The study of sequences defined by a first degree recursive relation, with the help of pocket calculator

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Abstract. In this paper we will show, how we can use a simple pocket calculator, to teach mathematics. Namely, a pocket calculator can be very useful to study the properties of sequences defined by first degree recursive relations, for example monotonicity, boundedness, and convergency, and so gain a deeper understanding.

Key words and phrases: sequence, bounded, monotonicity, limit, pocket calculator, convergency

ZDM Subject Classification: I30, N20, D10

Nowadays the calculator (or the calculators in mobile phones or in computers) can be used for other purposes than for ordinary arithmetic? We will show that we can use it either for studying the convergence of sequences, or for guessing the limit of a sequence. Moreover in the second part it plays a unique role, since it can create predictions regarding the limit of a sequence.

The theme described below was successfully tested by me on my mathematics classes, and I recommend it for anybody, to try it out because it is worth it!

The purpose of my experiments, was to study with the students the convergence of certain sequences, defined by a recursive relation. The students have previously gained the more important, necessary concepts and knowledge, the information related to monotonicity and boundedness and we hade started to calculate the limits of sequences. But we can say that we are only beginners in this domain.

In my opinion the whole mathematical analysis of High School especially the convergency of sequences and the calculate of their limits is hard to understand because a proper concrete illustrative model is missing. More exactly the phenomena related to infinite cannot really be modeled. We do not have any intuitive basis or support for this, only definitions, concepts, predefined rules and theorems. That's why I wondered how we can present this topic in such a way that the students can experience it actively. Then it came into my mind that with a manual calculator we can experience quickly and easily such concepts as monotonicity, boundedness and at last the concept of convergency. For this I decided, that I will try all that, which I'm going to share below. And I must admit that I did not regret it because according to feedback the students understood this topic better than if I had just written the lesson on the blackboard with chalk. But of course we presented and analyzed in details on the blackboard the following proofs, but only after we have tried the following experiments, and after that we discussed and proved our results.

Experiment 1: Let's type 2 into a pocket calculator and calculate its square root. Then calculate the square root of the result. We do this until we see the same number on the display. What can we say about the monotonicity, boundedness and limit of the sequence? How can we explain what we have seen?

Solution: On the display of a pocket calculator which can display only 8 characters, respectively we can see the following: 1.4141356; 1.1892071; 1.0905076; 1.0442737; 1.0218971; 1.0108892; 1.0054298; 1.0027112; 1.0013546; 1.0006770; 1.0003384; 1.0001691; 1.0000845; 1.0000422; 1.0000210; 1.0000104; 1.0000051; 1.0000025; 1.0000012; 1.0000005; 1.0000002; 1.0000001; **1**.; **1**.; **1**.; ….

The following properties can be read:

1) The resulting sequence is strictly decreasing. This we are going to prove. The members of the sequence in question are: $a_1=\sqrt{2}=2^{\frac{1}{2}}$, $a_2=\sqrt{a_1}=\sqrt{\sqrt{2}}=2^{\frac{1}{2^2}}$, $a_3=\sqrt{a_2}=\sqrt{\sqrt{\sqrt{2}}}=2^{\frac{1}{2^3}}$, and usually $a_{n+1}=\sqrt{a_n}=\overbrace{\sqrt{\sqrt{\sqrt{\ldots\sqrt{2}}}}}^{(n+1)-root\ sign}=2^{\frac{1}{2^{n+1}}}$ for $\forall n\geq 0$. It is easy to see that $a_1>\sqrt{a_1}=a_2$, hence if we use the method of mathematical induction and we assume that $a_{n-1}>a_n$, then after the $a_{n+1}-a_n=\sqrt{a_n}-\sqrt{a_{n-1}}=\frac{a_n-a_{n-1}}{\sqrt{a_n}+\sqrt{a_{n-1}}}$ formula immediately follows that $a_n>a_{n+1}$, namely the sequence is indeed strictly decreasing. Another way to prove the monotonicity of the sequence, is if we write $\frac{a_{n+1}}{a_n}=\frac{2^{\frac{1}{2^{n+1}}}}{2^{\frac{1}{2^n}}}=2^{-\frac{1}{2^n}}<2^0=1$. This is shorter than the previous demonstration, which we presented to help to acquiring the proof technique.

2) Because the sequence is strictly decreasing it has a best upper bound. This way

$a_n\leq a_1=\sqrt{2},\ \forall n\geq 1$.

After what we can see on the calculator we can have a presumption that the best lower bound is 1, namely $1\leq a_n$, $\forall n\geq 1$. This we can prove because $a_n=2^{\frac{1}{2^n}}>2^0=1$ it is obvious in the case of $\forall n\geq 1$. Accordingly $a_n\in\left(1,\sqrt{2}\right]$ in the case of $\forall n\geq 1$.

3) Since the sequence $\left(a_n\right)_{n\geq 1}$ is strictly decreasing and bounded, hence from the theorem of Weierstrasse the sequence is convergent, so has a limit. On the display of the pocket calculator we sense this, as after executing a certain number of operations the display we shows 1. This, therefore means that $\lim_{n\to\infty}a_n=1$, which is indeed true, because $\lim_{n\to\infty}2^{\frac{1}{2^n}}=1$ obviously. At the same time we have illustrated the theorem which says that the best lower bound of a monotonic decreasing sequence is its limit, in our case 1.

4) Furthermore it has to be explained that after executing a certain number of operations why there always appears 1 on the display of the calculator. We can find out the reason for this when we are doing these calculations on a calculator that has more than 8 digits on its display. Where after 1,0000000 there can appear digits which are not a 0. Therefore with an 8 digit calculator we could only approximate the limit of the sequence which was 1 with seven decimals.

Experiment 2: The task is the same as in Experiment 1, but with the difference that in the place of 2 we can pick an arbitrary number $a > 0$.

Solution: Executing the calculations in specific cases and observing the shown values, we can see that after a number of steps the number 1 is shown again. 1 can be reached faster or slower that is to say even the "speed of convergence" changes. Every other result and the proof are the same as shown before. Summing up we proved that in the case of every $a > 0$ $\lim_{n\to\infty}\overbrace{\sqrt{\sqrt{\sqrt{\ldots\sqrt{a}}}}}^{n-root\ sign}=1$ i.e. $\sqrt{\sqrt{\sqrt{\ldots\sqrt{a}}}}=1$, where the number of the root signs is infinitely big.

Experiment 3: Let's type 2 into a pocket calculator and calculate its square root. Multiply the result by two and calculate its square root again. We shall repeat this operation until we get the same number on the display. What can we say about the monotonicity, boundedness and limit of the sequence? How can we explain what we have seen?

Solution: On the display of a pocket calculator which can only display 8 characters, respectively we can see the following: 1.4141356; 1.6817928; 1.8340080; 1,9152065; 1.9571441; 1.9784560; 1.9891988; 1.9945921; 1.9972942; 1.9986466; 1.9993232; 1.9996615; 1.9998307; 1.9999153; 1.9999576; 1.9999758; 1.9999884; 1.9999947; 1.9999973; 1.9999986; 1.9999993; 1.9999996; 1.9999998; 1.9999999; **2; 2; 2;….** The following properties can be read:

1) The resulting sequence is strictly increasing. We are going to prove this! The members of the sequence in question are: $a_1=\sqrt{2}=2^{\frac{1}{2}}$, $a_2=\sqrt{2a_1}=\sqrt{2\sqrt{2}}=2^{\frac{1}{2}+\frac{1}{4}}$, $a_3=\sqrt{2a_2}=\sqrt{2\sqrt{2\sqrt{2}}}=2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$, and usually $a_{n+1}=\sqrt{2a_n}=\overbrace{\sqrt{2\sqrt{2\sqrt{2\ldots\sqrt{2}}}}}^{(n+1)-root\ sign}=2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^{n+1}}}=2^{1-\frac{1}{2^{n+1}}}$ for $\forall n\geq 0$..

It is easy to see that on the basis of the relations $\sqrt{2}=a_1<\sqrt{2a_1}=\sqrt{2\sqrt{2}}=a_2$, and $a_{n+1}-a_n=\sqrt{2a_n}-\sqrt{2a_{n-1}}=\frac{2(a_n-a_{n-1})}{\sqrt{2a_n}+\sqrt{2a_{n-1}}}$ we can prove by mathematical induction that the sequence is indeed strictly increasing.

2) Since the sequence is strictly increasing it has a lower best bound, and thus in the case of $a_n\geq a_1=\sqrt{2}$, for $\forall n\geq 1$. On the basis of the calculator's display we have a suspicion that the best upper bound is 2, that is in the case of $a_n<2$, for $\forall n\geq 1$ This can be proved with the help of mathematical induction, as $a_1=\sqrt{2}<2$, and if we suppose that $a_n<2$, then $a_{n+1}=\sqrt{2a_n}<\sqrt{2\cdot 2}=2$. So $a_n\in\left[\sqrt{2},2\right)$ for $\forall n\geq 1$.

3) Since the sequence $(a_n)_{n\geq 1}$ is strictly increasing and limited, on the basis of Weierstrasse theorem the sequence is convergent, that it has a limit. This can be seen on the calculator getting the number 2 on its display after a number of operations have been done. This means that the $\lim_{n\to\infty} a_n = 2$, because $\lim_{n\to\infty} 2^{1-\frac{1}{2^n}} = 2$ is clear. At the same time the theorem is illustrated, according to which the best upper bound of a monotonous increasing sequence is the limit of the sequence, in our case this is 2.

Experiment 4: The problem is the same as in the first two experiments, only instead of 2 we take another number *a*> 0.

Solution: Executing the calculations in specific cases and observing the shown values, we can see that after a number of steps the number *a* is shown again. Every other result and the proof is the same as it was shown before only $\lim_{n\to\infty} a_n = a$ changes according to $a_1=\sqrt{a}$. Summing up we proved that in the case of every *a*> 0 , $\lim_{n\to\infty} \overbrace{\sqrt{a\sqrt{a\sqrt{\ldots a\sqrt{a}}}}}^{n-root\ sign} = a$ namely $\sqrt{a\sqrt{a\sqrt{\ldots a\sqrt{a}}}} = a$, where the number of the root signs is infinitely large.

Experiment 5: Let's type 2 into a pocket calculator and calculate its square root. Add 2 to the result and calculate its square root again. We shall repeat this operation until we get the same number on the display. What can we say about the monotonicity, boundedness and limit of the sequence? How can we explain what we have seen?

Solution: On the display of a pocket calculator which can only display 8 characters, in turns, we can see the following: 1.414135; 1.8477590; 1.9615705; 1.9903694; 1.9975909; 1.9993976; 1.9998494; 1.9999623; 1.9999905; 1.9999976; 1.9999994; 1.9999998; 1.9999999; **2**; **2**;**2**;….

1) The resulting sequence is strictly increasing, which we shall prove. The members of the sequence in question are: $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+a_1} = \sqrt{2+\sqrt{2}}$, $a_3 = \sqrt{2+a_2} = \sqrt{2+\sqrt{2+\sqrt{2}}}$, and usually $a_{n+1} = \sqrt{2+a_n} = \overbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}^{(n+1)-root\ sign}$ for $\forall n \geq 0$. We can see that in this case we get only a recursive relationship, we cannot write algebraically the general member, as in the previous cases. It is easy to see that on the basis of the relations $a_1 = \sqrt{2} < \sqrt{2+\sqrt{2}} = a_2$, and $a_{n+1} - a_n = \sqrt{2+a_n} - \sqrt{2+a_{n-1}} = \dfrac{a_n - a_{n-1}}{\sqrt{2+a_n} + \sqrt{2+a_{n-1}}}$ we can prove by mathematical induction that the sequence is strictly increasing.

2) Since the sequence is strictly increasing it has a best lower bound, and thus $a_n \geq a_1 = \sqrt{2}$, for $\forall n \geq 1$. On the basis of the calculator's display we believe that the best lower bound is 2, namely $a_n < 2$, for $\forall n \geq 1$. This can easily be proved by mathematical induction, because $a_1 = \sqrt{2} < 2$, and if we suppose that $a_n < 2$, then $a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2$. So, $a_n \in \left[\sqrt{2}, 2\right)$ for $\forall n \geq 1$.

3) Because the sequence $(a_n)_{n\geq 1}$ is strictly increasing and bounded, on the basis of Weierstrasse's theorem the sequence is convergent, i.e. it has a limit. This can be seen on the calculator getting the number 2 on its display after a number of operations have been completed. This means that the $\lim_{n\to\infty} a_n = 2$, which is true, because if $\lim_{n\to\infty} a_n = x$, then $\lim_{n\to\infty} a_{n-1} = x$ is also true. The limit satisfied $a_{n+1} = \sqrt{2+a_n}$ so $x = \sqrt{2+x} \Leftrightarrow x^2 - x - 2 = 0$ giving x= 2, because

x=-1 $\notin \left[\sqrt{2}, 2\right)$. Therefore $\lim_{n\to\infty} a_n = 2$ which means, that $\lim_{n\to\infty} \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{n-root\ sign} = 2$ or otherwise $\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2+...}}}} = 2$ where the number of root-signs is infinitely large.

<u>Experiment 6</u>:

If we take the number 6 instead of 2 in the exercise before, what can we say about the monotonicity, boundedness and limit of the sequence? How can we explain what we have seen?

Solution: The resulting sequence is $a_1 = \sqrt{6}$, $a_2 = \sqrt{6+a_1} = \sqrt{6+\sqrt{6}}$, $a_3 = \sqrt{6+a_2} = \sqrt{6+\sqrt{6+\sqrt{6}}}$, and in general $a_{n+1} = \sqrt{6+a_n} = \underbrace{\sqrt{6+\sqrt{6+\sqrt{6+...+\sqrt{6}}}}}_{(n+1)-root\ sign}$ for $n \geq 0$. We are going to prove the same way as before that the sequence is strictly increasing. Therefore the best lower bound is the first member, namely $a_n \geq a_1 = \sqrt{6}$, for $\forall n \geq 1$. This time we can see the best upper bound on the display is 3. This we can easily prove by mathematical induction, namely $a_1 = \sqrt{6} < 3$, and if we suppose, that $a_n < 3$, then $a_{n+1} = \sqrt{6+a_n} < \sqrt{6+3} = 3$. Therefore $a_n \in \left[\sqrt{6}, 3\right)$ for $\forall n \geq 1$. Since the $(a_n)_{n\geq 1}$ sequence is strictly increasing and bounded, on the basis of Weierstrasse theorem the sequence is convergent, that is it has a limit. This can be seen on the display of the calculator getting the number 3 after a number of operations. This means that $\lim_{n\to\infty} a_n = 3$, in the same way as before $x = \sqrt{6+x} \Leftrightarrow x^2 - x - 6 = 0$ which has a positive root of x= 3.

<u>Experiment 7</u>:

If we take the number 12 instead of 6 in the previous example, what can we say about the monotonicity, boundedness and limit of the sequence? How can we explain what we have seen?

Solution: The resulting sequence is $a_1 = \sqrt{12}$, $a_2 = \sqrt{12 + a_1} = \sqrt{12 + \sqrt{12}}$, $a_3 = \sqrt{12 + a_2} = \sqrt{12 + \sqrt{12 + \sqrt{12}}}$,and usually $a_{n+1} = \sqrt{12 + a_n} = \overbrace{\sqrt{12 + \sqrt{12 + \sqrt{12 + ... + \sqrt{12}}}}}^{(n+1)-root\ sign}$ for $\forall n \geq 1$. We are going prove in the same way as before that the sequence is strictly increasing. Therefore the best lower bound is the first member, namely $a_n \geq a_1 = \sqrt{12}$, for $\forall n \geq 1$. This time we can see the best upper bound on the display is 4. This can be easily proved by mathematical induction, namely $a_1 = \sqrt{12} < 4$ and if we suppose that $a_n < 4$ then $a_{n+1} = \sqrt{12 + a_n} < \sqrt{12 + 4} = 4$. Thus $a_n \in \left[\sqrt{12}, 4\right)$ for $\forall n \geq 1$. Because the sequence $(a_n)_{n \geq 1}$ is strictly increasing and bounded, the theorem of Weierstrasse says the sequence is convergent, so has a limit. On the display of the pocket calculator we see this, after executing a certain number of operations on the display we can see 4. This therefore means that $\lim_{n \to \infty} a_n = 4$, in the same way as it was shown before $x = \sqrt{12 + x} \Leftrightarrow x^2 - x - 12 = 0$ it has a positive root and that is $x = 4$.

<u>Remark 1</u>:
The reader probably ask himself what kind of number should we choose besides 2, 6 and 12 so that $\lim_{n \to \infty} a_n = k$ is a natural number. The answer to this question is not that difficult, because the preceding sequences the $a_1 = \sqrt{a}$, $a_{n+1} = \sqrt{a + a_n}$, $a > 0$ were defined recursively, and if we consider the limit here, then we can see that $k = \sqrt{a + k} \Leftrightarrow a = k^2 - k = (k-1)k$ result. Thus, if $a_1 = \sqrt{a}$ and $a \in \{1 \cdot 2;\ 2 \cdot 3;\ 3 \cdot 4; ...; (k-1)k; ...\}_{k \in N^* \setminus \{1\}}$ then the limits of the sequence in question are going to be 2, 3, 4, …, k, … . Similar experiments like the previous ones can be done.

<u>Experiment 8</u>: Let's type 1 into a pocket calculator and calculate its square root. Add 1 to the result and calculate its square root again. We shall repeat this operation until we get the same number on the display. What can we say about the monotonicity, boundedness and limit of the sequence? How can we explain what we have seen?
Solution: On the display of a pocket calculator which can only display 8 characters, we see the following: 1; 1.414135; 1.5537739, 1.5980531; 1.6118477; 1.6161212; 1.6174427; 1.6178512; 1.6179775; 1.6180165; 1.61802859; 1.6180323; 1.6180334; 1.6180338; **1.6180339; 1.6180339; 1.6180339;** …. This time for a "constant" number we didn't get an integer value, but the number 1.6180339, which is of course only a seven decimal approximation.

Our sequence this time is $a_1 = \sqrt{1}$, $a_2 = \sqrt{1 + a_1} = \sqrt{1 + \sqrt{1}} = \sqrt{2}$, $a_3 = \sqrt{1 + a_2} = \sqrt{1 + \sqrt{1 + \sqrt{1}}}$, and usually $a_{n+1} = \sqrt{1 + a_n} = \overbrace{\sqrt{1 + \sqrt{1 + \sqrt{1 + ... + \sqrt{1}}}}}^{(n+1)-root\ \mathrm{sign}}$ for $\forall n \geq 0$. According to the computed values the sequence is strictly increasing, which we can prove by mathematical induction similarly to previous cases with $a_{n+1}-a_n=\sqrt{1+a_n}-\sqrt{1+a_{n-1}}=\dfrac{a_n-a_{n-1}}{\sqrt{1+a_n}+\sqrt{1+a_{n-1}}}$ relation. Since the sequence is strictly increasing, the best lower bound is the first member, namely $a_n \geq a_1=1$, for $\forall n \geq 1$. Our guess regarding the upper bound is that the an approximate value of the upper bound is 1.6180339… If this proves correct, then the sequence would be monotonic and bounded, thus there would exist $\lim\limits_{n\to\infty} a_n = x$, which on the basis of $a_{n+1}=\sqrt{1+a_n}$ gives $x=\sqrt{1+x} \Leftrightarrow x^2-x-1=0$ which has a positive root of $x=\dfrac{1+\sqrt{5}}{2}=\mathbf{1.6180339}...$ agreeing with the calculator. Thus, the best upper bound of the sequence is the number $\dfrac{1+\sqrt{5}}{2}=\mathbf{1.6180339}...$ (this is also called the golden ratio), and this we can prove by induction, if we assume, that $a_n<\dfrac{1+\sqrt{5}}{2}$, then $a_{n+1}=\sqrt{1+a_n}<\sqrt{1+\dfrac{1+\sqrt{5}}{2}}=\dfrac{1+\sqrt{5}}{2}$.

<u>Remark 2</u>:
The preceding sequences $a_1=\sqrt{a}$, $a_{n+1}=\sqrt{a+a_n}$, *a> 0* were defined by a recursive relation. It can be easily proved, that this sequence is strictly increasing and bounded, more precisely $a_n \in \left[\sqrt{a}, \dfrac{1+\sqrt{4a+1}}{2}\right)$. This is why it is convergent and if $\lim\limits_{n\to\infty} a_n = x$, then this gives $x=\sqrt{a+x} \Leftrightarrow x^2-x-a=0$, which has a positive root $x=\dfrac{1+\sqrt{4a+1}}{2}$. Based on Remark 1, this limit will only be an integer number for $a\in N^*$ if $a=(k-1)k$ and $k\in N^*\setminus\{1\}$, in every other case it will be an irrational number.

<u>Exercise 9</u>: Let’s type the reciprocal of 1 into a pocket calculator and add 1 to it. To the reciprocal of the result let’s add again 1, and to the reciprocal of this result let’s add again 1. We shall repeat this operation until the number on the display dues not change. What can we say about the monotonicity, boundedness and limit of the sequence? How can we explain what we have seen?

Solution: On the display of a pocket calculator which can display only 8 characters, in turns, we can see the following: 2; 1.5000000; 1.6666666; 1.6000000; 1.6250000; 1.6153846; 1.6190476; 1.6176470; 1.6181818; 1.6179775; 1.6180555; 1.6180257; 1.6180371; 1.6180328; 1.6180344; 1.6180338; 1.6180340; **1.6180339; 1.6180339; 1.6180339;** …

We defined the sequence as: $a_1 = 1$, $a_2 = \frac{1}{a_1} + 1$, $a_3 = \frac{1}{a_2} + 1$, and in general $a_{n+1} = \frac{1}{a_n} + 1$, for $\forall n \geq 1$. If we follow the numbers on the display we can see, that the sequence is neither increasing nor decreasing, but $a_1 < a_3 < a_5 < ... < a_{2k-1} < ...$ and $a_2 > a_4 > a_6 > ... > a_{2k} > ...$i.e. the odd members of the sequence form a subsequence which is strictly increasing, and the other subsequence formed by the even members is strictly decreasing. We are going to prove this result. By the recursion $a_{n+2} - a_n = \left(\frac{1}{a_{n+1}} + 1\right) - \left(\frac{1}{a_{n-1}} + 1\right) = \frac{1}{a_{n+1}a_{n-1}a_n a_{n-2}}(a_n - a_{n-2})$ using mathematical induction. Regarding boundedness we can prove that $1 < a_{2k-1} < \frac{1+\sqrt{5}}{2} < a_{2k} < 2$. Therefore the sequence is bounded, thus under Weierstrasse's theorem it is convergent, and so has a limit. Let be $\lim_{n\to\infty} a_n = x$, so based on the $a_{n+1} = \frac{1}{a_n} + 1$ recursion $x = \frac{1}{x} + 1 \Leftrightarrow x^2 - x - 1 = 0$, which has a positive root $x = \frac{1+\sqrt{5}}{2} \approx 1.6180339\ldots$thr so-called "golden ratio". Which we got, on the display of the calculator, with a 7 decimal precession as a limit.

<u>Remark 3</u>:

As in the previous experiment, in the place of the $a_1 = 1$ we would put any arbitrary number. The sequence will still be defined by $a_{n+1} = \frac{1}{a_n} + a$ recursion, and according to what we have proved before, the same results can be derived with $\frac{1+\sqrt{5}}{2}$ replaced by $\frac{a+\sqrt{a^2+4}}{2}$.

<u>Remark 4</u>:

It is well known that we define the Fibonacci sequence as: $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for $\forall\ n \geq 1$. We can produce a sequence of the ratios of two successive members from the Fibonacci sequence, namely $\frac{f_{k+1}}{f_k} = a_k$ for $\forall k \geq 1$. Then $f_{n+2} = f_{n+1} + f_n \Leftrightarrow a_{n+1} = \frac{1}{a_n} + 1$ that is to say, that we have generated the sequence from the previous experiment. Based on that experiment, the sequence produced of the ratios of two successive members from the Fibonacci sequence is convergent and its limit is $\frac{1+\sqrt{5}}{2}$.

Finally we notice, that the convergence of numerous other sequences defined by first degree recursive relation can be studied with a pocket calculator. For the interested reader we recommend, a study of the convergence of the sequences defined by first degree recursive relation below with the help of the model shown before:

1) $a_1 = a \geq \frac{3}{4}$ and $a_{n+1} = \sqrt{4a_n - 3}$ for $\forall n \geq 1$.

2) $a_1 = a > 0$, $k > 0$, $a_{n+1} = \sqrt{a_n} + k$ for $\forall n \geq 1$.

3) $a_1 > 0$, $k > 0$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{k}{a_n}\right)$ for $\forall n \geq 1$.

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