Teorema lui Cauchy

1. enuntul teoremei
2. demonstratia teoremei
3. interpretare geometrica
4. aplicatii

1. ENUNTUL TEOREMEI

Fie f si g doua functii, f,g:[a,b]→R, cu proprietatile:

a) f si g continue pe [a,b]
b) f si g derivabile pe (a,b)
c) g'(x)=0

atunci g(a)=g(b) si (∃) cel putin un punct c∈(a,b) a.i.

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

2. DEMONSTRATIA TEOREMEI

$$\left.\begin{array}{l}\textbf{P.p.a. ca } \mathbf{g(a)=g(b)}\\ \textbf{g continua pe [a,b]}\\ \textbf{g der. pe (a,b)}\end{array}\right\}\Rightarrow (\exists)\textbf{ cel putin un punct } \mathbf{c\in(a,b)} \textbf{ a.i. } \mathbf{g'(c)=0}$$

$\mathbf{dar\ g'(x)\neq 0 \Rightarrow}$ **contradictie.**

$\mathbf{\Rightarrow g(a)\neq g(b).}$

Fie $\mathbf{h(x)=f(x)+k\cdot g(x);}$ **k - constanta reala a.i.** $\mathbf{h(a)=h(b)}$

$\mathbf{\Rightarrow f(a)+k\cdot g(a)=f(b)+k\cdot g(b)}$

$\mathbf{\Rightarrow f(a)-f(b)=k\cdot g(b)-k\cdot g(a)}$

$\mathbf{\Rightarrow f(a)-f(b)=k(g(b)-g(a))}$

$$\mathbf{\Rightarrow k=-\frac{f(b)-f(a)}{g(b)-g(a)}}$$

$$\left.\begin{array}{l}\mathbf{h\ \ continua\ \ pe\ [a,b]}\\ \mathbf{h\ \ derivabila\ \ pe\ (a,b)}\\ \mathbf{h(a)=h(b)}\end{array}\right\}\Rightarrow (\exists)\ \mathbf{cel\ putin\ \ un\ \ punct\ \ c\in(a,b)\ a.i.\ h'(c)=0;}$$

$$\mathbf{h'(x)=f'(x)+k\cdot g'(x)}$$

$$\mathbf{h'(c)=f'(c)+k\cdot g'(c)}$$

$$\Rightarrow \mathbf{f'(c)+k\cdot g'(c)=0}$$

$$\Rightarrow \mathbf{k=-\frac{f'(c)}{g'(c)}}$$

$$\Rightarrow \mathbf{\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}}$$

3.INTERPRETARE GEOMETRICA

Pantele celor doua drepte sunt proportionale cu pantele tangentelor duse la graficul functiei in punctul c corespunzator.

4. APLICATII

1. Sa se aplice TEOREMA LUI CAUCHY in cazul functiilor :

$$f:[-2,5]\to R$$

$$f(x)=\begin{cases}\sqrt{x+3} & ;-2\le x<1\\ \frac{x}{4}+\frac{7}{4} & ;1\le x\le 5\end{cases}$$

$$g:[-2,5]\to R \qquad g(x)=x.$$

Functia f este continua pe [-2,1) si [1,5] ca functie elementara.

Se pune problema in $x=1$

$$\lim_{x\to 1;x<1} f(x)=\lim_{x\to 1;x<1}\sqrt{x+3}=2$$

$$\lim_{x\to 1;x>1} f(x)=\lim_{x\to 1;x>1}\frac{x+7}{4}=2$$

$$f(1)=2 \Rightarrow f \text{ este continua pe } [-2,5]$$

Functia f este derivabila pe [-2,1) si [1,5] ca functie elementara .

Se pune problema in $x=1$

$$f'_S(1)=\lim_{x\to 1;x<1}\frac{f(x)-f(1)}{x-1}=\frac{1}{4}$$

$$f'_D(1)=\lim_{x\to 1;x<1}\frac{f(x)-f(1)}{x-1}=\frac{1}{4}$$

$\Rightarrow$ **f este derivabila pe[-2,5] .**

Functia g este continua si derivabila pe [-2,5] ca f. elementara.

$$g'(x) = 1 \neq 0$$

$$\xrightarrow{t.cauchy} (\exists) \text{cel putin un } c \in (-2,5) \text{ a.i. } \frac{f'(c)}{g'(c)} = \frac{f(5)-f(-2)}{g(5)-g(-2)}$$

$$f'(x) = \begin{cases} \frac{1}{2\sqrt{x-3}} & ; x \in (-2,1) \\ \frac{1}{4} & ; x \in [1,5) \end{cases}$$

$$g'(x) = 1$$

$$\frac{f'(c)}{g'(c)} = \frac{2}{7}$$

Cazul 1 $: x \in (-2,1)$

$$\Rightarrow f'(c) = \frac{1}{2\sqrt{c-3}} \;;\; g'(c) = 1 \Rightarrow \frac{1}{2\sqrt{c-3}} = \frac{2}{7}$$

$$4\sqrt{c-3} = 7 \Rightarrow 16(c-3) = 49 \Rightarrow c = \frac{97}{16} \in (-2,1)$$

Cazul 2 $: x \in (1,5)$

$$\Rightarrow f'(c) = \frac{1}{4} \;;\; g'(c) = 1 \Rightarrow \frac{1}{4} = \frac{2}{7} \text{ (F)}$$

$$\Rightarrow c = \frac{97}{16}$$

2. Sa se aplice Teorema lui Cauchy pentru functiile $f,g:[1,e]\to R$, $f(x)=\ln(x)$ si $g(x)=2x-1$, determinand punctul c corespunzator. Similar pentru functiile $f,g:\left[\frac{\pi}{6},\frac{\pi}{3}\right]\to R$

$f(x)=\sin(x)$ si $g(x)=\cos(x)$.

- $f:[1,e]\to R$ $f(x)=\ln(x)$

  $g:[1,e]\to R$ $g(x)=2x-1$

f si g sunt continue pe $[1,e]$ ca functii elementare

f si g sunt derivabile pe $(1,e)$ ca functii elementare

$g'(x)=(2x-1)'=2$

$\xrightarrow{\text{t. cauchy}}$ $(\exists)$ cel putin un punct $c\in(1,e)$ a.i. $\frac{f'(c)}{g'(c)}=\frac{f(e)-f(1)}{g(e)-g(1)}=\frac{1}{2e-2}$

$f'(x)=\frac{1}{x}$; $g'(x)=2$

$\frac{f'(c)}{g'(c)}=\frac{1}{2c}=\frac{1}{2e-2}\Rightarrow 2c=2(e-1)\Rightarrow c=e-1.$

- $f:\left[\frac{\pi}{6};\frac{\pi}{3}\right]\to R$ $f(x)=\sin(x)$

  $g:\left[\frac{\pi}{6};\frac{\pi}{3}\right]\to R$ $g(x)=\cos(x)$

$\mathbf{f\ si\ g\ sunt\ continue\ pe} \left[\frac{\pi}{6};\frac{\pi}{3}\right] \mathbf{ca\ f\ elementare}$

$\mathbf{f\ si\ g\ sunt\ der.\ pe} \left[\frac{\pi}{6};\frac{\pi}{3}\right] \mathbf{ca\ f\ elementare}$

$\mathbf{g'(x) = -sin(x)\ ;\ sin(x) \neq 0 \Rightarrow x \neq k\pi}$

$\xrightarrow{\mathbf{t.\ cauchy}}$ $\mathbf{(\exists)\ cel\ putin\ un\ punct\ c} \in \left(\frac{\pi}{6};\frac{\pi}{3}\right)$

$$\mathbf{a.i.}\ \frac{f'(c)}{g'(c)} = \frac{f\left(\frac{\pi}{3}\right) - f\left(\frac{\pi}{6}\right)}{g\left(\frac{\pi}{3}\right) - g\left(\frac{\pi}{6}\right)} = -1$$

$$\frac{f'(c)}{g'(c)} = \frac{\cos(c)}{-\sin(c)} = -ctg(c) = -1 \Rightarrow c = \frac{\pi}{4}$$

3. Sa se studieze valabilita tea teremei lui Cauchy si sa se determine valoarea punctului c :

$$f:[0,3] \to R\ \ f(x) = \begin{cases} \frac{x^3}{3} - x^2 + 1, & x \in (1,3] \\ -x + \frac{4}{3}, & x \in [0,1] \end{cases}$$

$$g:[0,3] \to R\ \ g(x) = x$$

Functia f este continua si derivabila pe [0,1) si (1,3]
ca functie elementara.
Se pune problema in $x=1$.

Continuitatea

$$\lim_{x\to 1;x<1} f(x) = \lim_{x\to 1;x<1} -x+\frac{4}{3} = 1/3$$

$$\lim_{x\to 1;x>1} f(x) = \lim_{x\to 1;x>1} \frac{x^3}{3} - x^2 + 1 = 1/3$$

$f(1)=1/3 \Rightarrow$ f este continua pe [0,3].

Derivabilitatea

$$f'_S(1) = \lim_{x\to 1;x<1} \frac{f(x)-f(1)}{x-1} = -1$$

$$f'_D(1) = \lim_{x\to 1;x>1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1;x>1} \frac{x^3-3x^2+2}{3(x-1)} = -1$$

$\Rightarrow$ f este der. pe (0,3).
Functia g este continua si derivabila pe [0,3]
ca functie elementara.

$g'(x)=1 \overset{t.\ cauchy}{\Rightarrow} (\exists)$ cel putin un punct $c\in(0,3)$

$$\text{a.i. } \frac{f'(c)}{g'(c)} = \frac{f(3)-f(0)}{g(3)-g(0)} = -1/9$$

$$f'(x)=\begin{cases}x^2-2x & x\in(1,3)\\ -1 & x\in(0,1)\end{cases}$$

$$g'(x)=1$$

$\text{Cazul } 1: x\in(1,3)$

$$f'(c)=c^2-2c\ ;\ g'(c)=1\Rightarrow \frac{f'(c)}{g'(c)}=c^2-2c=-1/9$$

$$\Rightarrow 9c^2-18c+1=0$$

$$\Delta=288\ \Rightarrow c_1=\frac{3+2\sqrt{2}}{3}\in(1,3)$$

$\text{Cazul } 2: x\in(0,1)$

$$f'(c)=-1\quad g'(c)=1\ \Rightarrow -1=-1\,(A)$$

$$\Rightarrow c=\frac{3+2\sqrt{2}}{3}$$