THE ANALOGUE AND GENERALIZATION OF ONE THEOREM OF GERGONNE

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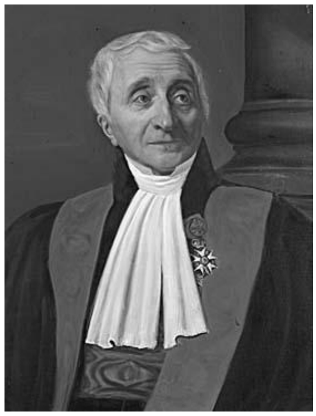
Abstract. In this work we are going to prove the spatial analogue of one plane geometrical theorem of Gergonne, relying on the analogue concepts and results of plane and spatial geometry. After that we are going to define one genaralization of the theorem and we are going to prove it. .

Key words and phrases : triangle, circle, distance, straight line, plane, tetrahedron, sphere.
ZDM Subject Classification: G10, G70.



Gergonne, Joseph Diaz (Nancy, France, 1771 June 19th - Montpellier, 1859 May 4th .) He was a French Mathematician, who studied at the College de Nancy and after that he taught private students. During the French revolution he joined the army and in 1795 he became the professor of Mathematics in Nimes at the recently estabilished École Centrale. He was impressed and inspired by Monge who was the director of École Polytechnique in Paris at this time. In 1810 he founded his own periodical with the title *Annales de mathématiques pures et appliquées,* which appeared until 1830 and many popular Mathematicians of his age published in it. He dealt especially with Geometry, but he published work on other domains of Mathematics, too. In 1816 he became the leader of Astronomy department at Montpellier university, in 1830 he became the Rector of the university and he taught here until 1844 when he retired.

His best known result is the solution of Appollonius's problem. In projective geometry he introduced the concept of Duality. In primary mathematics the so called Gergonne point is related to his name. This is the following: if the sides *AB, BC* and *CA* of the triangle ABC are touched by the circle inscribed in *A', B', C'* points, then *AA', BB', CC'* straight lines are running together and that G point was named Gergonne point in honour of the scientist. .

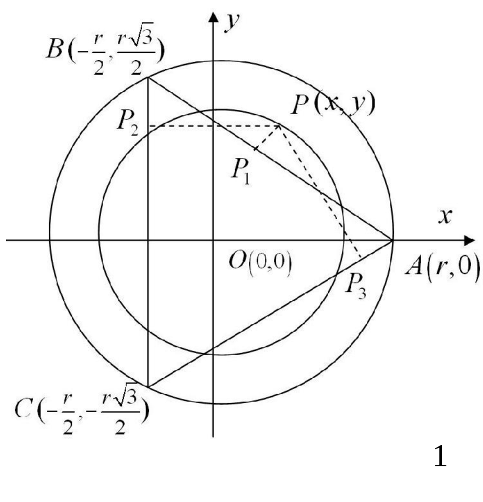
Of course many other results and theorems keep Gergonne's name, one of them for example is the following:

Theorem: To each point of the circle inscribed the regular triangle is true that the square sum of the distances from the sides of the triangle is constant.

We can read about the theorem and two different proofs in [1]and [2] where the proof is done with the tools of the classical plane geometry.

Starting from this theorem a bit more genaral form of the theorem can be defined which is the following:

Proposition 1 : Let's define the set of those P plane points for which the sum of the squares of the distances measured from the sides of the ABC regular triangle is constant. In the following we would be interested in a solution which can be extended to space, too with analogy and which can be made general to the case of the regular polygon. Because of this during our solution we should use tools, procedures and methods which have their spatial analogues or which can be made general. Let's look at such a solution!



Solution:Let's arrange the regular triangle ABC in the rectangular coordinate system in such a way that the coordinates of its vertexes should be the followings:

$A(r,0)$, $B\left(-\frac{r}{2},\frac{r\sqrt{3}}{2}\right)$ and $C\left(-\frac{r}{2},-\frac{r\sqrt{3}}{2}\right)$, where *r> 0* is a fixed point (as it is shown in the figure above). The P_1,P_2,P_3 are in order the projection of P point drawn perpendicular to the *AB, BC, CA* sides of the triangle ABC so the sum in point is : $PP_1^2+PP_2^2+PP_3^2=k$ (*) , where k> 0 is a given constant.

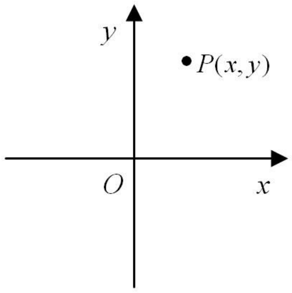
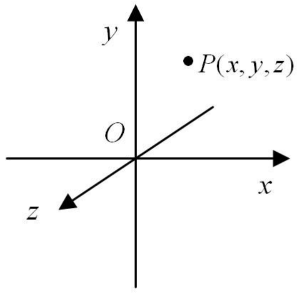
For the analogue extension and genaralization of the proposition we use the tools of coordinate geometry. It is known that the equation of the straight line crossing the two (x_1,y_1) and (x_2,y_2) points is the following: $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$. According to this if the equation of *AB, BC, CA* side straight lines is in order d_{AB},d_{BC},d_{CA}, then: $d_{AB}:\sqrt{3}\cdot x+3y-r\cdot\sqrt{3}=0$, $d_{BC}:2x+r=0$, $d_{CA}:-\sqrt{3}\cdot x+3y+r\cdot\sqrt{3}=0$.

If the coordinates of P variable point are *P*(*x,y*), then according to the distance formula between the point and the straight line it can be written that: $PP_1=\frac{\left|\sqrt{3}\cdot x+3y-r\cdot\sqrt{3}\right|}{\sqrt{12}}$,

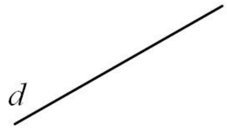
$PP_2=\frac{|2x+r|}{2}$, $PP_3=\frac{\left|-\sqrt{3}\cdot x+3y+r\cdot\sqrt{3}\right|}{\sqrt{12}}$.

Now if we write PP_1, PP_2, and PP_3 expressions into the (*) relation , then after countings we get that $x^2+y^2=\frac{2}{3}k^2-\frac{r^2}{2}$, which is the equation of such a circle, which has O(0,0) central point and $R=\sqrt{\frac{2}{3}k^2-\frac{r^2}{2}}$ ray. . (The circle can be with real or imaginary ray or it can degenerate into the O point.). So the wanted geometrical place is such a circle the central point of which is identical with the central point of the triangle and its ray is $R=\sqrt{\frac{2}{3}k^2-\frac{r^2}{2}}$, and in fact this circle is concentric either with the circle inscribed or roundscribed and contains those.

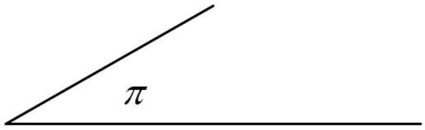
In order to define and prove the analogue proposition we should compare the knowledge used to prove the previous proposition and the corresponding spatial knowledge which are in fact analogue concept pairs:

Point in the plane	**Point in the space**
	

Straight line in the plane **Plane in the space**

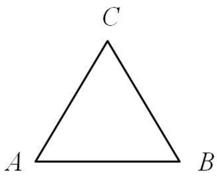


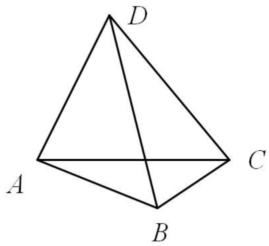
$d: ax + by + c = 0$



$\pi: ax + by + cz + d = 0$

Regular Triangle **Regular Tetrahedron**





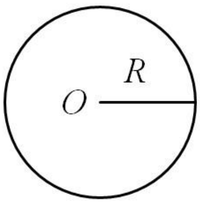
The equation of the straight line in plane **The equation of plane in space**

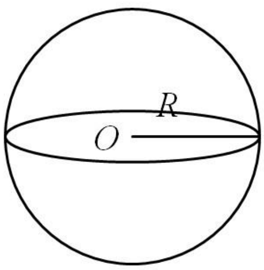
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

The distance of the point from a straight line **The distance of the point from a plane**

$$d(P,e) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \qquad d(P,\pi) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Circle in the plane **Sphere in the space**



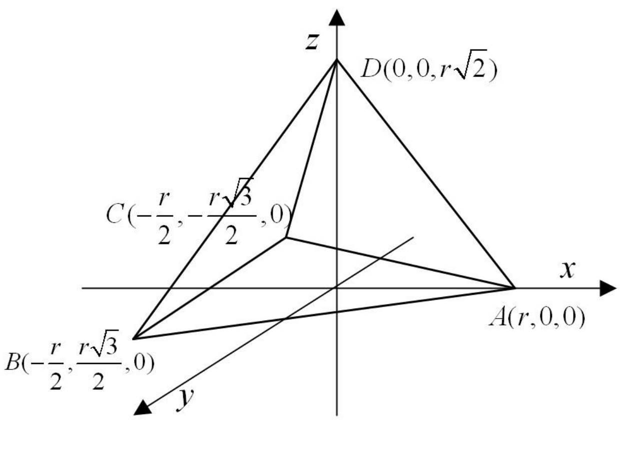


$$(x-a)^2 + (y-b)^2 = r^2 \qquad (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

According to these analogue concept and relation pairs we will be able to define and prove the spatial analogue of Proposition 1:

Proposition 2: Let's define the set of those spatial P points for which the sum of the squares of the distances measured from the lateral face of a regular tetrahedron *ABCD* is constant!

Solution: Let's arrange the regular *ABCD* tetrahedron into the rectangular coordinate



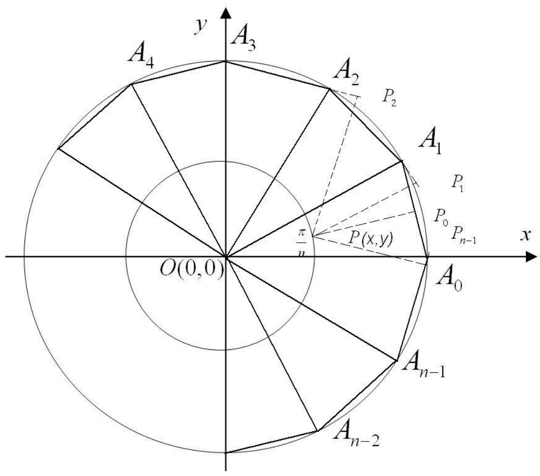
system in such a way that the coordinates of its wertexes to be the followings:
$A(r,0,0)$, $B\left(-\frac{r}{2},\frac{r\sqrt{3}}{2},0\right)$, $C\left(-\frac{r}{2},-\frac{r\sqrt{3}}{2},0\right)$ and $D(0,0,r\sqrt{2})$ where *r> 0* is a fixed number (as you can see in the figure). The P_1,P_2,P_3,P_4 are in order the projections of P point drawn perpendicular to the planes (ABC), (BCD), (CDA), (DAB) of regular *ABCD* tetrahedron , so the sum in point is: $PP_1^2+PP_2^2+PP_3^2+PP_4^2=k$ (*) , where k> 0 is a given constant. Now we can write the equation of the planes defined by the three points, which are the followings:
(ABC): z= 0; (BCD): $2\sqrt{2}\cdot x-z+r\sqrt{2}=0$; (CDA): $\sqrt{6}\cdot x-3\sqrt{2}\cdot y+\sqrt{3}\cdot z-r\sqrt{6}=0$ and (DAB) : $\sqrt{6}\cdot x+3\sqrt{2}\cdot y+\sqrt{3}\cdot z-r\sqrt{6}=0$. If the coordinates of P spatial variable point are *P*(*x*,*y*,*z*), then according to the distance formula between the point and the plane it can be written that:

$PP_1=|z|$; $PP_2=\frac{|2\sqrt{2}\cdot x-z+r\sqrt{2}|}{3}$; $PP_3=\frac{|\sqrt{6}\cdot x-3\sqrt{2}\cdot y+\sqrt{3}\cdot z-r\sqrt{6}|}{3\sqrt{3}}$; and $PP_4=\frac{|\sqrt{6}\cdot x+3\sqrt{2}\cdot y+\sqrt{3}\cdot z-r\sqrt{6}|}{3\sqrt{3}}$. Now according to the $PP_1^2+PP_2^2+PP_3^2+PP_4^2=k$ relation after countings we get that, $x^2+y^2+\left(z-\frac{r\sqrt{2}}{4}\right)^2=\frac{3}{4}k^2-\frac{3}{8}r^2$, which is the equation of such a sphere, the central point of which is exactly the central point of the tetrahedron $(0,0,\frac{r\sqrt{2}}{4})$ and its ray is $R=\sqrt{\frac{3}{4}k^2-\frac{3}{8}r^2}$. And with this we have already defined the discussed geometrical place in space too.

In the following steps we turn to the generalization of Proposition 1, in such a way hat instead of the regular triangle we take a regular polygon with *n* sides:

Proposition 3: Let's define the set of those P points in plane for which the sum of the squares of the distances measured from the sides of $A_0A_1A_2...A_{n-1}$ regular polygon with *n*-sides is constant!

Solution: We arrange the discussed regular polygon with ***n*** sides in the coordinate system according to the next figure. As the roots of $z^n=r^n$ binom-equation (r> 0) are $z_k=r\cdot\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right)$ for every $k\in\{0,1,...,n-1\}$ are the wertexes of the regular polygon with ***n*** sides which can be roundscribed and which has its central point the ***origo*** and its ray ***r,*** that's why the coordinates of A_k points are $A_k(x_k,y_k)$, where $x_k=r\cdot\cos\frac{2k\pi}{n}$ and $y_k=r\cdot\sin\frac{2k\pi}{n}$ for every $k\in\{0,1,...,n-1\}$. Let's write the equation of the sides A_kA_{k+1} where $A_n\equiv A_0$. These are the following:



$a_k\cdot x+y+c_k=0$, where $a_k=ctg\frac{(2k+1)\pi}{n}$ (1) and $c_k=-r\cdot\cos\frac{\pi}{n}:\sin\frac{(2k+1)\pi}{n}$ (2) for every $k\in\{0,1,...,n-1\}$. In this way according to the relation $PP_0^2+PP_1^2+...+PP_{n-1}^2=\kappa$ it can be written that:

$$\kappa^2 = \sum_{k=0}^{n-1} PP_k^2 = \sum_{k=0}^{n-1} \frac{\left(a_k \cdot x + y + c_k\right)^2}{1+a_k^2}$$ that is $A\cdot x^2 + B\cdot y^2 + 2C\cdot xy + 2D\cdot x + 2E\cdot y + F = k^2$ (3)

where $A = \sum_{k=0}^{n-1} \cos^2 \frac{(2k+1)\pi}{n}$, $B = \sum_{k=0}^{n-1} \sin^2 \frac{(2k+1)\pi}{n}$, $C = \sum_{k=0}^{n-1} \sin \frac{(2k+1)\pi}{n} \cdot \cos \frac{(2k+1)\pi}{n}$,

$D = \sum_{k=0}^{n-1} \frac{a_k c_k}{1+a_k^2}$, $E = \sum_{k=0}^{n-1} \frac{c_k}{1+a_k^2}$, $D = \sum_{k=0}^{n-1} \frac{c_k^2}{1+a_k^2}$ where the values of a_k, c_k are given by the (1) and (2) relations. In the following steps we are giong to prove that:

$$(a)\ A = B = \frac{1}{2}n \qquad (b)\ C = D = E = 0 \qquad (c)\ F = n\cdot r^2 \cdot \cos^2 \frac{\pi}{n}$$

For the proof we should notice that in the sense of Viéte relations for the roots of the equation $z^n = r^n$ is true that $z_0 + z_1 + ... + z_{n-1} = 0$, that is $\sum_{k=0}^{n-1} r\cdot\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) = 0$,

that's why $\sum_{k=0}^{n-1} \cos\frac{2k\pi}{n} = \sum_{k=0}^{n-1} \sin\frac{2k\pi}{n} = 0$ (4).

According to the Viéte relations also $\sum_{0\le i<j\le n}^{n-1} z_i z_j = 0$ as $\sum_{k=0}^{n-1} z_k = 0$, that's why it is also true that

$\sum_{k=0}^{n-1} z_k^2 = 0$, from which $\sum_{k=0}^{n-1}\left(\cos^2\frac{2k\pi}{n} - \sin^2\frac{2k\pi}{n}\right) = 0$ and $\sum_{k=0}^{n-1} \sin\frac{(2k+1)\pi}{n} \cdot \cos\frac{(2k+1)\pi}{n} = 0$

(5). According to the basic formula $\sin^2 t + \cos^2 t = 1$, from (5) comes that

$\sum_{k=0}^{n-1}\left(1 - \sin^2\frac{(2k+1)\pi}{n}\right) = \sum_{k=0}^{n-1} \sin^2\frac{(2k+1)\pi}{n}$, in this way $\sum_{k=0}^{n-1} \sin^2\frac{(2k+1)\pi}{n} = \sum_{k=0}^{n-1} \cos^2\frac{(2k+1)\pi}{n} = \frac{1}{2}n$.

Moreover

$$A = \sum_{k=0}^{n-1} \cos^2\left(\frac{2k\pi}{n} + \frac{\pi}{n}\right) = \sum_{k=0}^{n-1}\left(\cos\frac{\pi}{n}\cos\frac{2k\pi}{n} - \sin\frac{\pi}{n}\sin\frac{2k\pi}{n}\right)^2 = \frac{1}{2}n\left(\sin^2\frac{\pi}{n} + \cos^2\frac{\pi}{n}\right) = \frac{1}{2}n$$

(6). In the similar way it can be seen that $B = \frac{1}{2}n$. And using aslo the formulas

$\sin(p+q) = \sin p \cos q + \sin q \cos p$ and $\cos(p+q) = \cos p \cos q - \sin q \sin p$ for the values

$p = \frac{2k\pi}{n}$, $q = \frac{\pi}{n}$ we can immediately get that C= 0 .

Then in order to define the values of D, E, F besides the relation (4) we also use the formulas $ctgt = \cos t : \sin t$ and $1 + ctg^2 t = 1 : \sin^2 t$ according to which if we write the values of a_k, c_k from (2) we immediately get the following relations:

$D = -r\cos\frac{\pi}{n} \cdot \sum_{k=0}^{n-1} \cos\frac{(2k+1)\pi}{n} = -r\cos\frac{\pi}{n}\left(\cos\frac{\pi}{n} \cdot \sum_{k=0}^{n-1}\cos\frac{2k\pi}{n} - \sin\frac{\pi}{n} \cdot \sum_{k=0}^{n-1}\sin\frac{2k\pi}{n}\right) = 0$ and

$$E = -r\cos\frac{\pi}{n} \cdot \sum_{k=0}^{n-1} \sin\frac{(2k+1)\pi}{n} = -r\cos\frac{\pi}{n}\left(\cos\frac{\pi}{n} \cdot \sum_{k=0}^{n-1}\sin\frac{2k\pi}{n} + \sin\frac{\pi}{n} \cdot \sum_{k=0}^{n-1}\cos\frac{2k\pi}{n}\right) = 0$$

also $F = r^2 \cdot \sum_{k=0}^{n-1} \cos^2\frac{\pi}{n} = n\cdot r^2 \cdot \cos^2\frac{\pi}{n}$. So according to the results of (a), (b), (c) the equation

under (3) will be the equation $x^2 + y^2 = 2\left(\frac{k^2}{n} - r^2 \cdot \cos^2\frac{\pi}{n}\right)$ which is the equation of a circle with central point O(0,0), the ray of which is $R_n = \sqrt{2\left(\frac{k^2}{n} - r^2 \cdot \cos^2 \frac{\pi}{n}\right)}$. We can see that in the case *n=3* we get back the $R_3 = \sqrt{\frac{2}{3}k^2 - \frac{r^2}{2}}$, already found ray.

In order to extend the presented analogue or generalized results we can consider whether it will remain valid in the case of spatial regular geometric solids? The answer to this is not so easy, because if in the plane there is a regular polygon with ***n*** optional number, the number of regular geometrical solids in space is limited to the tetrahedron, hexahedron (cube), octahedron, dodecahedron, icosahedron. For the regular tetrahedron we have already proved the analogue result and in this way we can easily prove for the case of the cube, but for the other three regular geometrical solids the countings are more difficult, that's why we do not deal with it in this work. .

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