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FROM MATCHSTICK PUZZLES TO ISOPERIMETRIC PROBLEMS

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Abstract. In this paper, we are going to present a way from a simple matchstick puzzle to isoperimetric problems. This is a much more interesting, simple and instructive way to learn the basics of isoperimetry. The elements of mathematical history used in the paper are meant to make the topics described more colorful and interesting. One of the important features of the paper is that all the issues are discussed using elementary mathematics, so that it would be easily accessible to anyone.

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MSC Subject Classification: 97G30

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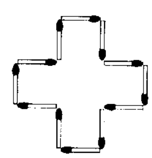
Introduction

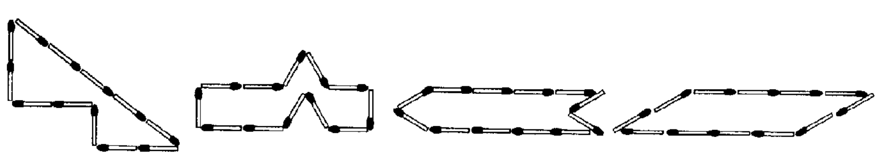
There are probably few of our readers who have not met any kind of matchstick puzzle in their lives. Nowadays, these have become more and more fashionable and ingenious, as we can also try to take action in order to solve them. In most cases we have to move a matchstick (or sticks) and replace it, we have to add to or take away from them so that we come to a given result or situation. And this is not always easy; often a puzzle has more than one solution, with one being more ingenious than the others. The starting point of this topic is simply a matchstick puzzle.

A matchstick puzzle and its solution

We have formed a cross in this Figure from 12 matchsticks. Its area is 5 matchstick squares. Is it possible to use all the 12 matchsticks without any overlapping to form a shape whose area is only 4 matchstick squares, but whose perimeter, however, remains 12 matchstick units?



The answer is: yes. The puzzle has several solutions, and here are some of them:



Starting from this puzzle, we could even make many other shapes, whose areas we can measure. Here are some of these shapes:

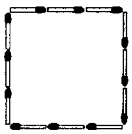
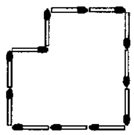
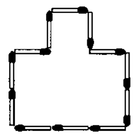
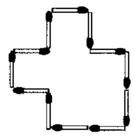
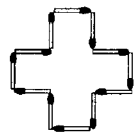
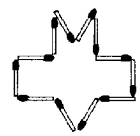
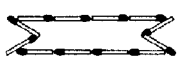







Figure 1 Figure 2 Figure 3 Figure 4 Figure 5 Figure 6



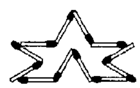


Figure 7 Figure 8 Figure 9 Figure 10 Figure 11

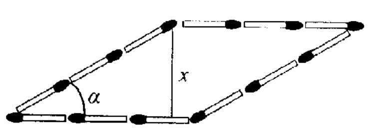
The areas of the polygons in Figures 1–11 are, in sequence: 9; 8; 7; 6; 5; 4; $4-(\sqrt{3})/2 \approx 3{,}14$; 3, $3-(\sqrt{3})/2 \approx 2{,}14$; 2; $\sqrt{3} \approx 1{,}73$ matchstick squares. The area and the shape of the forms made can change in many other ways. We can, of course, formulate some questions regarding this aspect:

1) What size could the area of the smallest possible shape of 12 matchsticks in perimeter be?
2) What size could the area of the largest possible shape of 12 matchsticks in perimeter be?
3) Within these limits, what are the exact area values that the shapes of 12 matches in perimeter would result in?

Attempts and endeavors

While designing the shapes of these 11 figures, it is hardly possible that the idea by which we could decide whether we have reached the smallest area of polygon occurs to us. We have to choose another way to experiment. The parallelogram to be found in the fourth solution of the matchstick puzzle presented, or another parallelogram may be able to provide us with a hint. And, in a moment, we shall see why that is so.

Let us imagine a 3×3 parallelogram instead of the 3×3 square of Figure 1. Let us regard it as "articulately movable" in its four corners (and only there). This is when we can change the size of the acute angle α of the parallelogram, or that of its height x. If we choose the latter option, the area of the parallelogram will be $T = 3\cdot x$ matchstick squares.



Let us consider the function defined by the formula $f\colon (0;3] \to (0;9]$, $f(x) = 3 \cdot x$. We can easily check that for any number expressing a square measure of $T \in (0;9]$ there is a number $x \in (0;3]$, to which $T = 3 \cdot x$; and this existing value x is $x = T/3 \in (0;3]$, counted on a pre-elected T. Actually, in this way we have checked the surjectivity of function f. We can draw three important conclusions:

1° By decreasing the height of an “articulately movable” parallelogram x, we can arrive to a T which may be as small a (positive) number as possible, while the perimeter of it remains 12 matchsticks long.

2° By selecting any area $T \in (0;9]$ in advance, we can elect the height of the “articulately movable” parallelogram so that its area would exactly be T. That is, according to the values of the variable x, the parallelogram of a perimeter of 12 matchsticks and of an area of T can be made in case of any area $T \in (0;9]$. (And it will not even be a concave polygon as the ones we have seen in Figures 1–11.)

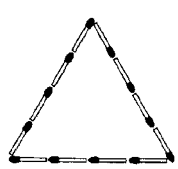
3° We can make the “articulately movable” parallelogram of the largest area if $x = 3$, that is, the parallelogram is a 3×3 square.

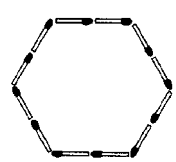
What shape should we make in order to get the largest possible area? If we take notice of the fact stated in the previous three conclusions, namely that in those conditions the largest area was that of a 3×3 square, we may soon realize that as the square is a regular quadrilateral shape, why not make other regular polygons out of the 12 matchsticks.

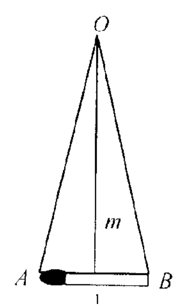
It is easy to demonstrate that an equilateral triangle can be made whose side-length is of 4 units. The area of this is

$$T_3 = \frac{a^2 \cdot \sqrt{3}}{4} = \frac{4^2 \cdot \sqrt{3}}{4} = 4 \cdot \sqrt{3} \approx 6{,}92,$$

which is much less than 9.







Let us make a regular convex polygon with more than 4 sides! For instance, we can use the 12 matchsticks to make a regular hexagon with a length of 2 units along one side. The area of this is equal with the area of six regular triangles having a length of 2 units along one side, that is,

$$T_6 = 6 \cdot \frac{a^2 \cdot \sqrt{3}}{4} = 6 \cdot \frac{2^2 \cdot \sqrt{3}}{4} = 6 \cdot \sqrt{3} \approx 10{,}38,$$

and this is already more than 9.

So, it would be good if we could make a regular convex polygon with even more sides, more precisely, with the most possible sides!

The 12 matchsticks could be used to make a regular dodecagon of 1 unit side-length. Let us call the centre of this O, and two of its consecutive vertices A and B. Let its length drawn from O be m. $\angle AOB = 30°$, consequently

$$m = \frac{1}{2} \cdot \frac{1}{\operatorname{tg} 15°} = \frac{1}{2(2-\sqrt{3})} = \frac{2+\sqrt{3}}{2}.$$

Therefore, the area of the regular dodecagon in question is

$$T_{12} = 12 \cdot \frac{1 \cdot m}{2} = 3(2+\sqrt{3}) \approx 11{,}19.$$

And this is even more than that of the value of T_6. Naturally, we may ask the question: can we get a number even greater than T_{12}? Our intuition suggests this is not probable. How could we prove it, though? Not too easily, as by our analysis, we fall into a mathematical topic of a less elementary level, that of the so called isoperimetric problem. Let us, then, have a look into the parts of this field which may be important to us now.

A short review of the planar isoperimetric problem

The word “isoperimetric” is a compound of the constituents ‘iso’ (permanent) and ‘perimeter’. To put it simple, the problem is the following:

a) What is the area which could be surrounded with a rope of a given length?

b) Which of the plane figures delimited by a rope of a certain length has the largest area?

c) Which plane figure, having a given perimeter, has the largest area?

These questions only differ in a formal aspect, in the way they are phrased, for the answer can be given to all by the same means.

The planar isoperimetric theorem is the following:

Theorem 1. *Among all planar regions with the same perimeter, the circle encloses the greatest area.*

The theorem *sounds* simple. However, there does not yet exist an elementary proof been brought to it yet. Before we write about the historical aspects of the problem and the theorem, let us phrase two of the isoperimetric theorems of polygons:

Theorem 2. *If we have n matchsticks, then the largest area is enclosed by a regular n-sided polygon.*

Theorem 3. *Among polygons with the same given perimeter, there is none with the largest area. In other words, from the regular polygons with the same perimeter, having n and, respectively $n+1$ sides, the latter has a greater area.*

This latter statement is also founded on some intuition, for if we start to draw regular polygons with a greater and greater number of sides within a circle, their area will get "closer and closer" to that of the circle without reaching it, as according to Theorem 1.

First and foremost, let us look at a little historical review.

As the legend holds, the origins of the isoperimetric problem are connected to Dido's name, who was the daughter of King of Tyre. She married her uncle, Acerbas, who was soon murdered for his extraordinary fortune. Hereupon, Dido fled to Cyprus, wherefrom she sailed on to the African coast close to Sicily. She visited the ruler of the area, and told him that she would like to purchase as much land on the coast, as she would be able to enclose with the hide of a bull. The ruler smilingly agreed to the request of the beautiful queen, what is more, he even generously granted her a huge hide of a bull. The clever Dido cut the hide into thin strips, and by binding these, she got such a long string that she was able to enclose a much larger area on the coast (and reaching into the sea) that the ruler would have imagined. This is how she founded the flourishing city of Carthage, the queen of which she soon became.

Zenodorus was a Greek mathematician who had already dealt with isoperimetric shapes in 150 BC, and he proved 14 theorems making part of this topic, proving among others theorem 2 above. Unfortunately, his book entitled *Isoperimetric shapes* was destroyed, but his results were made public and proven by Pappus of Alexandria in around 300 AD.

A number of famous mathematicians dealt with this topic in the modern times. A few well-known names are: Descartes (1596–1650), Jacob Bernoulli (1645–1705), Johann Bernoulli (1667–1748), Euler (1707–1783), Lagrange (1736–1813). Undoubtedly, the one whose work was the consummation of all earlier results was Swiss mathematician Jacob Steiner (1796–1863). He synthesized earlier results, adding new ideas to the subject. However, all of them (including Zenodorus) regarded it as obvious and did not prove the fact that this problem does have a proven solution. Dirichlet (1805–1859) was the first to observe the deficiency in the proof of the isoperimetric theorem; and it was Weierstrass (1815–1892) who finally eliminated this deficiency in 1870, when he properly proved the famous extreme nature of the circle. His entire work resulted in revolutionary changes from a mathematical point of view. A number of other mathematicians dealt with the different proofs of the problem, but there have not yet been any truly elementary confirmations.

Answers to the formulated questions

In the course of our experiments, we have proven that, by using 12 matchsticks, *any* shape of *the smallest possible area* can be made (while retaining a perimeter of 12 matchstick units)—specifically, this is an "articulately movable" parallelogram. What is more, one can make parallelograms having an area of any $T \in (0;9]$ matchstick squares. Beside the methods presented, this can also be reached by changing the angle α of the "articulately movable" parallelogram, as $T = 3 \cdot x = 9 \sin \alpha$, and, for symmetrical reasons, it is enough to examine the case $\alpha \in (0; 90°]$, when $\sin \alpha$ takes all the values from the interval $(0; 1]$. The final answer to forming a polygon of the maximum area is given by Theorems 2 and 3: the largest area can indeed be formed by a 12-sided regular convex polygon studied during our experiments, and this will be of $3(2+\sqrt{3})$ matchstick squares, as the dodecagon is the regular polygon with the most sides which it is possible to make out of 12 matchsticks. Consequently, the area of the possible polygons of 12 matchsticks in perimeter falls in the interval $(0;\ 3(2+\sqrt{3})]$. However, we can offer no answer to the question whether any polygon having an area of $T \in (9;\ 3(2+\sqrt{3})]$ matchstick squares can be made. Since we believe that, however elementary the answer to that may be, its proof would not be a short one, we leave the eager reader to examine this problem him/herself.

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