x,y)$ and $Q(a,0)$ be the ends of the connecting rod as shown in the picture. The axis of rotation of the crankshaft passes through the origin of the xy-plane and is perpendicular to it. The point P where the rod is connected to crankshaft moves on a circle with radius 5 cm and center at the origin. The point Q where the rod is connected to the piston moves along the positive x-axis. θ is the angle between the ray OP and the positive x-axis.



The question is:

$$a = 11 \text{ cm} \;\text{ and }\; \frac{da}{dt} = -1200 \text{ cm/sec} \implies \frac{d\theta}{dt} = ?$$

We have

$$x^2 + y^2 = 5^2 \tag{I}$$

and

$$(x - a)^2 + y^2 = 14^2 \; . \tag{II}$$

At the moment in question $a = 11$ cm. Substituting this in (**I**) and (**II**) we obtain $x^2 + y^2 = 5^2$ and $(x - 11)^2 + y^2 = 14^2$. Subtracting the second equation from the first gives $22x - 11^2 = 5^2 - 14^2$, and solving for x we get $x = -25/11$ cm. Then $y = 20\sqrt{6}/11$ cm.

Differentiating (**I**) and (**II**) with respect to time t we obtain

$$x\,\frac{dx}{dt} + y\,\frac{dy}{dt} = 0 \tag{III}$$

and

$$(x - a) \cdot \left(\frac{dx}{dt} - \frac{da}{dt}\right) + y\,\frac{dy}{dt} = 0 \; . \tag{IV}$$

At the moment in question $a = 11$ cm, $\dfrac{da}{dt} = v = -1200$ cm/sec, $x = -25/11$ cm and $y = 20\sqrt{6}/11$ cm. Substituting these in (**III**) and (**IV**) we get

$$-5\,\frac{dx}{dt} + 4\sqrt{6}\,\frac{dy}{dt} = 0 \tag{V}$$

and

$$-146\,\frac{dx}{dt} + 20\sqrt{6}\,\frac{dy}{dt} = -146v \; . \tag{VI}$$

Subtracting 5 times (**V**) from (**VI**) we find $-121\dfrac{dx}{dt} = -146\,v$, and hence $\dfrac{dx}{dt} = \dfrac{146}{121}\,v$. Substituting this back in (**V**) gives $\dfrac{dy}{dt} = \dfrac{365}{242\sqrt{6}}\,v$.

Now we are ready to compute $\dfrac{d\theta}{dt}$. Since $\tan\theta = y/x$, differentiation gives

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2}$$

and using $\sec^2\theta = 1 + \tan^2\theta = 1 + (y/x)^2$ we obtain

$$\frac{d\theta}{dt} = \frac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2 + y^2} \; .$$

Plugging $x = -\dfrac{25}{11}$ cm, $y = \dfrac{20\sqrt{6}}{11}$ cm, $\dfrac{dx}{dt} = \dfrac{146}{121}\,v$, and $\dfrac{dy}{dt} = \dfrac{365}{242\sqrt{6}}\,v$ in this formula gives $\dfrac{d\theta}{dt} = -\dfrac{73}{110\sqrt{6}}\,v = \dfrac{1460\sqrt{6}}{11}$ radian/sec. $\square$

Remark: $\dfrac{1460\sqrt{6}}{11}$ radian/sec is $\dfrac{1460\sqrt{6}}{11}\cdot\dfrac{60}{2\pi}$ rpm or approximately 3105 rpm.

Remark: This problem has a shorter solution if we use the law of cosines. Start with $x^2+5^2-2\cdot 5\,x\cos\theta = 14^2$ and differentiate with respect to t to obtain

$$\frac{d\theta}{dt} = \frac{5\cos\theta - 11}{5\sin\theta}\,v\,.$$

Put $x = 11$ cm in the first equation to find $\cos\theta = -5/11$ and then $\sin\theta = 4\sqrt{6}/11$. Now substituting these in the second equation gives the answer.

20. Determine how fast the length of an edge of a cube is changing at the moment when the length of the edge is 5 cm and the volume of the cube is decreasing at a rate of 100 cm^3/sec.

Solution: Let a denote the length of an edge of the cube, and V denote the volume of the cube. Then we have $V = a^3$. Differentiating with respect to time t gives $\dfrac{dV}{dt} = 3a^2\dfrac{da}{dt}$. Substituting $\dfrac{dV}{dt} = -100$ cm^3/sec and $a = 5$ cm for the moment in question, we obtain $\dfrac{da}{dt} = -\dfrac{4}{3}$ cm/sec. Therefore the length of the edge is decreasing at a rate of $\dfrac{4}{3}$ cm/sec at that moment. $\square$

21. We measure the radius and the height of a cone with 1% and 2% errors, respectively. We use these data to compute the volume of the cone. Estimate the percentage error in volume.

Solution: Let r, h, and V be the radius, the height and the volume of the cone, respectively.

$$V = \frac{\pi}{3}r^2h \Longrightarrow dV = \frac{2\pi}{3}rh\,dr + \frac{\pi}{3}r^2dh \Longrightarrow \frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}\,.$$

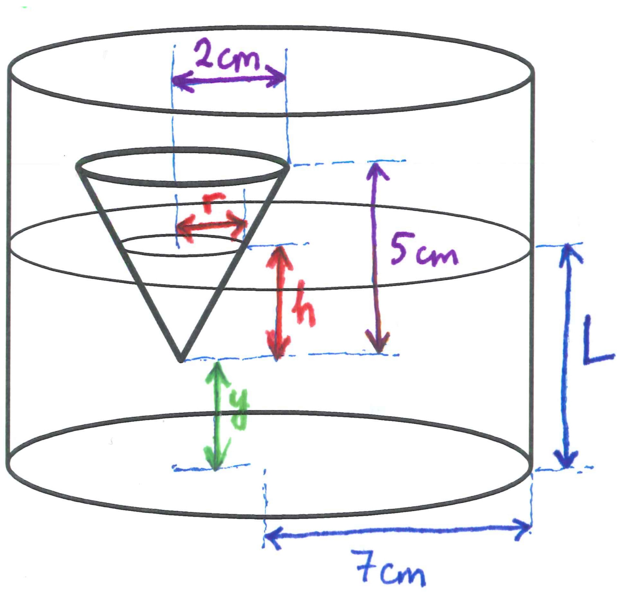
Since the error in r is 1% we have $\left|\dfrac{dr}{r}\right| \le 1\%$. Similarly $\left|\dfrac{dh}{h}\right| \le 2\%$. Now using the triangle inequality we obtain

$$\left|\frac{dV}{V}\right| = \left|2\frac{dr}{r} + \frac{dh}{h}\right| \le 2\left|\frac{dr}{r}\right| + \left|\frac{dh}{h}\right| \le 2\cdot 1\% + 2\% = 4\%\,.$$

The error in volume is 4%. $\square$

22. A cone of radius 2 cm and height 5 cm is lowered point first into a tall cylinder of radius 7 cm that is partially filled with water. Determine how fast the depth of the water is changing at the moment when the cone is completely submerged if the cone is moving with a speed of 3 cm/s at that moment.

Solution: Let r and h be the radius and the height of the part of the cone that is under the water level, respectively. Let L be the depth of the water in the cylinder and let y be the vertical distance from the tip of the cone to the bottom of the cylinder. Let V_0 be the volume of the water.



Then

$$V_0 = \pi \cdot 7^2 \cdot L - \frac{\pi}{3} r^2 h = \pi \cdot 7^2 \cdot L - \frac{\pi}{3}\left(\frac{2}{5}h\right)^2 h = 49\pi L - \frac{4\pi}{75}h^3$$

where we used the fact that $r/h = 2/5$.

Now differentiating this with respect to time t gives

$$0 = \frac{d}{dt}V_0 = 49\pi\frac{dL}{dt} - \frac{4\pi}{25}h^2\frac{dh}{dt} .$$

In particular at the moment when the cone is completely submerged we have $h = 5$ cm and

$$49\frac{dL}{dt} = 4\frac{dh}{dt} .$$

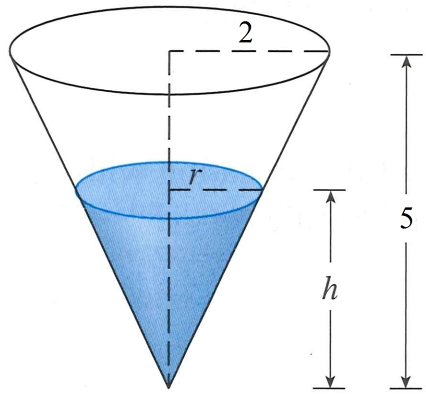
On the other hand, at the same moment

$$h = L - y \Longrightarrow \frac{dh}{dt} = \frac{dL}{dt} - \frac{dy}{dt} \Longrightarrow \frac{dh}{dt} = \frac{dL}{dt} + 3$$

because $dy/dt = -3$ cm/s as the cone is being lowered at a speed of 3 cm/s.

From these two equations we obtain $dL/dt = 4/15$ cm/s. In other words, the depth of the water is increasing at a rate of $4/15$ cm/s at that moment. $\square$

23. A water tank has the shape of an upside-down cone with radius 2 m and height 5 m. The water is running out of the tank through a small hole at the bottom. Assume that the speed of the water flowing through the hole is proportional to the square root of the depth of the water in the tank.



a. In this part, suppose that the water is running out at a rate of 3 m^3/min when the depth of the water in the tank is 4 m. Find the rate at which the water level is changing at this moment.

b. In this part, suppose that the water level is falling at a rate of 1/3 m/min when the tank is full. Find the rate at which the water level is changing when the depth of the water in the tank is 4 m.

c. In this part, suppose that it takes 3 minutes for the depth of the water to decrease from 5 m to 4 m. Find how long it takes for the full tank to completely drain.

Solution: Let r and h denote the radius and the height of the cone formed by the water, and let V denote the volume of the water. Using the fact that $r/h = 2/5$, we obtain

$$V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}\left(\frac{2}{5}h\right)^2 h = \frac{4\pi}{75}h^3$$

and hence:

$$\frac{dV}{dt} = \frac{4\pi}{25}h^2\frac{dh}{dt}$$

In part **(a)**, we are given that $dV/dt = -3\ \mathrm{m}^3/\mathrm{min}$ when $h = 4$ m. Substituting these in the equation above and solving for dh/dt we obtain $dh/dt = -75/(64\pi)$ m/min. Hence the water level is falling at a rate of $75/(64\pi)$ m/min in part **(a)**.

Now we will use the condition that the speed of the water flowing through the hole is proportional to the square root of the depth of the water in the tank. This means

$$\frac{dV}{dt} = -k\sqrt{h}$$

for some positive constant k. Substituting this in the previous equation we obtain:

$$-k\sqrt{h} = \frac{4\pi}{25}h^2\frac{dh}{dt} \tag{$\circledast$}$$

In part **(b)**, we are given that $dh/dt = -1/3$ m/min when $h = 5$ m. Substituting these in (⊛) we obtain:

$$-k\sqrt{5} = \frac{4\pi}{25}\,5^2\left(-\frac{1}{3}\right)$$

On the other hand, (⊛) gives

$$-k\sqrt{4} = \frac{4\pi}{25}\,4^2\,\frac{dh}{dt}$$

when $h = 4$ m. Now solving for dh/dt from these two equations gives $dh/dt = -5\sqrt{5}/24$ m/min. Hence the water level is falling at a rate of $5\sqrt{5}/24$ m/min in part **(b)**.

For part **(c)**, we rewrite (⊛) in the form

$$\frac{4\pi}{25}\,h^{3/2}\,dh = -k\,dt$$

and integrate to obtain

$$\frac{8\pi}{125}\,h^{5/2} = -kt + C$$

for some constant C. In other words, $h^{5/2} = -at + b$ for some constants a and b. As $h = 5$ m when $t = 0$ min, we have $b = 5^{5/2}$. Using the condition in part **(c)** that $h = 4$ m when $t = 3$ min we obtain $4^{5/2} = -a \cdot 3 + 5^{5/2}$, so $a = (5^{5/2} - 4^{5/2})/3$. Finally, the tank is empty when $h = 0$ and this happens when $t = b/a = 3 \cdot 5^{5/2}/(5^{5/2} - 4^{5/2})$ min. Hence the tank drains in $3/(1 - (4/5)^{5/2})$ minutes in part **(c)**. □

24. Find the absolute maximum value and the absolute minimum value of $f(x) = x^{4/3} - x - x^{1/3}$ on the interval $[-1, 6]$.

Remark: How to find the absolute maximum and the absolute minimum values of a **continuous** function f on a **closed** interval $[a, b]$ of **finite length**:

i. Compute f'.

ii. Find the critical points of f in (a, b).

iii. Add the endpoints a and b to this list.

iv. Compute the value of f at each point in the list.

v. The largest value is the the absolute maximum and the smallest value is the absolute minimum of f on $[a, b]$.

Solution: $f'(x) = \frac{4}{3}x^{1/3} - 1 - \frac{1}{3}x^{-2/3}$. The derivative is not defined at $x = 0$, therefore $x = 0$ is a critical point. Next we solve $f'(x) = 0$.

In the equation $\frac{4}{3}x^{1/3} - 1 - \frac{1}{3}x^{-2/3} = 0$ we let $z = x^{1/3}$ to obtain the equation $4z^3 - 3z^2 - 1 = 0$. Since $z = 1$ is a root, we have the factorization $4z^3 - 3z^2 - 1 = (z-1)(4z^2 + z + 1)$. As the quadratic factor has no real roots, $z = 1$ is the only solution. Therefore $x = z^3 = 1^3 = 1$, which belongs to the interval $[-1, 6]$, is the only other critical point.

We have $f(0) = 0$, $f(1) = -1$, $f(-1) = 3$, and $f(6) = 5 \cdot 6^{1/3} - 6$. Observe that $5 \cdot 6^{1/3} - 6 > 3 \iff 5 \cdot 6^{1/3} > 9 \iff 5^3 \cdot 6 > 9^3 \iff 725 > 721$.

We conclude that the absolute maximum and minimum values of $f(x) = x^{4/3} - x - x^{1/3}$ on the interval $[-1, 6]$ are $5 \cdot 6^{1/3} - 6$ and -1, respectively. $\square$

25. Find the absolute maximum and the absolute minimum values of $f(x) = \dfrac{x+1}{x^2 + x + 9}$ on the interval $[0, \infty)$.

Remark: When looking for the absolute maximum and the absolute minimum values of a continuous function on an interval that is not necessarily closed or of finite length, a modified version of the algorithm above can be used.

- In *Step iv*, if an endpoint does not belong to the interval, then we compute the appropriate one-sided limit of the function at that point instead of the value of the function.
- In *Step v*, if the largest value (which can be ∞) occurs only as a limit, then we conclude that there is no absolute maximum. Similarly for the smallest value.

Solution: We compute $f'(x) = -\dfrac{x^2 + 2x - 8}{(x^2 + x + 9)^2}$. The roots of $f'(x) = 0$ are $x = -4$ and $x = 2$. Only $x = 2$ is in the interval $[0, \infty)$. So our list is 0, 2, and ∞.

$f(0) = \frac{1}{9}$, $f(2) = \frac{1}{5}$, and $\lim_{x \to \infty} f(x) = 0$. Since $\frac{1}{5} > \frac{1}{9} > 0$, the absolute maximum is $\frac{1}{5}$ and there is no absolute minimum. $\square$

26. Suppose that a function f satisfies $f'(x) = f(x/2)$ for all x and $f(0) = 1$. Show that if $f(x_0) = 0$ for some $x_0 > 0$, then there is x_1 such that $0 < x_1 < x_0$ and $f(x_1) = 0$.

Solution: Since f is differentiable, f is continuous. Suppose that $f(x_0) = 0$ for some $x_0 > 0$. We apply the Mean Value Theorem to f on $[0, x_0]$ to conclude that there is c such that $0 < c < x_0$ and $f'(c) = \dfrac{f(x_0) - f(0)}{x_0 - 0} = -\dfrac{1}{x_0} < 0$. Then $f'(x) = f(x/2)$ gives $f(c/2) = f'(c) < 0$. We also have $f(0) = 1 > 0$. We apply the Intermediate Value Theorem to f on $[0, c/2]$ to conclude that there is x_1 such that $0 < x_1 < c/2$ and $f(x_1) = 0$. As $0 < x_1 < c/2 < c < x_0$ we are done. $\square$

Remark: With a little bit more work it can be shown that $f(x) > 1$ for all $x > 0$.

27. Show that if f is a twice-differentiable function such that $f(0) = 1$, $f'(0) = -1$, $f(1) = 2$, $f'(1) = 5$, and $f''(x) \geq 0$ for all x, then $f(x) \geq 1/3$ for all $0 \leq x \leq 1$.

Solution: Let $x_1 < x_2$ be in $[0, 1]$. The Mean Value Theorem applied to the function f' on the interval $[x_1, x_2]$ says that there is a point c in (x_1, x_2) such that $\dfrac{f'(x_2) - f'(x_1)}{x_2 - x_1} = f''(c)$. As $f''(c) \geq 0$ we have $f'(x_1) \leq f'(x_2)$. In particular, we have $-1 = f'(0) \leq f'(x) \leq f'(1) = 5$ for all x in $(0, 1)$.

Let $0 < x < 1$. By the Mean Value Theorem applied to the function f on the interval $[0, x]$, there exists a point c_1 in $(0, x)$ such that $\dfrac{f(x) - f(0)}{x - 0} = f'(c_1)$. Since $f'(c_1) \geq -1$, it follows that

$$f(x) \geq -x + 1 \qquad (*)$$

for $0 < x < 1$. Similarly, applying the Mean Value Theorem to the function f on the interval $[x, 1]$, we see that there exists a point c_2 in $(x, 1)$ such that $\dfrac{f(1) - f(x)}{1 - x} = f'(c_2)$. Now using the fact that $f'(c_2) \leq 5$ we conclude that

$$f(x) \geq 5x - 3 \qquad (**)$$

for $0 < x < 1$.

Adding 5 times the inequality **(∗)** to the inequality **(∗∗)** we get $f(x) \geq 1/3$ for all x in $[0, 1]$. $\square$

Remark: In fact, it can be shown that $f(x) > 1/3$ for all x in $[0, 1]$.

28. Sketch the graph of $y = 5x^{2/3} - 2x^{5/3}$.

Solution: $y' = \dfrac{10}{3}x^{-1/3} - \dfrac{10}{3}x^{2/3} = \dfrac{10}{3}x^{-1/3}(1-x)$. Therefore $y' > 0$ on $(0,1)$, and $y' < 0$ on $(-\infty,0)$ and $(1,\infty)$.

$y'' = -\dfrac{10}{9}x^{-4/3} - \dfrac{20}{9}x^{-1/3} = -\dfrac{20}{9}x^{-4/3}(x+1/2)$. Therefore $y'' > 0$ on $(-\infty,-1/2)$, and $y'' < 0$ on $(-1/2,0)$ and $(0,\infty)$.

These give us the following table of signs and shapes.

x		$-1/2$		0		1	
y'	$-$	$\vert$	$-$	$\wr$	$+$	0	$-$
y''	$+$	0	$-$	$\wr$	$-$	$\vert$	$-$
	◟	$\uparrow$	◝	$\uparrow$	◜	$\uparrow$	◝
		inf.pt.		*loc.min.*		*loc.max.*	

We compute the y-coordinates of the important points to get $(0,0)$ for the local minimum, $(1,3)$ for the local maximum, and $(-1/2, 3\sqrt[3]{2})$ for the inflection point. Note that since $\sqrt[3]{2} > 1$, we have $3\sqrt[3]{2} > 3$. Also note that the function is continuous at $x = 0$, but

$$\lim_{x\to 0^+} y' = \lim_{x\to 0^+}\left(\frac{10}{3}x^{-1/3}(1-x)\right) = \infty$$

and

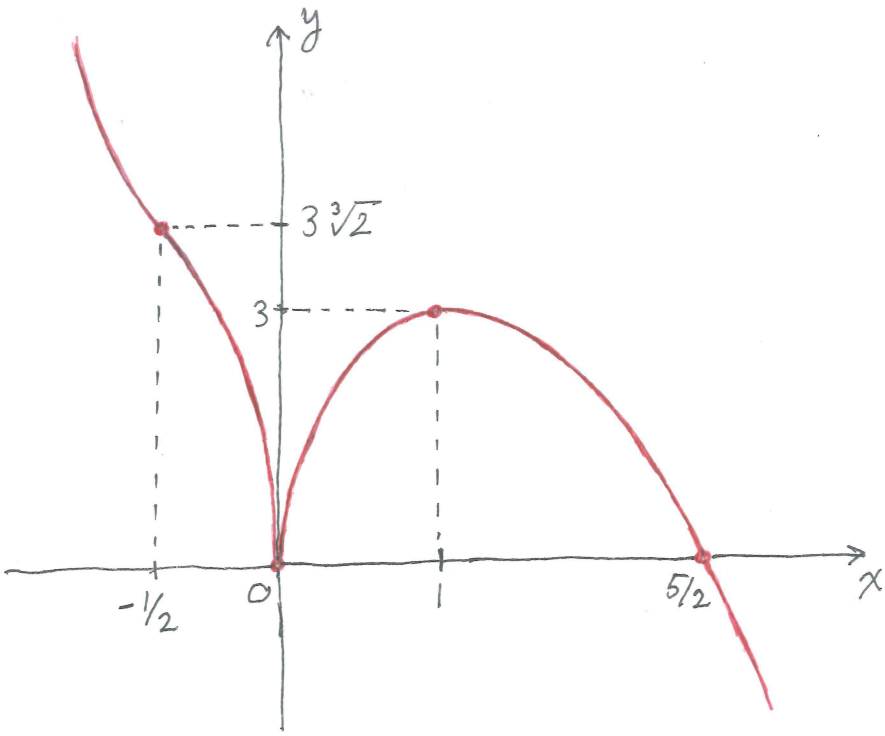
$$\lim_{x\to 0^-} y' = \lim_{x\to 0^-}\left(\frac{10}{3}x^{-1/3}(1-x)\right) = -\infty\,.$$

Therefore $(0,0)$ is a cusp.

Finally we find the x-intercepts: $y = 0 \Rightarrow 5x^{2/3} - 2x^{5/3} = 0 \Rightarrow 2x^{2/3}(5-2x) = 0 \Rightarrow x = 0$ or $x = 5/2$.

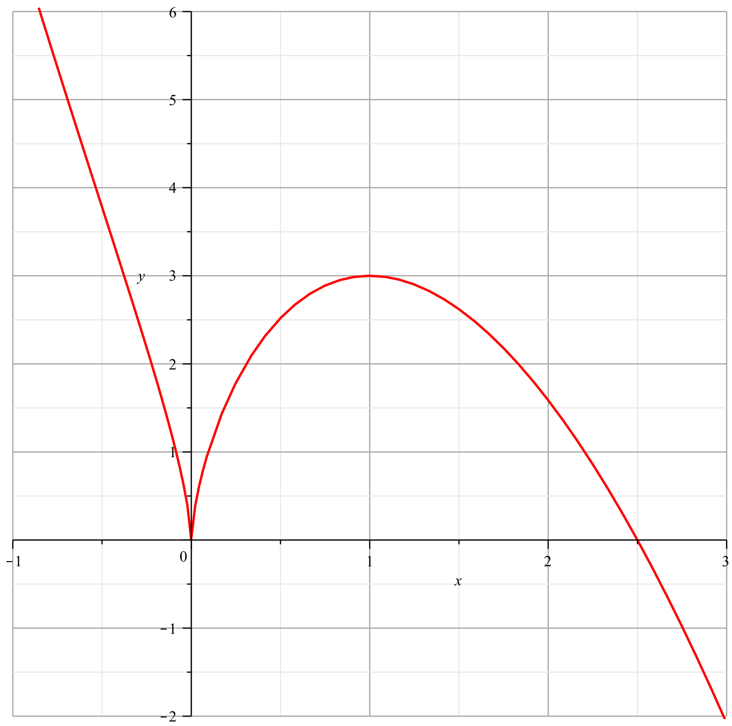
(*The graph is on the next page.*)

Now we use these data to draw the graph:



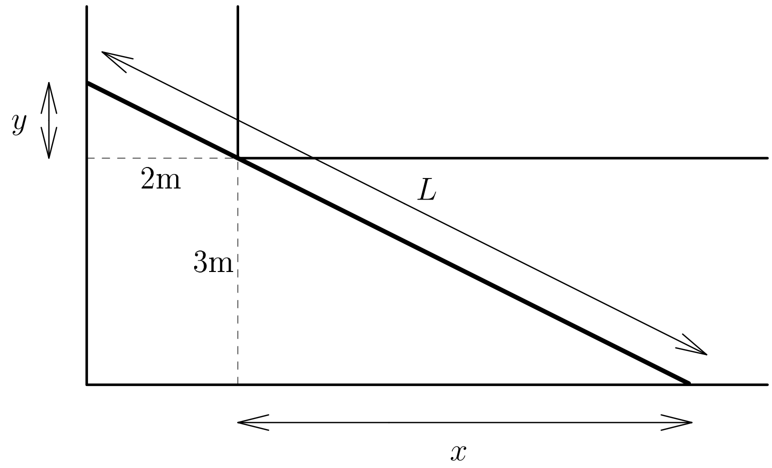
$\square$

Remark: Here is the same graph drawn by *Maple*:



29. Two corridors meet at a corner. One of the corridors is 2 m wide and the other one is 3 m wide. What is the length of the longest ladder that can be carried horizontally around this corner?

Solution: Length of the longest ladder will be equal to the absolute *minimum* value of L in the picture.



Using the relation $3/x = y/2$ we obtain

$$L = (x^2 + 9)^{1/2} + (4 + y^2)^{1/2} = (x^2 + 9)^{1/2} + (4 + (6/x)^2)^{1/2} \ .$$

Therefore we have to minimize

$$L = (x^2 + 9)^{1/2} \cdot \left(1 + \frac{2}{x}\right) \quad \text{for} \quad 0 < x < \infty \ .$$

We first look at the critical points:

$$\frac{dL}{dx} = (x^2 + 9)^{-1/2} \cdot x \cdot \left(1 + \frac{2}{x}\right) + (x^2 + 9)^{1/2} \cdot \left(-\frac{2}{x^2}\right) = 0$$
$$\Longrightarrow x + 2 - \left(2 + \frac{18}{x^2}\right) = 0 \Longrightarrow x = 18^{1/3} \text{ m} \Longrightarrow L = (2^{2/3} + 3^{2/3})^{3/2} \text{ m}$$

Since $\lim_{x \to 0^+} L = \infty$ and $\lim_{x \to \infty} L = \infty$, the value at the critical point is indeed the absolute minimum.

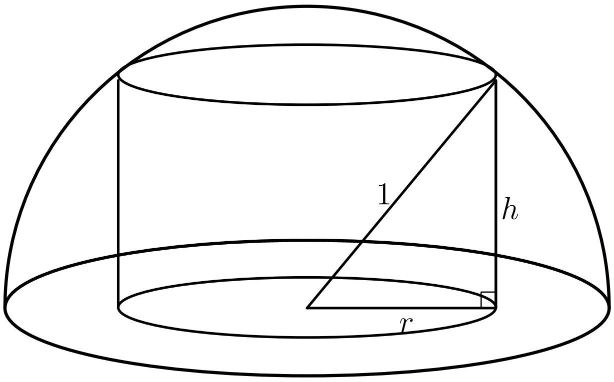
Hence the length of the longest ladder that can be carried around this corner is $(2^{2/3} + 3^{2/3})^{3/2}$ m. □

Remark: $(2^{2/3} + 3^{2/3})^{3/2}$ m is approximately 7.02 m.

30. Find the maximum possible total surface area of a cylinder inscribed in a hemisphere of radius 1.

Solution: Let S be the total surface area of the cylinder, $S = 2\pi r^2 + 2\pi rh$. We have $h = (1-r^2)^{1/2}$. Hence we want to maximize

$$S = 2\pi r^2 + 2\pi r(1-r^2)^{1/2} \quad \text{for} \quad 0 \le r \le 1\,.$$



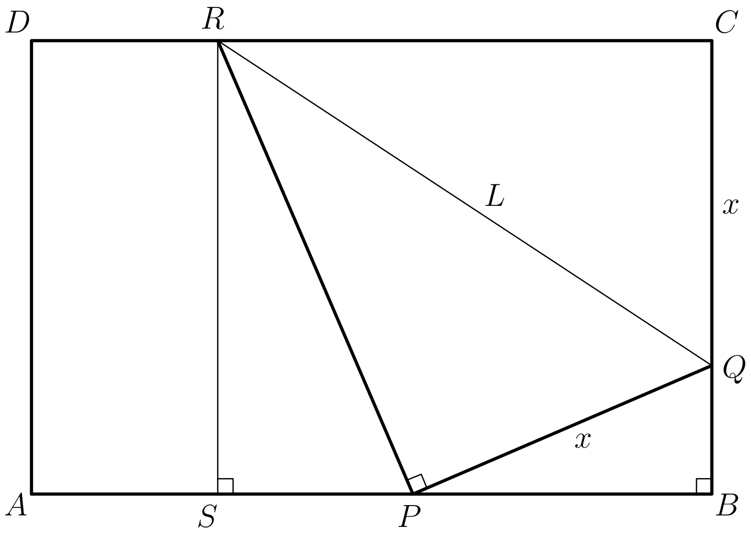
First we find the critical points: $\dfrac{1}{2\pi}\dfrac{dS}{dr} = 2r + (1-r^2)^{1/2} - r^2(1-r^2)^{-1/2} = 0 \implies 2r(1-r^2)^{1/2} = 2r^2 - 1 \implies 4r^2(1-r^2) = (2r^2-1)^2 \implies 8r^4 - 8r^2 + 1 = 0 \implies r^2 = \dfrac{8 \pm \sqrt{32}}{16}$. Note that $2r(1-r^2)^{1/2} = 2r^2 - 1$ implies $r^2 \ge \dfrac{1}{2}$, and therefore $r^2 = \dfrac{2+\sqrt{2}}{4}$ and $r = \dfrac{\sqrt{2+\sqrt{2}}}{2}$.

We get $S = (1+\sqrt{2})\pi$ at the critical point $r = \dfrac{\sqrt{2+\sqrt{2}}}{2}$, $S = 0$ at the endpoint $r = 0$, and $S = 2\pi$ at the endpoint $r = 1$. As $\sqrt{2} > 1$ we have $1 + \sqrt{2} > 2$, and the absolute maximum value occurs at the critical point.

The maximum possible surface area of the cylinder is $(1+\sqrt{2})\pi$. □

31. A fold is formed on a 20 cm × 30 cm rectangular sheet of paper running from the short side to the long side by placing a corner over the long side. Find the minimum possible length of the fold.

Solution: Let $ABCD$ be the sheet of paper and let P be the point on the edge AB where the corner C is folded over. The fold runs from Q on the edge BC to R on the edge CD. Let S be the projection of R on to the edge AB. Let L be the length of the fold QR and let $x = CQ$.



Then $PQ/PB = RP/RS$ by the similarity of the triangles PBQ and RSP. Hence $RP = 20x/\sqrt{x^2 - (20-x)^2} = 20x/\sqrt{40x - 400}$ and $L^2 = RQ^2 = RP^2 + PQ^2 = 400x^2/(40x - 400) + x^2 = x^3/(x - 10)$. The largest possible value of x is 20 cm. The smallest possible value of x occurs when $RP = 30$ cm; that is when $20x/\sqrt{40x - 400} = 30$ or $x = 45 - 15\sqrt{5}$ cm.

Therefore we want to minimize L where $L^2 = x^3/(x - 10)$ for $45 - 15\sqrt{5} \le x \le 20$.

Differentiating $2L dL/dx = 3x^2/(x - 10) - x^3/(x - 10)^2$ and setting $dL/dx = 0$ we obtain $x = 15$ cm as the only critical point in the domain. For $x = 15$ cm we have $L = 15\sqrt{3}$ cm.

At the endpoint $x = 20$ cm we have $L = 20\sqrt{2}$ cm and at the endpoint $x = 45 - 15\sqrt{5}$ cm we have $L = 15\sqrt{18 - 6\sqrt{5}}$ cm.

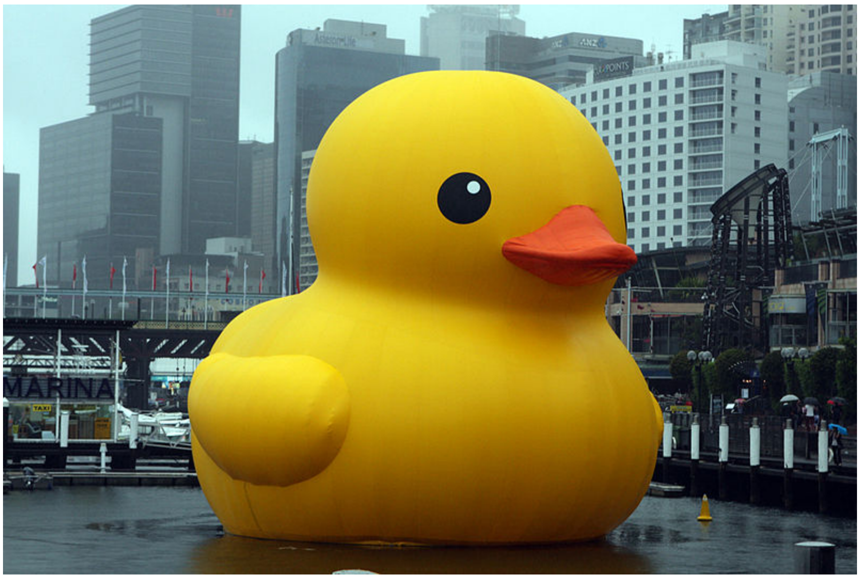
Since $15\sqrt{3}$ cm is the smallest of these values, we conclude that this is the smallest possible value for the length of the fold. □

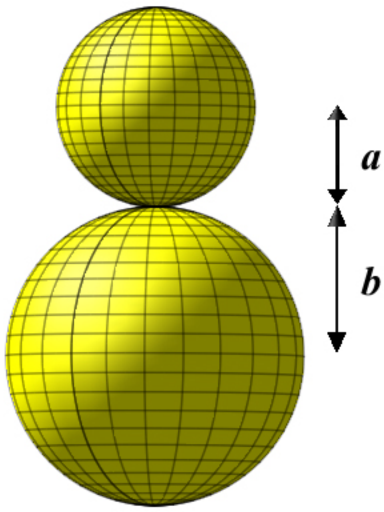
Remark: If the question is posed for a $w \times \ell$ sheet of paper with $w \le 2\sqrt{2}\ell/3$, then the length of the shortest possible fold is $3\sqrt{3}w/4$. Therefore, for a A4 size paper with $w = 210$ mm and $\ell = 297$ mm, the shortest fold is approximately 273 mm; and for a letter size paper with $w = 8.5$ in and $\ell = 11$ in, the shortest fold is approximately 11.04 in.

32. The Rubber Duck is a sculpture designed by Florentijn Hofman and constructed from PVC. For the purposes of this question, we consider the Rubber Duck to consist of a spherical head of radius a and a spherical body of radius b. The research shows that the *cuteness* $\mathscr{K}$ of the Rubber Duck is given by

$$\mathscr{K} = \begin{cases} \dfrac{a}{b}\left(1 - \dfrac{a}{b}\right)(a+b) & \text{if } 0 \le a < b, \\ 0 & \text{if } 0 < b \le a. \end{cases}$$

Find the dimensions of the cutest Rubber Duck with a total surface area of $400\pi\,\mathrm{m}^2$.





Solution: The total surface area of the Rubber Duck is $4\pi a^2 + 4\pi b^2$. Hence $4\pi a^2 + 4\pi b^2 = 400\pi$ m^2, giving $b^2 = 100 - a^2$. Therefore,

$$\mathscr{K} = \begin{cases} \dfrac{a(100-2a^2)}{100-a^2} & \text{if } 0 \le a \le 5\sqrt{2}, \\ 0 & \text{otherwise,} \end{cases}$$

and we have to find where the absolute maximum value of

$$\mathscr{K} = \frac{100a - 2a^3}{100 - a^2}$$

on the interval $0 \le a \le 5\sqrt{2}$ occurs.

We first find the critical points. As

$$\frac{d\mathscr{K}}{da} = \frac{(100-6a^2)(100-a^2) - (100a - 2a^3)(-2a)}{(100-a^2)^2},$$

$d\mathscr{K}/da = 0$ gives $a^4 - 250a^2 + 5000 = 0$. Now using the quadratic formula we obtain

$$a^2 = \frac{250 \pm \sqrt{250^2 - 4 \cdot 5000}}{2} = 125 \pm 25\sqrt{17},$$

which gives us four solutions

$$a = 5\sqrt{5+\sqrt{17}},\ 5\sqrt{5-\sqrt{17}},\ -5\sqrt{5+\sqrt{17}},\ -5\sqrt{5-\sqrt{17}}\,.$$

Only the first two of these are positive and the first one is greater than $5\sqrt{2}$. On the other hand, $0 < 5\sqrt{5-\sqrt{17}} < 5\sqrt{2}$ as $3 < \sqrt{17} < 5$. We conclude that $a = 5\sqrt{5-\sqrt{17}}$ is the only critical point.

We have

$$\mathscr{K} = 10\sqrt{5-\sqrt{17}} \cdot \frac{\sqrt{17}-3}{\sqrt{17}-1} > 0$$

for $a = 5\sqrt{5-\sqrt{17}}$. As $\mathscr{K} = 0$ at the endpoints $a = 0$ and $a = 5\sqrt{2}$ of the interval, this is the absolute maximum value.

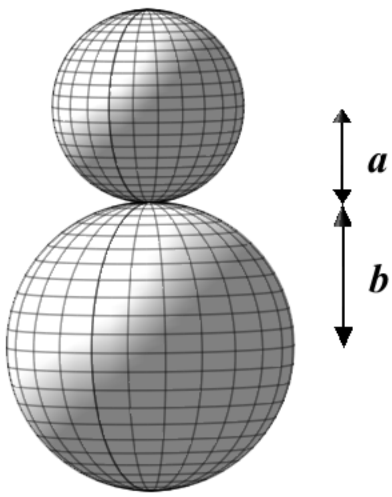
Therefore the cutest Rubber Duck has $a = 5\sqrt{5-\sqrt{17}}$ m and $b = 5\sqrt{\sqrt{17}-1}$ m. $\square$

33. A snowman is an anthropomorphic sculpture made from snow as well as some pieces of coal, a carrot, a hat and a scarf. For the purposes of this question, we consider a snowman to consist of a spherical head of radius a and a spherical body of radius b, and we also assume that the snow does not melt and its density does not change while it is being sculpted.

The research shows that the *cuteness* $\mathscr{K}$ of a snowman is given by

$$\mathscr{K} = \begin{cases} \left(\dfrac{a}{b}\right)^2 \left(1 - \dfrac{a}{b}\right)(a^2+ab+b^2) & \text{if } 0 \le a < b, \\ 0 & \text{if } 0 < b \le a. \end{cases}$$

Find the dimensions of the cutest snowman that can be built with $4\pi/3\,\mathrm{m}^3$ of snow.



Solution: The total volume of the snowman is $\dfrac{4\pi}{3}a^3 + \dfrac{4\pi}{3}b^3$. Hence $\dfrac{4\pi}{3}a^3 + \dfrac{4\pi}{3}b^3 =$

$\dfrac{4\pi}{3}$ m^3, giving $b^3 = 1 - a^3$. Therefore,

$$\mathscr{K} = \begin{cases} \dfrac{a^2(1-2a^3)}{1-a^3} & \text{if } 0 \le a \le \dfrac{1}{\sqrt[3]{2}}, \\ 0 & \text{otherwise,} \end{cases}$$

and we have to find where the absolute maximum value of

$$\mathscr{K} = \frac{a^2 - 2a^5}{1-a^3}$$

on the interval $0 \le a \le \dfrac{1}{\sqrt[3]{2}}$ occurs.

We first find the critical points. As

$$\frac{d\mathscr{K}}{da} = \frac{(2a-10a^4)(1-a^3)-(a^2-2a^5)(-3a^2)}{(1-a^3)^2},$$

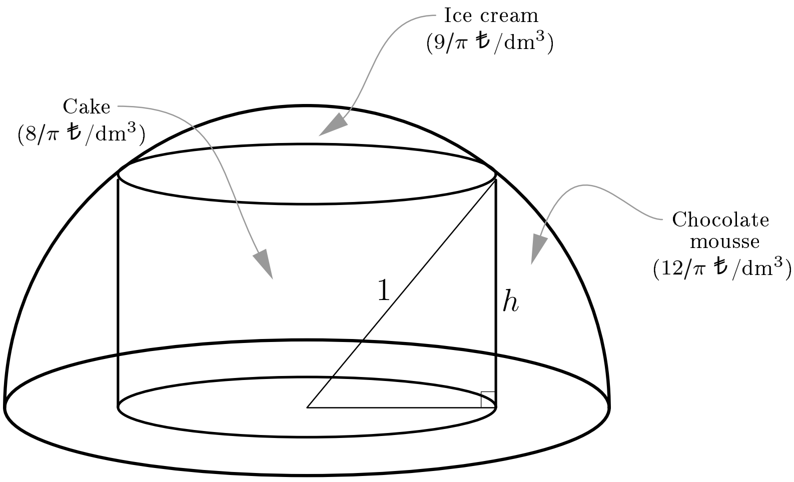
$d\mathscr{K}/da = 0$ gives $4a^7 - 9a^4 + 2a = 0$. As we have $4a^7 - 9a^4 + 2a = a(a^3-2)(4a^3-1)$, the critical points are $a = 0$, $a = \sqrt[3]{2}$ and $a = 1/\sqrt[3]{4}$. Hence the only critical point in the interval we are interested in is $a = 1/\sqrt[3]{4}$.

We have $\mathscr{K} = 1/(3\sqrt[3]{2}) > 0$ for $a = 1/\sqrt[3]{4}$. As $\mathscr{K} = 0$ at the endpoints $a = 0$ and $a = \dfrac{1}{\sqrt[3]{2}}$ of the interval, this is the absolute maximum value.

Therefore the cutest snowman has $a = 1/\sqrt[3]{4}$ m and $b = \sqrt[3]{3/4}$ m. □

34. A dessert in the shape of a hemisphere with radius 1 dm is made by baking a cylindrical cake of height h, and topping it with a spherical cap of ice cream and surrounding it with a hemispherical ring of chocolate mousse as shown in the figure. If the cake costs $8/\pi$ ₺/dm^3, the ice cream costs $9/\pi$ ₺/dm^3 and the chocolate mousse costs $12/\pi$ ₺/dm^3, determine the value of h for **(a)** the least expensive and **(b)** the most expensive dessert that can be made.

You may use the fact that the volume of a hemispherical ring of height h is $2\pi h^3/3$.



Solution: We have

$$
\begin{aligned}
\text{Cost} &= \frac{12}{\pi}\cdot(\text{Volume of the ring}) + \frac{8}{\pi}\cdot(\text{Volume of the cylinder}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \frac{9}{\pi}\cdot(\text{Volume of the cap}) \\
&= \frac{12}{\pi}\cdot\frac{2\pi}{3}h^3 + \frac{8}{\pi}\cdot\pi(1-h^2)h + \frac{9}{\pi}\cdot\left(\frac{2\pi}{3}\cdot 1^3 - \frac{2\pi}{3}h^3 - \pi(1-h^2)h\right) \\
&= 6 - h + 3h^3
\end{aligned}
$$

and hence we want to find the absolute maximum and the absolute minimum values of

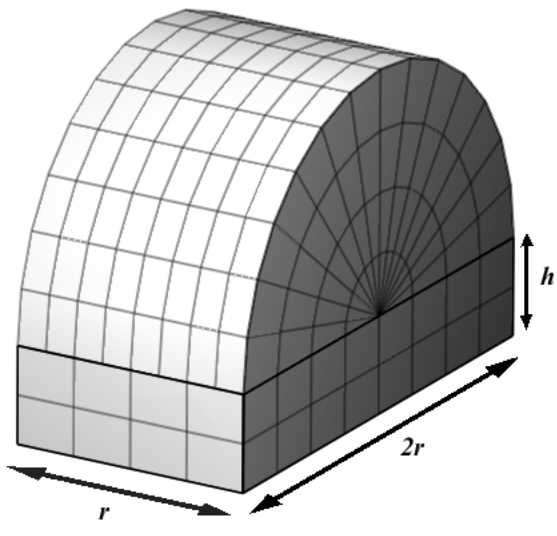
$$\text{Cost} = 6 - h + 3h^3 \quad \text{for} \quad 0 \le h \le 1 \,.$$

We first find the critical points. As $d(\text{Cost})/dh = -1 + 9h^2$, setting this equal to zero gives $h = 1/3$ dm and $h = -1/3$ dm. Only the first of these are in the interval $[0, 1]$, and at that point the cost is $52/9$ ₺.

Next we look at the endpoints of the interval. $h = 0$ dm gives Cost = 6 ₺, and $h = 1$ dm gives Cost = 8 ₺.

Therefore the least expensive dessert has $h = 1/3$ dm, and the most expensive dessert has $h = 1$ dm. □

35. We want to build a greenhouse that has a half cylinder roof of radius r and height r mounted horizontally on top of four rectangular walls of height h as shown in the figure. We have $200\pi\,\text{m}^2$ of plastic sheet to be used in the construction of this structure. Find the value of r for the greenhouse with the largest possible volume we can build.



Solution: We have

$$200\pi\text{m}^2 = \text{Total Surface Area} = \underbrace{2\cdot(r+2r)h}_{\text{walls}} + \underbrace{2\cdot\frac{1}{2}\pi r^2}_{\text{half-disks}} + \underbrace{\frac{1}{2}\cdot 2\pi r\cdot r}_{\textit{top}} = 6rh + 2\pi r^2$$

and hence $h = \dfrac{\pi}{3}\left(\dfrac{100}{r} - r\right)$. In particular, $0 < r \le 10$.

Let V denote the volume of the greenhouse. Then

$$V = \underbrace{r \cdot 2r \cdot h}_{bottom} + \underbrace{\frac{1}{2} \cdot \pi r^2 \cdot r}_{top} = 2r^2 h + \frac{\pi}{2} r^3$$

and substituting h in terms of r, the problem becomes:

$$\text{Maximize } \; V = \frac{200\pi}{3} r - \frac{\pi}{6} r^3 \; \text{ for } \; 0 < r \le 10\,.$$

We first find the critical points in the interval $(0, 10]$:

$$\frac{dV}{dr} = \frac{200\pi}{3} - \frac{\pi}{2} r^2 = 0 \Longrightarrow r = \pm \frac{20}{\sqrt{3}}\,\mathrm{m}$$

As $\sqrt{3} < 2$, neither of these satisfies $0 < r \le 10$, and there are no critical points in the interval.

Next we look at the endpoints of the interval $(0, 10]$: At $r = 0\,\mathrm{m}$ we have $\lim_{r\to 0^+} V = 0\,\mathrm{m}^3$, and at $r = 10\,\mathrm{m}$ we have $V = 500\pi\,\mathrm{m}^3$.

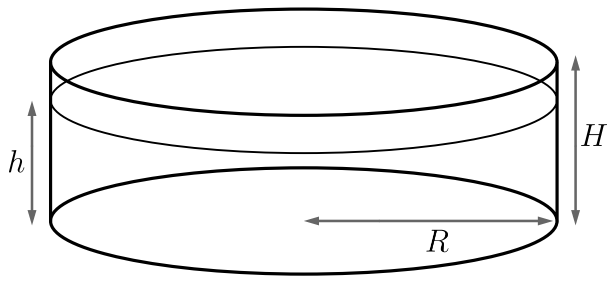
Therefore the maximum possible volume is 500π m^3 and occurs when $r = 10$ m (and $h = 0$ m). ☐

36. A pool, like the one in front of the Faculty of Science Building A, loses water from its sides and its bottom due to seepage, and from its top due to evaporation. For a pool with radius R and depth H in meters, the rate of this loss in m^3/hour is given by an expression of the form

$$aR^2 + bR^2h + cRh^2$$

where h is the depth of the water in meters, and a, b, c are constants independent of R, H and h. Due to this loss, water must be pumped into the pool to keep it at the same level even when the drains are closed.

Suppose that $a = 1/300$ m/hour and $b = c = 1/150$ 1/hour. Find the dimensions of the pool with a volume of $45\pi\,\mathrm{m}^3$ which will require the water to be pumped at the slowest rate to keep it completely full.



Solution: From $\pi R^2 H = 45\pi$ we have $H = 45/R^2$. When the pool is full, the water is lost at the rate

$$L = aR^2 + bR^2H + cRH^2 = \frac{1}{300}R^2 + \frac{1}{150}R^2 \cdot \frac{45}{R^2} + \frac{1}{150}R \cdot \left(\frac{45}{R^2}\right)^2.$$

Hence we want to

$$\text{Minimize } L = \frac{1}{300}R^2 + \frac{3}{10} + \frac{27}{2} \cdot \frac{1}{R^3} \text{ for } 0 < R < \infty.$$

We first find the critical points:

$$\frac{dL}{dR} = \frac{1}{150}R - \frac{81}{2}\frac{1}{R^4} = 0 \Longrightarrow R = 3 \cdot 5^{2/5}\,\text{m}$$

As $\lim\limits_{R\to 0^+} L = \infty$ and $\lim\limits_{R\to\infty} L = \infty$ at the endpoints of the interval, the absolute minimum value of L occurs at $R = 3 \cdot 5^{2/5}$ m, and the corresponding depth is $H = 45/(3 \cdot 5^{2/5})^2 = 5^{1/5}$ m.

Hence the pool that requires the water to be pumped at the slowest rate has radius $R = 3 \cdot 5^{2/5}$ m and depth $H = 45/(3 \cdot 5^{2/5})^2 = 5^{1/5}$ m. □

37. Let f be a continuous function.

a. Find $f(4)$ if $\displaystyle\int_0^{x^2} f(t)\,dt = x\sin\pi x$ for all x.

b. Find $f(4)$ if $\displaystyle\int_0^{f(x)} t^2\,dt = x\sin\pi x$ for all x.

Solution: **a.** $\displaystyle\int_0^{x^2} f(t)\,dt = x\sin\pi x \Longrightarrow \frac{d}{dx}\int_0^{x^2} f(t)\,dt = \frac{d}{dx}(x\sin\pi x) \overset{\text{FTC1}}{\Longrightarrow}$ $f(x^2)\cdot 2x = \sin\pi x + x\cdot\pi\cos\pi x$. Now letting $x = 2$ we get $f(4) = \pi/2$.

b. $\displaystyle\int_0^{f(x)} t^2\,dt = x\sin\pi x \Longrightarrow \frac{t^3}{3}\Big]_0^{f(x)} = x\sin\pi x \Longrightarrow \frac{f(x)^3}{3} = x\sin\pi x$. Hence $x = 4$ gives $f(4) = 0$. □

Remark: One might ask if such functions exist. In part **(b)**, $f(x) = \sqrt[3]{3x\sin\pi x}$ is the unique function satisfying the given condition.

In part **(a)**, $f(x) = \dfrac{\sin\pi\sqrt{x}}{2\sqrt{x}} + \dfrac{\pi}{2}\cos\pi\sqrt{x}$ for $x > 0$, and $f(0) = \pi$ by continuity; but it can be anything for $x < 0$ so long as it is continuous.

38. Compute $\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(0,0)}$ if y is a differentiable function of x satisfying the equation:

$$\int_0^{x+y} e^{-t^2}\,dt = xy$$

Solution: Differentiating the equation with respect to x we obtain:

$$\begin{gathered}
\int_0^{x+y} e^{-t^2}\,dt = xy \\
\Big\Downarrow \; d/dx \\
\frac{d}{dx}\int_0^{x+y} e^{-t^2}\,dt = \frac{d}{dx}(xy) \\
\Big\Downarrow \; \text{FTC1} \\
e^{-(x+y)^2}\cdot\frac{d}{dx}(x+y) = \frac{d}{dx}(xy) \\
\Big\Downarrow \\
e^{-(x+y)^2}\left(1+\frac{dy}{dx}\right) = y + x\frac{dy}{dx} \qquad (\circledast) \\
\Big\Downarrow \; x=0, y=0 \\
\frac{dy}{dx} = -1 \;\text{ at }\; (x,y)=(0,0)
\end{gathered}$$

Now differentiating ($\circledast$) with respect to x again we get:

$$\begin{gathered}
e^{-(x+y)^2}\left(1+\frac{dy}{dx}\right) = y + x\frac{dy}{dx} \\
\Big\Downarrow \; d/dx \\
\frac{d}{dx}\left(e^{-(x+y)^2}\left(1+\frac{dy}{dx}\right)\right) = \frac{d}{dx}\left(y + x\frac{dy}{dx}\right) \\
\Big\Downarrow \\
e^{-(x+y)^2}(-2(x+y))\left(1+\frac{dy}{dx}\right)^2 + e^{-(x+y)^2}\frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} \\
\Big\Downarrow \; x=0, y=0, dy/dx=-1 \\
\frac{d^2y}{dx^2} = -2 \;\text{ at }\; (x,y)=(0,0)
\end{gathered}$$

$\square$

39. Suppose that f is a continuous function satisfying

$$f(x) = x \int_0^x f(t)\,dt + x^3$$

for all x, and c is a real number such that $f(c) = 1$. Express $f'(c)$ in terms of c only.

Solution: We have:

$$\frac{d}{dx} f(x) = \int_0^x f(t)\,dt + x\frac{d}{dx}\int_0^x f(t)\,dt + 3x^3 \overset{\text{FTC1}}{\underset{\downarrow}{=}} \int_0^x f(t)\,dt + xf(x) + 3x^2$$

Now substituting $x = c$ and using the facts that $f(c) = c\int_0^c f(t)\,dt + c^3$ and $f(c) = 1$ we obtain:

$$f'(c) = \int_0^c f(t)\,dt + cf(c) + 3c^2 = \frac{f(c) - c^3}{c} + c + 3c^2 = \frac{1}{c} + c + 2c^2$$

□

Remark: It can be shown that $f(x) = 2x(e^{x^2/2} - 1)$ is the only function that satisfies the given condition.

40. Evaluate the limit $\displaystyle\lim_{x\to 0} \frac{\int_0^x \sin(xt^3)\,dt}{x^5}$.

Solution: We first make the change of variable $t = x^{-1/3}u$, $dt = x^{-1/3}du$, to obtain:

$$\int_0^x \sin(xt^3)\,dt = x^{-1/3}\int_0^{x^{4/3}} \sin(u^3)\,du$$

Now we have:

$$\begin{aligned}
\lim_{x\to 0}\frac{\int_0^x \sin(xt^3)\,dt}{x^5} &= \lim_{x\to 0}\frac{x^{-1/3}\int_0^{x^{4/3}} \sin(u^3)\,du}{x^5} \\
&= \lim_{x\to 0}\frac{\int_0^{x^{4/3}} \sin(u^3)\,du}{x^{16/3}} \\
&\overset{\text{L'H}}{\underset{\downarrow}{=}} \lim_{x\to 0}\frac{\frac{d}{dx}\int_0^{x^{4/3}} \sin(u^3)\,du}{16/3\,x^{13/3}} \\
&\overset{\text{FTC1}}{\underset{\downarrow}{=}} \lim_{x\to 0}\frac{\sin(x^4)\cdot 4/3\,x^{1/3}}{16/3\,x^{13/3}} \\
&= \lim_{x\to 0}\frac{\sin(x^4)\cdot 4/3\,x^{1/3}}{16/3\,x^{13/3}} \\
&= \frac{1}{4}\lim_{x\to 0}\frac{\sin(x^4)}{x^4} \\
&= \frac{1}{4}\cdot 1 = \frac{1}{4}
\end{aligned}$$

□

41. Suppose that f is a twice-differentiable function satisfying $f(0) = -2$, $f'(0) = 11$, $f''(0) = -8$, $f(2) = 5$, $f'(2) = -3$, $f''(2) = 7$; and also suppose that the function

$$g(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

has a critical point at $x = 2$. Determine whether the critical point of g at $x = 2$ is a local minimum, a local maximum or neither.

Solution: We first observe that

$$\begin{aligned}\frac{d}{dx}g(x) &= \frac{d}{dx}\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\\ &\overset{\overset{\text{FTC1}}{\downarrow}}{=} -\frac{1}{x^2}\int_0^x f(t)\,dt + \frac{1}{x}\frac{d}{dx}\int_0^x f(t)\,dt\\ &= -\frac{1}{x^2}\int_0^x f(t)\,dt + \frac{1}{x}f(x)\end{aligned}$$

and as $g'(2) = 0$ we must have $-1/4\int_0^2 f(t)\,dt + 1/2\,f(2) = 0$, and hence $\int_0^2 f(t)\,dt = 2f(2) = 10$.

Now differentiating a second time we obtain that:

$$\begin{aligned}\frac{d^2}{dx^2}g(x) &= \frac{d}{dx}\left(-\frac{1}{x^2}\int_0^x f(t)\,dt + \frac{1}{x}f(x)\right)\\ &\overset{\overset{\text{FTC1}}{\downarrow}}{=} \frac{2}{x^3}\int_0^x f(t)\,dt - \frac{1}{x^2}\frac{d}{dx}\int_0^x f(t)\,dt - \frac{1}{x^2}f(x) + \frac{1}{x}f'(x)\\ &= \frac{2}{x^3}\int_0^x f(t)\,dt - \frac{1}{x^2}f(x) - \frac{1}{x^2}f(x) + \frac{1}{x}f'(x)\end{aligned}$$

Substituting $x = 2$ in this we get

$$g''(2) = \frac{1}{4}\int_0^2 f(t)\,dt - \frac{1}{2}f(2) + \frac{1}{2}f'(2) = -\frac{3}{2} < 0$$

and conclude that g has a local maximum at $x = 2$. □

42. Suppose that f is a continuous and positive function on $[0, 5]$, and the area between the graph of $y = f(x)$ and the x-axis for $0 \le x \le 5$ is 8. Let $A(c)$ denote the area between the graph of $y = f(x)$ and the x-axis for $0 \le x \le c$, and let $B(c)$ denote the area between the graph of $y = f(x)$ and the x-axis for $c \le x \le 5$. Let $R(c) = A(c)/B(c)$. If $R(3) = 1$ and $\left.\frac{dR}{dc}\right|_{c=3} = 7$, find $f(3)$.

Solution: We have $A(3) + B(3) = 8$ and $R(3) = 1 \Longrightarrow A(3) = B(3)$, implying $A(3) = B(3) = 4$.

As

$$A(c) = \int_0^c f(t)\,dt \quad \text{and} \quad B(c) = \int_c^5 f(t)\,dt\,,$$

differentiating these with respect to c and using the Fundamental Theorem of Calculus Part 1 we obtain $A'(c) = f(c)$ and $B'(c) = -f(c)$. In particular, $A'(3) = f(3)$ and $B'(3) = -f(3)$.

On the other hand,

$$\frac{d}{dc}R(c) = \frac{d}{dc}\frac{A(c)}{B(c)} = \frac{A'(c)B(c) - A(c)B'(c)}{B(c)^2}$$

and letting $c = 3$ gives

$$7 = \frac{d}{dc}R(c)\Big|_{c=3} = \frac{A'(3)B(3) - A(3)B'(3)}{B(3)^2} = \frac{f(3)B(3) + A(3)f(3)}{B(3)^2} = \frac{f(3)}{2}$$

and hence $f(3) = 14$. □

43. Let f be a continuous function and let

$$g(x) = \int_{-1}^{1} f(t)\,|x-t|\,dt\,.$$

Express $g''(x)$ in terms of $f(x)$ for $-1 < x < 1$.

Solution: For $-1 < x < 1$ we rewrite the definition of $g(x)$ as follows:

$$\begin{aligned} g(x) &= \int_{-1}^{x} f(t)\,(x-t)\,dt + \int_{x}^{1} f(t)\,(t-x)\,dt \\ &= x\int_{-1}^{x} f(t)\,dt - \int_{-1}^{x} f(t)t\,dt + \int_{x}^{1} f(t)t\,dt - x\int_{x}^{1} f(t)\,dt \end{aligned}$$

Now we differentiate using the FTC1 to obtain:

$$\begin{aligned} g'(x) &= \int_{-1}^{x} f(t)\,dt + x\frac{d}{dx}\int_{-1}^{x} f(t)\,dt - \frac{d}{dx}\int_{-1}^{x} f(t)t\,dt \\ &\qquad + \frac{d}{dx}\int_{x}^{1} f(t)t\,dt - \int_{x}^{1} f(t)\,dt - x\frac{d}{dx}\int_{x}^{1} f(t)\,dt \\ &= \int_{-1}^{x} f(t)\,dt + xf(x) - f(x)x - f(x)x - \int_{x}^{1} f(t)\,dt + xf(x) \\ &= \int_{-1}^{x} f(t)\,dt - \int_{x}^{1} f(t)\,dt \end{aligned}$$

Using FTC1 again we compute $g''(x)$ to find

$$\begin{aligned} g''(x) &= \frac{d}{dx}\int_{-1}^{x} f(t)\,dt - \frac{d}{dx}\int_{x}^{1} f(t)\,dt \\ &= f(x) + f(x) \\ &= 2f(x) \end{aligned}$$

for $-1 < x < 1$. □

44. Suppose that a continuous function f satisfies the equation

$$f(x) = x^2 - x + (1-x)\int_0^x t^2 f(t)\,dt + x\int_x^1 (t-t^2)f(t)\,dt$$

for all x. Express $f''(1/2)$ in terms of $A = f(1/2)$.

Solution: We have

$$\begin{aligned} f'(x) &= 2x - 1 - \int_0^x t^2 f(t)\,dt + (1-x)x^2 f(x) + \int_x^1 (t-t^2)f(t)\,dt - x(x-x^2)f(x) \\ &= 2x - 1 - \int_0^x t^2 f(t)\,dt + \int_x^1 (t-t^2)f(t)\,dt \end{aligned}$$

for all x, where we used the Fundamental Theorem of Calculus Part 1 twice:

$$\begin{aligned} \frac{d}{dx}\int_0^x t^2 f(t)\,dt &\overset{\text{FTC1}}{=} x^2 f(x) \\ \frac{d}{dx}\int_x^1 (t-t^2)f(t)\,dt &\overset{\text{FTC1}}{=} -(x-x^2)f(x) \end{aligned}$$

Differentiating again and using FTC1 similarly we obtain:

$$f''(x) = 2 - x^2 f(x) - (x-x^2)f(x) = 2 - xf(x)$$

for all x. This gives $f''(1/2) = 2 - 1/2 \cdot f(1/2) = 2 - A/2$. □

45. Evaluate the limit $\lim\limits_{n\to\infty}\left(n\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+3)^2} + \cdots + \frac{1}{(4n-1)^2}\right)\right)$.

Solution: Consider the function $f(x) = \dfrac{1}{x^2}$ on the interval $[2,4]$. If we divide this interval into n subintervals of equal length $\dfrac{2}{n}$ using the points $x_k = 2 + \dfrac{2k}{n}$, $0 \le k \le n$, and choose our sample points to be the midpoints $c_k = 2 + \dfrac{2k-1}{n}$, $1 \le k \le n$, of these subintervals, the Riemann sum for these data becomes

$$\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f\left(2 + \frac{2k-1}{n}\right)\cdot\frac{2}{n} = \sum_{k=1}^{n}\frac{2n}{(2n+2k-1)^2}$$

and the definition of the definite integral gives

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2n}{(2n+2k-1)^2} = \int_2^4 \frac{dx}{x^2} = -\left.\frac{1}{x}\right]_2^4 = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}.$$

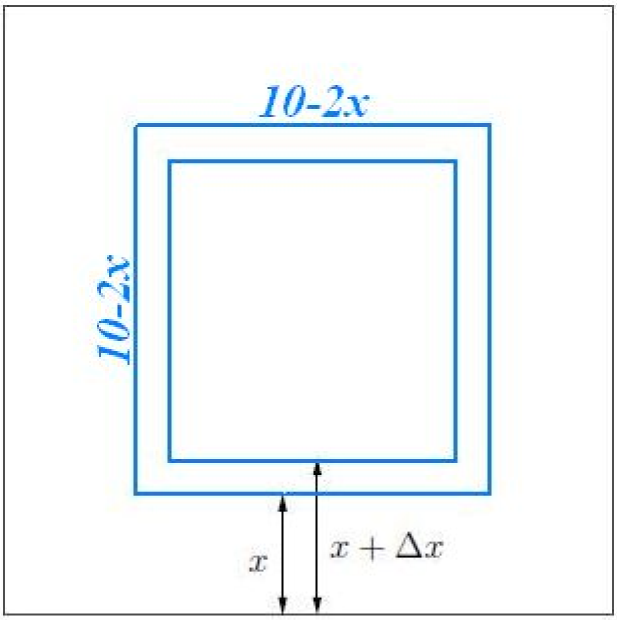
Therefore

$$\lim_{n\to\infty}\left(n\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+3)^2} + \cdots + \frac{1}{(4n-1)^2}\right)\right) = \frac{1}{8}.$$

□

46. When pirates retire, they live on the Square Island which has the shape of a square with 10 hectometer (=hm) long sides. Having lived all their lives on it, the retired pirates want to be as far away from the sea as possible. As a result, the pirate population density $p(x)$ at a point on the Square Island is proportional to the distance x of the point from the shore and reaches its largest value of 15 pirate/hm^2 at the center of the island. Find the total number N of the pirates on the island.

Solution: Firstly, we have $p(x) = 3x$. Then we observe that the points on the island whose distance to the shore are x hm lie on a square whose sides are parallel to the sides of the island and $10-2x$ hm long.



Now consider another square which consists of points whose distances to the shore are $x+\Delta x$ hm for some small positive Δx, and choose a x^* which lies between x and $x+\Delta x$. Then the number of pirates living in the strip between these two squares is approximately $4\cdot(10-2x)\cdot p(x^*)\cdot\Delta x$.

Hence N can be approximated by the Riemann sum

$$N \approx \sum_{i=1}^{n} 4 \cdot (10 - 2x_i) \cdot p(x_i^*) \cdot \Delta x_i$$

for a partition $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 5$ of the interval $[0,5]$ and sample points x_i^* in $[x_{i-1}, x_i]$ for $1 \le i \le n$, and in the limit we obtain:

$$N = \int_0^5 4(10 - 2x)p(x)\,dx$$

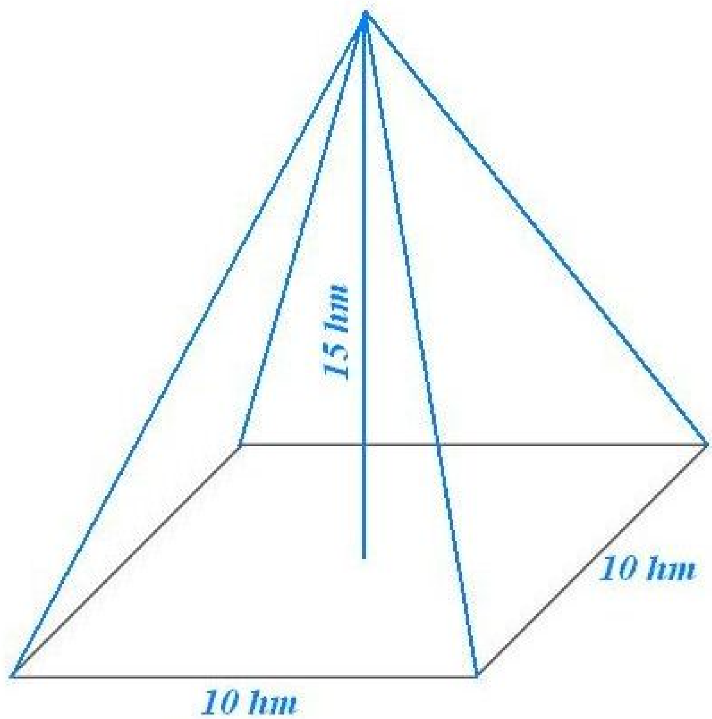
This gives:

$$N = \int_0^5 4(10 - 2x)(3x)\,dx = 24 \int_0^5 (5x - x^2)\,dx = 24\left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_0^5 = 500$$

There are 500 pirates living on the island. □

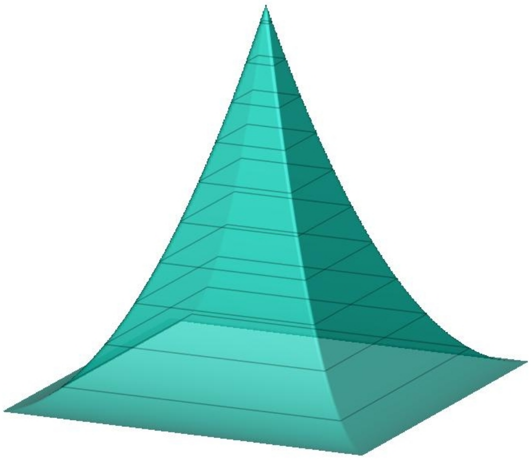
Remark: Imagine that the entire landscape of the Square Island consists of just a mountain and the altitude of a point that is horizontally x hm away from the shore is $p(x)$ hm. Then the island has the shape of a square pyramid with a 10 hm by 10 hm base and 15 hm height, and N is now the volume of the island. Hence we have:

$$N = \frac{1}{3}(\text{height})(\text{area of the base}) = \frac{1}{3} \cdot 15 \cdot 10^2 = 500$$



Remark: Compare this example with **Example 30** in **Part 2**.

47. An island has the shape of a 10 hm×10 hm square and its landscape consists of a mountain whose height h at a horizontal distance x from the shore is given by $h = x^2$ where both h and x are measured in hectometers (=hm). Let V be the volume of the mountain.



a. Express V as an integral with respect to h by considering cutting the mountain into slices as shown in *Figure a*.

b. Express V as an integral with respect to x by considering cutting the mountain into shells as shown in *Figure b*.

Figure a:

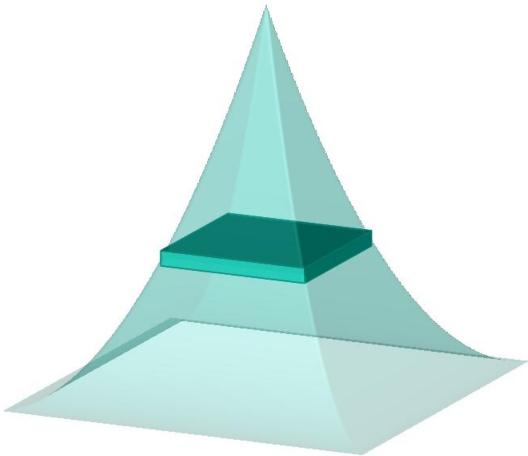
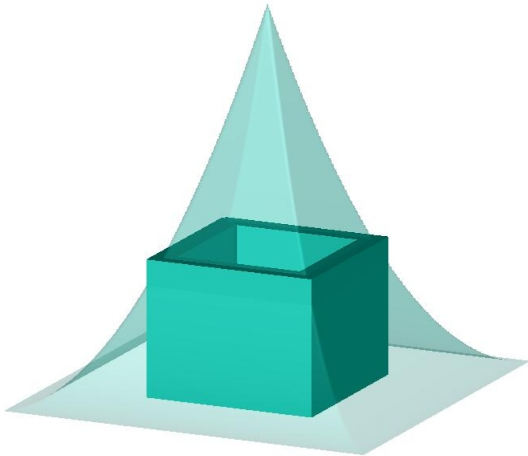


Figure b:



c. Compute V.

Solution: a. A horizontal slice at a height h is approximately a prism with a square base of side length $10 - 2x = 10 - 2\sqrt{h}$ and height Δh, and therefore its volume is approximately $(10-2x)^2\,\Delta h = (10-2\sqrt{h})^2\,\Delta h$. Hence the volume V can be approximated by the Riemann sum

$$V \approx \sum_{i=1}^{n}(10-2\sqrt{h_i^*})^2\,\Delta h_i$$

for a partition $0 = h_0 < h_1 < \cdots < h_{n-1} < h_n = 25$ of the interval $[0,25]$ and sample points h_i^* in $[h_{i-1}, h_i]$ for $1 \le i \le n$, and in the limit we obtain:

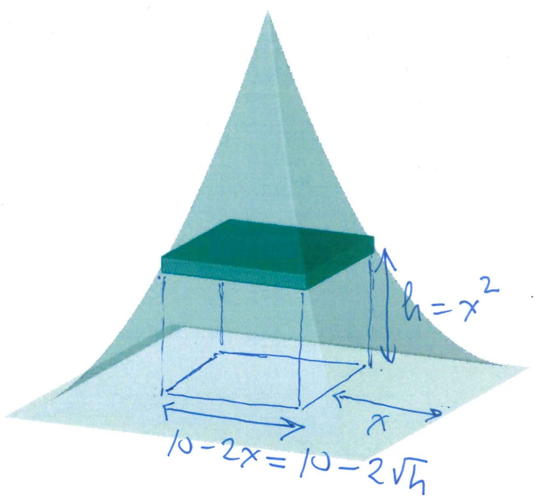
$$V = \int_0^{25}(10-2\sqrt{h})^2\,dh$$

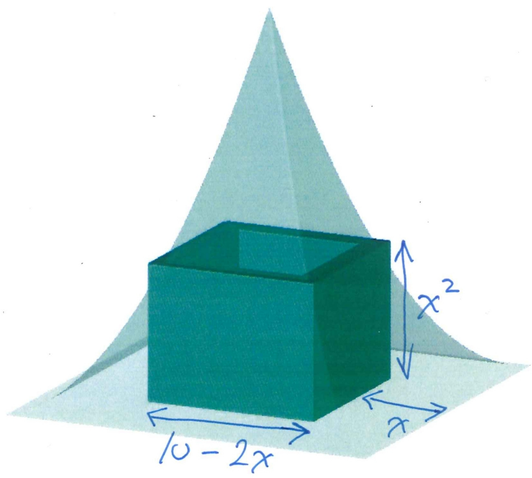
b. A shell at a horizontal distance x from the shore consists of four pieces each of which is approximately a rectangular prism of dimensions $x^2 \times (10-2x) \times \Delta x$ where Δx is the thickness of the shell. Therefore its volume is approximately $4\,(10-2x)\,x^2\,\Delta x$. Hence the volume V can be approximated by the Riemann sum

$$V \approx \sum_{i=1}^{n} 4\,(10-2x_i^*)\,(x_i^*)^2\,\Delta x_i$$

for a partition $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 5$ of the interval $[0,5]$ and sample points x_i^* in $[x_{i-1}, x_i]$ for $1 \le i \le n$, and in the limit we obtain:

$$V = \int_0^5 4\,(10-2x)\,x^2\,dx$$





c. We use the integral in **Part c** to compute the volume:

$$V = \int_0^5 4\,(10-2x)\,x^2\,dx = 8\int_0^5 (5x^2 - x^3)\,dx = 8\left[\frac{5}{3}x^3 - \frac{1}{4}x^4\right]_0^5 = \frac{1250}{3}\ \text{hm}^3$$

□

48. Evaluate the following integrals.

a. $\displaystyle\int x\sin(x^2)\cos(x^2)\,dx$

b. $\displaystyle\int_0^1 x\sqrt{1-x}\,dx$

Solution: a. Let $u = \sin(x^2)$. Then $du = 2x\cos(x^2)\,dx$, and

$$\int x\sin(x^2)\cos(x^2)\,dx = \frac{1}{2}\int u\,du = \frac{1}{2}\cdot\frac{u^2}{2} + C = \frac{1}{4}\sin^2(x^2) + C\,.$$

b. Let $u = 1-x$. Then $du = -dx$, and $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 0$. Therefore:

$$\int_0^1 x\sqrt{1-x}\,dx = \int_1^0 (1-u)u^{1/2}\,(-du) = \int_0^1 (u^{1/2} - u^{3/2})\,du$$

$$= \left[\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right]_0^1 = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$$

□

Remark: In part **(a)**, if we let $u = \cos(x^2)$, then we obtain the answer

$$\int x\sin(x^2)\cos(x^2)\,dx = -\frac{1}{4}\cos^2(x^2) + C\,;$$

and if we first observe that $\sin(x^2)\cos(x^2) = \dfrac{1}{2}\sin(2x^2)$, and then let $u = \sin(2x^2)$, then we obtain the answer

$$\int x\sin(x^2)\cos(x^2)\,dx = -\frac{1}{8}\cos(2x^2) + C\,.$$

These are all correct answers. If we write

$$\begin{aligned}\int x\sin(x^2)\cos(x^2)\,dx &= \frac{1}{4}\sin^2(x^2) + C_1\\ \int x\sin(x^2)\cos(x^2)\,dx &= -\frac{1}{4}\cos^2(x^2) + C_2\\ \int x\sin(x^2)\cos(x^2)\,dx &= -\frac{1}{8}\cos(2x^2) + C_3\end{aligned}$$

then $C_2 = C_1 + \dfrac{1}{4}$ and $C_3 = C_1 + \dfrac{1}{8}$.

49. Show that $\displaystyle\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{a}{2}$ for any positive continuous function on $[0,a]$.

Solution: Let $I = \displaystyle\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx\,.$

Let $u = a - x$. Then $du = -dx$, $x = 0 \Rightarrow u = a$, $x = a \Rightarrow u = 0$, and

$$I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}\,(-du) = \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx\,.$$

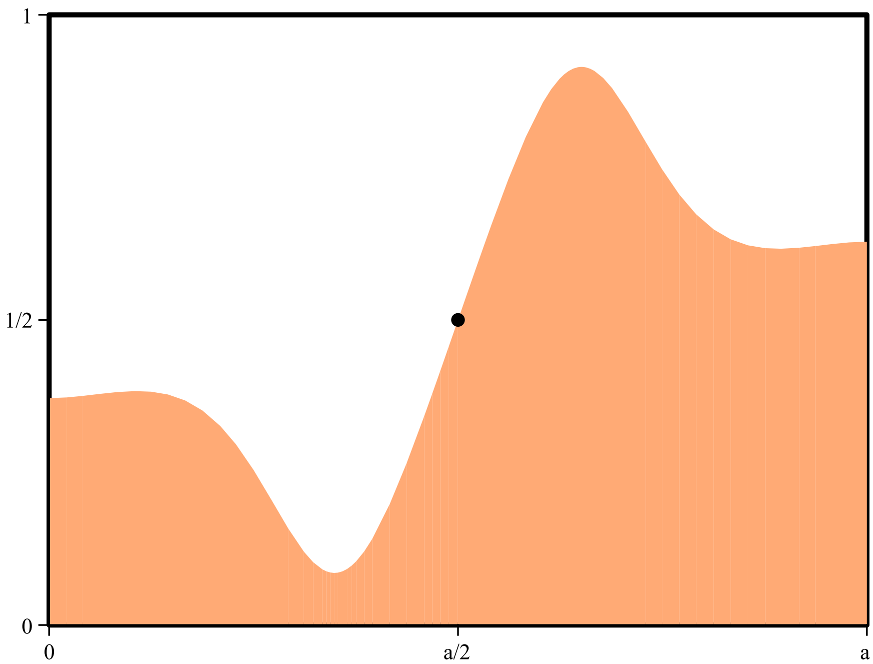
Therefore

$$\begin{aligned}2I &= \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx + \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx\\ &= \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,dx\\ &= \int_0^a dx = a\end{aligned}$$

and $I = \dfrac{a}{2}$. □

(*See the remark on the next page.*)

Remark: Here is an explanation of what is going on with no integral signs: Consider the rectangle with a vertex at the origin, and sides along the positive x- and the y-axes with lengths a and 1, respectively. The graph of $y = f(x)/(f(x) + f(a-x))$ is symmetric with respect to the center $(a/2, 1/2)$ of this rectangle, and therefore divides it into two regions of equal area. Since the area of the rectangle is a and I is the area of the lower half, we have $I = \dfrac{a}{2}$.



50. Let R be the region bounded by the parabola $y = x - x^2$ and the x-axis, and let V be the volume of the solid generated by revolving R about the x-axis.

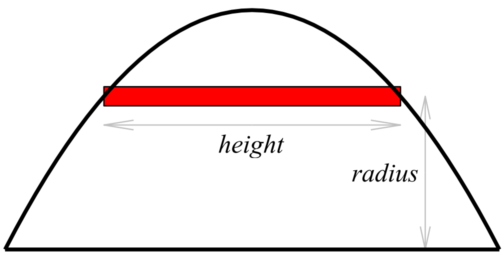
a. Express V as an integral using the disk method. (*Do not compute!*)

b. Express V as an integral using the cylindrical shell method. (*Do not compute!*)

Solution: The parabola intersects the x-axis at $x = 0$ and $x = 1$. Therefore we have

$$V = \pi \int_0^1 R(x)^2\,dx = \int_0^1 (x - x^2)^2\,dx$$

for part **(a)**.

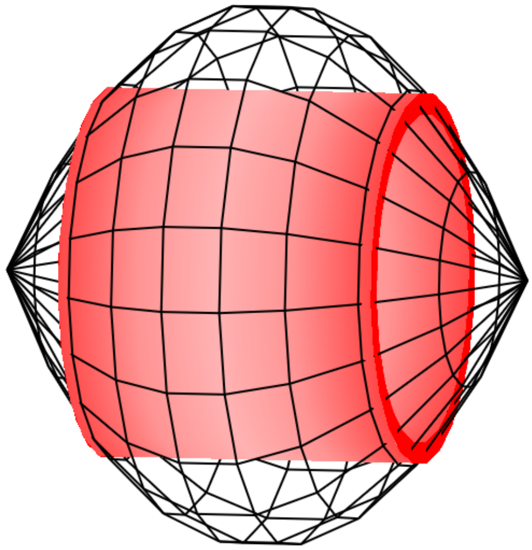


For part **(b)**, the cylindrical shell method gives a y-integral,

$$V = 2\pi \int_c^d (\text{radius of the shell})(\text{height of the shell})\,dy$$

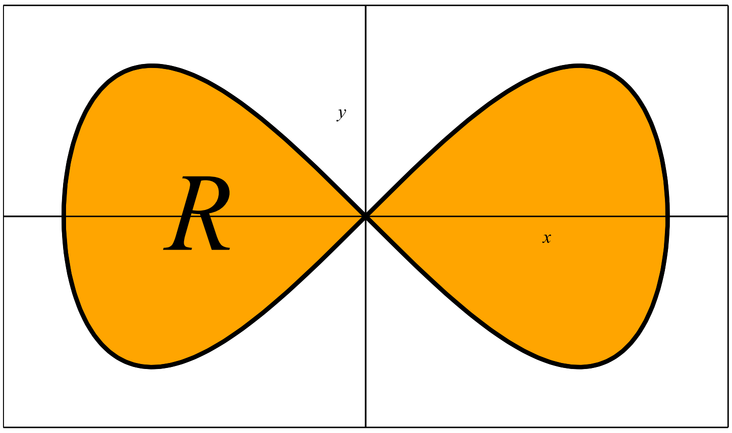
as the region is revolved about the x-axis. We have $c = 0$ and $d = 1/4$, the y-coordinate of the highest point of the parabola. The radius of the shell is the vertical distance from the red rectangle in the figure to the x-axis, which is y. The height of the shell is the horizontal length of the rectangle; that is, the difference between the x-coordinates of the right and the left sides of the rectangle. By solving the equation $y = x - x^2$ for x, we find these values as $x = \dfrac{1+\sqrt{1-4y}}{2}$ and $x = \dfrac{1-\sqrt{1-4y}}{2}$, respectively. Hence

$$V = 2\pi \int_0^{1/4} y \left(\frac{1+\sqrt{1-4y}}{2} - \frac{1-\sqrt{1-4y}}{2} \right) dy \, .$$



□

51. Let R be the region bounded by the curve $y^2 = x^2 - x^4$. Let V be the volume obtained by rotating R about the x-axis. Let W be the volume obtained by rotating R about the y-axis.



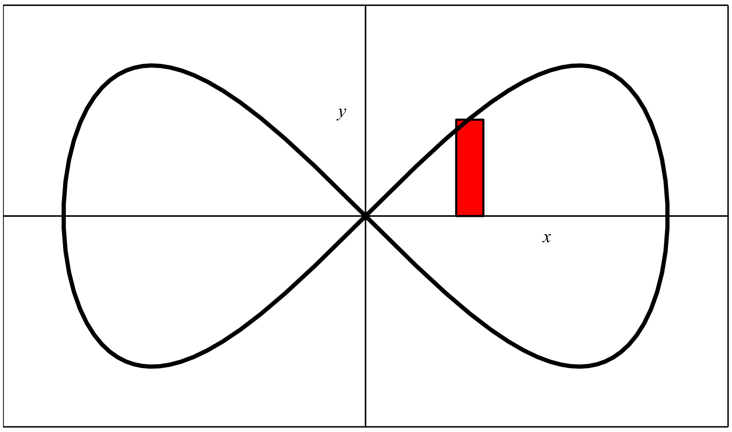
a. Express V using both the disk method and the cylindrical shells method.

b. Express W using both the disk method and the cylindrical shells method.

c. Compute V and W.

Solution: By symmetry, V is 2 times the volume obtained by revolving the portion of R lying in the first quadrant about the x-axis. In the disk method, the red vertical rectangles, when revolved about the x-axis, form the disks that are used in the computation of V, and therefore the radii of the disks are given by $\sqrt{x^2 - x^4}$. Hence:

$$V = 2 \cdot \pi \int_0^1 (\text{radius of disk})^2\, dx = 2 \cdot \pi \int_0^1 (\sqrt{x^2 - x^4})^2\, dx$$



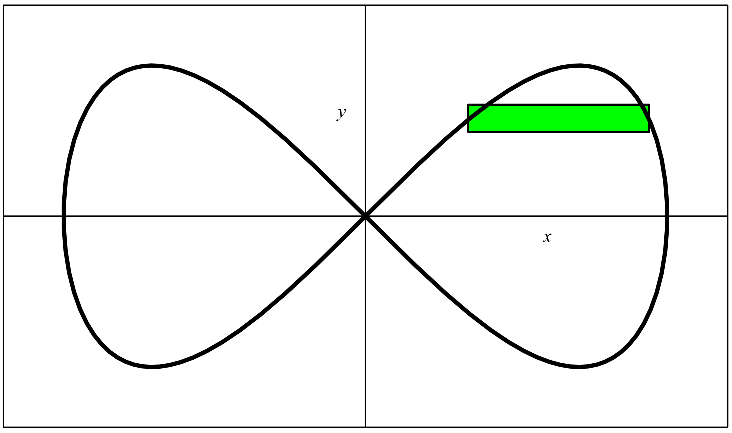
Again by symmetry, W is 2 times the volume generated by revolving the portion of R lying in the first quadrant about the y-axis. This time in the washer method, the green horizontal rectangles, when revolved about the y-axis, form the washers that are used in the computation of W.

To find the outer and the inner radii of the washers we have to solve $y^2 = x^2 - x^4$ for x. Applying the quadratic formula to $(x^2)^2 - x^2 + y^2 = 0$ we obtain $x^2 = (1 \pm \sqrt{1 - 4y^2})/2$, and this gives us

$$x = \sqrt{(1+\sqrt{1-4y^2})/2} \ \text{ and } \ x = \sqrt{(1-\sqrt{1-4y^2})/2}$$

as the two nonnegative solutions. Hence:

$$\begin{aligned} W &= 2 \cdot \pi \int_0^{1/2} \left((\text{outer radius of washer})^2 - (\text{inner radius of washer})^2\right) dy \\ &= 2 \cdot \pi \int_0^{1/2} \left(\left(\sqrt{\frac{1+\sqrt{1-4y^2}}{2}}\right)^2 - \left(\sqrt{\frac{1-\sqrt{1-4y^2}}{2}}\right)^2\right) dy \end{aligned}$$



Now we consider the cylindrical shells method for both volumes. When revolved about the y-axis the red vertical rectangles generate the cylindrical shells that are used in the computation of W, and therefore the heights of these shells are given by $\sqrt{x^2 - x^4}$. Hence:

$$W = 2 \cdot 2\pi \int_0^1 (\text{radius of shell})(\text{height of shell})\, dx = 2 \cdot 2\pi \int_0^1 x\sqrt{x^2 - x^4}\, dx$$

When revolved about the x-axis the green horizontal rectangles generate the cylindrical shells that are used in the computation of V. Therefore the heights of these shells are given by $\sqrt{(1+\sqrt{1-4y^2})/2} - \sqrt{(1-\sqrt{1-4y^2})/2}$. Hence:

$$\begin{aligned} V &= 2 \cdot 2\pi \int_0^{1/2} (\text{radius of shell})(\text{height of shell})\, dy \\ &= 2 \cdot 2\pi \int_0^{1/2} y\left(\sqrt{\frac{1+\sqrt{1-4y^2}}{2}} - \sqrt{\frac{1-\sqrt{1-4y^2}}{2}}\right) dy \end{aligned}$$

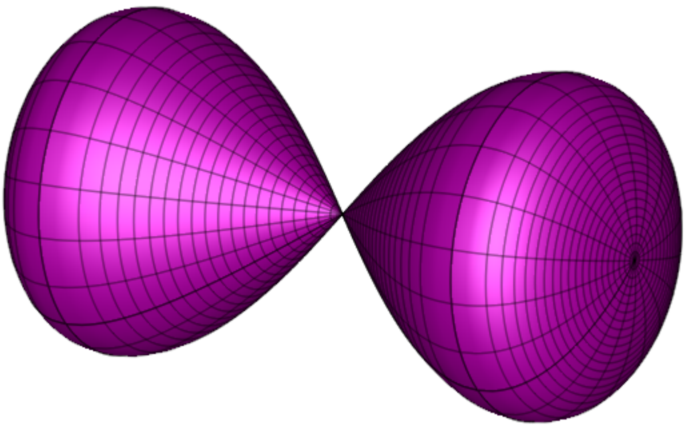
Finally we compute the volumes. To compute V we use the integral we obtained with the disk method,

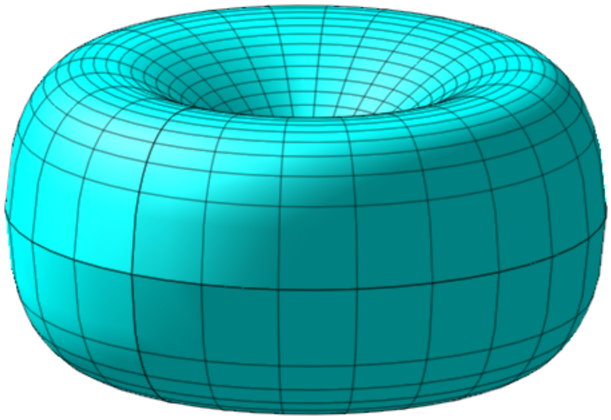
$$V = 2 \cdot \pi \int_0^1 (x^2 - x^4)\, dx = 2\pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{4\pi}{15}$$

and to compute W we use the integral we obtained with the washer method

$$\begin{aligned}
W &= 2\cdot\pi\int_0^{1/2}\left(\frac{1+\sqrt{1-4y^2}}{2}-\frac{1-\sqrt{1-4y^2}}{2}\right)dy \\
&= 2\pi\int_0^{1/2}\sqrt{1-4y^2}\,dy \\
&= 2\pi\int_0^{\pi/2}\cos\theta\cdot\frac{1}{2}\cos\theta\,d\theta \\
&= \pi\int_0^{\pi/2}\cos^2\theta\,d\theta \\
&= \pi\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta \\
&= \pi\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{\pi/2} \\
&= \frac{\pi^2}{4}
\end{aligned}$$

where we made the change of variable $y=\frac{1}{2}\sin\theta$, $dy=\frac{1}{2}\cos\theta\,d\theta$. □





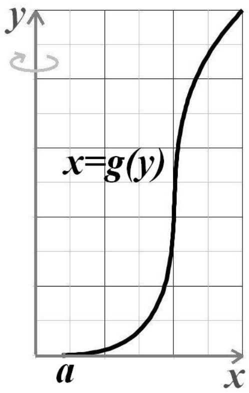
Remark: Compare this example with **Example 36** in **Part 2**.

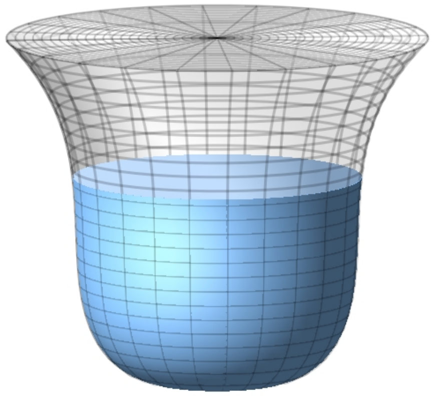
52. A water tank has a bottom consisting of a disk of radius a with $0\le a<3$, and a side surface having the shape generated by revolving the graph of a continuous nonnegative function $x=g(y)$ for $0\le y\le 5$ with $g(0)=a$ and $g(5)=3$ about the y-axis where all units are in meters. Assume that:

① As water runs out of a small hole at the bottom of tank, the speed of the water flowing through the hole at any moment is proportional to the square root of the depth of the water in the tank at that moment.

② The function g and the constant a are chosen in such a way that the depth of the water changes at a constant rate at all times.

Find the volume of the tank.





Solution: Let $V(h)$ denote the volume of the water when the depth is h. Then:

$$V(h) = \pi \int_0^h g(y)^2 \, dy$$

Differentiating this with respect to time t and using the Fundamental Theorem of Calculus, Part 1 we obtain:

$$\frac{dV}{dt} = \pi g(h)^2 \frac{dh}{dt}$$

Using the condition 1 which says that $dV/dt = -k_1\sqrt{h}$ for some positive constant k_1 and the condition 2 which says that $dh/dt = -k_2$ for some positive constant k_2, we conclude that $g(h) = k \cdot h^{1/4}$ for some positive constant k. In particular, $a = g(0) = 0$, and $g(5) = 3$ gives $k = 3/5^{1/4}$. Therefore $g(h) = 3h^{1/4}/5^{1/4}$.

Hence the volume of the tank is:

$$V(5) = \pi \int_0^5 \left(\frac{3}{5^{1/4}} y^{1/4} \right)^2 dy = \frac{9\pi}{5^{1/2}} \cdot \frac{y^{3/2}}{3/2} \Bigg]_0^5 = 30\pi \text{ m}^3$$

□

53. Let V be the volume of the water-dropper shown in the figure on the next page which has the shape obtained by revolving the curve $x^4 + y^4 = 1$ about the line $x = -5/2$ where all units are in centimeters.

a. Express V as an integral using the cylindrical shells method.

b. Express V as an integral using the washer method.

c. Show that the improper integral $\int_0^1 u^{-3/4}(1-u)^{1/4}\,du$ converges.

d. Express V in terms of $A = \int_0^1 u^{-3/4}(1-u)^{1/4}\,du$.

Solution: a. The radius and height of the cylindrical shells are $x - (-5/2)$ and $(1-x^4)^{1/4} - (-(1-x^4)^{1/4})$, respectively. Hence:

$$V = 2\pi \int_{-1}^{1} (\text{radius}) \cdot (\text{height})\,dx = 2\pi \int_{-1}^{1} \left(x + \frac{5}{2}\right) \cdot \left((1-x^4)^{1/4} - (-(1-x^4)^{1/4}\right)\,dx$$

b. The outer and inner radii of the washers are $5/2 + (1-y^4)^{1/4}$ and $5/2 - (1-y^4)^{1/4}$, respectively. Hence:

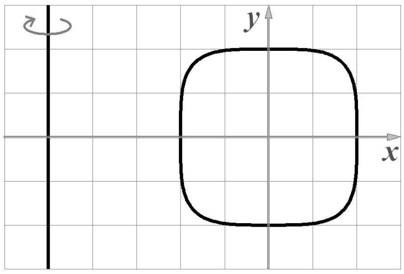
$$\begin{aligned} V &= \pi \int_{-1}^{1} \left((\text{outer radius})^2 - (\text{inner radius})^2\right) dy \\ &= \pi \int_{-1}^{1} \left(\left(\frac{5}{2} + (1-y^4)^{1/4}\right)^2 - \left(\frac{5}{2} - (1-y^4)^{1/4}\right)^2\right) dy \end{aligned}$$

c. We have $0 < u \le 1 \Longrightarrow 0 \le 1-u < 1 \Longrightarrow 0 \le (1-u)^{1/4} < 1 \Longrightarrow 0 \le u^{-3/4}(1-u)^{1/4} < u^{-3/4}$ for all $0 < u \le 1$. Since $\int_0^1 u^{-3/4}\,du$ is convergent (because $p = 3/4 < 1$), so is $\int_0^1 u^{-3/4}(1-u)^{1/4}\,du$ by the Comparison Test.

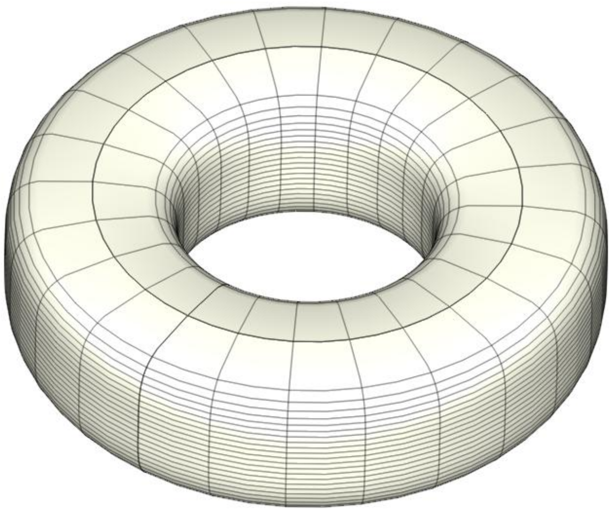
d. We use the integral in part **(b)** and symmetry

$$\begin{aligned} V &= 10\pi \int_{-1}^{1} (1-y^4)^{1/4}\,dy \\ &= 20\pi \int_{0}^{1} (1-y^4)^{1/4}\,dy \\ &= 20\pi \int_{0}^{1} (1-u)^{1/4} \cdot \frac{1}{4} u^{-3/4}\,du \\ &= 5\pi \int_{0}^{1} u^{-3/4}(1-u)^{1/4}\,du \\ &= 5\pi A \end{aligned}$$

as well as the change of variable $y = u^{1/4}$, $dy = 1/4 \cdot u^{-3/4}\,du$. $\square$



y
x



54. Find the absolute maximum and the absolute minimum values of $f(x) = (x^2 - 3)e^x$ on the interval $[-2, 2]$.

Solution: $f'(x) = 2xe^x + (x^2 - 3)e^x = (x^2 + 2x - 3)e^x$. We want to solve $f'(x) = 0$.

$$(x^2 + 2x - 3)e^x = 0 \Longrightarrow x^2 + 2x - 3 = 0 \Longrightarrow x = 1, -3\,.$$

Since -3 does not belong to the interval $[-2, 2]$, the only critical point is $x = 1$. Hence we are going to compute f at the points $x = 1, -2,\ 2$.

$f(1) = -2e$, $f(-2) = e^{-2}$, and $f(2) = e^2$. Since $e > 1$ we have $e^2 > e^{-2} > -2e$. Therefore the absolute maximum is e^2 and the absolute minimum is $-2e$. $\square$

55. Find the absolute maximum and the absolute minimum values of $x^{1/x}$.

Solution: Let $y = x^{1/x}$. Then $\ln y = \dfrac{\ln x}{x}$. Differentiating this with respect to x we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\ln y = \frac{d}{dx}\frac{\ln x}{x} = \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2} \Longrightarrow \frac{d}{dx}x^{1/x} = x^{1/x}\,\frac{1 - \ln x}{x^2}\,.$$

Since $1 - \ln x > 0$ for $0 < x < e$ and $1 - \ln x < 0$ for $x > e$, the absolute maximum value of $x^{1/x}$ occurs at $x = e$ and is $e^{1/e}$. $x^{1/x}$ has no absolute minimum value. $\square$

Remark: Although the reasoning above does not require it, let us also look at what happens at the endpoints of the domain. Since

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{L'H}}{\downarrow}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0\,,$$

we have $\lim\limits_{x\to\infty} x^{1/x} = \lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} e^{\ln y} = e^0 = 1$. On the other hand, $\lim\limits_{x\to 0^+} x^{1/x} = 0$ as $0^\infty = 0$.

56. Evaluate the limit $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} 2^{k/n}\right)$.

Solution: $\displaystyle\sum_{k=1}^{n} 2^{k/n}\,\frac{1}{n}$ is a Riemann sum $\displaystyle\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ for $f(x) = 2^x$ on $[0, 1]$ for the partition $x_k = \dfrac{k}{n}$, $0 \le k \le n$, and the sample points $c_k = \dfrac{k}{n}$, $1 \le k \le n$. Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} 2^{k/n}\right) = \int_0^1 2^x\,dx = \frac{2^x}{\ln 2}\bigg]_0^1 = \frac{1}{\ln 2}\,.$$

$\square$

Remark: Alternatively, we can use the formula for the sum of a finite geometric series

$$\sum_{k=1}^{n} 2^{k/n} = \frac{2^{(n+1)/n} - 2^{1/n}}{2^{1/n} - 1} = \frac{2^{1/n}}{2^{1/n} - 1} ,$$

and then

$$\lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} 2^{k/n} \right) = \lim_{n\to\infty} \frac{2^{1/n}}{n(2^{1/n} - 1)} = \lim_{t\to 0^+} \frac{t2^t}{2^t - 1} \overset{\text{L'H}}{\underset{\downarrow}{=}} \lim_{t\to 0^+} \frac{2^t + t \cdot \ln 2 \cdot 2^t}{\ln 2 \cdot 2^t} = \frac{1}{\ln 2} .$$

57. Evaluate the limit $\lim_{x\to 0} \left(\frac{\sin x}{x} \right)^{1/x^2}$.

Solution: Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1}{x^2} = \infty$, this is limit has the indeterminate form 1^∞.

Let $y = \left(\frac{\sin x}{x} \right)^{1/x^2}$. Then $\ln y = \frac{\ln\left(\frac{\sin x}{x}\right)}{x^2}$. As $x \to 0$, this will have the indeterminate form $\frac{0}{0}$ and we can use L'Hôpital's Rule.

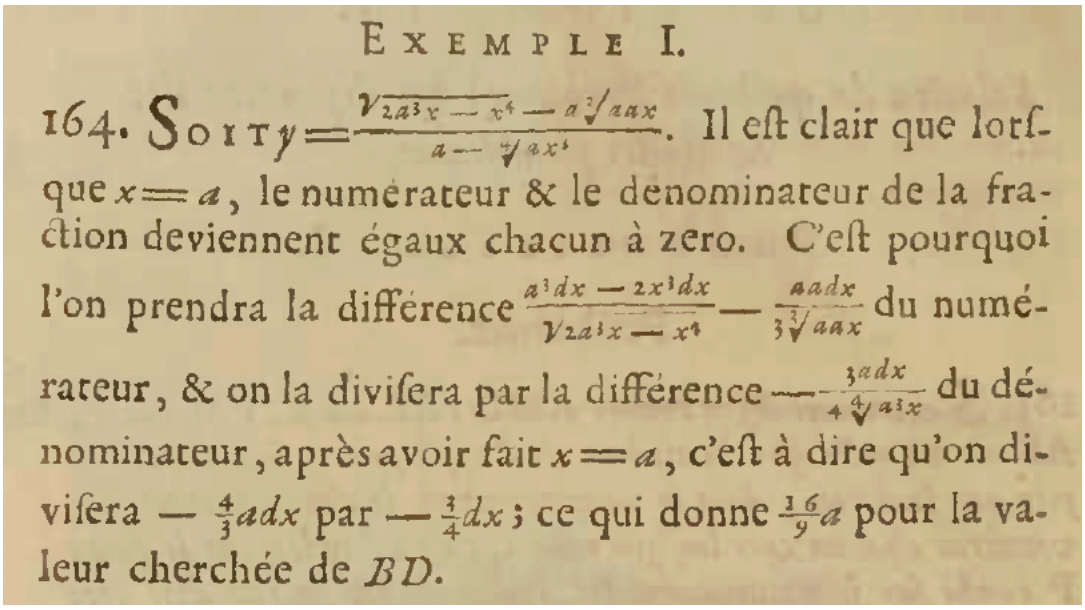
$$\begin{aligned}
\lim_{x\to 0} \ln y &= \lim_{x\to 0} \frac{\ln(\sin x) - \ln x}{x^2} \\
&\overset{\text{L'H}}{\underset{\downarrow}{=}} \lim_{x\to 0} \frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{2x} = \lim_{x\to 0} \frac{x\cos x - \sin x}{2x^2 \sin x} \\
&\overset{\text{L'H}}{\underset{\downarrow}{=}} \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{4x\sin x + 2x^2 \cos x} = \lim_{x\to 0} \frac{-\sin x}{4\sin x + 2x\cos x} \\
&= \lim_{x\to 0} \frac{-\frac{\sin x}{x}}{4\frac{\sin x}{x} + 2\cos x} = \frac{-1}{4+2} = -\frac{1}{6}
\end{aligned}$$

Here applications of L'Hôpital's Rule are indicated with L'H. Then

$$\lim_{x\to 0} \left(\frac{\sin x}{x} \right)^{1/x^2} = \lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{-1/6}$$

using the continuity of the exponential function. □

Remark: The first example of an application of L'Hôpital's Rule in Guillaume François Antoine Marquis de L'Hôpital's book *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* of 1696:



EXEMPLE I.

164. SOIT $y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$. Il eſt clair que lorſque $x = a$, le numerateur & le denominateur de la fraction deviennent égaux chacun à zero. C'eſt pourquoi l'on prendra la différence $\frac{a^3dx - 2x^3dx}{\sqrt{2a^3x - x^4}} - \frac{aadx}{3\sqrt[3]{aax}}$ du numérateur, & on la diviſera par la différence $-\frac{3adx}{4\sqrt[4]{a^3x}}$ du dénominateur, après avoir fait $x = a$, c'eſt à dire qu'on diviſera $-\frac{4}{3}adx$ par $-\frac{3}{4}dx$; ce qui donne $\frac{16}{9}a$ pour la valeur cherchée de BD.

There is a typo. Can you find it?

58. Evaluate $\lim\limits_{x\to 0} \dfrac{\cos(2x) - e^{-2x^2}}{\sin^4 x}$.

Solution:

$$
\begin{aligned}
\lim_{x\to 0} \frac{\cos(2x) - e^{-2x^2}}{\sin^4 x} &= \lim_{x\to 0} \left(\frac{\cos(2x) - e^{-2x^2}}{x^4} \cdot \frac{x^4}{\sin^4 x} \right) \\
&= \lim_{x\to 0} \frac{\cos(2x) - e^{-2x^2}}{x^4} \cdot \left(\lim_{x\to 0} \frac{x}{\sin x} \right)^4 \\
&= \lim_{x\to 0} \frac{\cos(2x) - e^{-2x^2}}{x^4} \\
&\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-2\sin(2x) + 4xe^{-2x^2}}{4x^3} \\
&\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-4\cos(2x) + 4e^{-2x^2} - 16x^2e^{-2x^2}}{12x^2} \\
&\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{8\sin(2x) - 48xe^{-2x^2} + 64x^3e^{-2x^2}}{24x} \\
&= \lim_{x\to 0} \left(\frac{\sin(2x)}{3x} - 2e^{-2x^2} + \frac{8}{3}x^2e^{-2x^2} \right) \\
&= \frac{2}{3} - 2 + 0 = -\frac{4}{3}
\end{aligned}
$$

□

Remark: It is easier to solve this problem using the Taylor series, which will be seen in Calculus

II:

$$\lim_{x\to 0}\frac{\cos(2x)-e^{-2x^2}}{\sin^4 x}$$

$$= \lim_{x\to 0}\frac{\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\cdots\right)-\left(1+(-2x^2)+\frac{(-2x^2)^2}{2!}+\frac{(-2x^2)^3}{3!}+\cdots\right)}{\left(x-\frac{x^3}{3!}+\cdots\right)^4}$$

$$= \lim_{x\to 0}\frac{-\frac{4}{3}x^4+\frac{56}{45}x^6+\cdots}{x^4-\frac{2}{3}x^6+\cdots}$$

$$= \lim_{x\to 0}\frac{-\frac{4}{3}+\frac{56}{45}x^2+\cdots}{1-\frac{2}{3}x^2+\cdots}$$

$$= -\frac{4}{3}$$

59. Find the value of the constant a for which the limit $\lim_{x\to 0}\frac{\sin(x+ax^3)-x}{x^5}$ exists and compute the limit for this value of a.

Solution: We have

$$\lim_{x\to 0}\frac{\sin(x+ax^3)-x}{x^5}$$

$$\overset{\text{L'H}}{\underset{\downarrow}{=}} \lim_{x\to 0}\frac{\cos(x+ax^3)(1+3ax^2)-1}{5x^4}$$

$$\overset{\text{L'H}}{\underset{\downarrow}{=}} \lim_{x\to 0}\frac{-\sin(x+ax^3)(1+3ax^2)^2+\cos(x+ax^3)(6ax)}{20x^3}$$

$$\overset{\text{L'H}}{\underset{\downarrow}{=}} \lim_{x\to 0}\frac{-\cos(x+ax^3)(1+3ax^2)^3-\sin(x+ax^3)\cdot 3(1+3ax^2)(6ax)+\cos(x+ax^3)(6a)}{60x^2}\,.$$

The numerator of the fraction inside this last limit goes to $-1+6a$ as $x\to 0$. Therefore the limit does not exist unless $a=1/6$. If $a=1/6$ then

$$\lim_{x\to 0}\frac{\sin(x+x^3/6)-x}{x^5}$$

$$= \lim_{x\to 0}\frac{-\cos(x+x^3/6)(1+x^2/2)^3-\sin(x+x^3/6)\cdot 3(1+x^2/2)x+\cos(x+x^3/6)}{60x^2}$$

is the sum of

$$\lim_{x\to 0}\frac{\cos(x+x^3/6)(1-(1+x^2/2)^3)}{60x^2}$$

$$= -\lim_{x\to 0}\frac{\cos(x+x^3/6)(3/2+3x^2/4+x^4/8)}{60} = -\frac{1}{40}$$

and

$$\lim_{x\to 0}\frac{-\sin(x+x^3/6)\cdot 3(1+x^2/2)x}{60x^2}$$

$$= -\lim_{x\to 0}\frac{\sin(x+x^3/6)}{x+x^3/6}\cdot\lim_{x\to 0}\frac{(1+x^2/2)(1+x^2/6)}{20} = -\frac{1}{20}\,.$$

Hence

$$\lim_{x\to 0} \frac{\sin(x + x^3/6) - x}{x^5} = -\frac{3}{40} .$$

$\square$

Remark: Once again there are shorter ways of doing this. If we use the Taylor series then

$$\begin{aligned}\sin(x + ax^3) &= (x + ax^3) - \frac{(x + ax^3)^3}{3!} + \frac{(x + ax^3)^5}{5!} - \cdots \\ &= x + \left(a - \frac{1}{6}\right) x^3 + \left(\frac{1}{120} - \frac{1}{2}\, a\right) x^5 + \cdots\end{aligned}$$

and it is immediate that the limit exists exactly when $a = 1/6$ and then its value is $-3/40$.

60. Let $b > a > 0$ be constants. Find the area of the surface generated by revolving the circle $(x - b)^2 + y^2 = a^2$ about the y-axis.

Solution: We have

$$\text{Surface Area} = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

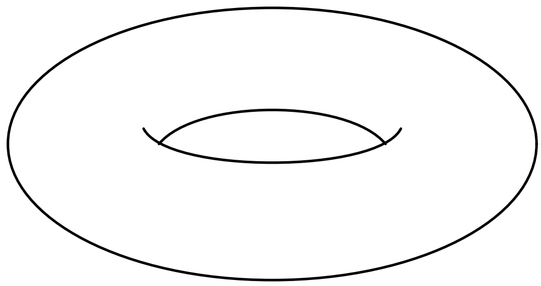
for a surface generated by revolving a curve about the y-axis. We have $x = b + \sqrt{a^2 - y^2}$ and $x = b - \sqrt{a^2 - y^2}$ for the right and left halves of the circle, respectively. Then

$$\frac{dx}{dy} = \mp \frac{y}{\sqrt{a^2 - y^2}} \implies \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \frac{a}{\sqrt{a^2 - y^2}} .$$

Hence

$$\begin{aligned}\text{Surface Area} = 2\pi \int_{-a}^{a} (b + \sqrt{a^2 - y^2})\, \frac{a}{\sqrt{a^2 - y^2}}\, dy& \\ + 2\pi \int_{-a}^{a} (b - \sqrt{a^2 - y^2})\, \frac{a}{\sqrt{a^2 - y^2}}\, dy& \\ = 4\pi ab \int_{-a}^{a} \frac{1}{\sqrt{a^2 - y^2}}\, dy& \\ = 4\pi ab \arcsin\left(\frac{y}{a}\right)\Big]_{-a}^{a}& \\ = 4\pi ab\,(\arcsin 1 - \arcsin(-1)) = 4\pi ab \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 4\pi^2 ab&\end{aligned}$$

$\square$



Remark: The surface generated by revolving a circle about a line (in the same plane) that does not intersect it is called a *torus*.

61. Find $y(1)$ if $\dfrac{dy}{dx} = xy^2$ and $y(0) = 1$.

Solution: We have

$$\frac{dy}{dx} = xy^2 \Longrightarrow \frac{dy}{y^2} = x\,dx \Longrightarrow \int \frac{dy}{y^2} = \int x\,dx \Longrightarrow -\frac{1}{y} = \frac{1}{2}x^2 + C\,.$$

Then $y(0) = 1 \Longrightarrow -1 = C$. Hence $y = \dfrac{2}{2 - x^2}$. This in turn gives $y(1) = 2$. □

Remark: What would your answer be if the question asked $y(2)$?

62. Nitrogen dioxide is a reddish-brown gas that contributes to air pollution, and also gives the smog its color. Under sunlight it decomposes producing other pollutants, one of which is ozone. As nitrogen dioxide decomposes, its density decreases at a rate proportional to the square of the density. Suppose that the density of nitrogen dioxide Q is $1/2$ grams per liter at time $t = 0$ minutes and $6/25$ grams per liter at time $t = 3$ minutes. Find Q when $t = 15$ minutes. (Assume that no new nitrogen dioxide is added to the environment.)

Solution: We have $dQ/dt = -kQ^2$ where k is a positive constant. Then $dQ/Q^2 = -k\,dt$ and integrating we obtain $-1/Q = -kt + C$ where C is a constant.

Letting $t = 0$ minutes we find $-1/(1/2) = -1/Q(0) = C$, hence $C = -2$ liters per gram. Now letting $t = 3$ min gives $-1/(6/25) = -1/Q(3) = -k \cdot 3 - 2$ and hence $k = 13/18$ liters per gram per minute. Finally we obtain $Q = 18/(13t + 36)$.

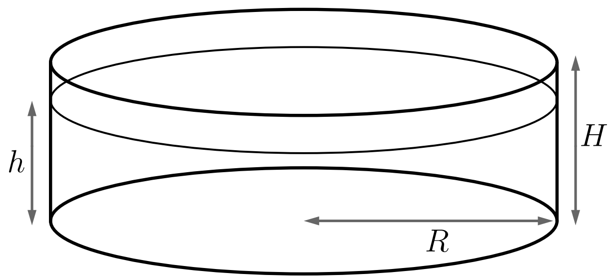
Using this we compute the density of nitrogen dioxide as $Q(15) = 6/77$ grams per liter after 15 minutes. □

63. A pool, like the one in front of the Faculty of Science Building A, loses water from its sides and its bottom due to seepage. For a pool with radius R and depth H in meters, the rate of this loss in m^3/hour is given by

$$\frac{dV}{dt} = -aR^2h - bRh^2$$

where V is the volume of the water in cubic meters, t is the time in hours, h is the depth of the water in meters, and a and b are constants independent of R, H and h.

Consider a pool with $H = 1$ m, $R = 6$ m and $a = b = \pi/500$ 1/hour.



a. Find the depth of the water h as a function of time t if $h = 1$ m when $t = 0$ hour.

b. The term aR^2h on the right side of the equation represents the rate of loss due to seepage through the bottom of the pool. Find the total volume of the water that seeps through the bottom while the initially full pool completely empties.

Solution: a. We have:

$$\begin{aligned}
V = \pi R^2 h \Longrightarrow \pi R^2 \dot{h} &= -\frac{\pi}{500} R^2 h - \frac{\pi}{500} R h^2 \\
\Longrightarrow 6^2 \cdot \dot{h} &= -\frac{1}{500} 6^2 \cdot h - \frac{1}{500} 6 \cdot h^2 \\
\Longrightarrow \frac{6\,dh}{h(h+6)} &= -\frac{dt}{500} \\
\Longrightarrow \left(\frac{1}{h} - \frac{1}{h+6}\right) dh &= -\frac{dt}{500} \\
\Longrightarrow \ln h - \ln(h+6) &= -\frac{t}{500} + C \\
\Longrightarrow \frac{h}{h+6} &= Ae^{-t/500}
\end{aligned}$$

Substituting $h = 1$ m when $t = 0$ hours in this last equation we find $A = 1/7$. Hence $h = 6/(7e^{t/500} - 1)$ for $t \geq 0$.

b. The volume of the water lost through the bottom of the pool is

$$\begin{aligned}
\text{Volume} &= \int_{t=0}^{t=\infty} a\, R^2\, h\, dh = \frac{\pi}{500} 6^2 \int_0^\infty \frac{6}{7e^{t/500} - 1}\, dt \\
&= 216\pi \int_0^\infty \frac{e^{-t/500}\, dt/500}{7 - e^{-t/500}} = 216\pi \int_6^7 \frac{du}{u} = 216\pi \ln(7/6)\, \text{m}^3
\end{aligned}$$

where we used the substitution $u = 7 - e^{-t/500}$, $du = e^{-t/500}\, dt/500$. □

64. Some students believe that Bilkent Math 101 exams get more difficult as time passes. This is in fact true. The difficulty level $H(t)$ of these exams satisfies the differential equation

$$\frac{dH}{dt} = H^a$$

with the initial condition $H(0) = 1$, where t is time measured from Fall 1986 in academic years and a is a constant whose value is a secret.

There is a quatrain in Nostradamus's *Les Propheties* which can be interpreted to be about Bilkent Math 101 exams.

a. According to one interpretation, the exams will become infinitely difficult in Fall 2021. Accepting this interpretation, find the difficulty level of the exams in Fall 2016.

b. According to another interpretation, the exams will be twice as difficult in Fall 2021 as they were in Fall 1986. Show that a must satisfy $-8 < a < -7$ if this is the case.

Solution: a. We have $H^{-a}\,dH = dt$. If $a = 1$, then integration gives $\ln|H| = t + C$ for some constant C, and the initial condition $H(0) = 1$ implies that $C = 0$. Hence $|H| = e^t$, and by continuity we have $H = e^t$. On the other hand, if $a \neq 1$, then by integration we obtain $H^{-a+1}/(-a+1) = t + C$ for some constant C. Using the condition that $H(0) = 1$ we get $C = 1/(-a+1)$ and hence $H = ((1-a)t+1)^{1/(1-a)}$.

We want $\lim_{t\to 35^-} H = \infty$. Among these solutions this is possible only if $(1-a)\cdot 35+1 = 0$ and $1/(1-a) < 0$. So we must have $a = 36/35$. Therefore $H = (1-t/35)^{-35}$. In particular, $H(30) = 7^{35}$, and the exams of Fall 2016 are 7^{35} times as difficult as those of Fall 1986.

b. This time we want $H(35) = 2$. Since $e^{35} > e > 2$, this is not possible if $a = 1$. Therefore $H = ((1-a)t+1)^{1/(1-a)}$ for some constant $a \neq 1$ and $((1-a)\cdot 35+1)^{1/(1-a)} = 2$. In other words, $36 - 35a = 2^{1-a}$ and $a \neq 1$.

Consider the function $g(a) = 36 - 35a - 2^{1-a}$. We have $f'(a) = -35 + \ln 2\cdot 2^{1-a}$, which is 0 only when $a = \log_2(2(\ln 2)/35)$. Therefore, by Rolle's Theorem, f can have at most one more zero beside $a = 1$. On the other hand, $f(-8) = -196 < 0$ and $f(-7) = 25 > 0$, and f must have a zero between -8 and -7 by the Intermediate Value Theorem. We conclude that $H(35) = 2$ is possible for exactly one value of a and this value lies in the interval $(-8,-7)$. □

65. Suppose that f is a function that has a continuous second derivative and that satisfies $f(0) = 4$, $f(1) = 3$, $f'(0) = 5$, $f'(1) = 7$, $f''(0) = 8$ and $f''(1) = 11$. Show that:

$$\int_0^1 f(x)f''(x)\,dx \leq 1$$

Solution: We first do an integration by parts with $u = f(x)$, $dv = f''(x)\,dx$, hence $du = f'(x)\,dx$, $v = f'(x)$, to obtain:

$$\int_0^1 f(x)f''(x)\,dx = \Big[f(x)f'(x)\Big]_0^1 - \int_0^1 (f'(x))^2\,dx$$

The first term on the right is equal to $f(1)f'(1) - f(0)f'(0) = 3\cdot 7 - 4\cdot 5 = 1$, and the second term is nonnegative as $(f'(x))^2 \geq 0$. The result follows. □

Remark: It can be shown that:

$$\int_0^1 f(x)f''(x)\,dx < 0$$

66. Evaluate the following integrals.

a. $\displaystyle\int \sin^3 x\,\sin 2x\,dx$

b. $\displaystyle\int_1^e \frac{\ln x}{\sqrt{x}}\,dx$

Solution: a. We use the identity $\sin 2x = 2\sin\cos x$ to obtain

$$\begin{aligned}
\int \sin^3 x\,\sin 2x\,dx &= \int \sin^3 x\,2\sin x\,\cos x\,dx \\
&= 2\int \sin^4 x\,\cos x\,dx \\
&= 2\int u^4\,du \\
&= \frac{2}{5}u^5 + C \\
&= \frac{2}{5}\sin^5 x + C
\end{aligned}$$

after the substitution $u = \sin x$, $du = \cos x\,dx$.

b. The substitution $u = \sqrt{x}$, $du = dx/(2\sqrt{x})$ gives:

$$\begin{aligned}
\int_1^e \frac{\ln x}{\sqrt{x}}\,dx &= 4\int_1^{\sqrt{e}} \ln u\,du \\
&= 4\Big[u\ln u - u\Big]_1^{\sqrt{e}} \\
&= 4\left(\sqrt{e}\ln\sqrt{e} - \sqrt{e} + 1\right) \\
&= 4 - 2\sqrt{e}
\end{aligned}$$

□

Remark: There are other ways of doing these. Here are some:

Solution: a. Use the double-angle formula $\sin^2 x = (1-\cos 2x)/2$ and then the substitution $u = 1-\cos 2x$, $du = 2\sin 2x\,dx$:

$$\begin{aligned}
\int \sin^3 x\,\sin 2x\,dx &= \int \left(\frac{1-\cos 2x}{2}\right)^3 \sin 2x\,dx \\
&= \frac{1}{4\sqrt{2}}\int u^{3/2}\,du \\
&= \frac{1}{10\sqrt{2}}u^{5/2} + C \\
&= \frac{1}{10\sqrt{2}}(1-\cos 2x)^{5/2} + C
\end{aligned}$$

Or you can run this by simply saying $u = \sin^2 x$, $du = 2\sin x\cos x\,dx = \sin 2x\,dx$ and:

$$\begin{aligned}
\int \sin^3 x\,\sin 2x\,dx &= \int u^{3/2}\,du \\
&= \frac{2}{5}u^{5/2} + C \\
&= \frac{2}{5}\sin^5 x + C
\end{aligned}$$

Or use the identity $\sin^3 x = (3\sin x - \sin 3x)/4$ and the trigonometric product to sum formulas:

$$\begin{aligned}
\int \sin^3 x \, \sin 2x \, dx &= \int \frac{1}{4} (3\sin x - \sin 3x) \, \sin 2x \, dx \\
&= \frac{1}{4} \int (3 \sin x \sin 2x - \sin 3x \sin 2x) \, dx \\
&= \frac{1}{8} \int \big(3(\cos x - \cos 3x) - (\cos x - \cos 5x)\big) \, dx \\
&= \frac{1}{8} \int (\cos 5x - 3\cos 3x + 2\cos x) \, dx \\
&= \frac{1}{40} \sin 5x - \frac{1}{8} \sin 3x + \frac{1}{4} \sin x + C
\end{aligned}$$

Or, if you are willing to go complex, use the identity $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$:

$$\begin{aligned}
\int \sin^3 x \, \sin 2x \, dx &= \int \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 \frac{e^{2ix} - e^{-2ix}}{2i} \, dx \\
&= \frac{1}{16} \int (e^{5ix} + e^{-5ix} - 3e^{3ix} - 3e^{-3ix} + 2e^{ix} + 2e^{-ix}) \, dx \\
&= \frac{1}{16} \left(\frac{e^{5ix} - e^{-5ix}}{5i} - \frac{e^{3ix} - e^{-3ix}}{i} + 2 \, \frac{e^{ix} - e^{-ix}}{i} \right) + C \\
&= \frac{1}{40} \sin 5x - \frac{1}{8} \sin 3x + \frac{1}{4} \sin x + C
\end{aligned}$$

b. Do integration by parts:

$$\begin{aligned}
\int_1^e \frac{\ln x}{\sqrt{x}} \, dx &= \int_1^e \ln x \, d(2\sqrt{x}) \\
&= \big[2\sqrt{x} \, \ln x\big]_1^e - \int_1^e 2\sqrt{x} \, \frac{dx}{x} \\
&= 2\sqrt{e} - 2 \int_1^e \frac{dx}{\sqrt{x}} \\
&= 2\sqrt{e} - 4\big[\sqrt{x}\big]_1^e \\
&= 2\sqrt{e} - 4\sqrt{e} + 4 \\
&= 4 - 2\sqrt{e}
\end{aligned}$$

Or do the other integration by parts first,

$$\begin{aligned}
\int_1^e \frac{\ln x}{\sqrt{x}} \, dx &= \int_1^e \frac{1}{\sqrt{x}} \, d(x \, \ln x - x) \\
&= \left[\frac{x \, \ln x - x}{\sqrt{x}}\right]_1^e + \frac{1}{2} \int_1^e \frac{x \, \ln x - x}{x^{3/2}} \, dx \\
&= 1 + \frac{1}{2} \int_1^e \frac{\ln x}{\sqrt{x}} \, dx - \big[\sqrt{x}\big]_1^e \\
&= 2 - \sqrt{e} + \frac{1}{2} \int_1^e \frac{\ln x}{\sqrt{x}} \, dx
\end{aligned}$$

and then from

$$\frac{1}{2} \int_1^e \frac{\ln x}{\sqrt{x}} \, dx = 2 - \sqrt{e}$$

solve for

$$\int_1^e \frac{\ln x}{\sqrt{x}}\, dx = 4 - 2\sqrt{e}\,.$$

Or first do the substitution $x = e^u$, $dx = e^u\, du$, and then do an integration by parts:

$$\begin{aligned}
\int_1^e \frac{\ln x}{\sqrt{x}}\, dx &= \int_0^1 \frac{u}{\sqrt{e^u}}\, e^u\, du \\
&= \int_0^1 u\, e^{u/2}\, du \\
&= \int_0^1 u\, d(2e^{u/2}) \\
&= \left[2ue^{u/2}\right]_0^1 - 2\int_0^1 e^{u/2}\, du \\
&= 2e^{1/2} - 4\left[e^{u/2}\right]_0^1 \\
&= 2e^{1/2} - 4e^{1/2} + 4 \\
&= 4 - 2\sqrt{e}
\end{aligned}$$

□

67. Evaluate the following integrals:

a. $\int e^{\sqrt{x}}\, dx$

b. $\int \sqrt{1 - x^2}\, dx$

c. $\int \frac{dx}{(x^2 + 1)^2}$

Solution: a. We first do a change of variable $x = t^2$, $dx = 2tdt$,

$$\int e^{\sqrt{x}}\, dx = 2\int te^t\, dt$$

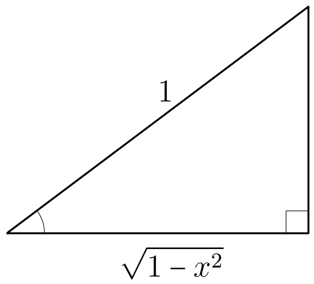
and then do an integration by parts, $u = t, dv = e^t\, dt \Longrightarrow du = dt, v = e^t$:

$$\begin{aligned}
&= 2te^t - 2\int e^t\, dt \\
&= 2te^t - 2e^t + C \\
&= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C
\end{aligned}$$

b. We use the trigonometric substitution $x = \sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = \cos\theta\, d\theta$

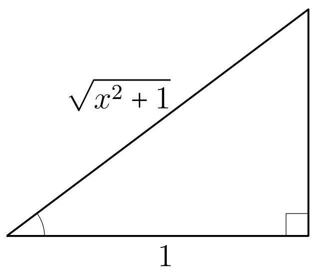
and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ as $\cos\theta \geq 0$ for $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$.

$$\begin{aligned}
\int \sqrt{1-x^2}\,dx &= \int \cos\theta \cdot \cos\theta\,d\theta \\
&= \int \cos^2\theta\,d\theta \\
&= \frac{1}{2} \int (1+\cos 2\theta)\,d\theta \\
&= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C \\
&= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\,\cos\theta + C \\
&= \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C
\end{aligned}$$



c. We use the trigonometric substitution $x = \tan\theta$, $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$ and $x^2+1 = \tan^2\theta + 1 = \sec^2\theta$.

$$\begin{aligned}
\int \frac{dx}{(x^2+1)^2} &= \int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta \\
&= \int \cos^2\theta\,d\theta \\
&= \frac{1}{2} \int (1+\cos 2\theta)\,d\theta \\
&= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C \\
&= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\,\cos\theta + C \\
&= \frac{1}{2}\arctan x + \frac{1}{2}\cdot\frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}} + C \\
&= \frac{1}{2}\arctan x + \frac{1}{2}\cdot\frac{x}{x^2+1} + C
\end{aligned}$$



□

Remark: Other methods can also be used. For instance, the integral in part **(b)** can be done using integration by parts.

68. Evaluate the following integrals:

a. $\displaystyle\int_{-1/2}^{1/2} \sqrt{\frac{1-x}{1+x}} \arcsin x \, dx$

b. $\displaystyle\int_{-1/2}^{1/2} \frac{dx}{x+\sqrt{1-x^2}}$

c. $\displaystyle\int_{0}^{\infty} \frac{dx}{1+e^x}$

Solution: **a.** We first re-write the integral as follows:

$$\int_{-1/2}^{1/2} \sqrt{\frac{1-x}{1+x}} \arcsin x \, dx = \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}} \arcsin x \, dx + \int_{-1/2}^{1/2} \frac{-x}{\sqrt{1-x^2}} \arcsin x \, dx$$

The first integral on the right vanishes as the integrand is odd and the integration interval is symmetric about the origin. We do integration by parts for the second integral with $u = \arcsin x$ and $dv = \dfrac{-x}{\sqrt{1-x^2}}\,dx$, and hence $du = \dfrac{1}{\sqrt{1-x^2}}$ and $v = \sqrt{1-x^2}$:

$$\int_{-1/2}^{1/2} \arcsin x \frac{-x}{\sqrt{1-x^2}}\,dx = \Big[\arcsin x \sqrt{1-x^2}\Big]_{-1/2}^{1/2} - \int_{-1/2}^{1/2} dx = \frac{\pi}{2\sqrt{3}} - 1$$

Therefore:

$$\int_{-1/2}^{1/2} \sqrt{\frac{1-x}{1+x}} \arcsin x \, dx = \frac{\pi}{2\sqrt{3}} - 1$$

b. We start by changing variables $t = \sin\theta$, $dt = \cos\theta\, d\theta$.

$$\begin{aligned}
\int_{-1/2}^{1/2} \frac{dx}{x+\sqrt{1-x^2}} &= \int_{-\pi/6}^{\pi/6} \frac{\cos\theta\, d\theta}{\sin\theta + \cos\theta} \\
&= \int_{-\pi/6}^{\pi/6} \frac{\cos\theta(\cos\theta - \sin\theta)}{\cos^2\theta - \sin^2\theta}\, d\theta \\
&= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \frac{1+\cos 2\theta - \sin 2\theta}{\cos 2\theta}\, d\theta \\
&= \frac{1}{2}\int_{-\pi/6}^{\pi/6} (\sec 2\theta + 1 - \tan 2\theta)\, d\theta \\
&= \frac{1}{2}\left[\frac{1}{2}\ln|\tan 2\theta + \sec 2\theta| + \theta - \frac{1}{2}\ln|\sec 2\theta|\right]_{-\pi/6}^{\pi/6} \\
&= \frac{1}{2}\ln(\sqrt{3}+2) + \frac{\pi}{6}
\end{aligned}$$

c. Let $u = e^{-x} + 1$, $du = -e^{-x}\,dx$. Then

$$\int \frac{dx}{1+e^x} = \int \frac{e^{-x}\,dx}{e^{-x}+1} = -\int \frac{du}{u} = -\ln|u| + C = -\ln(1+e^{-x}) + C$$

and from this

$$\int_0^\infty \frac{dx}{1+e^x} = \lim_{c\to\infty} \int_0^c \frac{dx}{1+e^x} = -\lim_{c\to\infty} [\ln(1+e^{-x})]_0^c = -\lim_{c\to\infty} (\ln(1+e^{-c}) - \ln 2) = \ln 2$$

follows. $\square$

Remark: Here is another way of doing part **(c)**. Let $u = e^x$, $du = e^x\,dx$. Then

$$\begin{aligned}\int \frac{dx}{1+e^x} &= \int \frac{e^x\,dx}{e^x+(e^x)^2} \\ &= \int \frac{du}{u+u^2} \\ &= \int \left(\frac{1}{u} - \frac{1}{1+u}\right) du \\ &= \ln|u| - \ln|1+u| + C \\ &= x - \ln(1+e^x) + C\end{aligned}$$

and from this

$$\begin{aligned}\int_0^\infty \frac{dx}{1+e^x} &= \lim_{c\to\infty} \int_0^c \frac{dx}{1+e^x} \\ &= \lim_{c\to\infty} [x - \ln(1+e^x)]_0^c \\ &= \lim_{c\to\infty} (c - \ln(1+e^c) + \ln 2) \\ &= \lim_{c\to\infty} (-\ln(e^{-c}) - \ln(1+e^c) + \ln 2) \\ &= \lim_{c\to\infty} (-\ln(e^{-c}+1) + \ln 2) \\ &= \ln 2\end{aligned}$$

follows.

69. Evaluate the improper integral

$$\int_0^\infty \frac{dx}{(ax+1)(x^2+1)}$$

where a is a positive constant.

Solution: We have the partial fraction decomposition

$$\frac{1}{(ax+1)(x^2+1)} = \frac{1}{a^2+1}\left(\frac{a^2}{ax+1} - \frac{ax-1}{x^2+1}\right).$$

Hence:

$$\begin{aligned}\int \frac{dx}{(ax+1)(x^2+1)} &= \frac{1}{a^2+1}\left(\int \frac{a^2}{ax+1}\,dx - \int \frac{ax-1}{x^2+1}\,dx\right) \\ &= \frac{1}{a^2+1}\left(a\ln|ax+1| - \frac{a}{2}\ln(x^2+1) + \arctan(x)\right) + C\,.\end{aligned}$$

Therefore:

$$\begin{aligned}
\int_0^\infty \frac{dx}{(ax+1)(x^2+1)} &= \lim_{c\to\infty} \int_0^c \frac{dx}{(ax+1)(x^2+1)} \\
&= \frac{1}{a^2+1} \lim_{c\to\infty} \left[a\ln|ax+1| - \frac{a}{2}\ln(x^2+1) + \arctan(x) \right]_0^c \\
&= \frac{1}{a^2+1} \lim_{c\to\infty} \Bigg(a\ln(ac+1) - \frac{a}{2}\ln(c^2+1) + \arctan c \\
&\qquad\qquad - a\ln 1 + \frac{a}{2}\ln 1 - \arctan 0 \Bigg) \\
&= \frac{a}{a^2+1} \lim_{c\to\infty} \ln\left(\frac{ac+1}{\sqrt{c^2+1}}\right) + \frac{1}{a^2+1}\lim_{c\to\infty} \arctan c \\
&= \frac{1}{a^2+1}\left(a\ln a + \frac{\pi}{2}\right)
\end{aligned}$$

as $\lim_{c\to\infty} \dfrac{ac+1}{\sqrt{c^2+1}} = a$ and $\lim_{c\to\infty} \arctan c = \dfrac{\pi}{2}$. □

70. The curve $y = 1/x$, $x \geq 1$, is revolved about the x-axis to generate a surface S and the region between the curve and the x-axis for $x \geq 1$ is revolved about the x-axis to generate a solid D.

a. Show that D has finite volume.

b. Show that S has infinite area.

Solution: a. Using the disk method we obtain

$$\text{Volume} = \pi \int_1^\infty R(x)^2\,dx = \pi \int_1^\infty \frac{dx}{x^2}\,.$$

Since $p = 2 > 1$, this integral converges.

b. The surface area formula gives

$$\text{Surface Area} = 2\pi \int_1^\infty y\sqrt{1+(y')^2}\,dx = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx\,.$$

We have $\dfrac{1}{x}\sqrt{1+\dfrac{1}{x^4}} \geq \dfrac{1}{x} \geq 0$ for $x \geq 1$, and $\displaystyle\int_1^\infty \frac{dx}{x} = \infty$. The surface area is infinite by the Direct Comparison Test. □

Remark: We want to get the surface S painted for a reasonable, finite price. We offer the job to *Painter1* and *Painter2*.

Painter1 says: "It cannot be done. S has infinite area, it cannot be painted with finite amount of paint."

Painter2 says: "It can be done. D has finite volume. We can fill the inside of S with volume(D) cubic units of paint and let the excess paint run out."

Whom should we believe?

71. Let n be a nonnegative integer. Show that $\displaystyle\int_0^\infty t^n e^{-t}\,dt = n!$.

Solution: We use induction on n.

- Let $n = 0$. Then $\displaystyle\int_0^\infty e^{-t}\,dt = 1 = 0!$.
- Let $n > 0$ and assume that $\displaystyle\int_0^\infty t^{n-1}e^{-t}\,dt = (n-1)!$. Integration by parts gives

$$\begin{aligned}\int_0^\infty t^n e^{-t}\,dt &= \lim_{c\to\infty}\int_0^c t^n e^{-t}\,dt\\ &= \lim_{c\to\infty}\left([-t^n e^{-t}]_0^c + n\int_0^c t^{n-1}e^{-t}\,dt\right)\\ &= \lim_{c\to\infty}[-t^n e^{-t}]_0^c + n\int_0^\infty t^{n-1}e^{-t}\,dt\\ &= n\int_0^\infty t^{n-1}e^{-t}\,dt\\ &= n\cdot(n-1)!\\ &= n!\end{aligned}$$

where $\lim_{c\to\infty} c^n e^{-c} = 0$ can be seen after n applications of L'Hôpital's Rule. □

Remark: The *Gamma function* is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

for $x > 0$. It can be shown that the improper integral on the right converges if and only if $x > 0$. Note that 0 is also a "bad point" besides ∞ for $0 < x < 1$.

A calculation similar to the one in the solution above shows that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$. This relation $\Gamma(x) = \Gamma(x+1)/x$ can be used repeatedly to define the Gamma function for all real numbers which are not nonpositive integers.

Since $\Gamma(n+1) = n!$ for all nonnegative integers n, we can use the Gamma function to define the factorials of all real numbers which are not negative integers by $x! = \Gamma(x+1)$. In particular,

$$\left(-\frac{1}{2}\right)! = \Gamma(1/2) = \int_0^\infty t^{-1/2}e^{-t}\,dt = 2\int_0^\infty e^{-u^2}\,du = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}\,.$$

The volume of an n-dimensional ball with radius r is $\pi^{n/2}r^n/(n/2)!$. Check this formula for $n = 1$ (the interval $[-r, r]$), $n = 2$ (the disk with radius r), and $n = 3$ (the sphere with radius r). The case $n = 4$ will be seen in Calculus II.

72. Show that $\displaystyle\int_0^\infty \frac{\ln x}{x^2+1}\,dx = 0$.

Solution: First of all, by definition,

$$\int_0^\infty \frac{\ln x}{x^2+1}\,dx = \int_0^1 \frac{\ln x}{x^2+1}\,dx + \int_1^\infty \frac{\ln x}{x^2+1}\,dx$$

and the improper integral on the left converges if and only if both basic improper integrals on the right converge.

Consider $\displaystyle\int_1^\infty \frac{\ln x}{x^2+1}\,dx$. Note that $0 \le \dfrac{\ln x}{x^2+1} \le \dfrac{\ln x}{x^2}$ for $x \ge 1$. On the other hand,

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^2}\,dx &= \int_0^\infty te^{-t}\,dt \\
&= \lim_{c\to\infty}\int_0^c te^{-t}\,dt \\
&= \lim_{c\to\infty}\int_0^c t\,d(-e^{-t}) \\
&= \lim_{c\to\infty}\left(\left[-te^{-t}\right]_0^c + \int_0^c e^{-t}\,dt\right) \\
&= \lim_{c\to\infty}\left(-ce^{-c} - \left[e^{-t}\right]_0^c\right) \\
&= \lim_{c\to\infty}\left(-ce^{-c} - e^{-c} + 1\right) \\
&= 1
\end{aligned}$$

where we used the substitution $x = e^t$, $dx = e^t\,dt$, followed by the definition of the improper integral, then an integration by parts and finally the limit

$$\lim_{c\to\infty} ce^{-c} = \lim_{c\to\infty}\frac{c}{e^c} \overset{\text{L'H}}{=} \lim_{c\to\infty}\frac{1}{e^c} = 0\,.$$

Therefore by the Direct Comparison Test the improper integral $\displaystyle\int_1^\infty \frac{\ln x}{x^2+1}\,dx$ converges.

Now the convergence of $\displaystyle\int_0^1 \frac{\ln x}{x^2+1}\,dx$ follows as

$$\int_0^1 \frac{\ln x}{x^2+1}\,dx = \int_\infty^1 \frac{\ln(1/u)}{(1/u)^2+1}\cdot\frac{-du}{u^2} = -\int_1^\infty \frac{\ln u}{1+u^2}\,du$$

where we used the change of variable $x = 1/u$, $dx = -du/u^2$, and this also gives:

$$\begin{aligned}
\int_0^\infty \frac{\ln x}{x^2+1}\,dx &= \int_0^1 \frac{\ln x}{x^2+1}\,dx + \int_1^\infty \frac{\ln x}{x^2+1}\,dx \\
&= -\int_1^\infty \frac{\ln x}{x^2+1}\,dx + \int_1^\infty \frac{\ln x}{x^2+1}\,dx = 0
\end{aligned}$$

□

Remark: Instead of computing $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$, one can show its convergence using the comparison $0 \le \dfrac{\ln x}{x^2} \le \dfrac{1}{x^{3/2}}$ for $x \ge 1$ and the fact that $\displaystyle\int_1^\infty \frac{dx}{x^{3/2}}$ is convergent as $p = 3/2 > 1$.

Here the fact that $\dfrac{\ln x}{x^2} \le \dfrac{1}{x^{3/2}}$ for $x \ge 1$ can be seen as follows: Consider $f(x) = \sqrt{x} - \ln x$ on $[1, \infty)$. Then $f'(x) = 1/(2\sqrt{x}) - 1/x = (\sqrt{x} - 2)/(2x)$ and $x = 4$ is the only critical point. Since $f'(x) < 0$ for $x < 4$ and $f'(x) > 0$ for $x > 4$, $f(4) = \sqrt{4} - \ln 4 = 2 - 2\ln 2 = 2(1 - \ln 2) > 0$ must be the absolute minimum value of f on $[1, \infty)$, and we are done.

73.* Determine whether the improper integral $\displaystyle\int_0^\infty \frac{dx}{e^x - e^{-x}}$ converges or diverges.

Solution: By definition,

$$\int_0^\infty \frac{dx}{e^x - e^{-x}} = \int_0^1 \frac{1}{e^x - e^{-x}} + \int_1^\infty \frac{dx}{e^x - e^{-x}}$$

and the given integral converges if and only if both of the integrals on the right hand side converge.

Let us consider $\displaystyle\int_0^1 \frac{dx}{e^x - e^{-x}}$ first. Since we have the linearization

$$e^x - e^{-x} \approx (1 + x) - (1 - x) = 2x\,,$$

centered at $x = 0$, we expect $1/(e^x - e^{-x})$ to behave like $1/(2x)$ near the "bad point" 0, and therefore this integral to diverge.

In fact, we have

$$L = \lim_{x\to 0^+} \frac{1/(e^x - e^{-x})}{1/x} = \lim_{x\to 0^+} \frac{x}{e^x - e^{-x}} \overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{1}{e^x + e^{-x}} = \frac{1}{2}\,.$$

Since $0 < L < \infty$ and $\displaystyle\int_0^1 \frac{dx}{x}$ is divergent (because $p = 1 \ge 1$), we conclude that $\displaystyle\int_0^1 \frac{dx}{e^x - e^{-x}}$ diverges by the Limit Comparison Test. Therefore $\displaystyle\int_0^\infty \frac{dx}{e^x - e^{-x}}$ diverges too. □

Remark: The other improper integral on the right hand side converges. We have

$$L = \lim_{x\to\infty} \frac{1/(e^x - e^{-x})}{e^{-x}} = \lim_{x\to\infty} \frac{1}{1 - e^{-2x}} = 1\,.$$

Since $0 < L < \infty$ and

$$\int_1^\infty e^{-x}\,dx = \lim_{c\to\infty} \int_1^c e^{-x}\,dx = \lim_{c\to\infty} [-e^{-x}]_1^c = \lim_{c\to\infty} (-e^{-c} + e^{-1}) = e^{-1} < \infty\,,$$

we conclude that $\displaystyle\int_1^\infty \frac{dx}{e^x - e^{-x}}$ converges by the Limit Comparison Test.

74.* Determine whether the improper integral $\displaystyle\int_0^\infty \frac{1 - e^{-1/x}}{\sqrt{x}}\,dx$ converges or diverges.

*Examples marked red are not part of the Fall 2016 Syllabus.
*Examples marked red are not part of the Fall 2016 Syllabus.

Solution: By definition,

$$\int_0^\infty \frac{1-e^{-1/x}}{\sqrt{x}}\,dx = \int_0^1 \frac{1-e^{-1/x}}{\sqrt{x}}\,dx + \int_1^\infty \frac{1-e^{-1/x}}{\sqrt{x}}\,dx$$

and the given integral converges if and only if both integrals on the right hand side converge.

On one hand, since

$$L = \lim_{x\to 0^+} \frac{\dfrac{1-e^{-1/x}}{\sqrt{x}}}{\dfrac{1}{\sqrt{x}}} = \lim_{x\to 0^+} (1-e^{-1/x}) = 1$$

is a positive real number, and $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ converges as $p = 1/2 < 1$; we conclude that $\displaystyle\int_0^1 \frac{1-e^{-1/x}}{\sqrt{x}}\,dx$ converges by the Limit Comparison Test.

On the other hand, since

$$L = \lim_{x\to\infty} \frac{\dfrac{1-e^{-1/x}}{\sqrt{x}}}{\dfrac{1}{x^{3/2}}} = \lim_{x\to\infty} \frac{1-e^{-1/x}}{1/x} = \lim_{t\to 0^+} \frac{1-e^{-t}}{t} \overset{\text{L'H}}{=} \lim_{t\to 0^+} \frac{e^{-t}}{1} = 1$$

is a positive real number, and $\displaystyle\int_1^\infty \frac{dx}{x^{3/2}}$ converges as $p = 3/2 > 1$; we conclude that $\displaystyle\int_1^\infty \frac{1-e^{-1/x}}{\sqrt{x}}\,dx$ also converges by the Limit Comparison Test.

Hence $\displaystyle\int_0^\infty \frac{1-e^{-1/x}}{\sqrt{x}}\,dx$ converges. $\square$

Part 2: Multi-variable Functions

1. Consider the point $P(3,-5,1)$ and the line $L: x = 2t-1\,,\ y = -t+2\,,\ z = -2t\,;\ -\infty < t < \infty\,.$

a. Find the equation of the plane passing through P perpendicular to L.

b. Find the equation of the plane passing through P and containing L.

Solution: a. The line L is parallel to the vector $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. Therefore we can take the normal vector of the plane to be $\mathbf{n} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. Then the equation of the plane is $2\cdot(x-3) + (-1)\cdot(y-(-5)) + (-2)\cdot(z-1) = 0$, or $2x - y - 2z = 9$.

b. A normal $\mathbf{n}$ to the plane containing the line L and the point P will be perpendicular to $\mathbf{v}$, and it will also be perpendicular to $\overrightarrow{PQ}$ where Q is any point on the line. We can take $Q(-1,2,0)$, the point corresponding to $t = 0$, and then $\overrightarrow{PQ} = -4\mathbf{i} + 7\mathbf{j} - \mathbf{k}$. Now we can take the normal vector $\mathbf{v}$ to be

$$\mathbf{n} = \mathbf{v} \times \overrightarrow{PQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ -4 & 7 & -1 \end{vmatrix} = 15\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}\,,$$

or in fact $\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Then the equation of the plane is

$$3\cdot(x-3) + 2\cdot(y-(-5)) + 2\cdot(z-1) = 0\,,$$

or $3x + 2y + 2z = 1$. □

2. Consider the plane $\mathscr{P}: 3x - 4y + z = 10$, and the points $P(2,3,-1)$ and $Q(1,2,2)$.

a. Find the equation of the line passing through P perpendicular to $\mathscr{P}$.

b. Find the equation of the plane passing through P and Q perpendicular to $\mathscr{P}$.

Solution: a. Since $\mathbf{n} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ is normal to the plane, it will be parallel to any line perpendicular to the plane. Hence we can take $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, and the equation of the line is $x = 3t + 2$, $y = -4t + 3$, $z = t - 1$; $-\infty < t < \infty$.

b. A plane perpendicular to $\mathscr{P}$ will have a normal $\mathbf{n}'$ perpendicular to the normal $\mathbf{n} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ of $\mathscr{P}$. Also a plane containing the points P and Q will have a normal $\mathbf{n}'$ perpendicular to $\overrightarrow{PQ} = -\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Therefore we can take $\mathbf{n}'$ to be

$$\mathbf{n}' = \mathbf{n} \times \overrightarrow{PQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 1 \\ -1 & -1 & 3 \end{vmatrix} = -11\mathbf{i} - 10\mathbf{j} - 7\mathbf{k}\,,$$

or rather $\mathbf{n}' = 11\mathbf{i} + 10\mathbf{j} + 7\mathbf{k}$. This gives the equation of the plane as

$$11 \cdot (x-2) + 10 \cdot (y-3) + 7 \cdot (z-(-1)) = 0\,,$$

or $11x + 10y + 7z = 45$. □

3. Find a parametric equation of the line L that *intersects* both of the lines

$$L_1: \quad x = 2t-1, \qquad y = -t+2, \qquad z = 3t+1$$

and

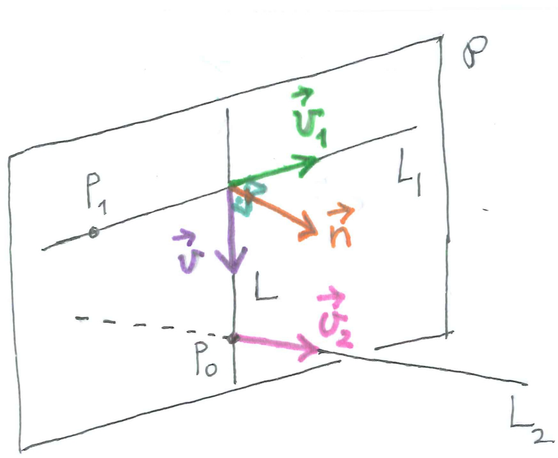
$$L_2: \quad x = s+5, \qquad y = 2s+3, \qquad z = -s$$

perpendicularly.

Solution: $\mathbf{v}_1 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v}_2 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ are the velocity vectors of the lines L_1 and L_2, respectively. Hence,

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 2 & -1 \end{vmatrix} = -5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$$

is a velocity vector for L. So we may take $\mathbf{v} = \mathbf{i} - \mathbf{j} - \mathbf{k}$.



Then

$$\mathbf{n} = \mathbf{v} \times \mathbf{v}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & -1 & 3 \end{vmatrix} = -4\mathbf{i} - 5\mathbf{j} + \mathbf{k}$$

is normal to the plane $\mathscr{P}$ containing the lines L and L_1. As $P_1(-1,2,1)$ is in $\mathscr{P}$, an equation of $\mathscr{P}$ is

$$-4 \cdot (x-(-1)) + (-5) \cdot (y-2) + 1 \cdot (z-1) = 0\,,$$

or $4x + 5y - z = 5$. At the point P_0 of intersection of $\mathscr{P}$ and L_2, s satisfies:

$$4 \cdot (s+5) + 5 \cdot (2s+3) - (-s) = 5$$

Hence $s = -2$. Substituting this back in the equations of L_2 gives $P_0(3,-1,2)$. Therefore an equation of L is:

$$x = t+3\,, \qquad y = -t-1\,, \qquad z = -t+2\,; \qquad (-\infty < t < \infty)\,.$$

□

4. Let c be a constant. Show the angle between the position and the velocity vectors along the curve $\mathbf{r} = e^{ct}\cos t\,\mathbf{i} + e^{ct}\sin t\,\mathbf{j}$, $-\infty < t < \infty$, is constant.

Solution: We have

$$\mathbf{r} = e^{ct}\cos t\,\mathbf{i} + e^{ct}\sin t\,\mathbf{j}$$

and

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (c\,e^{ct}\cos t - e^{ct}\sin t)\mathbf{i} + (c\,e^{ct}\sin t + e^{ct}\cos t)\mathbf{j}\,.$$

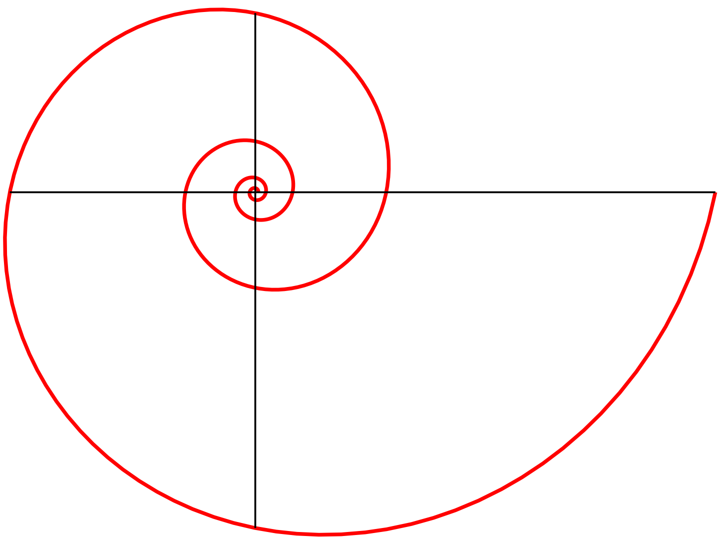
Then

$$\begin{aligned}
|\mathbf{r}| &= ((e^{ct}\cos t)^2 + (e^{ct}\sin t)^2)^{1/2} = e^{ct}\,,\\
|\mathbf{v}| &= ((c\,e^{ct}\cos t - e^{ct}\sin t)^2 + (c\,e^{ct}\sin t + e^{ct}\cos t)^2)^{1/2} = \sqrt{c^2+1}\,e^{ct}\,,\\
\mathbf{r}\cdot\mathbf{v} &= e^{ct}\cos t\cdot(c\,e^{ct}\cos t - e^{ct}\sin t) + e^{ct}\sin t\cdot(c\,e^{ct}\sin t + e^{ct}\cos t) = c\,e^{2t}\,.
\end{aligned}$$

Therefore, if θ is the angle between $\mathbf{r}$ and $\mathbf{v}$, we have

$$\cos\theta = \frac{\mathbf{r}\cdot\mathbf{v}}{|\mathbf{r}|\,|\mathbf{v}|} = \frac{c\,e^{2t}}{e^{ct}\cdot\sqrt{c^2+1}\,e^{ct}} = \frac{c}{\sqrt{c^2+1}}\,,$$

and we conclude that θ is constant. $\square$



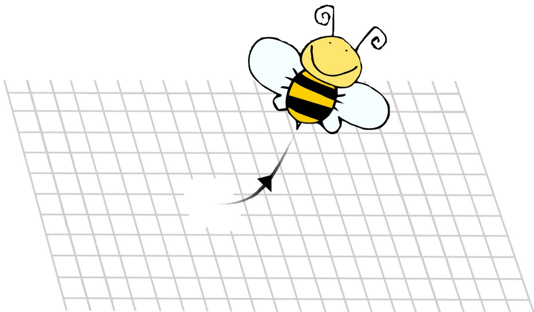
5. In the xyz-space where a flyscreen lies along the plane with the equation

$$2x + y - 2z = 1\ ,$$

the trajectory of a bee as a function of time t is given by

$$\mathbf{r} = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$$

for $-\infty < t < \infty$.



a. Find all times t when the bee is flying parallel to the screen.

b. Find all times t when the bee is flying perpendicular to the screen.

c. There are holes in the screen through which the bee passes. Find the coordinates of all of these holes.

Solution: a. The normal vector of the plane along which the screen lies is $\mathbf{n} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, and the velocity vector of the bee is $\mathbf{v} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$. The bee is flying parallel to the screen whenever these two vectors are perpendicular to each other; in other words, whenever $\mathbf{n} \cdot \mathbf{v} = 0$. As $\mathbf{n} \cdot \mathbf{v} = 2 + 2t - 6t^2$, we want $3t^2 - t - 1 = 0$. Hence $t = (1 \pm \sqrt{13})/6$ are the times when the bee is flying parallel to the screen.

b. The bee is flying perpendicular to the screen whenever $\mathbf{n}$ and $\mathbf{v}$ are parallel to each other; in other words, whenever $1/2 = 2t/1 = 3t^2/(-2)$. This implies $t^2 = -1/3$ which is not possible. Therefore there is no moment when the bee is flying perpendicular to the screen.

c. Substituting the coordinates $x = t$, $y = t^2$, $z = t^3$ of the position of the bee into the equation $2x + y - 2z = 1$ of the plane we obtain $2t^3 - t^2 - 2t + 1 = 0$. As $2t^3 - t^2 - 2t + 1 = (2t - 1)(t - 1)(t + 1)$, the fly is in the plane of the screen when $t = 1/2$, $t = 1$, $t = -1$. These times correspond to the points $(x, y, z) = (1/2, 1/4, 1/8)$, $(1, 1, 1)$ and $(-1, 1, -1)$. $\square$

6. a. Show that $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^6+y^2} = 0$.

b. Show that $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^6+y^2}$ does not exist.

Solution: a. We have $0 \le y^2 \le x^6 + y^2$ for all (x,y). Hence

$$0 \le \left|\frac{xy^2}{x^6+y^2}\right| = |x| \cdot \frac{y^2}{x^6+y^2} \le |x| \cdot 1 = |x|$$

for all $(x,y) \ne (0,0)$. Since $\lim_{(x,y)\to(0,0)} |x| = 0$, the Sandwich Theorem gives

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^6+y^2} = 0 .$$

b. The limit along the x-axis is

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the } x\text{-axis}}} \frac{xy}{x^6+y^2} = \lim_{x\to 0} \frac{x\cdot 0}{x^6+0^2} = \lim_{x\to 0} 0 = 0 ,$$

whereas the limit along the $y = x$ line is

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the line } y=x}} \frac{xy}{x^6+y^2} = \lim_{x\to 0} \frac{x\cdot x}{x^6+x^2} = \lim_{x\to 0} \frac{1}{x^4+1} = 1 .$$

Since these two limits are different, the two-variable limit $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^6+y^2}$ does not exist by the Two-Path Test. □

Remark: *Sertöz Theorem* tells exactly when such limits exist:

Let a and b be nonnegative real numbers, and c and d be positive real numbers.

- If $\dfrac{a}{c} + \dfrac{b}{d} > 1$, then $\lim_{(x,y)\to(0,0)} \dfrac{|x|^a|y|^b}{|x|^c+|y|^d} = 0$.
- If $\dfrac{a}{c} + \dfrac{b}{d} \le 1$, then $\lim_{(x,y)\to(0,0)} \dfrac{|x|^a|y|^b}{|x|^c+|y|^d}$ does not exist.

Let a and b be nonnegative integers, and c and d be positive even integers.

- If $\dfrac{a}{c} + \dfrac{b}{d} > 1$, then $\lim_{(x,y)\to(0,0)} \dfrac{x^a y^b}{x^c+y^d} = 0$.
- If $\dfrac{a}{c} + \dfrac{b}{d} \le 1$, then $\lim_{(x,y)\to(0,0)} \dfrac{x^a y^b}{x^c+y^d}$ does not exist.

A proof of the first version can be found in `http://www.fen.bilkent.edu.tr/~otekman/calc2/sertoztheorem.pdf`. A proof of the n-variable case of the second version can be found in `http://sertoz.bilkent.edu.tr/depo/limit.pdf`.

7. Determine all values of the constant $\alpha > 0$ for which the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{|x|^3 + |y|^\alpha}$$

exists.

Solution: We observe that

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the curve } x=y^3}} \frac{x^2y^3}{|x|^3+|y|^\alpha} = \lim_{y\to 0} \frac{(y^3)^2y^3}{|y^3|^3+|y|^\alpha} = \lim_{y\to 0}\frac{y}{|y|}\cdot \lim_{y\to 0}\frac{1}{1+|y|^{\alpha-9}} \,.$$

$\lim\limits_{y\to 0}\dfrac{1}{1+|y|^{\alpha-9}}$ is 1 if $\alpha > 9$ and 1/2 if $\alpha = 9$. On the other hand $\lim\limits_{y\to 0}\dfrac{y}{|y|}$ does not exist. Therefore the limit of $\dfrac{x^2y^3}{|x|^3+|y|^\alpha}$ along the curve $x = y^3$ does not exist, and consequently the two-variable limit $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y^3}{|x|^3+|y|^\alpha}$ does not exist either for $\alpha \geq 9$.

Now we will show that the limit is 0 if $\alpha < 9$. We have

$$0 \leq \left|\frac{x^2y^3}{|x|^3+|y|^\alpha}\right| = \frac{(|x|/|y|^{\alpha/3})^2}{(|x|/|y|^{\alpha/3})^3+1}\cdot|y|^{3-\alpha/3} \leq |y|^{3-\alpha/3}$$

for all $(x,y) \neq (0,0)$. Here we used the fact that if $t \geq 1$, then $t^2 \leq t^3$ and hence $\dfrac{t^2}{t^3+1} \leq 1$, and if $0 < t < 1$, then $t^2 < 1$ gives $\dfrac{t^2}{t^3+1} \leq 1$. Since $\lim\limits_{(x,y)\to(0,0)}|y|^{3-\alpha/3} = 0$ for $\alpha < 9$, $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y^3}{|x|^3+|y|^\alpha} = 0$ follows by the Sandwich Theorem. □

8. Determine all values of the positive constant k for which the limit

$$\lim_{(x,y)\to(0,0)}\frac{|x|}{(x^2+y^2)^k}$$

exists.

Solution: The limit exists for $k < \dfrac{1}{2}$, and does not exist for $k \geq \dfrac{1}{2}$.

Suppose $k < \dfrac{1}{2}$. We have $0 \leq x^2 \leq x^2+y^2$ for all (x,y), and hence

$$0 \leq \frac{|x|}{(x^2+y^2)^k} \leq |x|^{1-2k}\left(\frac{x^2}{x^2+y^2}\right)^k \leq |x|^{1-2k}$$

for all $(x,y) \neq (0,0)$. Since $1-2k>0$, we have $|x|^{1-2k} \to 0$ as $(x,y) \to (0,0)$. Therefore

$$\lim_{(x,y)\to(0,0)} \frac{|x|}{(x^2+y^2)^k} = 0$$

by the Sandwich Theorem.

Suppose $k = \dfrac{1}{2}$. Then

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the } x\text{-axis}}} \frac{|x|}{(x^2+y^2)^{1/2}} = \lim_{x\to0} \frac{|x|}{(x^2+0^2)^{1/2}} = \lim_{x\to0} 1 = 1\,,$$

and

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the } y\text{-axis}}} \frac{|x|}{(x^2+y^2)^{1/2}} = \lim_{y\to0} \frac{|0|}{(0^2+y^2)^{1/2}} = \lim_{x\to0} 0 = 0\,.$$

Since these limits are different, the two-variable limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x|}{(x^2+y^2)^{1/2}}$ does not exist by the Two-Path Test.

Suppose $k > 1/2$. Then

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the } x\text{-axis}}} \frac{|x|}{(x^2+y^2)^{k}} = \lim_{x\to0} \frac{|x|}{(x^2+0^2)^{k}} = \lim_{x\to0} |x|^{1-2k} = \infty$$

does not exist. Therefore the two-variable limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x|}{(x^2+y^2)^{k}}$ does not exist either. $\square$

9. Let

$$f(x,y) = \begin{cases} \dfrac{x^a y^b}{x^4+y^6} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

where a and b are nonnegative integers. In each of **(a-e)**, determine whether there exist values of a and b for which f satisfies the given condition.

a. $f(x,y)$ is continuous at $(0,0)$.

b. $f(x,y)$ goes to 1 as (x,y) approaches $(0,0)$ along the line $y=x$, and $f(x,y)$ goes to -1 as (x,y) approaches $(0,0)$ along the line $y=-x$.

c. $f(x,y)$ goes to 0 as (x,y) approaches $(0,0)$ along any line through the origin, and the limit $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

d. $f(x,y)$ goes to 0 as (x,y) approaches $(0,0)$ along any line through the origin except the y-axis, and $f(x,y)$ goes to 1 as (x,y) approaches $(0,0)$ along the y-axis.

e. $f_x(0,0)$ and $f_y(0,0)$ exist, and $f(x,y)$ is not differentiable at $(0,0)$.

Solution: a. If $a = 4$ and $b = 1$, then f is continuous at $(0,0)$. This follows from the Sandwich Theorem as

$$0 \le |f(x,y)| = \left|\frac{x^4 y}{x^4 + y^6}\right| \le \frac{x^4}{x^4 + y^6} \cdot |y| \le 1 \cdot |y| \le |y|$$

for $(x,y) \ne 0$ implies that the limit of $f(x,y)$ at $(0,0)$ is $0 = f(0,0)$.

b. Let $a = 3$ and $b = 1$. Then

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the line } y=x}} f(x,y) = \lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^4}{x^4 + x^6} = \lim_{x\to 0} \frac{1}{1 + x^2} = 1$$

and

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the line } y=-x}} f(x,y) = \lim_{x\to 0} f(x,-x) = \lim_{x\to 0} \frac{-x^4}{x^4 + x^6} = \lim_{x\to 0} \frac{-1}{1 + x^2} = -1\,.$$

c. Let $a = 2$ and $b = 3$. Then we have

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the line } y=mx}} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{m^3 x^5}{x^4 + m^6 x^6} = \lim_{x\to 0} \frac{m^3 x}{1 + m^6 x^2} = 0$$

as well as

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the } y\text{-axis}}} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} 0 = 0\,.$$

However,

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the curve } y=x^{2/3}}} f(x,y) = \lim_{x\to 0} f(x,x^{2/3}) = \lim_{x\to 0} \frac{x^4}{x^4 + x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2} \ne 0$$

and hence the limit of $f(x,y)$ at $(0,0)$ does not exist by the 2-Path Test.

d. Let $a = 0$ and $b = 6$. Then we have

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the line } y=mx}} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{m^6 x^6}{x^4 + m^6 x^6} = \lim_{x\to 0} \frac{m^6 x^2}{1 + m^6 x^2} = 0$$

and

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the } y\text{-axis}}} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{y^6}{y^6} = \lim_{y\to 0} 1 = 1\,.$$

e. If $a = 1$ and $b = 1$, then $f(x,y)$ is not differentiable at $(0,0)$ as it is not even continuous there. This can be seen by considering its limit along the line $y = x$ which does not exist. On the other hand, $f_x(0,0)$ and $f_y(0,0)$ are both 0 as f is identically zero on both axes. □

Remark: The complete lists of ordered pairs (a,b) of nonnegative integers that satisfy the given conditions are as follows:

a. all (a,b) with $3a+2b>12$

b. $(3,1)$, $(1,3)$

c. $(1,4)$, $(2,3)$

d. $(0,6)$

e. $(0,7)$, $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(2,1)$, $(2,2)$, $(2,3)$, $(2,4)$, $(3,1)$, $(3,2)$, $(4,1)$

10. In Genetics, *Fisher's Equation*,

$$\frac{\partial p}{\partial t} = p\,(1-p) + \frac{\partial^2 p}{\partial x^2}$$

describes the spread of an advantageous allele in a population with uniform density along a 1-dimensional habitat, like a shoreline, as a result of both reproduction and dispersion of the offspring. Here $p(x,t)$ is the frequency of the allele as a function of the position x and the time t.

Find all possible values of the pair of constants (a,b) for which the function

$$p(x,t) = \frac{1}{(1+e^{ax+bt})^2}$$

satisfies the Fisher's Equation.

Solution: We have

$$\begin{aligned} p_t &= -2(1+e^{ax+bt})^{-3}\cdot e^{ax+bt}\cdot b \\ p_x &= -2(1+e^{ax+bt})^{-3}\cdot e^{ax+bt}\cdot a \\ p_{xx} &= 6(1+e^{ax+bt})^{-4}\cdot (e^{ax+bt}\cdot a)^2 - 2(1+e^{ax+bt})^{-3}\cdot e^{ax+bt}\cdot a^2 \\ p(1-p) &= (1+e^{ax+bt})^{-4}\cdot (2e^{ax+bt} + (e^{ax+bt})^2) \end{aligned}$$

and substitution these in the Fisher's Equation gives

$$-2b(1+e^{ax+bt}) = 2 + e^{ax+bt} + 6a^2 e^{ax+bt} - 2a^2(1+e^{ax+bt})$$

or:

$$2a^2 - 2b - 2 = (4a^2 + 2b + 1)e^{ax+bt}$$

As the left hand side if this equality is constant, so must be the right hand side. This is possible only if e^{ax+bt} is constant or $4a^2+2b+1=0$. e^{ax+bt} is constant only if $a=0$ and $b=0$, and for these values the equation becomes $-2=1$ which is not possible. In the second case, the equation becomes $2a^2-2b-2=0$. Adding $4a^2+2b+1=0$ and $2a^2-2b-2=0$ we find $6a^2=1$ and hence $b=-5/6$. Therefore $(a,b)=(1/\sqrt{6},-5/6)$ and $(-1/\sqrt{6},-5/6)$ are the only values for which the given function satisfies the Fisher's Equation. □

11. Assume that z and w are differentiable functions of x and y satisfying the equations $xw^3 + yz^2 + z^3 = -1$ and $zw^3 - xz^3 + y^2w = 1$. Find $\dfrac{\partial z}{\partial x}$ at $(x, y, z, w) = (1, -1, -1, 1)$.

Solution: Differentiating the equations with respect to x and keeping in mind that z and w depend on x, but y does not; we obtain

$$w^3 + x \cdot 3w^2 w_x + y \cdot 2z z_x + 3z^2 z_x = 0\,,$$

and

$$z_x w^3 + z \cdot 3w^2 w_x - z^3 - x \cdot 3z^2 z_x + y^2 w_x = 0\,.$$

Substituting $x = 1$, $y = -1$, $z = -1$, $w = 1$, we get $5z_x + 3w_x = -1$ and $2z_x + 2w_x = 1$. Solving for z_x we find

$$\frac{\partial z}{\partial x} = -\frac{5}{4} \quad \text{at} \quad (x, y, z, w) = (1, -1, -1, 1)\,.$$

□

12. Once upon a time there was an xy-plane. The temperature at each point of this xy-plane changed as time passed. Bugs roamed this xy-plane. Each bug had a device that measured the temperature in real time, and showed the rate of change of temperature with respect to time on its screen.

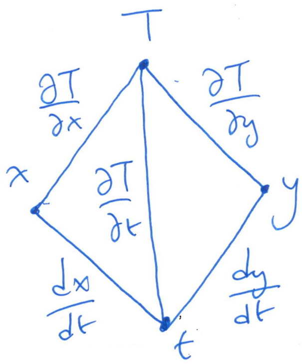
One day four of these bugs met at a point P_0. At the moment they met:

- The first bug was moving with velocity $\mathbf{v}_1 = 2\mathbf{i} + \mathbf{j}$ m/s and its device was showing 1°C/s.
- The second bug was moving with velocity $\mathbf{v}_2 = \mathbf{i} - 5\mathbf{j}$ m/s and its device was showing −1°C/s.
- The third bug was moving with velocity $\mathbf{v}_3 = \mathbf{i} + \mathbf{j}$ m/s and its device was showing 2°C/s.
- The fourth bug was sitting still.

What was the device of the fourth bug showing?

Solution:

T = temperature
t = time



$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial t}$$

We have $\vec{v} = \left.\frac{dx}{dt}\right|_{t_0} \vec{i} + \left.\frac{dy}{dt}\right|_{t_0} \vec{j}$.

Let $a = \frac{\partial T}{\partial x}$; $b = \frac{\partial T}{\partial y}$, $c = \frac{\partial T}{\partial t}$ at (P_0, t_0)

Bug 1: $2a + b + c = 1$ $\quad a = -1$

Bug 2: $a - 5b + c = -1$ $\quad -5b + c = 0$ $\quad c = \frac{5}{2}$ °C/s

Bug 3: $a + b + c = 2$ $\quad b + c = 3$

The fourth bug's device is showing $\frac{5}{2}$°C/s. $\square$

13. Let $z = f(x, y)$ be a differentiable function such that

$$f(3,3) = 1, \qquad f_x(3,3) = -2, \qquad f_y(3,3) = 11,$$

$$f(2,5) = 1, \qquad f_x(2,5) = 7, \qquad f_y(2,5) = -3.$$

Suppose w is a differentiable function of u and v satisfying the equation

$$f(w, w) = f(uv, u^2 + v^2)$$

for all (u, v). Find $\dfrac{\partial w}{\partial u}$ at $(u, v, w) = (1, 2, 3)$.

Solution: Differentiating the identity

$$f(w, w) = f(uv, u^2 + v^2)$$

with respect to u gives

$$f_x(w,w)\frac{\partial w}{\partial u} + f_y(w,w)\frac{\partial w}{\partial u} = f_x(uv, u^2+v^2)\frac{\partial(uv)}{\partial u} + f_y(uv, u^2+v^2)\frac{\partial(u^2+v^2)}{\partial u}$$

by the Chain Rule. Hence

$$\left(f_x(w,w)+f_y(w,w)\right)\frac{\partial w}{\partial u}=f_x(uv,u^2+v^2)\,v+f_y(uv,u^2+v^2)\,2u$$

which leads to

$$\left(f_x(3,3)+f_y(3,3)\right)\frac{\partial w}{\partial u}=2f_x(2,5)+2f_y(2,5)$$

after substituting $(u,v,w)=(1,2,3)$. Now using $f_x(3,3)=-2$, $f_y(3,3)=11$, $f_x(2,5)=7$, and $f_y(2,5)=-3$, we conclude that

$$\frac{\partial w}{\partial u}=\frac{8}{9}\quad\text{at}\quad(u,v,w)=(1,2,3)\,.$$

□

14. Let $u=x+y+z$, $v=xy+yz+zx$, $w=xyz$, and suppose that $f(u,v,w)$ is a differentiable function satisfying $f(u,v,w)=x^4+y^4+z^4$ for all (x,y,z). Find $f_u(2,-1,-2)$.

Solution: We can take $(x,y,z)=(1,-1,2)$ as this gives $(u,v,w)=(2,-1,-2)$.

Differentiating $f(u,v,w)=x^4+y^4+z^4$ with respect to x, y, z, respectively, we obtain:

$$\begin{aligned} f_u\cdot u_x+f_v\cdot v_x+f_w\cdot w_x&=4x^3\\ f_u\cdot u_y+f_v\cdot v_y+f_w\cdot w_y&=4y^3\\ f_u\cdot u_z+f_v\cdot v_z+f_w\cdot w_z&=4z^3 \end{aligned}$$

Now using $u=x+y+z$, $v=xy+yz+zx$, $w=xyz$, these give:

$$\begin{aligned} f_u\cdot 1+f_v\cdot(y+z)+f_w\cdot yz&=4x^3\\ f_u\cdot 1+f_v\cdot(x+z)+f_w\cdot xz&=4y^3\\ f_u\cdot 1+f_v\cdot(x+y)+f_w\cdot xy&=4z^3 \end{aligned}$$

Substituting $(x,y,z)=(1,-1,2)$ we get:

$$\begin{aligned} f_u+f_v-2f_w&=4\\ f_u+3f_v+2f_w&=-4\\ f_u-f_w&=32 \end{aligned}$$

Subtracting 3 times the first equation from the second gives $-2f_u+8f_w=-16$, and adding 8 times the third equation to this gives $6f_u=240$. So $f_u(2,-1,-2)=40$. □

Remark: There are such f. In fact, $f(u,v,w)=u^4-4u^2+2v^2+4uw$ is one. Also note that for a given (u,v,w), the corresponding (x,y,z) must be the roots of $T^3-uT^2+vT-w=0$ and hence is determined up to a permutation of its entries, making the answer independent of the choice.

15. Let $z=f(x,y)$ be a twice-differentiable function and $x=r\cos\theta$, $y=r\sin\theta$. Show that

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2}\,.$$

Solution: If $z = F(x, y)$ is a differentiable function of x and y, then by the chain rule we have

$$\frac{\partial z}{\partial r} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial r} = F_x \cdot \cos\theta + F_y \cdot \sin\theta\,, \qquad (\boldsymbol{i})$$

and similarly,

$$\frac{\partial z}{\partial \theta} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial \theta} = F_x \cdot (-r\sin\theta) + F_y \cdot (r\cos\theta)\,. \qquad (\boldsymbol{ii})$$

We use $(\boldsymbol{i})$ with $F = f$ to obtain

$$\frac{\partial z}{\partial r} = f_x \cdot \cos\theta + f_y \cdot \sin\theta\,. \qquad (\mathbf{A})$$

Then

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}(f_x) \cdot \cos\theta + \frac{\partial}{\partial r}(f_y) \cdot \sin\theta\,.$$

To compute $\frac{\partial}{\partial r}(f_x)$ and $\frac{\partial}{\partial r}(f_y)$ we use $(\boldsymbol{i})$ with $F = f_x$ and $F = f_y$, respectively:

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= (f_{xx}\cos\theta + f_{xy}\sin\theta)\cos\theta + (f_{yx}\cos\theta + f_{yy}\sin\theta)\sin\theta \\ &= f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta \qquad (\mathbf{B})\end{aligned}$$

Similarly, using $(\boldsymbol{ii})$ with $F = f$ gives

$$\frac{\partial z}{\partial \theta} = f_x \cdot (-r\sin\theta) + f_y \cdot (r\cos\theta)\,,$$

and differentiating this with respect to θ again gives

$$\begin{aligned}\frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial}{\partial \theta}(f_x) \cdot (-r\sin\theta) + f_x \cdot \frac{\partial}{\partial \theta}(-r\sin\theta) \\ &\qquad + \frac{\partial}{\partial \theta}(f_y) \cdot (r\cos\theta) + f_y \cdot \frac{\partial}{\partial \theta}(r\cos\theta) \\ &= (f_{xx}(-r\sin\theta) + f_{xy}r\cos\theta)(-r\sin\theta) + f_x(-r\cos\theta) \\ &\qquad + (f_{yx}(-r\sin\theta) + f_{yy}r\cos\theta)(r\cos\theta) + f_y(-r\sin\theta) \\ &= f_{xx}r^2\sin^2\theta - 2f_{xy}r^2\cos\theta\sin\theta + f_{yy}r^2\cos^2\theta \\ &\qquad - r(f_x\cos\theta + f_y\ \sin\theta) \qquad (\mathbf{C})\end{aligned}$$

where we used $(\boldsymbol{ii})$ with $F = f_x$ and $F = f_y$.

Now if we add $(\mathbf{B})$, $1/r$ times $(\mathbf{A})$, and $1/r^2$ times $(\mathbf{C})$, we obtain $f_{xx} + f_{yy}$. □

16. Suppose that $f(x, y)$ is a twice-differentiable function with continuous derivatives satisfying

$$f\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right) = f(x, y)$$

for all $(x, y) \neq (0, 0)$. Find $f_{xx}(3/10, 1/10)$ if $f_x(3, 1) = -8$, $f_y(3, 1) = 7$, $f_{xx}(3, 1) = 2$, $f_{xy}(3, 1) = 5$, $f_{yy}(3, 1) = -4$.

Solution: We have:

$$
\begin{aligned}
f_x(x,y) &= \frac{\partial}{\partial x} f\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \\
&= f_x\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) \\
&\qquad + f_y\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) \\
&= f_x\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} \\
&\qquad + f_y\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \frac{-2xy}{(x^2+y^2)^2}
\end{aligned}
$$

Differentiating this a second time we obtain:

$$
\begin{aligned}
f_{xx}(x,y) &= \frac{\partial}{\partial x}\left(f_x\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \frac{y^2-x^2}{(x^2+y^2)^2}\right. \\
&\qquad \left. + f_y\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \frac{-2xy}{(x^2+y^2)^2}\right) \\
&= f_{xx}\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)^2 \\
&\qquad + f_{xy}\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \frac{-2xy}{(x^2+y^2)^2} \cdot \frac{y^2-x^2}{(x^2+y^2)^2} \\
&\qquad + f_x\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \frac{\partial}{\partial x}\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right) \\
&\qquad + f_{yx}\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \frac{y^2-x^2}{(x^2+y^2)^2} \cdot \frac{-2xy}{(x^2+y^2)^2} \\
&\qquad + f_{yy}\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \left(\frac{-2xy}{(x^2+y^2)^2}\right)^2 \\
&\qquad + f_y\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \frac{\partial}{\partial x}\left(\frac{-2xy}{(x^2+y^2)^2}\right) \\
&= f_{xx}\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)^2 \\
&\qquad + 2f_{xy}\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \frac{y^2-x^2}{(x^2+y^2)^2} \cdot \frac{-2xy}{(x^2+y^2)^2} \\
&\qquad + f_{yy}\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \left(\frac{-2xy}{(x^2+y^2)^2}\right)^2 \\
&\qquad + f_x\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \frac{2x(x^2-3y^2)}{(x^2+y^2)^3} \\
&\qquad + f_y\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}
\end{aligned}
$$

Now letting $(x,y) = (3/10, 1/10)$ gives:

$$\begin{aligned} f_{xx}(3/10,1/10) &= f_{xx}(3,10)\cdot(-8)^2 + 2f_{xy}(3,10)\cdot(-8)\cdot(-6) + f_{yy}(3,10)\cdot(-6)^2 \\ &\qquad + f_x(3,10)\cdot 36 + f_y(3,10)\cdot 52 \\ &= 2\cdot 64 + 2\cdot 5\cdot 48 + (-4)\cdot 36 + (-8)\cdot 36 + 7\cdot 52 \\ &= 540 \end{aligned}$$

□

17. Consider the following conditions for a differentiable function $f(x,y)$:

❶ $f(2,1) = 8$

❷ An equation for the tangent line to the level curve $f(x,y) = 8$ in the xy-plane at the point $(2,1)$ is $3x - 5y = 1$

Let $\mathscr{P}$ be the tangent plane to the graph of $z = f(x,y)$ at the point $(2,1,8)$.

In each of the parts **(a-e)** below a ❸$^{\text{rd}}$ condition is given. Determine whether

- there is no function satisfying the conditions ❶-❸,
- there are functions satisfying the conditions ❶-❸, but they do not all have the same tangent plane $\mathscr{P}$, or
- there are functions satisfying the conditions ❶-❸ and all of these functions have the same tangent plane $\mathscr{P}$. (In this case find an equation of $\mathscr{P}$ too.)

a. ❸ $f(3,2) = 11$

b. ❸ $f_x(2,1) = -1$

c. ❸ $\left.\dfrac{d}{dt} f(t^2+1, t^3)\right|_{t=1} = 6$

d. ❸ The line with parametric equations $x = 4t+2$, $y = 2t+1$, $z = t+8$, $(-\infty < t < \infty)$, lies in $\mathscr{P}$

e. ❸ The line with parametric equations $x = -t+2$, $y = 2t+1$, $z = t+8$, $(-\infty < t < \infty)$, is perpendicular to $\mathscr{P}$

Solution: The condition ❷ implies that $f_x(2,1) = 3c$ and $f_y(2,1) = -5c$ for some constant c. c is uniquely determined by the condition ❸ in parts **(b-d)** and there is no such c in part **(e)**:

b. If $f_x(2,1) = -1$ gives $c = -1/3$. Hence $f_y(2,1) = 5/3$. Therefore all functions satisfying the conditions ❶-❸ have the same tangent plane $\mathscr{P}$. An equation for this common tangent plane $\mathscr{P}$ is $3x - 5y + 3z = 25$, and an example of a function satisfying the given conditions is $f(x,y) = -x + 5/3\,y + 25/3$.

c. As we have

$$\frac{d}{dt}f(t^2+1,t^3) = f_x(t^2+1,t^3)\cdot 2t + f_y(t^2+1,t^3)\cdot 3t^2$$

by the Chain Rule, substituting $t = 1$ we obtain $3c\cdot 2+(-5c)\cdot 3 = 6$ and $c = -2/3$. Hence $f_x(2,1) = -2$ and $f_y(2,1) = 10/3$. Therefore all functions satisfying the conditions ❶-❸ have the same tangent plane $\mathscr{P}$. An equation for this common tangent plane $\mathscr{P}$ is $6x - 10y + 3z = 26$, and an example of a function satisfying the given conditions is $f(x,y) = -2x + 10/3\,y + 26/3$.

d. As the tangent plane $\mathscr{P}$ will have an equation of the form $z = 3c\cdot(x-2)+(-5c)\cdot(y-1)+8$, for the line with parametric equations $x = 4t+2$, $y = 2t+1$, $z = t+8$, $(-\infty < t < \infty)$, to lie in $\mathscr{P}$ we must have $1 = (3c)\cdot 4+(-5c)\cdot 2$ and $c = 1/2$. Hence $f_x(2,1) = 3/2$ and $f_y(2,1) = -5/2$. Therefore all functions satisfying the conditions ❶-❸ have the same tangent plane $\mathscr{P}$. An equation for this common tangent plane $\mathscr{P}$ is $3x - 5y - 2z = -15$, and an example of a function satisfying the given conditions is $f(x,y) = 3/2\,x - 5/2\,y + 15/2$.

e. As the tangent plane $\mathscr{P}$ will have an equation of the form $3c\cdot(x-2)+(-5c)\cdot(y-1)-(z-8) = 0$, for the line with parametric equations $x = -t+2$, $y = 2t+1$, $z = t+8$, $(-\infty < t < \infty)$, to be perpendicular to $\mathscr{P}$ we must have the the normal vector $\mathbf{n} = 3c\mathbf{i}-5c\mathbf{j}-\mathbf{k}$ of the plane and the velocity vector $\mathbf{v} = -\mathbf{i}+2\mathbf{j}+\mathbf{k}$ of the line to be parallel. This requires $3c/(-1) = (-5c)/2 = -1/1$, which is not possible. Therefore there is no function f satisfying the conditions ❶-❸.

Finally, in part **(a)**, both of the functions $f(x,y) = 3(x-2)-5(y-1)+8+5(x-2)^2$ and $f(x,y) = -3(x-2)+5(y-1)+8+(x-2)^2$ satisfy the conditions ❶-❸, but equations of their graphs' tangent planes $\mathscr{P}$ at $(2,1,8)$ are $z = 3x-5y+7$ and $z = -3x+5y+9$, respectively. Therefore, in this case there are functions satisfying the conditions ❶-❸, but they do not all have the same tangent plane $\mathscr{P}$. □

18. Let $f(x,y) = x^3y - xy^2 + cx^2$ where c is a constant. Find c if f increases fastest at the point $P_0(3,2)$ in the direction of the vector $\mathbf{A} = 2\mathbf{i}+5\mathbf{j}$.

Solution: $\nabla f = (3x^2y-y^2+2cx)\mathbf{i}+(x^3-2xy)\mathbf{j} \Longrightarrow (\nabla f)_{P_0} = (50+6c)\mathbf{i}+15\mathbf{j}$. Since f increases the fastest at P_0 in the direction of $(\nabla f)_{P_0}$, $\mathbf{A} = 2\mathbf{i}+5\mathbf{j}$ must be a positive multiple of $(\nabla f)_{P_0}$. Hence $(50+6c)/2 = 15/5 \Longrightarrow c = -22/3$. Finally we check that $c = -22/3$ gives $(\nabla f)_{P_0} = 6\mathbf{i}+15\mathbf{j} = 3\mathbf{A}$ which is indeed a *positive* multiple of $\mathbf{A}$. □

19. Find a vector that is tangent to the intersection curve of the surfaces $x^2+y^2+z^2 = 9$ and $z = xy$ at the point $P_0(1,2,2)$.

Solution: Let $f(x,y,z) = x^2+y^2+z^2$ and $g(x,y,z) = xy - z$. Then the given surfaces are level surfaces of f and g.

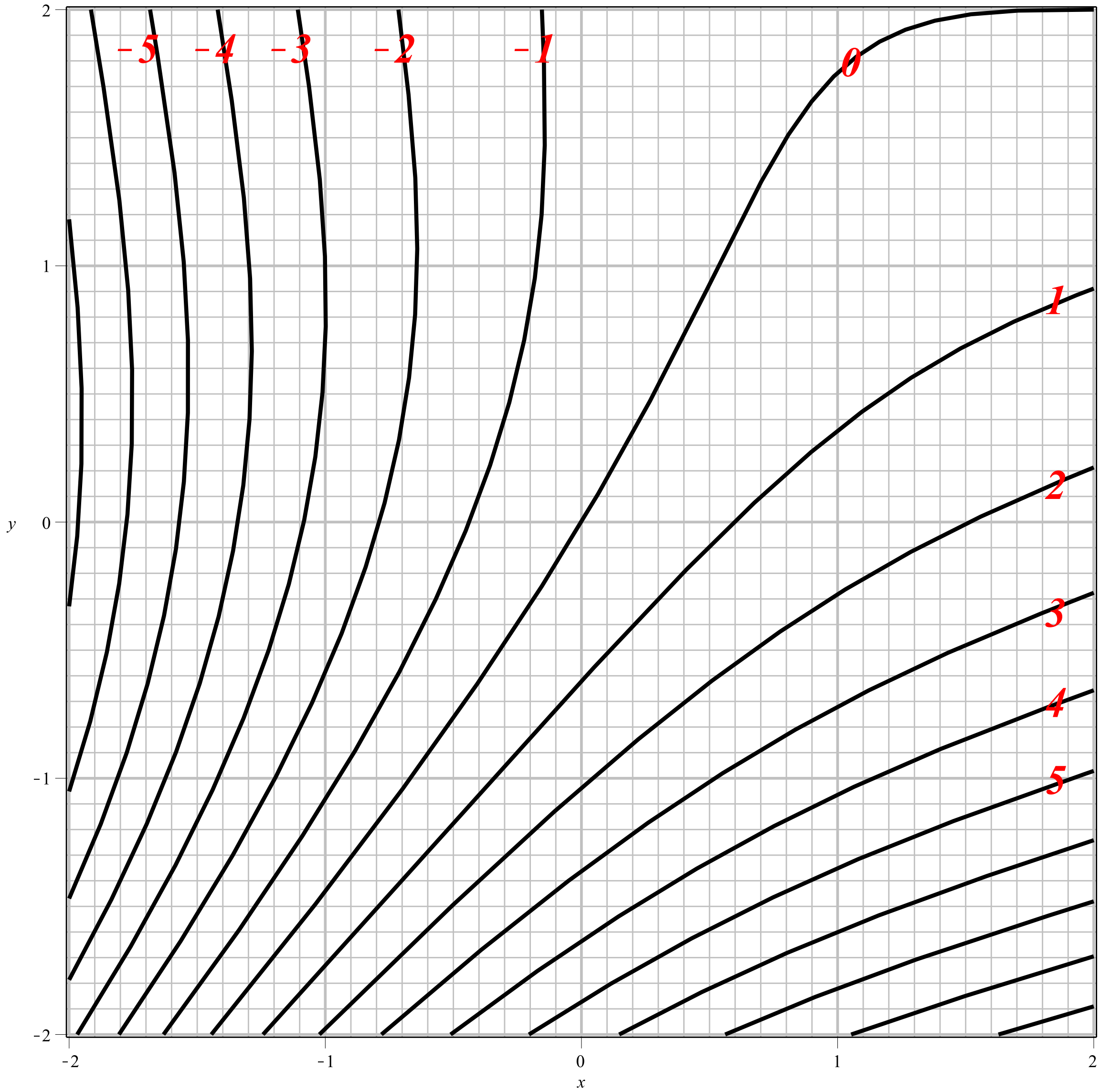
We have $\nabla f = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} \Longrightarrow (\nabla f)_{P_0} = 2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$ and $\nabla g = y\mathbf{i}+x\mathbf{j}-\mathbf{k} \Longrightarrow (\nabla g)_{P_0} = 2\mathbf{i}+\mathbf{j}-\mathbf{k}$. $(\nabla f)_{P_0}$ is normal to the surface defined by $x^2+y^2+z^2 = 9$ and

$(\nabla g)_{P_0}$ is normal to the surface defined by $z = xy$. Therefore,

$$(\nabla f)_{P_0} \times (\nabla g)_{P_0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 4 \\ 2 & 1 & -1 \end{vmatrix} = -8\mathbf{i} + 10\mathbf{j} - 6\mathbf{k},$$

or any multiple of it, is tangent to both of these surfaces, and hence, to their curve of intersection at P_0. $\square$

20. In the figure below some of the level curves and the corresponding values of a nice function $f(x, y)$ are shown.

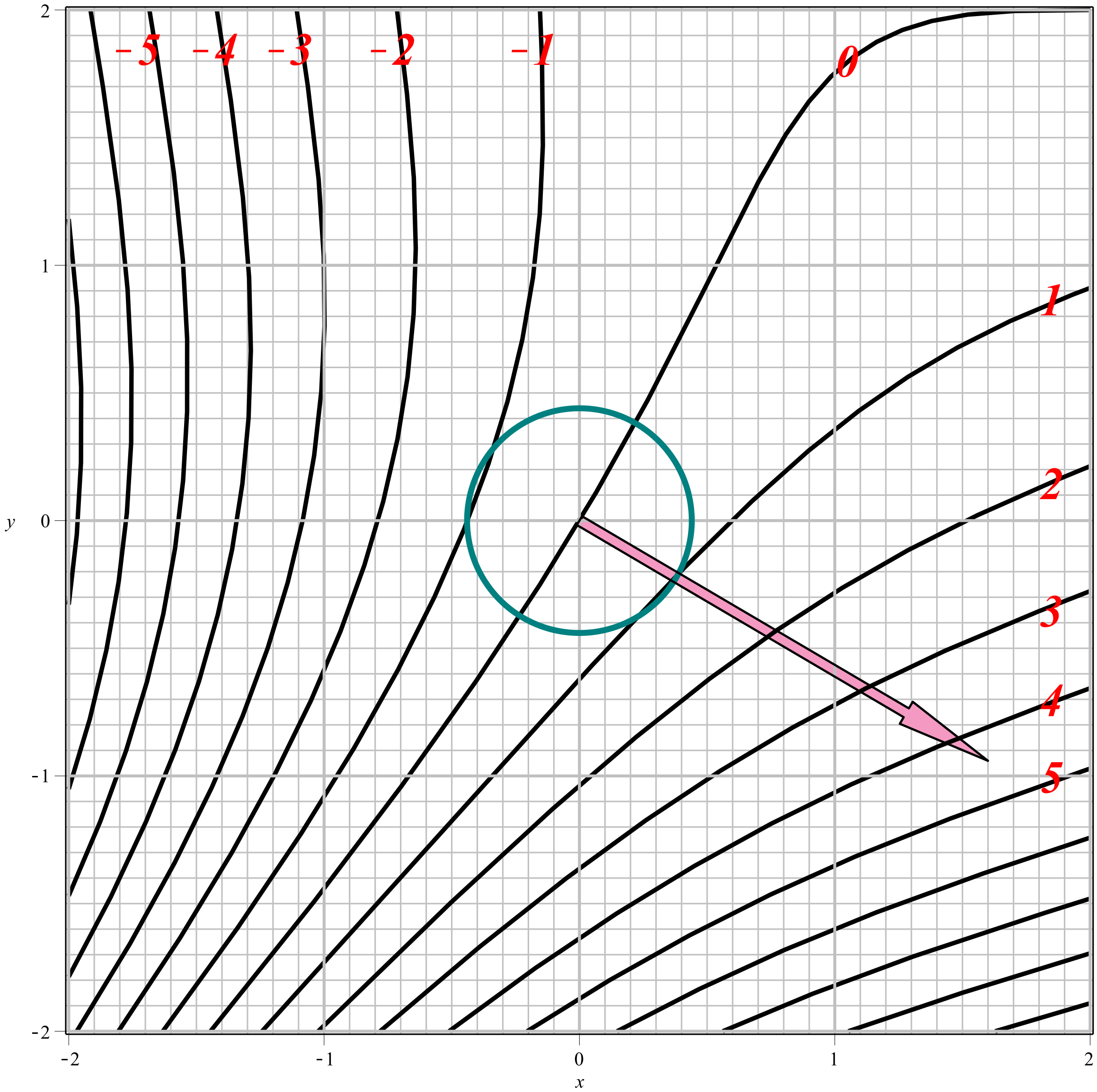


a. Draw $\nabla f(0,0)$ as best you can on the figure.

b. Determine the signs of the derivatives f_x, f_y, f_{xx}, f_{yy}, f_{xy} at the origin.

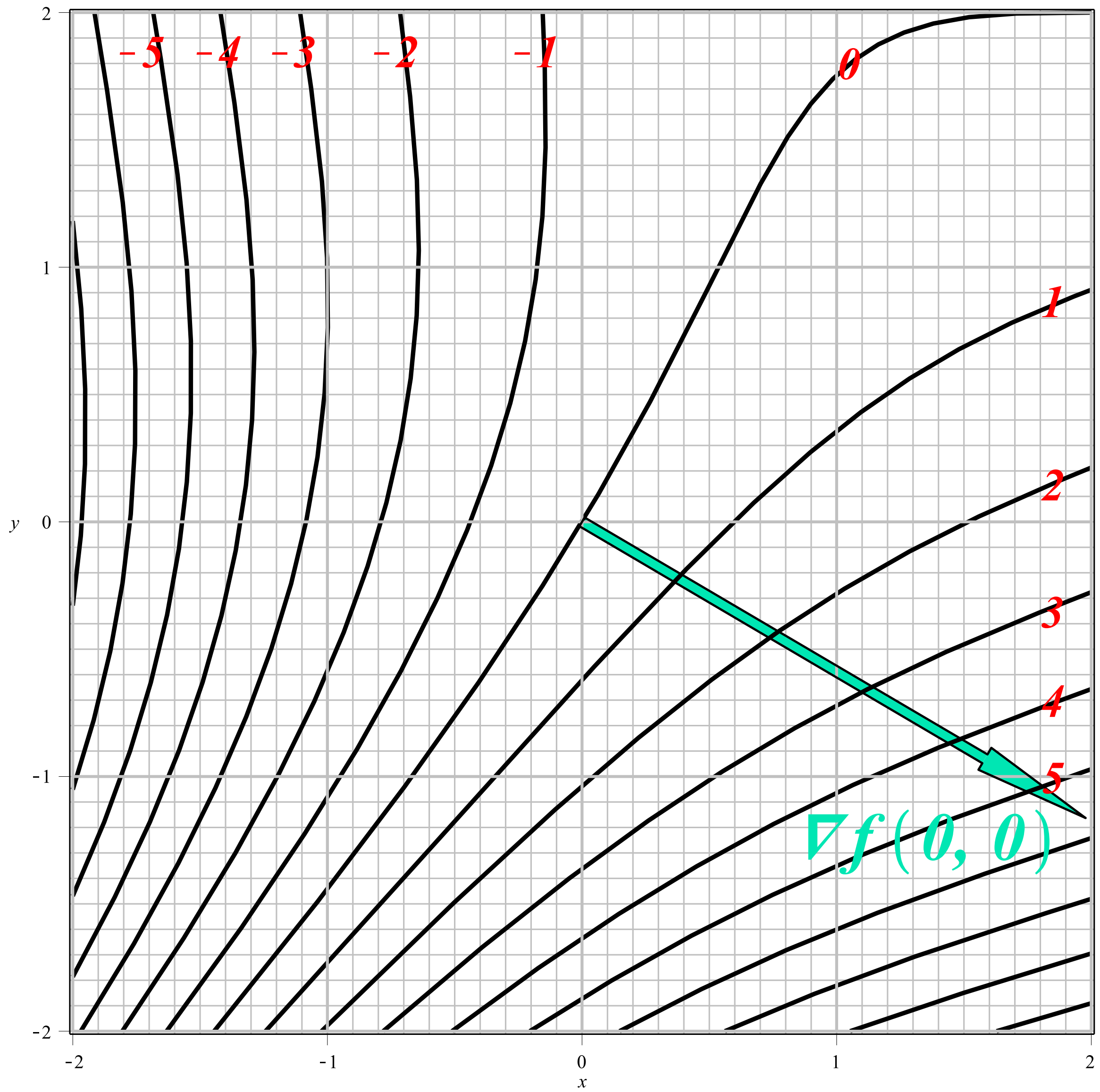
Remark: This is an approximation problem and it should be solved under the assumption that we are dealing with a function for which good approximations can be made using only the given data.

Solution: a. We know that $\nabla f(0,0)$ is perpendicular to the level curve of f passing through $(0,0)$ and points in the direction f increases, hence lies along the pink vector in the figure.



We also know that the length of $\nabla f(0,0)$ is the rate of change of f with respect to distance in this direction. On the figure using the teal circle we measure that the pink vector crosses the level curve $f = 1$ at a distance of approximately 0.44 units from the origin. Hence along this direction f increases at an rate of $1/0.44 \approx 2.27$ in this direction. So now we draw a vector of this length in this direction as shown in the next figure and this is our approximate $\nabla f(0,0)$.

(*The figure is on the next page.*)

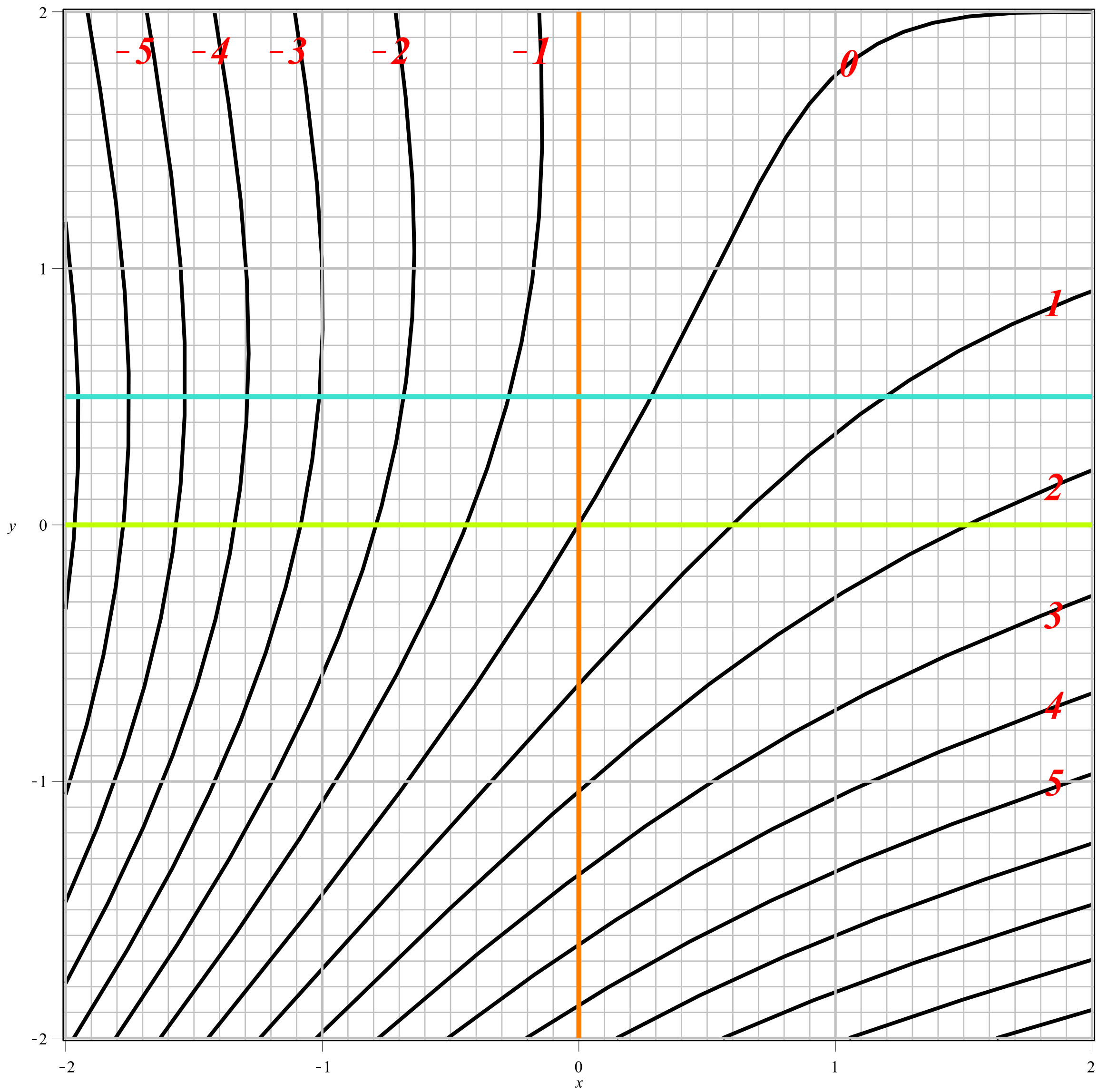


b. Considering the components of $\nabla f(0,0)$ in part **(a)** it can be seen that $f_x(0,0) > 0$ and $f_y(0,0) < 0$. This can also be done in the following way: As we move along the x-axis (the lime line in the figure on the next page) to the right, we meet level curves belonging to higher values of f; hence $f_x(0,0) > 0$. Similarly, as we move along the y-axis (the orange line) upwards, we meet level curves belonging to smaller values of f values; hence $f_y(0,0) < 0$.

Moreover, note that as we move in the positive direction along both axes, the points where we meet the level curves of f belonging to values with the same difference become farther apart. This means that the absolute value of the rate of change of f is getting smaller. This in turn means $f_{xx}(0,0) < 0$ as $f_x(0,0) > 0$, and $f_{yy}(0,0) > 0$ as $f_y(0,0) < 0$.

(*Part **(b)** is continued on the next page.*)

Finally, we compare the rate of changes along the lime and turquoise lines. The level curves intersect the turquoise line at points farther apart than the points they intersect the lime line. Hence f_x, which is positive, is getting smaller as we move upwards along y-axis. In other words, $f_{xy}(0,0) < 0$.



$\square$

Remark: It is also possible to find approximate values of these derivatives using the figure. In fact, those of $f_x(0,0)$ and $f_y(0,0)$ can already be read off as the components of $\nabla f(0,0)$.

21. Let a be a constant. Find and classify all critical points of $f(x,y) = x^3 - 3axy + y^3$.

Solution: At a critical point $f_x = 3x^2 - 3ay = 0$ and $f_y = -3ax + 3y^2 = 0$. If $a = 0$, then $3x^2 = 0 \Rightarrow x = 0$ and $3y^2 = 0 \Rightarrow y = 0$, and $(0,0)$ is the only critical point. If $a \neq 0$, then the first equation gives $y = x^2/a$, and substituting this in the second equation we get $x^4 - a^3x = 0$ whose solutions are $x = 0$ and $x = a$. Now using $y = x^2/a$, we get $(0,0)$ and (a,a) as the critical points.

We compute the discriminant:

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yz} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -3a \\ -3a & 6y \end{vmatrix}$$

As $\Delta(a,a) = 27a^2$ and $f_{xx}(a,a) = 6a$, (a,a) is a local minimum for $a > 0$ and a local maximum for $a < 0$. On the other hand, $\Delta(0,0) = -9a^2$ implies that $(0,0)$ is a saddle point for $a \neq 0$.

Now we look at the sole critical point $(0,0)$ in the case $a = 0$. As $\Delta(0,0) = 0$, the second derivative test fails in this case. If we restrict $f(x,y) = x^3 + y^3$ to the x-axis we get $f(x,0) = x^3$. Since this single variable function does not have a local maximum or minimum at $x = 0$, $f(x,y)$ cannot have a local maximum or minimum at $(0,0)$ either. We conclude that $(0,0)$ is a saddle point when $a = 0$. □

22. Find the absolute maximum and minimum values of the function $f(x,y) = 2x^3 + 2xy^2 - x - y^2$ on the unit disk $D = \{(x,y) : x^2 + y^2 \leq 1\}$.

Solution: We first find the critical points of $f(x,y)$ in the interior of D. At a critical point we have $f_x = 6x^2 + 2y^2 - 1 = 0$ and $f_y = 4xy - 2y = 0$. The second equation implies that $y = 0$ or $x = 1/2$. Substituting these into the first equation we obtain $y = \pm 1/\sqrt{6}$ in the first case, and no solution in the second case. Therefore the critical points are $(x,y) = (1/\sqrt{6}, 0)$ and $(-1/\sqrt{6}, 0)$. Note that both of these points lie in the interior of D.

Now we look at the boundary of D, that is, the unit circle $x^2 + y^2 = 1$. We can solve y as $y = \pm\sqrt{1-x^2}$, $-1 \leq x \leq 1$. These solutions correspond to the upper and lower semicircles. Then $f(x, \pm\sqrt{1-x^2}) = x^2 + x - 1$ for $-1 \leq x \leq 1$, and $\frac{d}{dx} f(x, \pm\sqrt{1-x^2}) = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$. This gives the critical points $(x,y) = (-1/2, \sqrt{3}/2)$ and $(-1/2, -\sqrt{3}/2)$ of the restriction of f to the boundary of D. We must also include the endpoints $x = -1$ and $x = 1$, in other words, the points $(x,y) = (1,0)$ and $(-1,0)$ in our list.

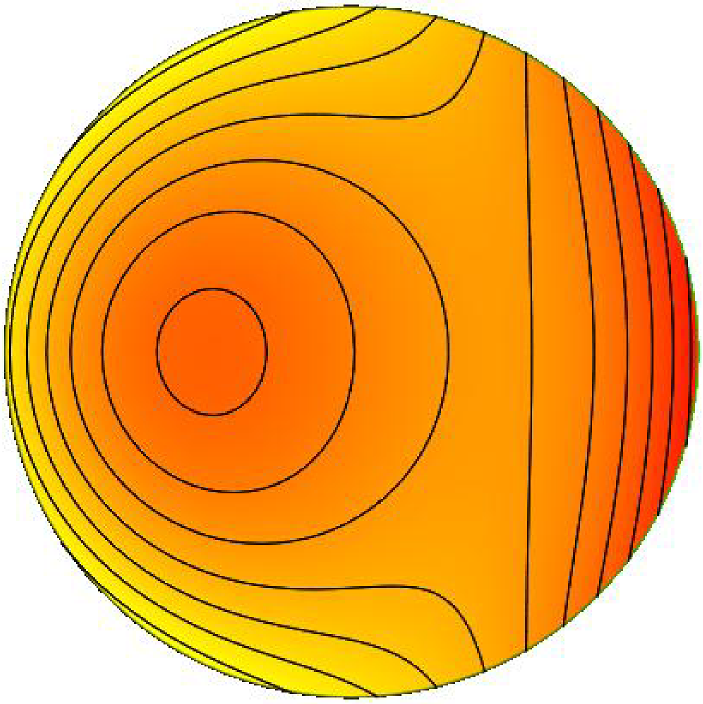
Hence the absolute maximum and the absolute minimum of f on D occur at some of the points

$$(1/\sqrt{6}, 0),\ (-1/\sqrt{6}, 0),\ (-1/2, \sqrt{3}/2),\ (-1/2, -\sqrt{3}/2),\ (1,0),\ (-1,0)\,.$$

The values of f at these points are

$$-\frac{1}{3}\sqrt{\frac{2}{3}},\ \frac{1}{3}\sqrt{\frac{2}{3}},\ -\frac{5}{4},\ -\frac{5}{4},\ 1, -1,$$

respectively. Therefore the absolute maximum value is 1 and the absolute minimum value is $-5/4$. $\square$



Remark: Here are two more ways of dealing with the critical points of the restriction of f to the boundary of D.

In the first one we parametrize the boundary, which is the unit circle, by $x = \cos t$, $y = \sin t$, $-\infty < t < \infty$. Then

$$\frac{d}{dt} f(\cos t, \sin t) = \frac{d}{dt}(\cos^2 t + \cos t - 1) = -2\cos t \sin t - \sin t = 0 \Rightarrow \cos t = -\frac{1}{2} \quad \text{or} \quad \sin t = 0\,,$$

and these give us $(x, y) = (-1/2, \sqrt{3}/2), (-1/2, -\sqrt{3}/2), (1, 0), (-1, 0)$.

In the second we use the Lagrange Multipliers Method for the boundary $\overbrace{x^2 + y^2}^{g(x,y)} = 1$.

$$\left.\begin{array}{rcl} \nabla f & = & \lambda \nabla g \\ g & = & 1 \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} f_x & = & \lambda g_x \\ f_y & = & \lambda g_y \\ g & = & 1 \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcll} 6x^2 + 2y^2 - 1 & = & \lambda\, 2x & ① \\ 4xy - 2y & = & \lambda\, 2y & ② \\ x^2 + y^2 & = & 1 & ③ \end{array}\right.$$

② gives $y = 0$ or $x = (1+\lambda)/2$. If $y = 0$, then ③ gives $x = \pm 1$ (and $\lambda = \pm 5/2$). On the other hand, if $x = (1 + \lambda)/2$, then from ① and ③, $4x^2 - 2\lambda x + 1 = 0 \Rightarrow x = -1/2$ and $y = \pm\sqrt{3}/2$. Therefore the points $(x, y) = (-1/2, \sqrt{3}/2), (-1/2, -\sqrt{3}/2), (1, 0), (-1, 0)$ are added to the list.

23. Find the absolute maximum and minimum values of the function

$$f(x,y) = 2(x^2+y^2-1)^2 + x^2 - y^2$$

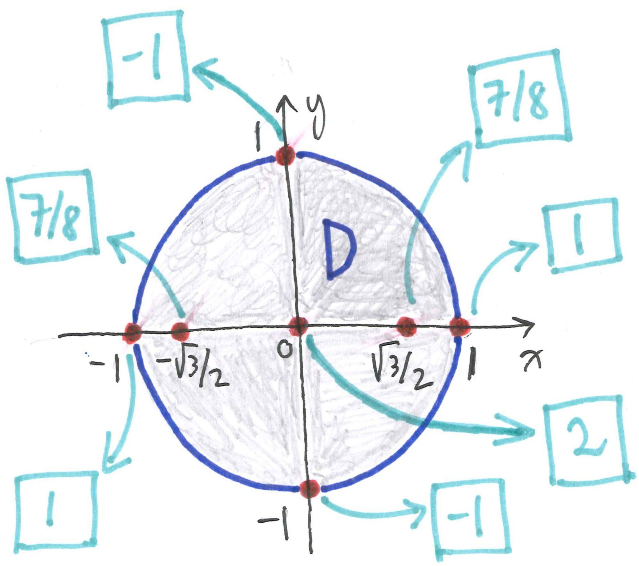
on the unit disk $D = \{(x,y) : x^2+y^2 \le 1\}$.

Solution: We first find the critical points of f in the interior of D. We want:

$$f_x = 4(x^2+y^2-1)\cdot 2x + 2x = 0 \quad \text{and} \quad f_y = 4(x^2+y^2-1)\cdot 2y - 2y = 0\,.$$

The first equation gives $x = 0$ or $4x^2+4y^2-3 = 0$, and the second one gives $y = 0$ or $4x^2+4y^2-5 = 0$. Therefore $x = 0$ and $y = 0$, or $4x^2+4y^2-3 = 0$ and $y = 0$, or $x = 0$ and $4x^2+4y^2-5 = 0$, or $4x^2+4y^2-3 = 0$ and $4x^2+4y^2-5 = 0$. In the last case there is no solution, and the first three cases leads to the points $(0,0)$, $(\pm\sqrt{3}/2, 0)$ and $(0, \pm\sqrt{5}/2)$. Only the first three of these critical points lie in the interior of D as for the last two $x^2+y^2 = 5/4 > 1$.

Next we consider the restriction of f to the boundary of D which is the unit circle. As on the boundary $y = \pm\sqrt{1-x^2}$, we consider the functions $f(x, \pm\sqrt{1-x^2}) = 2x^2-1$ on the interval $-1 \le x \le 1$. As $(d/dx)f(x, \pm\sqrt{1-x^2}) = 4x$, the only critical point of both of these restrictions occur when $x = 0$. This leads the points $(x,y) = (0,1)$ and $(0,-1)$. On the other hand, for both functions the endpoints $x = 1$ and $x = -1$ of the interval $-1 \le x \le 1$ give the points $(x,y) = (1,0)$ and $(-1,0)$.



Now we find the values of f at these seven points:

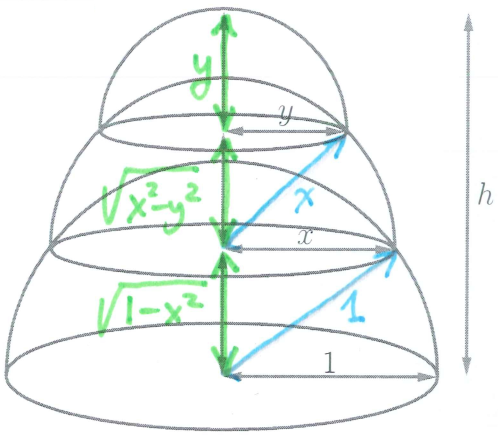
$$f(0,0) = 2$$
$$f(\sqrt{3}/2, 0) = f(-\sqrt{3}/2) = 7/8$$
$$f(0,1) = f(0,-1) = -1$$
$$f(1,0) = f(-1,0) = 1$$

Therefore the absolute maximum and minimum values of f on D are 2 and -1, respectively. □

Remark: The boundary can also be dealt with using the parametrization $x = \cos t$, $y = \sin t$, $(-\infty < t < \infty)$, of the unit circle. Then the restriction of f to the unit circle gives the function $f(\cos t, \sin t) = \cos 2t$ on the interval $-\infty < t < \infty$. $(d/dt) f(\cos t, \sin t) = -2 \sin 2t = 0$ means that t is an integer multiple of $\pi/2$ and this gives the critical points $(x, y) = (\pm 1, 0)$ and $(0, \pm 1)$ for the restriction of f to the boundary.

Remark: Consider the function $g(x, y) = 2(x^2 + y^2 - 1)^2 + 2xy$ on the unit disk D. This is the same function as f rotated 45° counterclockwise. Hence the absolute maximum value of g on D will be 2 assumed at the point $(x, y) = (0, 0)$, and the absolute minimum value -1 at the points $(x, y) = (1/\sqrt{2}, 1/\sqrt{2})$ and $(x, y) = (-1/\sqrt{2}, -1/\sqrt{2})$. It is a good exercise to solve this problem directly.

24. Three hemispheres with radiuses 1, x and y, where $1 \geq x \geq y \geq 0$, are stacked on top of each other as shown in the figure. Find the largest possible value of the total height h.



Solution: We want to maximize

$$h(x,y) = \sqrt{1-x^2} + \sqrt{x^2-y^2} + y$$

on the closed and bounded region $D = \{(x,y) : 0 \le y \le x \le 1\}$.

We first find the critical points of h in the interior of D. We want

$$h_x = -\frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{x^2-y^2}} = 0 \ \text{ and } \ h_y = -\frac{y}{\sqrt{x^2-y^2}} + 1 = 0\,.$$

From the second equation we obtain $y^2 = x^2 - y^2 \implies x^2 = 2y^2 \implies x = \sqrt{2}y$ where we used the fact that $x > 0$ and $y > 0$. Now substituting this in the first equation gives $1 - 2y^2 = 2y^2 - y^2 \implies 3y^2 = 1 \implies y = 1/\sqrt{3}$ as $y > 0$ and hence $x = \sqrt{2/3}$. So the only critical point of h is $(x,y) = (\sqrt{2/3}, 1/\sqrt{3})$ and it lies in D as $0 \le 1/\sqrt{3} \le \sqrt{2/3} \le 1$.

Now we consider the restriction of h to the boundary of D.

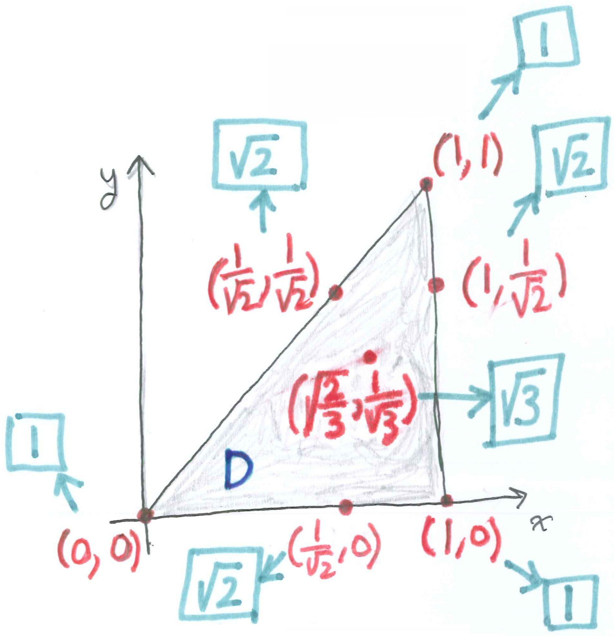
Side 1: On the bottom edge of the triangle, we have $y = 0$ and $0 \le x \le 1$, and therefore we are considering the function $h(x,0) = \sqrt{1-x^2} + x$ for $0 \le x \le 1$.

$$\frac{d}{dx}h(x,0) = -\frac{x}{\sqrt{1-x^2}} + 1 = 0 \implies x^2 = 1 - x^2 \implies 2x^2 = 1 \implies x = \frac{1}{\sqrt{2}}$$

as $x > 0$. Taking the endpoints $x = 0$ and $x = 1$ into account, *Side 1* gives us the points $(x,y) = (1/\sqrt{2}, 0), (0,0), (1,0)$.

Side 2: On the right edge of the triangle, we have $x = 1$ and $0 \le y \le 1$, and therefore we are considering the function $h(1,y) = \sqrt{1-y^2} + y$ for $0 \le y \le 1$. As in the case of *Side 1* this leads to the points $(x,y) = (1, 1/\sqrt{2}), (1,0), (1,1)$.

Side 3: On the top edge of the triangle, we have $y = x$ and $0 \le x \le 1$, and therefore we are considering the function $h(x,x) = \sqrt{1-x^2} + x$ for $0 \le x \le 1$. Once again as in the case of *Side 1* we obtain the points $(x,y) = (1/\sqrt{2}, 1/\sqrt{2}), (0,0), (1,1)$.



Now we find the values of h at these seven points:

$$\begin{aligned} &h(0,0) = h(1,0) = h(1,1) = 1 \\ &h(1/\sqrt{2},0) = h(1,1/\sqrt{2}) = f(1/\sqrt{2},1/\sqrt{2}) = \sqrt{2} \\ &h(\sqrt{2/3},1/\sqrt{3}) = \sqrt{3} \end{aligned}$$

Therefore the maximum possible total height of the three hemispheres is $\sqrt{3}$. □

Remark: Here is a single variable argument which solves the problem for any number hemispheres: Suppose the maximum possible height for k hemispheres is $\sqrt{k}$ for some $k \geq 1$. Then for $k+1$ hemispheres, where the second one from the bottom has radius r, the maximum possible height will be $H(r) = \sqrt{k}r + \sqrt{1-r^2}$ for $0 \leq r \leq 1$. The value of $H(r)$ is 1 and $\sqrt{k}$ at the endpoints $r = 0$ and $r = 1$, respectively; and its value is $\sqrt{k+1}$ at its only critical point $r = \sqrt{k/(k+1)}$. Therefore we conclude inductively that the maximum possible height for n hemispheres is $\sqrt{n}$ for $n \geq 1$.

25. Find the absolute maximum and the absolute minimum of $f(x,y,z) = x^3 + yz$ on the unit sphere $x^2+y^2+z^2 = 1$.

Solution: We will use the Lagrange Multipliers Method. Let $g(x,y,z) = x^2+y^2+z^2-1$. Then

$$\left.\begin{array}{rcl} \nabla f &=& \lambda \nabla g \\ g &=& 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} f_x &=& \lambda g_x \\ f_y &=& \lambda g_y \\ f_z &=& \lambda g_z \\ g &=& 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcll} 3x^2 &=& \lambda 2x & (1) \\ z &=& \lambda 2y & (2) \\ y &=& \lambda 2z & (3) \\ x^2+y^2+z^2 &=& 1 & (4) \end{array}\right.$$

From (2) and (3) we obtain $z = 4\lambda^2 z$, and hence $z = 0$ or $\lambda = 1/2$ or $\lambda = -1/2$.

- If $z = 0$, then (3) gives $y = 0$, and then (4) gives $x = \pm 1$. In this case we have the points $(\pm 1, 0, 0)$.
- If $\lambda = 1/2$, then from (1) and (3), $3x^2 = x$ and $y = z$. Therefore we either have $x = 0$ in which case $y = z = \pm 1/\sqrt{2}$ by (4), or we have $x = 1/3$ and then $y = z = \pm 2/3$ again by (4). In this case the critical points are $(0, \pm 1/\sqrt{2}, \pm 1/\sqrt{2})$ and $(1/3, \pm 2/3, \pm 2/3)$.
- If $\lambda = -1/2$, then from (1) and (3), $3x^2 = -x$ and $y = -z$. A reasoning similar to the previous case gives the points $(0, \pm 1/\sqrt{2}, \mp 1/\sqrt{2})$ and $(-1/3, \pm 2/3, \mp 2/3)$.

Hence the critical points are $(\pm 1,0,0), (0,\pm 1/\sqrt{2},\pm 1/\sqrt{2}), (1/3,\pm 2/3,\pm 2/3), (0,\pm 1/\sqrt{2},\mp 1/\sqrt{2}), (-1/3,\pm 2/3,\mp 2/3)$, and the values of f at these points are $\pm 1, 1/2, -1/2, 13/27, -13/27$, respectively. Therefore the absolute maximum is 1 and the absolute minimum is -1. □

26. Evaluate the following integrals:

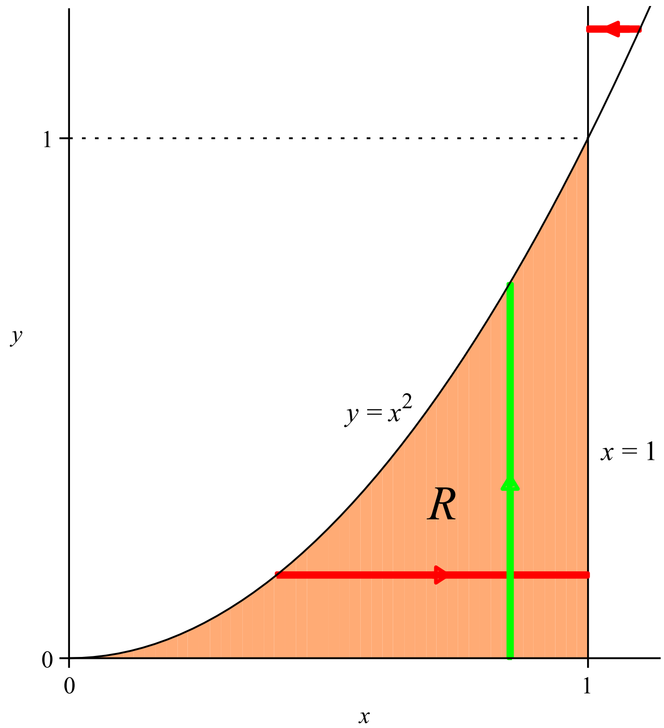
a. $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \sin(\pi x^3)\, dx\, dy$

b. $\displaystyle\iint_R y^2 e^{-x^2}\, dA$ where $R = \{(x,y) : 0 \le y \le x\}$

c. $\displaystyle\int_0^2 \int_0^{\sqrt{2y-y^2}} \frac{xy}{x^2+y^2}\, dx\, dy$

d. $\displaystyle\int_0^\infty \int_0^\infty \frac{dy\, dx}{(x^2+y^2)^2+1}$

Solution: a. We will first express the iterated integral as a double integral and then reverse the order of integration. The x-integral goes from $x = \sqrt{y}$ to $x = 1$ as shown by the red line segments in the figure in the xy-plane. Then the y-integral goes from $y = 0$ to $y = 1$.



Therefore the intervals of the x-integral trace out the region R bounded by the parabola $y = x^2$, the line $x = 1$, and the x-axis. Note that the x-integrals are always from left to right in the interval $0 \le y \le 1$. Hence we have

$$\int_0^1 \int_{\sqrt{y}}^1 \sin(\pi x^3)\, dx\, dy = \iint_R \sin(\pi x^3)\, dA\,.$$

Now we express this double integral as an iterated integral with the y-integral first. The green line segment in the figure shows the interval of integration for the y-integral which goes from $y = 0$ to $y = x^2$. Then:

$$
\begin{aligned}
\int_0^1 \int_{\sqrt{y}}^1 \sin(\pi x^3)\,dx\,dy &= \iint_R \sin(\pi x^3)\,dA \\
&= \int_0^1 \int_0^{x^2} \sin(\pi x^3)\,dy\,dx \\
&= \int_0^1 \sin(\pi x^3)\; y\Big]_{y=0}^{y=x^2}\,dx \\
&= \int_0^1 \sin(\pi x^3) x^2\,dx \\
&= -\frac{1}{3\pi}\cos(\pi x^3)\Big]_0^1 \\
&= \frac{2}{3\pi}
\end{aligned}
$$

b. We integrate with respect to y first to obtain

$$
\begin{aligned}
\iint_R y^2 e^{-x^2}\,dA &= \int_0^\infty \int_0^x y^2 e^{-x^2}\,dy\,dx \\
&= \frac{1}{3}\int_0^\infty x^3 e^{-x^2}\,dx \\
&= \frac{1}{6}\int_0^\infty t e^{-t}\,dt \\
&= \frac{1}{6}
\end{aligned}
$$

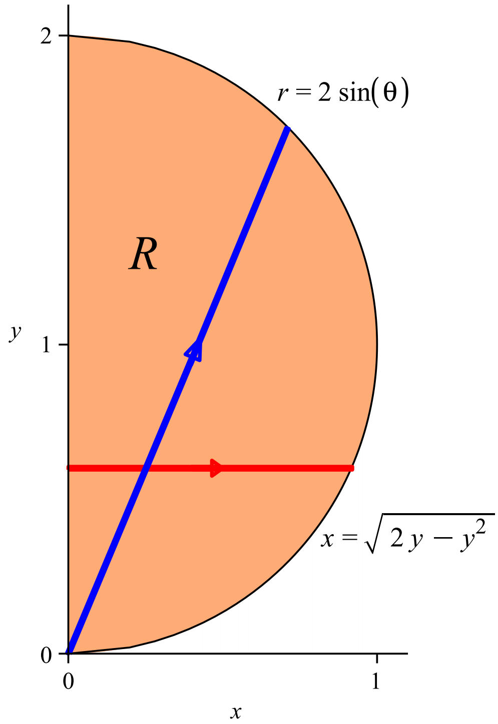
where we used the integration by parts

$$
\begin{aligned}
\int_0^\infty t e^{-t}\,dt &= \int_0^\infty t\,d(-e^{-t}) \\
&= \lim_{c\to\infty}\left([-te^{-t}]_0^c + \int_0^c e^{-t}\,dt\right) \\
&= -\lim_{c\to\infty}\frac{c}{e^c} - \lim_{c\to\infty}[e^{-t}]_0^c \\
&\overset{\text{L'H}}{\downarrow}{=} -\lim_{c\to\infty}\frac{1}{e^c} - \lim_{c\to\infty}(e^{-c}-1) \\
&= 1
\end{aligned}
$$

in the last step.

c. This time we will use the polar coordinates. To do so we first determine the region of integration R.

In the iterated integral the x-integral goes from $x = 0$ to $x = \sqrt{2y - y^2}$ as shown by the red line segment in the figure. Since $x = \sqrt{2y - y^2} \implies x^2 = 2y - y^2 \implies x^2 + (y-1)^2 = 1^2$, $x = \sqrt{2y - y^2}$ gives the right semicircle of the circle $x^2 + (y-1)^2 = 1$. On the other hand, the y-integral goes from $y = 0$ to $y = 2$. Therefore R is right half of the disk $x^2 + (y-1)^2 \le 1$. Note that the polar equation of the circle $x^2 + (y-1)^2 = 1$ is $r = 2\sin\theta$, and to obtain the right semicircle we vary θ from $\theta = 0$ to $\theta = \pi/2$.



Hence:

$$
\begin{aligned}
\int_0^2 \int_0^{\sqrt{2y-y^2}} \frac{xy}{x^2+y^2}\,dx\,dy &= \iint_R \frac{xy}{x^2+y^2}\,dA \\
&= \int_0^{\pi/2} \int_0^{2\sin\theta} \frac{r\cos\theta \cdot r\sin\theta}{r^2}\, r\,dr\,d\theta \\
&= \int_0^{\pi/2} \int_0^{2\sin\theta} \sin\theta\cos\theta\, r\,dr\,d\theta \\
&= \int_0^{\pi/2} \sin\theta\cos\theta \left[\frac{r^2}{2}\right]_{r=0}^{r=2\sin\theta} dr\,d\theta \\
&= \int_0^{\pi/2} 2\sin^3\theta\cos\theta\,d\theta \\
&= \left.\frac{\sin^4\theta}{2}\right]_0^{\pi/2} \\
&= \frac{1}{2}
\end{aligned}
$$

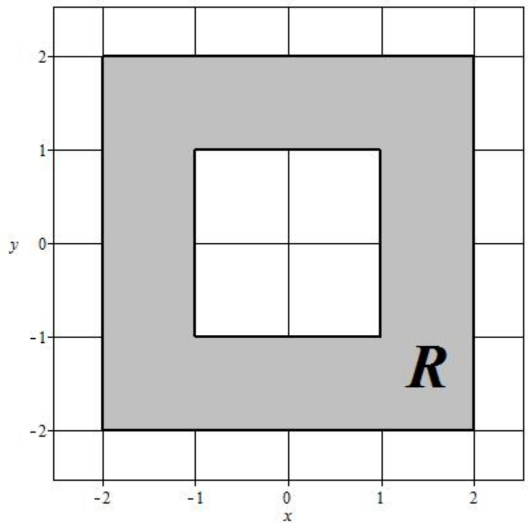
d. Again we use the polar coordinates. This time the integration region is $R = \{(x,y) : x \geq 0 \text{ and } y \geq 0\}$, that is, the first quadrant.

$$
\begin{aligned}
\int_0^\infty \int_0^\infty \frac{dy\,dx}{(x^2+y^2)^2+1} &= \iint_R \frac{1}{(x^2+y^2)^2+1}\,dA \\
&= \int_0^{\pi/2} \int_0^\infty \frac{r\,dr\,d\theta}{r^4+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\pi/2} \frac{1}{2} \lim_{c\to\infty} \left[\arctan(r^2)\right]_{r=0}^{r=c} \, d\theta \\
&= \int_0^{\pi/2} \frac{\pi}{4} \, d\theta \\
&= \frac{\pi^2}{8}
\end{aligned}
$$

□

27. Evaluate the double integral $\iint_R \frac{1}{(x^2+y^2)^2} \, dA$ where R is the region shown in the figure.



Solution: Let R' be the portion of the region lying in the first quadrant between the line $y = x$ and the x-axis. By symmetry we have:

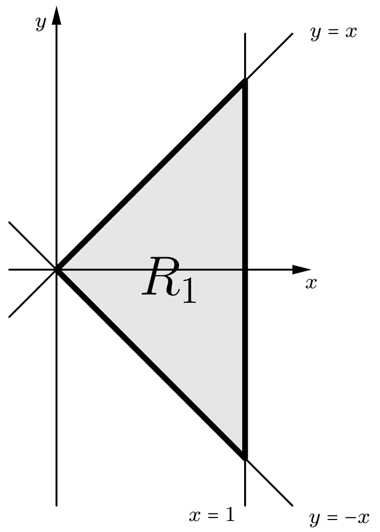
$$
\begin{aligned}
\iint_R \frac{1}{(x^2+y^2)^2} \, dA &= 8 \iint_{R'} \frac{1}{(x^2+y^2)^2} \, dA \\
&= 8 \int_0^{\pi/4} \int_{\sec\theta}^{2\sec\theta} \frac{1}{(r^2)^2} \, r \, dr \, d\theta \\
&= 8 \int_0^{\pi/4} \left[-\frac{1}{2r^2} \right]_{r=\sec\theta}^{r=2\sec\theta} d\theta \\
&= 3 \int_0^{\pi/4} \cos^2\theta \, d\theta \\
&= 3 \int_0^{\pi/4} \frac{1+\cos 2\theta}{2} \, d\theta \\
&= \frac{3}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/4} \\
&= \frac{3\pi}{8} + \frac{3}{4}
\end{aligned}
$$

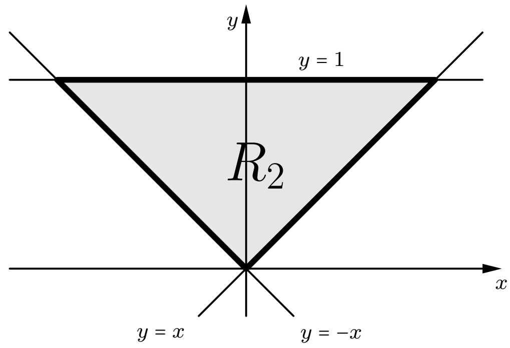
□

28. Evaluate the following integrals.

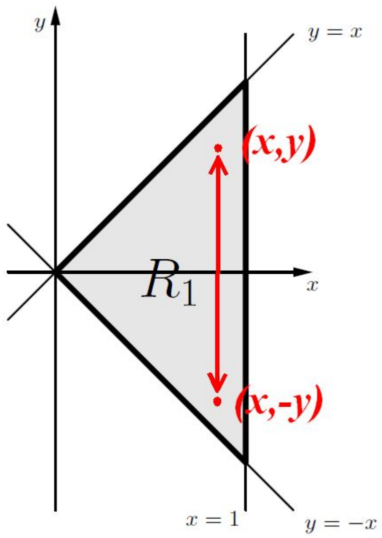
a. $\displaystyle\iint_{R_1} x^2 y\cos(\pi y^5/2)\,dA$ where R_1 is the region shown in the figure.

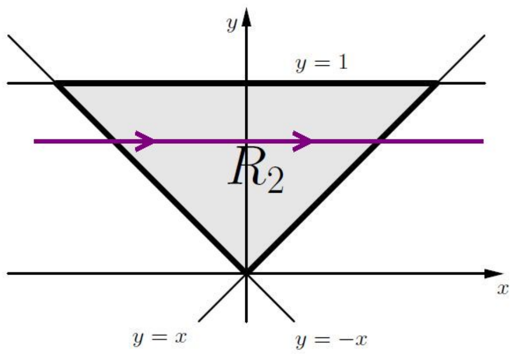
b. $\displaystyle\iint_{R_2} x^2 y\cos(\pi y^5/2)\,dA$ where R_2 is the region shown in the figure.





Solution: a. Note that the region R_1 is symmetric with respect to the x-axis, but the function $f(x,y) = x^2y\cos(\pi y^5/2)$ changes sign under reflection with respect to the x-axis: $f(x,-y) = -f(x,y)$. Therefore $\displaystyle\iint_{R_1} x^2y\cos(\pi y^5/2)\,dA = 0$.





b. We integrate with respect to x, then y:

$$\iint_{R_2} x^2y\cos(\pi y^5/2)\,dA = \int_0^1\int_{-y}^{y} x^2y\cos(\pi y^5/2)\,dx\,dy = \int_0^1 \left[\frac{1}{3}x^3y\cos(\pi y^5/2)\right]_{x=-y}^{x=y} dy$$

$$= \frac{2}{3}\int_0^1 y^4\cos(\pi y^5/2)\,dy = \frac{2}{3}\left[\frac{2}{5\pi}\sin(\pi y^5/2)\right]_0^1 = \frac{4}{15\pi}$$

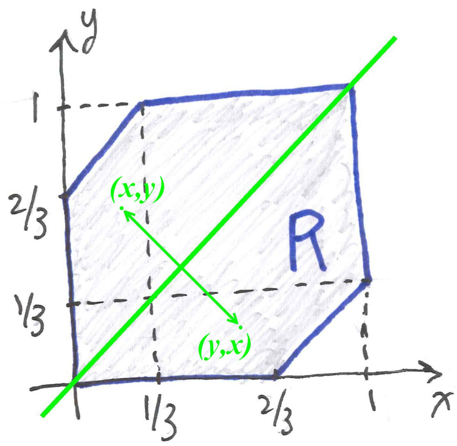
□

29. Evaluate the following integral $\iint_R (1+x-y)\,dA$ where $R = \{(x,y) : |x-y| \le 2/3 \text{ and } 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$.

Solution: We first observe that

$$\iint_R (1+x-y)\,dA = \iint_R dA + \iint_R (x-y)\,dA$$

and then note that the second integral on the right is zero because the region R is symmetric with respect to the line $y = x$ whereas the function $f(x,y) = x-y$ changes sign under the reflection with respect to this line: $f(y,x) = -f(x,y)$.



On the other hand the first on the right is just the area of R, that is $1^2 - (1/3)^2 = 8/9$. Hence:

$$\iint_R (1+x-y)\,dA = \iint_R dA + \iint_R (x-y)\,dA = \frac{8}{9} + 0 = \frac{8}{9}$$

□

30. When pirates retire, they live on the Square Island

$$S = \{(x,y) : |x| \le 5 \text{ and } |y| \le 5\}$$

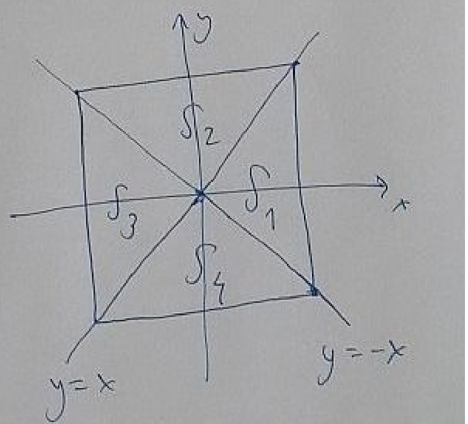
in the Sea of xy-plane where all distances are measured in hectometers (=hm). Having lived all their lives on it, the retired pirates want to be as far away from the sea as possible. As a result, the pirate population density $p(x,y)$ at a point (x,y) on the Square Island is proportional to the distance of the point from the shore and reaches its largest value of 15 pirate/hm^2 at the center of the island. Find the total number N of pirates living on the Square Island.

Solution:

$$p(x,y) = \left\{\begin{array}{ll} 15-3x & \text{if } -x \le y \le x \\ 15-3y & \text{if } -y \le x \le y \\ 15+3x & \text{if } x \le y \le -x \\ 15+3y & \text{if } y \le x \le -y \end{array}\right\} = 15 - \frac{3}{2}\cdot\left(|x-y|+|x+y|\right)$$

$$N = \iint_S p(x,y)\, dA$$

$$= 4 \iint_{S_1} p(x,y)\, dA \quad \text{(by symmetry!)}$$



$$= 4 \int_0^5 \int_{-x}^{x} (15-3x)\, dy\, dx$$

$$= 4 \int_0^5 \left[(15-3x)y\right]_{y=-x}^{y=x} dx = 8 \int_0^5 (15x-3x^2)\, dx = 8\left[\frac{15}{2}x^2 - x^3\right]_0^5 = 4\cdot 5\cdot 25 = 500 \text{ pirates}$$

$\square$

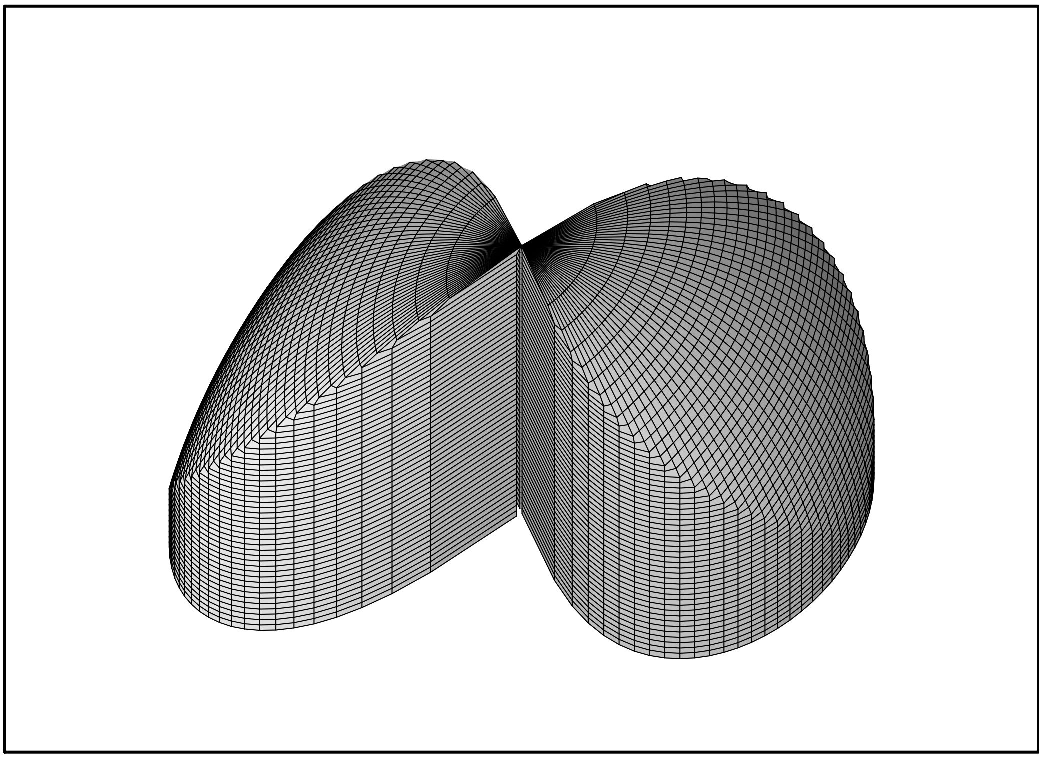
Remark: Compare this example with **Example 46** in **Part 1**.

31. The region enclosed by the lemniscate $r^2 = 2\cos(2\theta)$ in the plane is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume.

Solution: Let R be the the region enclosed by the lemniscate in the first quadrant. By symmetry the volume is

$$
\begin{aligned}
&4\iint_R \sqrt{2-r^2}\,dA = 4\int_0^{\pi/4}\int_0^{\sqrt{2\cos 2\theta}} \sqrt{2-r^2}\,r\,dr\,d\theta \\
&= 4\int_0^{\pi/4}\left[-\frac{1}{3}(2-r^2)^{3/2}\right]_{r=0}^{r=\sqrt{2\cos 2\theta}} d\theta = \frac{4}{3}\int_0^{\pi/4}\left(2^{3/2} - (2-2\cos 2\theta)^{3/2}\right) d\theta \\
&= \frac{8\sqrt{2}}{3}\int_0^{\pi/4}\left(1 - (2\sin^2\theta)^{3/2}\right) d\theta = \frac{2\sqrt{2}\pi}{3} - \frac{32}{3}\int_0^{\pi/4}\sin^3\theta\,d\theta \\
&= \frac{2\sqrt{2}\pi}{3} - \frac{32}{3}\int_0^{\pi/4}(1-\cos^2\theta)\sin\theta\,d\theta = \frac{2\sqrt{2}\pi}{3} - \frac{32}{3}\int_{1/\sqrt{2}}^{1}(1-u^2)\,du \\
&= \frac{2\sqrt{2}\pi}{3} - \frac{32}{3}\left[u - \frac{u^3}{3}\right]_{1/\sqrt{2}}^{1} = \frac{2\sqrt{2}\pi}{3} - \frac{32}{3}\left(\frac{2}{3} - \frac{5}{6\sqrt{2}}\right) \\
&= \frac{2\sqrt{2}\pi}{3} - \frac{64}{9} + \frac{40\sqrt{2}}{9}\,.
\end{aligned}
$$

□

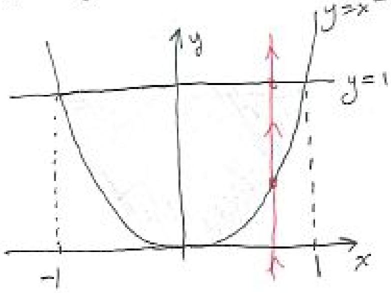


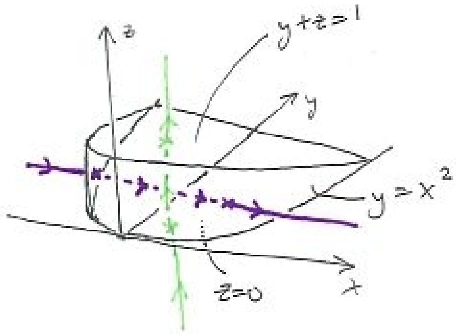
32. Let D be the region in space bounded by the plane $y + z = 1$ on the top, the parabolic cylinder $y = x^2$ on the sides, and the xy-plane at the bottom. Express the volume V of the region D in terms of iterated integrals with orders of integration **(a)** $dz\,dy\,dx$ and **(b)** $dx\,dy\,dz$.

Solution:

a.

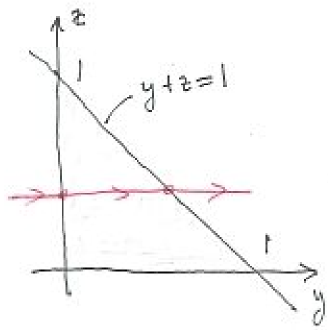
$$V = \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} dz\,dy\,dx$$





b.

$$V = \int_{0}^{1} \int_{0}^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz$$



$\square$

33. Consider the triple integral $\iiint_D \frac{1}{(x^2+y^2+z^2)^2}\, dV$ where D is the region bounded by the cylinder $x^2+y^2=1$ on the sides and by the hemisphere $z=\sqrt{4-x^2-y^2}$ at the bottom. Express this integral in terms of iterated integrals in

a. Cartesian coordinates,

b. cylindrical coordinates,

c. spherical coordinates, and

d. evaluate the integral in the coordinate system of your choice.

Solution: We observe that the projection of D to the xy-plane is the unit disk $x^2+y^2 \le 1$. A vertical line passing through a point of the unit disk enters the region at a point on the hemisphere and remains in the region from there on. The equations of the hemisphere $z=\sqrt{4-x^2-y^2}$ and the cylinder $x^2+y^2=1$ are $z=\sqrt{4-r^2}$ and $r=1$, respectively, in cylindrical coordinates. The answers to parts **(a)**,

$$\iiint_D \frac{1}{(x^2+y^2+z^2)^2}\, dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{4-x^2-y^2}}^{\infty} \frac{1}{(x^2+y^2+z^2)^2}\, dz\, dy\, dx\,,$$

and **(b)**,

$$\iiint_D \frac{1}{(x^2+y^2+z^2)^2}\, dV = \int_0^{2\pi}\int_0^1\int_{\sqrt{4-r^2}}^{\infty} \frac{1}{(r^2+z^2)^2}\, r\, dz\, dr\, d\theta\,,$$

immediately follow from these observations.

To do part **(c)** we further observe that

- the equations of the hemisphere and the cylinder are $\rho=2$ and $\rho\sin\phi=1$, respectively,
- a ray starting at the origin enters the region at a point on the hemisphere and leaves the region at a point on the cylinder, and
- such a ray intersects the region exactly when $0 \le \phi \le \pi/6$. (The ray passes through a point on the intersection circle when $\phi=\pi/6$.)

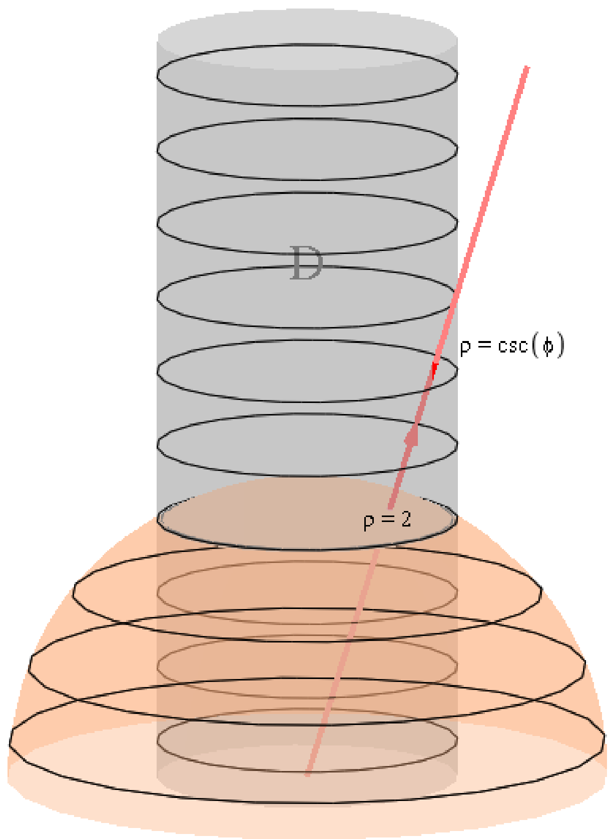
Therefore,

$$\iiint_D \frac{1}{(x^2+y^2+z^2)^2}\, dV = \int_0^{2\pi}\int_0^{\pi/6}\int_2^{\csc\phi} \frac{1}{\rho^4}\, \rho^2\sin\phi\, d\rho\, d\phi\, d\theta\,.$$

We use this last iterated integral for the computation **(d)**:

$$\begin{aligned}
\iiint_D \frac{1}{(x^2+y^2+z^2)^2}\, dV &= \int_0^{2\pi}\int_0^{\pi/6}\int_2^{\csc\phi} \frac{1}{\rho^2}\, \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/6} \left[-\frac{1}{\rho}\right]_{\rho=2}^{\rho=\csc\phi} \sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/6} \left(-\sin\phi+\frac{1}{2}\right) \sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/6} \left(-\frac{1-\cos 2\phi}{2}+\frac{\sin\phi}{2}\right) d\phi\, d\theta \\
&= \int_0^{2\pi} \left[-\frac{\phi}{2}+\frac{\sin 2\phi}{4}-\frac{\cos\phi}{2}\right]_{\phi=0}^{\phi=\pi/6} d\theta \\
&= \int_0^{2\pi} \left(-\frac{\pi}{12}+\frac{\sqrt{3}}{8}-\frac{\sqrt{3}}{4}+\frac{1}{2}\right) d\theta \\
&= \pi\left(1-\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)
\end{aligned}$$

□



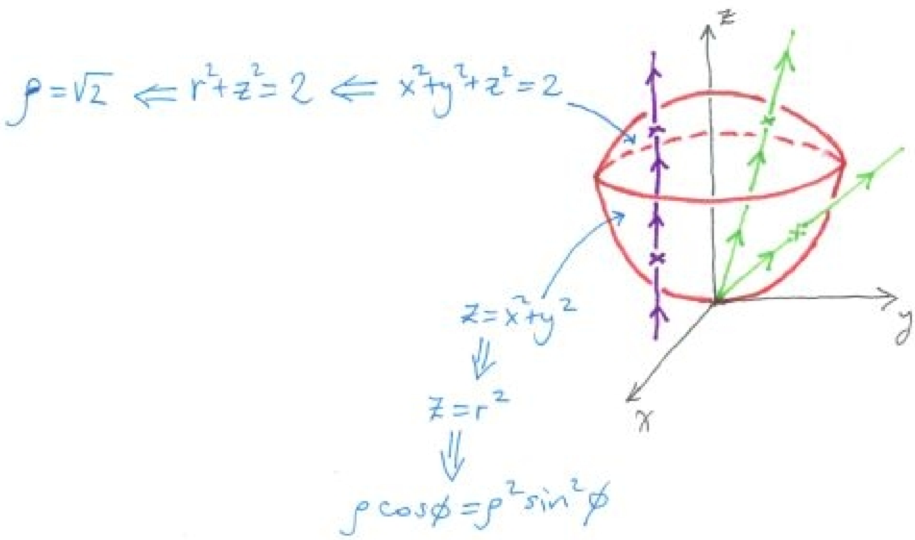
34. Let D be the region in space bounded on the top by the sphere $x^2 + y^2 + z^2 = 2$ and on the bottom by the paraboloid $z = x^2 + y^2$. Express the volume V of D in terms of iterated integrals in the **(a)** Cartesian, **(b)** cylindrical, and **(c)** spherical coordinates.

Solution: Note that the projection of the curve of intersection of the sphere and the paraboloid is the unit circle $x^2 + y^2 = 1$ in the xy-plane. This curve also bounds the projection of the solid D to the xy-plane. Hence we have

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dz\, dy\, dx$$

and

$$V = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r\, dz\, dr\, d\theta\,.$$



Now observe that a ray starting at the origin leaves D through the sphere if the angle it makes with the positive z-axis is less than $\pi/4$, whereas it does so through the paraboloid if this angle is between $\pi/4$ and $\pi/2$. Hence:

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta + \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\cos\phi/\sin^2\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

□

35. Consider the iterated integral

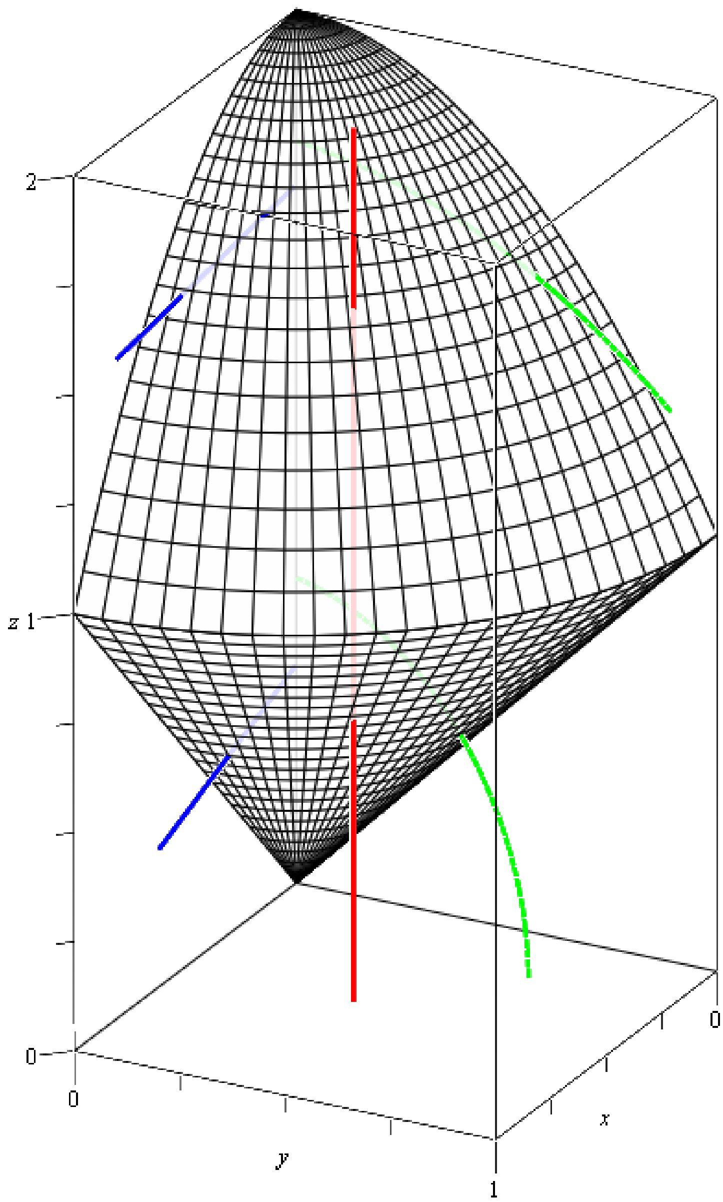
$$\int_0^{\pi/2}\int_0^1\int_r^{2-r^2} dz\,dr\,d\theta$$

in cylindrical coordinates.

a. Change the order of integration into $dr\,dz\,d\theta$.

b. Express the integral in spherical coordinates with order of integration $d\phi\,d\rho\,d\theta$.

Solution: a. $z = r$ is the cone $z^2 = x^2 + y^2$, and $z = 2 - r^2$ is the paraboloid $z = 2 - x^2 - y^2$. These surfaces intersect along a circle that is also the curve of intersection of the cylinder $r = 1$ and the horizontal plane $z = 1$. Since $0 \le r \le 1$ and $0 \le \theta \le \pi/2$, the integration region D is the region in the first octant bounded by the paraboloid on the top and the cone at the bottom.



When we integrate with respect to r first, we will be moving along a curve on which z and θ are constant. These two conditions describe a horizontal plane and a plane containing the z-axis, respectively. Therefore the r-integration is along a ray perpendicular to the z-axis like the blue ones in the figure. Such a ray starts on the z-axis in D, and leaves D through the cone if $0 \le z \le 1$, and through the paraboloid if $1 \le z \le 2$. Therefore,

$$\int_0^{\pi/2}\int_0^1\int_r^{2-r^2} dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1\int_0^z dr\,dz\,d\theta + \int_0^{\pi/2}\int_1^2\int_0^{\sqrt{2-z}} dr\,dz\,d\theta$$

where we used the fact that $z = 2 - r^2$ and $r \ge 0 \Longrightarrow r = \sqrt{2-z}$.

b. We have

$$\int_0^{\pi/2}\int_0^1\int_r^{2-r^2} dz\,dr\,d\theta = \iiint_D \frac{1}{r}\,dV$$

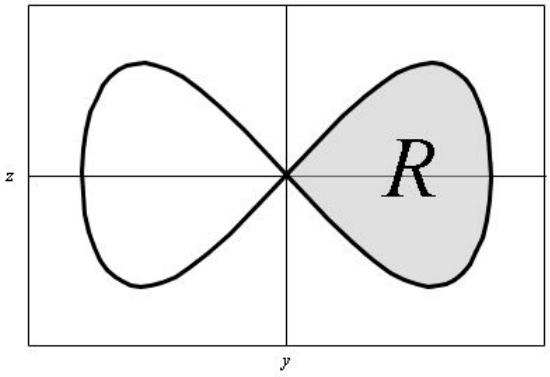
as $dV = r\,dz\,dr\,d\theta$ in cylindrical coordinates where D is the region described in part **(a)**.

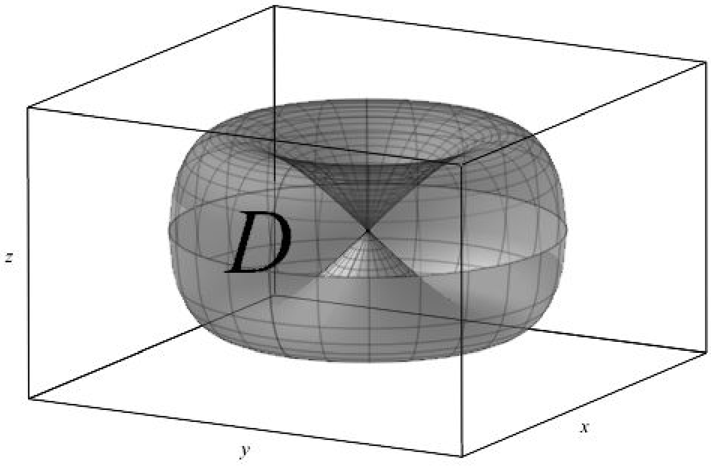
Since we want to integrate with respect to ϕ first, we will be moving along the curves on which ρ and θ are constant. These two conditions describe a sphere with center at the origin and a half-plane whose spine is the z-axis, respectively. Therefore the ϕ-integration takes place along a vertical semicircle subtended by a diameter along the z-axis and with center at the origin like the green ones in the figure. Such a semicircle starts on the positive z-axis in D, and leave the region of integration intersecting the cone for $0 \le \rho \le \sqrt{2}$ and the paraboloid for $\sqrt{2} \le \rho \le 2$. The upper half of the cone $z = r$ has the equation $\phi = \pi/4$ in spherical coordinates. On the other hand, $z = 2 - r^2 \Longrightarrow \rho\cos\phi = 2 - (\rho\sin\phi)^2 \Longrightarrow \cos\phi = (1 + \sqrt{4\rho^2 - 7})/(2\rho)$ for a point on the paraboloid as $\phi \le \pi/4$. Therefore,

$$\begin{aligned}\iiint_D \frac{1}{r}\,dV &= \iiint_D \frac{1}{\rho\sin\phi}\,dV \\ &= \int_0^{\pi/2}\int_0^{\sqrt{2}}\int_0^{\pi/4} \rho\,d\phi\,d\rho\,d\theta \\ &\quad + \int_0^{\pi/2}\int_{\sqrt{2}}^2\int_0^{\arccos((1+\sqrt{4\rho^2-7})/(2\rho))} \rho\,d\phi\,d\rho\,d\theta\end{aligned}$$

where we substituted $dV = \rho^2 \sin\phi\; d\phi\,d\rho\,d\theta$ for the volume element in spherical coordinates. $\square$

36. The region R bounded by the curve $z^2 = y^2 - y^4$ in the right half of the yz-plane is rotated about the z-axis to obtain a solid D in the xyz-space. Express the volume V of D in terms of iterated integrals in **(a)** the Cartesian, **(b)** the cylindrical and **(c)** the spherical coordinates.





Solution: Since the surface bounding the the solid D is obtained by revolving a curve about the z-axis, its equation in the cylindrical coordinates will not depend on θ. Since $r = y$ in the yz-plane and the curve has the equation $z^2 = y^2 - y^4$, we conclude that $z^2 = r^2 - r^4$ is the equation of the surface in the cylindrical coordinates. From this

$$V = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{r^2-r^4}}^{\sqrt{r^2-r^4}} r \, dz \, dr \, d\theta$$

follows as the projection of D to the xy-plane is the unit disk.

The equation $z^2 = r^2 - r^4$ in the cylindrical coordinates transforms to $z^2 = (x^2 + y^2) - (x^2 + y^2)^2$ in the Cartesian coordinates, and once again using the fact that the projection of D to the xy-plane is the unit disk, one now obtains:

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{(x^2+y^2)-(x^2+y^2)^2}}^{\sqrt{(x^2+y^2)-(x^2+y^2)^2}} dz \, dy \, dx$$

The spherical coordinates require a little bit more work. First note that the equation $z^2 = r^2 - r^4$ in the cylindrical coordinates now gives $(\rho \cos\phi)^2 = (\rho \sin\phi)^2 - (\rho \sin\phi)^4$, or $\rho = \sqrt{\sin^2\phi - \cos^2\phi}/\sin^2\phi$, in the spherical coordinates. Next note that for $\sin^2\phi - \cos^2\phi = -\cos 2\phi$ to be nonnegative, ϕ must be between $\pi/4$ and $3\pi/4$ in the interval $[0, \pi]$. Therefore:

$$V = \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_0^{\sqrt{\sin^2\phi - \cos^2\phi}/\sin^2\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

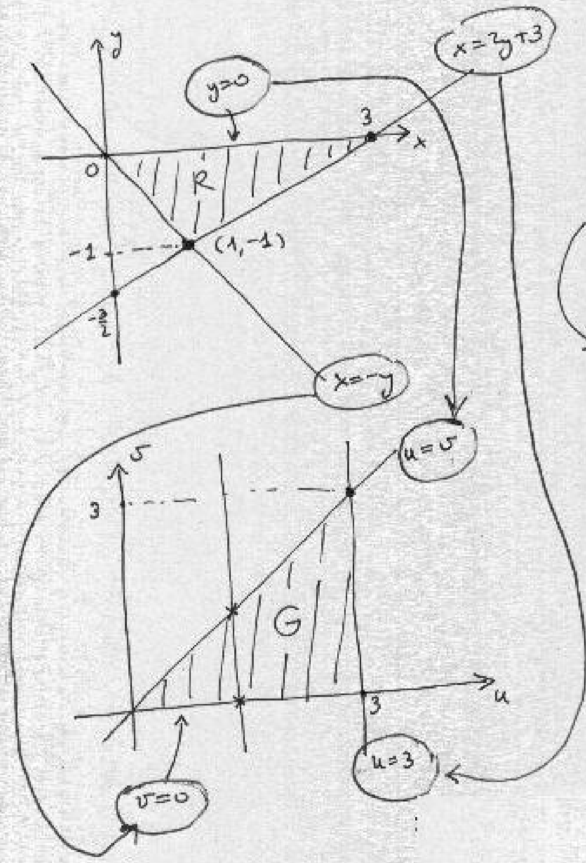
□

Remark: Compare this example with **Example 51** in **Part 1**.

37. Evaluate the iterated integral $\displaystyle\int_{-1}^{0}\int_{-y}^{2y+3} \frac{x+y}{(x-2y)^2}\, e^{x-2y}\,dx\,dy\,.$

Solution:

④ $$I = \int_{-1}^{0}\int_{-y}^{2y+3} \frac{x+y}{(x-2y)^2}\, e^{x-2y}\,dx\,dy = \iint_R \frac{x+y}{(x-2y)^2}\, e^{x-2y}\,dA$$



We'll try the following change of variables:

$$\begin{cases} u = x-2y \\ v = x+y \end{cases} \qquad \begin{cases} x = \dfrac{u+2v}{3} \\[2mm] y = \dfrac{v-u}{3} \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9}+\frac{2}{9} = \frac{1}{3}$$

$$I = \iint_G \frac{v}{u^2}\, e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA = \frac{1}{3}\int_0^3\int_0^u \frac{v}{u^2}\, e^u\,dv\,du = \frac{1}{6}\int_0^3 e^u\,du = \frac{e^3-1}{6}\,.$$

□

38. Let (u,v) and (x,y) be two coordinate systems. Show that $\dfrac{\partial(x,y)}{\partial(u,v)}\cdot\dfrac{\partial(u,v)}{\partial(x,y)} = 1\,.$

Solution: By definition,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u\,y_v - x_v\,y_u$$

and

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x\,v_y - u_y\,v_x \quad .$$

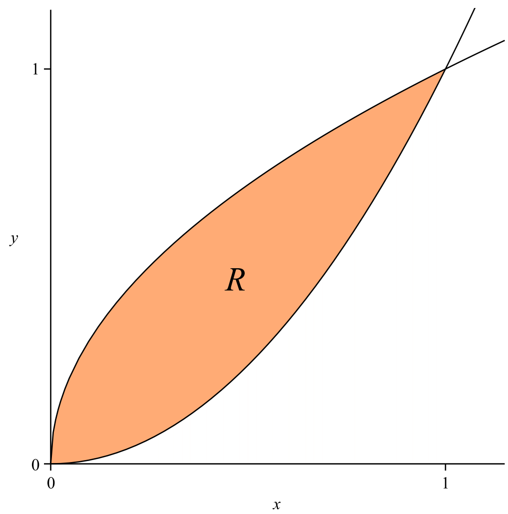
Therefore,

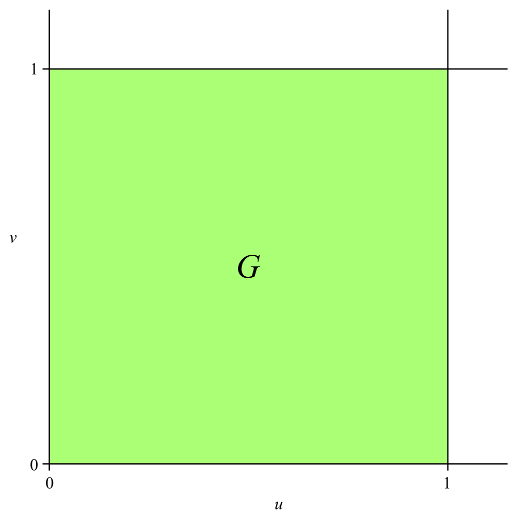
$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)} &= (x_u\,y_v - x_v\,y_u)(u_x\,v_y - u_y\,v_x)\\
&= x_u\,u_x\,y_v\,v_y + x_v\,v_x\,y_u\,u_y - x_u\,u_y\,y_v\,v_x - x_v\,u_x\,y_u\,v_y\\
&= x_u\,u_x\,y_v\,v_y + x_v\,v_x\,y_u\,u_y - x_u\,u_y\,y_v\,v_x - x_v\,u_x\,y_u\,v_y\\
&\quad + x_v\,v_x\,y_v\,v_y + x_u\,u_x\,y_u\,u_y - x_v\,v_x\,y_v\,v_y - x_u\,u_x\,y_u\,u_y\\
&= (x_u\,u_x + x_v\,v_x)\,(y_u\,u_y + y_v\,v_y)\\
&\qquad - (x_u\,u_y + x_v\,v_y)\,(y_u\,u_x + y_v\,v_x)\\
&= x_x\,y_y - x_y\,y_x\\
&= 1\cdot 1 - 0\cdot 0\\
&= 1
\end{aligned}$$

□

39. Evaluate the double integral $\displaystyle\iint_R e^{x^2/y}\,dA$ where $R = \{(x,y) : x^2 \le y \le \sqrt{x}\}$.

Solution: We change the coordinates to $u = y^2/x$, $v = x^2/y$. In this coordinate system the region of integration becomes $G = \{(u,v) : 0 \le u \le 1 \text{ and } 0 \le v \le 1\}$.





We have

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -y^2/x^2 & 2y/x \\ 2x/y & -x^2/y^2 \end{vmatrix} = -3 \Longrightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = -\frac{1}{3}\,,$$

and the change of variables formula gives

$$\iint_R e^{x^2/y}\,dx\,dy = \iint_G e^v\,\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv = \int_0^1\int_0^1 e^v\left|-\frac{1}{3}\right|\,dv\,du = \frac{e-1}{3}\,.$$

□

Remark: We can multiply differentials. The rules are $du\,du = 0 = dv\,dv$ and $dv\,du = -du\,dv$ in any uv-coordinate system. For instance, when changing from Cartesian to polar coordinates, we have

$$\begin{aligned} dx\,dy &= d(r\,\cos\theta)\,d(r\,\sin\theta) \\ &= (\cos\theta\,dr - r\,\sin\theta\,d\theta)(\sin\theta\,dr + r\,\cos\theta\,d\theta) \\ &= -r\,\sin^2\theta\,d\theta\,dr + r\,\cos^2\theta\,dr\,d\theta \\ &= r(\sin^2\theta + \cos^2\theta)\,dr\,d\theta \\ &= r\,dr\,d\theta \end{aligned}$$

and in **Example 39** we have

$$\begin{aligned} du\,dv &= d(y^2/x)\,d(x^2/y) \\ &= (2y/x\,dy - y^2/x^2\,dx)(2x/y\,dx - x^2/y^2\,dy) \\ &= 4\,dy\,dx + dx\,dy \\ &= -4\,dx\,dy + dx\,dy \\ &= -3\,dx\,dy\,, \end{aligned}$$

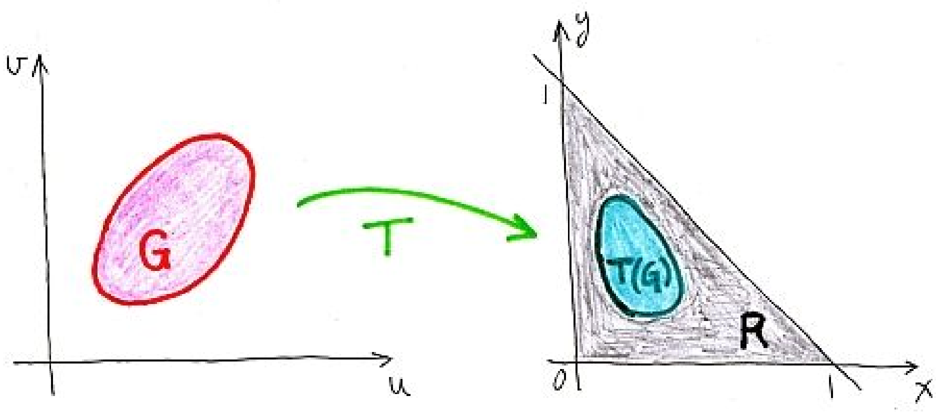
hence $dx\,dy = -\dfrac{1}{3}\,du\,dv$. This can be used to keep track of how the area element changes under a coordinate change, but note that the sign of the factor in front must be corrected by hand so that it is positive on the region of integration.

40. Consider the transformation $T: x = \dfrac{u}{u+v+1}\,,\; y = \dfrac{v}{u+v+1}$. Show that there is a constant C such that the inequality

$$\iint_G \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \le C$$

holds for all regions G contained in the first quadrant of the uv-plane.

Solution: Observe that $x = \dfrac{u}{u+v+1} \ge 0$, $y = \dfrac{v}{u+v+1} \ge 0$ and $x+y = \dfrac{u+v}{u+v+1} \le 1$ for $u \ge 0$ and $v \ge 0$. We also have $u = \dfrac{x}{1-x-y}$ and $v = \dfrac{y}{1-x-y}$. Therefore T maps the first quadrant of the uv-plane into the triangle $R = \{(x,y) : x+y \le 1, x \ge 0, y \ge 0\}$ in a one-to-one manner.



Hence:

$$\iint_G \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_{T(G)} dx\,dy = (\text{Area of } T(G)) \le (\text{Area of } R) = \frac{1}{2}$$

□

Remark: One can also do a straightforward computation.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} (v+1)/(u+v+1)^2 & -u/(u+v+1)^2 \\ -v/(u+v+1)^2 & (u+1)/(u+v+1)^2 \end{vmatrix} = \frac{1}{(u+v+1)^3}$$

and hence

$$\iint_G \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \le \int_0^\infty \int_0^\infty \frac{du\,dv}{(u+v+1)^3} = \frac{1}{2}\int_0^\infty \frac{dv}{(v+1)^2} = \frac{1}{2}.$$

41. Compute the Jacobian $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}$ where (ρ,ϕ,θ) are the spherical coordinates and (x,y,z) are the Cartesian coordinates.

Solution: We have $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. Therefore

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix} \\
&= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= (-\rho\sin\phi\sin\theta)\begin{vmatrix} \sin\phi\sin\theta & \rho\cos\phi\sin\theta \\ \cos\phi & -\rho\sin\phi \end{vmatrix} \\
&\qquad - (\rho\sin\phi\cos\theta)\begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta \\ \cos\phi & -\rho\sin\phi \end{vmatrix} \\
&\qquad\qquad + 0\cdot\begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta \end{vmatrix} \\
&= (-\rho\sin\phi\sin\theta)(-\rho\sin\theta)(\sin^2\phi+\cos^2\phi) \\
&\qquad - (\rho\sin\phi\cos\theta)(-\rho\cos\theta)(\sin^2\phi+\cos^2\phi) \\
&\qquad\qquad + 0 \\
&= \rho^2\sin\phi\,(\sin^2\theta+\cos^2\theta) \\
&= \rho^2\sin\phi
\end{aligned}$$

where we used the cofactor expansion with respect to the third column. □

Part 3: Sequences and Series

1. Let $a_1 = 1$, $a_2 = a_3 = 2$, $a_4 = a_5 = a_6 = 3$, $a_7 = a_8 = a_9 = a_{10} = 4$, and so on. That is, $a_n : 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, \dots$. What is a_{2017}?

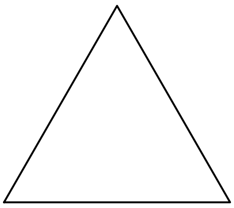
> **Solution:** $a_n = k$ if $1 + 2 + \cdots + (k-1) < n \le 1 + 2 + \cdots + k$. In other words, $a_n = k$ if $k(k-1)/2 < n \le k(k+1)/2$. Since for $k = 64$, $k(k-1)/2 = 2016$ and $k(k+1)/2 = 2080$, we have $a_{2017} = 64$. $\square$

Remark: A more explicit formula $a_n = \left\lceil \dfrac{\sqrt{8n+1} - 1}{2} \right\rceil$ can be obtained.

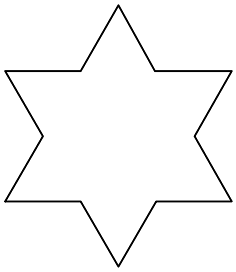
2. *Curve*$_1$ is an equilateral triangle with unit sides. For $n \ge 2$, *Curve*$_n$ is obtained from *Curve*$_{n-1}$ by replacing the middle third of every edge with the other two sides of the outward pointing equilateral triangle sitting on it. Let L_n be the length of *Curve*$_n$ and A_n be the area of the region enclosed by *Curve*$_n$. Find L_n, A_n, $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} A_n$.

> **Solution:** Let e_n and d_n denote the number of edges and the length of each edge of *Curve*$_n$, respectively. Since $e_n = 4e_{n-1}$ and $d_n = d_{n-1}/3$ for $n \ge 2$, and $e_1 = 3$ and $d_1 = 1$, we find that $e_n = 3 \cdot 4^{n-1}$ and $d_n = 1/3^{n-1}$ for $n \ge 1$. It follows that $L_n = e_n d_n = 3 \cdot (4/3)^{n-1}$ for $n \ge 1$, and $\lim_{n\to\infty} L_n = \lim_{n\to\infty} 3 \cdot (4/3)^{n-1} = \infty$.
>
> On the other hand, the n^{th} region is obtained by adjoining e_{n-1} equilateral triangles of side length $d_{n-1}/3$ to the $(n-1)^{\text{st}}$ region. Therefore,
>
> $$A_n = A_{n-1} + e_{n-1} \cdot \frac{\sqrt{3}\,(d_{n-1}/3)^2}{4} = A_{n-1} + 3 \cdot 4^{n-2} \cdot \frac{\sqrt{3}}{4} \left(\frac{1}{3^{n-1}} \right)^2$$
>
> and then
>
> $$A_n = \frac{1}{4\sqrt{3}} \left(\left(\frac{4}{9}\right)^{n-2} + \left(\frac{4}{9}\right)^{n-3} + \cdots + 1 \right) + A_1$$
>
> for $n \ge 2$. Since $A_1 = \dfrac{\sqrt{3}}{4}$, this gives
>
> $$A_n = \frac{3\sqrt{3}}{20} \left(1 - \left(\frac{4}{9}\right)^{n-1} \right) + \frac{\sqrt{3}}{4}$$
>
> for $n \ge 1$. We obtain
>
> $$\lim_{n\to\infty} A_n = \frac{3\sqrt{3}}{20} + \frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{5}\,.$$

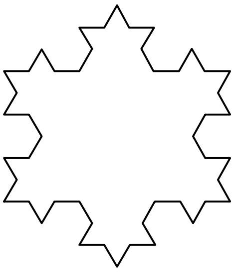
$Curve_1$:



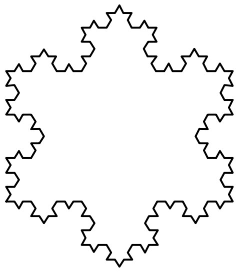
$Curve_2$:



$Curve_3$:



$Curve_4$:



□

Remark: A magician puts 1 ball into an empty box at $t = 0$ sec, takes out 1 ball at $t = 1/2$ sec, puts 2 balls at $t = 3/4$ sec, takes out 1 ball at $t = 7/8$ sec, puts 3 balls at $t = 15/16$ sec, takes out 1 ball at $t = 31/32$ sec, and so on. Then she challenges you to guess how many balls there are in the box at $t = 1$ sec.

You ask for the advice of your friends.

> *Friend1* says: "The net result of $2k + 1$st and $2k + 2$nd steps for $k \geq 1$ is to put at least one more ball into the box. There are infinitely many balls in the box at $t = 1$ sec."
>
> *Friend2* says: "Imagine that the balls are labeled as **ball1**, **ball2**, **ball3**, and so on with invisible ink which only the magician can see. Then she puts **ball1** at $t = 0$ sec, takes out **ball1** at $t = 1/2$ sec, puts **ball2** and **ball3** at $t = 3/4$ sec, takes out **ball2** at $t = 7/8$ sec, puts **ball4**, **ball5**, and **ball6** at $t = 15/16$ sec, takes out **ball3** at $t = 31/32$ sec, and so on. For any given k, **ball*k*** is not in the box at time $t = 1$ sec, because it was taken out at time $t = 1 - 1/2^{2k-1}$ sec. There are no balls in the box at time $t = 1$ sec."

What do you think?

3. Let the sequence $\{a_n\}$ be defined by $a_1 = 1$ and $a_n = \dfrac{1}{1 + a_{n-1}}$ for $n \geq 1$. Show that the sequence converges and find its limit.

Solution: Observe that if $0 < a_n < a_{n+1}$, then $0 < 1 + a_n < 1 + a_{n+1}$ and $a_{n+1} = 1/(1 + a_n) > 1/(1 + a_{n+1}) = a_{n+2}$. Similarly, if $a_n > a_{n+1} > 0$, then $1 + a_n > 1 + a_{n+1} > 0$ and $a_{n+1} = 1/(1 + a_n) < 1/(1 + a_{n+1}) = a_{n+2}$. Since $a_1 = 1 > 1/2 = a_2$, it follows that

$$1 = a_1 > a_3 > a_5 > \cdots > a_6 > a_4 > a_2 = \frac{1}{2}\,.$$

Therefore, the sequence $\{a_{2n}\}_{n=1}^{\infty}$ is increasing and bounded from above by 1, and the sequence $\{a_{2n-1}\}_{n=1}^{\infty}$ is decreasing and bounded from below by 1/2. By the Monotonic Sequence Theorem, then both of these sequences are convergent.

Let $\lim\limits_{n\to\infty} a_{2n-1} = L$ and $\lim\limits_{n\to\infty} a_{2n} = M$. Taking the limit of $a_{2n+1} = 1/(1+a_{2n})$ as $n \to \infty$ we obtain $L = 1/(1 + M)$, and taking the limit of $a_{2n} = 1/(1 + a_{2n-1})$ as $n \to \infty$ we obtain $M = 1/(1 + L)$. From $M + ML = 1$ and $LM + L = 1$ it follows that $M = L$. Therefore the sequence $\{a_n\}_{n=1}^{\infty}$ converges to $L = M$.

Finally, $L^2 + L - 1 = 0$ gives $L = (\sqrt{5} - 1)/2$ or $L = -(\sqrt{5} + 1)/2$. Since $0 < a_n$ for all $n \geq 1$, we must have $0 \leq L$. Hence $L = (\sqrt{5} - 1)/2$ is the limit and

$$\lim_{n\to\infty} a_n = \frac{\sqrt{5} - 1}{2}\,.$$

□

Remark: This result is sometimes expressed symbolically in the form:

$$\frac{\sqrt{5}-1}{2} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}$$

Remark: Note that $x_n = F_n/F_{n+1}$ for $n \geq 1$ where $F_n : 1, 1, 2, 3, 5, 8, \ldots$ is the Fibonacci sequence. So we showed that $\lim\limits_{n\to\infty} F_n/F_{n+1} = (\sqrt{5} - 1)/2$.

Remark: It can be showed with a little bit more work that any sequence satisfying the given recursion relation converges to $(\sqrt{5} - 1)/2$ if $a_1 \neq -(1 + \sqrt{5})/2$ and $a_1 \neq -F_{n+1}/F_n$ for any $n \geq 1$. On the other hand, if $a_1 = -(1 + \sqrt{5})/2$, then the sequence is constant; and if $a_1 = -F_{n+1}/F_n$ for some $n \geq 1$, then $a_n = -1$ and a_{n+1} is undefined.

4. Let the sequence $\{x_n\}$ be defined by $x_0 = 2$ and $x_n = \dfrac{x_{n-1}}{2} + \dfrac{1}{x_{n-1}}$ for $n \geq 1$.

a. Find the limit of this sequence assuming it exists.

b. Show that the limit exists.

Solution: a. We assume that the limit $L = \lim x_n$ exists. Then

$$x_n = \frac{x_{n-1}}{2} + \frac{1}{x_{n-1}} \text{ for } n \geq 1 \Longrightarrow x_n x_{n-1} = \frac{x_{n-1}^2}{2} + 1 \text{ for } n \geq 1$$

$$\Longrightarrow \lim(x_n x_{n-1}) = \lim \frac{x_{n-1}^2}{2} + 1 \Longrightarrow L^2 = \frac{L^2}{2} + 1 \Longrightarrow L^2 = 2$$

as $\lim x_{n-1} = \lim x_n = L$. Therefore $L = \sqrt{2}$ or $L = -\sqrt{2}$.

$x_0 = 2 > 0$ and if $x_n > 0$ then $x_{n+1} = x_n/2 + 1/x_n > 0 + 0 = 0$. Hence by induction $x_n > 0$ for all $n \geq 0$. It follows that $L = \lim x_n \geq 0$. We conclude that the limit is $\sqrt{2}$.

b. In part **(a)** we proved that $x_n > 0$ for all $n \geq 0$. Therefore $\{x_n\}$ is bounded from below.

Now we will show by induction on n that $\sqrt{2} < x_{n+1} < x_n$ for all $n \geq 0$.

- Let $n = 0$. Since $x_0 = 2$, $x_1 = 3/2$, and $x_1^2 = 9/4 > 2$, we have $\sqrt{2} < x_1 < x_0$.
- Let $n > 0$ and assume that $\sqrt{2} < x_{n+1} < x_n$. Then

$$x_{n+1} - x_{n+2} = \frac{x_{n+1}}{2} - \frac{1}{x_{n+1}} = \frac{x_{n+1}^2 - 2}{2x_{n+1}} > 0$$

  and hence $x_{n+2} < x_{n+1}$. On the other hand,

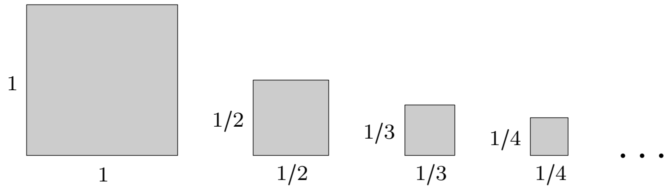
$$x_{n+2}^2 - 2 = \left(\frac{x_{n+1}}{2} + \frac{1}{x_{n+1}}\right)^2 - 2 = \left(\frac{x_{n+1}^2 - 2}{2x_{n+1}}\right)^2 > 0$$

  and since $x_{n+2} > 0$, we have $x_{n+2} > \sqrt{2}$.

Therefore the sequence is decreasing.

Since the sequence is bounded from below and decreasing, we conclude that it converges by the Monotonic Sequence Theorem. □

5. We have a $1/n \times 1/n$ square for each positive integer n. For each of **(a-c)**, determine whether it is possible or not to place these squares in the xy-plane in such a way that they completely cover the given set.



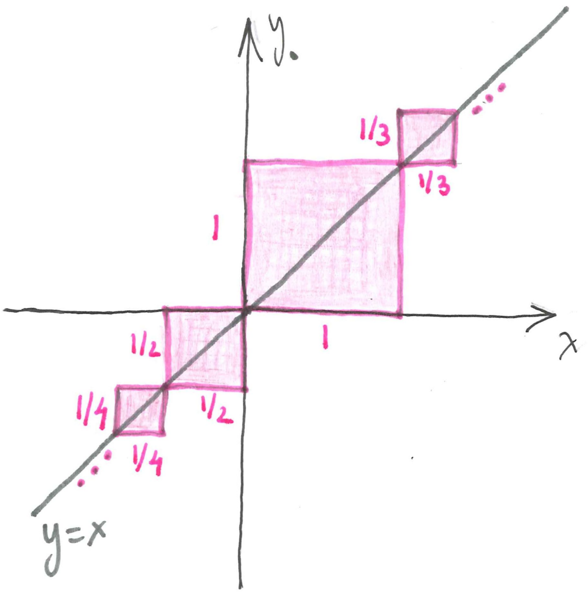
a. The entire xy-plane

b. The line defined by the equation $y = x$

c. The region between the graph of $y = e^{-x}$ and the x-axis for $x \geq 0$

Solution: a. The sum of the areas of the squares is $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This series is convergent as it is the p-series with $p = 2 > 1$. (In fact, its sum is $\pi^2/6$.) On the other hand, the entire plane has infinite area. Hence it is not possible to cover the entire plane using these squares.

b. We place the odd-numbered squares along the half of the line lying in the first quadrant and the even-numbered ones along the half lying in the third quadrant as shown in the figure. As $\sum_{k=1}^{\infty} \frac{\sqrt{2}}{2k} = \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \frac{1}{k} = \infty$ and $\sum_{k=1}^{\infty} \frac{\sqrt{2}}{2k-1} > \sum_{k=1}^{\infty} \frac{\sqrt{2}}{2k} = \infty$, the entire line is covered by the squares.



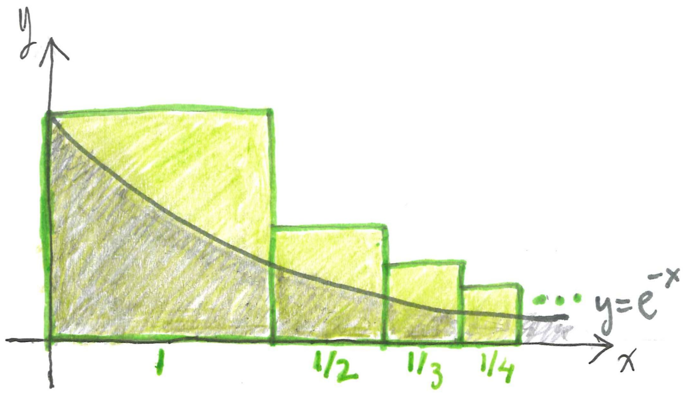
c. We place the squares along the positive x-axis as shown in the figure. Then the squares extend along the entire positive axis as $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. On the other hand, by the Integral Test Inequality we have

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}>\int_1^{n+1}\frac{dx}{x}=\ln(n+1)$$

for all $n\geq 1$. (This inequality can be obtained by comparing the area covered by rectangles erected on the unit intervals along the interval $[1,n+1]$ with upper left corners on the graph of $y=1/x$ and the area under the graph on this interval.) Hence

$$e^{-x}\leq e^{-(1+1/2+1/3+\cdots+1/n)}\leq\frac{1}{n+1}$$

for $\sum_{k=1}^{n}\frac{1}{k}\leq x\leq\sum_{k=1}^{n+1}\frac{1}{k}$ for all $n\geq 0$. In other words, the top side of the nth rectangle never lies lower than the graph of $y=e^{-x}$. Therefore we can cover the region between the graph of $y=e^{-x}$ and the positive x-axis with the squares.



□

6. For each of the series in **(a-d)**, determine whether there exists a positive integer n such that the nth partial sum s_n of the series satisfies the condition $2014\leq s_n\leq 2015$.

a. $\sum_{n=1}^{\infty}\frac{1}{2^n}$ **b.** $\sum_{n=1}^{\infty}5^n$ **c.** $\sum_{n=1}^{\infty}\left(\frac{2999}{3000}\right)^n$ **d.** $\sum_{n=1}^{\infty}\frac{1}{n}$

Solution: Note that all terms of these series are positive, and therefore, their partial sums form increasing sequences. This will be used repeatedly in the following.

a. We have

$$s_n=\sum_{k=1}^{n}\frac{1}{2^k}<\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1/2}{1-1/2}=1<2014$$

for all $n\geq 1$, where we used the geometric series sum formula, and hence there are no partial sums lying between 2014 and 2015.

b. This time we have

$$s_n=\sum_{k=1}^{n}5^k\leq\sum_{k=1}^{4}5^k=5+25+125+625=780<2014$$

for $n \le 4$, and

$$s_n = \sum_{k=1}^{n} 5^k \ge \sum_{k=1}^{5} 5^k = 5 + 25 + 125 + 625 + 3125 = 3905 > 2015$$

for $n \ge 5$. Therefore no s_n lies between 2014 and 2015.

c. Every term of this series is between 0 and 1, hence its partial sums increase in steps smaller than 1 starting with $s_1 = 2999/3000 < 2014$. Moreover,

$$\sum_{n=1}^{\infty} \left(\frac{2999}{3000}\right)^n = \frac{2999/3000}{1 - 2999/3000} = 2999 > 2015$$

by the geometric series sum formula. Hence there is at least one partial sum s_n satisfying the condition $2014 \le s_n \le 2015$.

d. The same reasoning as in part **(c)** works, this time with the observation that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty > 2015\,,$$

to give the existence of at least one partial sum s_n satisfying the condition $2014 \le s_n \le 2015$. □

7. Determine whether each of the following series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} (2^{1/n} - 2^{1/(n+1)})$ **b.** $\displaystyle\sum_{n=2}^{\infty} \frac{n \ln n}{3^n}$ **c.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \cos\left(\frac{\pi}{n}\right)$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ **e.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}\sin n}$ **f.** $\displaystyle\sum_{n=1}^{\infty} \frac{5^n - 2^n}{7^n - 6^n}$

g. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$ **h.** $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

Solution: a. We have $s_n = \displaystyle\sum_{k=1}^{n} (2^{1/k} - 2^{1/(k+1)}) = \sum_{k=1}^{n} 2^{1/k} - \sum_{k=1}^{n} 2^{1/(k+1)} = 2 - 2^{1/(n+1)}$. Hence $\displaystyle\lim_{n\to\infty} s_n = 1$. Therefore the series converges and $\displaystyle\sum_{n=1}^{\infty} (2^{1/n} - 2^{1/(n+1)}) = 1$.

b. We have $a_n = n \ln n / 3^n$ and

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)\ln(n+1)/3^{n+1}}{n \ln n / 3^n} = \frac{1}{3} \cdot \lim_{n\to\infty} \frac{n+1}{n} \cdot \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n}$$

$$= \frac{1}{3} \cdot 1 \cdot \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \overset{\text{L'H}}{=} \frac{1}{3} \cdot \lim_{x\to\infty} \frac{1/(x+1)}{1/x} = \frac{1}{3} \cdot 1 = \frac{1}{3}\,.$$

Since $\rho = 1/3 < 1$, the series $\displaystyle\sum_{n=2}^{\infty} \frac{n \ln n}{3^n}$ converges by the Ratio Test.

c. Since $a_n = (-1)^{n+1}\cos(\pi/n)$, we have $|a_n| = \cos(\pi/n)$ for $n \geq 2$, and $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \cos(\pi/n) = \cos 0 = 1$. Therefore $\lim_{n\to\infty} a_n \neq 0$, and the series diverges by the nth Term Test.

d. We have $a_n = n^n/n!$ and

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e\,.$$

Since $\rho = e > 1$, the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges by the Ratio Test.

e. We observe that

$$L = \lim_{n\to\infty} \frac{1/(n + \sqrt{n}\sin n)}{1/n} = \lim_{n\to\infty} \frac{1}{1 + \dfrac{\sin n}{\sqrt{n}}} = \frac{1}{1+0} = 1\,.$$

Since $0 < L < \infty$ and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we conclude that the series $\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}\sin n}$ diverges by the Limit Comparison Test.

f. The geometric series $\sum_{n=1}^{\infty} (5/7)^n$ converges as $r = 5/7 \Longrightarrow |r| = 5/7 < 1$. We also have $L = \lim_{n\to\infty} \frac{(5^n - 2^n)/(7^n - 6^n)}{5^n/7^n} = \lim_{n\to\infty} \frac{1 - (2/5)^n}{1 - (6/7)^n} = \frac{1-0}{1-0} = 1$. Since $0 < L < \infty$, the series $\sum_{n=1}^{\infty} \frac{5^n - 2^n}{7^n - 6^n}$ converges by the Limit Comparison Test.

g. Since $\ln x$ is an increasing function on $(0,\infty)$, $\ln(\ln x)$ is an increasing function on $(1,\infty)$. Both $\ln x$ and $\ln(\ln x)$ are positive on $[3,\infty)$. Therefore $x \ln x \ln(\ln x)$ is positive and increasing on $[3,\infty)$ as it is the product of three positive and increasing functions. This in turn implies that $1/(x\ln x \ln(\ln x))$ is a positive and decreasing function on $[3,\infty)$. Since it is also continuous, we can apply the Integral Test. The improper integral

$$\int_3^{\infty} \frac{dx}{x \ln x \ln(\ln x)} = \int_{\ln(\ln 3)}^{\infty} \frac{du}{u}$$

diverges, where we used the change of variable $u = \ln(lnx), du = dx/(x\ln x)$. Hence the series $\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$ diverges.

h. Consider the series $\sum_{n=1}^{\infty} \frac{1 + \sin n}{n^2}$. Since $0 \leq 1 + \sin n \leq 2$ for $n \geq 1$, we have $0 \leq \frac{1+\sin n}{n^2} \leq \frac{2}{n^2}$ for $n \geq 1$. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as $p = 2 > 1$. Therefore the series $\sum_{n=1}^{\infty} \frac{2}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Then the series $\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$ converges by the Direct Comparison Test. Since both $\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge, the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$, which is their difference, also converges. □

Remark: Other tests can be used too. Here are some examples:

In part **(a)**, the Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also works.

In part **(b)**, the Limit Comparison Test with the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ also works.

In part **(d)**, the Root Test also works where we use the fact that $\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}$.

Remark: The following was a bonus problem on *Moodle* in Spring 2010 Math 102 course.

Problem K: In class we showed that $0.\overline{12} = 0.121212\ldots = 12/99$ as an application of the geometric series.

We learn the following trick in elementary school: If $x = 0.121212\ldots$, then $100x = 12.121212\ldots$; their difference gives $99x = 12$ and therefore $x = 12/99$. Of course when we see this in elementary school, no one talks about convergence.

That is what we are going to do in this problem. We will forget about convergence as we know it, and take a trip to Tersonia.

In Tersonia after they teach the students about integers and rational numbers, they come to the decimal representations of numbers. As you know our decimal expansions have the form

$$\pm d_n d_{n-1} \ldots d_2 d_1 d_0 . d_{-1} d_{-2} \ldots$$

where each d_i is in $\{0, 1, \ldots, 9\}$. We can have infinitely many nonzero digits after the decimal point, but we must have only finitely many nonzero digits before the decimal point. In Tersonia they do just the opposite. Their decimal expansions have the form

$$\ldots t_3 t_2 t_1 t_0 . t_{-1} t_{-2} \ldots t_{-n}$$

where each t_i is in $\{0, 1, \ldots, 9\}$. Note that there is no minus sign. They can have infinitely many nonzero digits before the decimal point, but they can only have finitely many digits after the decimal point.

Take a few minutes to convince yourself that Tersonians can add and multiply their decimal expansions just like we do.

Why no minus sign? Well, because Tersonians don't need it. "Negative" numbers are already there. For instance, consider the number $y = \overline{12}.0 = \ldots 121212.0$. Then $100y = \ldots 121200.0$ and $-99y = 12$. Therefore $y = -12/99$. So in fact $\overline{12}.0$ is a "negative" number.

Here are some problems from *Tersonian Elementary School Mathematics Book*:

a. $\dfrac{1}{2} = ?$ **b.** $\dfrac{1}{3} = ?$ **c.** $\dfrac{1}{7} = ?$ **d.** $-1 = ?$

e. Find two nonzero numbers A and B such that $AB = 0$.

Part **(e)** was later turned into a programming challenge. A Java applet that computes the last n digits of A and B when their last digits are given can be found at `http://www.fen.bilkent.edu.tr/~otekman/calc2/ters.html`.

8. Determine the smallest of the real numbers A, B, C, D, E where :

$$A = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \qquad B = \sum_{n=1}^{\infty} \frac{1}{n2^n} \qquad C = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \qquad D = \sum_{n=1}^{\infty} \frac{n}{3^n} \qquad E = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)}$$

Solution: Firstly we have

$$A = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{2}{3}$$

by the geometric series sum formula. Next we observe that

$$B = \sum_{n=1}^{\infty} \frac{1}{n2^n} > \frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} = \frac{2}{3} \quad \text{and} \quad D = \sum_{n=1}^{\infty} \frac{n}{3^n} > \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} = \frac{2}{3}$$

as all the remaining terms of these series are positive. Finally,

$$E = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)} > 1 - \frac{1}{3\cdot 3} = \frac{8}{9} > \frac{2}{3} \quad \text{and} \quad C = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} < 1 - \frac{1}{2!} + \frac{1}{3!} = \frac{2}{3}$$

by the Alternating Series Estimate, which can be applied to these series as $\{1/(3^n \cdot (2n+1))\}_{n=0}^{\infty}$ and $\{1/n!\}_{n=1}^{\infty}$ are decreasing sequences with limit 0.

Hence C is the smallest. $\square$

Remark: One can also observe that $B = \ln 2$, $C = 1 - 1/e$, $D = 3/4$, and $E = \pi/(2\sqrt{3})$ as

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad \text{for} \quad -1 < x \le 1\,,$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all} \quad x\,,$$

$$\frac{x}{(1-x)^2} = x\,\frac{d}{dx}\,\frac{1}{1-x} = x\,\frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n\,x^n \quad \text{for} \quad |x| < 1\,, \text{ and}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for} \quad |x| \le 1\,.$$

Therefore, $C < A < B < D < E$.

9. Consider $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{5^n(n^2+1)}$.

a. Show that $\frac{4}{3} < f(3) < \frac{3}{2}$.

b. Show that $\frac{3}{4} < f(-3) < \frac{4}{5}$.

Solution: a. We have:

$$f(3) = \sum_{n=0}^{\infty} \frac{3^n}{5^n(n^2+1)} > 1 + \frac{3}{5\cdot 2} + \frac{3^2}{5^2\cdot 5} = \frac{343}{250} > \frac{4}{3}$$

and:

$$\begin{aligned} f(3) = \sum_{n=0}^{\infty} \frac{3^n}{5^n(n^2+1)} &= 1 + \frac{3}{5\cdot 2} + \sum_{n=2}^{\infty} \frac{3^n}{5^n(n^2+1)} \\ &< \frac{13}{10} + \frac{1}{5}\sum_{n=2}^{\infty} \frac{3^n}{5^n} = \frac{13}{10} + \frac{1}{5}\cdot\frac{(3/5)^2}{1-3/5} = \frac{37}{25} < \frac{3}{2} \end{aligned}$$

b. We have

$$f(-3) = \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{5^n(n^2+1)} < 1 - \frac{3}{5\cdot 2} + \frac{3^2}{5^2\cdot 5} = \frac{193}{250} < \frac{4}{5}$$

and

$$f(-3) = \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{5^n(n^2+1)} > 1 - \frac{3}{5\cdot 2} + \frac{3^2}{5^2\cdot 5} - \frac{3^3}{5^3\cdot 10} = \frac{469}{625} > \frac{3}{4}$$

by the Alternate Series Estimate as $\left\{\left(\frac{3}{5}\right)^n \cdot \frac{1}{n^2+1}\right\}_{n=0}^{\infty}$ is a decreasing sequence and $\lim_{n\to\infty}\left(\frac{3}{5}\right)^n \cdot \frac{1}{n^2+1} = 0$. □

10. Determine whether the sum of the series $\sum_{n=0}^{\infty} \frac{(-4)^n}{n!(n+1)!}$ is positive or negative.

Solution: Let $b_n = \frac{4^n}{n!(n+1)!}$ for $n \geq 0$. Then:

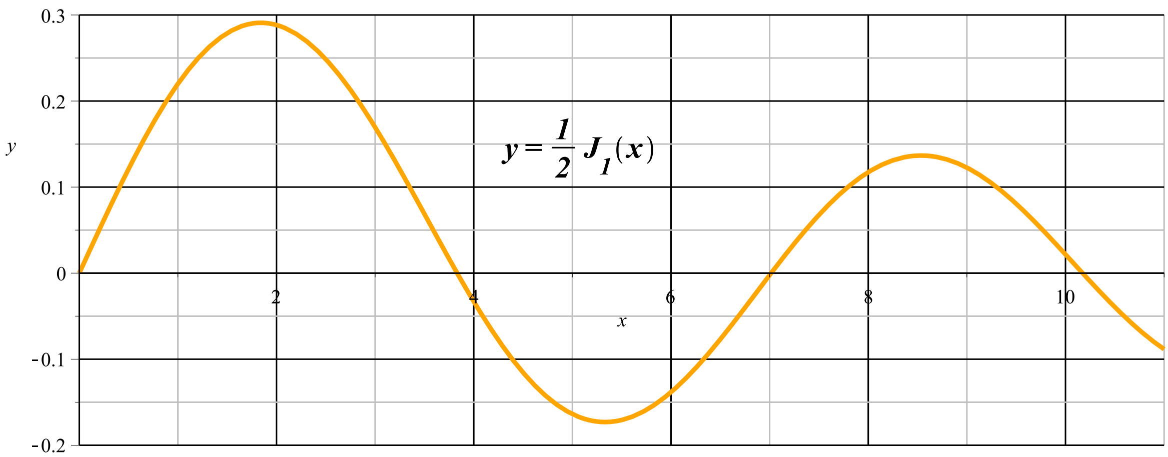
- $b_n > 0$ for all $n \geq 0$.
- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{4^n}{n!(n+1)!} = \lim_{n\to\infty} \frac{4^n}{n!} \cdot \lim_{n\to\infty} \frac{1}{(n+1)!} = 0$ as both limits are 0, the first one being one of the "*Useful Limits*".
- $b_n \geq b_{n+1} \iff \frac{4^n}{n!(n+1)!} \geq \frac{4^{n+1}}{(n+1)!(n+2)!} \iff (n+2)(n+1) \geq 4 \iff n \geq 1$.

Therefore the series satisfies the conditions of the Alternating Series Test. In particular, it converges. Moreover,

$$S = \sum_{n=0}^{\infty} \frac{(-4)^n}{n!(n+1)!} = 1 - \frac{4}{1!2!} + \frac{4^2}{2!3!} - \frac{4^3}{3!4!} + \frac{4^4}{4!5!} - \cdots = \underbrace{1 - 2 + \frac{4}{3} - \frac{4}{9}}_{s_3=-1/9} + \underbrace{\frac{4}{45}}_{b_4} - \cdots$$

and, by the Alternating Series Estimate, $|S - s_3| < b_4$. In other words, $|S - (-1/9)| < 1/45$, and S is negative. $\square$

Remark: $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^n}{n!(n+1)!}$ is $\frac{1}{2}J_1(4)$, where $J_1(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+1}}{n!(n+1)!}$ is the Bessel function of the first kind of order 1, and its value is approximately -0.03302166401.



11. Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{(2^n+1)(n^2+1)}$.

Solution: We will give two different solutions.

Solution 1: Here we will use the Ratio Test.

We have $a_n = (-1)^n \dfrac{x^n}{(2^n+1)(n^2+1)}$ and

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \\
&= \lim_{n\to\infty} \frac{\left|(-1)^{n+1} \dfrac{x^{n+1}}{(2^{n+1}+1)((n+1)^2+1)}\right|}{\left|(-1)^n \dfrac{x^n}{(2^n+1)(n^2+1)}\right|} \\
&= \lim_{n\to\infty} \frac{\dfrac{|x|^{n+1}}{(2^{n+1}+1)((n+1)^2+1)}}{\dfrac{|x|^n}{(2^n+1)(n^2+1)}} \\
&= \lim_{n\to\infty} \left(\frac{1+2^{-n}}{2+2^{-n}} \cdot \frac{n^2+1}{(n+1)^2+1}\right)|x| \\
&= \frac{|x|}{2}.
\end{aligned}$$

If $|x| < 2$, then $\rho = |x|/2 < 1$ and the power series converges absolutely by the Ratio Test. On the other hand, if $|x| > 2$, then $\rho = |x|/2 > 1$ and the power series diverges by the Ratio Test.

It follows by the definition of the radius of convergence that $R = 2$.

Solution 3: Here we will use the radius of convergence formulas.

We have $c_n = \dfrac{(-1)^n}{(2^n+1)(n^2+1)}$, and the radius of convergence formula gives

$$\begin{aligned}
\frac{1}{R} &= \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{1/((2^{n+1}+1)((n+1)^2+1))}{1/((2^n+1)(n^2+1))} \\
&= \lim_{n\to\infty} \left(\frac{1+2^{-n}}{2+2^{-n}} \cdot \frac{n^2+1}{(n+1)^2+1}\right) = \frac{1}{2}.
\end{aligned}$$

Therefore $R = 2$. □

12. Consider the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2^n+1)(n^2+1)}$.

a. Find the radius of convergence of the power series.

b. Determine whether the power series converges or diverges at the right endpoint of its interval of convergence.

c. Determine whether the power series converges or diverges at the left endpoint of its interval of convergence.

Solution: a. We have $a_n = (-1)^n\dfrac{x^{2n+1}}{(2^n+1)(n^2+1)}$ and

$$\begin{aligned}
\rho &= \lim_{n\to\infty}|a_n|^{1/n}\\
&= \lim_{n\to\infty}\left(\frac{|x|^{2n+1}}{(2^n+1)(n^2+1)}\right)^{1/n}\\
&= \lim_{n\to\infty}\frac{|x|^2\cdot|x|^{1/n}}{2\cdot(1+2^{-n})^{1/n}\cdot n^{1/n}\cdot(1+n^{-2})^{1/n}}\\
&= \frac{|x|^2}{2}.
\end{aligned}$$

If $|x|<\sqrt{2}$, then $\rho=|x|^2/2<1$ and the power series converges absolutely by the Root Test; and if $|x|>\sqrt{2}$, then $\rho=|x|^2/2>1$ and the power series diverges by the Root Test. Therefore $R=\sqrt{2}$.

b. At $x=\sqrt{2}$ we have

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2^n+1)(n^2+1)} &= \sum_{n=0}^{\infty}(-1)^n\frac{(\sqrt{2})^{2n+1}}{(2^n+1)(n^2+1)}\\
&= \sqrt{2}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(2^n+1)(n^2+1)}.
\end{aligned}$$

Consider the corresponding absolute value series $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{(2^n+1)(n^2+1)}$. Since $0<\dfrac{2^n}{(2^n+1)(n^2+1)}<\dfrac{1}{n^2}$ for all $n\geq 1$ and the p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ with $p=2>1$ converges, $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{(2^n+1)(n^2+1)}$ converges by the Direct Comparison Test; and then the power series at $x=\sqrt{2}$ converges absolutely by the Absolute Convergence Test.

c. At $x=-\sqrt{2}$ we have

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2^n+1)(n^2+1)} &= \sum_{n=0}^{\infty}(-1)^n\frac{(-\sqrt{2})^{2n+1}}{(2^n+1)(n^2+1)}\\
&= \sqrt{2}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^n}{(2^n+1)(n^2+1)}.
\end{aligned}$$

This is just −1 times the series we considered in part **(b)**, and therefore it converges absolutely. □

13. Determine the radius of convergence and the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{3n + (-1)^{n+1}} .$$

Also determine the type of convergence at each point of the interval of convergence.

Solution: As $c_n = \dfrac{1}{3n + (-1)^{n+1}}$, the formulas for the radius of convergence gives

$$\frac{1}{R} = \lim \frac{|c_{n+1}|}{|c_n|} = \lim \frac{|1/(3(n+1) + (-1)^{n+2})|}{|1/(3n + (-1)^{n+1})|} = \lim \frac{3 + (-1)^{n+1}/n}{3 + 3/n + (-1)^{n+2}/n} = 1 .$$

Therefore $R = 1$. Hence we have absolute convergence for $|x| < 1$ and divergence for $|x| > 1$.

At $x = 1$ we have

$$\sum_{n=0}^{\infty} \frac{x^n}{3n + (-1)^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{3n + (-1)^{n+1}} .$$

Since

$$L = \lim_{n\to\infty} \frac{1/(3n + (-1)^{n+1})}{1/n} = \lim_{n\to\infty} \frac{1}{3 + (-1)^{n+1}/n} = \frac{1}{3}$$

is a positive real number and the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the power series diverges by the Limit Comparison Test at $x = 1$.

At $x = -1$ we have

$$\sum_{n=0}^{\infty} \frac{x^n}{3n + (-1)^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3n + (-1)^{n+1}} .$$

Let $u_n = \dfrac{1}{3n + (-1)^{n+1}}$.

i. $u_n = \dfrac{1}{3n + (-1)^{n+1}} > 0$ for all $n \geq 1$.

ii. $0 < 3n + (-1)^{n+1} \leq 3n + 1 < 3n + 2 \leq 3(n + 1) + (-1)^{n+2}$ for all $n \geq 1$. Hence $u_n > u_{n+1}$ for $n \geq 1$.

iii. $\displaystyle\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{3n + (-1)^{n+1}} = 0$.

It follows that the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3n + (-1)^{n+1}}$ converges by the Alternating Series Test. Its absolute value series is the same as the series $\displaystyle 1 + \sum_{n=1}^{\infty} \frac{1}{3n + (-1)^{n+1}}$ and we showed that this series diverges. Hence the power series converges conditionally at $x = -1$.

To summarize, the radius of convergence is $R = 1$, the interval of convergence is $[-1, 1)$, the power series converges absolutely at every point of $(-1, 1)$, and it converges conditionally at $x = -1$. $\square$

14. Consider the power series $f(x) = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} x^n$.

a. Show that the radius of convergence of the power series is $R = 1$.

b. Determine the behavior of this power series at the endpoints of its interval of convergence.

c. Show that $2(1-x)f'(x) = f(x)$ for $|x| < 1$.

d. Solve this differential equation to show that $f(x) = \frac{1}{\sqrt{1-x}}$ for $|x| < 1$.

e. Show that

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} \cdot \frac{x^{2n+1}}{2n+1}$$

for $|x| < 1$.

Solution: a. We have $c_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$ for $n \geq 1$. We use the radius of convergence formula

$$\frac{1}{R} = \lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n \to \infty} \frac{\dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)(2n+2)}}{\dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}} = \lim_{n \to \infty} \frac{2n+1}{2n+2} = 1$$

to obtain $R = 1$.

b. At $x = 1$ we have the series $1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$. Since

$$\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} = \frac{3}{2} \cdot \frac{5}{4} \cdot \cdots \cdot \frac{2n-1}{2n-2} \cdot \frac{1}{2n} > \frac{1}{2n}$$

for $n > 1$ and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we conclude that the series at $x = 1$ diverges by the Direct Comparison Test.

At $x = -1$ we obtain the alternating series $1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$ with $u_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$. We have ***(i)*** $u_n > 0$ for all $n \geq 0$, and we also have ***(ii)***

$u_n > u_{n+1}$ for all $n \geq 0$ as $u_n/u_{n+1} = (2n+2)/(2n+1) > 1$ for $n \geq 0$. On the other hand,

$$u_n^2 = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot \dots \cdot (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot \dots \cdot (2n)^2} = \frac{1 \cdot 3}{2^2} \cdot \frac{3 \cdot 5}{4^2} \cdot \dots \cdot \frac{(2n-3)(2n-1)}{(2n-2)^2} \cdot \frac{2n-1}{2n} \cdot \frac{1}{2n} < \frac{1}{2n}$$

for $n > 1$. Therefore $0 < u_n < 1/\sqrt{2n}$ for $n > 1$, and the Sandwich Theorem gives ***(iii)*** $\lim_{n\to\infty} u_n = 0$. We conclude that the series at $x = -1$ converges by the Alternating Series Test. The convergence is conditional as we have already seen that the absolute value series diverges.

c. For $|x| < 1$ we have

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)} x^n.$$

Differentiating this we get

$$f'(x) = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n-2) \cdot 2} x^{n-1}$$

for $|x| < 1$. Therefore

$$2xf'(x) = x + \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n-2)} x^n$$

and

$$2f'(x) = 1 + \frac{3}{2} x + \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)} x^n$$

for $|x| < 1$. Taking the difference of these two, we obtain

$$\begin{aligned}
2(1-x)f'(x) &= 1 + \frac{1}{2} x + \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n-2)} \cdot \left(\frac{2n+1}{2n} - 1 \right) x^n \\
&= 1 + \frac{1}{2} x + \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n-2)} \cdot \frac{1}{2n} x^n \\
&= 1 + \frac{1}{2} x + \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)} x^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)} x^n \\
&= f(x)
\end{aligned}$$

for $|x| < 1$.

d. For $|x| < 1$,

$$\begin{aligned} 2(1-x)f'(x) = f(x) &\Longrightarrow \frac{f'(x)}{f(x)} = \frac{1}{2(1-x)} \\ &\Longrightarrow \ln|f(x)| = -\frac{1}{2}\ln|1-x| + C \ \text{ for some constant } C \\ &\Longrightarrow f(x) = \frac{A}{\sqrt{1-x}} \ \text{ for some constant } A\,. \end{aligned}$$

Now substituting $x = 0$ gives $1 = f(0) = A$. Therefore, $f(x) = \dfrac{1}{\sqrt{1-x}}$ for $|x| < 1$.

e. We have

$$\frac{1}{\sqrt{1-x}} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}\, x^n$$

for $|x| < 1$. Substituting x^2 for x gives

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}\, x^{2n}$$

for $|x^2| < 1$. Integrating this we obtain

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} \cdot \frac{x^{2n+1}}{2n+1} + C$$

for $|x| < 1$. Substituting $x = 0$ gives $0 = \arcsin 0 = C$, and hence $C = 0$. Thus

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} \cdot \frac{x^{2n+1}}{2n+1}$$

for $|x| < 1$. $\square$

Remark: It can be shown that

$$\frac{1}{\sqrt{1-x}} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}\, x^n$$

for $-1 \le x < 1$, and

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} \cdot \frac{x^{2n+1}}{2n+1}$$

for $|x| \le 1$.

15. Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n(2x-1)^{3n+1}}{5^n}$.

a. Find the radius of convergence of the power series.

b. Find the sum of the power series explicitly.

Solution: a. $a_n = \dfrac{n(2x-1)^{3n+1}}{5^n}$ gives

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \\
&= \lim_{n\to\infty} \frac{|(n+1)(2x-1)^{3(n+1)+1}/5^{n+1}|}{|n(2x-1)^{3n+1}/5^n|} \\
&= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{|2x-1|^3}{5} \\
&= \frac{|2x-1|^3}{5}.
\end{aligned}$$

If $|2x-1|^3/5 < 1$, then $\rho < 1$ and the series converges by the Ratio Test; and if $|2x-1|^3/5 > 1$, then $\rho > 1$ and the series diverges by the Ratio Test. Since $|2x-1|^3/5 < 1 \iff \left|x - \dfrac{1}{2}\right| < \dfrac{\sqrt[3]{5}}{2}$, it follows that the radius of convergence of the power series is $R = \sqrt[3]{5}/2$.

b. We know that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $|x| < 1$. Differentiating this we obtain

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$

for $|x| < 1$. Now we replace x with $(2x-1)^3/5$ to get

$$\sum_{n=1}^{\infty} \frac{n(2x-1)^{3(n-1)}}{5^{n-1}} = \frac{1}{\left(1 - \dfrac{(2x-1)^3}{5}\right)^2}$$

for $|(2x-1)^3/5| < 1$. Finally we multiply by $(2x-1)^4/5$ to obtain

$$\sum_{n=1}^{\infty} \frac{n(2x-1)^{3n+1}}{5^n} = \frac{(2x-1)^4/5}{\left(1 - \dfrac{(2x-1)^3}{5}\right)^2} = \frac{5}{4} \cdot \frac{(2x-1)^4}{(4x^3 - 6x^2 + 3x - 3)^2}$$

for $|x - 1/2| < \sqrt[3]{5}/2$. □

16. Consider the function defined by:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!\,2^{n(n-1)/2}}$$

a. Find the domain of f.

b. Evaluate the limit $\lim\limits_{x\to 0} \dfrac{f(x) - e^x}{1 - \cos x}$.

c. Show that $f(2) < e + \dfrac{3}{2}$.

d. Show that $f(-2) < 0$.

Solution: a. Since $c_n = \dfrac{1}{n!\,2^{n(n-1)/2}}$, the radius of convergence formula gives

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{1/((n+1)!\,2^{(n+1)n/2})}{1/(n!\,2^{n(n-1)/2})} = \lim_{n\to\infty} \frac{1}{(n+1)2^n} = 0\ .$$

Therefore $R = \infty$. This means that the domain of f is $(-\infty, \infty)$.

b. Using

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!\,2^{n(n-1)/2}} = 1 + x + \frac{x^2}{2\cdot 2!} + \frac{x^3}{2^3\cdot 3!} + \cdots$$

we obtain

$$\begin{aligned}
\lim_{x\to 0} \frac{f(x) - e^x}{1 - \cos x} &= \lim_{x\to 0} \frac{\left(1 + x + \dfrac{x^2}{2\cdot 2!} + \dfrac{x^3}{2^3\cdot 3!} + \cdots\right) - \left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots\right)}{1 - \left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots\right)} \\
&= \lim_{x\to 0} \frac{-\dfrac{1}{4}x^2 - \dfrac{7}{48}x^3 - \cdots}{\dfrac{1}{2}x^2 - \dfrac{1}{24}x^4 + \cdots} = \lim_{x\to 0} \frac{-\dfrac{1}{4} - \dfrac{7}{48}x - \cdots}{\dfrac{1}{2} - \dfrac{1}{24}x^2 + \cdots} \\
&= \frac{-\dfrac{1}{4}}{\dfrac{1}{2}} = -\frac{1}{2}\ .
\end{aligned}$$

c. We have

$$f(2) = \sum_{n=0}^{\infty} \frac{2^n}{n!\,2^{n(n-1)/2}} = \sum_{n=0}^{\infty} \frac{1}{n!\,2^{n(n-3)/2}} = 1 + 2 + \frac{2}{2!} + \frac{1}{3!} + \sum_{n=4}^{\infty} \frac{1}{n!\,2^{n(n-3)/2}}\ .$$

On the other hand,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \sum_{n=4}^{\infty} \frac{1}{n!}\ .$$

Since $n(n-3) > 0$ for $n \geq 4$, we have $\dfrac{1}{n!\, 2^{n(n-3)/2}} < \dfrac{1}{n!}$ for $n \geq 4$. Hence $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n!\, 2^{n(n-3)/2}} < \sum_{n=4}^{\infty} \frac{1}{n!}$. Therefore, $f(2) - e < \dfrac{3}{2}$.

d. This time we have

$$\begin{aligned} f(-2) &= \sum_{n=0}^{\infty} \frac{(-2)^n}{n!\, 2^{n(n-1)/2}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^{n(n-3)/2}} \\ &= 1 - 2 + \frac{2}{2!} + \sum_{n=3}^{\infty} \frac{(-1)^n}{n!\, 2^{n(n-3)/2}} \\ &= \sum_{n=3}^{\infty} \frac{(-1)^n}{n!\, 2^{n(n-3)/2}} \\ &= -\frac{1}{3!} + \frac{1}{2^2 \cdot 4!} - \frac{1}{2^5 \cdot 5!} + \frac{1}{2^9 \cdot 6!} - \cdots . \end{aligned}$$

As $(n+1)2^{n-1} > 1$ for $n \geq 3$, we have $-\dfrac{1}{n!\, 2^{n(n-3)/2}} + \dfrac{1}{(n+1)!\, 2^{(n+1)(n-2)/2}} < 0$ for $n \geq 3$ too. Therefore, $f(-2) < -\dfrac{1}{3!} + \dfrac{1}{2^2 \cdot 4!} = -\dfrac{5}{32} < 0$. □

17. Estimate $\displaystyle\int_0^2 e^{-x^2}\, dx$ with error less than 0.01.

Solution: We have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

for all x. Therefore,

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots$$

for all x, and integration gives

$$\int_0^2 e^{-x^2}\, dx = 2 - \frac{2^3}{3} + \frac{2^5}{2!\,5} + \cdots + (-1)^n \frac{2^{2n+1}}{n!\,(2n+1)} + \cdots .$$

Let $u_n = \dfrac{2^{2n+1}}{n!\,(2n+1)}$. Then:

i. $u_n = \dfrac{2^{2n+1}}{n!\,(2n+1)} > 0$ for all $n \geq 0$.

ii. $2n^2 - 3n - 1 > 0$ for $n \geq 2 \Longrightarrow (n+1)(2n+1) > 2^2(2n+1)$ for $n \geq 2$. Hence $u_n > u_{n+1}$ for $n \geq 2$.

iii. $\displaystyle\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{4^n}{n!} \cdot \frac{2}{2n+1} \right) = 0$.

The series satisfies the conditions of Alternating Series Test for $n \geq 2$. Since $2^{23}/(23 \cdot 11!) = 32768/3586275 \approx 0.009 < 0.01$, it follows by the Alternating Series Estimate that the sum

$$\int_0^2 e^{-x^2}\,dx \approx 2 - \frac{2^3}{3} + \frac{2^5}{2!\,5} + \cdots - \frac{2^{19}}{9!\,19} + \frac{2^{21}}{10!\,21} = \frac{12223758182}{13749310575} \approx 0.89$$

approximates $\displaystyle\int_0^2 e^{-x^2}\,dx$ with error less than 0.01. □

18. Find the exact value of $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n(2n+1)}$.

Solution: On one hand we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{4^n(2n+1)} &= \sum_{n=0}^{\infty} \frac{(1/4)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(1/2)^{2n}}{2n+1} = 2\sum_{n=0}^{\infty} \frac{(1/2)^{2n+1}}{2n+1} \\ &= 2\left(1/2 + \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} + \cdots\right).\end{aligned}$$

On the other hand we have

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

for $-1 < x \leq 1$. In particular, $x = 1/2$ gives

$$\ln\left(\frac{3}{2}\right) = \ln\left(1 + \frac{1}{2}\right) = 1/2 - \frac{(1/2)^2}{2} + \frac{(1/2)^3}{3} - \frac{(1/2)^4}{4} + \frac{(1/2)^5}{5} - \cdots,$$

and $x = -1/2$ gives

$$\ln\left(\frac{1}{2}\right) = \ln\left(1 - \frac{1}{2}\right) = -1/2 - \frac{(1/2)^2}{2} - \frac{(1/2)^3}{3} - \frac{(1/2)^4}{4} - \frac{(1/2)^5}{5} - \cdots.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{1}{4^n(2n+1)} = \ln\left(\frac{3}{2}\right) - \ln\left(\frac{1}{2}\right) = \ln 3.$$

□

Part 4: Vector Analysis

1. Find the value of the line integral

$$\oint_C (3x^2y^2 + y)\,dx + 2x^3y\,dy$$

where C is the cardioid $r = 1 + \cos\theta$ parameterized counterclockwise.

Solution: We use the Green's Theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

where R is the region enclosed by the simple closed curve C. Therefore

$$\begin{aligned}
\oint_C \overbrace{(3x^2y^2 + y)}^{M}\,dx + \overbrace{2x^3y}^{N}\,dy &= \iint_R \left(\frac{\partial}{\partial x}\overbrace{(2x^3y)}^{N} - \frac{\partial}{\partial y}\overbrace{(3x^2y^2 + y)}^{M} \right) dA \\
&= \iint_R \big(6x^2y - (6x^2y + 1)\big)\,dA \\
&= -\iint_R dA \\
&= -\int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta \\
&= -\int_0^{2\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=1+\cos\theta} d\theta \\
&= -\frac{1}{2}\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)\,d\theta \\
&= -\frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta \\
&= -\frac{1}{2}\cdot\frac{3}{2}\cdot 2\pi \\
&= -\frac{3\pi}{2}.
\end{aligned}$$

□

Remark We used the *Circulation-Curl Form* of the Green's Theorem, but the computation is exactly the same with the *Flux-Divergence Form*:

$$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

In fact, when expressed in terms of components and coordinates, both forms of the Green's Theorem can be summarized and most easily remembered as

$$\oint_C \omega = \iint_R d\omega$$

where $\omega = A\,dx + B\,dy$ is a differential form and $d\omega = dA\,dx + dB\,dy$. The multiplication of differentials here is as described in the **Remark** following **Example 39** in **Part 2**. Indeed, if $\omega = M\,dx + N\,dy$, then

$$\begin{aligned} d\omega = dM\,dx + dN\,dy &= \left(\frac{\partial M}{\partial x}\,dx + \frac{\partial M}{\partial y}\,dy\right)dx + \left(\frac{\partial N}{\partial x}\,dx + \frac{\partial N}{\partial y}\,dy\right)dy \\ &= \frac{\partial M}{\partial y}\,dy\,dx + \frac{\partial N}{\partial x}\,dx\,dy = -\frac{\partial M}{\partial y}\,dx\,dy + \frac{\partial N}{\partial x}\,dx\,dy = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy\,, \end{aligned}$$

and if $\omega = M\,dy - N\,dx$, then

$$\begin{aligned} d\omega = dM\,dy - dN\,dx &= \left(\frac{\partial M}{\partial x}\,dx + \frac{\partial M}{\partial y}\,dy\right)dy - \left(\frac{\partial N}{\partial x}\,dx + \frac{\partial N}{\partial y}\,dy\right)dx \\ &= \frac{\partial M}{\partial x}\,dx\,dy - \frac{\partial N}{\partial y}\,dy\,dx = \frac{\partial M}{\partial x}\,dx\,dy + \frac{\partial N}{\partial y}\,dx\,dy = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy \end{aligned}$$

where we used $dx\,dx = 0 = dy\,dy$ and $dy\,dx = -dx\,dy$. Unlike in the case of change of variables, now we must always get a $dx\,dy$ under the double integral and we must not get rid of the sign.

2. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where

$$\mathbf{F} = \frac{-y}{4x^2+9y^2}\,\mathbf{i} + \frac{x}{4x^2+9y^2}\,\mathbf{j}$$

and C is the unit circle parametrized in the counterclockwise direction.

Solution: Observe that

$$\begin{aligned} \operatorname{curl}\mathbf{F} &= \frac{\partial}{\partial x}\left(\frac{x}{4x^2+9y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{4x^2+9y^2}\right) \\ &= \frac{1}{4x^2+9y^2} - \frac{8x^2}{(4x^2+9y^2)^2} + \frac{1}{4x^2+9y^2} - \frac{18y^2}{4x^2+9y^2} = 0 \end{aligned}$$

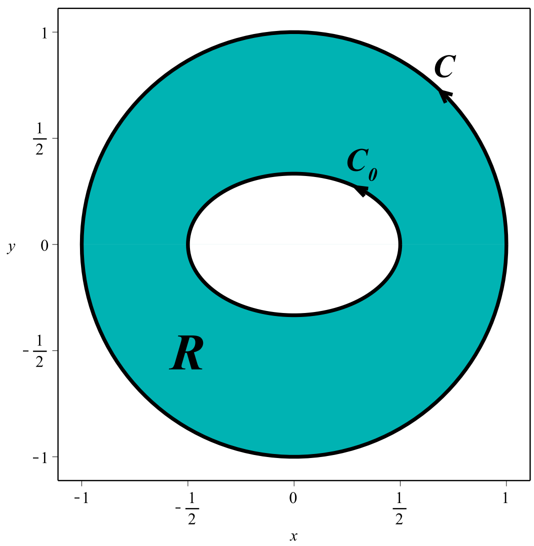
at all points $(x,y) \neq (0,0)$.

Consider $C_0 : \mathbf{r} = (\cos t)/2\,\mathbf{i} + (\sin t)/3\,\mathbf{j}$, $0 \le t \le 2\pi$, the counterclockwise parametrization of the ellipse $4x^2 + 9y^2 = 1$. Let R be the region lying inside the unit circle and outside this ellipse.

Since $\operatorname{curl}\mathbf{F} = 0$ at all points of R, $\iint_R \operatorname{curl}\mathbf{F}\,dA = 0$. Therefore by the generalized Green's Theorem,

$$\begin{aligned} \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_{C_0} \mathbf{F}\cdot d\mathbf{r} = \oint_{C_0} \frac{-y\,dx + x\,dy}{4x^2+9y^2} \\ &= \int_{t=0}^{t=2\pi} \frac{-(\sin t)/3\;d((\cos t)/2) + (\cos t)/2\;d((\sin t)/3)}{\cos^2 t + \sin^2 t} \\ &= \frac{1}{6}\int_0^{2\pi} dt = \frac{1}{6}\cdot 2\pi = \frac{\pi}{3}\,. \end{aligned}$$

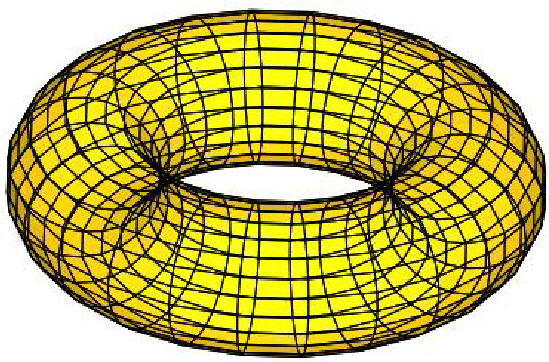
□



3. Find the surface area of the parameterized torus

$$\mathbf{r} = (a + b\cos u)\cos v\,\mathbf{i} + (a + b\cos u)\sin v\,\mathbf{j} + b\sin u\,\mathbf{k}\,,$$

$0 \le u \le 2\pi$, $0 \le v \le 2\pi$, where $0 < b < a$ are constants.



Solution: We have:

$$\begin{aligned}
\mathbf{r}_u &= -b\sin u\cos v\,\mathbf{i} - b\sin u\sin v\,\mathbf{j} + b\cos u\,\mathbf{k}\\
\mathbf{r}_v &= -(a + b\cos u)\sin v\,\mathbf{i} + (a + b\cos u)\cos v\,\mathbf{j}\\
\mathbf{r}_u \times \mathbf{r}_v &= -b(a + b\cos u)\cos v\cos u\,\mathbf{i} - b(a + b\cos u)\sin v\cos u\,\mathbf{j} - b(a + b\cos u)\sin u\,\mathbf{k}\\
|\mathbf{r}_u \times \mathbf{r}_v| &= b(a + b\cos u)(\cos^2 v\cos^2 u + \sin^2 v\cos^2 u + \sin^2 u)^{1/2} = b(a + b\cos u)
\end{aligned}$$

Therefore

$$\begin{aligned}
\text{Surface Area} &= \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\,dA = \iint_R b(a + b\cos u)\,dA\\
&= \int_0^{2\pi}\int_0^{2\pi} b(a + b\cos u)\,du\,dv = ab\cdot 2\pi\cdot 2\pi = 4\pi^2 ab
\end{aligned}$$

where $R = \{(u, v) : 0 \le u \le 2\pi \text{ and } 0 \le v \le 2\pi\}$. $\square$

4. Find the area of the portion S of the cylinder $x^2 + y^2 = 2y$ that lies inside the sphere $x^2 + y^2 + z^2 = 4$.

Solution: Let $F(x, y, z) = x^2 + y^2 - 2y$. Then the cylinder is given by $F(x, y, z) = 0$. We will use the projection to the yz-plane. This projection is 2-1 from the portion of the cylinder inside the sphere to the region R given by the inequalities $0 \le 2y \le 4 - z^2$ in the yz-plane, and by symmetry

$$\text{Surface Area of } S = 2 \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA$$

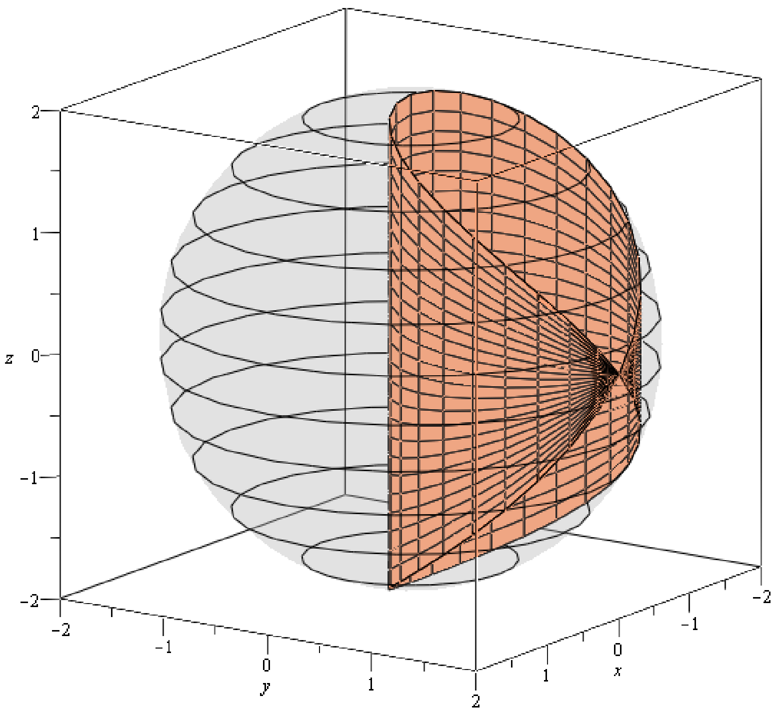
where $\mathbf{p} = \mathbf{i}$. We have

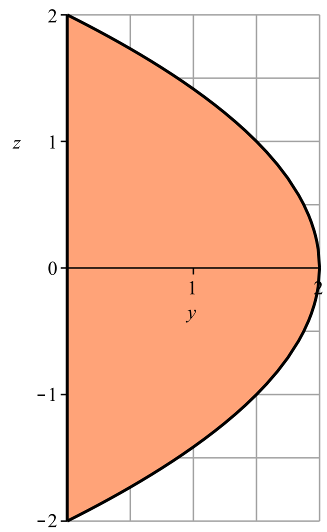
$$\begin{aligned} \nabla F &= 2x\mathbf{i} + (2y - 2)\mathbf{j} \\ |\nabla F| &= (4x^2 + (2y-2)^2)^{1/2} = (4x^2 + 4y^2 - 8y + 4)^{1/2} = 2 \ \text{ on } S \\ \nabla F \cdot \mathbf{p} &= \nabla F \cdot \mathbf{i} = 2x \end{aligned}$$

and therefore

$$\begin{aligned} \text{Surface Area} &= 2 \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA = 2 \iint_R \frac{2}{|2x|}\, dA = 2 \iint_R \frac{1}{|x|}\, dA \\ &= 2 \iint_R \frac{1}{\sqrt{2y - y^2}}\, dA \qquad \text{as } |x| = \sqrt{2y - y^2} \text{ on } S \\ &= 2 \int_0^2 \int_{-\sqrt{4-2y}}^{\sqrt{4-2y}} \frac{1}{\sqrt{2y - y^2}}\, dz\, dy = 2 \cdot 2 \int_0^2 \frac{\sqrt{4 - 2y}}{\sqrt{2y - y^2}}\, dy \\ &= 4 \cdot \sqrt{2} \int_0^2 \frac{1}{\sqrt{y}}\, dy = 4\sqrt{2} \cdot 2\sqrt{2} = 16\,. \end{aligned}$$

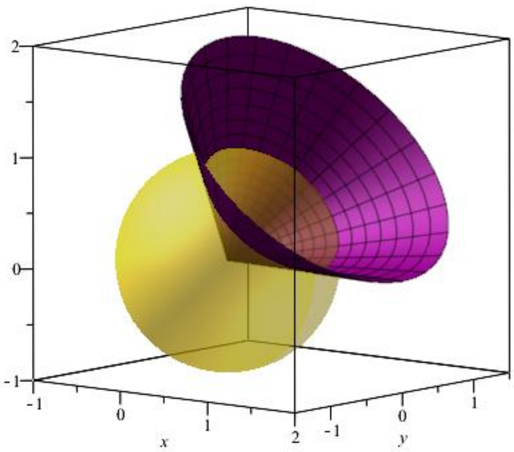
□

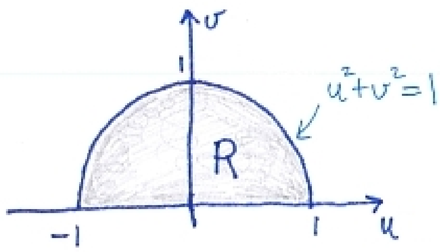




5. Consider the parametrized surface $S : \mathbf{r} = u^2\mathbf{i} + \sqrt{2}\,uv\mathbf{j} + v^2\mathbf{k}, -\infty < u < \infty\,, 0 \le v < \infty\,.$ Find the area of the portion of the surface S that lies inside the unit ball $x^2 + y^2 + z^2 \le 1$.

Solution: We have $x = u^2$, $y = \sqrt{2}uv$ and $z = v^2$ on the surface. Therefore, $x^2 + y^2 + z^2 \le 1$ means $(u^2)^2 + (\sqrt{2}uv)^2 + (v^2)^2 \le 1$. In other words, $(u^2 + v^2)^2 \le 1$, or $u^2 + v^2 \le 1$. Hence the part of the surface lying inside the sphere is the image of the region R, the upper half of the unit disk, in the uv-plane.





To compute the surface area we first compute:

$$
\begin{aligned}
\mathbf{r}_u &= 2u\mathbf{i} + \sqrt{2}v\mathbf{j}\\
\mathbf{r}_v &= \sqrt{2}u\mathbf{j} + 2v\mathbf{k}\\
\mathbf{r}_u \times \mathbf{r}_u &= 2\sqrt{2}v^2\mathbf{i} - 4uv\mathbf{j} + 2\sqrt{2}u^2\mathbf{k}\\
|\mathbf{r}_u \times \mathbf{r}_u| &= ((2\sqrt{2}v^2)^2 + (4uv)^2 + (2\sqrt{2}u^2)^2)^{1/2} = 2\sqrt{2}(u^2 + v^2)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\text{Surface Area} &= \iint_R |\mathbf{r}_u \times \mathbf{r}_u|\; du\, dv\\
&= 2\sqrt{2} \iint_R (u^2 + v^2)\; du\, dv\\
&= 2\sqrt{2} \int_0^\pi \int_0^1 r^2 \cdot r\; dr\, d\theta\\
&= \frac{1}{\sqrt{2}} \int_0^\pi d\theta\\
&= \frac{\pi}{\sqrt{2}}
\end{aligned}
$$

where we used the polar coordinates in the uv-plane. □

Remark: There is a shorter way of solving this problem which does not use Calculus. The given parametrization maps the upper half-uv-plane onto the half-cone given by the equation $y^2 = 2xz$, and the conditions $x \ge 0$ and $z \ge 0$ in a one-to-one manner (except on the u-axis). This half-cone has its vertex at the origin, its axis lies along the bisector of the positive x- and z-axes, and it has an opening angle of 45°. The portion of this half-cone cut off by the unit sphere is the lateral surface of a right cone with slant height $\ell = 1$ and radius $r = 1/\sqrt{2}$, hence with area $\pi r \ell = \pi/\sqrt{2}$.

6. Verify Stokes's Theorem for the vector field $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{k}$ and the surface S, where S is the portion of the paraboloid $z = x^2 + y^2$ satisfying $z \le 3$, with the unit normal vector field $\mathbf{n}$ pointing away from the z-axis.

Solution: S is bounded by the curve $C : \mathbf{r} = \sqrt{3}\cos t\,\mathbf{i} - \sqrt{3}\sin t\,\mathbf{j} + 3\,\mathbf{k}$, $0 \le t \le 2\pi$. Note that this parametrization is consistent with the direction of $\mathbf{n}$. The circulation of $\mathbf{F}$ around C is

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \oint_C (y\,\mathbf{i} + z\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})\\ &= \int_{t=0}^{t=2\pi} (-\sqrt{3}\sin t\,\mathbf{i} + 3\,\mathbf{k})\cdot(d(\sqrt{3}\cos t)\,\mathbf{i} + d(-\sqrt{3}\sin t)\,\mathbf{j} + d(3)\,\mathbf{k})\\ &= \int_0^{2\pi} (-\sqrt{3}\sin t)(-\sqrt{3}\sin t)\,dt = 3\int_0^{2\pi} \sin^2 t\,dt = 3\pi\end{aligned}$$

On the other hand, the paraboloid is a level surface of $f(x,y,z) = z - x^2 - y^2$, and $\nabla f = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ points in the opposite direction to $\mathbf{n}$, so we will choose the minus sign from $\pm$ in the flux integral. The projection of S into the xy-plane is the disk $R = \{(x,y) : x^2 + y^2 \le 3\}$. Finally,

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y & 0 & z\end{vmatrix} = -\mathbf{k}\,.$$

The flux of $\nabla\times\mathbf{F}$ across S is

$$\begin{aligned}\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma &= \iint_R \mathbf{F}\cdot\frac{\pm\nabla f}{|\mathbf{k}\cdot\nabla f|}\,dA\\ &= \iint_R (-\mathbf{k})\cdot(2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k})\,dA\\ &= \iint_R dA = \text{Area of } R = \pi(\sqrt{3})^2 = 3\pi\,.\end{aligned}$$

Hence $\displaystyle\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma = \oint_C \mathbf{F}\cdot d\mathbf{r}$. □

7. Verify Divergence Theorem for the vector field $\mathbf{F} = xz\,\mathbf{i} + yz\,\mathbf{j} + z^3\,\mathbf{k}$ and the region $D = \{(x,y,z) : x^2 + y^2 + z^2 \le 4\}$.

Solution: $\nabla\cdot\mathbf{F} = \partial(xz)/\partial x + \partial(yz)/\partial y + \partial(z^3)/\partial z = z + z + 3z^2 = 2z + 3z^2$, and

$$\begin{aligned}\iiint_D \nabla\cdot\mathbf{F}\,dV &= \iiint_D (2z + 3z^2)\,dV\\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} (2\rho\cos\phi + 3\rho^2\cos^2\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\ &= \int_0^{2\pi}\int_0^{\pi} (8\cos\phi + \frac{96}{5}\cos^2\phi)\,\sin\phi\,d\phi\,d\theta\\ &= \frac{64}{5}\int_0^{2\pi} d\theta = \frac{128\pi}{5}\,.\end{aligned}$$

To compute the outward flux through the sphere $S = \{(x,y,z) : x^2 + y^2 + z^2 = 4\}$, we divide it as the upper hemisphere $S_1 = \{(x,y,z) : x^2 + y^2 + z^2 = 4 \text{ and } x \geq 0\}$ and the lower hemisphere $S_2 = \{(x,y,z) : x^2 + y^2 + z^2 = 4 \text{ and } x \leq 0\}$, and project both onto the disk $R = \{(x,y) : x^2 + y^2 \leq 4\}$ in the xy-plane.

The sphere is a level surface of $f(x,y,z) = x^2 + y^2 + z^2$. $\nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$ points in the same direction as $\mathbf{n}$, the outward pointing unit normal vector field on S, so we will choose the plus sign from $\pm$ in the flux integral.

For the upper hemisphere we have

$$\begin{aligned}
\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \iint_R \mathbf{F}\cdot\frac{\pm\nabla f}{|\mathbf{k}\cdot\nabla f|}\,dA \\
&= \iint_R (xz\,\mathbf{i} + yz\,\mathbf{j} + z^3\,\mathbf{k})\cdot\frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{2z}\,dA \\
&= \iint_R (x^2 + y^2 + z^3)\,dA \\
&= \iint_R (x^2 + y^2 + (4 - x^2 - y^2)^{3/2})\,dA \\
&= \int_0^{2\pi}\int_0^2 (r^2 + (4 - r^2)^{3/2})\,r\,dr\,d\theta \\
&= \int_0^{2\pi} (4 + \frac{32}{5})\,d\theta = 8\pi + \frac{64\pi}{5}\,.
\end{aligned}$$

A similar computation for the lower hemisphere gives

$$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,d\sigma = -8\pi + \frac{64\pi}{5}\,.$$

Therefore

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,d\sigma + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,d\sigma \\
&= (8\pi + \frac{64\pi}{5}) + (-8\pi + \frac{64\pi}{5}) = \frac{128\pi}{5}\,,
\end{aligned}$$

and $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV\,.$ □