Internal analogies between isomorphic geometric structures

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Abstract. One of the most important aims of learning and teaching Mathematics is solving problems. This cannot be imagined without analogical thinking. The word analog has a Greek origin, meaning similar, the same as something else from certain points of view. In the case of systems *A* and *B*, if the *a, b, c* and *k* essential characteristics refer to *A* and *a, b* and *c* essential characteristics refer to *B* as well, then according to the analogy of the characteristics of *A* and *B* we can state that *k* characteristics can be expected to refer to *B*, but it does not certainly refer to it. So if we say that two things are similar if they are the same from one point of view, then we say that two things are analog if their corresponding parts are in equal relations. György Pólya dealt with the analogy deeply in [8]-[10], where he emphasizes the importance of analogy in education and teaching, in problem solving, in discovery and simply in everyday thinking. He also emphasizes that analogical conclusion is risky as the consequence made by analogy is only probable and not sure, it needs to be proved. In my paper I am writing about analogies which are clear analogies, which are sure, because they appear among isomorphic structures. Throwing light upon and analysing these analogies, we can reveal their importance in solving geometrical problems.

Key words and phrases: analogy, isomorphism, structure, vector, complex.

ZDM Subject Classification: D40, D50, G10, G40, G70

Introduction

In modern Mathematics teaching, using analogies has a much importance in every domain of Mathematics. But analogy has a different importance in solving geometric problems. There we need very much to make conclusions with analogy, as this provides the basis of our guesses which are important for the solution.

In [3] the authors deal with interesting themes, with analogical conclusions and generalizations, they compare and contrast 2-D plane and 3-D space geometry and throw light upon the fact that if we replace given plane concepts with analog space concepts, with plane problems analog problems can be composed. These analogies are called external analogies and as we mentioned before, the conclusion is only probable, it still needs to be proved.

In my paper I examine analogies, which are internal and clear analogies, as the analogy appears in the frame of the same system or structure. So these are a clear and sure analogies, as they appears in the isomorphic classes of the same structure.

In the following we are trying to throw light upon why it is possible, when it is advantageous and how geometric problems can be solved with vectors, complex numbers (affixes), on the complex numerical plane of Gauss, or in $R \times R$ coordinate system, with analytic methods.

For this, first of all we throw light on geometric problems solved using vectorial methods or, using the tools of analytic coordinate geometry, or in a compex numerical plane; we can only talk about the differences in the so called ”language” or ’’model” so we only analyse which model is more advantageous, easier, more spectacular, or which solution is more instructive.

First of all we rewrite the more important parts of classical vector geometry theory into complex numerical plane, into affixes. We do the same with some parts of the results of the $R \times R$ coordinate geometry. In both cases we compare some important concepts and results.

After this in concrete cases we throw light upon the advantages of the one or the other method, its application fields, its limits or disadvantages.

The Isomorphy of real plane of Euclid and complex plane of Gauss

One important task of Algebra is examining the isomorph invariances. If we look at one set, (class) of algebraical stuctures, then in this set the isomorphism will probably make a relation which is:

- ***Reflexive,*** as the identical mapping is isomorphism
- ***Symmetrical*** as the isomorphic mappings have their inverses and that these are isomorphism as well.
- ***Transitive*** as the product of isomorphisms is an isomorphism as well

According to these, on the examined set of structures the isomorphism defines one eqivalence relation and that is called *isomorphia*.

The isomporhia -as an equivalence relation – defines one class of the set of structures.

An equivalence class contains those structures which are isomorphic with one another. These are the so called isomorphia classes.

From the point of view of abstract algebra the structures belonging to one isomorphia class *cannot be considered different.* This is the **PRINCIPLE OF ISOMORPHIA,** which was first composed by E. Seintz during his examinations on theory test in 1912. An isomorphia class is named as abstract algebraical structure. The concrete algebraical structure is one representative of one isomorphia class. In this case two representatives of the same class mean the same only a *''language adaptation",* a rewriting is necessary.

We emphasize the importance of these things for High School Mathematics with three representatives of one important and representative isomorphia class.

Interpreting on the (x, y) elements of $\boldsymbol{R^2 = R \times R}$ the so called Euclid numerical plane, the addition "as component" and scalar multiplication, keeping the common characteristics we get such a vector space with two dimensions which has one of its standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Looking at the algebraical form of $z = a + i \cdot b$ $(a, b \in R,\ i^2 = -1)$ C complex numbers we can interpret additon of "real part with real part" and "imaginary part with imaginary part" and the multiplication with scalar. According to the fulfilment of the axioms we also get a vector space with two dimensions and one of its basis is for example ***1*** and ***i***. We most often call this numerical plane the complex numerical plane of Gauss, using the name of its creator, which became popular in specialised bibliography as well.

Let us note with $\boldsymbol{V_2}$ the set of free vectors of the plane. Using knowledge of the triangle or parallelogram we can interpret the additon of two vectors, as well as the multiplicative operation with a scalar which fulfills the axioms. In this case we also interpret a vector space with two dimensions, and one of its basis is $\vec{i}$ and $\vec{j}$

For simplicity we only mark these three vector spaces with $\boldsymbol{R^2}$, ***C***, $\boldsymbol{V_2}$. It is known that **each vector space with n dimensions is isomorphic with $\boldsymbol{R^n}$ *vector space.*** So in our cases the three vector spaces with 2 dimensions make an isomorphia class, which have the three previous representatives.

(1) $\boldsymbol{V_2 \cong C}$ **(2)** $\boldsymbol{R^2 \cong C}$ **(3)** $\boldsymbol{V_2 \cong R^2}$ isomorphism.

What is more, these isomorphisms are distant preserving, so called isometries, as the following results are true:

If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any two points of $\boldsymbol{R^2}$, then according to the interpretation the distance between the two points is:

$$d_R^2(P_1, P_2) = |P_1P_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

If $P(z_1)$, $P(z_2)$, $z_1 = x_1 + i\cdot y_1$, and $z_2 = x_2 + i\cdot y_2$ are any two points of **C** complex numerical plane, then the distance between the two points is:

$$d_C(P_1, P_2) = |(x_1-x_2)+i\cdot(y_1-y_2)| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

If P_1, P_2 are any two points in $\{O, \mathbf{i}, \mathbf{j}\}$ vector space, as $\overline{P_1P_2} = \overline{OP_2} - \overline{OP_1} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j}$, because of this the distance between the two points is:

$$d_{V_2}(P_1, P_2) = \|P_1P\| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\ .$$

The isomorphism **(1)** makes possible the "translation" of the vectorial description of plane geometry into the language of complex numerical plane and vice versa**.**

The isomorphism **(2)** makes possible the "translation" of the description of coordinate geometry into the language of complex numerical plane and vice versa.

The isomorphism **(3**) makes possible the "translation" of the vectorial description of plane geometry into the algebraical language of the coordinate geometry and vice versa. In the following we are going to discuss the application possibilities and areas of the results provided by these isomorphisms.

The internal analogy of V_2 vector plane and C complex numerical plane

In the next steps, following only practical points of view, we compare the more important classical vector geometrical concepts and results with its corresponding analog concepts and results in the complex numerical plane.

According to the isomorphism **(1)** described above we correspond evidently one and only one complex number to each point of the plane. In the following we mark these with the corresponding lower case letters (ex. *a, b, c, m, n*, …) of the capitals (ex. *A, B, C, M, N*,…) which name the certain points. In any other cases we mark the exceptions in advance. These corresponding complex numbers are called the affixes of the given points.

More exactly if we call the $M(x, y)$ point the geometrical image of the $z = x + i\cdot y$ complex number, we call the complex number $z = x + i\cdot y$ the affixum of $M(x, y)$. Let us mark with P the set of of the points of Π plane, on which we have fixed *xOy* rectangular coordinate system.

We can easily see that in the case of every complex number $z = x + i\cdot y$ the function interpreted as $\phi : C \to R,\ \phi(z) = M(x, y)$ is a bijective mapping between **C** and **R** sets, because of this we call the Π plane (the elements of which we identified with the complex numbers by the ϕ function) complex numerical plane. In this way the *Ox* axis of the coordinate system is a real axis and the *Oy* axis is the imaginary axis.

In the *xOy* rectangular coordinate system with *O* origin we take a point *M*. Each point *M* corresponds to one and only one *m* complex number, so the vector $\overline{OM}$ corresponds to the complex number *m* which can be written : $\overline{OM} = m$. Then if *A* and *B* are two points of the previously mentioned plane, then as $\overline{AB} = \overline{OB} - \overline{OA}$, so the vector $\overline{AB}$ corresponds the complex number *(b- a)* , so we can identify it with that and write that: $\boxed{\overline{AB} = b - a}$, where $a, b \in C$. That is why the length of AB section is : $|\overline{AB}| = |b - a|$.

With the help of this connection we can rewrite the important vectorial results into complex numbers. In this way we get the important analog connections of $\boldsymbol{V_2}$ and ***C***.

1) If on the *(AB)* section *M* is a point to which $\dfrac{|\overline{AM}|}{|\overline{MB}|} = k > 0$, then:

$$\overline{OM} = \frac{\overline{OA} + k\overline{OB}}{1+k} \qquad \text{respectively} \quad m = \frac{a+kb}{1+k}$$

2) In the particular cases of *M* being exactly the midpoint, then:

$$\overline{OM} = \frac{\overline{OA} + \overline{OB}}{2} \qquad \text{respectively} \quad m = \frac{a+b}{2}$$

3) The M point is on *AB* if and only if there exist a $k \in R$ so that:

$$\overline{OM} = \frac{\overline{OA} + k\overline{OB}}{1+k} \qquad \text{respectively} \quad m = \frac{a+kb}{1+k}$$

4) The *ABCD* quadrilateral is a parallelogram exactly if:

$$\overline{OA} + \overline{OC} = \overline{OB} + \overline{OD} \qquad \text{respectively} \qquad a+c=b+d$$

5) If the centre of gravity of *ABC* triangle is *G*, then:

$$\overline{OG} = \frac{\overline{OA} + \overline{OB} + \overline{OC}}{3} \qquad \text{respectively} \qquad g = \frac{a+b+c}{3}$$

(We can observe that the position of the centre of the gravity does not depend on the position of O).

6) If the centre of the gravity of $A_1A_2...A_n$ n-angle is *G*, then:

$$\overline{OG} = \frac{\overline{OA_1} + \overline{OA_2} + ... + \overline{OA_n}}{n} \qquad \text{respectively} \qquad g = \frac{a_1 + a_2 + ... + a_n}{n}$$

7) If H is the orthocentre of ABC triangle then:

$$\overline{OH} = \overline{OA} + \overline{OB} + \overline{OC} \qquad \text{respectively} \quad h = a+b+c-2\cdot o$$

(We can notice that the position of the orthocentre depends on the position of O)

8) If the central point of the Euler-circle (the nine-point circle) is E, then:

$$\overline{OE} = \frac{\overline{OA} + \overline{OB} + \overline{OC}}{2} \quad \text{respectively} \quad e = \frac{a+b+c}{2} - o$$

(We can notice that the position of E point also depends on the affixum of O too!)

9) If I is the incentre of ABC triangle then:

$$\overrightarrow{OI} = \frac{BC\cdot\overrightarrow{OA} + CA\cdot\overrightarrow{OB} + AB\cdot\overrightarrow{OC}}{AB+BC+CA} \text{respecively} \quad i = \frac{BC\cdot a + CA\cdot b + AB\cdot c}{AB+BC+CA}$$

(We can notice that the position of I point does not depend on the position of O).

10) If *E, F, K* are in order the internal points of *AB, BC, CA* sides for which:

$AF \bigcap BK \bigcap CE = \{M\}$ ands $\frac{AE}{EB} = \frac{\alpha}{\beta}$; $\frac{BF}{FC} = \frac{\gamma}{\beta}$; $\frac{CK}{KA} = \frac{\alpha}{\gamma}$, then:

$$\overrightarrow{OM} = \frac{\alpha\cdot\overrightarrow{OA} + \beta\cdot\overrightarrow{OB} + \gamma\cdot\overrightarrow{OC}}{\alpha+\beta+\gamma} \qquad \text{respectively} \qquad m = \frac{\alpha\cdot a + \beta\cdot b + \gamma\cdot c}{\alpha+\beta+\gamma}$$

(We can notice that the position of M does not depend on the position of O).

11) If *E, F, K* are in order internal points of *AB, BC, CA* sides, then the *AF, BK, CE* line segments are concurent if:

$$\frac{\overline{AE}}{\overline{EB}} \cdot \frac{\overline{BF}}{\overline{FC}} \cdot \frac{\overline{CK}}{\overline{KA}} = 1 \qquad \text{respectively} \qquad \frac{e-a}{b-e} \cdot \frac{f-b}{c-f} \cdot \frac{k-c}{a-k} = 1$$

12)If $A_1B_1C_1$ and $A_2B_2C_2$ which the same orientation triangles are similar if and only if :

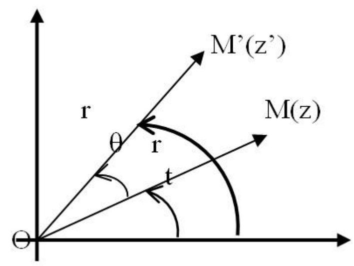
$$\frac{\overline{A_1B_1}}{\overline{A_2B_2}} = \frac{\overline{B_1C_1}}{\overline{B_2C_2}} = \frac{\overline{C_1A_1}}{\overline{C_2A_2}} \qquad \text{respectively} \qquad a_1\left(b_2-c_2\right)+b_1\left(c_2-a_2\right)+c_1\left(a_2-b_2\right)=0$$

Here we stop with the enumeration, but we emphasize the following: *each* vectorial connection can be rewritten into complex numbers using the connections of type $\boxed{\overline{AB} = b-a}$. With this many other important theoremas and results referring to many important points can be written. Many vectorial tasks and theoremas can be found for example in [4], [6] and [7].

Of course questions can arise that apart from the fact that between $\boldsymbol{V_2}$ and ***C*** there is a considerable internal analogy what is the use of pointing this out? The answer is that when: we are solving plane geometry problems we can choose which method is the most suitable. In many cases the vectorial solution is more suitable, but in many other cases the solution with the complex numbers is more advantageous. Why are the methods with complex numbers more suitable in some cases? We can realise it very easily if we read the followings.

Particular plane geometry characteristics on C set

We should notice that on the set of $\boldsymbol{V_2}$ vectors the operations which can be interpretted with vectors are: the addition and subtraction of vectors, the multiplication of the vectors with scalar, the scalar multiplication of the vectors, the vectorial multiplication of vectors, the mixed multiplication of vectors. The operations interpretted with the complex numbers are the following: addition, subtraction, multiplication, division, exponentiation, extraction of root.



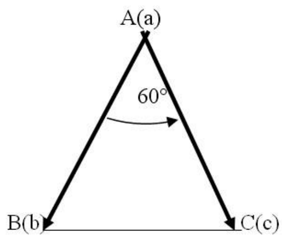
In addition complex numbers written in algebraical form $z = a + i \cdot b$ can also be written in trigonometrical form: $z = r(\cos t + i \sin t)$. With the complex numbers written in this way we can also do the previous operations. Among these operations the multiplication of complex numbers has a special importance. If $z = r(\cos t + i \sin t)$ and $z_\theta = \cos\theta + i\sin\theta$, then $z \cdot z_\theta = r(\cos t + i \sin t)\cdot(\cos\theta + i\sin\theta) = r(\cos(t+\theta) + i\sin(t+\theta)) = z'$.

This has a very important geometrical meaning. If we rotate the line *OM* around *O* through an angle of θ, then we get the line *OM'*, which can be written with the complex numbers as $\boxed{z' = z_\theta \cdot z}$. So a rotation around *O* means a multiplication with complex number with argument θ. If the central point of the rotation is not the origin *O,* but a point z_0 affixum point, then $z' - z_0 = z_\theta \cdot (z - z_0)$.

In geometrical problems in many cases are about the measurement of angles, so this result has great importance, and this result does not have a corresponding on in $\boldsymbol{V_2}$ plane.

The intoduced rotation has many other important consequences, let's see some of them:

1) The *ABC* triangle is equilateral if and only if, one of the following is true:



(a) $a-b=\varepsilon\cdot(c-b)$ or $b-c=\varepsilon\cdot(a-c)$ or $c-a=\varepsilon\cdot(b-a)$
where $\varepsilon=\cos 60°+i\sin 60°$
(b) $a-\varepsilon\cdot c+\varepsilon^2\cdot b=0$ or $b-\varepsilon\cdot a+\varepsilon^2\cdot c=0$ or $c-\varepsilon\cdot b+\varepsilon^2\cdot a=0$
where $\varepsilon=\cos 60°+i\sin 60°$
(c) $a+\tau\cdot b+\tau^2\cdot c=b+\tau\cdot c+\tau^2\cdot a=c+\tau\cdot a+\tau^2\cdot b=0$
where $\tau=\cos 120°+i\sin 120°$
(d) $a^2+b^2+c^2=ab+bc+ca$
2) The ABCD is a square if and only if
$a+c=b+d$ and one of the following affirmations is true:
(a) $a=(1-i)\cdot b+i\cdot c$ or $b=(1-i)\cdot c+i\cdot d$ or $c=(1-i)\cdot d+i\cdot a$ or $d=(1-i)\cdot a+i\cdot b$
where $i^2=-1$.
(b) $x+i\cdot y+i^2\cdot z+i^3\cdot t=0$ where *x, y, z, t* is the permutation of the comlplex numbers *a, b, c, d*
3) The $A_1A_2\ldots A_n$ is a regular polygon with *n* sides *(n > 3)* if and only if
$\boldsymbol{a_1+a_2\varepsilon+a_3\varepsilon^2+\ldots+a_n\varepsilon^{n-1}=0}$, where $\varepsilon^n=1$ one of the so called unit roots with n order and $\varepsilon\neq 1$.
4) *ABCD* is a symmetrical trapezoid if and only if: $a\cdot b=c\cdot d$
5) *ABCD* is a quadrilateral with perpendicular diagonals if and only if: $\frac{a}{b}=-\frac{c}{d}$
6) *ABCD* is a cyclic quadrilateral, if and only if $\frac{a-c}{a-d}:\frac{b-c}{b-d}\in R$

Apart from rotation the examination of the arguments of the complex numbers makes possible the statement of many other important special characteristics.
1) If $M_1(z_1)$ and $M_2(z_2)$ are two points of the complex numbers and O is the origin, then
$M_1OM_2\sphericalangle=\arg\frac{z_2}{z_1}$
2) If $M_1(z_1)$, $M_2(z_2)$ and $M_3(z_3)$are three points of the complex numerical plane and O is the origin, then:
$M_2M_1M_3\sphericalangle=\arg\frac{z_3-z_1}{z_2-z_1}$
3) If $M_1(z_1)$, $M_2(z_2)$, $M_3(z_3)$ are different points in pairs, they are collinear if and only if
$\frac{z_3-z_1}{z_2-z_1}\in R^*$
4) If $M_1(z_1)$, $M_2(z_2)$, $M_3(z_3)$ and $M_4(z_4)$ are different points from the origin and from one another, then $(M_1M_3,M_2M_4)\sphericalangle=\arg\frac{z_3-z_1}{z_4-z_2}$

The proof of the previous statements can be found in [1], [11] and [12] where many other important results can be found.
From the statements presented above we can realise that the solution with affixes has many advantages compared with vectorial solutions, especially if in our task we have a equilateral triangle, square, polygon, cyclic quadrilateral, parallelogram, or just the measurement of angles.

The internal analogy of R^2 real plane and C complex numerical plane

In the following we are going to compare the more important results of ***R^2*** and ***C*** coordinate geometry related to straightlines which emphasize the internal analogy between the two structures.

The analogy between the two structures comes from the fact that from the relations $z = x + iy$ and $\overline{z} = x - iy$, x and y can be expressed, as $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2i}$. In this way an equation of the ***R^2*** plane, with the form $f(x, y) = 0$ becomes an equation with the form $\phi\left(z,\overline{z}\right) = 0$.

In the following we are going to examine how the most important classical results can be written in complex numbers.

In the following we use $M(z_1), M(z_2), M(z_3)$ and $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_3 = x_3 + iy_3$

1) The distance between M_1, M_2 points is:

$d(M_1, M_2) = \sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2}$ respectively $d(M_1, M_2) = \sqrt{\left(z_2 - z_1\right)\cdot(\overline{z_2} - \overline{z_1})}$

2) The general form of the straight line is :

$ax + by + c = 0$ respectively $\alpha \cdot z + \overline{\alpha \cdot z} + \beta = 0$, where $\alpha = (a - ib):2$ and $\beta = c \in R$.

3) The slope of the straight is :

$m = \tan\theta = -\frac{b}{a}$ respectively $m = tan\theta = i\cdot\frac{\alpha+\overline{\alpha}}{\alpha-\overline{\alpha}}$ *or* $\theta = \frac{i}{2}\cdot\ln\left(\frac{\alpha}{\overline{\alpha}}\right)$

4) The equation of a straight line which goes across a certain point and has a certain slope is:

$y - y_0 = m(x - x_0)$ respectively $z - z_0 = m\cdot(\overline{z} - \overline{z_0})$

5) The equation of the straight which goes between two points is :

$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ y_2 & y_2 & 1 \end{vmatrix} = 0$ respectively $\begin{vmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \end{vmatrix} = 0$

6) The points M_1, M_2 and M_3 are collinear if:

$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ respectively $\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0$

7) The distance of a point $M(x_0, y_0)$, $z_0 = x_0 + iy_0$ from a straight line is :

d = $\frac{\left|a\cdot x_0 + b\cdot y_0 + c\right|}{\sqrt{a^2+b^2}}$ respectively **d** = $\frac{\left|\alpha\cdot z_0 + \overline{\alpha}\cdot\overline{z_0} + \beta\right|}{2\cdot\left|\alpha\right|}$

8) The area of the triangle made by M_1, M_2 and M_3 points is :

$T = \frac{1}{2}|\Delta|$ where $\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ respectively $T = |\Delta|$ where $\Delta = \frac{i}{4}\cdot\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix}$

9) The measure of an angle between two straightslines is :

$\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|$ respectively $\theta_1 - \theta_2 = \ln\left(\dfrac{\overline{\alpha_1}\alpha_2}{\alpha_1\alpha_2}\right)$

10) The mutual relation of two straightlines is :

a) parallel, if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$ respectively $\dfrac{\alpha_1}{\overline{\alpha_1}} = \dfrac{\alpha_2}{\overline{\alpha_2}}$

b) perpendicular if $a_1 a_2 + b_1 b_2 = 0$ respectively $\dfrac{\alpha_1}{\overline{\alpha_1}} + \dfrac{\alpha_2}{\overline{\alpha_2}} = 0$

c) concurrent, if $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$ respectively $\dfrac{\alpha_1}{\overline{\alpha_1}} \neq \dfrac{\alpha_2}{\overline{\alpha_2}}$

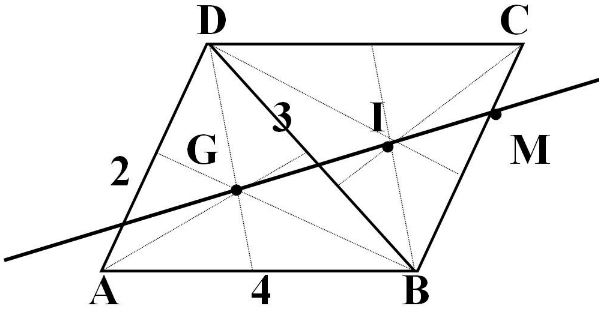
Many other coordinate geometrical results can be written in complex numbers, also the majority of the geometry of curved lines, as well as the transformations. But besides the analogy coming from the rewritings we still have the question, how much more advantageous is one method than the other while solving geometrical problems? The answer to this is the following: in the coordinate sysytem with (x, y) coordinates we have to to make twice as many operations than with only one affix, but on the other hand the analytical geometry solved with complex numbers can seem difficult. We can read about the application of the method for example in [5].

The illustration of the internal analogy of V_2, C, and R^2

We illustrate the analogy of V_2, C and $R \times R$ geometry with the help of a sample, according to which it will be seen how a considerable and perfect internal analogy there is among the three geometrical solutions. Of course we have to realise the analogical relations and results in order to be able to use them.

Sample problem: In the *ABCD* parallelogram *AB=4, BC=2, BD=3*. The G point is the centre of gravity of the *ABD* triangle, *I* the central point of the circle written in the triangle *BCD* and *M* that point of trisection of side *(BC)* which is the nearer to point C. We have to prove that *G*, *I* and *M* points are collinear.

The drawing of the problem is the following:



Vectorial Solution

We mark the arbitrary point *O* on the plane. We can write the following known vectorial relations:

(1) *ABCD* is a parallelogram, because of this $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{OD}$ from where $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OD} - \overrightarrow{OC}$

(2) In *ABD* triangle *G* is centre of gravity, that's why $\overrightarrow{OG}=\frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OD}}{3}$

(3) The central point of the circle written in triangle *BCD* (the point of intersection of bisections) and *BC=2, CD=4, DB=3*, that so

$$\overrightarrow{OI}=\frac{\overrightarrow{OD}\left|\overrightarrow{BC}\right|+\overrightarrow{OB}\left|\overrightarrow{CD}\right|+\overrightarrow{OC}\left|\overrightarrow{BD}\right|}{\left|\overrightarrow{BC}\right|+\left|\overrightarrow{CD}\right|+\left|\overrightarrow{BD}\right|}=\frac{2\overrightarrow{OD}+4\overrightarrow{OB}+3\overrightarrow{OC}}{2+4+3}=$$

$$=\frac{2\overrightarrow{OD}+4\overrightarrow{OB}+3\overrightarrow{OC}}{9}=\frac{4}{9}\overrightarrow{OB}+\frac{3}{9}\overrightarrow{OC}+\frac{2}{9}\overrightarrow{OD}$$

(4) The point *M* on the line *BC* is the trisect point which is nearer to *C* so

$$\overrightarrow{OM}=\frac{\overrightarrow{OB}+2\overrightarrow{OC}}{1+2}=\frac{1}{3}\overrightarrow{OB}+\frac{2}{3}\overrightarrow{OC}$$

(5) So for points *G, I, M* to be collinear it is enough if there exist a real number λ for which $\overrightarrow{OI}=\frac{\overrightarrow{OG}+\lambda\overrightarrow{OM}}{1+\lambda}=\frac{1}{1+\lambda}\overrightarrow{OG}+\frac{\lambda}{\lambda+1}\overrightarrow{OM}$

(6) According to (1) and (2) if we put our expression for $\overrightarrow{OA}$ into $\overrightarrow{OG}$ we get the following $\overrightarrow{OG}=\frac{2}{3}\overrightarrow{OB}+\frac{2}{3}\overrightarrow{OD}-\frac{1}{3}\overrightarrow{OC}$

(7) If now in (5) in the place of $\overrightarrow{OI}$, $\overrightarrow{OG}$, $\overrightarrow{OM}$ we write the vectors expressed by the results of (3), (4), (6) we get that $(\lambda-2)(\overrightarrow{OB}-3\overrightarrow{OC}+2\overrightarrow{OD})=0$ from which gives $\lambda=2$. So the wanted real number λ exists. So this also shows that beside the fact that *G, I, M* are collinear we also have the ratio *GI: IM=2: 1*.

Solution with complex numbers

We choose *a, b, c, d, g, i, m* complex numbers, affixes corresponding to points *A, B, C, D, G, I, M* .

(1) *ABCD is parallelogram,* then $a+c=b+d$ giving $a=b+d-c$

(2) In *ABD* triangle *G* is centre of gravity, so $g=\frac{a+b+c}{3}$

(3) The central point of the circle written into *BCD* triangle (the point of intersection of bisections) is *BC=2, CD=4, DB=3*, that's why $i=\frac{4b+3c+2d}{4+3+2}=\frac{4b+3c+2d}{9}$

(4) The *M* point on the line *BC* section is the point of trisection which is nearer to *C,* so $m=\frac{b+2c}{1+2}=\frac{b+2c}{3}$

(5) So for points *G, I, M* to be collinear it is enough if there exist a real number λ for which $i=\frac{g+\lambda m}{1+\lambda}=\frac{1}{1+\lambda}g+\frac{\lambda}{\lambda+1}m$

(6) If according to (1) and (2) we write the expression for *a* into g we get $g=\frac{2}{3}b+\frac{2}{3}d-\frac{1}{3}c$

(7) Now if in (5) in place of the *i, g, m* we write the expressions from the results of (3), (4), (6) we get that $(\lambda-2)(b-3c+2d)=0$, from which gives $\lambda=2$. So the real number λ exists. This also shows that besides the fact that *G, I, M* collinear we also have the ratio *GI: IM =2: 1*.

Coordinate geometrical solution

$A(x_A, y_A)$, $B(x_B, y_B)$, $A(x_C, y_C)$, $D(x_D, y_D)$, $G(x_G, y_G)$, $I(x_I, y_I)$, $M(x_M, y_M)$ are in order the coordinates of the points mentioned above. We can write the following known relation:

(1) The diagonals of ABCD parallelogram halve each other: $\frac{x_A + x_C}{2} = \frac{x_B + x_D}{2}$ and $\frac{y_A + y_C}{2} = \frac{y_B + y_D}{2}$, from which $x_A = x_B + x_D - x_C$ and $y_A = y_B + y_D - y_C$

(2) In triangle *ABD*, $G(x_G, y_G)$ is centre of gravity, so $x_G = \frac{x_A + x_B + x_D}{3}$ and $y_G = \frac{y_A + y_B + y_D}{3}$

(3) The I is the central point of the circle written in *BCD* triangle (the point of intersection of the bisections) and *BC=2, CD=4, DB=3*, so

$$x_I = \frac{|BC| \cdot x_D + |CD| \cdot x_B + |BD| \cdot x_C}{|BC| + |CD| + |BD|} = \frac{2x_D + 4x_B + 3x_C}{2+4+3} =$$

$$= \frac{2x_D + 4x_B + 3x_C}{2+4+3} = \frac{4}{9}x_B + \frac{3}{9}x_C + \frac{2}{9}x_D$$. Similarly $y_I = \frac{4}{9}y_B + \frac{3}{9}y_C + \frac{2}{9}y_D$.

(4) The point *M* on the line *BC* is the point of trisection which is nearer to *C,* so $x_M = \frac{x_B + 2x_C}{1+2} = \frac{1}{3}x_B + \frac{2}{3}x_C$ and $y_M = \frac{1}{3}y_B + \frac{2}{3}y_C$

(5) So for points *G, I, M* to be collinear it is enough if there exist a real number λ for which $x_I = \frac{x_G + \lambda x_M}{1+\lambda} = \frac{1}{1+\lambda}x_G + \frac{\lambda}{\lambda+1}x_M$ respectively $y_I = \frac{1}{1+\lambda}y_G + \frac{\lambda}{\lambda+1}y_M$

(6) From (1) and (2) we write the $A(x_A, y_A)$ coordinates into $G(x_G, y_G)$ we get that $x_G = \frac{2}{3}x_B + \frac{2}{3}x_D - \frac{1}{3}x_C$ respectively $y_G = \frac{2}{3}y_B + \frac{2}{3}y_D - \frac{1}{3}y_C$.

(7) Now if in (6) in the places of $G(x_G, y_G)$, $I(x_I, y_I)$, $M(x_M, y_M)$ we write the coordinates expressed according to the results of (3), (4), (6) we can find that $(\lambda - 2)(x_B - 3x_C + 2x_D) = 0$ respectively $(\lambda - 2)(y_B - 3y_C + 2y_D) = 0$ from which gives $\lambda = 2$. So the real number λ exists. This also shows that besides the fact that *G, I, M* are collinear we again have the ratio *GI: IM=2: 1*.

Conclusion

In the previous descriptions we demonstrated why there is such a perfect analogy in ***V_2* , *C*** and ***R^2*** geometry. We could see that it is not enough just taking into consideration that the three structures are isomorphic, but it is important to find the analogical concepts and relations among them. It is possible only if we rewrite the concepts and results of one geometry into the language of the other geometry. In this way we get the analogical relations, characteristics and results which can help us prove. If we discover these analogies, we have the possibility to choose which method is the most suitable, easiest, shorteset and the most instructive to solve a given task. We could also see that the vectorial results can very easily be written into the complex plane in this way we shortened them, and we can say that we turned the vectorial geometry into algebraical geometry. We could also see that in the complex numerical plane there are many results (especially those related to angles), which do not have their analogs in the vectorial or analytical results. This means that in such kinds of problems it is more advantageous to work in the complex plane, as this models more easily that particular problem. We could also see that classical coordinate geometry can be written into complex analytical geometry and the analogy is considerable here as well. While solving problems among the two models we should choose the one which stands closer to the characteristics of the problem. We could also see that the solving on complex numerical plane is shorter than solving in $R \times R$, because in the complex plane a point can be characterized with a complex number, while in the case of geometry $R \times R$ a point is characterized with a number pair (x, y), so in this case we have more variables.

In conclusion we can point out that becuse of the intrernal analogy of the three stuctures we have the possibility of finding more solutions, we can choose the shortest, the most instructive and the easiest method to prove. Among the three structures there is internal analogy and this is a clear, proved analogy (as an isomorphism proves this) so we can be sure that in our case that the analogy does not lead us to only a probable result, but to a result which is absolutely sure.

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