2.16 From k-Angle Numbers To Pell Type Equations

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This paper is continuation of [1], here we use some of those notations and result.

I. The generalization of the figured numbers. In [1] we could read about the so called figured numbers, studied by pitagorens, for example Triangle Numbers, Rectangle Numbers, Square Numbers, Gnom Numbers, Tetrahedron Numbers and Cube Numbers. In this paper I will write about the generalization of this types of numbers and about it's applications. In [2] we can read about the introduction of k-angle numbers. Draw regular k-angles, such that their side lenght be by turns 1,2,3 unit. In each polygon we note the vertexes and one side those point which divide the sides in unit segments. The first four 6-angle number is shown in figure 1.

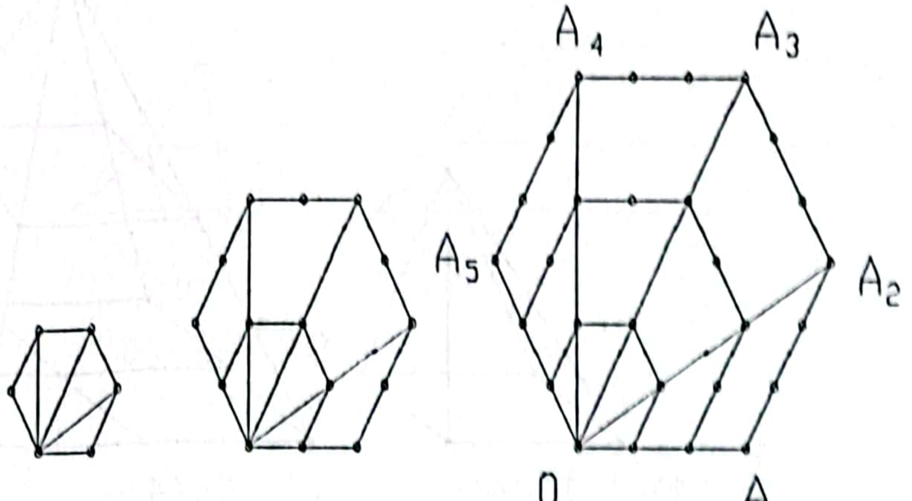

Fig. 1

By definition, the first k-angle number is 1, the second is k, the third k-angle numbers is the number of the marked points on the boundary and interior of the second k-angle. The n-th k-angle number is the number of the marked points on the boundary and interior of $(n-1)$-th regular k-angle. If we denote with $S_n(k)$ the n-th k-angle number then

i.)
$S_n(k) = (k-2)H_n - (k-3)n$, *or* $S_n(k) = \frac{1}{2}[(k-2)n^2 - (k-4)n]$
$\forall k, n \in N^*$.
If $k \geq 3$, then H_n is the n-th triangle number. We show a different proof from that in [2], a genuine, elementary one. During the proof, w follow the last draw of figure 1. but instead of the 6-angle, we think o $A_1A_2 \dots A_{k-1}$ regular k-angle, which we divide to $OA_1A_2, OA_2A_3, \dots, OA_{k-2}A_{k-1}$ disjoint triangels and those number exactly $k-2$. In this way we drop $S_n(k)$ in $k-2$ pieces of H_n, we ha to substract only the number of points situated on $OA_2, OA_3 \dots OA_{k-1}$ segments, because we count them twice. Thus $S_n(k) = (n-2)H_n - (k-3)n$, where $H_n = \frac{n(n+1)}{2}$. (see[1]) (The number of points situated on $OA_2, OA_3, \dots, OA_{k-1}$ segments is $(k-3)n$). It is obvious that for $k = 3$ and $k = 4$ from (i) we reobtain the n-th triangle number, respectivelly, the n-th square number. Ther are many posibilities for the special generalization of the k angle number. We can introduce for example the n-th k pyramide number. Note this with $G_n(k)$, for any $k, n \in N^*$. The first three 4-pyramide numbers are represented in figure 2.

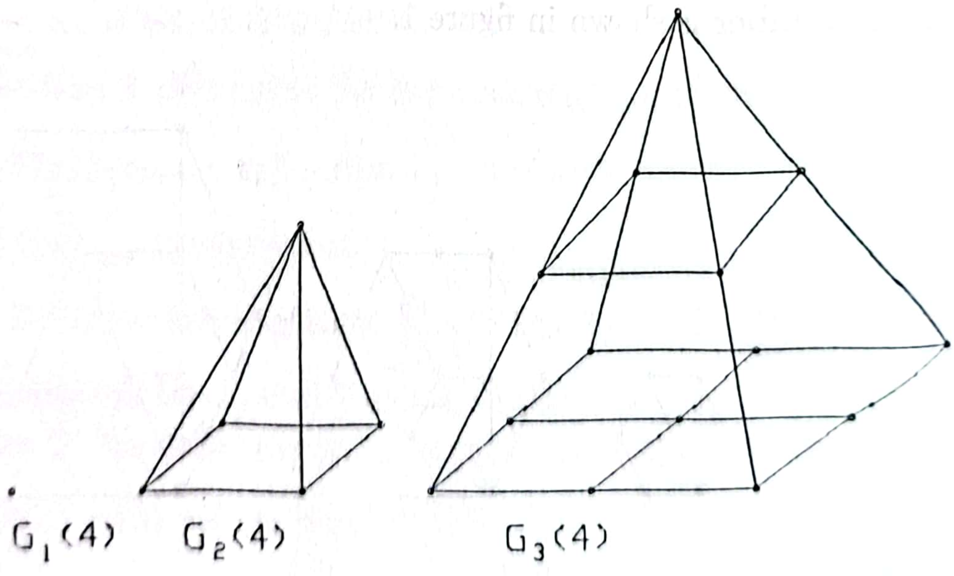

Fig. 2
It can be observed that $G_n(k) = S_1(k) + S_2(k) \cdots + S_n(k)$, for any $k, n \in N^*, k \geq 3$. The n-th k pyramide number is given by the following equations:
ii.)
$G_n(k) = (k-2)t_n - (k-3)H_n$, *or* $G_n(k) = \frac{n(n+1)}{6}[(k-2)n - k + 5]$

$\forall k, n \in N^*, k \geq 3$ (t is the n-th tetrahedron number). (see[1]). Based on (i), using the notations and results from [1]:

$$G_n(k) = \sum_{i=1}^{n} S_i(k) = (k-2)\sum_{i=1}^{n} H_i - (k-3)\sum_{i=1}^{n} =$$

$$(k-2)\sum_{i=1}^{n} \frac{i(i+1)}{2} - (k-3)\frac{n(n+1)}{2} = (k-2)\frac{n(n+1)(n+2)}{6} - (k-3)\frac{n(n+1)}{2}$$

from where (ii) immediately result. It is obvious that for $k = 3$ from (ii) we reobtain the formula of the n-th tetrahedron number, respectively $G_n(3) = h_n = \frac{n(n+1)(n+2)}{6}$. In our following generalizations it also can be introduced the n-th k prism number, such that m number $(m \in N^* - \{1\}) n-th$ k angle number matching together and covering each other in space by the following application:

iii.)

$H_n(k, m) := mS_n(k) = m\frac{(k-2)n^2-(k-4)n}{2}$, $\forall k, n \in N^* k \geq 3$. It is obvious that for $k = 4$ and $m = n$ $H_n(4; n) = n^3$ give the n-th cube number (the k_n). (see[1]). Following the principle of the regularity, it can be also defined the n-th with octahedron number. The first three octahedron is represented in figure 3.

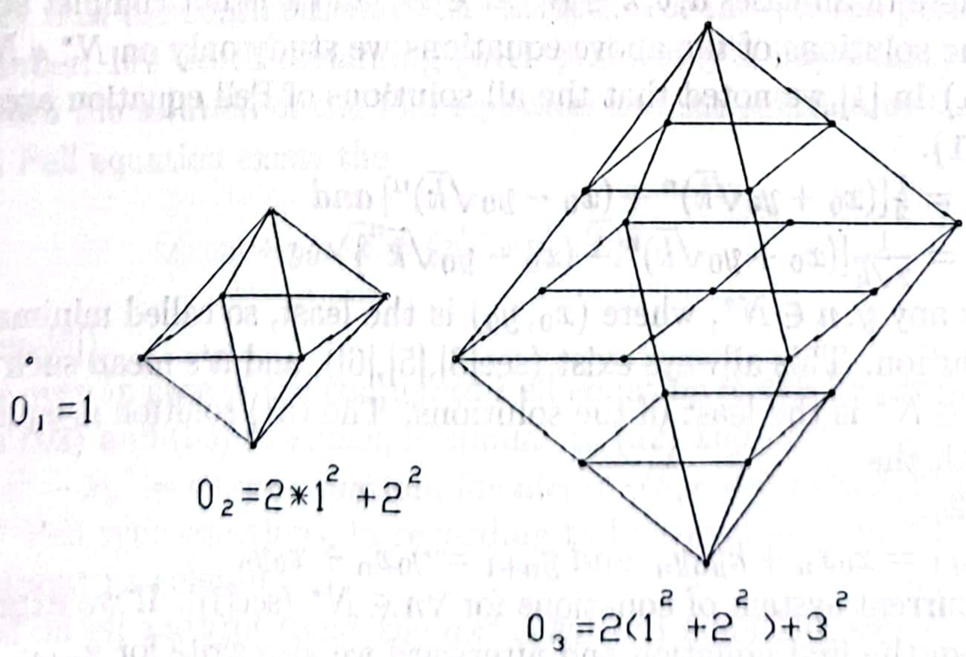

Fig. 3

It's obvious that we "faced" two 4-pyramide number, only that the contact balls would be count, only once. Thus the n-th octahedron number iv.)

(ii)

$$O_n = 2G_{n-1}(4) + h^2 \quad = \quad 2\frac{(n-1)n(2n-3)}{6} + n^2 = \frac{n(2n^2+1)}{3}$$

Surrely we can construct much more (perhaps multiple dimension) types of figural numbers. We entrust this creation to the interested reader.

II. About Pell-types equations

Next we examime shortly (mainly with practical character) the following diophantine equations very ramifying theory.

- (A) $x^2 - ky^2 = 1$ (Pell-equation)
- (B) $x^2 - ky^2 = -1$ (conjugate Pell-equation)
- (C) $x^2 - ky^2 = c$
- (D) $ax^2 - by^2 = 1$
- (E) $ax^2 - by^2 = c$

where in all cases $a, b, k \in N^*, C \in Z^*$ and k is not complet square. The solutions of the above equations we study only on $N^* * N^*$.

(A) In [1] we noted that the all solutions of Pell equation are given by

(a1).

$x_n = \frac{1}{2}[(x_0 + y_0\sqrt{k})^n + (x_0 - y_0\sqrt{k})^n]$ *and*

$y_n = \frac{1}{2\sqrt{k}}[(x_0 + y_0\sqrt{k})^n - (x_0 - y_0\sqrt{k}^n]$

for any $y, n \in N^*$, where (x_0, y_0) is the least, so called minimal or basic solution. This allways exist (see[3],[5],[6]), and it's mean such that $y_0 \in N^*$ is the least of the solutions. The (a1) solution is equivalent with the

(a2).

$x_{n+1} = x_0x_n + ky_0y_n$ *and* $y_{n+1} = y_0x_n + x_0y_n$

recurrent system of equations for $\forall n \in N^*$ (see[1]). If we express in from the first equation and afterward we also write for y_{n+1}, based on the second equation of (a2), (a2) is equivalent with the

(a3).

$x_{n+1} = 2x_0x_n - x_{n-1}$, *and* $y_{n+1} = 2x_0y_n - y_{n-1}$,

$x_1 = x_0^2 + ky_0^2$, $y_1 = 2x_0y_0$

recurent system of equations.

Practicaly is very important to find the (x_0, y_0) basic solution. In the best case we find it by qessing, but for example, for $(x^2 - 13y^2 = 1)$ equation isn't probable, that we guess the $x_0 = 649$ and $y_0 = 180$ basic solution.
It's interesting even today that Waclaw Sierpinski, pole mathematician (1882-1969) concluded that the $x^2 - 991y^2 = 1$ equation has no basic solution. But finally the basic solution turned out:
$x_0 = 379516400906811930638014896080$
$y_0 = 12055735790331359447442538767$
To find the basic solution J. Lagrange applied $\sqrt{k}$ the continous fraction method for $\sqrt{k}$ (see[5],[6],[7]). From practical viewpoint we can read about this in [3],[7],[8].
(B) The all solutions of the conjugate Pell equation, are given by **(b1).**
$x_n = \frac{1}{2}[(x'_0 + y'_0\sqrt{k})^{2n-1} + (x'_0 + y'_0\sqrt{k})^{2n-1}]$,
$y_n = \frac{1}{2\sqrt{k}}[(x'_0 + y'_0\sqrt{k})^{2n-1} - (x'_0 - y'_0\sqrt{k})^{2n-1}]$
formula, for any $n \in N^*$ (see[5],[6],[1]).
But here the essential difference is that the (x'_0, y'_0) basic solution exists, only if in the continous fraction extraction of the $\sqrt{k}$ the period is odd number. It's worth remarking (with practically is important) that between the solution of the Pell equation and the solution of the conjugate Pell equation exists the

$$x_0 + y_0\sqrt{k} = (x'_0 + y'_0\sqrt{k})^2$$

relation (see[6]).
Otherwise even in case of the conjugate Pell equation it can be deduced analogous (b2) and (b3) relations, is similar to (a2) and (a3).
(C) The $x^2 - ky^2 = c$ type equation, for $a|c| < \sqrt{k}, c \neq 0$ belong to the "clarified" Pell type equations, in regarding to its solvability. In [5] we can read about its solution.
(D) Based on [4] and [10] about the $ax^2 - by^2 = 1$ equation, we emphasize the following:
1. If $ab = k^2 (k \in N^* - 1)$, then $a = k_1^2$ and $b = k_2^2$ $(k_1, k_2 \in N^*)$ and thus
$k_1^2x^2 - k_2^2y^2 = 1 \Leftrightarrow (k_1x - k_2y)(k_1x + k_2y) = 1$,
so $1 < k_1x + k_2y = k_1x - k_2y = 1$
that is contradiction, namely this equation has no solution.

2. In so far as $ab \neq k^2 (k \in N^* - \{1\})$ execute the linear transformation for $x = x_0u + ky_0v$ and $y = y_0u + ax_0v$. From here
$1 = ax^2 - by^2 = a(x_0u + by_0v)^2 - b(y_0u + ax_0v)^2 =$
$= (ax_0^2 - by_0^2)(u^2 - abv^2) = u^2 - abv^2$
In so far as for the $ax^2 - by^2 = 1$ equation exists (x_0, y_0) basic solution, so the entire solution can be reduced to the solution of the $u^2 - abv^2 = 1$ Pell equation.
The hardest practical problem is that it doesn't exists yet a theory to decide which (D) type equation has basic solution, only particulary cases are studied (see[9],[10],[11]).
(E) To solve the discused equation we attempt the methode Lagrange (see[4],185pg) The essence of the methode is that we execute the $x = \alpha y + cz$ linear transformation, and we determine the $\alpha \in Z$ number, such that the free number is ± 1. After that we use the $y = \beta z + t$ linear transformation in order to determine the $\beta \in Z$ number, such that the coefficient of the z-t number is zero.
In so far as exists such $\alpha, \beta \in Z$ numbers, we obtain one of the (A), (B), or (C) equation. To avoid the abstractization we present this method in three Application .

Applications

Next we present some from the wide ranged applications of the k-angle numbers.

Application 1. Prove that ti find tha p-angle numbers which are in the same time q-angle numbers lead to the resolvation of the
$ax^2 - by^2 = c,\ a, b \in N^*,\ c \in Z$
equation $(p, q \in N, p \geq 3, q \geq 3)$

Proof. The n-th p-angle number: $S_n(p) = \frac{1}{2}[(p-2)n^2 - (p-4)n]$ The m-th q-angle number: $S_m(q) = \frac{1}{2}[(q-2)m^2 - (q-4)m]$ While $S_n(p) = \frac{4(p-2)^2n^2 - 4(p-2)(p-4)n}{8(p-2)} = \frac{[2(p-2)n-(p-4)]^2-(p-4)^2}{8(p-2)}$ therefore the $S_n(p) = s_m(q)$ become $\frac{[2(p-2)n-(p-4)]^2-(p-4)^2}{8(p-2)} = \frac{[2(q-2)m-(q-4)]^2-(q-4)^2}{8(q-2)}$ from where after the operations we obtain the
(1.) $ax^2 - by^2 = c$ equation, where $a = q-2, b = p-2, c = (q-p)(2p+2q-pq), x = 2(p-2)n - (p-4), y = 2(q-2)m - (q-4)$

Application 2. Prove that, all six angle number is also three angle number.

Proof 1. The m-th three angle number respectively the m-th six angle number formula is $S_n(3) = \frac{n(n+1)}{2}$ resprctively $S_m(6) = \frac{4m^2-2m}{2}$ and

the equality on the basis of $S_n(3) = S_m(6)$ reaction of the application 1.(1.) (for $p = 3,\ q = 6$ *or* $p = 6,$ *and* $q = 3$ values), follow the $4x^2 - y^2 = 0$ equation, where $x = 2n + 1$ and $y = 8m - 2$. Thus based on $(2x - y)(2x + y) = 0$, $y = 2x$, namely
$8m - 2 = 4n + 2 \Leftrightarrow n = 2m - 1$. Thus the m-th six angle number corespond with the $(2m - 1)$-th three angle number.

Proof 2. This last result we can immediately obtain if we observe that
$S_m(6) = \frac{4m^2-2m}{2} = \frac{(2m-1)^2+(2m-1)}{2} = \frac{(2m-1)(2m-1+1)}{2} = S_{2m-1}(3)$

Remark. It can be observed that in the (1.) equation from the application 1. $c = 0$ is true only in the previous case, namely if $p = 3, q = 6$ or $p = 6, q = 3$ (obviosly $p = q$ is excluded). That is trully right, since $c = 0$ and $p \neq q$, $2p + 2q - pq = 0 \Leftrightarrow p = 2 + \frac{4}{q-2} \in N$, from where $q - 2 \in \{1, 2, 4\}$, thuse $q \in \{3, 4, 6\}$. If $q = 3 \Rightarrow p = 6$, if $q = 4 \Rightarrow p = 4 \Rightarrow p = q$, absurd, if $q = 6 \Rightarrow p = 3$.

Application 3. Determine those five angle numbers, which are also three angle numbers.

Solution The m-th five angle number: $S_m(5) = \frac{3m^2-m}{2}, m \in N^*$ The n-th three angle number: $S_n(3) = \frac{n(n+1)}{2}, n \in N^*$ The $S_m(5) = S_n(3)$, according to the transformation of (1.) (for $p = 5$ and $q = 3$) and $a = 1$, $b = 3$, $c = -2$, $x = 6n - 1$ and $y = 2m + 1$ equation (1) result the

(i)

$x^2 - 3y^2 = -2$

To solve this equation we use the Lagrange method. Execute the $x = \alpha y - 2z$ (2) linear transformation such that the free number of the equation (i) is +1 or −1. After computing result

(ii)

$\alpha^2 y^2 - 4\alpha yz + 4z^2 - 3y^2 = -2$ *or* $\frac{\alpha^2-3}{2} y^2 - 2\alpha yz + 2z^2 = -1$

We determine the $\alpha \in Z$ value such that $\frac{\alpha^2-3}{2} \in Z$. It can be observe that the $\alpha = 3$ value correspond thus (ii) become

(iii)

$3y^2 - 6yz + 2z^2 = -1$

Now we execute the $y = \beta z + t$ (3) linear transformation such that the coefficient of the yz form member is zero. After the transformation (iv) become

(iv)

$(3\beta^2 - 6\beta)z^2 + (6\beta - 6)zt + 3t^2 = -1$

In so far as $\beta = 1$, so the (iv) equation become

(v)
$z^2 - 3t^2 = 1$
Pell equation. The basic solution $z_0 = 2$ and $t_0 = 1$, therefore based on (a1), the solution are
(4.1). $z_r = \frac{1}{2}[(2+\sqrt{3})^r + (2-\sqrt{3})^r]$
(4.2). $t_r = \frac{1}{2\sqrt{3}}[(2+\sqrt{3})^r - (2-\sqrt{3})^r]$
for any $r \in N^*$. According to (2) and (3) transformations
(5). $x_r = 3y_r - 2z_r$
and
(6). $y_r = z_r + t_r$
from where
(7.1). $x_r = z_r + 3t_r$
(7.2). $y_r = z_r + t_r$
for any $r \in N$. Therefore on the base of (4.1) and (4.2) the ecuation (i) solutions are
(8.1). $x_r = \frac{(3+\sqrt{3})(2+\sqrt{3})^r - (3-\sqrt{3})(2-\sqrt{3})^r}{2\sqrt{3}}$
(8.2). $y_r = \frac{(1+\sqrt{3})(2+\sqrt{3})^r - (1-\sqrt{3})(2-\sqrt{3})^r}{2\sqrt{3}}$
for any $r \in N$. The final answer to the problem we obtain from the :
(9.1). $6n - 1 = x_r$
(9.2). $2m + 1 = y_r$
equation. That's solvy we write for the (v) equation the (a3) type reccurent's that:
(10.1). $z_r = 4z_{r-1} - z_{r-2}$
(10.2). $t_r = 4t_{r-1} - t_{r-2}$
for any $r \in N^* - \{1\}$, and $z_0 = 2, z_1 = 7; t_0 = 1, t_1 = 4$ (ex. from (4.1) and (4.2)). From (7.1), (10.1) and (10.2)
$x_r = z_r + 3t_r = 4z_{r-1} - z_{r-2} + 3(4t_{r-1} - t_{r-2}) =$
$= 4(z_{r-1} + 3t_{r-1}) - (z_{r-2} + 3t_{r-2}) = 4x_{r-1} - x_{r-2}$
and we compute in the same way for (7.2). Thus
(11.1). $x_r = 4x_{r-1} - x_{r-2}$
(11.2). $y_r = 4y_{r-1} - y_{r-2}$
as well as (7.1), (7.2) and from (8.1), (8.2),
$x_0 = 5, x_1 = 19; y_0 = 3, y_1 = 11$
It's obvious that the members of the $(y_r)r$ sequence are all odd numbers, this for example from (11.2) can be proved by induction. Thus the (9.2) condition is fulfilled for every $r \in N$. Now write in order some member of the $(x_r)r$ sequence:

(12). 5, 19, 71, 265, 989, ...
It can be obsreved that $x_{2k} = M6 - 1$, till
(13). $x_{2k+1} = M6 + 1$
for any $k \in N$. Thus using the mathematical induction from (11.1) $x_{2k+2} = 4x_{2k+1} - x_{2k}$, for any $k \in N$, and considering the statements from (13). as true, we can write in order that:
$x_{2k+2} = 4(M6 + 1) - (M6 - 1) = M6 + 5M6 - 1$
Thus the solution of (9.1) and (9.2) are:
(14.1). $n_k = \frac{x_{2k}+1}{6}$
(14.2). $m_k = \frac{y_{2k}-1}{2}$
$\forall k \in N$, and so, those numbers that are also five angle numbers and also three angle numbers are:
(15). $S_k(5,3) = \frac{n_k(n_k+1)}{2} = \frac{3m_k^2 - m_k}{2}$
$\forall k \in N$.
We obtain the values of x_{2k} and y_{2k} immediately from (11.1) and (11.2), the values of n_k and m_k from (14.1) and (14.2), but the (8.1) and (8.2) for $r = 2k$ give the result in closed form.
Some example for $k \in 0, 1, 2$
$\frac{1(1+1)}{2} = 1 = \frac{3 \cdot 1^2 - 1}{2}; \frac{20 \cdot 21}{2} = 210 = \frac{3 \cdot 12^2 - 12}{2};$
$\frac{285 \cdot 286}{2} = 40755 = \frac{3 \cdot 165^2 - 165}{2}$
namely, 1, 210, 40755 are also three angle and five angle numbers.

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